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Ex 10.1

Given, $\{r_t, t \in \mathbb{Z}\}$ is a weakly stationary process

$r_0 = 1$

$r_1 = r_{-1} = \rho$

$r_t = 0, \ |t| > 1$

r	...	0	0	ρ	1	ρ	0	0
t	...	-3	-2	-1	0	1	2	3 ...

Using definition 6.1.13, if $\{r_t, t \in \mathbb{Z}\}$ is the acf of a stationary stochastic process, then

$$r_t = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{it\omega} f(\omega)\, d\omega \quad \text{for } t \in \mathbb{Z}$$

Also, as per remark 6.1.14,

if $\sum_{t=-\infty}^{\infty} |r_t| < \infty$

then spectral density $f(\omega) = \sum_{t=-\infty}^{\infty} r_t e^{-it\omega}$

Now $\sum_{t=-\infty}^{\infty} |r_t| = |0| + |0| \cdots + |0| + |\rho| + |1| + |\rho| + |0| + |0| \cdots |0|$

$= 1 + 2|\rho| < \infty$

$$\Rightarrow f(\omega) = \sum_{t=-\infty}^{\infty} r_t e^{-it\omega}$$

$$= 0 + 0 \cdots + r_{-1} e^{-i(-1)\omega} + r_0 e^{-i(0)\omega} + r_1 e^{-i(1)\omega} + 0 \cdots + 0$$

$$= \rho e^{i\omega} + 1 + \rho e^{-i\omega} = 1 + \rho(e^{i\omega} + e^{-i\omega})$$

Now according to Euler's formula

$$e^{i\omega} = \cos(\omega) + i \sin(\omega)$$

$$\Rightarrow 1 + \rho(e^{i\omega} + e^{-i\omega}) = 1 + \rho(\cos(\omega) + i\sin(\omega) + \cos(-\omega) + i\sin(-\omega))$$

$$= 1 + \rho(\cos(\omega) + \cos(\omega) + \cancel{i\sin(\omega)} - \cancel{i\sin(\omega)})$$

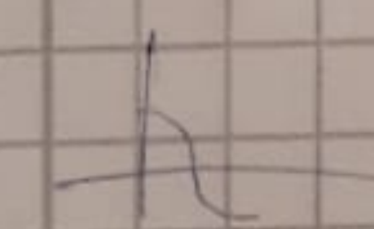
$$= 1 + 2\rho \cos\omega$$

also per remark 6.1.14, $f(\omega) \geq 0 \ \forall \omega \in [-\pi, \pi]$

$\Rightarrow \quad 1 + 2\rho \cos\omega \geq 0$

$\Rightarrow \quad |\rho| \leq \frac{1}{2}$ since $\cos(\omega) \in [-1, 1] \ \forall \omega \in [-\pi, \pi]$

$\Rightarrow \{\gamma_t, t \in \mathbb{Z}\}$ can not be the acf of a weakly stationary process if $\rho > \frac{1}{2}$



Ex 10.3

a) Given process $\{X_t = \varepsilon_t + b\,\varepsilon_{t-1},\ t \in \mathbb{Z}\}$, ε_t is white noise

to prove that this is a generalized linear process we need to prove that

i) $\{X_t, t \in \mathbb{Z}\}$ is a stationary process

ii) ~~$\{X_t\}$~~ $X_t = \sum_{k=-\infty}^{\infty} b_k\, \varepsilon_{t-k}$, $t \in \mathbb{Z}$

i.e. it is an infinite linear combination of white noise random variables.

to prove i) we need to show $\rightarrow E[X_t] = \mu \quad \forall t \in \mathbb{Z}$

$\rightarrow E[X_t^2] < \infty \quad \forall t \in \mathbb{Z}$

$\rightarrow \operatorname{cov}[X_s, X_{s+t}] = \gamma_t \quad \forall s,t \in \mathbb{Z}$

$\longrightarrow E[X_t] = E[\varepsilon_t + b\,\varepsilon_{t-1}] = E[\varepsilon_t] + b\,E[\varepsilon_{t-1}]$

$\Rightarrow E[X_t] = 0 + b(0) = 0$

as it is given that ε_t is white noise and we know $E[\varepsilon_t] = 0$

$\longrightarrow E[X_t^2] = \operatorname{var}[X_t] + E[\cancel{X_t}]^2$

$= \operatorname{var}(\varepsilon_t) + b^2 \operatorname{var}(\varepsilon_{t-1})$

(and $\operatorname{var}(\varepsilon_t) = \operatorname{var}(\varepsilon_{t-1}) = \sigma_\varepsilon^2$)

$= (1+b^2)\,\sigma_\varepsilon^2$

and $(1+b^2) < \infty$ and $\sigma_\varepsilon^2 < \infty$

$\Rightarrow E[X_t^2] = (1+b^2)\,\sigma_\varepsilon^2 < \infty$

$\longrightarrow \operatorname{cov}(X_s, X_{s+t}) = E[(X_s - E[\cancel{X_s}])(X_{s+t} - E[\cancel{X_{s+t}}])]$

$= E[X_s \cdot X_{s+t}]$

for $t = -1$, $\operatorname{cov}(X_s, X_{s-1}) = E[X_s \cdot X_{s-1}] = E[(\varepsilon_s + b\,\varepsilon_{s-1})(\varepsilon_{s-1} + b\,\varepsilon_{s-2})]$

$= E[\cancel{\varepsilon_s \varepsilon_{s-1}}] + b^2 E[\cancel{\varepsilon_{s-1} \varepsilon_{s-2}}] + b\,E[\varepsilon_{s-1}^2]$

$+ b\,E[\cancel{\varepsilon_s \varepsilon_{s-2}}]$

(as ε_t are uncorrelated)

$= b\,E[\varepsilon_{s-1}^2] = b\,\sigma_\varepsilon^2$

for $t=1$, $E[X_s X_{s+1}] = E[(\varepsilon_s + \theta \varepsilon_{s-1})(\varepsilon_{s+1} + \theta \varepsilon_s)]$

$$= E[\cancel{\varepsilon_s \varepsilon_{s+1}}]^{0} + \theta E[\varepsilon_s^2] + \theta E[\cancel{\varepsilon_{s-1} \varepsilon_{s+1}}]^{0} + \theta^2 E[\cancel{\varepsilon_{s-1}\varepsilon_s}]^{0}$$

$$= \theta E[\varepsilon_s^2] = \theta \sigma_\varepsilon^2$$

for $t=0$, $E[X_s X_s] = E[X_s^2] = var(X_s) = (1+\theta^2)\sigma_\varepsilon^2$

~~$\Rightarrow \gamma(t) = cov[X_s, X_{s+t}] = (1+\theta^2)\sigma_\varepsilon^2, t=0$~~

~~$, t$~~

~~$\theta\sigma_\varepsilon^2$~~

for $t > 1$
$t < 1$

$$E[X_s X_{s+t}] = E[(\varepsilon_s + \theta\varepsilon_{s-1})(\varepsilon_{s+t} + \theta \varepsilon_{s+t-1})]$$

$$= E[\varepsilon_s \varepsilon_{s+t} + \theta \varepsilon_s \varepsilon_{s+t-1} + \theta \varepsilon_{s-1}\varepsilon_{s+t} + \theta^2 \varepsilon_{s-1}\varepsilon_{s+t-1}]$$

$$= 0 + 0 + 0 + 0 = 0$$

$$\Rightarrow \gamma_t = cov(X_s, X_{s+t}) = \begin{cases} (1+\theta^2)\sigma_\varepsilon^2 & , t=0 \\ \theta\sigma_\varepsilon^2 & , t = \pm 1 \\ 0 & , \text{otherwise} \end{cases}$$

Hence $X_t = \varepsilon_t + \theta \varepsilon_{t-1}$ is a (weakly) stationary process

Now, to prove ii) we can show that $X_t = \varepsilon_t + \theta \varepsilon_{t-1}$ is a linear combination of ε_t's

as $X_t = \sum_{k=0}^{\infty} b_k \varepsilon_{t-k}, t \in \mathbb{Z}$

as $X_t = \dots$ ~~$+ X_t$~~ $+ b_0 \varepsilon_{t-0} + b_1 \varepsilon_{t-1} + \dots$

~~where all other than~~ where $b_k = 0$ for $k \neq 0$, $k \neq 1$

~~Hence~~

$b_0 = 1$

and $b_1 = \theta \;\forall \theta \in R$

Hence $\{X_t, t \in \mathbb{Z}\}$ a generalized linear process $\forall\, \theta \in \mathbb{R}$

b) $\rho_t = \frac{\gamma_t}{\gamma_0}$

$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta \cancel{\sigma_\varepsilon^2}}{(1+\theta^2)\cancel{\sigma_\varepsilon^2}} = \boxed{\frac{\theta}{1+\theta^2}}$

c) in markdown file

d) given $\sigma_\varepsilon^2 = 1$

for $\theta = 1/3$, $\rho_t = \begin{cases} 1 & , t = 1 \\ \frac{1/3}{1+1/9} = 0.3 & , t = \pm 1 \\ 0 & , \text{otherwise} \end{cases}$

for $\theta = 3$, $\rho_t = \begin{cases} 1 & , t = 0 \\ \frac{3}{1+9} = 0.3 & , t = \pm 1 \\ 0 & , \text{otherwise} \end{cases}$

Hence these 2 distributions have the same autocorrelation for different θ values and can't be distinguished without knowing the law of the white noise exercise 2 all moments of the normal distribution are calculated by the use of moment generating functions

$$X_t = \begin{cases} \varepsilon_t & \text{, if } t \text{ is even} \\ \frac{\varepsilon_{t-1}^2 - 1}{\sqrt{2}} & \text{if } t \text{ is odd} \end{cases}$$

$$\mathbb{E}(X_t) = \begin{cases} \mathbb{E}(\varepsilon_t) = 0 & \text{, if } t \text{ even} \\ \mathbb{E}\left(\frac{\varepsilon_{t-1}^2 - 1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \cdot \left(\underbrace{\mathbb{E}(\varepsilon_{t-1}^2)}_{\substack{=1 \\ \text{as } \varepsilon_{t-1} \sim \mathcal{N}(0,1)}} - 1 \right) = 0 & \text{, if } t \text{ odd} \end{cases}$$

$$\operatorname{var}(X_t) = \begin{cases} \operatorname{var}(\varepsilon_t) = 1 < \infty & \text{, } t \text{ even} \\ \operatorname{var}\left(\frac{\varepsilon_{t-1}^2 - 1}{\sqrt{2}}\right) = \frac{1}{2} \underbrace{\operatorname{var}(\varepsilon_{t-1}^2)}_{= \underbrace{\mathbb{E}(\varepsilon_{t-1}^4)}_{=3} - \underbrace{\mathbb{E}(\varepsilon_{t-1}^2)^2}_{=1}} = \frac{1}{2} \cdot 2 = 1 < \infty & \text{, } t \text{ odd} \end{cases}$$

it remains to be shown that the X_t are uncorrelated but not independent

let $s \neq t$ show: $\operatorname{cov}(X_s, X_t) = 0$

s,t both even: claim follows from ind. of $\varepsilon_t, \varepsilon_s$

s,t both odd: " " " " " " " and measurability of the function $x \mapsto \frac{x^2 - 1}{\sqrt{2}}$

one even, one odd:

$$\operatorname{cov}(X_s, X_t) = \mathbb{E}(X_s \cdot X_t) = \mathbb{E}\left(\frac{\varepsilon_{s-1}^2 - 1}{\sqrt{2}} \cdot \varepsilon_t\right) = \frac{1}{\sqrt{2}}\left(\mathbb{E}(\varepsilon_{s-1}^2 \varepsilon_t) - \mathbb{E}(\varepsilon_t)\right)$$

two cases: 1. $t \neq s-1 \Rightarrow \mathbb{E}(\varepsilon_{s-1}^2 \varepsilon_t) - \mathbb{E}(\varepsilon_t) \overset{\text{ind.}}{=} \mathbb{E}(\varepsilon_{s-1}^2)\underbrace{\mathbb{E}(\varepsilon_t)}_{=0} - \underbrace{\mathbb{E}(\varepsilon_t)}_{=0} = 0$

2. $t = s-1 \Rightarrow$ " $= \underbrace{\mathbb{E}(\varepsilon_t^3)}_{=0} - \underbrace{\mathbb{E}(\varepsilon_t)}_{=0} = 0$

$\Rightarrow$ The X_t are uncorrelated

$\Rightarrow$ X_t are white noise

suppose the X_t are independent:

$$X_2 = \varepsilon_2 , \quad X_3 = \frac{\varepsilon_2^2 - 1}{\sqrt{2}}$$

$\{X_2 \in [0,1]\}$ and $\{X_3 \in [-\frac{1}{\sqrt{2}}, 0]\}$ both obviously have prob < 1

$$\Rightarrow P(\{X_2 \in [0,1]\}) > P(\{X_2 \in [0,1]\}) \cdot P(\{X_3 \in [-\tfrac{1}{\sqrt{2}}, 0]\})$$

$$\overset{\text{ind.}}{=} P(\{X_2 \in [0,1]\} \cap \{X_3 \in [-\tfrac{1}{\sqrt{2}}, 0]\}) = P(\{X_2 \in [0,1]\}) \quad ↯$$

$x \mapsto \frac{x^2-1}{\sqrt{2}}$ is monotone
calculate both interval limits

$\Rightarrow$ X_t are not independent and therefore no strict white noise exercise 4

suppose there exists a stationary solution:

$$\mathrm{var}(X_t - \alpha^{n+1} X_{t-n-1}) = \mathrm{var}(\alpha X_{t-1} + \varepsilon_t - \alpha^{n+1} X_{t-n-1})$$

$$= \dots = \mathrm{var}(\alpha^{n+1} X_{t-n-1} + \varepsilon_t + \dots + \varepsilon_{t-n} - \alpha^{n+1} X_{t-n-1})$$

$$= \mathrm{var}(\varepsilon_t + \dots + \varepsilon_{t-n}) \underset{\substack{\uparrow \\ \varepsilon_t \text{ white} \\ \text{noise}}}{=} \mathrm{var}(\varepsilon_t) + \dots + \mathrm{var}(\varepsilon_{t-n}) = n \cdot \sigma_\varepsilon^2 \xrightarrow{n \to \infty} \infty$$

by assumpt. of stationarity: $\mathbb{E}(X_t^2) < \infty$, $\mathbb{E}(X_t) = \mathbb{E}(X_{t-n-1}) \overset{\substack{\text{wlog by shifting} \\ \text{the process by a constant} \\ \downarrow}}{=} 0$

if we show $\mathbb{E}(X_t^2) \geq \mathrm{var}(X_t - \alpha^{n+1} X_{t-n-1})$ for all n we get a contradiction

to $\mathbb{E}(X_t^2) < \infty$ and therefore our claim