

EE5609: Matrix Theory

Assignment-4

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Abstract—This document contains a proof to show the given equation represents two parallel lines.

Download the python codes from

<https://github.com/pranaya14014/EE5609/tree/master/Assignment4/code>

and latex-tikz codes from

<https://github.com/pranaya14014/EE5609/tree/master/Assignment4>

1 PROBLEM

Prove that the equation

$$x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0 \quad (1.0.1)$$

represents two parallel lines.

2 THEORY

The second order general equation of the form

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.1)$$

can be written as:

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.2)$$

where,

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \mathbf{u}^T = (d \quad e) \quad (2.0.3)$$

Pair of straight line equation in vector form by,

$$\mathbf{n}_1^T \mathbf{x} = \mathbf{c}_1 \quad (2.0.4)$$

$$\mathbf{n}_2^T \mathbf{x} = \mathbf{c}_2 \quad (2.0.5)$$

3 CONDITION TO BE PARALLEL

Two lines are said to be parallel if they have same slopes or the angle between them is zero.

4 SOLUTION

The given equation (2.0.2) can be written using (1.0.1) as

$$\mathbf{x}^T \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 2 & 6 \end{pmatrix} \mathbf{x} - 5 = 0 \quad (4.0.1)$$

$$\mathbf{V} = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \quad f = -5 \quad (4.0.2)$$

Equation (1.0.1) represents pair of straight line as,

$$D = \begin{vmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \\ 2 & 6 & -5 \end{vmatrix} = 0 \quad (4.0.3)$$

Equating the product of (2.0.4) and (2.0.5) with (4.0.1)

$$(\mathbf{n}_1^T \mathbf{x} - \mathbf{c}_1)(\mathbf{n}_2^T \mathbf{x} - \mathbf{c}_2) = \mathbf{x}^T \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 2 & 6 \end{pmatrix} \mathbf{x} - 5 \quad (4.0.4)$$

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} \quad (4.0.5)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} 2 \\ 6 \end{pmatrix} \quad (4.0.6)$$

$$c_1 c_2 = -5 \quad (4.0.7)$$

The slopes of the lines can be given by roots of the equation,

$$cm^2 + 2bm + a = 0 \quad (4.0.8)$$

$$m_i = \frac{-b \pm \sqrt{-|\mathbf{V}|}}{c} \quad (4.0.9)$$

$$\mathbf{n}_i = k_i \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (4.0.10)$$

From (4.0.1) in (4.0.8)

$$9m^2 + 6m + 1 = 0 \quad (4.0.11)$$

Using (4.0.2) we get

$$|\mathbf{V}| = \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 0 \quad (4.0.12)$$

Substituting the values in (4.0.9),

$$m_i = \frac{-3 \pm 0}{9} \quad (4.0.13)$$

$$m_1 = m_2 = \frac{-1}{3} \quad (4.0.14)$$

$$\mathbf{n}_1 = k_1 \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \quad (4.0.15)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \quad (4.0.16)$$

Using (4.0.5) we get,

$$k_1 k_2 = 9 \quad (4.0.17)$$

Taking $k_1 = 3$ and $k_2 = 3$ we get

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} \quad (4.0.18)$$

$$\mathbf{n}_2 = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} \quad (4.0.19)$$

Verifying \mathbf{n}_1 and \mathbf{n}_2 by computing the convolution by representing \mathbf{n}_1 as Toeplitz matrix,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix} \quad (4.0.20)$$

Finding the Angle between the lines,

$$\theta = \cos^{-1} \left(\frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \right) \quad (4.0.21)$$

$$\mathbf{n}_1^T \mathbf{n}_2 = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 10 \quad (4.0.22)$$

$$\|\mathbf{n}_1\| = \sqrt{10} \quad \|\mathbf{n}_2\| = \sqrt{10} \quad (4.0.23)$$

Substituting (4.0.22) and (4.0.23) in (4.0.21) we get,

$$\theta = \cos^{-1}(1) \quad (4.0.24)$$

$$\theta = 0^\circ \quad (4.0.25)$$

From (4.0.14) and (4.0.25) shows the given equation (1.0.1) represents two parallel lines. Hence proved.

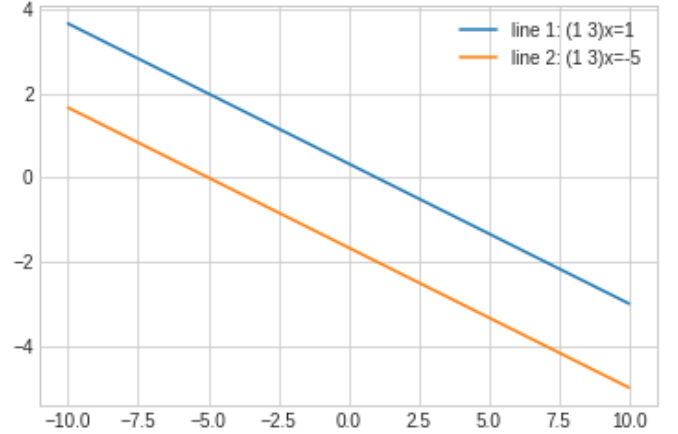


Fig. 0: Pair of straight lines plot generated using python