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Ex2.1 a) for regression through origin $m(x) = \beta x$

The mean squared deviation $D(\beta) = \sum_{i=1}^{N} (y_i - \beta x_i)^2$

$\Rightarrow$ ~~$\sum y_i^2$~~

to minimize $D(\beta)$ set $\frac{d D(\beta)}{d\beta} = 0$

~~$\Rightarrow \sum x_i y_i$~~

$$\frac{d D(\beta)}{d\beta} = \sum -2x_i (y_i - \beta x_i) = 0$$

$$\Rightarrow \sum x_i y_i - \sum \beta x_i^2 = 0$$

$$\Rightarrow \boxed{\hat{\beta} = \frac{\sum_{i=1}^{N} x_i y_i}{\sum_{i=1}^{N} x_i^2}}$$

b) $MSE(\hat{\beta}) = var(\hat{\beta}) + [(bias(\hat{\beta})]^2$ (Lemma 1.2.2)

$bias(\hat{\beta}) = E(\hat{\beta}) - \beta$

$$E(\hat{\beta}) = E\left[\frac{\sum^{N} x_i y_i}{\sum_i^N x_i^2}\right] = \left(\frac{1}{\sum x_i^2}\right) E\left(\sum x_i y_i\right)$$

x_i's are constant

$$= \frac{\sum E(x_i y_i)}{\sum x_i^2} = \frac{\sum x_i E(y_i)}{\sum x_i^2}$$

(Now $E(y_i) = m(x_i) = \beta x_i$)

$$= \frac{\sum x_i \beta x_i}{\sum x_i^2} = \frac{\beta \cancel{\sum x_i^2}}{\cancel{\sum x_i^2}}$$

$$= \beta$$

$\Rightarrow$ $\text{bias}(\hat{\beta}) = \beta - \beta = 0$ $\Rightarrow$ unbiased

$\Rightarrow$ $MSE(\hat{\beta}) = var(\hat{\beta})$

$$= var\left(\frac{\sum x_i y_i}{\sum x_i^2}\right)$$

(as x_i's are constants, we take the square out)

$$= \left(\frac{1}{\sum x_i^2}\right)^2 var\left(\sum x_i y_i\right)$$

$$= \left(\frac{1}{\sum x_i^2}\right)^2 \sum var(x_i y_i)$$

$$= \left(\frac{1}{\sum x_i^2}\right)^2 \sum x_i^2 var(y_i)$$

$$= \frac{var(y_i)}{\sum_{i=1}^{N} x_i^2} = \frac{var(\beta x_i + z_i)}{\sum_{i=1}^{N} x_i^2}$$

$$= \frac{var(\beta x_i) + var(z_i)}{\sum_{i=1}^{N} x_i^2}$$

(Now, $var(\beta x_i) = 0$ as β and x_i are constants)

$$= \frac{0 + var(z_i)}{\sum_{i=1}^{N} x_i^2}$$

$$\Rightarrow \boxed{MSE(\hat{\beta}) = \frac{\sigma^2}{\sum_{i=1}^{N} x_i^2}}$$

Ex 2.3

$$L(\theta|Z) = p_\theta(Z) \quad , \quad Z = Z_1, \dots, Z_N \text{ iid}$$

for $\mathcal{L}(Z) = N(\mu, \sigma^2)$, $p(z) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(z-\mu)^2}{\sigma^2}}$

$$\Rightarrow L(\theta|Z) = p(z_1, \dots, z_n) = \prod_{i=1}^{n} p(z_i)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{z_i-\mu}{\sigma^2}\right)^2}$$

$$\Rightarrow L(\mu, \sigma^2|Z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(-\frac{1}{2\sigma^2}\right)\sum (z_i-\mu)^2} \quad (1)$$

$$\lambda = \frac{\max_{\sigma_0^2, \mu_0} L(\sigma_0^2, \mu_0|Z)}{\max_{\sigma^2, \mu^2} L(\sigma^2, \mu|Z)} = \frac{\max_{\sigma_0^2, \mu_0^2} L(\sigma_0^2, \mu_0|Z)}{L(\hat{\sigma}_N, \mu_N|Z)}$$

partially differentiating (1) wrt. σ^2 for $\mu_0 \in \Theta_0$ of H_0

we get $\sigma_0^2 = \cancel{\frac{\sum (z_i-\mu)^2}{N}} \frac{\sum (z_i-\mu_0)^2}{N}$ (2)

(same as in Ex 1.2.10)

$$\Rightarrow \lambda = \frac{L\left(\frac{\sum (z_i-\mu_0)^2}{N}, \mu_0 \,|\, Z\right)}{L\left(\frac{\sum (z_i-\bar{z})^2}{N}, \bar{z} \,|\, Z\right)}$$

$$\Rightarrow \lambda = \frac{\frac{1}{\sqrt{2\pi\left(\frac{\sum (z_i-\mu_0)^2}{N}\right)}} e^{-\frac{1}{2\left(\frac{\sum (z_i-\mu_0)^2}{N}\right)}\sum (z_i-\mu_0)^2}}{\frac{1}{\sqrt{2\pi\left(\frac{\sum (z_i-\bar{z})^2}{N}\right)}} e^{\frac{1}{2\left(\frac{\sum (z_i-\bar{z})^2}{N}\right)}\sum (z_i-\bar{z})^2}}$$

$$= \left(\sqrt{\frac{\sum (z_i-\bar{z})^2}{\sum (z_i-\mu_0)^2}}\right)\left(\frac{\cancel{e^{-\frac{N}{2}}}}{\cancel{e^{-\frac{N}{2}}}}\right) = \sqrt{\frac{\sum (z_i-\bar{z})^2}{\sum (z_i-\mu_0)^2}} \quad (3)$$

Now $\sum (z_i-\bar{z})^2 - \sum (z_i-\mu_0)^2$

$$= \sum \cancel{z_i^2} + \sum \bar{z}^2 - \sum 2 z_i \bar{z} - \sum (\cancel{z_i^2} + \mu_0^2 - 2 z_i \mu)$$

$$= \sum \bar{z}^2 - \sum 2 z_i \bar{z} - \sum \mu_0^2 + 2 \sum \mu_0 z_i$$

$$= N\bar{z}^2 - 2(N\bar{z})(\bar{z}) - N\mu_0^2 + 2\mu N \bar{z} \qquad \left[\sum z_i = N\bar{z}\right]$$

$$= N\bar{z}^2 - 2N\bar{z}^2 - N\mu_0^2 + 2\mu_0 n \bar{z}$$

$$= -N\bar{z}^2 - N\mu^2 + 2\mu_0 N\bar{z}$$

$$= -(N\bar{z}^2 + N\mu_0^2 - 2\mu N\bar{z})$$

$$= -N(\bar{z}-\mu_0)^2$$

$$\Rightarrow \sum (z_i-\bar{z})^2 + N(\bar{z}-\mu_0)^2 = \sum (z_i-\mu_0)^2$$

substituting in (3)

$$\lambda = \sqrt{\frac{\sum (z_i-\bar{z})^2}{\sum (z_i-\bar{z})^2 + N(\bar{z}-\mu_0)^2}} = \sqrt{\frac{1}{1 + \frac{N(\bar{z}-\mu_0)^2}{\sum (z_i-\bar{z})^2}}}$$

$$= \sqrt{\frac{1}{1 + \left(\frac{N}{N-1}\right)\frac{(\bar{z}-\mu_0)^2}{\underbrace{\frac{\sum (z_i-\bar{z})^2}{(n-1)}}_{s^2}}}} = \sqrt{\frac{1}{1 + \left(\frac{N}{N-1}\right)\frac{(\bar{z}-\mu_0)^2}{s^2}}}$$

$$= \sqrt{\frac{1}{1 + \left(\frac{1}{N-1}\right)\left(\frac{\sqrt{N}(\bar{z}-\mu_0)}{s}\right)^2}}$$

$$\Rightarrow \lambda = \sqrt{\frac{1}{1 + \left(\frac{1}{N-1}\right)\cancel{(\ )}^2 \left(T_N(z)\right)^2}}$$

$$\left(\text{as } T_N(z) = \sqrt{N}\,\frac{(\bar{z} - \mu_0)}{S}\right)$$

Now $\lambda \geq C_\alpha$

$$\Rightarrow \lambda^2 \geq C_\alpha^2$$

$$\Rightarrow \frac{1}{1 + \frac{T_N^2}{N-1}} \geq C_\alpha^2$$

$$\Rightarrow \frac{1}{C_\alpha^2} \geq 1 + \frac{T_N^2}{N-1}$$

$$\Rightarrow T_N^2 \leq \left(\frac{1}{C_\alpha^2} - 1\right)(N-1)$$

$\Rightarrow$ ~~$T_N \leq \sqrt{\ }$~~

lets just call the RHS as some constant C_β

$$\Rightarrow T_N^2 \leq C_\beta$$

$$\Rightarrow T_N \leq C_\beta \text{ and } T_N \geq -C_\beta$$

$\Rightarrow$ this is same as the t-test with a different critical value

$$C_\beta = \sqrt{\left(\frac{1}{C_\alpha^2} - 1\right)(N-1)}$$