EE5609: Matrix Theory Assignment-17

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***Abstract*—This document contains a proof for linear operators.**

Download the latex-tikz codes from

https://github.com/pranaya14014/EE5609/tree/master/Assignment17

1 Problem

Let $E_1, ..., E_k$ be linear operators on the space $\mathbf{V}$ such that $E_1 + ... + E_k = \mathbf{I}$.
a) Prove that if $E_iE_j = 0$ for $i \neq j$,then $E_i^2 = E_i$ for each i.
b) In the case $k = 2$, prove the converse of (a). That is, if $E_1 + E_2 = \mathbf{I}$ and $E_1^2 = E_1$, $E_2^2 = E_2$, then $E_1E_2 = 0$

2 Solution

a) From the given,

$$E_1 + ... + E_k = \mathbf{I} \tag{2.0.1}$$

$$E_i = \mathbf{I} - E_1 - ... - E_{i-1} - E_{i+1} - ... - E_k \tag{2.0.2}$$

$$E_i^2 = E_i\left(\mathbf{I} - E_1 - ... - E_{i-1} - E_{i+1} - ... - E_k\right) \tag{2.0.3}$$

$$E_i^2 = E_i - E_iE_1.. - E_iE_{i-1} - E_iE_{i+1} - .. - E_iE_k \tag{2.0.4}$$

$$\implies E_i^2 = E_i - \sum_{i\neq j} E_iE_j \tag{2.0.5}$$

substituting $E_iE_j = 0$ for $i \neq j$ in the above equation we get,

$$E_i^2 = E_i - 0 \tag{2.0.6}$$

$$\implies E_i^2 = E_i \tag{2.0.7}$$

Hence proved if $E_iE_j = 0$ for $i \neq j$, then $E_i^2 = E_i$

b) Using,

$$E_1 + E_2 = \mathbf{I} \tag{2.0.8}$$

Multiplying both sides by E_1,

$$E_1(E_1 + E_2) = E_1 \tag{2.0.9}$$

$$\implies E_1^2 + E_1E_2 = E_1 \tag{2.0.10}$$

Substituting $E_1^2 = E_1$ in (2.0.8) we get,

$$\implies E_1 + E_1E_2 = E_1 \tag{2.0.11}$$

$$\implies E_1E_2 = 0 \tag{2.0.12}$$

Similarly multiplying on both sides of (2.0.8)and substituting $E_2^2 = E_2$ we get,

$$\implies E_1E_2 + E_2^2 = E_2 \tag{2.0.13}$$

$$\implies E_1E_2 + E_2 = E_2 \tag{2.0.14}$$

$$\implies E_1E_2 = 0 \tag{2.0.15}$$

Hence proved from (2.0.12) and (2.0.15) that if $E_1 + E_2 = \mathbf{I}$ and $E_1^2 = E_1$, $E_2^2 = E_2$, then $E_1E_2 = 0$