Ex 4.2 Pranaya Tomar & Martin Memmesheimer

a) We know, $R^2 = \frac{SS_{reg}}{TSS} = \frac{\sum_{j=1}^{N} (\hat{Y}_j - \bar{Y}_N)^2}{\sum_{j=1}^{N} (Y_j - \bar{Y}_N)^2}$

$\Rightarrow \underline{R^2 \geq 0}$ as it is positive (square)

Also, $R^2 = 1 - \frac{RSS}{TSS} = 1 - \frac{\sum_{j=1}^{N} \hat{Z}_j^2}{\sum_{j=1}^{N} (Y_j - \bar{Y}_N)^2}$

and again, $\frac{\sum_{j=1}^{N} \hat{Z}_j^2}{\sum_{j=1}^{N} (Y_j - \bar{Y}_N)^2} \geq 0$ as it is a positive square

$\Rightarrow \underline{1 - \frac{RSS}{TSS} \leq 1}$

$\Rightarrow 0 \leq R^2 \leq 1$

b) if $R^2 = 0$

$\Rightarrow \frac{SS_{reg}}{TSS} = 0 \Rightarrow SS_{reg} = 0 \Rightarrow \sum_{j=1}^{N} (\hat{Y}_j - \bar{Y}_N)^2 = 0$

$\Rightarrow$ now sum of non negative terms can be zero only if the terms are all individually 0.

$\Rightarrow (\hat{Y}_j - \bar{Y}_N)^2 = 0 \quad \forall j = 1 \dots N$

$\Rightarrow \hat{Y}_j = \bar{Y}_N \quad \forall j = 1 \dots N$

if $\hat{Y}_j = \bar{Y}_N \quad \forall j = 1 \dots N$

$\Rightarrow SS_{reg} = \sum_{j=1}^{N} (\hat{Y}_j - \bar{Y}_N)^2 = \sum_{j=1}^{N} (\hat{Y}_j - \hat{Y}_j)^2 = 0$

$\Rightarrow R^2 = \frac{0}{\sum_{j=1}^{N} (Y_j - \bar{Y}_N)^2} = 0$

$\Rightarrow R^2 = 0 \Leftrightarrow \hat{Y}_j = \bar{Y}_N \quad \forall j = 1 \dots N$

c) $R^2 = 1$ iff $R^2 = 1$

$\Rightarrow \frac{SS_{reg}}{TSS} = 1 \Rightarrow SS_{reg} = TSS$

$\Rightarrow \sum_{j=1}^{N} (\hat{Y}_j - \bar{Y}_N)^2 = \sum_{j=1}^{N} (Y_j - \bar{Y}_N)^2$

$\Rightarrow \sum_{j=1}^{N} (\hat{Y}_j^2 + \cancel{\bar{Y}_N^2} - 2\hat{Y}_j \bar{Y}_N) = \sum_{j=1}^{N} (Y_j^2 + \cancel{\bar{Y}_N^2} - 2 Y_j \bar{Y}_N)$

$\Rightarrow \sum_{j=1}^{N} (\hat{Y}_j^2 - Y_j^2) = 2\bar{Y}_N \sum_{j=1}^{N} (\hat{Y}_j - Y_j)$

$\Rightarrow \sum_{j=1}^{N} (\hat{Y}_j - Y_j)(\hat{Y}_j + Y_j) - 2\bar{Y}_N \sum_{j=1}^{N} (\hat{Y}_j - Y_j) = 0$

$\Rightarrow \sum_{j=1}^{N} (\hat{Y}_j - Y_j)(\hat{Y}_j + Y_j - 2\bar{Y}_N) = 0$

$\Rightarrow \hat{Y}_j = Y_j \;\forall\, j = 1 \dots N$

If $\hat{Y}_j = Y_j \;\forall\, j = 1 \dots N$

$\Rightarrow SS_{reg} = \sum_{j=1}^{N} (\hat{Y}_j - \bar{Y}_N)^2 = \sum_{j=1}^{N} (Y_j - \bar{Y}_N)^2$

and we know $TSS = \sum_{j=1}^{N} (Y_j - \bar{Y}_N)^2$

$\Rightarrow SS_{reg} = TSS$

$\Rightarrow R^2 = \frac{SS_{reg}}{TSS} = 1$

Hence $R^2 = 1 \iff \hat{Y}_j = Y_j \;\forall\, j = 1 \dots N$

d) $\|\hat{Z}\|_2 = \sqrt{\hat{Z}_1^2 + \cdots + \hat{Z}_N^2}$ and here ≥ 0

Now $R^2 = 1 - \frac{RSS}{TSS}$

and from a) we know that $R^2 \geq 0$

$$\Rightarrow 1 - \frac{RSS}{TSS} \geq 0$$

$$\Rightarrow RSS \leq TSS$$

$$\Rightarrow \sum_{j=1}^{N} \hat{Z}_j^2 \leq \sum_{j=1}^{N} (Y_j - \bar{Y}_N)^2$$

$$\Rightarrow \|\hat{Z}\|_2^2 \leq \sum_{j=1}^{N} (Y_j - \bar{Y}_N)^2$$

$$\Rightarrow \|\hat{Z}\|_2 \leq \sqrt{\sum_{j=1}^{N} (Y_j - \bar{Y}_N)}$$

Hence $0 \leq \|\hat{Z}\|_2 \leq \sqrt{\sum_{j=1}^{N} (Y_j - \bar{Y}_N)^2}$

ex 3

a) $$\frac{\partial W_N}{\partial b_1} = -2 \cdot \sum_{i=1}^{N} \frac{Y_i - b_1 - b_2 x_i}{\rho_i^2} \overset{!}{=} 0$$

$$\Leftrightarrow b_1 \cdot \sum_{i=1}^{N} \frac{1}{\rho_i^2} = \sum_{i=1}^{N} (Y_i - b_2 x_i) / \rho_i^2$$

$$\Rightarrow \hat{b}_1 = \frac{\sum_{i=1}^{N} (Y_i - \hat{b}_2 x_i)/\rho_i^2}{\sum_{i=1}^{N} \frac{1}{\rho_i^2}} \quad (1)$$

$$\frac{\partial W_N}{\partial b_2} = -2 \sum_{i=1}^{N} \frac{x_i (Y_i - b_1 - b_2 x_i)}{\rho_i^2} \overset{!}{=} 0$$

$$\Leftrightarrow \sum_{i=1}^{N} \frac{x_i (Y_i - b_1 - b_2 x_i)}{\rho_i^2} = 0$$

by using (1)

$$\sum_{i=1}^{N} \frac{x_i Y_i}{\rho_i^2} - \sum_{i=1}^{N} \frac{1}{\rho_i^2} x_i \cdot \left(\frac{\sum_{i=1}^{N} \frac{Y_i}{\rho_i^2} - b_2 \sum_{i=1}^{N} \frac{x_i}{\rho_i^2}}{\sum_{i=1}^{N} \frac{1}{\rho_i^2}} \right) - b_2 \sum_{i=1}^{N} \frac{x_i^2}{\rho_i^2} = 0$$

$$\Leftrightarrow b_2 \left(\frac{\left(\sum_{i=1}^{N} \frac{x_i}{\rho_i^2}\right)^2}{\sum_{i=1}^{N} \frac{1}{\rho_i^2}} - \sum_{i=1}^{N} \frac{x_i^2}{\rho_i^2} \right) = -\sum_{i=1}^{N} \frac{x_i Y_i}{\rho_i^2} + \frac{\sum_{i=1}^{N} \frac{x_i}{\rho_i^2} \cdot \sum_{i=1}^{N} \frac{Y_i}{\rho_i^2}}{\sum_{i=1}^{N} \frac{1}{\rho_i^2}}$$

$$\Rightarrow \hat{b}_2 = \frac{-\sum_{i=1}^{N} \frac{x_i Y_i}{\rho_i^2} + \frac{\sum_{i=1}^{N} \frac{x_i}{\rho_i^2} \cdot \sum_{i=1}^{N} \frac{Y_i}{\rho_i^2}}{\sum_{i=1}^{N} \frac{1}{\rho_i^2}}}{\left(\frac{\left(\sum_{i=1}^{N} \frac{x_i}{\rho_i^2}\right)^2}{\sum_{i=1}^{N} \frac{1}{\rho_i^2}} - \sum_{i=1}^{N} \frac{x_i^2}{\rho_i^2} \right)}$$

$$= \frac{\sum_{i=1}^{N} \frac{1}{\rho_i^2} \cdot \left(-\sum_{i=1}^{N} \frac{x_i Y_i}{\rho_i^2}\right) + \sum_{i=1}^{N} \frac{x_i}{\rho_i^2} \cdot \sum_{i=1}^{N} \frac{Y_i}{\rho_i^2}}{\left(\sum_{i=1}^{N} \frac{x_i}{\rho_i^2}\right)^2 - \sum_{i=1}^{N} \frac{1}{\rho_i^2} \cdot \sum_{i=1}^{N} \frac{x_i^2}{\rho_i^2}}$$

b) $$\hat{Y}_i = \rho_i^{-1} Y_i = \rho_i^{-1} b_1 + \rho_i^{-1} b_2 x_i + \rho_i^{-1} Z_i$$

$$= a_1 + a_2 x_i + \hat{Z}_i \quad \text{with } a_1 := \rho_i^{-1} b_1, \; a_2 := \rho_i^{-1} b_2$$

and $\hat{Z}_i \sim \mathcal{N}(0, \sigma^2)$ since $\mathbb{E}(\hat{Z}_i) = \rho_i^{-1} \mathbb{E}(Z_i) = 0$

and $\mathrm{var}(\hat{Z}_i) = \frac{1}{\rho_i^2} \mathrm{var}(Z_i) = \frac{1}{\rho_i^2} \rho_i^2 \sigma^2 = \sigma^2$

Therefore by 2.2.2. LS-estimators coincide with ML-estimators.

A is the design Matrix of the transformed (LRG) since $A^T A$ is positiv definite all eigenvalues are greater zero

$\Rightarrow Ker(A^T A) = \{0\} \Rightarrow A^T A$ is invertible

by 2.2.3 we get our estimators for a_1 and a_2. $\square$