RBE 510 – Multi-Robot Systems
Lecture 6: Sensing and Sensor Placement

Kevin Leahy

September 9, 2025

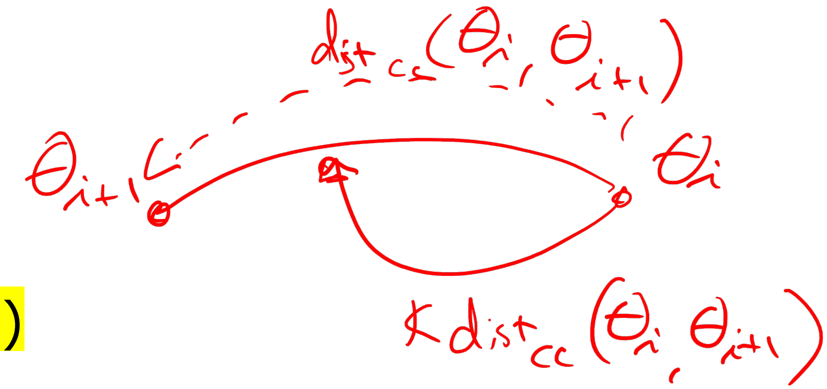
Admin

- HW 1 solutions and grades posted
  - Looks good overall. Questions?
- HW 2 due Friday midnight
  - Questions?
- Second paper posted this afternoon
  - Next presentation on Tuesday, September 16
- Office hours this week Thursday (virtual)

$$\theta_i(l+1) = (1-K)\theta_i(l) + K\theta_{i+1}(l)$$

robot index

time index



HW 2 Tips

- Your step function should update the physical state of your robot using the value computed with the **`ctl`** function.

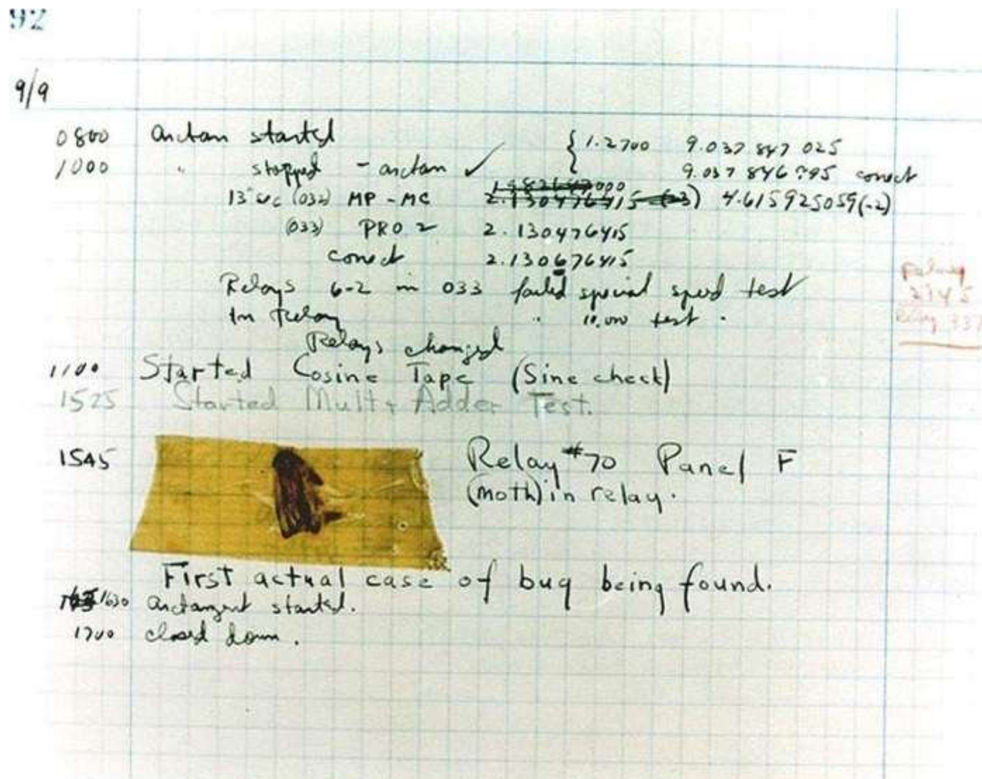
$\Theta_i(t+1) = step(\Theta_i, ctl(\Theta_i))$

- You should constrain the robot's state to be between **`0`** and **`2*pi`**. The modulo operator (**`%`**) should help with this.

- It is possible to encounter errors when the magnitude of the value computed by **`ctl`** gets too small. I recommend setting a minimum value for the output of **`ctl`** (say, **`0.001`**), and not allowing the output to be smaller than that value

- Play around with time horizon if agents are not quite converging

Today



9/9
0800 Anctan started
1000 stopped - anctan
(033) PRO 2 2.130476415
correct 2.130676415
Relays 6-2 in 033 failed special speed test
Relays changed
1100 Started Cosine Tape (Sine check)
1525 Started Mult + Adder Test.
1545 Relay #70 Panel F
(moth) in relay.
First actual case of bug being found.
1700 closed down.

Today

- Sensing introduction

- Sensing models

- Static sensor placement
  - Complexity
  - Information theory
  - Submodularity
  - Approaches to optimization

- **Goal: build up some models of sensing and understand the problems of complexity**

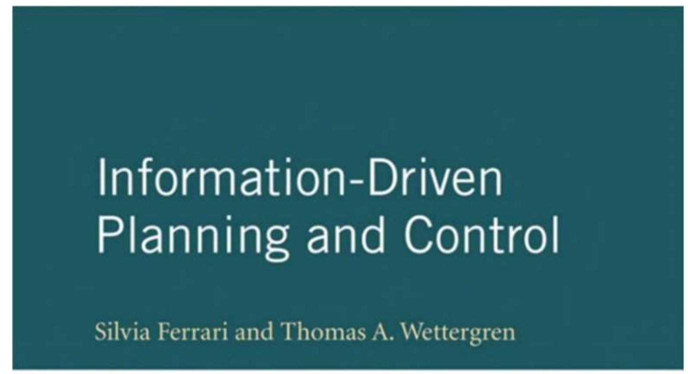
Overview: Multi-robot Sensing and Estimation

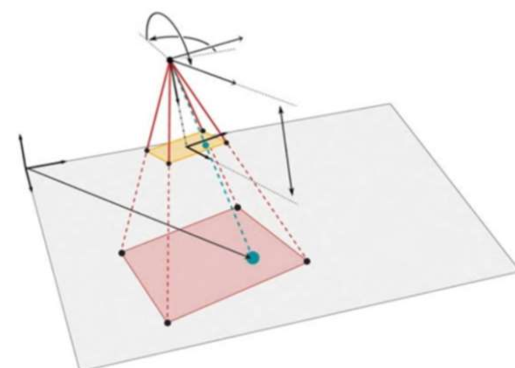
Switching Gears

- Lots of motion and control before
- Now looking at sensing and estimation
- Given a sensing objective and n robots:
  - Where should I put the robots to maximize what I sense?
  - How should I move them?
  - How do I quantify sensing quality?

Good resources

- Lecture notes will include citations for important resources
- Includes the books we've been using, as well as research papers
- Information-Driven Planning and Control (2021) is a great reference for diving in further





Motivation

The resulting distribution is shown in Figure 5.

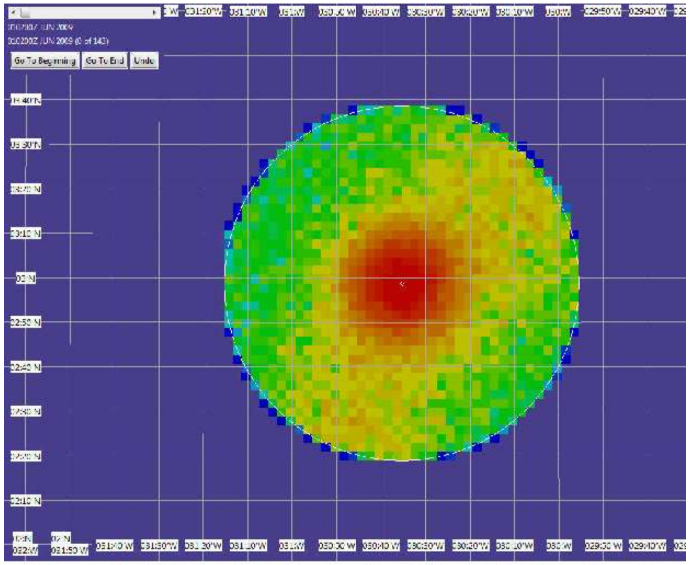


Figure 5. PDF for Impact Location Prior to Surface Search

We will focus primarily on sensing the environment and objects in it

Problem Characteristics

- Quality to be sensed:
  – Static
  – Dynamic

- Tasks:
  – Coverage
  – Search

- We are especially interested in stochasticity in sensing and how to manage it

Roadmap for the next few lectures

- Static sensing – get used to objectives, main difficulties, and prime our intuition
- Coverage – adding motion into the sensor placement problem
- Search – adding motion into the agents *and* the environment
  - This will help us segue into problems of decision-making in later lectures

Probability and Sensing

Probability Terms

- Sample space S of all possible outcomes of an experiment

- A subset of outcomes is an event $\mathcal{E}$

- Example:
  - Coin flip: $S = \{heads, tails\}$; $\mathcal{E}_1 = heads$, $\mathcal{E}_2 = tails$
  - 2 coin flips: $S = \{hh, ht, th, tt\}$; $\mathcal{E}_1 = tt, \ldots$ $\mathcal{E}_2 = \{ht, th\}$

- Note: any subset of outcomes is an event

$\hookrightarrow 2^{|S|}$ possibilities

Axioms of Proability

1. Non-negativity: $P(\mathcal{E}) \geq 0$

2. Normalization: $P(S) = 1$

3. Additivity: $P(\mathcal{E}_1 \cup \mathcal{E}_2) = P(\mathcal{E}_1) + P(\mathcal{E}_2)$ for $\mathcal{E}_1 \cap \mathcal{E}_2 = \{\emptyset\}$ (mutually exclusive events)

Mutually exclusive events have $P(\mathcal{E}_1 \cap \mathcal{E}_2) = 0$

Consequences of the Axioms

1. $P(\mathcal{E}^C) = 1 - P(\mathcal{E})$, where $\mathcal{E}^C = S \setminus \mathcal{E}$

2. $P(\mathcal{E}_1 \cup \mathcal{E}_2) = P(\mathcal{E}_1) + P(\mathcal{E}_2) - P(\mathcal{E}_1 \cap \mathcal{E}_2)$

3. $P(\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3) = \underbrace{P(\mathcal{E}_1) + P\left(\mathcal{E}_1^C \cap \mathcal{E}_2\right)}_{P(\mathcal{E}_1 \cup \mathcal{E}_2)} + P\left(\mathcal{E}_1^C + \mathcal{E}_2^C + \mathcal{E}_3\right)$

Example and Review (Ferrari and Wettergen 4.1)

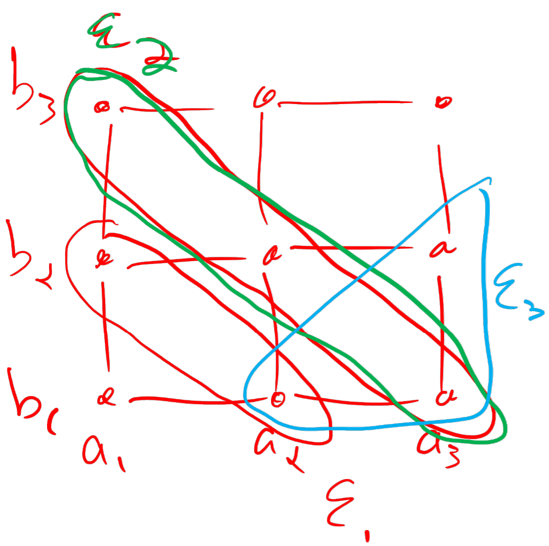
2 sensors a, b :

1: detect $\quad A := \{a_1, a_2, a_3\}$ outcomes

2: false alarm

3: no detection $\quad B := \{b_1, b_2, b_3\}$

$S = \underline{A \times B} = \{a_1 b_1, a_1 b_2, \dots a_3 b_3\}$ $\quad$ m outcomes $\quad$ n sensors $\quad m^n = |S|$



$\mathcal{E}_1$: 1 detection + 1 false alarm

$\mathcal{E}_1 \cap \mathcal{E}_2 = \{\emptyset\}$ $\quad P(\mathcal{E}_1 \cap \mathcal{E}_2) = 0$

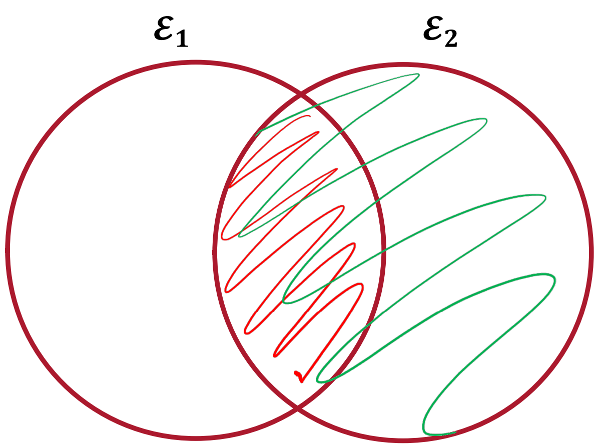
$\mathcal{E}_2 \cap \mathcal{E}_3 \neq \{\emptyset\}$

Example and Review (Ferrari and Wettergen 4.1)

Conditional Probability

- Relationship among probabilities of different events can be represented using conditional probability

- For two events $\mathcal{E}_1$ and $\mathcal{E}_2$, conditional probability of $\mathcal{E}_1$ given $\mathcal{E}_2$:



$$P(\mathcal{E}_1 \mid \mathcal{E}_2) = \frac{P(\mathcal{E}_1 \cap \mathcal{E}_2)}{P(\mathcal{E}_2)}$$ in red
in green

Conditional Probability

- We can treat $\mathcal{E}_2$ as the *prior*

- Then, we can write

$$P(\mathcal{E}_1 \cap \mathcal{E}_2) = P(\mathcal{E}_1 | \mathcal{E}_2) P(\mathcal{E}_2)$$
$$= P(\mathcal{E}_2 | \mathcal{E}_1) P(\mathcal{E}_1)$$

Independence, Conditional Independence

- Consider events $\mathcal{E}_1$ and $\mathcal{E}_2$

- They are independent iff

$$P(\mathcal{E}_1 \cap \mathcal{E}_2) = P(\mathcal{E}_1)P(\mathcal{E}_2)$$

- They are conditionally independent given $\mathcal{E}_3$ iff

$$P(\,\mathcal{E}_1 \cap \mathcal{E}_2 \mid \mathcal{E}_3\,) = P(\,\mathcal{E}_1 \mid \mathcal{E}_3\,)P(\,\mathcal{E}_2 \mid \mathcal{E}_3\,)$$

Conditional Probability

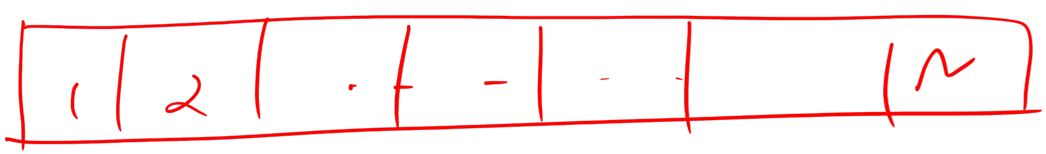
- Consider $\mathcal{E}$ occurs only if $\mathcal{E}_1, \ldots, \mathcal{E}_n$ all occur
- $P(\mathcal{E}) = P(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \cdots \cap \mathcal{E}_n) = P(\mathcal{E}_1) \dfrac{P(\mathcal{E}_1 \cap \mathcal{E}_2)}{P(\mathcal{E}_1)} \cdots \cdots \dfrac{P(\bigcap_{i=1}^{N} \mathcal{E}_i)}{P(\bigcap_{i=1}^{N-1} \mathcal{E}_i)}$

$$= P(\mathcal{E}_1) P(\mathcal{E}_2 \mid \mathcal{E}_1) \cdots P\left(\mathcal{E}_n \mid \bigcap_{i=1}^{n-1} \mathcal{E}_i\right)$$

Conditional Probability

- If S is partitioned into $\mathcal{E}_1, \ldots, \mathcal{E}_n$ disjoint events = mutually exclusive
- For any other event X in the environment
- $P(X) = P(X \cap \mathcal{E}_1) + \cdots + P(X \cap \mathcal{E}_n)$

$$= P(X \mid \mathcal{E}_1) P(\mathcal{E}_1) + \ldots + P(X \mid \mathcal{E}_n) P(\mathcal{E}_n)$$



Bayes' Rule

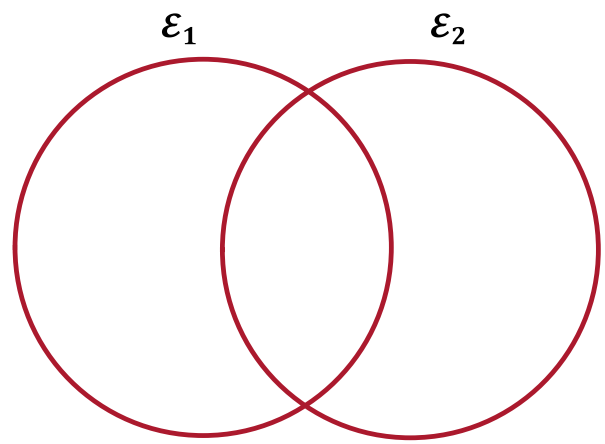
prior

$$P(\mathcal{E}_2|\mathcal{E}_1) = \frac{\overbrace{P(\mathcal{E}_1|\mathcal{E}_2)}^{\text{measurement likelihood}}P(\mathcal{E}_2)}{\underbrace{P(\mathcal{E}_1)}_{\text{total probability}}}$$

or

normalization factor

$$= \eta P(\mathcal{E}_1|\mathcal{E}_2)P(\mathcal{E}_2)$$



Bayes Rule for Multiple Events

- For $\mathcal{E}_1, \ldots, \mathcal{E}_n$, mutually exclusive events, with only one true event
- Probability of event $\mathcal{E}_i$ given X can be obtained as

- $P(\mathcal{E}_i \mid X) = \frac{P(X|\mathcal{E}_i)}{P(X)} = \dfrac{P(X|\mathcal{E}_i)P(\mathcal{E}_i)}{\sum_{i=1}^{N} P(X|\mathcal{E}_i)P(\mathcal{E}_i)}$

Example and Review (Ferrari and Wettergen 4.2)

Example and Review (Ferrari and Wettergen 4.2)

Sensing Models and Belief

Classification

- Classification: inferring value of hidden categorical variable from multiple (possibly heterogeneous) sensor measurements

- For us

$$P(\underbrace{X = x}_{\text{something about environment}} \mid \underbrace{Y = y})$$

(i.e. target location)

measurements we obtain

Sensor Model

- Let’s put a sensor in location q in one of two locations, s_1 or s_2
- $S = \{s_1, s_2\}$
- There is a target in one of the two locations but we don’t know which
- How to characterize the sensor?

Sensor Model

- Measurement likelihood

α, β typically small

max α = max β = 0.5

$P(y \mid x)$

- If we put the sensor where the target is
  - $P(y = 1 \mid q = s) = (1 - \beta)$
  - $P(y = 0 \mid q = s) = \beta$

missed detection / false negative

- If we put it somewhere else
  - $P(y = 1 \mid q \neq s) = \alpha$

false positive / false alarm

  - $P(y = 0 \mid q \neq s) = (1 - \alpha)$

Representing Prior Knowledge

- If we know there is *exactly one* target, then we can represent probability it is in s_1 or s_2

- $P(s = s_1) = b(s_1)$
- $P(s = s_2) = b(s_2) = 1 - b(s_1)$

- We know $P(s = s_1) + P(s = s_2) = 1$

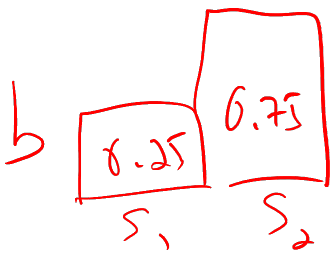
- This is a discrete probability distribution we call the **belief**

Belief

- Belief can be updated with Bayes' rule

$$\underbrace{P(S=s\mid Y=y;q)}_{\text{posterior}} = \frac{\overbrace{P(Y=y\mid S=s;q)}^{\text{measurement likelihood}}\,\overbrace{P(S=s)}^{\text{prior}}}{\underbrace{P(Y=y;q)}_{\text{total probability}}}$$

Updating Belief Example



$\alpha = \beta = 0.05$; measure 1 at loc s_1

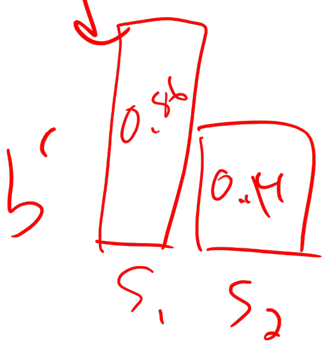
$$b'(s_1) = \frac{P(y=1 \mid q=s_1)\, b(s_1)}{P(y=1)} = \frac{P(y=1 \mid q=s_1)\, b(s_1)}{P(y=1 \mid q=s_1)\, b(s_1) + P(y=1 \mid q \neq s)\, b(s_2)}$$

$(1-\beta)$

α

$$= \frac{0.95(0.25)}{0.95(0.25) + 0.05(0.75)}$$

$\simeq 86\%$



Belief

- Target is in one of 2 locations

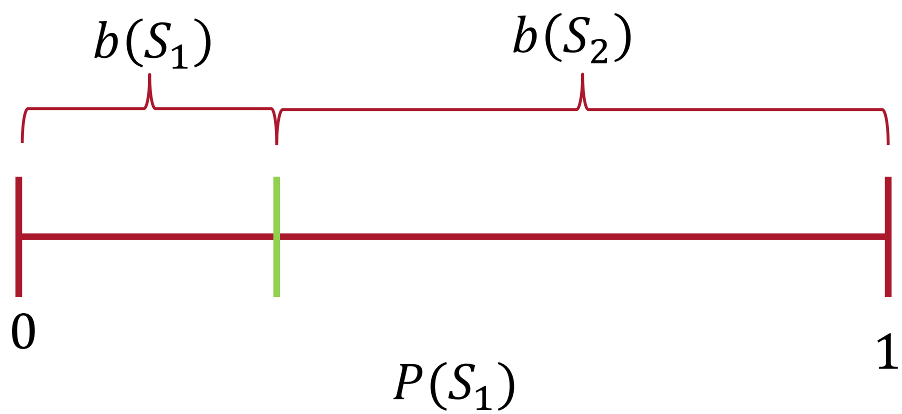
- $b(S_1) = P(S = S_1)$
- $b(S_2) = P(S = S_2)$

- $\sum_s b(s) = 1 \rightarrow b(S_1) + b(S_2) = 1$
- $b(S_2) = 1 - b(S_1)$

- Our entire belief is capture by $b(S_1)$

Belief

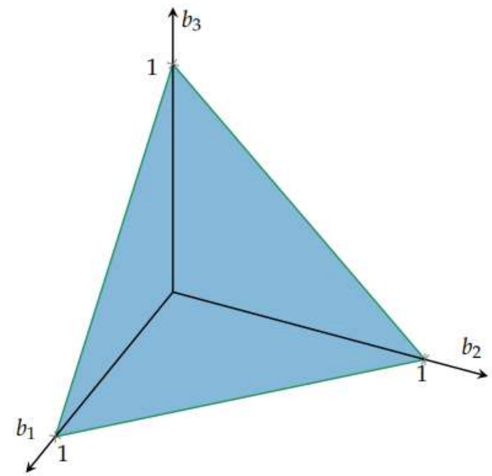
- $\sum_s b(s) = 1 \rightarrow b(S_1) + b(S_2) = 1$
- $b(S_2) = 1 - b(S_1)$



- In 2D, belief is a simple partition

Belief is a simplex

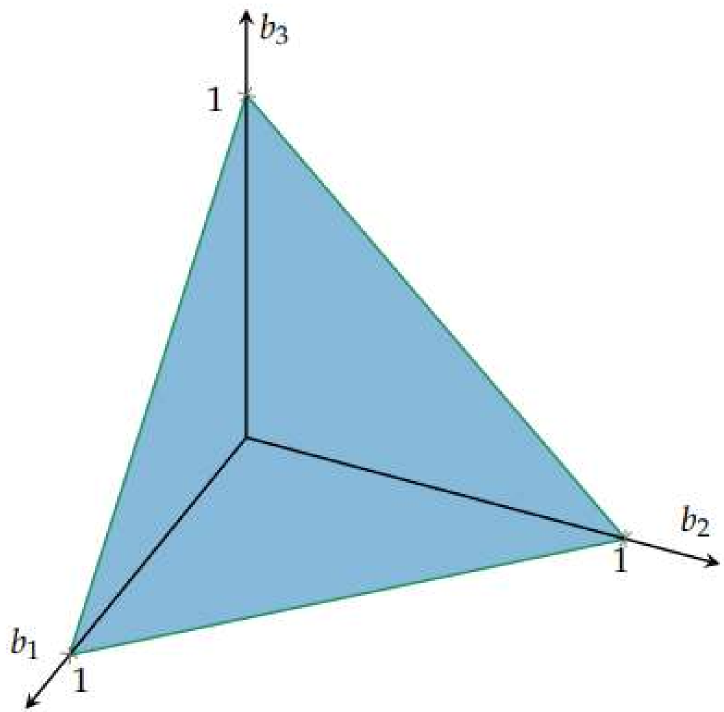
- In 3D (and in general) a point in discrete belief space lies on a ***simplex:***
  - $b(s) \geq 0$ for all $s \in S$
  - $\sum_s b(s) = 1$



- Handy vector form:
  - $\boldsymbol{b} \geq \mathbf{0}$
  - $\mathbf{1}^T \boldsymbol{b} = \mathbf{1}$

- For an n-dimensional space, a simplex is an $n-1$-dimensional space embedded in n dimensions

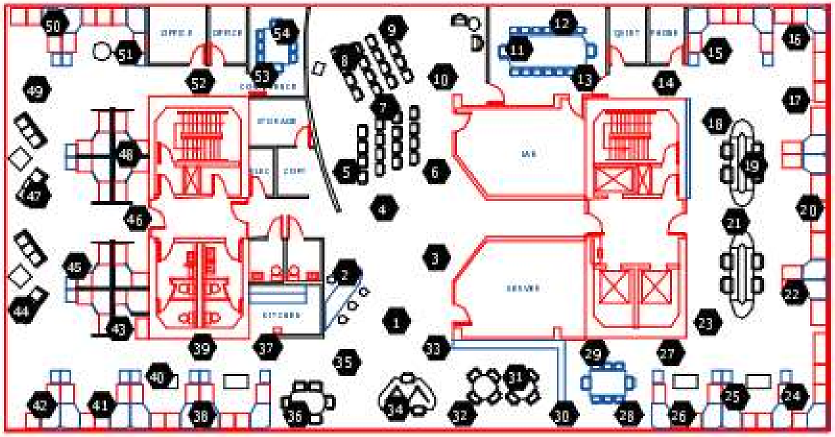
3D Simplex

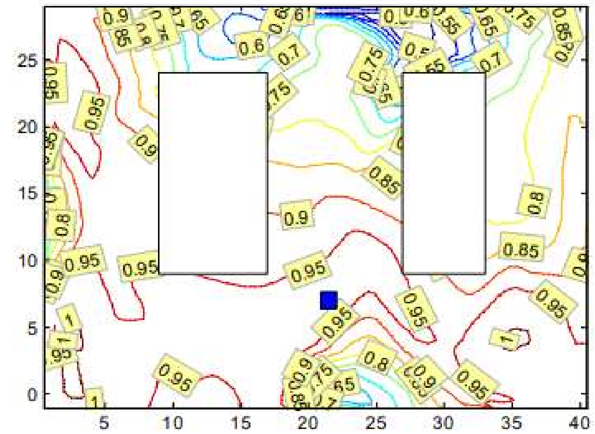


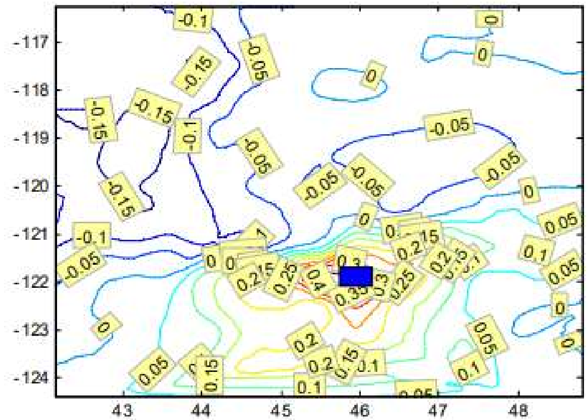
- If $b_1 = 1$ then $b_2 = b_3 = 0$
- Same if b_2 or b_3 equal 1
- If $b_1 = 0$, then it is reduced to the 2D partition between b_2 and b_3
- Same if b_2 or b_3 equal 0

Static Sensor Placement

Examples



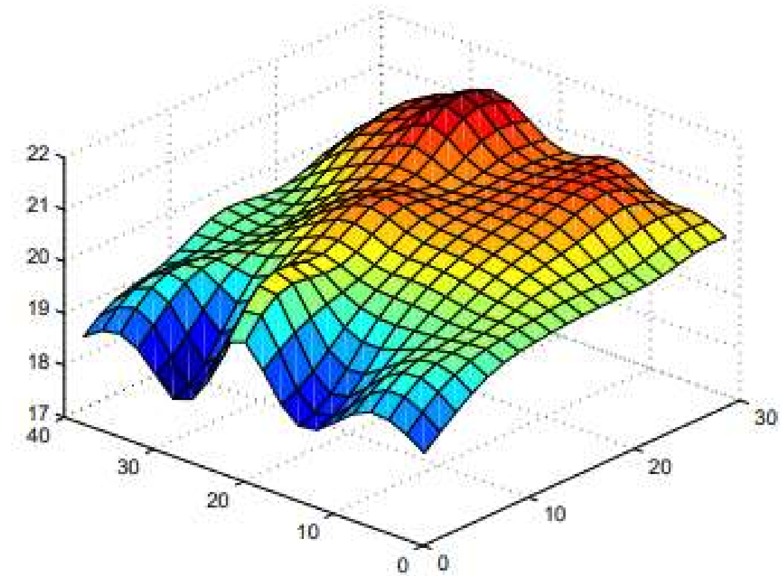




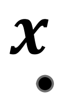
Problem Definition

- Given
  - Background probability distribution
  - Robot sensor model
  - Estimated impact of measurements on probability distribution
  - Objective function over the distribution

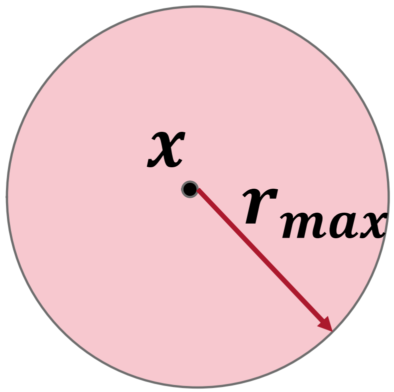
- Place a set of robots in locations to maximize your objective function



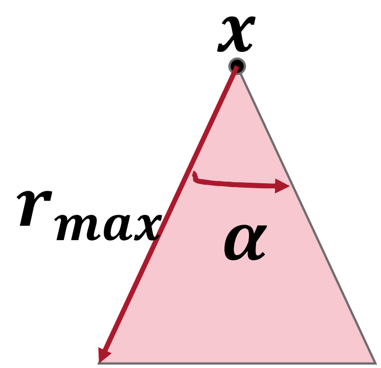
Field-of-View (FoV) Sensors



Point

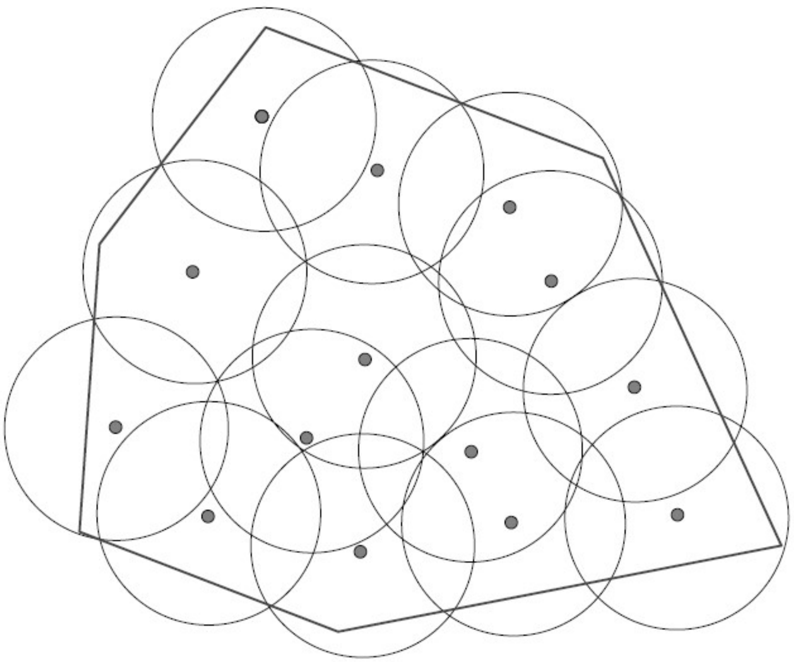


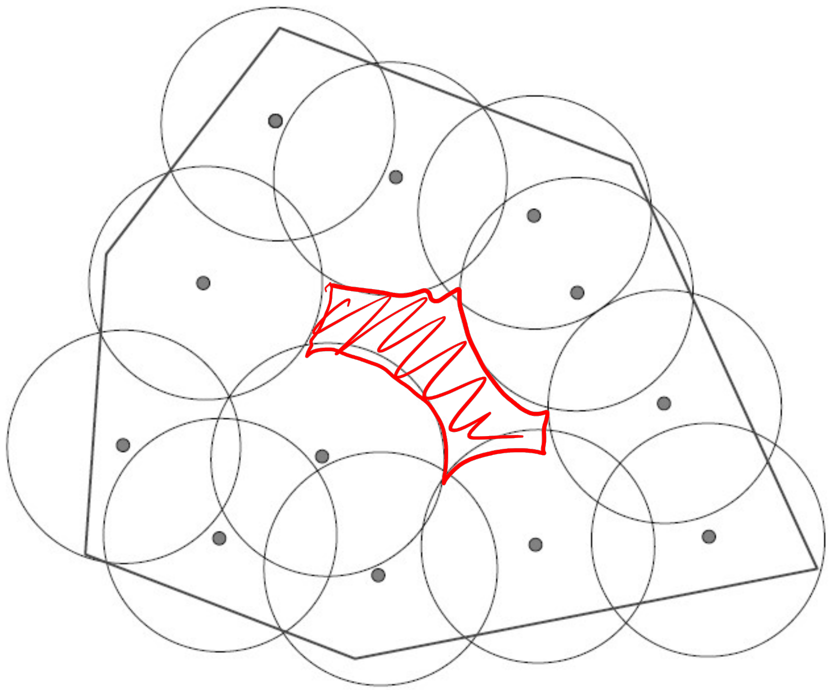
Omnidirectional



Directional

Covering the Environment





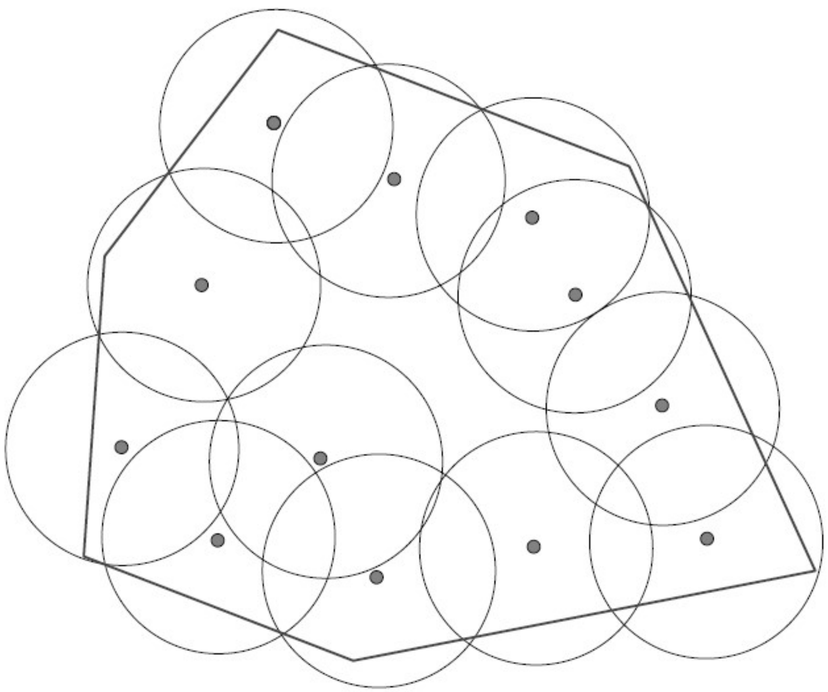
Coverage

- For uniform coverage (i.e., all areas equally “interesting”)

- Coverage criterion is $\frac{Total\ area\ of\ sensor\ footprints}{Total\ ROI}$

- Can be weighted by “interest”

Incomplete Coverage



- If we can't cover the whole environment
  - Where is the best set of places to put our limited number of sensors?
  - Is there a number of sensors that is "good enough" without placing all available sensors?
- Discretize and solve?

Combinatorics

- Given
  - n discrete locations
  - k robots
- Pick a subset of k locations that maximize your objective
- How to solve?
- Let's try brute force…

Combinatorics

- How many options are there to try?
- $\binom{n}{k}$
- Equivalent to $\frac{n!}{k!(n-k)!}$
- How big is this?

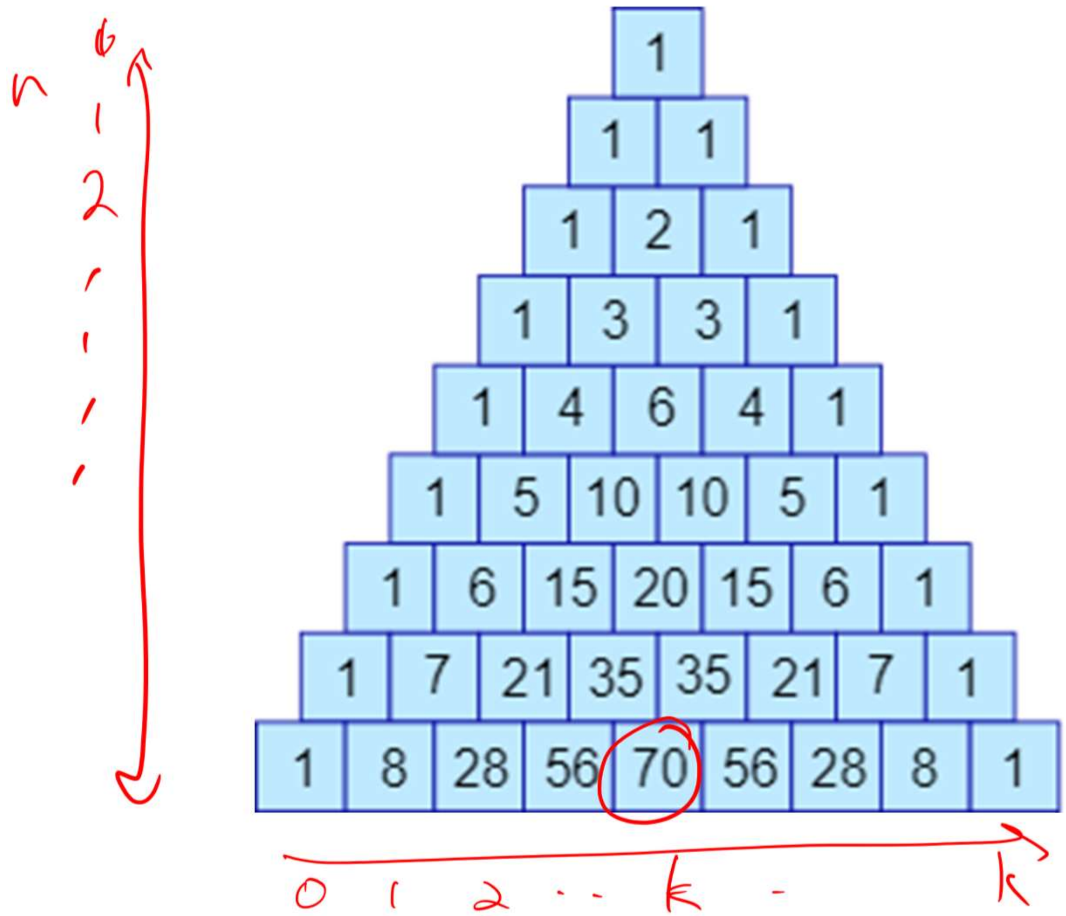
Binomial Coefficient – How many combinations?

- Let's start with $n = 2$
  - $k = 0$ 1
  - $k = 1$ 2
  - $k = 2$ 1

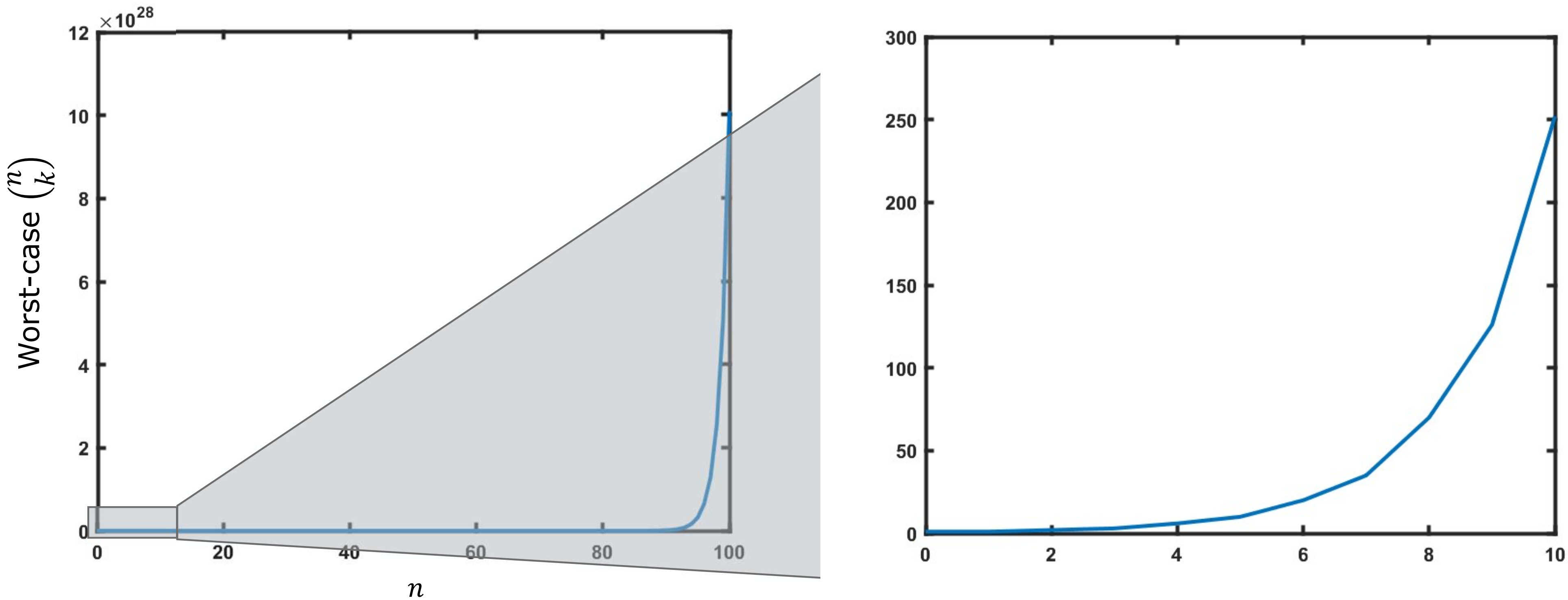
$\{s_1, s_2\}$

- How about $n = 5$
  - $k = 0$ 1
  - $k = 1$ 5
  - $k = 2$
  - $k = 3$
  - $k = 4$
  - $k = 5$

Pascal’s Triangle



Combinatorics



A way forward

- What can we do?
  - We know that in general we can’t get to optimality
- But maybe heuristics can help?
- In an ideal world, we can bound our heuristic
  - Within a certain distance of the optimal solution
  - Let’s try!

Information Theory

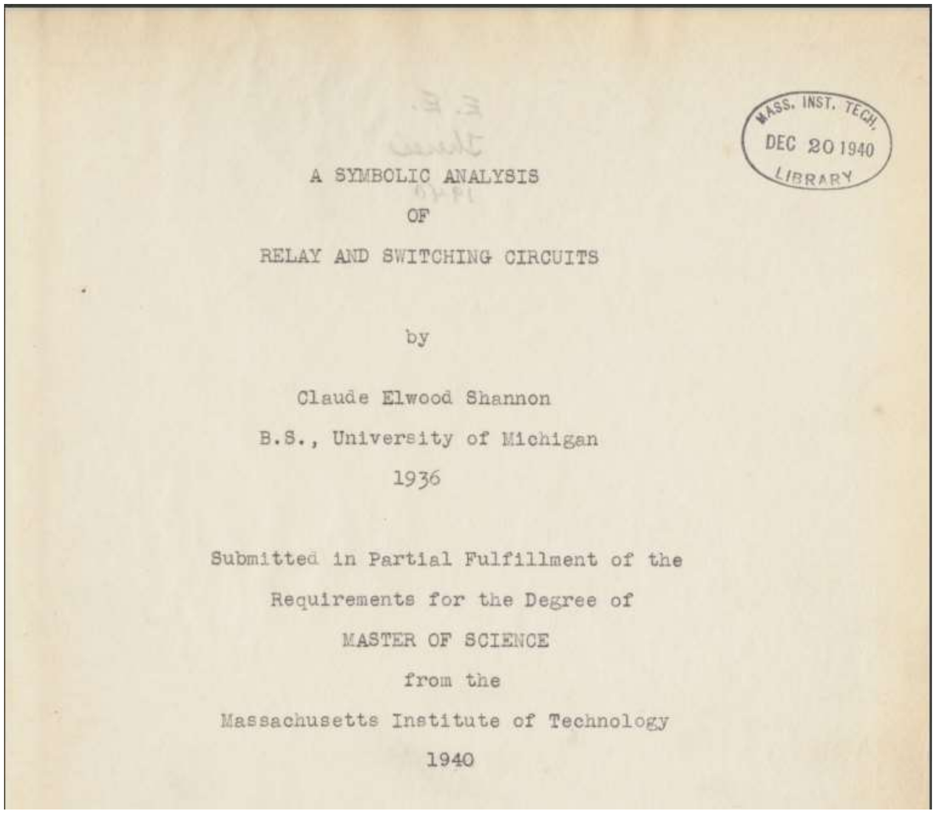
Sensing Objective and Information Theory

- What's a good sensing objective? How do we know we've "done it"?
- Our *uncertainty* about what we're sensing is low enough
- But we can't change the randomness, how do we change uncertainty?

Background – Claude Shannon

- "Father of the information age"
- Pioneer in CS, Information Theory, AI
- Data compression and decoding (codecs, zip files, etc)
- Wireless (and wired) digital communication

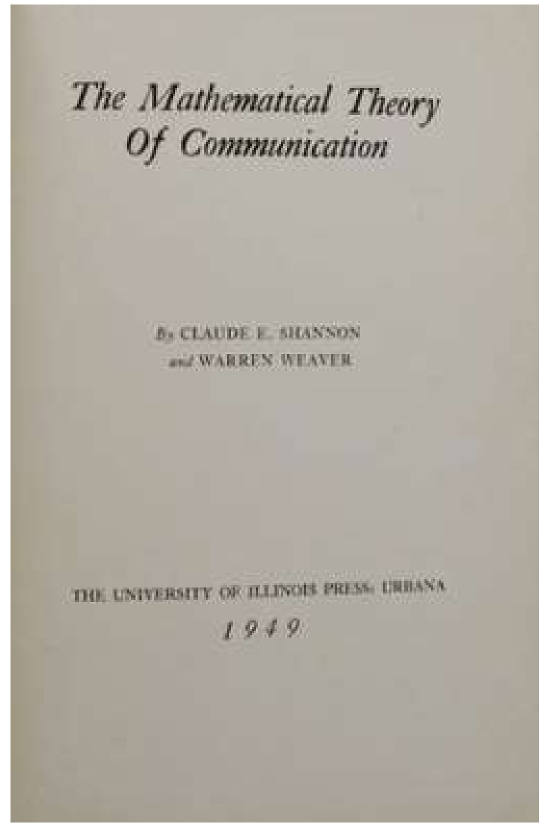
Works of Claude Shannon


A SYMBOLIC ANALYSIS
OF
RELAY AND SWITCHING CIRCUITS

by

Claude Elwood Shannon
B.S., University of Michigan
1936

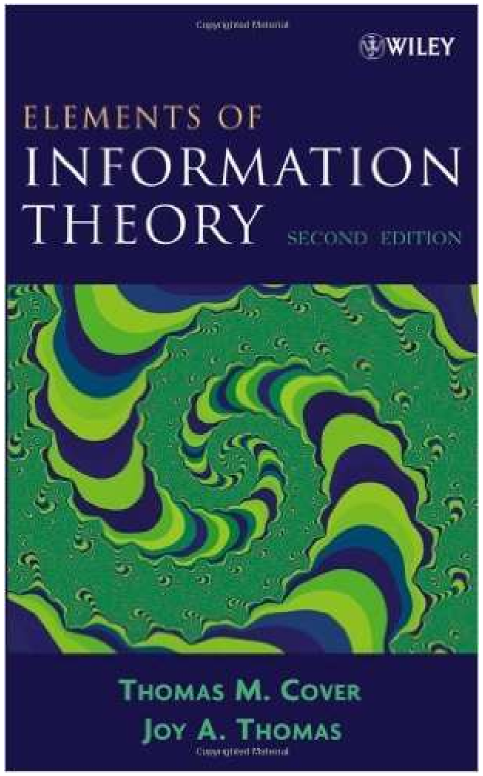
Submitted in Partial Fulfillment of the
Requirements for the Degree of
MASTER OF SCIENCE
from the
Massachusetts Institute of Technology
1940


The Mathematical Theory Of Communication

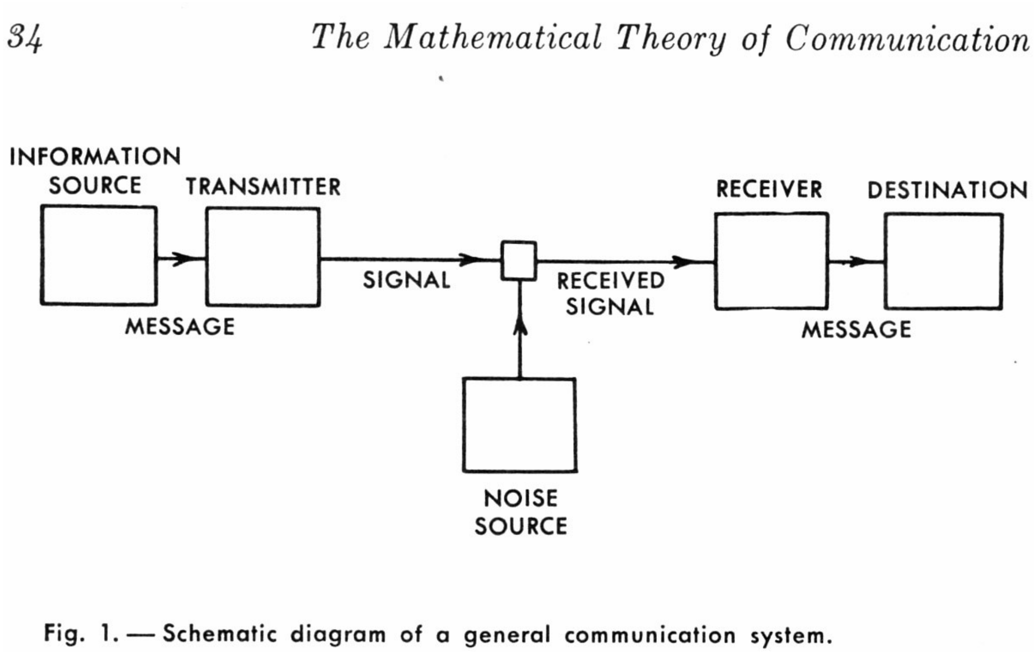
By CLAUDE E. SHANNON
and WARREN WEAVER

THE UNIVERSITY OF ILLINOIS PRESS · URBANA
1 9 4 9

Best Resource

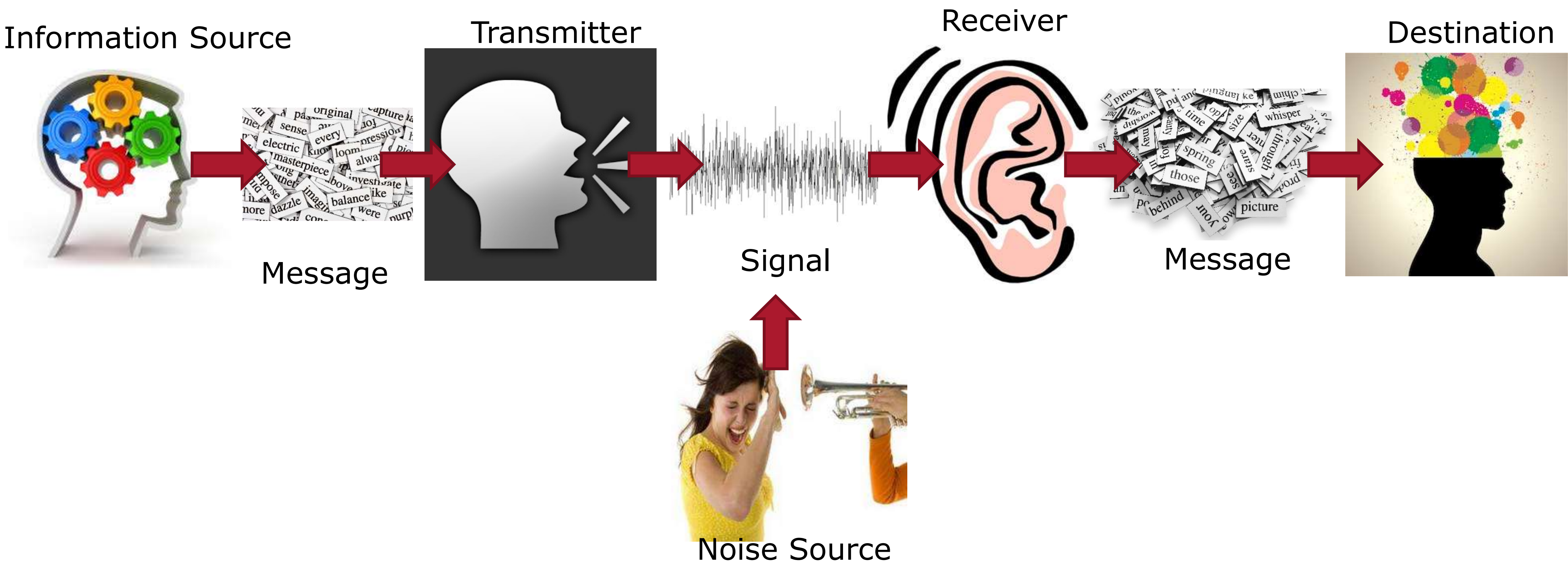


Motivation: Models of Communication

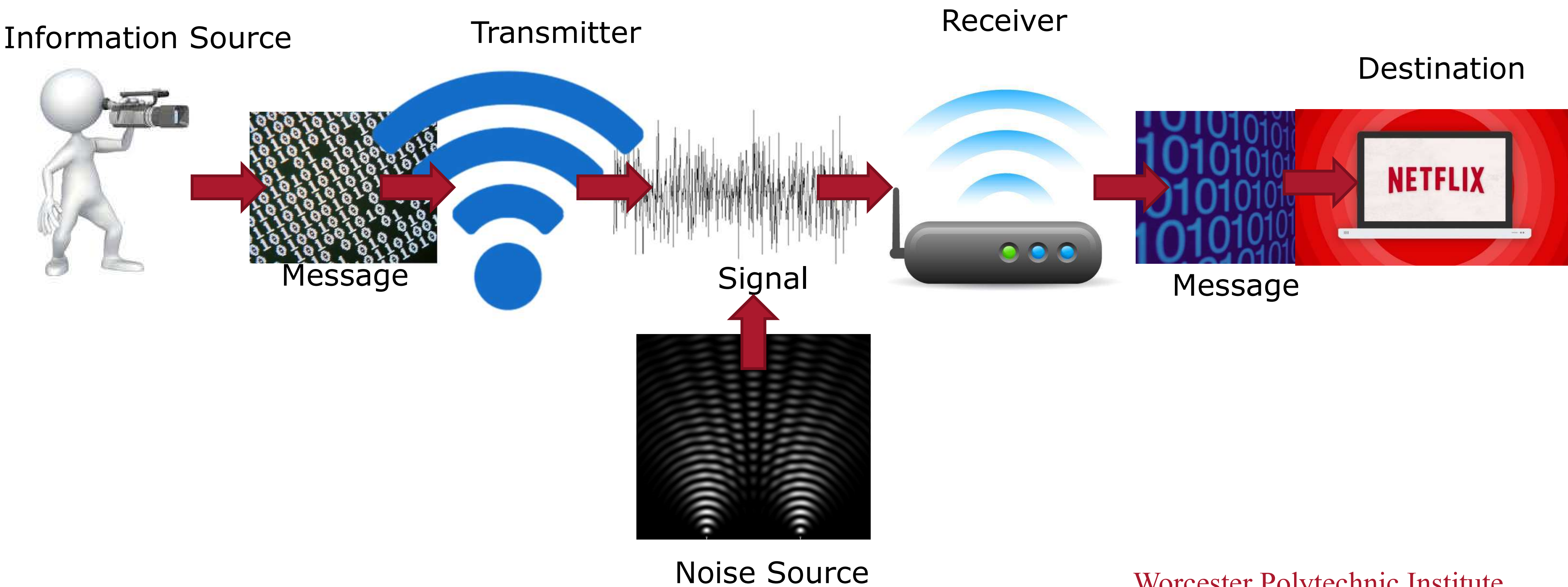


Shannon 194

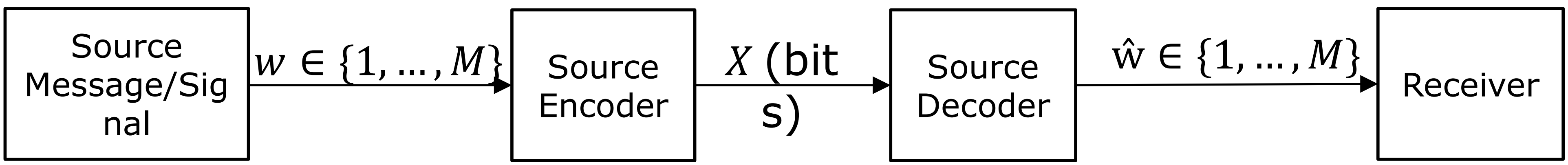
Example: Speech



Example: Video Streaming



Source Coding



Shannon 194

Motivating Example: Coin Flip

$\{0,1\}$

Motivating Example: Coin Flip

- Coin 1:
  - $P(x = 1) = 0.4$; $P(x = 0) = 0.6$

- Coin 2:
  - $P(x = 1) = 0.8$; $P(x = 0) = 0.2$

- Is one coin *more random* than the other?

- Can we generalize the difference between more than just biased coins?

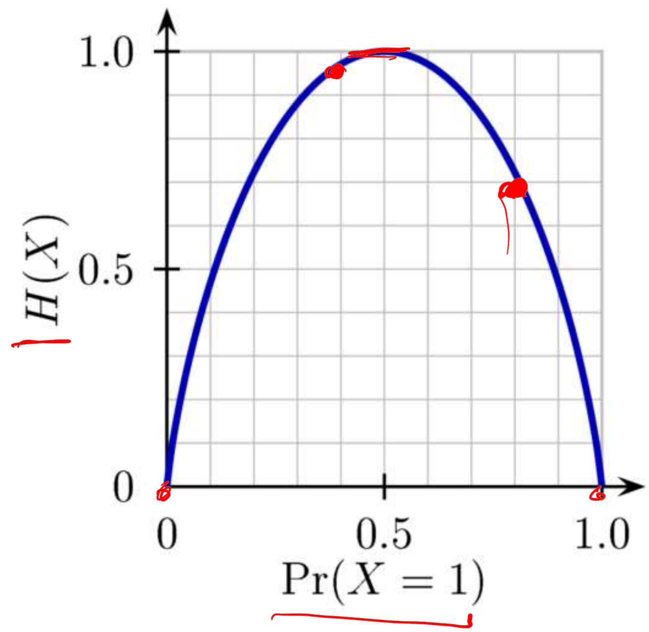
Entropy

- For a discrete R.V. $X \sim p_X(x)$ **entropy** is defined as:

$$H(X) = E[-\log_2(p_{X(x)})] = -\Sigma_{x \in X} p(x) \log_2 p(x)$$

- Quantifies "randomness" or "uncertainty" or "information content" of PMF
- Measured in bits
- Can also be measured in nats for $\log_e$ or other bases
- N.B. entropy is a function of a *probability distribution*
- N.B.2 this and other information measures can be extended to continuous RVs

Binary entropy



C_1: 0.4, 0.6
C_2: 0.8, 0.2

P heads

Some interesting properties

1. $H(X) \geq 0$

2. Maximized for uniform distribution: $H(X) = \log_2 |X|$

3. For uniform distribution, $H(X)$ increases monotonically in $|X|$

4. Any move toward uniformity increases $H(X)$