EE5609: Matrix Theory
Assignment-19

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Abstract

This document contains a solution for a quadratic form of given matrix.

Download the latex-tikz codes from

https://github.com/pranaya14014/EE5609/blob/master/Assignment19

1 PROBLEM

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

and define for $x, y, z \in \mathbb{R}$

$$\mathbf{Q}(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2}$$

Which of the following are True?

1. The matrix of second order partial derivatives of the quadratic form of $\mathbf{Q}$ is $2\mathbf{A}$.
2. The rank of the quadratic form of $\mathbf{Q}$ is 2
3. The signature of the quadratic form $\mathbf{Q}$ is $(+ + 0)$
4. The quadratic form $\mathbf{Q}$ takes the value 0 for some non-zero vector (x, y, z)

2 DEFINITIONS

Quadratic Form of a matrix	Let $\mathbf{V}$ be a vector space over $\mathbb{R}$. $\mathbf{A}$ be a symmetric matrix $n \times n$. Quadratic form on $\mathbf{V}$ is a real function, ($\mathbf{F} : \mathbf{V} \to \mathbb{R}$) defined as $F(x) = \mathbf{x}\mathbf{A}\mathbf{x}^T = \sum_{i,j=1}^{n} a_{ij}x_i x_j$ where $\mathbf{x} \in \mathbf{V}$
Signature of Quadratic form	The signature of quadratic form is (n_+, n_-, n_0) where n_+ is the number of positive entries, n_- is number of negative entries and n_0 is number of zero's in a_{ii}
Rank of quadratic form	Rank of quadratic form is the rank of its matrix which is maximum number of linearly independent rows/columns of a matrix

TABLE 1: Definitions

3 SOLUTION

Option 1	**The matrix of second order partial derivatives of the quadratic form of Q is 2A.**
Solution	$\mathbf{Q}(x,y,z)= \begin{pmatrix} x & y & z\end{pmatrix}\mathbf{A}\begin{pmatrix} x \\ y \\ z\end{pmatrix} = \begin{pmatrix} x & y & z\end{pmatrix}\begin{pmatrix} x+2y \\ -2z \\ z\end{pmatrix} = x^2+z^2+2xy-2yz$ First order partial derivaties: $\frac{\partial \mathbf{Q}}{\partial x} = 2x+2y \quad \frac{\partial \mathbf{Q}}{\partial y} = 2x-2z \quad \frac{\partial \mathbf{Q}}{\partial z} = 2z-2y$ Second order partial derivatives of: $\frac{\partial^2 \mathbf{Q}}{\partial x^2} = 2 \quad \frac{\partial^2 \mathbf{Q}}{\partial y^2} = 0 \quad \frac{\partial^2 \mathbf{Q}}{\partial z^2} = 2$ $\frac{\partial^2 \mathbf{Q}}{\partial x \partial y} = \frac{\partial^2 \mathbf{Q}}{\partial y \partial x} = 2 \quad \frac{\partial^2 \mathbf{Q}}{\partial x \partial z} = \frac{\partial^2 \mathbf{Q}}{\partial z \partial x} = 0 \quad \frac{\partial^2 \mathbf{Q}}{\partial y \partial z} = \frac{\partial^2 \mathbf{Q}}{\partial z \partial y} = -2$ Matrix of second order partial derivatives $\mathbf{Q}$: $\begin{pmatrix} \frac{\partial^2 \mathbf{Q}}{\partial x^2} & \frac{\partial^2 \mathbf{Q}}{\partial x \partial y} & \frac{\partial^2 \mathbf{Q}}{\partial x \partial z} \\ \frac{\partial^2 \mathbf{Q}}{\partial y \partial x} & \frac{\partial^2 \mathbf{Q}}{\partial y^2} & \frac{\partial^2 \mathbf{Q}}{\partial y \partial z} \\ \frac{\partial^2 \mathbf{Q}}{\partial z \partial x} & \frac{\partial^2 \mathbf{Q}}{\partial z \partial y} & \frac{\partial^2 \mathbf{Q}}{\partial z^2} \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix} \neq 2\mathbf{A}$ Hence, **Option 1** is not correct.
Option 2	**The rank of the quadratic form of Q is 2**
Solution	From above we have quadratic form of $\mathbf{Q} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix}$ Echelon form reduction: $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix} \xleftrightarrow{R_1=\frac{1}{2}} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix} \xleftrightarrow{R_2\to R_2-2R_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$ $\xleftrightarrow{R_2\to \frac{-1}{2}R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix} \xleftrightarrow{R_3\to R_3+2R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \xleftrightarrow{R_3\to \frac{1}{4}R_3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $\xleftrightarrow{R_1\to R_1-R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_2\to R_2-R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Rank = Number of non-zero rows = $3 \neq 2$ Hence, **Option 2** is not correct.
Option 3	**The signature of the quadratic form Q is** $(++0)$
Solution	From above we have quadratic form of $\mathbf{Q} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix}$ Finding eigen values: $\left\|\mathbf{Q}-\lambda\mathbf{I}\right\| = \begin{pmatrix} 2-\lambda & 2 & 0 \\ 2 & -\lambda & -2 \\ 0 & -2 & 2-\lambda \end{pmatrix}$ $\implies (2-\lambda)\left(-2\lambda+\lambda^2+4\right)+8=0$ $\implies \lambda^3-4\lambda^2-4\lambda+16=0$ $\lambda_1=4 \quad \lambda_2=2 \quad \lambda_3=-2$ Signature = $(n_+,n_-,n_0) = (2,1,0) \neq (++0)$ Hence, **Option 3** is not correct.
Option 4	**The quadratic form Q takes the value 0 for some non-zero vector** (x,y,z)
Solution	From above we have quadratic form of $\mathbf{Q} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix}$

	we can see that few elements are zero even though the vectors are non-zero. Therefore, **Option 4** is correct.

TABLE 2: Solution