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Ex 9.1

To show the model is a GLM we need the following

i) $Y_1 \dots Y_N$ are independent (satisfied as given in question)

ii) $Y_1 \dots Y_N$ have distributions belonging to the same exponential family in canonical form

i.e. $p_\theta(x) = s(x) t(\theta) e^{a(x) c(\theta)}$

$$= e^{\{a(x) c(\theta) + d(\theta) + f(x)\}}$$

where $c(\theta) = \theta$

iii) The relationship b/w the predictor variables

$\xi_i^T = (x_{i1} \dots x_{ip})$, $i = 1 \dots N$

and the response variable's expectation is given by a link function g

s.t. $\xi_i^T b = g(E(Y_i))$ $i = 1 \dots N$

a) Given, $\mu_j = E[Y_j] = b_1 e^{j b_2}$

$$\Rightarrow \log(E[Y_j]) = \log(b_1 e^{j b_2})$$

$$= \log b_1 + \log e^{(j b_2)}$$

$$= \log b_1 + j b_2$$

$$= (1 \;\; j) \begin{pmatrix} \log b_1 \\ b_2 \end{pmatrix} \quad \left[\text{given } \xi_i = (1 \;\; j)^T \right]$$

$$\Rightarrow \log(E[Y_j]) = \xi_i^T b^* \quad \text{where } b^* = \begin{pmatrix} \log b_1 \\ b_2 \end{pmatrix}$$

(reparameterization)

~~$\Rightarrow g = \log($~~

$$\Rightarrow \boxed{g(z) = \log(z)}$$

$$P(X=k) = \binom{k+r-1}{k} (1-p)^r p^k \qquad k \in \mathbb{N}_0$$

$$= \binom{k+r-1}{k} e^{\log (1-p)^r p^k}$$

$$P_p(X=k) = \binom{k+r-1}{k} e^{[r \log(1-p) + k \log p]}$$

(r is known and fixed)

$$= e^{[r \log(1-p) + k \log p + \log(\binom{k+r-1}{k})]}$$

reparameterize as $p^* = \log p$

$\Rightarrow p = e^{p^*}$

$$= e^{[r \log(1-e^{p^*}) + k p^* + \log(\binom{k+r-1}{k})]}$$

where $a(k) = k$

$c(p^*) = p^*$ (canonical)

$d(p^*) = r \log(1-e^{p^*})$ [r is a fixed constant]

$f(k) = \log\left(\binom{k+r-1}{k}\right)$

is a GLIM with

natural parameter $\boxed{p^* = \log p}$

c) $E[Y_j] = \mu_j = (\xi_j^T b)^2$

Given $b_1 + b_2 > 1$; $b_1, b_2 > 0$

$\Rightarrow$ ~~$\xi_j^T b = \pm\sqrt{\mu_j}$~~ $\Rightarrow \xi_i^T b = \pm\sqrt{E[Y_j]}$

but given $\xi_j = (1, j)^T$; $j = 1 \dots N$

and $b_1, b_2 > 0$

$\Rightarrow \xi_i^T b > 0$

$\Rightarrow \xi_i^T b = \sqrt{E[Y_j]}$

$$\boxed{g(z) = z^{1/2}}$$

$$f_\lambda(x) = \frac{\lambda^x}{x!} e^{-\lambda}$$

[poisson distribution]

$$= \frac{e^{\log \lambda^x} e^{-\lambda}}{x!} = \frac{e^{x \log \lambda - \lambda}}{x!} = \left(\frac{1}{x!}\right)\left(e^{-\lambda}\right)\left(e^{(x) \log \lambda}\right)$$

now reparametrize $\lambda^* = \log \lambda$

$$= \left(\frac{1}{x!}\right)\left(e^{-e^{\lambda^*}}\right)\left(e^{x \lambda^*}\right)$$

$\Rightarrow s(x) = \frac{1}{x!}$

$t(\lambda^*) = e^{-e^{\lambda^*}}$

$a(x) = x$

$c(\lambda^*) = \lambda^*$ (canonical)

is a GLM with $\boxed{\text{natural parameter } \lambda^* = \log \lambda}$

c) $E[Y_j] = \cancel{\mu_j} = z_j^T b$

$$\boxed{g(z) = z}$$

$$p_{x_{min}\lambda}(x) = \frac{\lambda\, x_{min}^{\lambda}}{x^{\lambda+1}}$$

given $x_{min} = 1$

$$= e^{\log\left(\frac{\lambda x_{min}^{\lambda}}{x^{\lambda+1}}\right)} = e^{\left[\cancel{\lambda \log x_{min}}^{\,0} + \log\lambda - (\lambda+1)\log x\right]}$$

reparametrize $\lambda = -\lambda^* - 1$

$$= e^{\left[\log(-\lambda^*-1) + \lambda^* \log x\right]}$$

where

$d(\lambda^*) = \log(-\lambda^*-1)$

$c(\lambda^*) = \lambda^*$ (canonical)

$a(x) = \log x$

$f(x) = 0$

is a GLM, with $\boxed{\text{natural parameter } \lambda^* = -\lambda - 1}$

d) $E[Y_j] = \mu_j = \begin{cases} \sqrt{z_i^T b} & \text{if } z_i^T b \geq \frac{N^2}{4} \\ \frac{2}{N} & \end{cases}$

$$\Rightarrow \boxed{g(z) = \begin{cases} z^2 & \text{if } z \geq N^2/4 \\ \frac{Nz}{2} & \text{else} \end{cases}}$$

$$p_{\alpha\beta}(x) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \mathbb{I}(x>0) \quad , \alpha \text{ is known}$$

$$= e^{\log\left[\frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \mathbb{I}(x>0)\right]}$$

$$\Rightarrow p_{\beta}(x) = e^{[\alpha \log\beta - \log(\Gamma(\alpha)) + (\alpha-1)\log x - \beta x]}$$

$$\cancel{= e^{\left[\frac{\log\beta - \frac{\log(\Gamma(\alpha))}{\alpha} + \frac{(\alpha-1)}{\alpha}\log x - \frac{\beta x}{\alpha}}{1/\alpha}\right]}}$$

$$\cancel{= e^{\left[\frac{\log\beta - \frac{\beta x}{\alpha}}{1/\alpha} + (\alpha-1)\log x - \log(\Gamma(\alpha))\right]}}$$

$$\cancel{= e^{\left[\frac{\log\beta - \frac{\beta x}{\alpha}}{1/\alpha} + (\alpha-1)\log x - \log[\Gamma(\alpha)]\right]}}$$

~~reparametrize $\beta^* = \frac{\beta}{\alpha} \Rightarrow \beta = \beta^* \alpha$~~

$$= e^{[\alpha\log\beta - \log(\Gamma(\alpha)) + [(\alpha-1) \; -\frac{1}{\beta}]\begin{bmatrix}\log(x) \\ x\end{bmatrix}]}$$

~~where $d(\beta) = \alpha \log\beta - \log(\Gamma(\alpha))$~~

~~$a(x) = \begin{bmatrix}\log(x) \\ x\end{bmatrix}$~~

~~$c(\beta) = [(\alpha-1) \; -\frac{1}{\beta}]$~~

reparametrize $\beta^* = [(\alpha-1) \; -\frac{1}{\beta}] = [\beta_1^* \quad \beta_2^*]$

$\Rightarrow \quad \alpha = \beta_1^* + 1$

$\beta = -\dfrac{1}{\beta_2^*}$

$$= e^{[(\beta_1^*+1)\log(-1/\beta_2^*) - \log\Gamma(\beta_1^*+1) + [\beta_1^* \ \beta_2^*]\begin{bmatrix}\log(x) \\ x\end{bmatrix}]}$$

where $d(\beta^*) = (\beta_1^*+1)\log\left(-\dfrac{1}{\beta_2^*}\right) - \log(\Gamma(\beta_1^*+1))$

$a(x) = \begin{bmatrix}\log(x \\ x\end{bmatrix}$

$c(\beta^*) = \beta^* = [\beta_1^* \ \ \beta_2^*]$

(canonical)

is a GLIM with natural parameter $\boxed{\beta^* = [(\alpha-1) \ \ -\tfrac{1}{\beta}]}$

exercise 3

a) $X_t = a + b\,\varepsilon_t$

under the measurable fct $f(x) = a + bx$ with a, b constant

the independence of X_t follows from ind. of ε_t. Moreover by 1.1.8. a) we get $X_t \sim \mathcal{N}(a, b^2\sigma^2)$.

Independent normally distributed r.v. are jointly Gaussian

$\Rightarrow \mathcal{L}(X_{t_1}, \dots, X_{t_n}) = \mathcal{N}(\mu, \Sigma)$ with $\mu = (a, \dots, a)^T$ $\Sigma = b^2\sigma^2 I_n$

for arbitrary $t_1, \dots, t_n \in \mathbb{Z}$ therefore also for $t+t_1, \dots, t+t_n \in \mathbb{Z}$

$\Rightarrow X_t$ is strictly stationary

$$\mathbb{E}(X_t^2) = \mathbb{E}((a+b\varepsilon_t)^2) = \mathbb{E}(a^2 + ab\,\varepsilon_t + b^2\varepsilon_t^2) = a^2 + ab\,\mathbb{E}(\varepsilon_t) + b^2 \underbrace{\mathbb{E}(\varepsilon_t^2)}_{=\sigma^2}$$

$$= a^2 + b^2\sigma^2 < \infty$$

(1): that $\mathbb{E}(\varepsilon_t^2) = \sigma^2$ follows from the fact that $\mathbb{E}(Y^2) = 1$ for $Y \sim \mathcal{N}(0,1)$ which can be shown by using moment generating functions

6.1.6 $\Rightarrow X_t$ to be weakly stationary

the mean is $\mathbb{E}(X_t) = \mathbb{E}(a + b\varepsilon_t) = a$

autocovariance function: if $t \neq 0$: $r_t = 0$ because of the independence of X_t

if $t = 0$: $r_0 = \operatorname{var}(X_s) = \mathbb{E}(X_s^2) - \mathbb{E}(X_s)^2 = a^2 + b^2\sigma^2 - a^2 = b^2\sigma^2$

b) $\mathbb{E}(X_t) = \mathbb{E}(\varepsilon_t)\cos(ct) + \mathbb{E}(\varepsilon_{t-1})\sin(ct) = 0$

$$\mathbb{E}(X_t^2) = \mathbb{E}((\varepsilon_t\cos(ct) + \varepsilon_{t-1}\sin(ct))^2) = \cos^2(ct)\underbrace{\mathbb{E}(\varepsilon_t^2)}_{\substack{=\sigma^2 \\ \text{by (1)}}} + \cos(ct)\sin(ct)\underbrace{\mathbb{E}(\varepsilon_t\varepsilon_{t-1})}_{\overset{\text{ind.}}{=}\mathbb{E}(\varepsilon_t)\mathbb{E}(\varepsilon_{t-1})=0} + \sin^2(ct)\underbrace{\mathbb{E}(\varepsilon_{t-1}^2)}_{\substack{=\sigma^2 \\ \text{by (1)}}}$$

$$= \sigma^2(\cos^2(ct) + \sin^2(ct)) = \sigma^2 < \infty$$

let $t = 1$: $\operatorname{cov}(X_s, X_{s+1}) = \mathbb{E}(\varepsilon_s\cos(cs) \cdot \varepsilon_s\sin(c(s+1)) = \cos(cs)\sin(c(s+1)) \cdot \underbrace{\mathbb{E}(\varepsilon_s^2)}_{=\sigma^2}$ not ind.

of s $\Rightarrow X_t$ not weakly stationary. Since the second moments exist by 6.1.6 X_t cannot be strictly stationary since otherwise it would be weakly stationary.

$\square$

exercise 4

X_t, Y_t weakly stationary + uncorrelated

$\mathbb{E}(Z_t) = \mathbb{E}(X_t + Y_t) = \mathbb{E}(X_t) + \mathbb{E}(Y_t) = \mu_X + \mu_Y \quad \forall t$

$\mathbb{E}(Z_t^2) = \mathbb{E}(X_t^2 + X_t Y_t + Y_t^2) = \underbrace{\mathbb{E}(X_t^2)}_{<\infty} + \mathbb{E}(X_t Y_t) + \underbrace{\mathbb{E}(Y_t^2)}_{<\infty}$

$r_s^Z = \operatorname{cov}(Z_s, Z_{s+t}) = \operatorname{cov}(X_s + Y_s, X_{s+t} + Y_{s+t})$

$= \operatorname{cov}(X_s, X_{s+t}) + \operatorname{cov}(Y_s, Y_{s+t}) + \underbrace{\operatorname{cov}(X_s, Y_{s+t}) + \operatorname{cov}(Y_s, X_{s+t})}_{=0 \text{ due to uncorrelatedness}}$

$= r_s^X + r_s^Y$

so if Z_t is a weakly stationary process then r_s^Z is its acf

By 6.1.10 to show that Z_t is weakly stationary it suffices to show that

r_t^Z is symmetric + pos def.

$r_t^Z = r_t^X + r_t^Y \overset{6.1.10}{=} r_{-t}^X + r_{-t}^Y = r_{-t}^Z \Rightarrow r_t^Z$ is symmetric

$\sum_{s,t=1}^{n} \bar{z}_t r_{t-s}^Z z_s = \underbrace{\sum_{s,t=1}^{n} \bar{z}_t r_{t-s}^X z_s}_{\geq 0 \text{ (by 6.1.10)}} + \underbrace{\sum_{s,t=1}^{n} \bar{z}_t r_{t-s}^Y z_s}_{\geq 0 \text{ (by 6.1.10)}} \geq 0$ for arbitrary $n \geq 1$ $z_1, \dots, z_n \in \mathbb{C}$

$\Rightarrow r_t^Z$ pos definite

It's left to show $F_Z = F_X + F_Y$ which follows directly from the fact that $r_t^Z = r_t^X + r_t^Y$ and linearity of the integral

$\square$