EE5609: Matrix Theory
Assignment-8

Y.Pranaya
AI20MTECH14014

***Abstract*—This document contains a proof to show that upper-triangular matrix is invertible if and only if diagonal elements are not 0.**

Download the latex-tikz codes from

https://github.com/pranaya14014/EE5609/tree/master/Assignment8

1 Problem

An $n \times n$ matrix $\mathbf{A}$ is called upper-triangular if $\mathbf{A_{ij}} = 0$ for $i > j$, that is, if every entry below the main diagonal is 0. Prove that an upper-triangular (square) matrix is invertible if and only if every entry on its main diagonal is different from 0.

2 Explanation

If inverse of $\mathbf{A}$ exists then the linear system $\mathbf{Ax} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$

3 Solution

Considering $\mathbf{A}$ and $\mathbf{b}$,

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & ... & a_{1,(n-1)} & a_{1,n} \\ 0 & a_{2,2} & ... & a_{2,(n-2)} & a_{2,n} \\ 0 & 0 & ... & & . \\ . & & ... & & . \\ 0 & 0 & ... & a_{(n-1),(n-1)} & a_{(n-1),(n)} \\ 0 & 0 & ... & 0 & a_{n,n} \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_{n-1} \\ b_n \end{pmatrix} \tag{3.0.1}$$

hence $\mathbf{x}$ can be found using augmented matrix

$$\mathbf{x} = \mathbf{A|b} = \begin{pmatrix} a_{1,1} & a_{1,2} & ... & a_{1,(n-1)} & a_{1,n} & b_1 \\ 0 & a_{2,2} & ... & a_{2,(n-2)} & a_{2,n} & b_2 \\ 0 & 0 & ... & & . & . \\ . & & ... & & . & . \\ 0 & 0 & ... & a_{(n-1),(n-1)} & a_{(n-1),(n)} & b_{n-1} \\ 0 & 0 & ... & 0 & a_{n,n} & b_n \end{pmatrix} \tag{3.0.2}$$

$$\xleftrightarrow{R_n \rightarrow \frac{R_n}{a_{n,n}}} \begin{pmatrix} a_{1,1} & a_{1,2} & ... & a_{1,(n-1)} & a_{1,n} & b_1 \\ 0 & a_{2,2} & ... & a_{2,(n-2)} & a_{2,n} & b_2 \\ 0 & 0 & ... & & . & . \\ . & & ... & & . & . \\ 0 & 0 & ... & a_{(n-1),(n-1)} & a_{(n-1),(n)} & b_{n-1} \\ 0 & 0 & ... & 0 & 1 & \frac{b_n}{a_{n,n}} \end{pmatrix} \tag{3.0.3}$$

$$\xleftrightarrow{R_{n-1} \rightarrow \frac{1}{a_{(n-1),(n-1)}}\left(R(n-1) - a_{(n-1),(n)}R_n\right)}$$

$$\begin{pmatrix} a_{1,1} & a_{1,2} & ... & a_{1,(n-1)} & a_{1,n} & b_1 \\ 0 & a_{2,2} & ... & a_{2,(n-2)} & a_{2,n} & b_2 \\ 0 & 0 & ... & & . & . \\ . & & ... & & . & . \\ 0 & 0 & ... & 1 & 0 & \frac{b_{n-1}}{a_{(n-1),(n-1)}} - \frac{a_{(n-1),(n)}b_n}{a_{n,n}} \\ 0 & 0 & ... & 0 & 1 & \frac{b_n}{a_{n,n}} \end{pmatrix} \tag{3.0.4}$$

From (3.0.4) we get,

$$\mathbf{x_n} = \frac{b_n}{a_{n,n}} \tag{3.0.5}$$

$$\mathbf{x_{n-1}} = \frac{b_{n-1}}{a_{(n-1),(n-1)}} - \frac{a_{(n-1),(n)}b_n}{a_{n,n}} \tag{3.0.6}$$

Using (3.0.5) in (3.0.6) we get,

$$\mathbf{x_{n-1}} = \frac{b_{n-1} - a_{(n-1),(n)}\mathbf{x_{(n-1)}}}{a_{(n-1),(n-1)}} \tag{3.0.7}$$

similary if we continue row reduction for i^{th} row we get ,

$$\mathbf{x_i} = \frac{b_i - \sum_{j=i+1}^{n} \mathbf{x_i} a_{i,j}}{a_{i,i}} \tag{3.0.8}$$

Using (3.0.5), $\mathbf{x}$ will have a unique solution for any $\mathbf{b}$ if and only if $a_{n,n} \neq 0$ as if it is 0 then $\mathbf{x}$ will have either no solution or infinitely many solution. Similarly if we consider any i^{th} row using (3.0.8) we can say $a_{i,i} \neq 0$.
Hence proved