AMM Dynamic Fee Strategy: Equation-Centric Technical Report

Consolidated from `Strategy.sol` and `yq-v2` analyses

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Contents

1 Purpose and Scope

This report gives a deep, equation-first explanation of the strategy implementation and proposed extensions, consolidating:

- `research/strategy-sol-math-report.md`
- `research/yq-v2-math-report.md`

It focuses on:

1. precise variable definitions,
2. exact update equations and control flow,
3. mathematical interpretation of each term,
4. ranked improvement suggestions with equation forms.

2 Notation and Core Simulator Math

2.1 Symbols

`Symbol`	Meaning
x, y	AMM reserves of token X and token Y
k	Constant-product invariant, $k = xy$
s	AMM spot price, $s = y/x$ (Y per X)
f	Fee (WAD-scaled in Solidity, decimal in simulator math)
γ	Fee multiplier, $\gamma = 1 - f$
p	External fair price from GBM
$\hat{p}$	Internal fair-price estimate (`pHat`)
$\hat{\sigma}$	Volatility estimate (`sigmaHat`)
$\hat{\lambda}$	Trade-arrival estimate (`lambdaHat`)
$\hat{q}$	Trade-size estimate (`sizeHat`)
$\hat{\tau}$	Toxicity EMA (`toxEma`, denoted toxSignal)
a	Activity EMA (`actEma`)
d	Direction state (`dirState`), centered at 1 (WAD)

2.2 Objective

The scoring objective can be viewed as:

$$\mathbb{E}[\text{Edge}] = \mathbb{E}[\text{Retail Edge}] - \mathbb{E}[\text{Arbitrage Loss}] \tag{1}$$

This creates a strict trade-off:

- higher fees reduce arbitrage intensity,
- but higher fees reduce routed retail flow.

2.3 AMM and fee-on-input model

$$k = xy, \qquad s = \frac{y}{x}, \qquad \gamma = 1 - f. \tag{2}$$

When fees are taken on input, only a fraction γ of input moves reserves.

2.4 Arbitrage optimal trade sizes

For fair price p, reserves (x, y):

$$\text{If } s < p: \quad \Delta x^{\star}_{\text{out}} = x - \sqrt{\frac{k}{\gamma p}}, \tag{3}$$

$$\text{If } s > p: \quad \Delta x^{\star}_{\text{in}} = \frac{\sqrt{\frac{k\gamma}{p}} - x}{\gamma}. \tag{4}$$

These equations explain why higher fees can create larger stale-price bands.

2.5 Retail routing split (two AMMs)

For buy-side total order size Y, AMM $i \in \{1, 2\}$:

$$A_i = \sqrt{x_i \gamma_i y_i}, \qquad r = \frac{A_1}{A_2}, \tag{5}$$

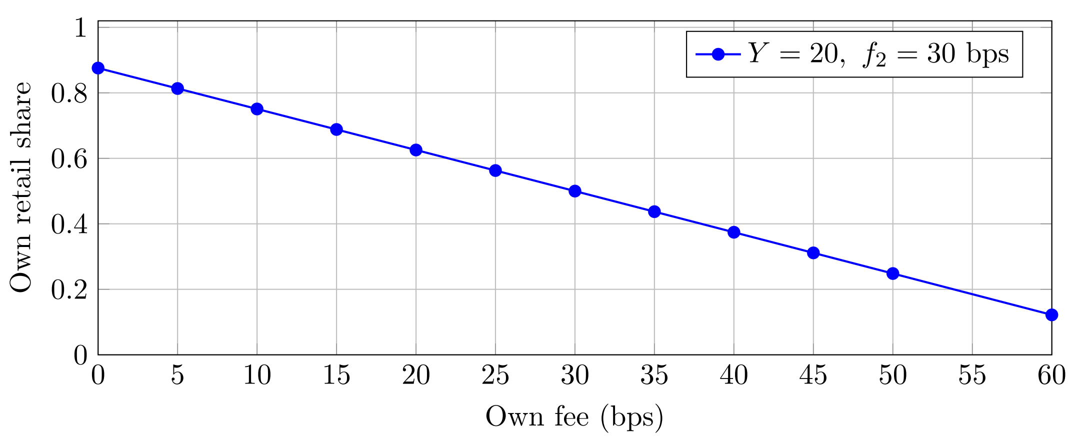
$$\Delta y_1 = \frac{r\,(y_2 + \gamma_2 Y) - y_1}{\gamma_1 + r\gamma_2}, \qquad \Delta y_2 = Y - \Delta y_1. \tag{6}$$

This split is nonlinear in fees; small fee changes can produce large flow-share changes.

2.6 Plot: flow share vs own fee

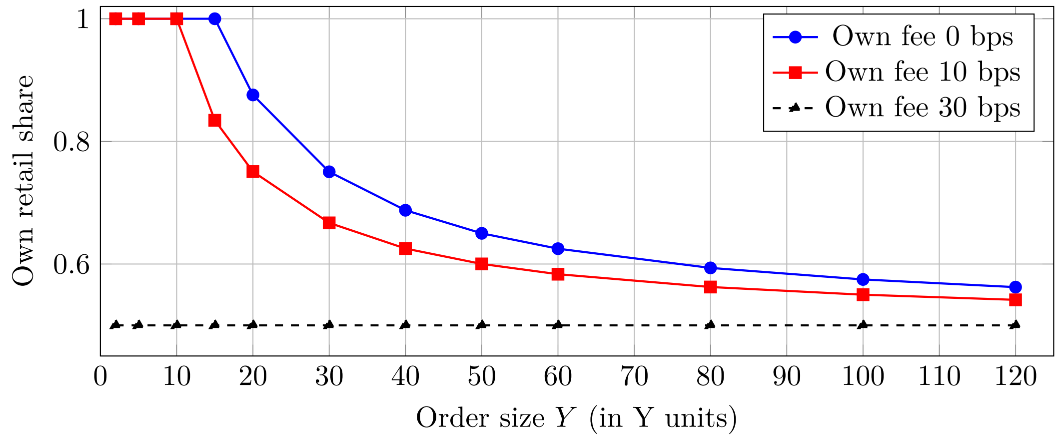
Assumptions for this plot:

- equal starting reserves ($x_1 = x_2 = 100$, $y_1 = y_2 = 10000$),
- competitor fixed at 30 bps,
- order size $Y = 20$ (near mean retail size).



Interpretation: even moving from 0 to 10 bps has a large share impact for typical trade sizes.

2.7 Plot: share vs order size (cannot always capture 100%)



Interpretation: with equal reserves and a 30 bps competitor, setting fee below 30 bps improves share but does not guarantee full capture for larger orders.

3 Deep Equation Walkthrough of `Strategy.sol`

3.1 State vector and initialization

Define the latent-state vector:

$$\mathbf{z}_t = \left[d_t,\ a_t,\ \hat{p}_t,\ \hat{\sigma}_t,\ \hat{\lambda}_t,\ \hat{q}_t,\ \hat{\tau}_t,\ n_t\right], \tag{7}$$

where n_t is within-step trade count.

Initialization sets:

$$d_0 = 1, \tag{8}$$

$$\hat{p}_0 = \frac{y_0}{x_0}, \tag{9}$$

$$\hat{\sigma}_0, \hat{\lambda}_0, \hat{q}_0 > 0 \text{ (nonzero priors for stability).} \tag{10}$$

3.2 Step-boundary decay and arrival update

Let Δt be elapsed steps since last observed timestamp. State decays are:

$$d_t \leftarrow 1 + (d_t - 1) \cdot \delta_d^{\Delta t} \quad \text{(or symmetric form below 1)}, \tag{11}$$

$$a_t \leftarrow a_t \cdot \delta_a^{\Delta t}, \tag{12}$$

$$\hat{q}_t \leftarrow \hat{q}_t \cdot \delta_q^{\Delta t}, \tag{13}$$

$$\hat{\tau}_t \leftarrow \hat{\tau}_t \cdot \delta_\tau^{\Delta t}. \tag{14}$$

Arrival-rate update from step count:

$$\lambda_{\text{inst}} = \min\left(\frac{n_{t-1}}{\Delta t},\ \lambda_{\max}\right), \tag{15}$$

$$\hat{\lambda}_t \leftarrow \rho_\lambda \hat{\lambda}_t + (1 - \rho_\lambda)\lambda_{\text{inst}}. \tag{16}$$

3.3 First-in-step indicator

$$\text{isNewStep} = \mathbf{1}\{timestamp_t > timestamp_{t-1}\}, \tag{17}$$

$$\text{firstInStep} = \mathbf{1}\{n_t = 0\} \text{ after new-step reset.} \tag{18}$$

This indicator is central for selective transition logic.

3.4 Implied price from executed trade side

Using previous quoted fee f_{used}:

$$\gamma_{\text{used}} = 1 - f_{\text{used}}, \tag{19}$$

$$p_t^{\text{impl}} = \begin{cases} s_t \gamma_{\text{used}}, & \text{if AMM buys X,} \\ s_t / \gamma_{\text{used}}, & \text{if AMM sells X.} \end{cases} \tag{20}$$

Relative deviation:

$$r_t = \frac{|p_t^{\text{impl}} - \hat{p}_t|}{\hat{p}_t}. \tag{21}$$

3.5 Adaptive gate and state updates

Shock gate:

$$g_t = \max\left(c_g \hat{\sigma}_t,\ g_{\min}\right). \tag{22}$$

Only if $r_t \le g_t$, update $\hat{p}$:

$$\hat{p}_t \leftarrow (1 - \alpha_t)\hat{p}_t + \alpha_t p_t^{\text{impl}}, \quad \alpha_t = \begin{cases} \alpha_{\text{step}}, & \text{firstInStep,} \\ \alpha_{\text{retail}}, & \text{otherwise.} \end{cases} \tag{23}$$

First-trade sigma update:

$$\hat{\sigma}_t \leftarrow \rho_\sigma \hat{\sigma}_t + (1 - \rho_\sigma) \min(r_t, r_{\max}). \tag{24}$$

3.6 Flow, direction, activity, and size

Trade intensity proxy:

$$q_t = \min\left(\frac{\text{amountY}_t}{\text{reserveY}_t},\ q_{\max}\right). \tag{25}$$

If q_t exceeds signal threshold:

$$\Delta d_t = \min(c_d q_t,\ \Delta d_{\max}), \tag{26}$$

$$d_t \leftarrow \begin{cases} \min(2,\ d_t + \Delta d_t), & \text{if AMM buys X,} \\ \max(0,\ d_t - \Delta d_t), & \text{otherwise,} \end{cases} \tag{27}$$

$$a_t \leftarrow \rho_a a_t + (1 - \rho_a) q_t, \tag{28}$$

$$\hat{q}_t \leftarrow \min(1,\ \rho_q \hat{q}_t + (1 - \rho_q) q_t). \tag{29}$$

3.7 Toxicity

$$\tau_t^{\text{raw}} = \frac{|s_t - \hat{p}_t|}{\hat{p}_t}, \qquad \tau_t = \min(\tau_t^{\text{raw}}, \tau_{\max}), \tag{30}$$

$$\hat{\tau}_t \leftarrow \rho_{\tau b} \hat{\tau}_t + (1 - \rho_{\tau b}) \tau_t. \tag{31}$$

3.8 Mid-fee decomposition

Flow-size interaction:

$$\phi_t = \hat{\lambda}_t \hat{q}_t. \tag{32}$$

Base:

$$f_t^{\text{base}} = f_0 + c_\sigma \hat{\sigma}_t + c_\lambda \hat{\lambda}_t + c_\phi \phi_t. \tag{33}$$

Symmetric widening:

$$f_t^{\text{mid}} = f_t^{\text{base}} + c_1 \hat{\tau}_t + c_2 \hat{\tau}_t^2 + c_3 \hat{\tau}_t^3 + c_a a_t + c_{\sigma\tau} \hat{\sigma}_t \hat{\tau}_t. \tag{34}$$

3.9 Directional and stale-sign asymmetry

Direction deviation:

$$\delta_t = |d_t - 1|. \tag{35}$$

Skew magnitude:

$$\kappa_t = c_\delta \delta_t + c_{\delta\tau} \delta_t \hat{\tau}_t. \tag{36}$$

Directional split:

$$(f_t^{bid}, f_t^{ask}) = \begin{cases} (f_t^{mid} + \kappa_t,\ \max(f_t^{mid} - \kappa_t, 0)), & d_t \geq 1, \\ (\max(f_t^{mid} - \kappa_t, 0),\ f_t^{mid} + \kappa_t), & d_t < 1. \end{cases} \tag{37}$$

Stale-sign shift (protect side up, attract side down):

$$\Delta_s = c_s \hat{\tau}_t, \qquad \Delta_a = \eta \Delta_s, \quad \eta > 1. \tag{38}$$

3.10 Trade-aligned toxicity boost

If trade side aligns with stale sign, add:

$$\Delta_{\text{trade}} = c_{\text{trade}}\ q_t \tag{39}$$

to the likely-protect side.

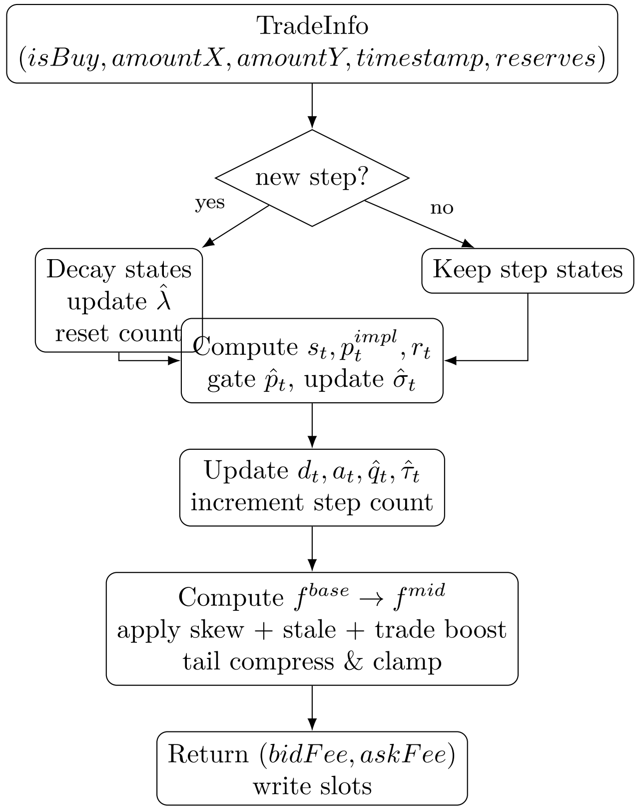
3.11 Tail compression

For knee k_f and slope $\beta \in (0, 1)$:

$$\mathcal{C}(f; \beta) = \begin{cases} f, & f \leq k_f, \\ k_f + \beta(f - k_f), & f > k_f. \end{cases} \tag{40}$$

This compresses high-end fee tails while preserving monotonicity.

3.12 Control-flow diagram



4 yq-v2 Extension: Sigma Momentum

‘yq-v2‘ adds previous-step sigma memory and a momentum term:

$$\Delta\hat{\sigma}_t = \max(\hat{\sigma}_t - \hat{\sigma}_{t-1}, 0), \tag{41}$$

$$f_t^{mid} \leftarrow f_t^{mid} + c_m \Delta\hat{\sigma}_t, \qquad c_m = 0.20. \tag{42}$$

Interpretation:

- catches volatility acceleration before the sigma EMA alone fully rises,
- should be small and selective to avoid retail-flow damage.

5 Consolidated Suggestions with Ratings

5.1 Rating scale

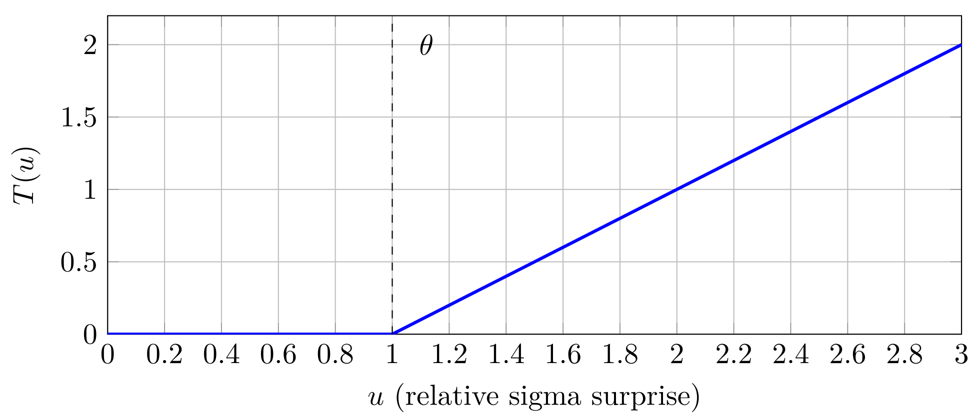
- 10: strongest mathematical fit for this simulator and architecture.
- 1: weak fit or structurally misaligned.

Suggestion	Equation form	Score	Core rationale
One-shot transition boost	$1_{\text{firstInStep}} \cdot c \cdot [\Delta\hat{\sigma}]_+$	9/10	avoids repeated same-step over-taxing
Relative sigma surprise	$c \cdot \left[\frac{[\Delta\hat{\sigma}]_+}{\max(\hat{\sigma}_{t-1},\epsilon)} - \theta\right]_+$	8/10	scale-invariant trigger
Toxicity-gated transition	$c \cdot trigger \cdot \min(1, \hat{\tau}/\tau^\star)$	8/10	protect only when stale risk is high
Calm attract anchor	if $\hat{\tau} < \tau_{low}$ and $a < a_{low}$ then $f_{attract} \le f_N + b$	8/10	preserve routing share
Regime + hysteresis	switch by T_{on}, T_{off} with $T_{off} < T_{on}$	7/10	stable mode switching
Reserve asymmetric penalty	signed inventory term with side-selective penalty	6/10	plausible but tune-sensitive
Intra-step escalation	repeated within-step widening	2/10	usually taxes retail too much

5.2 Plot: relative-sigma surprise trigger

Define:

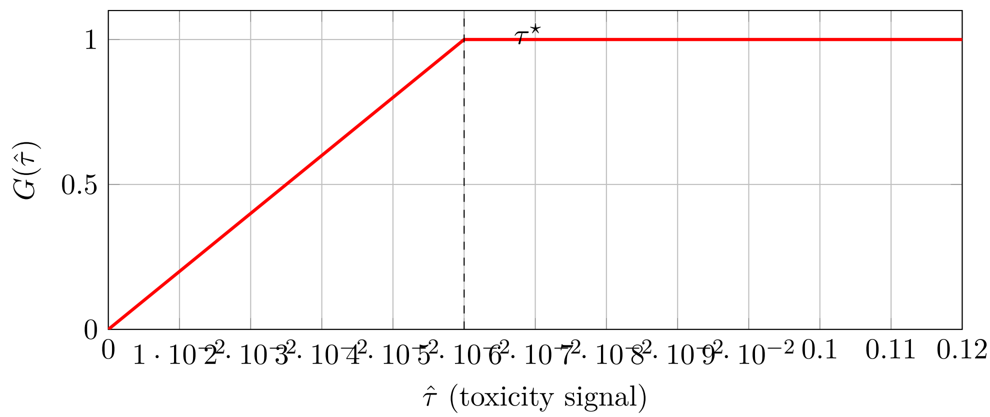
$$u = \frac{[\Delta\hat{\sigma}]_+}{\max(\hat{\sigma}_{t-1},\epsilon)}, \qquad T(u) = \max(u-\theta, 0). \tag{43}$$



5.3 Plot: toxicity gate

Define:

$$G(\hat{\tau}) = \min\left(1, \frac{\hat{\tau}}{\tau^\star}\right). \tag{44}$$

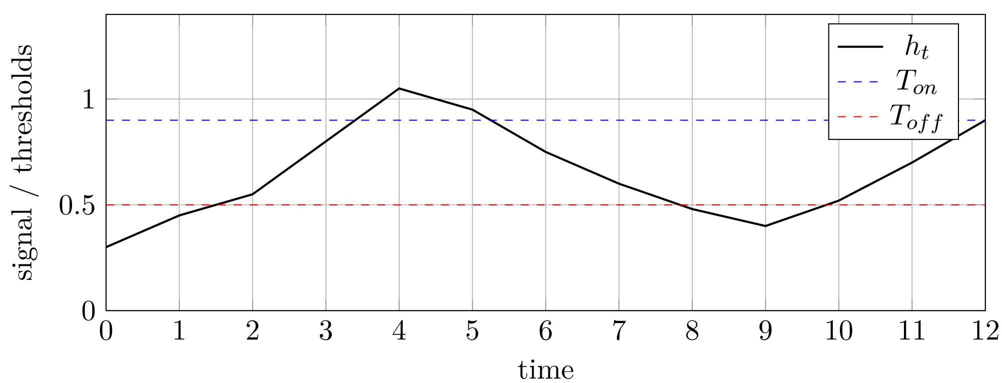


5.4 Hysteresis concept

With signal h_t:

$$\text{mode}_{t+1} = \begin{cases} \text{PROTECT}, & \text{if mode}_t = \text{CALM and } h_t \geq T_{on}, \\ \text{CALM}, & \text{if mode}_t = \text{PROTECT and } h_t \leq T_{off}, \\ \text{mode}_t, & \text{otherwise}, \end{cases} \tag{45}$$

with $T_{off} < T_{on}$.



6 Recommended Next Formula (yq-v3 candidate)

A compact, selective extension:

$$\Delta\hat{\sigma}_t = [\hat{\sigma}_t - \hat{\sigma}_{t-1}]_+, \quad u_t = \frac{\Delta\hat{\sigma}_t}{\max(\hat{\sigma}_{t-1}, \epsilon)}, \quad T_t = [u_t - \theta]_+, \quad G_t = \min\left(1, \frac{\hat{\tau}_t}{\tau^{\star}}\right), \tag{46}$$

$$\Delta f_t^{protect} = 1_{\text{firstInStep}} \cdot c_m \cdot T_t \cdot G_t. \tag{47}$$

Apply $\Delta f_t^{protect}$ primarily to the protect side, while retaining an attract-side calm cap:

$$\text{if } \hat{\tau}_t < \tau_{low} \text{ and } a_t < a_{low}, \text{ then } f_{attract} \leq f_N + b. \tag{48}$$

This is mathematically consistent with both reports:

- maximize selectivity in time (first-in-step),
- maximize selectivity in regime (relative surprise and toxicity gate),
- preserve retail competitiveness in calm states.

7 Equation-to-Variable Mapping

`Code variable`	Mathematical interpretation
`pHat`	$\hat{p}_t$: internal fair-price estimator
`sigmaHat`	$\hat{\sigma}_t$: volatility estimate from first-in-step returns
`lambdaHat`	$\hat{\lambda}_t$: estimated trade arrivals per step
`sizeHat`	$\hat{q}_t$: normalized trade-size state
`toxEma`	$\hat{\tau}_t$: stale-price toxicity signal
`actEma`	a_t: recent activity signal
`dirState`	d_t: directional pressure around neutral center 1
`stepTradeCount`	n_t: number of trades observed in current timestamp
`fBase`	base fee component from vol/arrival/flow-size
`fMid`	symmetric widened fee before bid/ask asymmetry
`skew`	direction and direction-toxicity asymmetry size

8 Final Takeaway

- `Strategy.sol` is already a strong near-frontier structure.
- The dominant failure mode is over-widening in low-risk states.
- Best next progress comes from **selective transition protection**, not globally higher fees.