EE5609: Matrix Theory Assignment-5

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Abstract—**This document contains a proof for Affine transformation of a given ellipse equation**

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1 PROBLEM

Show that, by a change of origin and the directions of the coordinate axes, the equation

$$\mathbf{x}^T \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 14 & 22 \end{pmatrix} \mathbf{x} + 27 = 0 \qquad (1.0.1)$$

can be transformed to

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} = 1 \qquad (1.0.2)$$

or

$$\mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} = 1 \qquad (1.0.3)$$

2 SOLUTION

The general second order equation can be expressed as follows,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \qquad (2.0.1)$$

Comparing (1.0.1) with (2.0.1),

$$\mathbf{V} = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \qquad (2.0.2)$$

$$\mathbf{u} = \begin{pmatrix} -7 \\ -11 \end{pmatrix} \qquad (2.0.3)$$

$$f = 27 \qquad (2.0.4)$$

Let **c** be the change in the origin. The equation (2.0.1) can be modified as

$$(\mathbf{x} + \mathbf{c})^T \mathbf{V}(\mathbf{x} + \mathbf{c}) + 2\mathbf{u}^T(\mathbf{x} + \mathbf{c}) + f = 0 \qquad (2.0.5)$$

Considering (2.0.5)

$$\implies (\mathbf{x} + \mathbf{c})^T \mathbf{V}(\mathbf{x} + \mathbf{c}) \qquad (2.0.6)$$

$$\implies \mathbf{x}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{x} + \mathbf{x}^T \mathbf{V} \mathbf{c} + \mathbf{c}^T \mathbf{V} \mathbf{c} \qquad (2.0.7)$$

In the above equation

$$\mathbf{c}^T \mathbf{V} \mathbf{x} = \mathbf{x}^T \mathbf{V} \mathbf{c} \qquad (2.0.8)$$

From (2.0.7) and (2.0.8) then (2.0.5) becomes

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{c}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{x} + 2\mathbf{u}^T \mathbf{c} + f = 0 \qquad (2.0.9)$$

Comparing (1.0.2) and (2.0.9)

$$2\mathbf{c}^T \mathbf{V} \mathbf{P} \mathbf{y} + 2\mathbf{u}^T \mathbf{P} \mathbf{y} = 0 \qquad (2.0.10)$$

$$\mathbf{c}^T \mathbf{V} \mathbf{P} \mathbf{y} = -\mathbf{u}^T \mathbf{P} \mathbf{y} \qquad (2.0.11)$$

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \qquad (2.0.12)$$

Substituting (2.0.2) and (2.0.3) in (2.0.12)

$$\mathbf{c} = \frac{-1}{24} \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} -7 \\ -11 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad (2.0.13)$$

Hence (2.0.9) becomes

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + \mathbf{c}^T \mathbf{V} \mathbf{c} + 2\mathbf{u}^T \mathbf{c} + f = 0 \qquad (2.0.14)$$

Substituting (2.0.2), (2.0.3) and (2.0.13) the above equation becomes

$$\mathbf{x}^T \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} -7 & -11 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 27 = 0 \qquad (2.0.15)$$

$$\mathbf{x}^T \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \mathbf{x} + 29 - 58 + 27 = 0 \qquad (2.0.16)$$

$$\mathbf{x}^T \mathbf{V} \mathbf{x} - 2 = 0 \qquad (2.0.17)$$

With change in the origin to point $\mathbf{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ but the $\mathbf{V}$ doesn't change.

$$|\mathbf{V}| = \begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} = 24 \qquad (2.0.18)$$

As $|\mathbf{V}| > 0$ it represents a ellipse.Hence $\mathbf{V}$ can be written as,

$$\mathbf{V} = \mathbf{PDP^T} \qquad (2.0.19)$$

The characteristic equation of $\mathbf{V}$ is given by

$$|\mathbf{V} - \mathbf{I}\lambda| = 0 \qquad (2.0.20)$$

$$\begin{vmatrix} 5 - \lambda & 1 \\ 1 & 5 - \lambda \end{vmatrix} = 0 \qquad (2.0.21)$$

$$\implies \lambda^2 - 10\lambda + 24 = 0 \qquad (2.0.22)$$

Hence the eigen vales are,

$$\lambda_1 = 4 \qquad (2.0.23)$$
$$\lambda_2 = 6 \qquad (2.0.24)$$

Hence diagonal vector is given by,

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} \qquad (2.0.25)$$

The eigen vector $\mathbf{p}$ is given by

$$\mathbf{Vp} = \lambda\mathbf{p} \qquad (2.0.26)$$
$$(\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \qquad (2.0.27)$$

For $\lambda_1 = 4$ the eigenvector is,

$$\mathbf{V} - \lambda_1\mathbf{I} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad (2.0.28)$$

$$\mathbf{p_1} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \qquad (2.0.29)$$

For $\lambda_1 = 6$ the eigenvector is,

$$\mathbf{V} - \lambda_1\mathbf{I} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \qquad (2.0.30)$$

$$\mathbf{p_1} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad (2.0.31)$$

Hence,

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad (2.0.32)$$

Substituting (2.0.25) and (2.0.32) in (2.0.19)

$$\mathbf{V} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad (2.0.33)$$

Hence substituting (2.0.33) in (2.0.17)

$$\mathbf{x^T} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \mathbf{x} = 2 \qquad (2.0.34)$$

$$\mathbf{y^T} \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} \mathbf{y} = 2 \qquad (2.0.35)$$

$$\mathbf{y^T} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{y} = 1 \qquad (2.0.36)$$

where $\mathbf{y}$ is given by Affine transformation

$$\mathbf{x} = \mathbf{Py} \qquad (2.0.37)$$
$$\mathbf{y} = \mathbf{P^Tx} \qquad (2.0.38)$$

The rotation matrix $\mathbf{P}$ can be given by,

$$\mathbf{P} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad (2.0.39)$$

Comparing (2.0.32) and (2.0.39)

$$\cos\theta = \frac{1}{\sqrt{2}} \qquad (2.0.40)$$

$$\theta = \frac{\pi}{4} \qquad (2.0.41)$$

But given the direction of coordinate axes changes so,

$$\theta = \pi + \frac{\pi}{4} \qquad (2.0.42)$$

Subtituting (2.0.42) in (2.0.39) we get

$$\mathbf{P} = \begin{pmatrix} \cos\left(\pi + \frac{\pi}{4}\right) & \sin\left(\pi + \frac{\pi}{4}\right) \\ -\sin\left(\pi + \frac{\pi}{4}\right) & \cos\left(\pi + \frac{\pi}{4}\right) \end{pmatrix} \qquad (2.0.43)$$

$$\mathbf{P} = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \qquad (2.0.44)$$

From (2.0.19) we find the diagonal matrix

$$\mathbf{D} = \mathbf{P^TVP} = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \qquad (2.0.45)$$

$$\mathbf{D} = \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix} \qquad (2.0.46)$$

Hence using (2.0.44), (2.0.46) and (2.0.19) in (2.0.17) we get.

$$\mathbf{x}^T \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \mathbf{x} = 2 \qquad (2.0.47)$$

using (2.0.38) the above equation becomes,

$$\mathbf{y}^T \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{y} = 2 \qquad (2.0.48)$$

$$\mathbf{y}^T \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{y} = 1 \qquad (2.0.49)$$

Hence from (2.0.36) and (2.0.49) proved that change of origin and the directions of the coordinate axes (1.0.1) can be tranformed to (1.0.2) or (1.0.3)
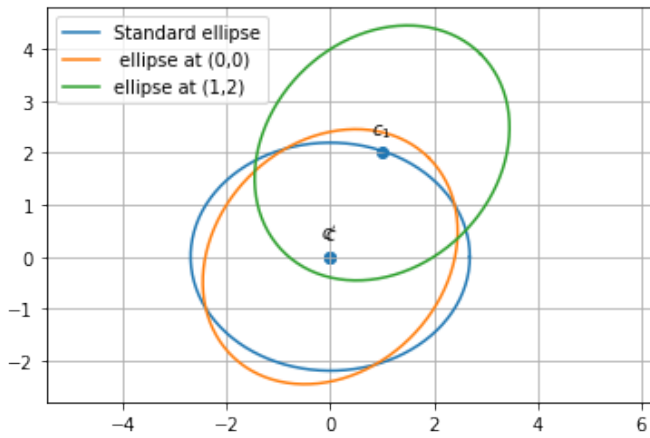

Fig. 0: Ellipse