RBE 510 – Multi-Robot Systems
Lecture 2: Consensus and Formations

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Today

- Consensus
  - Recap
  - Clarification and applications
  - Variations and extensions
- Formation Control

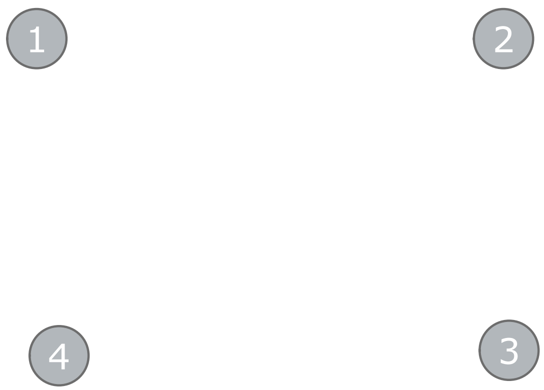
Admin

This Friday:

- HW0 Due
- HW1 Out
  - Start early
  - Not just pattern-matching
  - Includes programming (will discuss next lecture)

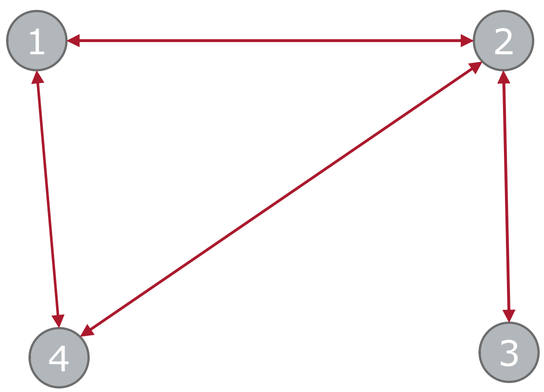
Recap

Linear Consensus Protocol



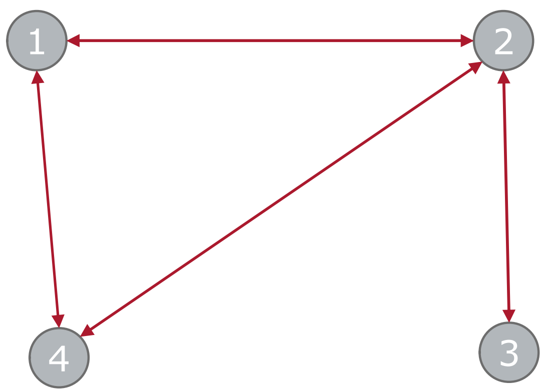
- We have I agents
  - Here, $I = \{1,2,3,4\}$
- Generic agent is agent i
  - Agent i has state x_i
  - Initial state of agent i is x_i^0
- Want all agents to agree on common state
  - e.g., states converge to $\bar{x} = \frac{1}{4}\sum_1^4 x_i$
- How?

Linear Consensus Protocol



- Agents need to share information
- Agents communicate if they are "close enough"
  - We'll talk more about this in future lectures
- Represent this as a graph
  - Edge between two nodes says that they can exchange information

Algebraic Graph Theory



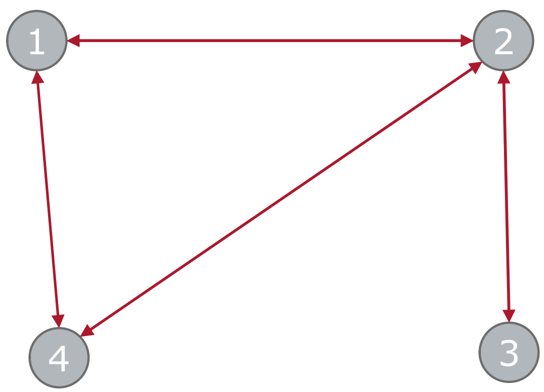
- We can use matrices to make our lives easier!

- Degree matrix

$$D = \begin{bmatrix} |\mathcal{N}_1| & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & |\mathcal{N}_n| \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & & & 0 \\ & 3 & & \\ & & 1 & \\ 0 & & & 2 \end{bmatrix}$$

Algebraic Graph Theory



- Adjacency relationship

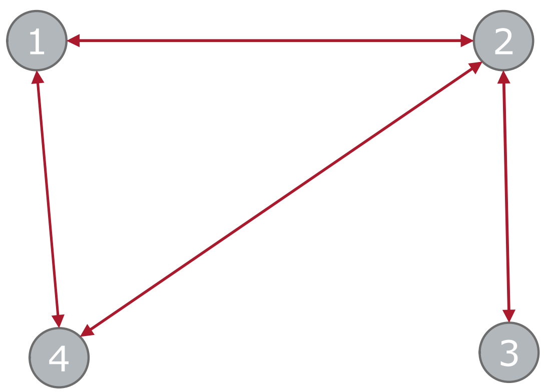
$$a_{ij} = \begin{cases} 1 & if\ (i,j) \in E \\ 0 & otherwise \end{cases}$$

- Adjacency matrix

$$A = \begin{bmatrix} 0 & \cdots & a_{ij} \\ \vdots & \ddots & \vdots \\ a_{ji} & \cdots & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Graph Laplacian



- *Laplacian matrix* of a graph is

$$L = D - A$$

- What is L for this graph?

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

Linear Consensus Protocol

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} \left(x_{j(t)} - x_{i(t)}\right)$$

$$x_i(t+1) = x(t) + \alpha \sum_{j \in \mathcal{N}_i} \left(x_j(t) - x_i(t)\right)$$

- A nice feature – agents only need information from their neighbors (i.e., *local* information)
- But we care about *global* behavior
- How can we link the two?

Agent view to Global View

$$\dot{x}_i(t) = \sum_{j \in N_i} \underbrace{a_{ij}(x_j - x_i)}$$

$$= \underbrace{\sum_{j \in N_i} a_{ij} x_j} - \underbrace{\sum_{j \in N_i} a_{ij}} x_i$$

$$- \deg(i) x_i + \underbrace{a_{i1} x_1 + a_{i2} x_2 \ldots}$$

$$\begin{bmatrix} x_1 = \sum_j a_{ij} x_j - \deg(1) x_1 \\ x_2 = \ldots \\ x_3 = \ldots \end{bmatrix}$$

$$= -D + A$$

$$= -L$$

Consensus Proofs

- Last time we proved (under some conditions):
  - Consensus converges
  - Converges to the average
  - Average is invariant/problem is convex

Resources for Consensus Lectures

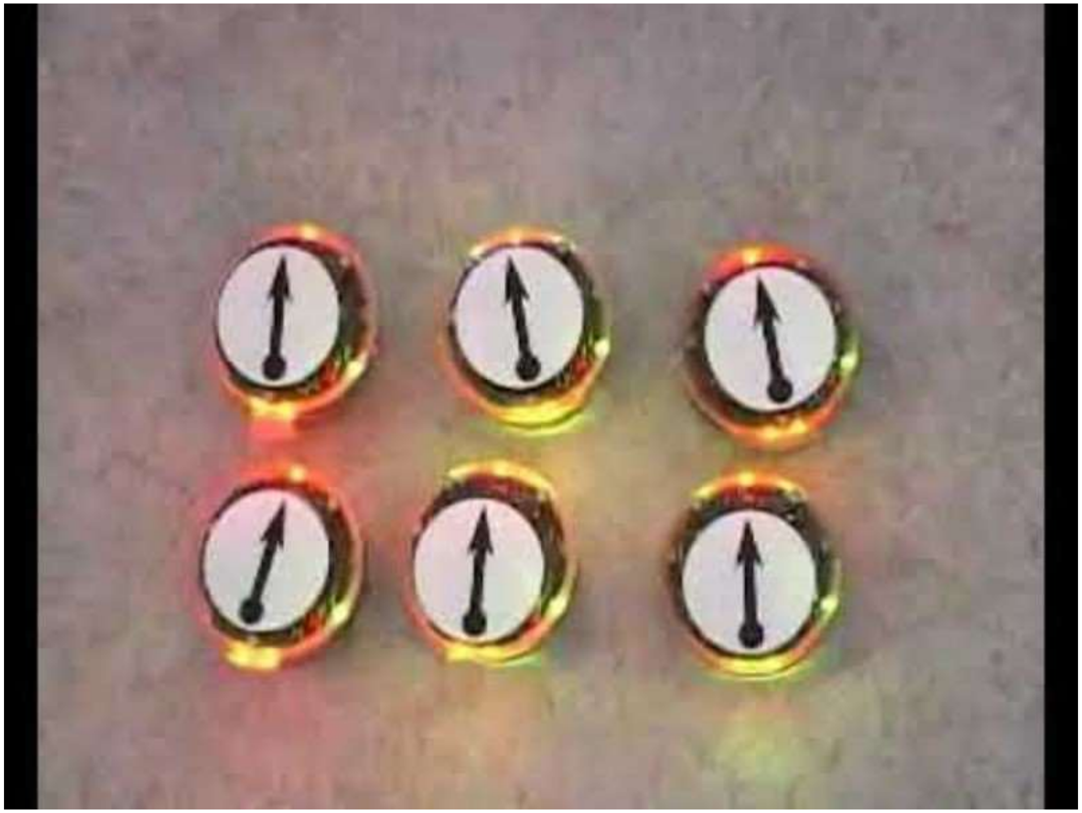
- There are lots of papers/resources/tutorials available for consensus
  - Mesbahi and Egerstedt *Graph Theoretic Methods in Multiagent Networks* Chapter 3, is a good general resource
- Today, we look at results from
  - Jadbabaie, Lin, and Morse "Coordination of Groups of Mobile Autonomous Agents Using Nearest Neighbor Rules" *IEEE Trans. on Automatic Control,* 2003 **9,521 citations**
  - Olfati-Saber and Murray "Consensus Problems in Networks of Agents with Switching Topology and Time-Delays" *IEEE Trans. on Automatic Control*, 2004 **13,497 citations**
  - Moreau "Stability of Multiagent Systems With Time-Dependent Communication Links" *IEEE Trans. on Automatic Control*, 2005 **3,148 citations**
- Also drawn from Mesbahi and Egerstedt, and Bullo, Cortes, and Martinez books

Consensus Extensions and Applications

Some Clarification

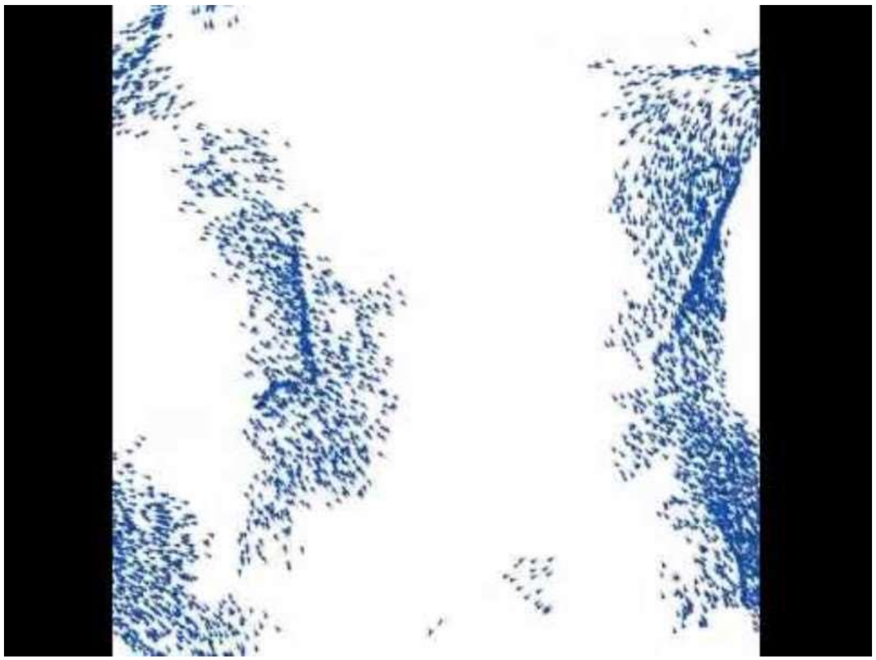
- Not just position! Applies to “processor state” or other information (like sensing)
- Will formalize next time when we discuss distributed algorithms more generally
- Is it useful for physical robots (i.e., will they collide?)
  - To discuss today somewhat
  - Also, generally get more connected with rendezvous, which I’ll ask you to think about for the homework

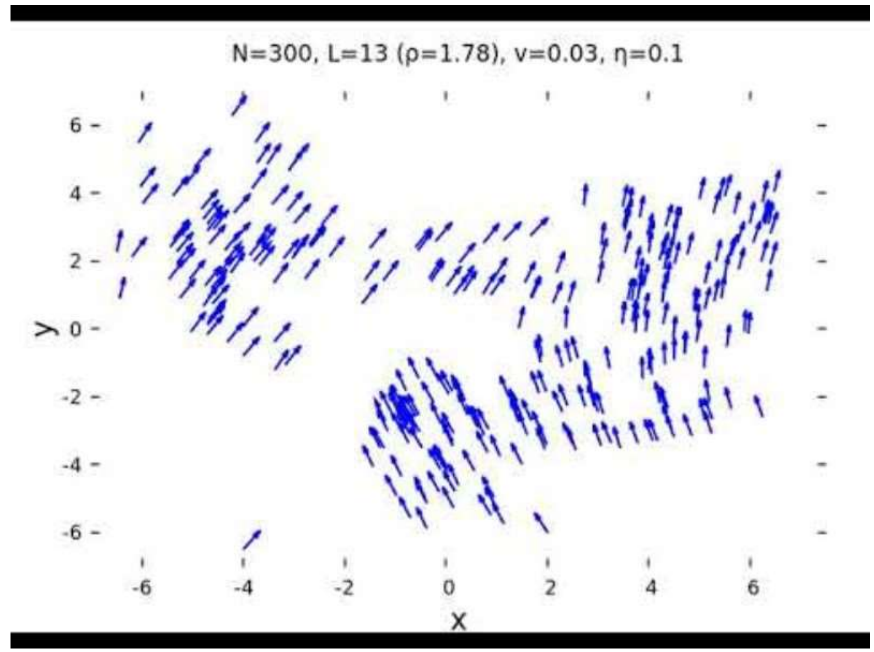
Heading Consensus



Rendezvous

Flocking





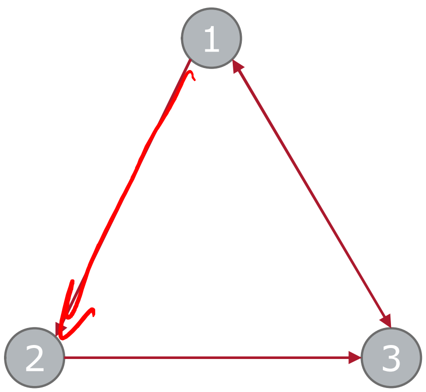
- Model developed by Vicsek (1995) in particle physics
- Jadbabaie et al. generalized this model in 2003

Consensus Applied

Assumptions

- We've made a lot of assumptions so far
  - Graph is undirected
  - Graph is time-invariant
  - Consensus is synchronous
  - No weighting on the graph
  - …
- Results exist for consensus that remove these assumptions (and many more)
- We'll cover only the first two cases

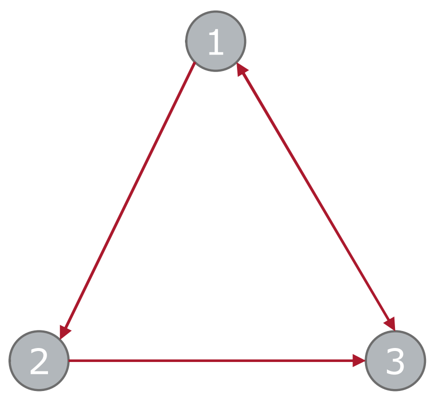
Consensus on Directed Graphs



- Directed graph has Laplacian $L = D - A$
- What is the adjacency matrix?

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

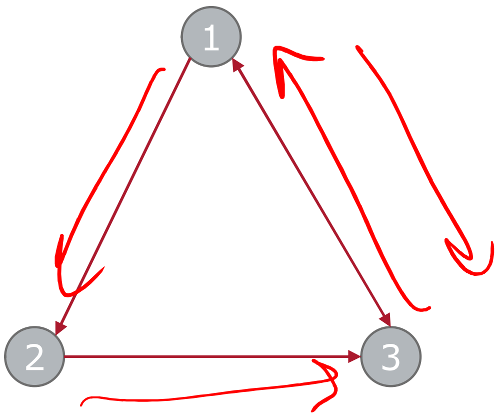
Consensus on Directed Graphs



- Directed graph has Laplacian $L = D - A$
- What is the out degree matrix?

$$D = \begin{bmatrix} 2 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$$

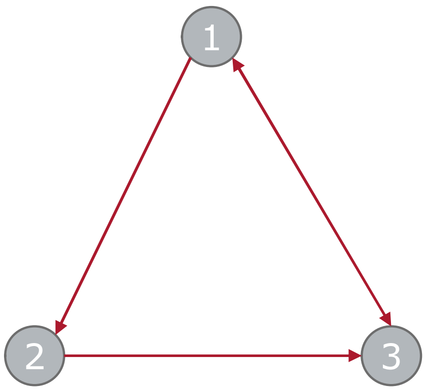
Consensus on Directed Graphs



- Directed graph has Laplacian $L = D - A$
- What is the Laplacian?

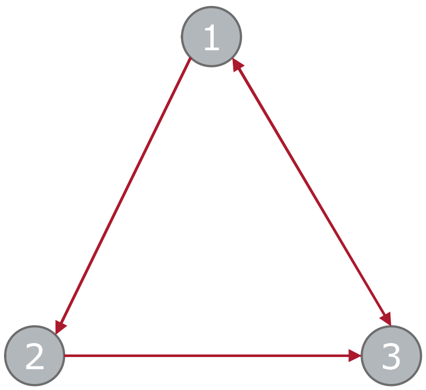
$$L = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

Consensus on Directed Graphs



- Directed graph has Laplacian $L = D - A$
- The graph is unbalanced
  - $\deg_{out}(v_i) \neq \deg_{in}(v_i)$ for all nodes
- Does it reach consensus?
- If so, to what?

Consensus on Directed Graphs



- Initial values
  - $x_1(0)$, $x_2(0)$, and $x_3(0)$

- Converges to
  - $x_i^* = \frac{[x_1(0)+x_{2(0)}+2x_3(0)]}{4}$

- What is happening here?

Convergence Proof

- Can’t use eigendecomposition as before. Why?
- Still, $0 = \lambda_1 \leq \lambda_2 \leq \cdots$
- All eigenvalues in closed LHP, so converges
- $\boldsymbol{x}^* = \mathbf{1}^{\mathrm{T}}\alpha$ is still a right eigenvector for λ_1, so it still converges to agreement for some $\alpha \in \mathbb{R}$

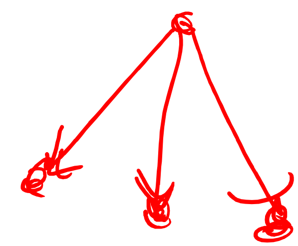
Prior Results

It has been shown that for G that is connected, applying

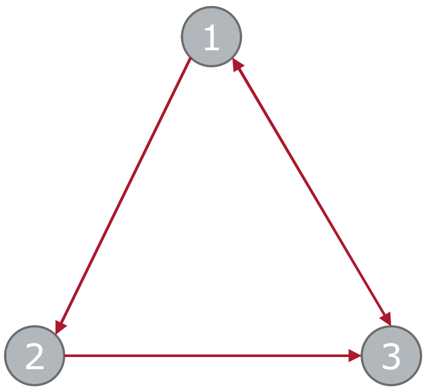
$$u_i(t) = \sum_{j \in \mathcal{N}_i} \left(x_j(t) - x_i(t) \right)$$

Converges to the average $\Leftrightarrow \sum_{i=1}^{n} u_i = 0$ (Saber and Murray 2003)

$$\mathbf{1}^T(-Lx) = 0$$



Condition Does Not Hold!



- $L = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

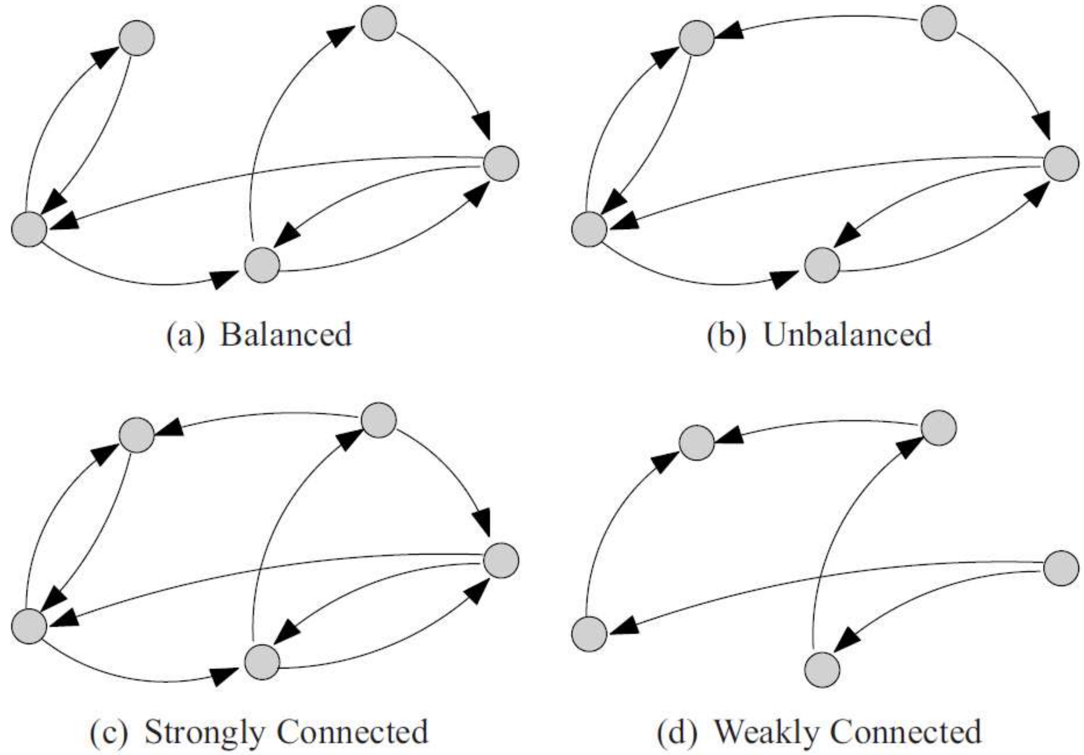
$$-Lx = \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\cancel{2x_1} + \cancel{x_2} + x_3 \\ -\cancel{x_2} + \cancel{x_3} \\ \cancel{x_1} - \cancel{x_3} \end{pmatrix}$$

$$= -x_1 + x_3 \neq 0$$

Graphs and Balance

- A graph is balanced iff:

$\deg_{out}(v_i) = \deg_{in}(v_i)\ \forall v_i \in V$



(a) Balanced

(b) Unbalanced

(c) Strongly Connected

(d) Weakly Connected

Theorem (Saber and Murray 2004)

For a graph $G = (V, E)$ the following are equivalent

1. G is balanced

   $L\mathbf{1} = 0$ because of rows

2. $\mathbf{1}^T L = 0$ columns

3. $\sum_{i=1}^{n} u_i = 0 \ \forall x \in \mathbb{R}^n$ when executing $u_i = \sum_{j \in \mathcal{N}_i} a_{ij}(x_j - x_i)$

What does this mean for convergence?

Proof 1 $\Leftrightarrow$ 2

G balanced $\iff \mathbb{1}^T L = 0$

$$\mathbb{1}^T L = \sum_i l_{j,i}$$

$$= \sum_{i \mid j \neq i} \left(l_{j,i} + l_{i,j} \right)$$

$$= \underbrace{-\deg_{in}(i)} + \underbrace{\deg_{out}(i)} = 0$$

Doesn't mean $L^T = L$

Proof 2 ⇔ 3

$$1^T L = 0 \iff \underbrace{\sum_i u_i = 0}$$

$$u = -Lx$$

$$\Rightarrow \sum_i u_i = 0 \ \forall x$$

$$\iff 1^T u = 0 \iff \underbrace{-(1^T L)x = 0 \ \forall x}$$

$$\Rightarrow 1^T L = 0$$

More Consisely

- The linear consensus protocol over a digraph converges to average consensus for every initial condition if and only if it is weakly connected and balanced.

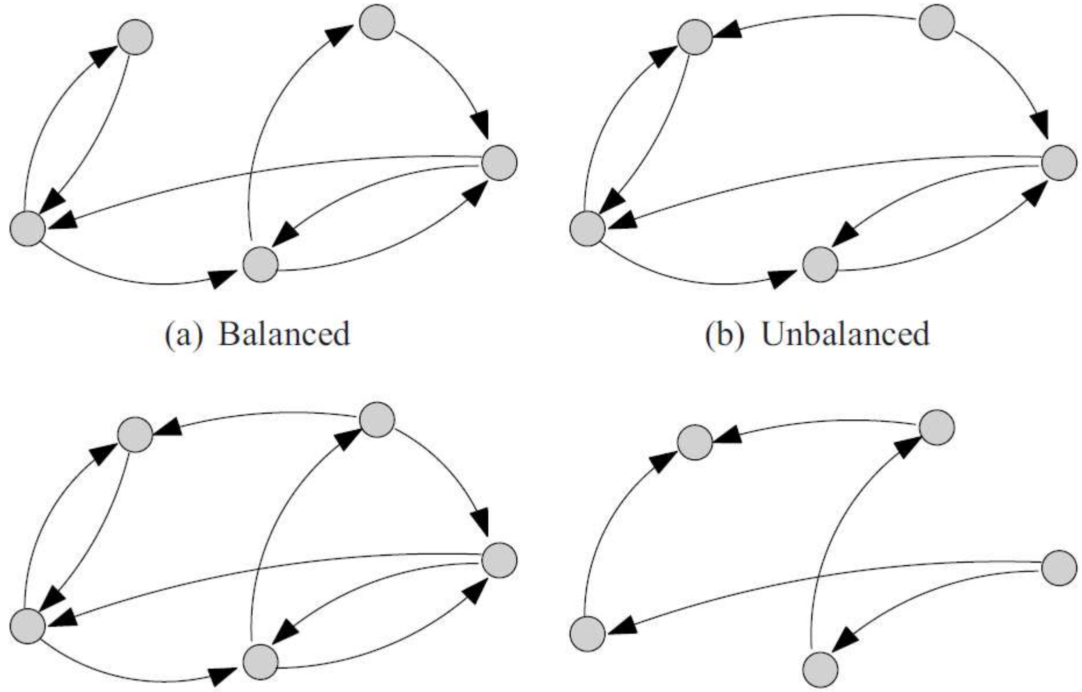
- Strongly connected if there is a directed path from every node to every other node

- Weakly connected if there is an undirected path from every node to every other node

- Weakly connected + balanced → strongly connected

More Concisely

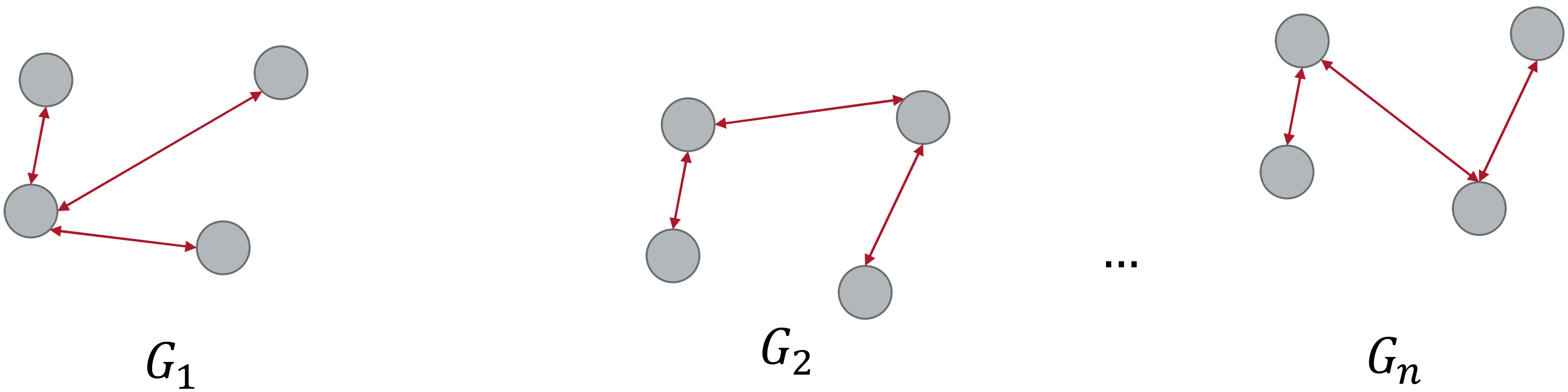
- Digraph converges to average iff
  - Balanced
  - Weakly connected



(a) Balanced (b) Unbalanced (c) Strongly Connected (d) Weakly Connected

Time-Varying Consensus and Other Ideas

Time-Varying Problem Set-Up



- A set of graphs Γ
- k^{th} graph is G_k
- Will we reach consensus? Under what conditions?

Time-Varying Topologies

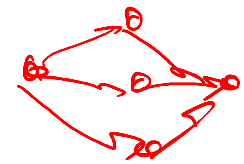
- Saber and Murray also proved that for a set of connected graphs, consensus still converges to the average
- This is a hybrid system!
- Non-trivial, but it makes sense if it requires that all graphs are connected and contracting

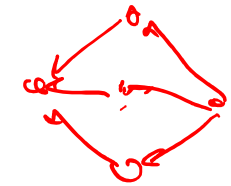
General Time-Varying Topologies

- This is harder! Non-linear and time-varying!

- Nonetheless, a proof was found for directed graphs by Moreau in 2005

General Time-Varying Topologies





- $v_i \in V$ is connected to $v_j \in V \setminus \{i\}$ if there is a path from i to j in the graph w.r.t. the direction of the edges

- For a sequence of graphs $G = (V, E(t))$ with $t \in \mathbb{N}$, a node $v_i \in V$ is connected to $v_j \in V \setminus \{i\}$ across interval $I \subseteq \mathbb{N}$ if it is connected to v_j for $G = \left(V, \cup_{t \in I} E(t)\right)$

- For a sequence of graphs $G = (V, E(t))$ if $\exists T \geq 0$ such that $\forall t_0 \in \mathbb{N}$ there is a node connected to all other nodes across $[t_0, t_0 + T]$ then the sequence converges as $t \to \infty$

One last view of consensus

- Define a function

$$\Psi_G(\boldsymbol{x}) = \frac{1}{2}(\boldsymbol{x}^T)L\boldsymbol{x} \qquad \frac{\partial}{\partial x}\frac{1}{2}x^2 = 2x$$

$$\frac{\partial \Psi}{\partial x} = \frac{1}{2}\left(Lx + (x^T L^T)\right) = \frac{1}{2}\left(Lx + L^T x\right)$$

$$= Lx \leftarrow \text{neg gradient}$$

$$\Psi_G = -\frac{1}{2}x^T \dot{x} = -\frac{1}{2}\sum_{i=1}^{n} x_i \dot{x}_i$$

$$= -\frac{1}{2}\sum_{i=1}^{n} x_i \sum_{j \in N_i}(x_i - x_j)$$

$$\Psi_G = -\frac{1}{2}\sum_i x_i \sum_{j\in N_i}(x_i - x_j)$$

for $\{i,j\} \in E$

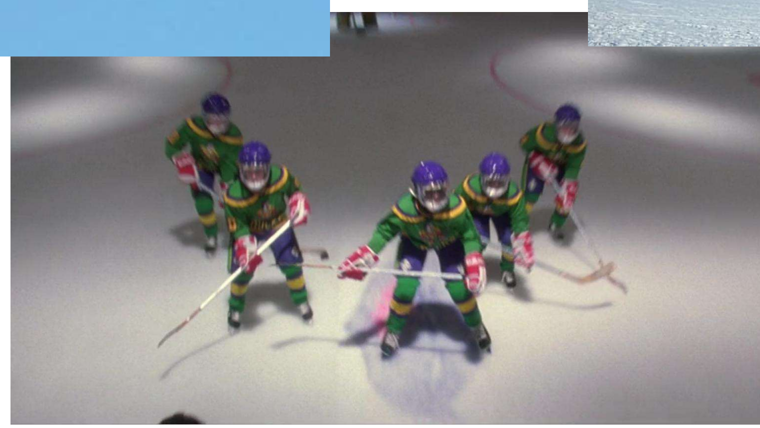
$$x_i(x_j - x_i) \; ; \quad x_j(x_i - x_j)$$

$$= -\frac{1}{2}\sum_{i,j\in E}(x_i - x_j)(x_j - x_i)$$

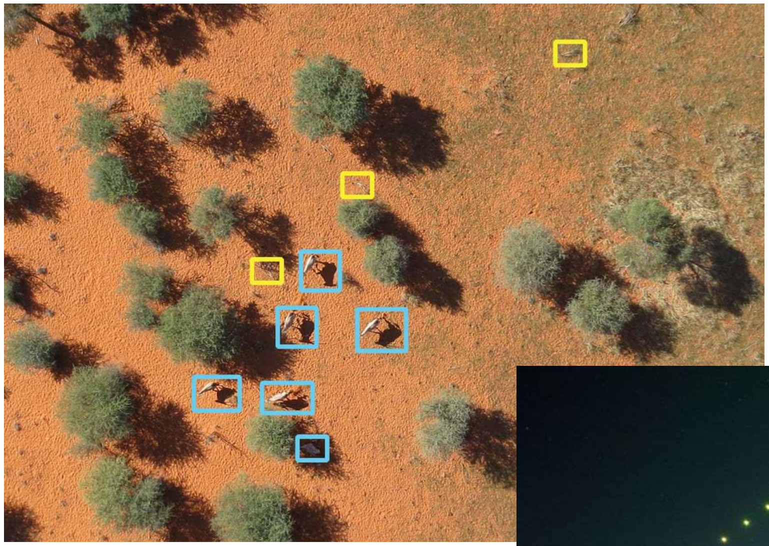
$$= \frac{1}{2}\sum_{\{i,j\}\in E}(x_i - x_j)^2$$

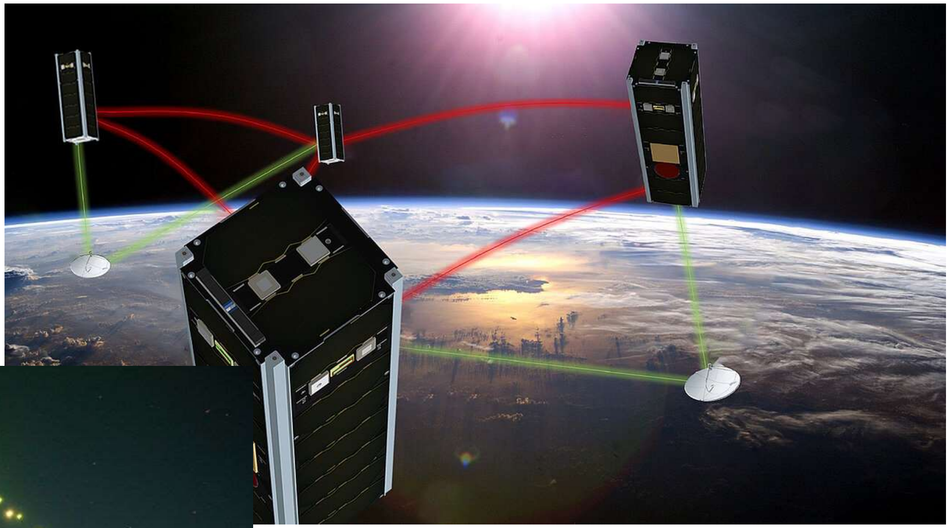
Formation Control

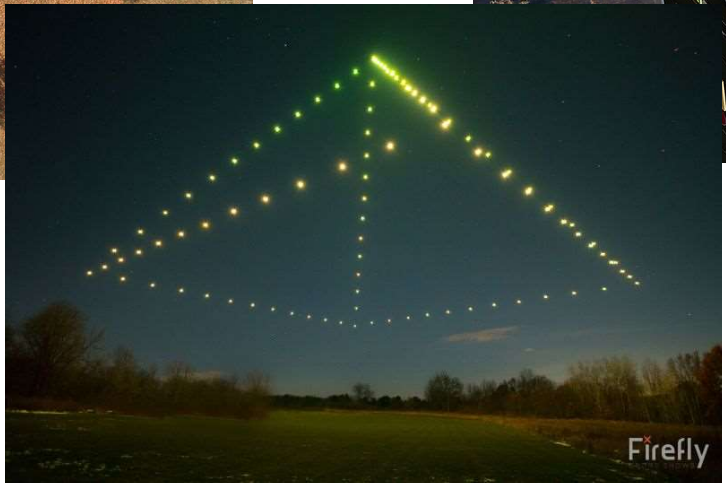
Formations



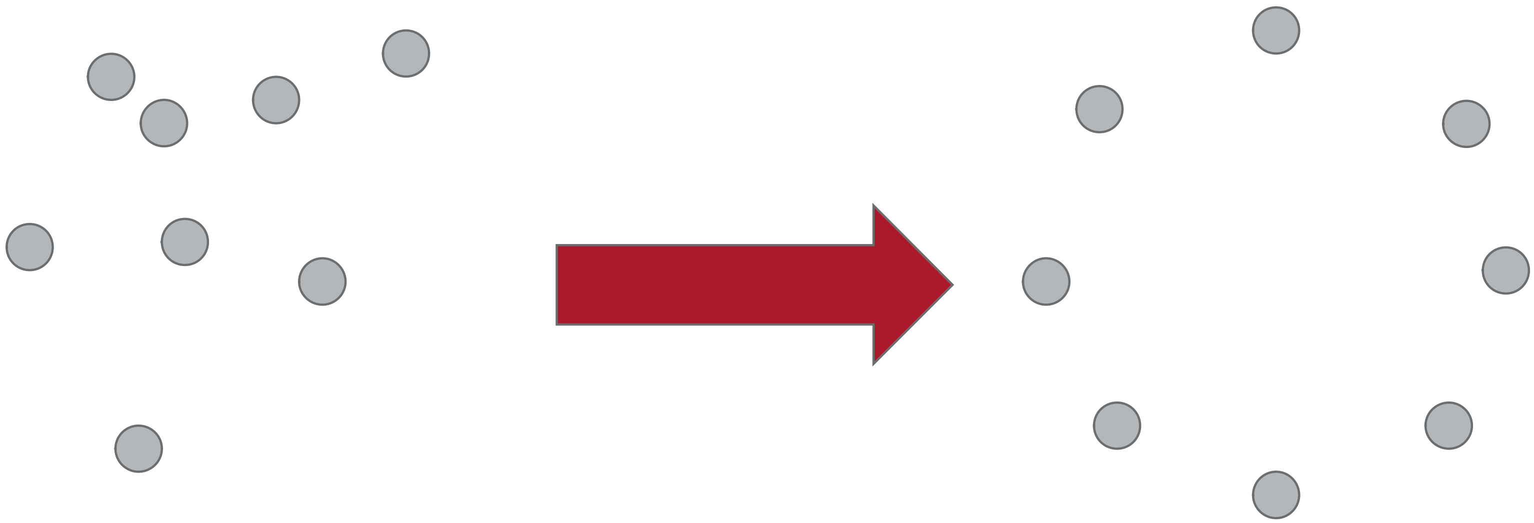
Formations with Robots







Formation Goal



- How to specify?
- Does it converge?
- Is convergence unique

Formation with Nearest-Neighbors

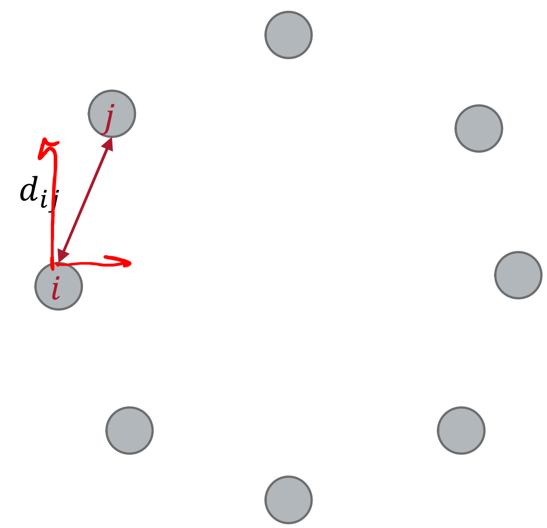
- Idea:
  - Specify inter-agent distances between pairs of agents

- d_{ij}: desired separation between agent i and agent j

- Proposed controller

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} a_{ij}\left(x_j - x_i - d_{ij}\right)$$

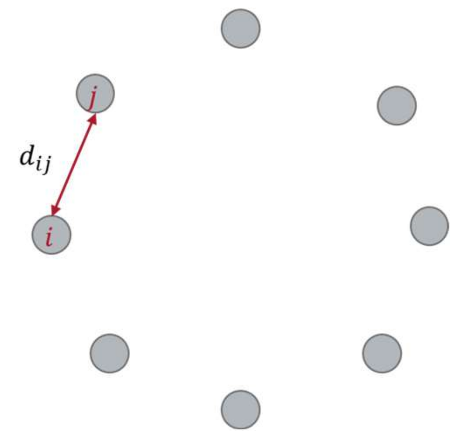
- Almost consensus. What is different about what agents need to know?



Formation with Nearest-Neighbors

- Proposed controller

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} a_{ij}\left(x_j - x_i - d_{ij}\right)$$



- Where is equilibrium?

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} a_{ij}\left(x_j - x_i - d_{ij}\right) = 0$$

- Guess:
  - $d_{ij} = -d_{ji}$ (and graph is undirected)
  - $x_j - x_i = d_{ij}$

- Is $d_{ij} = -d_{ji}$ *necessary* for an equilibrium?
- Is it sufficient?

Necessity: $d_{ij} = -d_{ji}$

- Equilibrium:

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} a_{ij}\underline{(x_j - x_i - d_{ij})} = 0$$

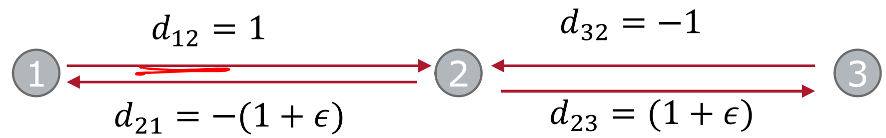
$$x_j - x_i = d_{ij} \quad \forall i$$

$$x_i - x_j = d_{ji}$$

$$\hookrightarrow -(x_j - x_i) = -d_{ij}$$

Insufficiency

- There are other equilibria!



$$\dot{x}_1 = (x_2 - x_1) - 1 = 0$$

$$\dot{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \dot{x}_2 = -(x_1 - x_2) + 1 \cancel{+ \varepsilon} + (x_3 - x_2) - (1 \cancel{+ \varepsilon}) = 0$$

$$\dot{x}_3 = x_2 - x_3 + 1 = 0$$

$$x_1 = x_2 - 1$$

$$x_2 = x_3 + x_1 - 2x_2 \longrightarrow x_2 = x_2 + 1 + x_2 - 1 - 2x_2 = 0$$

$$x_3 = x_2 + 1$$

System Level View

- $\dot{x}_i = \sum_{j \in \mathcal{N}_i} a_{ij} \left(x_j - x_i - d_{ij} \right)$
- $\dot{x}_i = \sum_{j \in \mathcal{N}_i} a_{ij} \left(x_j - x_i \right) - \sum_{j \in \mathcal{N}_i} a_{ij} d_{ij}$
- $\dot{\boldsymbol{x}} = -L\boldsymbol{x} + \mathrm{d}$
- Consensus was *linear*, this is *affine*

System-Level View

- $\dot{\boldsymbol{x}} = -L\boldsymbol{x} + \boldsymbol{d}$

- What are equilibria?

- $\dot{\boldsymbol{x}} = 0$
- $0 = -L\boldsymbol{x} + \boldsymbol{d}$
- $L\boldsymbol{x} = \boldsymbol{d}$

- Great, let's compute $\boldsymbol{x} = L^{-1}\boldsymbol{d}$

Problem!

- L is not invertible

- Why?
  - $\lambda_1 = 0,\ e_1 = [1,1,\dots,1]$
  - This directly implies that L is singular
  - Even more side note—rank of connected graph Laplacian is $n-1$

What now?

- $\dot{\boldsymbol{x}} = -L\boldsymbol{x} + \mathrm{d}$
- L is not invertible

- Two possibilities
  - No solution
  - Many solutions

- Let's examine what happens if we "guess" a solution

Candidate Solution x^*

- Assume x^* is a solution to $Lx^* + d$ $= 0$

$x^* \in \mathbb{R}^n$

$\alpha \in \mathbb{R}$

- What if we perturb it by a constant α?

- $\boldsymbol{x} = \boldsymbol{x}^* + \mathbf{1}\alpha;$ $\qquad \alpha \in \mathbb{R}$

0

- $L(\boldsymbol{x}^* + \mathbf{1}\alpha) = L\boldsymbol{x}^* + L\mathbf{1}\alpha$

$Lx^* = 0$

- $\therefore$ if $\exists$ one solution, $\exists$ infinitely many solutions

When is there a solution?

- We claimed that $d_{ij} = -d_{ji}$ was necessary condition
- But what about $\boldsymbol{d}$?
- Reminder, $\boldsymbol{d_i} = \sum_{j \in \mathcal{N}_i} a_{ij} d_{ij}$
- Suppose that $\mathbf{1}^T \boldsymbol{d} \neq 0$

$\sum u_i = 0$

$\sum d_i = 0$

$Lx = d$

$(Lx)^T \mathbf{1} = d^T \mathbf{1} \neq 0$

$x^T L^T \mathbf{1} \neq 0$

$L^T \mathbf{1} = 0 \Longrightarrow \mathbf{1}^T d = 0$

When is there a solution

- Suppose $\mathbf{1}^T \boldsymbol{d} = 0$
- Then
  - $\boldsymbol{d} \perp \mathbf{1}$
  - $\boldsymbol{d} \in range(L)$

- Proof by counterexample:

See previous slide

- Sum of all elements of $\boldsymbol{d}$ is zero!

Reducing to Consensus

- How to prove convergence?

$$\dot{\boldsymbol{x}} = -L\boldsymbol{x} + \boldsymbol{d}$$

- Change of variables:
- Let $\boldsymbol{x}^*$ be a solution

$\tilde{x} = x - x^*$;

$$\dot{x} = -Lx + d = -L(\tilde{x} - x^*) + d$$

$$= -L\tilde{x} + d - Lx^*$$

$$\dot{\tilde{x}} = \dot{x}$$

$$\dot{\tilde{x}} = -L\tilde{x}$$

Some observations

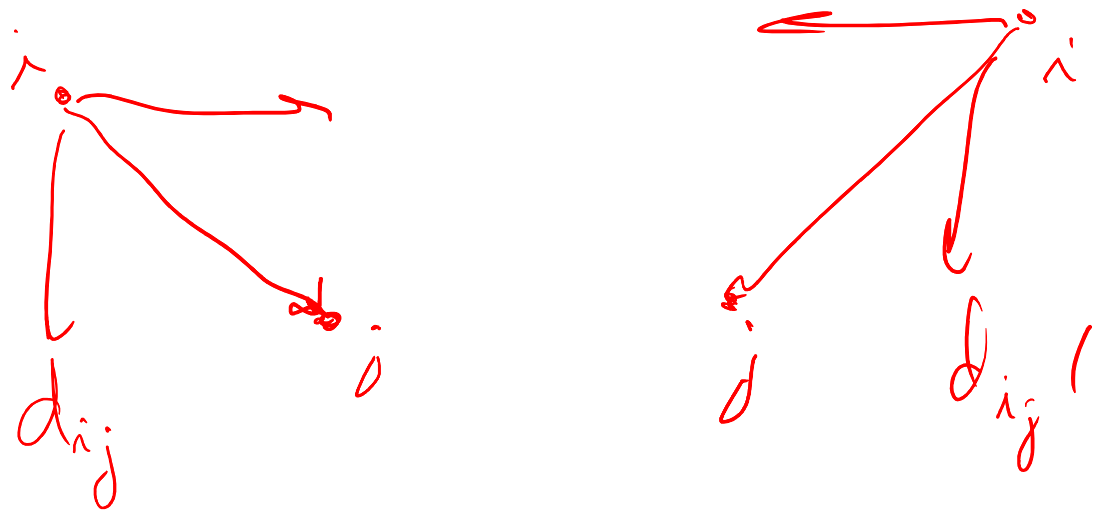
- Centroid is invariant again

$$\bar{x} = \frac{1}{n}\mathbb{1}^T x$$

$$\dot{\bar{x}} = \frac{1}{n}\mathbb{1}^T \dot{x} = \frac{1}{n}\left(-\mathbb{1}^T L x + \mathbb{1}^T u\right) = 0$$

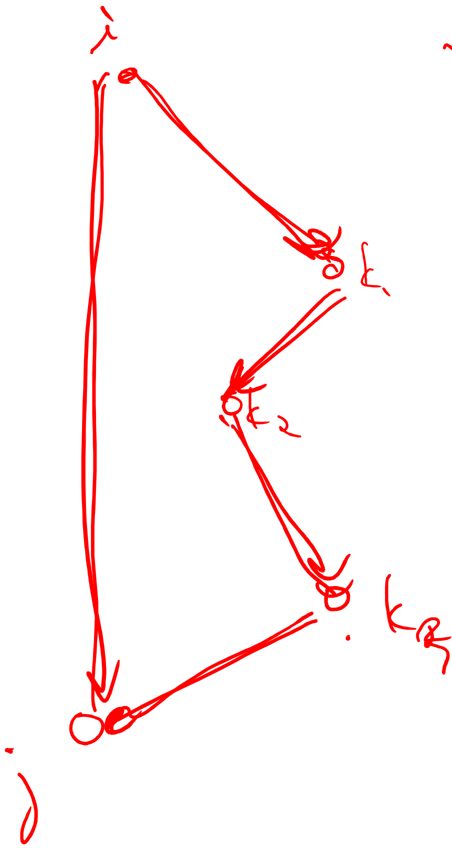
Some observations

- Formation can be translated but not rotated and remain a formation



$$d_{ij} \neq d_{ij}'$$

Some observations



$$x_j - x_i = (x_j - x_{k_3}) + (x_{k_3} - x_{k_2}) + \dots$$

What do Agents Need to Know?

- State of neighbors
- Desired distance
- Which neighbor is which!

- What don't they know?
  - Centroid
  - Therefore, where it will converge
  - That's what the infinite solutions mean! – Translation invariant

Wrap Up

Summary

- Consensus Recap
- Time-varying and weighted topologies
- Formation Control

Next Time

- HW0 Due
- HW1 Out
- Next time: detour to models and distributed algorithms
- Next Monday: how to control formations and abstractions