EE5609: Matrix Theory Assignment-4

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***Abstract*—This document contains a proof to show the given equation represents two parallel lines.**

Download the python codes from

https://github.com/pranaya14014/EE5609/tree/master/Assignment4/code

and latex-tikz codes from

https://github.com/pranaya14014/EE5609/tree/master/Assignment4

1 PROBLEM

Prove that the equation

$$x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0 \tag{1.0.1}$$

represents two parallel lines.

2 THEORY

The second order general equation of the form

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

can be written as:

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where,

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.3}$$

Pair of straight line equation in vector form by,

$$\mathbf{n_1}^T\mathbf{x} = \mathbf{c_1} \tag{2.0.4}$$

$$\mathbf{n_2}^T\mathbf{x} = \mathbf{c_2} \tag{2.0.5}$$

3 CONDITION TO BE PARALLEL

Two lines are said to be parallel if they have same slopes or the angle between them is zero.

4 SOLUTION

The given equation (1.0.1) can be written using (2.0.2) as

$$\mathbf{x}^T\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} 2 & 6 \end{pmatrix}\mathbf{x} - 5 = 0 \tag{4.0.1}$$

$$\mathbf{V} = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \quad f = -5 \tag{4.0.2}$$

Equation (1.0.1) represents pair of straight line as,

$$D = \begin{vmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \\ 2 & 6 & -5 \end{vmatrix} = 0 \tag{4.0.3}$$

Equating the product of (2.0.4) and (2.0.5) with (4.0.1)

$$(\mathbf{n_1}^T\mathbf{x} - \mathbf{c_1})(\mathbf{n_2}^T\mathbf{x} - \mathbf{c_2}) = \mathbf{x}^T\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} 2 & 6 \end{pmatrix}\mathbf{x} - 5 \tag{4.0.4}$$

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} \tag{4.0.5}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\begin{pmatrix} 2 \\ 6 \end{pmatrix} \tag{4.0.6}$$

$$c_1c_2 = -5 \tag{4.0.7}$$

The slopes of the lines can be given by roots of the equation,

$$cm^2 + 2bm + a = 0 \tag{4.0.8}$$

$$m_i = \frac{-b \pm \sqrt{-\left|\mathbf{V}\right|}}{c} \tag{4.0.9}$$

$$\mathbf{n_i} = k_i\begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{4.0.10}$$

From (4.0.1) in (4.0.8)

$$9m^2 + 6m + 1 = 0 \tag{4.0.11}$$

Using (4.0.2) we get

$$\left|\mathbf{V}\right| = \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 0 \tag{4.0.12}$$

Substituting the values in (4.0.9),

$$m_i = \frac{-3 \pm 0}{9} \tag{4.0.13}$$

$$m_1 = m_2 = \frac{-1}{3} \tag{4.0.14}$$

$$\mathbf{n_1} = k_1 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{4.0.15}$$

$$\mathbf{n_2} = k_2 \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{4.0.16}$$

Using (4.0.5) we get,

$$k_1 k_2 = 9 \tag{4.0.17}$$

Taking $k_1 = 3$ and $k_2 = 3$ we get

$$\mathbf{n_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{4.0.18}$$

$$\mathbf{n_2} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{4.0.19}$$

Verifying $\mathbf{n_1}$ and $\mathbf{n_2}$ by computing the convolution by representing $\mathbf{n_1}$ as Toeplitz matrix,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} \tag{4.0.20}$$

Finding the Angle between the lines,

$$\theta = \cos^{-1}\left(\frac{\mathbf{n_1}^T \mathbf{n_2}}{\|\mathbf{n_1}\| \, \|\mathbf{n_2}\|}\right) \tag{4.0.21}$$

$$\mathbf{n_1}^T \mathbf{n_2} = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 10 \tag{4.0.22}$$

$$\|\mathbf{n_1}\| = \sqrt{10} \quad \|\mathbf{n_2}\| = \sqrt{10} \tag{4.0.23}$$

Substituting (4.0.22) and (4.0.23) in (4.0.21) we get,

$$\theta = \cos^{-1}(1) \tag{4.0.24}$$

$$\theta = 0^\circ \tag{4.0.25}$$

From (4.0.14) and (4.0.25) shows the given equation (1.0.1) represents two parallel lines. Hence proved.

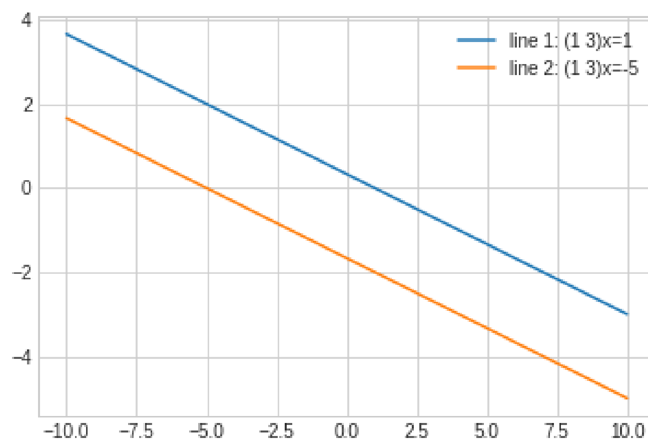

Fig. 0: Pair of straight lines plot generated using python