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Ex 5.1

a) Given, $\mathcal{L}(Z) = \mathcal{N}_N(0, \Sigma)$

and $Y = X\beta + Z$

$$\Rightarrow Y = IZ + X\beta$$

$$\Rightarrow \mathcal{L}(Y) = \mathcal{N}_N(I \cdot 0 + X\beta, I \Sigma I^T)$$

$$= \mathcal{N}_N(X\beta, \Sigma) \quad \text{using lemma 1.1.8}$$

$$\Rightarrow \mathcal{L}(BY) = \mathcal{N}_N(B X\beta + 0, B \Sigma B^T)$$

using lemma 1.1.8

$$\Rightarrow \mathcal{L}(\underbrace{BY}_{\tilde{Y}}) = \mathcal{L}(\underbrace{BX}_{\tilde{X}}\beta + \underbrace{BZ}_{\tilde{Z}}) = \mathcal{N}_N(BX\beta, B\Sigma B^T)$$

$$\Rightarrow \mathcal{L}(\tilde{Y}) = \mathcal{L}(\tilde{X}\beta + \tilde{Z}) = \mathcal{N}_N(\tilde{X}\beta, B\Sigma B^T)$$

$$\Rightarrow \mathcal{L}(\tilde{Z}) = \mathcal{L}(I\tilde{Y} - \tilde{X}\beta) = \mathcal{N}_N(I\tilde{X}\beta - \tilde{X}\beta, IB\Sigma B^T I^T)$$

$$\Rightarrow \mathcal{L}(\tilde{Z}) = \mathcal{N}_N(0, B\Sigma B^T)$$

using lemma 1.1.8

Hence $\tilde{Y} = \tilde{X}\beta + \tilde{Z}$ is the transformed model with residue $\tilde{Z}$, $\mathcal{L}(\tilde{Z}) = \mathcal{N}_N(0, B\Sigma B^T)$

so Assumption of $E(\tilde{Z}) = 0$ of LRG is satisfied

Also for covariance matrix $B\Sigma B^T$ we can find an orthogonal matrix B such that $B^T = B^{-1}$ (by property of orthogonal matrices) and hence $B\Sigma B^T = B^{-1}\Sigma B$ could result in a diagonal matrix. And for a specific value of B we could make sure that the diagonal elements are σ^2, hence fulfilling the assumption of $var(\tilde{Z}) = \sigma^2 = 1$

$\Rightarrow$ we need to find B such that the new covariance $\tilde{\Sigma}$

$$\tilde{\Sigma} = B^{-1} \Sigma B = I$$ (so that diagonal elements are 1)

or equivalently $\tilde{\Sigma} = B \Sigma B^T = I$

Now as an example, we can assume $B = B^T$ for some specific B.

$$\Rightarrow B \Sigma B^T = B \Sigma B = I$$

$$\Rightarrow B \Sigma = B^{-1}$$

$$\Rightarrow \Sigma = B^{-1} B^{-1} = B^{-2}$$

$$\Rightarrow B = \Sigma^{-1/2}$$

so, for $B = \Sigma^{-1/2}$ we can have the variance of the transformed residual $\tilde{Z}$ ~~z~~ to be 1.

so $var(\tilde{Z}) = \sigma^2 = 1$ for $B = \Sigma^{-1/2}$

b) Using Theorem 2.2.3

LS estimator of b, $\hat{b} = (X^T X)^{-1} X^T Y$

$$\Rightarrow \hat{b} = ((BX)^T BX)^{-1} (BX)^T BY$$

$$= (X^T B^T B X)^{-1} X^T B^T B Y$$

Now $B^T B = B^{-1} B = I$

$\Rightarrow$ B is orthogonal

$$\Rightarrow \boxed{\hat{b} = (X^T X)^{-1} X^T Y}$$

ex 2 a)

Claim: $D_i = \frac{1}{d\hat{s}^2} \cdot \sum_{j=1}^{N} (\hat{Y}_j - \hat{Y}_{j(i)})^2$

Proof: since both sides have the factor $\frac{1}{d\hat{s}^2}$ it suffices to show

$$\hat{z}_i^2 \cdot H_{ii} / (1 - H_{ii})^2 = \sum_{j=1}^{N} (\hat{Y}_j - \hat{Y}_{j(i)})^2$$

$$\sum_{j=1}^{N} (\hat{Y}_j - \hat{Y}_{j(i)})^2 = \| \hat{Y} - \hat{Y}_{(i)} \|^2$$

$$= \| X\hat{b} - X\hat{b}_{(i)} \|^2 = \| X(\hat{b} - \hat{b}_{(i)}) \|^2$$

$$= \| X \frac{1}{1-H_{ii}} (X^T X)^{-1} x_i^T \underbrace{(Y_i - x_i \hat{b})}_{\hat{z}_i} \|^2$$

$$= \hat{z}_i^2 \cdot \frac{1}{(1-H_{ii})^2} \| X \cdot (X^T X)^{-1} x_i^T \|^2$$

$$\overset{(1)}{=} \hat{z}_i^2 \cdot \frac{H_{ii}}{(1-H_{ii})^2}$$

(1) $\| X (X^T X)^{-1} x_i^T \|^2 = x_i \cdot \underbrace{(X^T X)^{-T} X^T \cdot X}_{= I} \cdot (X^T X)^{-1} x_i^T$

$= x_i (X^T X)^{-1} x_i^T = H_{ii}$

$\square$