EE5609: Matrix Theory Assignment-7

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***Abstract*—This document contains a solution to find the foot of a perpendicular from a point on the plane using Singular Value Decomposition (SVD).**

Download the python codes from

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https://github.com/pranaya14014/EE5609/tree/master/Assignment7/code
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and latex-tikz codes from

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https://github.com/pranaya14014/EE5609/blob/master/Assignment7
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1 PROBLEM

Find the foot of the perpendicular from $\begin{pmatrix}-5\\1\\3\end{pmatrix}$ on the plane $\begin{pmatrix}2 & -3 & 1\end{pmatrix}\mathbf{x} = 0$

2 SOLUTION

Let orthogonal vectors be $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix}a\\b\\c\end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{2.0.1}$$

$$\begin{pmatrix}a & b & c\end{pmatrix}\begin{pmatrix}2\\-3\\1\end{pmatrix} = 0 \tag{2.0.2}$$

$$\implies -5a + b + 3c = 0 \tag{2.0.3}$$

Let a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix}1\\0\\-2\end{pmatrix} \tag{2.0.4}$$

Let a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix}0\\1\\3\end{pmatrix} \tag{2.0.5}$$

From (2.0.4) and (2.0.5),

$$\mathbf{M} = \begin{pmatrix}1 & 0\\0 & 1\\-2 & 3\end{pmatrix} \tag{2.0.6}$$

Now solving the equation

$$\mathbf{Mx} = \mathbf{b} \tag{2.0.7}$$

Substituting the given point and (2.0.6) in (2.0.7)

$$\begin{pmatrix}1 & 0\\0 & 1\\-2 & 3\end{pmatrix}\mathbf{x} = \begin{pmatrix}-5\\1\\3\end{pmatrix} \tag{2.0.8}$$

Using the Singular value decomposition to solve (2.0.8) as follows,

$$\mathbf{M} = \mathbf{U\Sigma V}^T \tag{2.0.9}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{\Sigma}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}5 & -6\\-6 & 10\end{pmatrix} \tag{2.0.10}$$

$$\mathbf{MM}^T = \begin{pmatrix}1 & 0 & -2\\0 & 1 & 3\\-2 & 3 & 13\end{pmatrix} \tag{2.0.11}$$

Substituting (2.0.9) in (2.0.7)

$$\mathbf{U\Sigma V}^T\mathbf{x} = \mathbf{b} \tag{2.0.12}$$

$$\mathbf{x} = \mathbf{V\Sigma^{-1}U^T b} \tag{2.0.13}$$

where $\mathbf{\Sigma^{-1}}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{\Sigma}$. Now finding the eigen values of $\mathbf{MM}^T$

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{2.0.14}$$

$$\begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 3 \\ -2 & 3 & 13-\lambda \end{vmatrix} = 0 \tag{2.0.15}$$

$$\implies \lambda^3 - 15\lambda^2 + 14\lambda = 0 \tag{2.0.16}$$

Hence eigen values of $\mathbf{M}\mathbf{M}^T$,

$$\lambda_1 = 1 \quad \lambda_2 = 14 \quad \lambda_3 = 0 \tag{2.0.17}$$

Therefore eigen vectors of $\mathbf{M}\mathbf{M}^T$,

$$\mathbf{u_1} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \tag{2.0.18}$$

Normalizing the eigen vectors,

$$\mathbf{u_1} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad \mathbf{u_2} = \begin{pmatrix} \frac{-2}{\sqrt{182}} \\ \frac{3}{\sqrt{182}} \\ \frac{13}{\sqrt{182}} \end{pmatrix} \quad \mathbf{u_3} = \begin{pmatrix} \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix} \tag{2.0.19}$$

Hence from the above we get,

$$\mathbf{U} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{182}} & \frac{2}{\sqrt{14}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{182}} & \frac{-3}{\sqrt{14}} \\ 0 & \frac{13}{\sqrt{182}} & \frac{1}{\sqrt{14}} \end{pmatrix} \tag{2.0.20}$$

By computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{\Sigma}$ as,

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 14 \\ 0 & 0 \end{pmatrix} \tag{2.0.21}$$

Now calculating eigen values of $\mathbf{M}^T\mathbf{M}$

$$\left|\mathbf{M}^T\mathbf{M} - \lambda I\right| = 0 \tag{2.0.22}$$

$$\begin{vmatrix} 5-\lambda & -6 \\ -6 & 10-\lambda \end{vmatrix} = 0 \tag{2.0.23}$$

$$\implies \lambda^2 - 15\lambda + 14 = 0 \tag{2.0.24}$$

hence the eigen values of $\mathbf{M}^T\mathbf{M}$

$$\lambda_1 = 1 \quad \lambda_2 = 14 \tag{2.0.25}$$

Therefore eigen vectors $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v_1} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad \mathbf{v_2} = \begin{pmatrix} \frac{-2}{3} \\ 1 \end{pmatrix} \tag{2.0.26}$$

Normalizing the eigen vectors,

$$\mathbf{v_1} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} \quad \mathbf{v_2} = \begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix} \tag{2.0.27}$$

Hence $\mathbf{V}$ is given as,

$$\mathbf{V} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \tag{2.0.28}$$

Moore Pseudo inverse of Σ is,

$$\mathbf{\Sigma^{-1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{14}} & 0 \end{pmatrix} \tag{2.0.29}$$

Substituting (2.0.20), (2.0.28) and (2.0.29) in (2.0.13),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 0 \\ \frac{-2}{\sqrt{182}} & \frac{3}{\sqrt{182}} & \frac{13}{\sqrt{182}} \\ \frac{2}{\sqrt{14}} & \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{52}{\sqrt{182}} \\ \frac{-10}{\sqrt{1}} \end{pmatrix} \tag{2.0.30}$$

$$\mathbf{\Sigma^{-1}}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{14}} & 0 \end{pmatrix} \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{52}{\sqrt{182}} \\ \frac{-10}{\sqrt{14}} \end{pmatrix} = \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{26}{7\sqrt{13}} \end{pmatrix} \tag{2.0.31}$$

$$\mathbf{V}\mathbf{\Sigma^{-1}}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} \frac{-13}{\sqrt{13}} \\ \frac{26}{7\sqrt{13}} \end{pmatrix} = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix} \tag{2.0.32}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix} \tag{2.0.33}$$

Now verifying (2.0.33) using (2.0.7)

$$\mathbf{M}\mathbf{x} = \mathbf{b} \implies \mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{2.0.34}$$

Substituting (2.0.6), (2.0.10) and given point in (2.0.34)

$$\begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -11 \\ 10 \end{pmatrix} \tag{2.0.35}$$

$$\tag{2.0.36}$$

Solving the augmented matrix.

$$\begin{pmatrix} 5 & -6 & -11 \\ -6 & 10 & 10 \end{pmatrix} \xleftrightarrow{R_1 = \frac{R_1}{5}} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ -6 & 10 & 10 \end{pmatrix} \tag{2.0.37}$$

$$\xleftrightarrow{R_2 = R_2 + 6R_1} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ 0 & \frac{14}{5} & \frac{-16}{5} \end{pmatrix} \tag{2.0.38}$$

$$\xleftrightarrow{R_2 = \frac{5R_2}{14}} \begin{pmatrix} 1 & \frac{-6}{5} & \frac{-11}{5} \\ 0 & 1 & \frac{-8}{7} \end{pmatrix} \tag{2.0.39}$$

$$\xleftrightarrow{R_1 = R_1 + \frac{6R_2}{5}} \begin{pmatrix} 1 & 0 & \frac{-25}{7} \\ 0 & 1 & \frac{-8}{7} \end{pmatrix} \tag{2.0.40}$$

From (2.0.40) we get,

$$\mathbf{x} = \begin{pmatrix} \frac{-25}{7} \\ \frac{-8}{7} \end{pmatrix} \tag{2.0.41}$$

Hence from (2.0.33) and (2.0.41) the $\mathbf{x}$ is verified