EE5609: Matrix Theory
Assignment-8

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Abstract—**This document contains a proof to show that upper-triangular matrix is invertible if and only if diagonal elements are not 0.**

Download the latex-tikz codes from

https://github.com/pranaya14014/EE5609/tree/master/Assignment8

1 PROBLEM

An $n \times n$ matrix $\mathbf{A}$ is called upper-triangular if $\mathbf{A_{ij}} = 0$ for $i > j$, that is, if every entry below the main diagonal is 0. Prove that an upper-triangular (square) matrix is invertible if and only if every entry on its main diagonal is different from 0.

2 Properties of Determinants

Properties required for the proof:
1. If $\mathbf{A}$ is invertible then $\left|\mathbf{A}\right| \neq 0$
2. If the entries of a row(or column) in a square matrix are multiplied by a number k, then the determinant of resulting matrix is k$\left|\mathbf{A}\right|$.
3. $\left|\mathbf{I}\right| = 1$
4. The determinant value remains same if we do any operation like $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

3 SOLUTION

Considering $\mathbf{A}$, an upper triangular matrix and proceeding futher with properties of determinants

$$\left|\mathbf{A}\right| = \begin{vmatrix} a_{1,1} & a_{1,2} & ... & a_{1,(n-1)} & a_{1,n} \\ 0 & a_{2,2} & ... & a_{2,(n-2)} & a_{2,n} \\ 0 & 0 & ... & & . \\ . & & ... & & . \\ 0 & 0 & ... & a_{(n-1),(n-1)} & a_{(n-1),(n)} \\ 0 & 0 & ... & 0 & a_{n,n} \end{vmatrix} \tag{3.0.1}$$

$$\xleftrightarrow{R_{n-1} \to R_{n-1} - a_{(n-1),(n-1)} R_n} \begin{vmatrix} a_{1,1} & a_{1,2} & ... & a_{1,(n-1)} & a_{1,n} \\ 0 & a_{2,2} & ... & a_{2,(n-2)} & a_{2,n} \\ 0 & 0 & ... & & . \\ . & & ... & & . \\ 0 & 0 & ... & a_{(n-1),(n-1)} & 0 \\ 0 & 0 & ... & 0 & a_{n,n} \end{vmatrix} \tag{3.0.2}$$

Similarly performing the row operations on rest of the rows we get a diagonal matrix which is,

$$\left|\mathbf{A}\right| = \begin{vmatrix} a_{1,1} & 0 & ... & 0 & 0 \\ 0 & a_{2,2} & ... & 0 & 0 \\ 0 & 0 & ... & & . \\ . & & ... & & . \\ 0 & 0 & ... & a_{(n-1),(n-1)} & 0 \\ 0 & 0 & ... & 0 & a_{n,n} \end{vmatrix} \tag{3.0.3}$$

$$= a_{1,1} \begin{vmatrix} 1 & 0 & ... & 0 & 0 \\ 0 & a_{2,2} & ... & 0 & 0 \\ 0 & 0 & ... & & . \\ . & & ... & & . \\ 0 & 0 & ... & a_{(n-1),(n-1)} & 0 \\ 0 & 0 & ... & 0 & a_{n,n} \end{vmatrix} \tag{3.0.4}$$

$$= a_{1,1} a_{2,2} \begin{vmatrix} 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ 0 & 0 & ... & & . \\ . & & ... & & . \\ 0 & 0 & ... & a_{(n-1),(n-1)} & 0 \\ 0 & 0 & ... & 0 & a_{n,n} \end{vmatrix} \tag{3.0.5}$$

Similarly we get,

$$\left|\mathbf{A}\right| = a_{1,1} a_{2,2} ... a_{(n-1),(n-1)} a_{n,n} \begin{vmatrix} 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ 0 & 0 & ... & & . \\ . & & ... & & . \\ 0 & 0 & ... & 1 & 0 \\ 0 & 0 & ... & 0 & 1 \end{vmatrix} \tag{3.0.6}$$

$$= a_{1,1} a_{2,2} ... a_{(n-1),(n-1)} a_{n,n} \left|\mathbf{I_{n,n}}\right| \tag{3.0.7}$$

$$\left|\mathbf{A}\right| = a_{1,1}a_{2,2}...a_{(n-1),(n-1)}a_{n,n} \tag{3.0.8}$$

From (3.0.8) we see that $\left|\mathbf{A}\right|$ is the product of diagonal elements. If anyone of the diagonal element is zero then $\left|\mathbf{A}\right| = 0$ which implies $\mathbf{A}$ is not invertible.So, a matrix is invertible if and only if diagonal elements are not 0.
Hence Proved.