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EE5609: Matrix Theory Assignment-8

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Abstract—This document contains a proof to show that upper-triangular matrix is invertible if and only if diagonal elements are not 0.

Download the latex-tikz codes from

https://github.com/pranaya14014/EE5609/tree/master/Assignment8

1 PROBLEM

An $n \times n$ matrix **A** is called upper-triangular if $\mathbf{A_{ij}} = 0$ for i > j, that is, if every entry below the main diagonal is 0. Prove that an upper-triangular (square) matrix is invertible if and only if every entry on its main diagonal is different from 0.

2 Properties of Determinants

Properties required for the proof:

- 1. If **A** is invertible then $|\mathbf{A}| \neq 0$
- 2. If the entries of a row(or column) in a square matrix are multiplied by a number k, then the determinant of resulting matrix is $k|\mathbf{A}|$.
- 3. |I| = 1
- 4. The determinant value remains same if we do any operation like $R_i \rightarrow R_i + kR_j$ or $C_i \rightarrow C_i + kC_j$

3 SOLUTION

Considering A, an upper triangular matrix and proceeding futher with properties of determinants

$$|\mathbf{A}| = \begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,(n-1)} & a_{1,n} \\ 0 & a_{2,2} & \dots & a_{2,(n-2)} & a_{2,n} \\ 0 & 0 & \dots & & \ddots \\ & & \dots & & \ddots \\ 0 & 0 & \dots & a_{(n-1),(n-1)} & a_{(n-1),(n)} \\ 0 & 0 & \dots & 0 & a_{n,n} \end{vmatrix}$$
(3.0.1)

$$\xrightarrow{R_{n-1} \to R_{n-1} - a_{(n-1),(n-1)}R_n} \begin{vmatrix}
a_{1,1} & a_{1,2} & \dots & a_{1,(n-1)} & a_{1,n} \\
0 & a_{2,2} & \dots & a_{2,(n-2)} & a_{2,n} \\
0 & 0 & \dots & & & \\
\vdots & & \dots & & & \\
0 & 0 & \dots & a_{(n-1),(n-1)} & 0 \\
0 & 0 & \dots & 0 & a_{n,n}
\end{vmatrix}$$
(3.0.2)

Similarly performing the row operations on rest of the rows we get a diagonal matrix which is,

$$|\mathbf{A}| = \begin{vmatrix} a_{1,1} & 0 & \dots & 0 & 0 \\ 0 & a_{2,2} & \dots & 0 & 0 \\ 0 & 0 & \dots & & & \\ & & \dots & & & \\ 0 & 0 & \dots & a_{(n-1),(n-1)} & 0 \\ 0 & 0 & \dots & 0 & a_{n,n} \end{vmatrix}$$
(3.0.3)

$$= a_{1,1} \begin{vmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & a_{2,2} & \dots & 0 & 0 \\ 0 & 0 & \dots & & & \\ & & \dots & & & \\ 0 & 0 & \dots & a_{(n-1),(n-1)} & 0 \\ 0 & 0 & \dots & 0 & a_{n,n} \end{vmatrix}$$
(3.0.4)

$$= a_{1,1}a_{2,2}\begin{vmatrix} 1 & 0 & \dots & 0 & 0\\ 0 & 1 & \dots & 0 & 0\\ 0 & 0 & \dots & & & \\ & & \dots & & & \\ 0 & 0 & \dots & a_{(n-1),(n-1)} & 0\\ 0 & 0 & \dots & 0 & a_{n-1} \end{vmatrix}$$
(3.0.5)

Similarly we get,

$$|\mathbf{A}| = a_{1,1}a_{2,2}...a_{(n-1),(n-1)}a_{n,n}\begin{vmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{vmatrix}$$

$$= a_{1,1}a_{2,2}...a_{(n-1),(n-1)}a_{n,n}|\mathbf{I_{n,n}}|$$

$$(3.0.7)$$

$$|\mathbf{A}| = a_{1,1}a_{2,2}...a_{(n-1),(n-1)}a_{n,n}$$
 (3.0.8)

From (3.0.8) we see that $|\mathbf{A}|$ is the product of diagonal elements. If anyone of the diagonal element is zero then $|\mathbf{A}| = 0$ which implies \mathbf{A} is not invertible. So, a matrix is invertible if and only if diagonal elements are not 0.

Hence Proved.