EE5609: Matrix Theory Assignment-5

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***Abstract*—This document contains a proof to show the given equation represents two parallel lines.**

Download the python codes from

https://github.com/pranaya14014/EE5609/tree/master/Assignment5/code

and latex-tikz codes from

https://github.com/pranaya14014/EE5609/tree/master/Assignment5

1 Problem

Show that, by a change of origin and the directions of the coordinate axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 14 & 22 \end{pmatrix} \mathbf{x} + 27 = 0 \tag{1.0.1}$$

can be transformed to

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} = 1 \tag{1.0.2}$$

or

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} = 1 \tag{1.0.3}$$

2 Solution

The general second order equation can be expressed as follows,

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{2.0.1}$$

Comparing (1.0.1) with (2.0.1),

$$\mathbf{V} = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix} -7 \\ -11 \end{pmatrix} \tag{2.0.3}$$

$$f = 27 \tag{2.0.4}$$

Let $\mathbf{c}$ be the change in the origin. The equation (2.0.1) can be modified as

$$(\mathbf{x}+\mathbf{c})^{\mathbf{T}}\mathbf{V}(\mathbf{x}+\mathbf{c}) + 2\mathbf{u}^{\mathbf{T}}(\mathbf{x}+\mathbf{c}) + f = 0 \tag{2.0.5}$$

Considering (2.0.5)

$$\implies (\mathbf{x}+\mathbf{c})^{\mathbf{T}}\mathbf{V}(\mathbf{x}+\mathbf{c}) \tag{2.0.6}$$

$$\implies \mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + \mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{x} + \mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{c} + \mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{c} \tag{2.0.7}$$

In the above equation

$$\mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{x} = \mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{c} \tag{2.0.8}$$

From (2.0.7) and (2.0.8) then (2.0.5) becomes

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{x} + \mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{c} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{c} + f = 0 \tag{2.0.9}$$

Comparing (1.0.2) and (2.0.9)

$$2\mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{P}\mathbf{y} + 2\mathbf{u}^{\mathbf{T}}\mathbf{P}\mathbf{y} = 0 \tag{2.0.10}$$

$$\mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{P}\mathbf{y} = -\mathbf{u}^{\mathbf{T}}\mathbf{P}\mathbf{y} \tag{2.0.11}$$

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.0.12}$$

Substituting (2.0.2) and (2.0.3) in (2.0.12)

$$\mathbf{c} = \frac{-1}{24}\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} -7 \\ -11 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{2.0.13}$$

Hence (2.0.9) becomes

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + \mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{c} + 2\mathbf{u}^{\mathbf{T}}\mathbf{c} + f = 0 \tag{2.0.14}$$

Substituting (2.0.2), (2.0.3) and (2.0.13) the above equation becomes

$$\mathbf{x}^{\mathbf{T}}\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} -7 & -11 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 27 = 0 \tag{2.0.15}$$

$$\mathbf{x}^{\mathbf{T}}\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\mathbf{x} + 29 - 58 + 27 = 0 \tag{2.0.16}$$

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} - 2 = 0 \tag{2.0.17}$$

With change in the origin to point $\mathbf{c} = \begin{pmatrix}1\\2\end{pmatrix}$ but the $\mathbf{V}$ doesn't change.

$$\left|\mathbf{V}\right| = \begin{vmatrix}5 & 1\\1 & 5\end{vmatrix} = 24 \tag{2.0.18}$$

As $\left|\mathbf{V}\right| > 0$ it represents a ellipse.Hence $\mathbf{V}$ can be written as,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{\mathbf{T}} \tag{2.0.19}$$

The characteristic equation of $\mathbf{V}$ is given by

$$\left|\mathbf{V} - \mathbf{I}\lambda\right| = 0 \tag{2.0.20}$$

$$\begin{vmatrix}5-\lambda & 1\\1 & 5-\lambda\end{vmatrix} = 0 \tag{2.0.21}$$

$$\implies \lambda^2 - 10\lambda + 24 = 0 \tag{2.0.22}$$

Hence the eigen vales are,

$$\lambda_1 = 4 \tag{2.0.23}$$

$$\lambda_2 = 6 \tag{2.0.24}$$

Hence diagonal vector is given by,

$$\mathbf{D} = \begin{pmatrix}\lambda_1 & 0\\0\lambda_2 & \end{pmatrix} = \begin{pmatrix}4 & 0\\0 & 6\end{pmatrix} \tag{2.0.25}$$

The eigen vector $\mathbf{p}$ is given by

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.0.26}$$

$$(\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{2.0.27}$$

For $\lambda_1 = 4$ the eigenvector is,

$$\mathbf{V} - \lambda_1\mathbf{I} = \begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix} \tag{2.0.28}$$

$$\mathbf{p_1} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}\end{pmatrix} \tag{2.0.29}$$

For $\lambda_1 = 6$ the eigenvector is,

$$\mathbf{V} - \lambda_1\mathbf{I} = \begin{pmatrix}-1 & 1\\1 & -1\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix}-1 & 1\\0 & 0\end{pmatrix} \tag{2.0.30}$$

$$\mathbf{p_1} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix} \tag{2.0.31}$$

Hence,

$$\mathbf{P} = \begin{pmatrix}\mathbf{p_1} & \mathbf{p_2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} \tag{2.0.32}$$

Substituting (2.0.25) and (2.0.32) in (2.0.19)

$$\mathbf{V} = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}4 & 0\\0 & 6\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} \tag{2.0.33}$$

Hence substituting (2.0.33) in (2.0.17)

$$\mathbf{x}^{\mathbf{T}}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}4 & 0\\0 & 6\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\mathbf{x} = 2 \tag{2.0.34}$$

$$\mathbf{y}^{\mathbf{T}}\begin{pmatrix}4 & 0\\0 & 6\end{pmatrix}\mathbf{y} = 2 \tag{2.0.35}$$

$$\mathbf{y}^{\mathbf{T}}\begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}\mathbf{y} = 1 \tag{2.0.36}$$

where $\mathbf{y}$ is given by Affine transformation

$$\mathbf{x} = \mathbf{P}\mathbf{y} \tag{2.0.37}$$

$$\mathbf{y} = \mathbf{P}^{\mathbf{T}}\mathbf{x} \tag{2.0.38}$$

The rotation matrix $\mathbf{P}$ can be given by,

$$\mathbf{P} = \begin{pmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{pmatrix} \tag{2.0.39}$$

Comparing (2.0.32) and (2.0.39)

$$\cos\theta = \frac{1}{\sqrt{2}} \tag{2.0.40}$$

$$\theta = \frac{\pi}{4} \tag{2.0.41}$$

But given the direction of coordinate axes changes so,

$$\theta = \pi + \frac{\pi}{4} \tag{2.0.42}$$

Subtituting (2.0.42) in (2.0.39) we get

$$\mathbf{P} = \begin{pmatrix}\cos\left(\pi + \frac{\pi}{4}\right) & \sin\left(\pi + \frac{\pi}{4}\right)\\-\sin\left(\pi + \frac{\pi}{4}\right) & \cos\left(\pi + \frac{\pi}{4}\right)\end{pmatrix} \tag{2.0.43}$$

$$\mathbf{P} = \begin{pmatrix}\frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix} \tag{2.0.44}$$

From (2.0.19) we find the diagonal matrix

$$\mathbf{D} = \mathbf{P}^{\mathbf{T}}\mathbf{V}\mathbf{P} = \begin{pmatrix}\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}5 & 1\\1 & 5\end{pmatrix}\begin{pmatrix}\frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix} \tag{2.0.45}$$

$$\mathbf{D} = \begin{pmatrix}6 & 0\\0 & 4\end{pmatrix} \tag{2.0.46}$$

Hence using (2.0.44), (2.0.46) and (2.0.19) in (2.0.17) we get.

$$\mathbf{x^T}\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}\mathbf{x} = 2 \qquad (2.0.47)$$

using (2.0.38) the above equation becomes,

$$\mathbf{y^T}\begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}\mathbf{y} = 2 \qquad (2.0.48)$$

$$\mathbf{y^T}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{y} = 1 \qquad (2.0.49)$$

Hence from (2.0.36) and (2.0.49) proved that change of origin and the directions of the coordinate axes (1.0.1) can be tranformed to (1.0.2) or (1.0.3)