EE5609: Matrix Theory Assignment-5

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***Abstract*—This document contains a proof to show the given equation represents two parallel lines.**

Download the python codes from

https://github.com/pranaya14014/EE5609/tree/master/Assignment5/code

and latex-tikz codes from

https://github.com/pranaya14014/EE5609/tree/master/Assignment5

1 PROBLEM

Show that, by a change of origin and the directions of the coordinate axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 14 & 22 \end{pmatrix} \mathbf{x} + 27 = 0 \tag{1.0.1}$$

can be transformed to

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} = 1 \tag{1.0.2}$$

or

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} = 1 \tag{1.0.3}$$

2 SOLUTION

The general second order equation can be expressed as follows,

$$\mathbf{x}^\mathbf{T}\mathbf{V}\mathbf{x} + 2\mathbf{u}^\mathbf{T}\mathbf{x} + f = 0 \tag{2.0.1}$$

Comparing (1.0.1) with (2.0.1),

$$\mathbf{V} = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix} -7 \\ -11 \end{pmatrix} \tag{2.0.3}$$

$$f = 27 \tag{2.0.4}$$

let $\mathbf{c}$ be the change in the origin and $\mathbf{P}$ indicates the rotation of the axes. So, Affine transformation is given by,

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{2.0.5}$$

Given the direction of coordinate axes change so, $\theta = 180°$

$$\mathbf{P} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.0.6}$$

Subtituting (2.0.5) in (2.0.1)

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^\mathbf{T}\mathbf{V}(\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^\mathbf{T}(\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0 \tag{2.0.7}$$

Considering (2.0.7)

$$\implies (\mathbf{P}\mathbf{y} + \mathbf{c})^\mathbf{T}\mathbf{V}(\mathbf{P}\mathbf{y} + \mathbf{c}) \tag{2.0.8}$$

$$\implies \left(\mathbf{y}^\mathbf{T}\mathbf{P}^\mathbf{T}\mathbf{V} + \mathbf{c}^\mathbf{T}\mathbf{V}\right)(\mathbf{P}\mathbf{y} + \mathbf{c}) \tag{2.0.9}$$

$$\implies \mathbf{y}^\mathbf{T}\mathbf{P}^\mathbf{T}\mathbf{V}\mathbf{P}\mathbf{y} + \mathbf{y}^\mathbf{T}\mathbf{P}^\mathbf{T}\mathbf{V}\mathbf{c} + \mathbf{c}^\mathbf{T}\mathbf{V}\mathbf{P}\mathbf{y} + \mathbf{c}^\mathbf{T}\mathbf{V}\mathbf{c} \tag{2.0.10}$$

$$\implies \mathbf{y}^\mathbf{T}\mathbf{P}^\mathbf{T}\mathbf{V}\mathbf{P}\mathbf{y} + 2\mathbf{c}^\mathbf{T}\mathbf{V}\mathbf{P}\mathbf{y} + \mathbf{c}^\mathbf{T}\mathbf{V}\mathbf{c} \tag{2.0.11}$$

Subtituting (2.0.11) in (2.0.7)

$$\mathbf{y}^\mathbf{T}\mathbf{P}^\mathbf{T}\mathbf{V}\mathbf{P}\mathbf{y} + 2\mathbf{c}^\mathbf{T}\mathbf{V}\mathbf{P}\mathbf{y} + \mathbf{c}^\mathbf{T}\mathbf{V}\mathbf{c} + 2\mathbf{u}^\mathbf{T}\mathbf{P}\mathbf{y} + 2\mathbf{u}^\mathbf{T}\mathbf{c} + f = 0 \tag{2.0.12}$$

From (2.0.2)

$$\det \mathbf{V} = \begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} = 24 \tag{2.0.13}$$

As $\det \mathbf{V} > 0$ it represents a ellipse. The equation of ellipse will be of the form,

$$\mathbf{x}^\mathbf{T}\mathbf{V}\mathbf{x} + f = 0 \tag{2.0.14}$$

Comparing (2.0.12) and (2.0.14)

$$2\mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{P}\mathbf{y} + 2\mathbf{u}^{\mathbf{T}}\mathbf{P}\mathbf{y} = 0 \tag{2.0.15}$$

$$\mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{P}\mathbf{y} = -\mathbf{u}^{\mathbf{T}}\mathbf{P}\mathbf{y} \tag{2.0.16}$$

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.0.17}$$

Substituting (2.0.2) and (2.0.3) in (2.0.17)

$$\mathbf{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{2.0.18}$$

Hence (2.0.12) becomes

$$\mathbf{y}^{\mathbf{T}}\mathbf{P}^{\mathbf{T}}\mathbf{V}\mathbf{P}\mathbf{y} + \mathbf{c}^{\mathbf{T}}\mathbf{V}\mathbf{c} + 2\mathbf{u}^{\mathbf{T}}\mathbf{c} + f = 0 \tag{2.0.19}$$

Subtituting the values in (2.0.19)

$$\mathbf{y}^{\mathbf{T}}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{y} + \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} -7 & -11 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 27 = 0 \tag{2.0.20}$$

$$\mathbf{y}^{\mathbf{T}}\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\mathbf{y} + 29 - 58 + 27 = 0 \tag{2.0.21}$$

$$\mathbf{y}^{\mathbf{T}}\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\mathbf{y} - 2 = 0 \tag{2.0.22}$$

$$\mathbf{y}^{\mathbf{T}}\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\mathbf{y} = 2 \tag{2.0.23}$$