

EE5609: Matrix Theory

Assignment-19

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Abstract

This document contains a solution for a quadratic form of given matrix.

Download the latex-tikz codes from

<https://github.com/pranaya14014/EE5609/blob/master/Assignment19>

1 PROBLEM

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} \quad (1)$$

and define for $x, y, z \in \mathbb{R}$

$$\mathbf{Q}(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (2)$$

Which of the following are True?

1. The matrix of second order partial derivatives of the quadratic form of \mathbf{Q} is $2\mathbf{A}$.
2. The rank of the quadratic form of \mathbf{Q} is 2
3. The signature of the quadratic form \mathbf{Q} is $(+, +, 0)$
4. The quadratic form \mathbf{Q} takes the value 0 for some non-zero vector (x, y, z)

2 DEFINITIONS

Quadratic Form of a matrix	Let \mathbf{V} be a vector space over \mathbb{R} . \mathbf{A} be a symmetric matrix $n \times n$. Quadratic form on \mathbf{V} is a real function, $(\mathbf{F} : \mathbf{V} \rightarrow \mathbb{R})$ defined as $F(\mathbf{x}) = \mathbf{x}\mathbf{A}\mathbf{x}^T = \sum_{i,j=1}^n a_{ij}x_i x_j$ where $\mathbf{x} \in \mathbf{V}$
Signature of Quadratic form	The signature of quadratic form is (n_+, n_-, n_0) where n_+ is the number of positive entries, n_- is number of negative entries and n_0 is number of zero's in a_{ii}
Rank of quadratic form	Rank of quadratic form is the rank of its matrix which is maximum number of linearly independent rows/columns of a matrix

TABLE 1: Definitions

3 SOLUTION

Option 1	The matrix of second order partial derivatives of the quadratic form of Q is $2\mathbf{A}$.
Solution	$\mathbf{Q}(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} x+2y \\ -2z \\ z \end{pmatrix} = x^2 + z^2 + 2xy - 2yz$ <p>First order partial derivatives: $\frac{\partial \mathbf{Q}}{\partial x} = 2x + 2y$ $\frac{\partial \mathbf{Q}}{\partial y} = 2x - 2z$ $\frac{\partial \mathbf{Q}}{\partial z} = 2z - 2y$</p> <p>Second order partial derivatives of: $\frac{\partial^2 \mathbf{Q}}{\partial x^2} = 2$ $\frac{\partial^2 \mathbf{Q}}{\partial y^2} = 0$ $\frac{\partial^2 \mathbf{Q}}{\partial z^2} = 2$</p> $\frac{\partial^2 \mathbf{Q}}{\partial x \partial y} = \frac{\partial^2 \mathbf{Q}}{\partial y \partial x} = 2 \quad \frac{\partial^2 \mathbf{Q}}{\partial x \partial z} = \frac{\partial^2 \mathbf{Q}}{\partial z \partial x} = 0 \quad \frac{\partial^2 \mathbf{Q}}{\partial y \partial z} = \frac{\partial^2 \mathbf{Q}}{\partial z \partial y} = -2$ <p>Matrix of second order partial derivatives Q: $\begin{pmatrix} \frac{\partial^2 \mathbf{Q}}{\partial x^2} & \frac{\partial^2 \mathbf{Q}}{\partial x \partial y} & \frac{\partial^2 \mathbf{Q}}{\partial x \partial z} \\ \frac{\partial^2 \mathbf{Q}}{\partial y \partial x} & \frac{\partial^2 \mathbf{Q}}{\partial y^2} & \frac{\partial^2 \mathbf{Q}}{\partial y \partial z} \\ \frac{\partial^2 \mathbf{Q}}{\partial z \partial x} & \frac{\partial^2 \mathbf{Q}}{\partial z \partial y} & \frac{\partial^2 \mathbf{Q}}{\partial z^2} \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix} \neq 2\mathbf{A}$</p> <p>Hence, Option 1 is not correct.</p>
Option 2	The rank of the quadratic form of Q is 2
Solution	<p>From above we have quadratic form of Q = $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix}$</p> <p>Echelon form reduction: $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix} \xrightarrow{R_1 = \frac{1}{2}R_1} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$</p> $\xrightarrow{R_2 \rightarrow \frac{-1}{2}R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 + 2R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \xrightarrow{R_3 \rightarrow \frac{1}{4}R_3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $\xrightarrow{R_1 \rightarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ <p>Rank = Number of non-zero rows = 3 \neq 2</p> <p>Hence, Option 2 is not correct.</p>
Option 3	The signature of the quadratic form Q is $(+ + 0)$
Solution	<p>From above we have quadratic form of Q = $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix}$</p> <p>Finding eigen values: $\mathbf{Q} - \lambda \mathbf{I} = \begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & -\lambda & -2 \\ 0 & -2 & 2-\lambda \end{vmatrix}$</p> $\Rightarrow (2-\lambda)(-2\lambda + \lambda^2 + 4) + 8 = 0$ $\Rightarrow \lambda^3 - 4\lambda^2 - 4\lambda + 16 = 0$ $\lambda_1 = 4 \quad \lambda_2 = 2 \quad \lambda_3 = -2$ <p>Signature = $(n_+, n_-, n_0) = (2, 1, 0) \neq (+ + 0)$</p> <p>Hence, Option 3 is not correct.</p>
Option 4	The quadratic form Q takes the value 0 for some non-zero vector (x, y, z)
Solution	<p>From above we have quadratic form of Q = $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix}$</p>

we can see that few elements are zero even though the vectors are non-zero. Therefore, Option 4 is correct.

TABLE 2: Solution