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ex 3

a) Gauß-Markov holds for Z_i uncorrelated since $ZZ^t = \sigma^2 I_n$ is the only argument in the proof where independence was used (and it also holds in case of uncorrelatedness)

$\Rightarrow$ LSE are the Blue for ϕ, ρ

$$\hat{\phi} = (X^T X)^{-1} X^T Y_1$$
$$\hat{\rho} = (X^T X)^{-1} X^T Y_2$$

b) show Z_3 uncorrelated

$$\mathrm{cov}(Z_{3i}, Z_{3j}) = \mathrm{cov}(Z_{1i} + Z_{2i}, Z_{1j} + Z_{2j})$$

$$= \mathrm{cov}(Z_{1i}, Z_{1j}) + \underbrace{\mathrm{cov}(Z_{1i}, Z_{2j}) + \mathrm{cov}(Z_{2i}, Z_{1j})}_{=0 \text{ (since } Z_1, Z_2 \text{ uncorrelated)}} + \mathrm{cov}(Z_{2i}, Z_{2j})$$

$$= \begin{cases} \sigma_1^2 + \sigma_2^2 & \text{if } i=j \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow \mathrm{cov}(Z_3) = (\sigma_1^2 + \sigma_2^2) I_n$$

Gauß Markov $\Rightarrow$ LSE is blue for ψ $(\hat{\psi} = (X^T X)^{-1} X^T Y)$

and by calculation:

$$\hat{\psi} = (X^T X)^{-1} X^T Y = (X^T X)^{-1} X^T (k_1 Y_1 + k_2 Y_2)$$

$$= k_1 \underbrace{(X^T X)^{-1} X^T Y_1}_{=\hat{\phi}} + k_2 \underbrace{(X^T X)^{-1} X^T Y_2}_{=\hat{\rho}}$$

$$= k_1 \hat{\phi} + k_2 \hat{\rho}$$

$\square$

6.4

Given, $P(V_k = -k^\alpha) = P(V_k = k^\alpha) = \frac{1}{2k^{2\alpha}}$

$P(V_k = 0) = 1 - \frac{1}{k^{2\alpha}}$

$\Rightarrow P(V_k = -k^\alpha \cup V_k = k^\alpha \cup V_k = 0)$

$= \frac{1}{2k^{2\alpha}} + \frac{1}{2k^{2\alpha}} + 1 - \frac{1}{k^{2\alpha}}$

$= 1$

$\Rightarrow V_k \in \{-k^\alpha, k^\alpha, 0\}$

$\Rightarrow E[V_k] = \sum^{k} V_k P(V_k)$

$= \frac{-k^\alpha}{2k^{2\alpha}} + \frac{k^\alpha}{2k^{2\alpha}} + 0\left(1 - \frac{1}{k^{2\alpha}}\right)$

$\Rightarrow E[V_k] = 0$

$var(V_k) = E[(V_k - \cancelto{0}{E[V_k]})^2] = E[V_k^2]$

and $var(V_k) = \sum P(V_k)(V_k - \cancelto{0}{E[V_k]})^2$

$= \left(\frac{1}{2k^{2\alpha}}\right)(-k^\alpha)^2 + \left(\frac{1}{2k^{2\alpha}}\right)(k^\alpha)^2 + \left(1 - \frac{1}{k^{2\alpha}}\right)(0)^2$

$= \left(\frac{1}{2k^{2\alpha}}\right)(k^{2\alpha} + k^{2\alpha}) = \frac{\cancel{2k^{2\alpha}}}{\cancel{2k^{2\alpha}}} = 1$

$\Rightarrow var(V_k) = 1$

$$v_N^2 = var\left(\sum_{k=1}^{N} V_k\right) = \sum_{k=1}^{N} var(V_k) = \sum_{k=1}^{N} (1) = N$$

(as V_k's are independent)

$$\Rightarrow v_N^2 \longrightarrow \infty \quad (N \to \infty)$$

Now checking the Lindeberg condition

$$= \sum_{k=1}^{N} E\left[\frac{V_k^2}{v_N^2} \mathbb{1}_{(\delta,\infty)}\left(\frac{|V_k|}{v_N}\right)\right] \longrightarrow 0 \quad (N \to \infty) \quad \forall \delta > 0$$

moving v_N^2 out as it is not a random variable

$$= \sum_{k=1}^{N} \left(\frac{1}{v_N^2}\right) E\left[V_k^2 \mathbb{1}_{(\delta,\infty)}\left(\frac{|V_k|}{v_N}\right)\right]$$

indicator condition $\frac{|V_k|}{v_N} > \delta \Rightarrow |V_k| > \delta v_N$

$$= \sum_{k=1}^{N} \left(\frac{1}{v_N^2}\right) E\left[V_k^2 \mathbb{1}_{(\delta v_N,\infty)}\left(|V_k|\right)\right]$$

Now we already know $v_N^2 \longrightarrow \infty$

$\Rightarrow v_N \to \infty$

$\Rightarrow$ the indicator interval gets smaller and smaller $\Rightarrow$ its output tends to 0 as $\delta v_N \to \infty$

$$\Rightarrow E\left[V_k^2 \mathbb{1}_{(\delta v_N,\infty)}\left(|V_k|\right)\right] \longrightarrow 0$$

as $E[0] = 0$

Hence the numerator $\longrightarrow 0$ and the denominator $v_N^2 \longrightarrow \infty$

$$\Rightarrow \sum_{k=1}^{N} \left(\frac{1}{v_N^2}\right) E\left[V_k^2 \mathbb{1}_{(\delta v_N,\infty)}\left(|V_k|\right)\right] \longrightarrow 0$$

as $N \to \infty$