EE5609: Matrix Theory Assignment-10

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***Abstract*—This document checks the isomorphism of V of complex numbers on $\mathbb{R}^2$.**

Download the latex-tikz codes from

https://github.com/pranaya14014/EE5609/tree/master/Assignment10

1 PROBLEM

Let $\mathbf{V}$ be the set of complex numbers and let $\mathbb{F}$ be the field of real numbers. With the usual operations, $\mathbf{V}$ is a vector space over $\mathbb{F}$. Describe explicitly an isomorphism of this space onto $\mathbb{R}^2$.

2 EXPLANATION

Two vector spaces $\mathbf{V}$ and $\mathbf{W}$ over the same field $\mathbb{F}$ are isomorphic if there is a bijection $\mathbf{T} : \mathbf{V} \to \mathbf{W}$ which preserves addition and scalar multiplication, that is, for all vectors $\mathbf{u}, \mathbf{v} \in \mathbf{V}$, and all scalars $c \in \mathbf{F}$

$$\mathbf{T}(\mathbf{u}+\mathbf{v}) = \mathbf{T}(\mathbf{u}) + \mathbf{T}(\mathbf{v}) \tag{2.0.1}$$

$$\mathbf{T}(c\mathbf{u}) = c\mathbf{T}(\mathbf{v}) \tag{2.0.2}$$

The correspondence $\mathbf{T}$ is called an isomorphism onto $\mathbf{V}$ and $\mathbf{W}$.

3 SOLUTION

Let,

$$\mathbf{T} : \mathbf{V} \to \mathbb{R}^2 \tag{3.0.1}$$

$$\mathbf{T}(x+iy) = \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.0.2}$$

$$x, y \in \mathbb{R} \quad i \in \mathbb{C} \tag{3.0.3}$$

checking if (3.0.2) preserves multiplication and scalar multiplication. Let,

$$\mathbf{u} = x_1 + iy_1 \quad \mathbf{v} = x_2 + iy_2 \tag{3.0.4}$$

$$\mathbf{T}(\mathbf{u}+c\mathbf{v}) = \mathbf{T}((x_1+iy_1) + c(x_2+iy_2)) \tag{3.0.5}$$

$$= \mathbf{T}(x_1 + y_1 + cx_2 + icy_2) \tag{3.0.6}$$

$$= \mathbf{T}((x_1 + cx_2) + i(y_1 + cy_2)) \tag{3.0.7}$$

$$= \begin{pmatrix} x_1 + cx_2 \\ y_1 + cy_2 \end{pmatrix} \tag{3.0.8}$$

$$= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} cx_2 \\ cy_2 \end{pmatrix} \tag{3.0.9}$$

$$= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + c\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \tag{3.0.10}$$

$$= \mathbf{T}(\mathbf{u}) + c\mathbf{T}(\mathbf{v}) \tag{3.0.11}$$

Hence this is a Linear transformation and the inverse is $\mathbf{T}^{-1}(a, b) = (a + ib)$. Hence the two spaces are isomorphic.