RBE 510 – Multi-Robot Systems
Lecture 5: Pairwise Interactions

Kevin Leahy

05, September 2025

Admin

- HW 0 solutions/grades posted
- HW 1 due Today 11:59:59 PM
- HW 2 out this afternoon
- Office hours Wednesday 3-3:45 in UH 250 D

Admin

- First paper presentation on Monday

- Paper posted on canvas – read it!
  – Poonawala, Satici, Eckert, and Spong. “Collision-Free Formation Control with Decentralized Connectivity Preservation for Nonholonomic-Wheeled Mobile Robots” TCNS, 2015.

- Future papers – ML?

Today

- Clarifications
  - Homework 1 pointers
  - Rigidity

- Closing out consensus and formation control
  - Combining reference frame invariance with abstraction-based control
  - Generalizing the results for reference-frame invariance

Homework 1 Tips

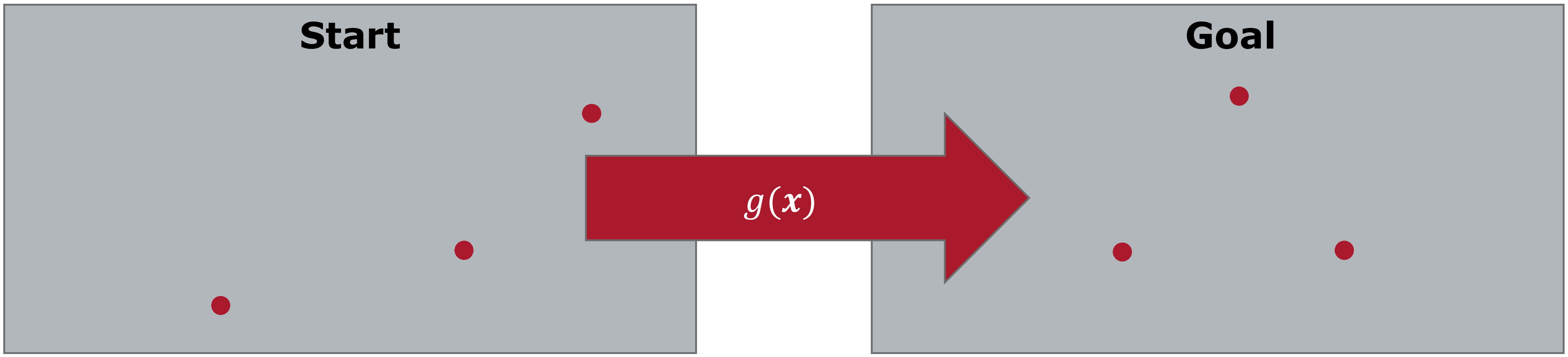
Question 1 – Consensus

- “compact form”

Question 2 – Stubborn Agent

- Show (analytically) ≈ prove

Modeling Multiple Robots



How to design $g(\boldsymbol{x})$?
What are its properties for stability?
Convergence?
To what?
Under what conditions?

$g_3(\boldsymbol{x})$

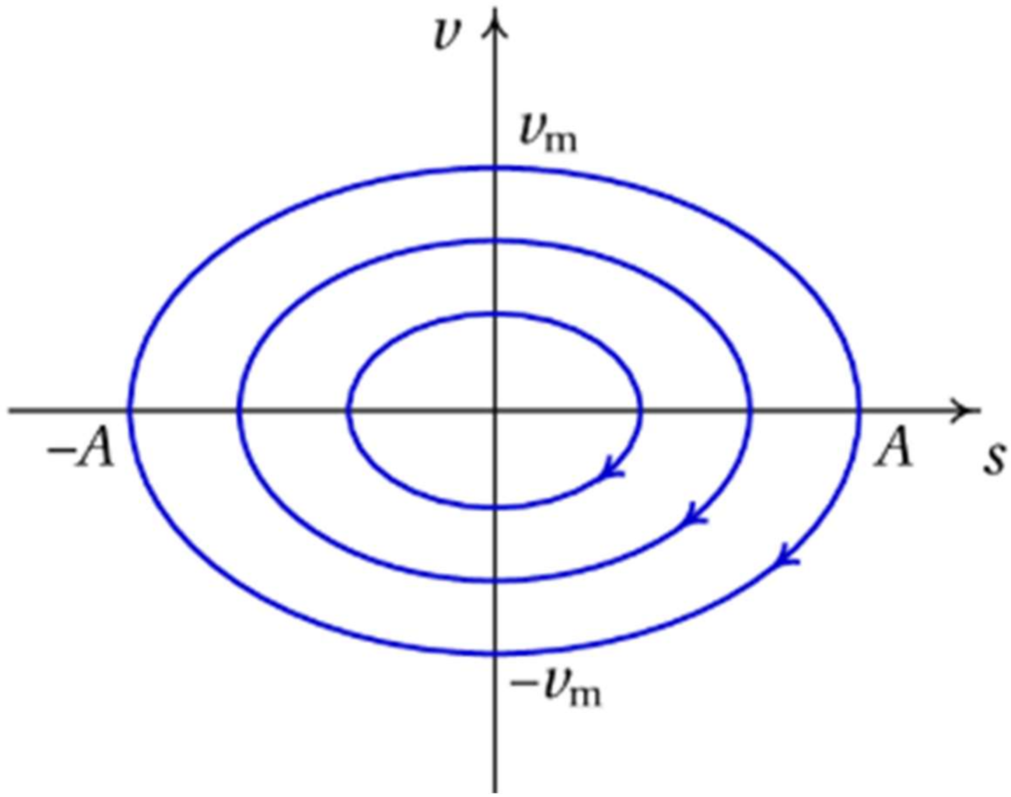
Question 2 (and in general)

- Dynamical system $\dot{x} = f(x, u)$
- Equlibrium
- Convergence

Question 2 (and in general)

- Existence of an equilibrium does not mean the system will reach it!
- Example, harmonic oscillator $\ddot{s} = -Cs$

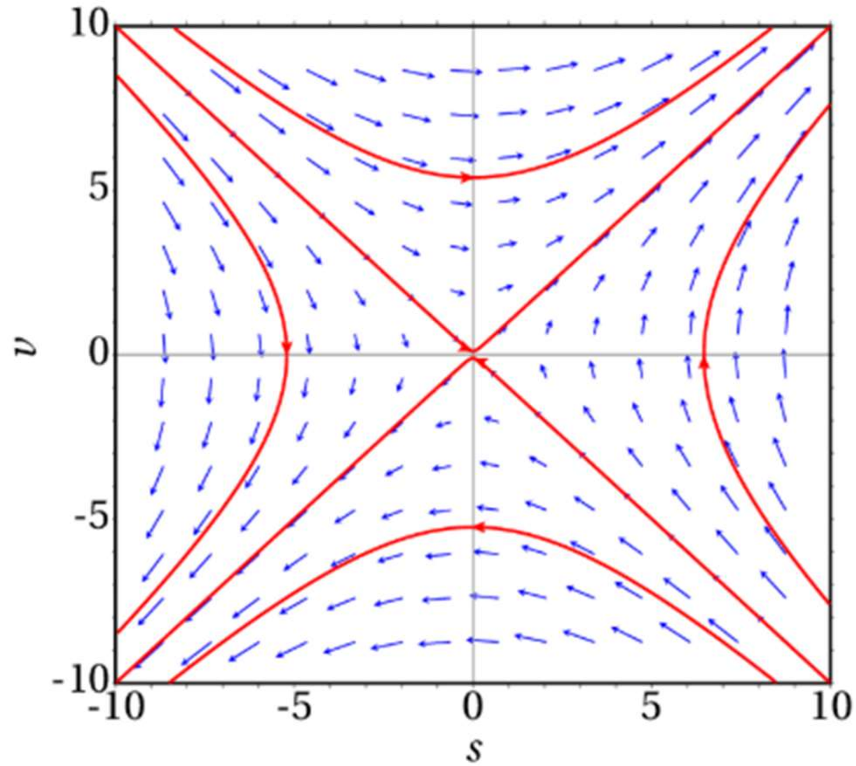
Harmonic Oscillator Phase Portrait



Other Example

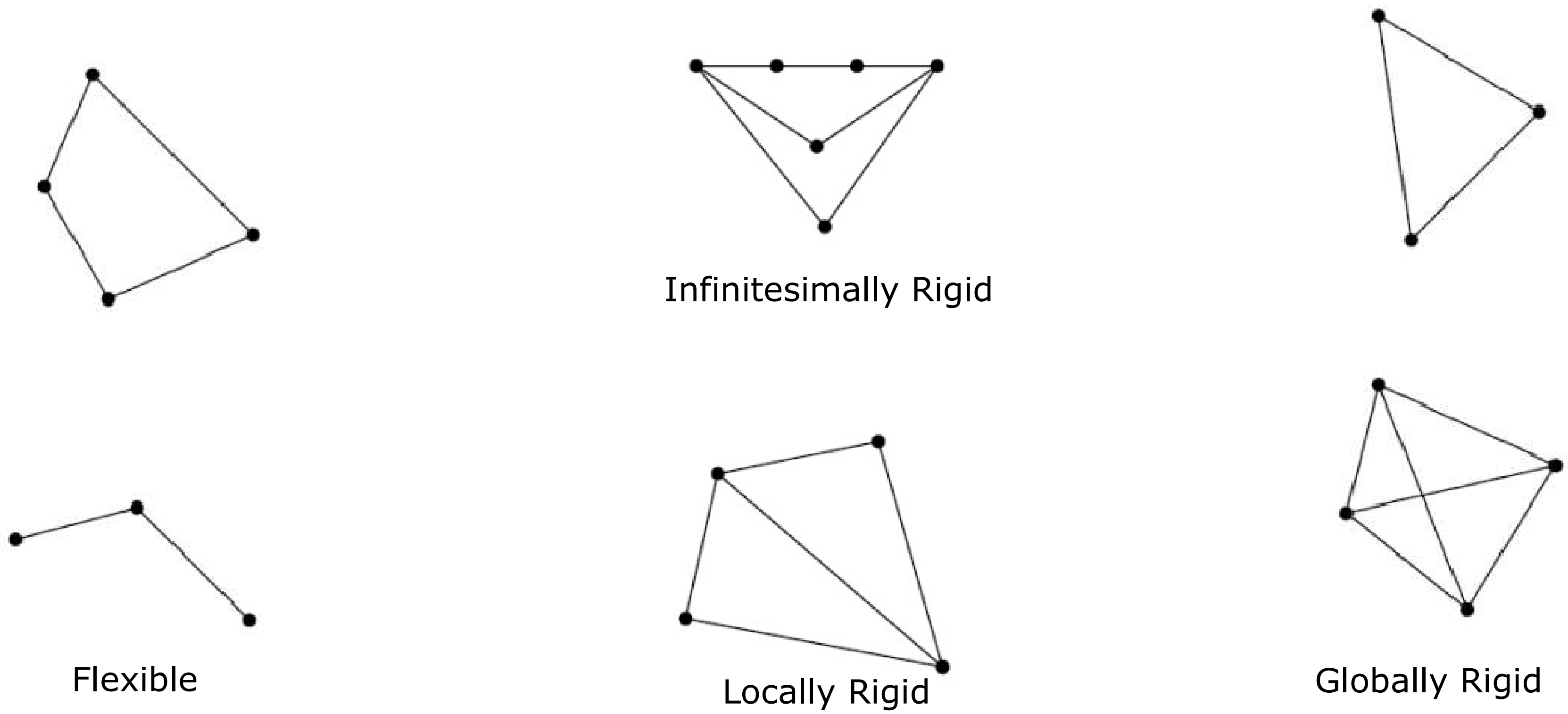
- Inverted oscillator $\ddot{s} = Cs$

Inverted Oscillator Phase Portrait



Clarification and Recap

Rigidity


Infinitesimally Rigid
Flexible
Locally Rigid
Globally Rigid

Recap

- Consensus-based control

$$\dot{x} = -Lx$$

  – Including different assumptions on topologies, information flow, etc.

- Formation control

$$\dot{x} = -Lx + d$$

  – Modified consensus

- Leader-follower

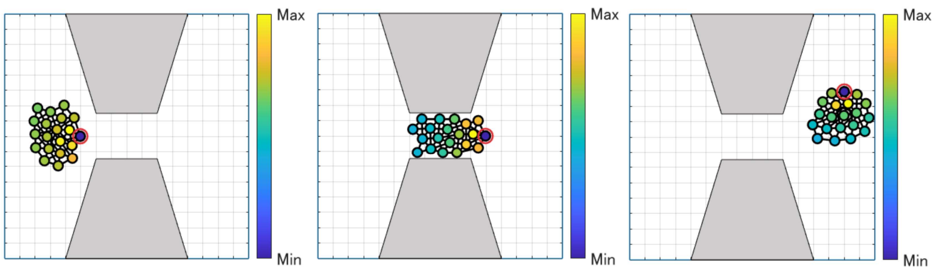
$$\begin{bmatrix} \dot{x_f} \\ \dot{x_l} \end{bmatrix} = -L \begin{bmatrix} x_f \\ x_l \end{bmatrix}$$

- Reference-frame invariant formations

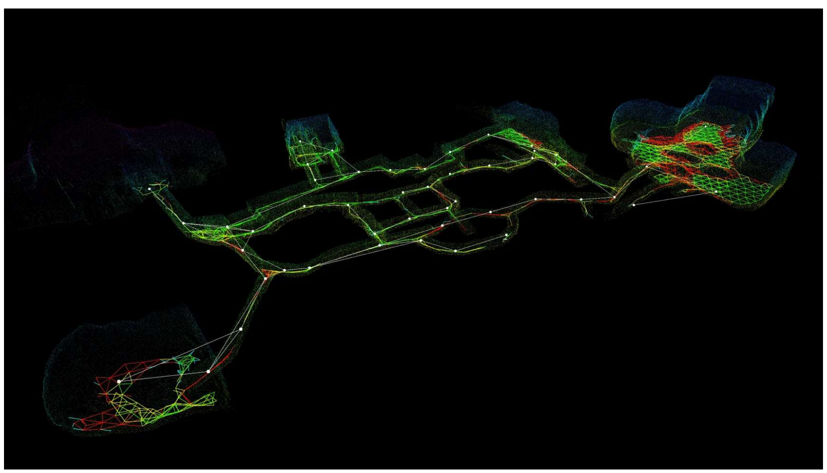
$$\dot{x_i} = \sum_{j \in \mathcal{N}_i} k_{ij} (\|p_i - p_j\| - d_{ij}) \frac{p_i - p_j}{\|p_i - p_j\|}$$

Putting It All Together

Motivation

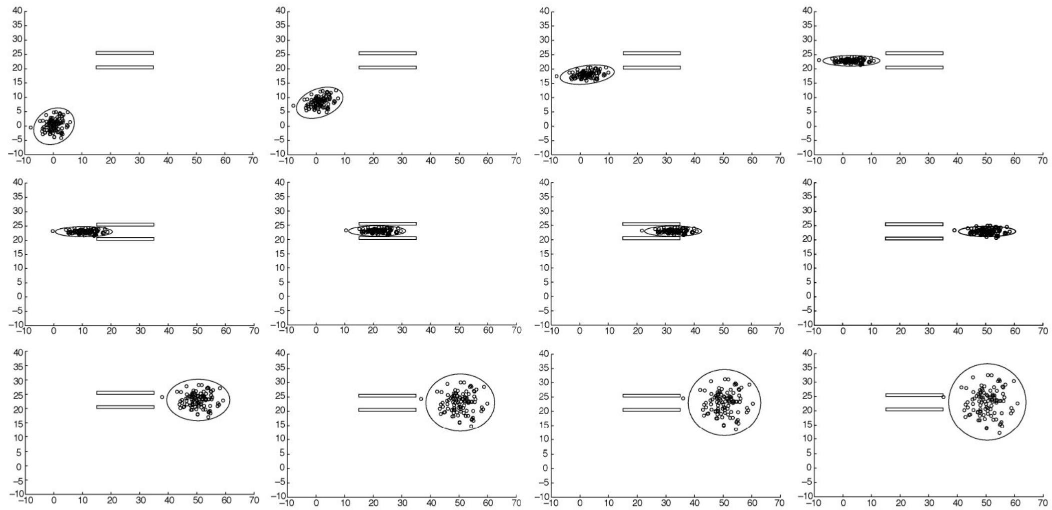


(**a**) t = 10 s (**b**) t = 70 s (**c**) t = 130 s



Recall

- Common centralized command and observer
- Automatic calculation of distributed individual control laws
- Control position, orientation, and shape of abstraction
- Requires absolute knowledge of position in a global reference frame



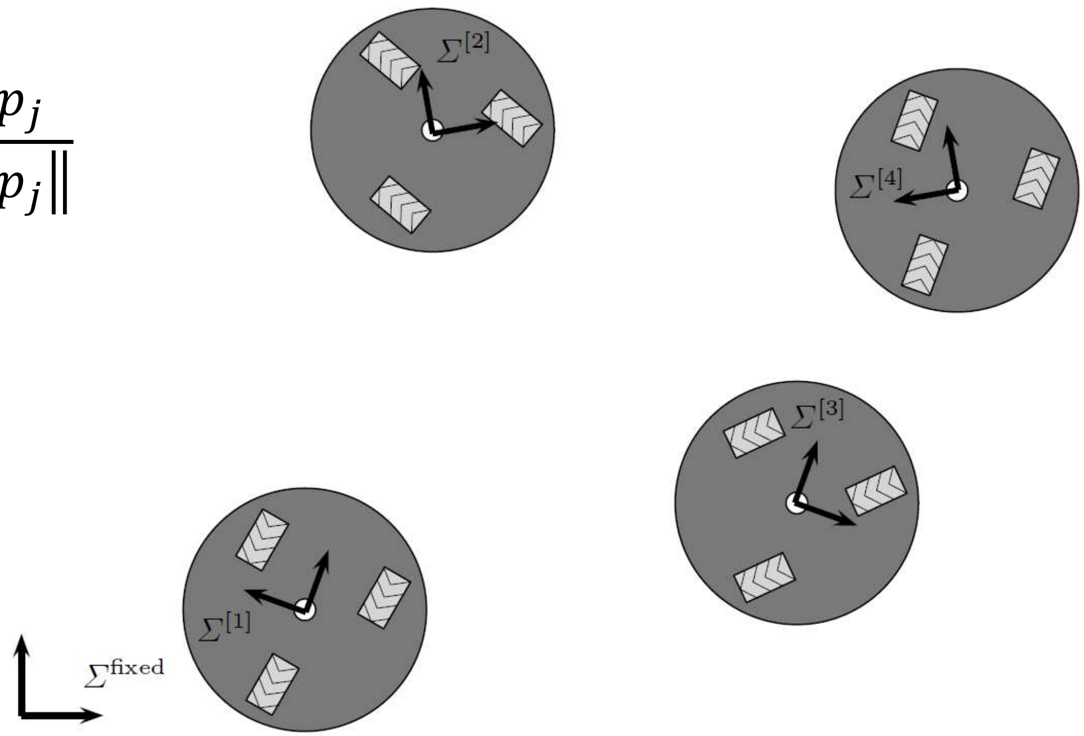
Recall

- Reference-frame invariant control laws

$$\dot{p}_i = \sum_{j:(i,j)\in E} \left(\|p_i - p_j\| - d_{ij}\right) \frac{p_i - p_j}{\|p_i - p_j\|}$$

- Use for consensus and formations with only local information

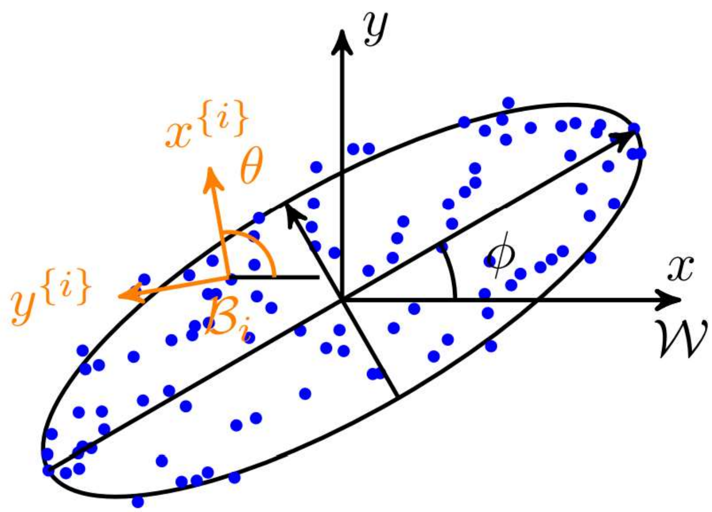
- Can we leverage this for abstraction control?



Observations

- Agents observe other agents in their own body frame
  - Relative position
  - Relative orientation
  - Agent i's observation is denoted o_i
  - Consists of all neighbors observed by its sensor

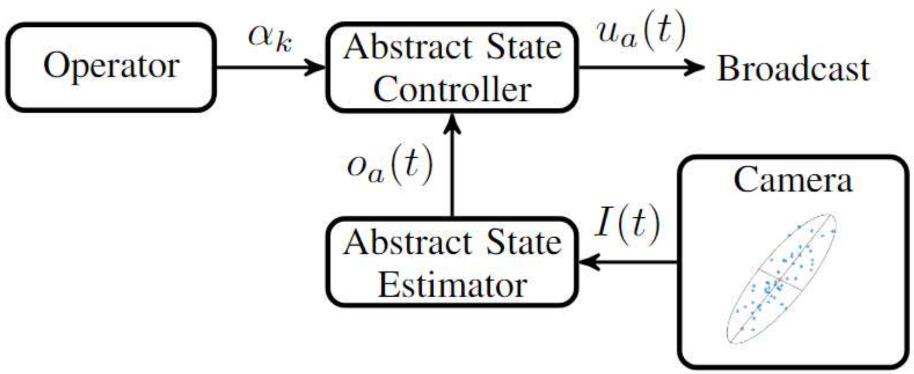
- Observer/Operator observes overall abstraction as μ, Σ, θ as discussed last time
  - Denoted o_a

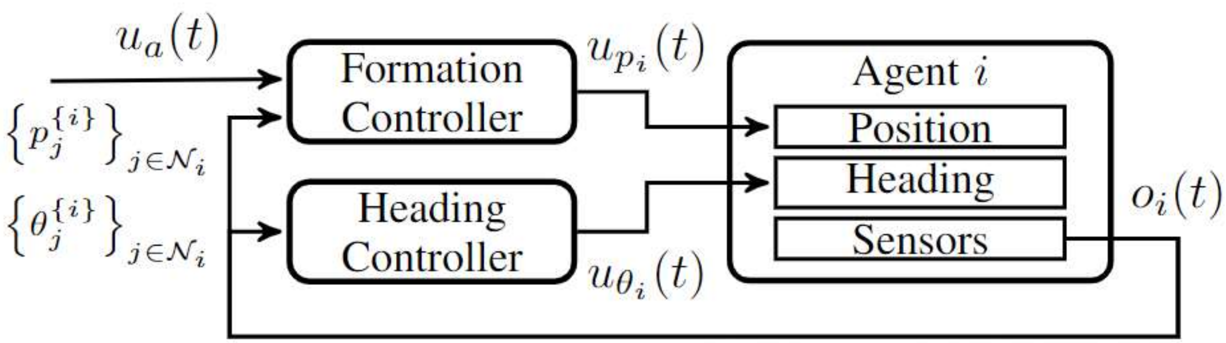


Goal

- Design global control law ν to map desired swarm behaviors to a control input u_a that is broadcast to the swarm
- Design a distributed control law $\pi: \mathrm{o_i} \times u_a \rightarrow u_i$ for each agent to satisfy the desired swarm behavior
- Such that
  - π depends only on o_i and u_a
  - π expressed in agents' body frames
  - No common global reference frame is required
  - No pairwise communication is required

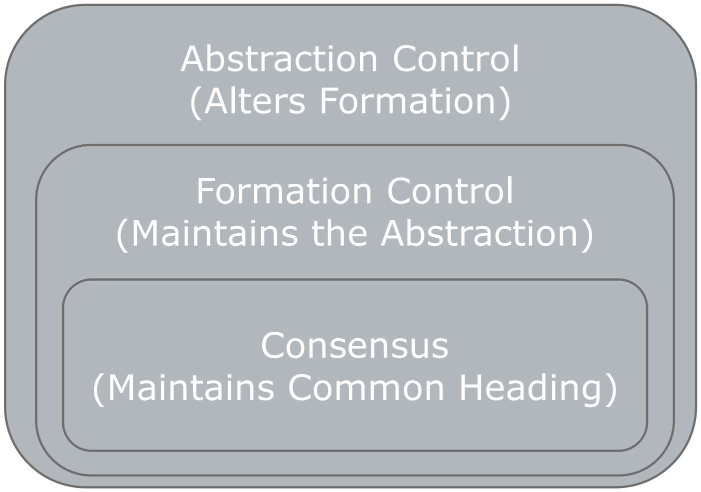
Architecture





Control Hierarchy

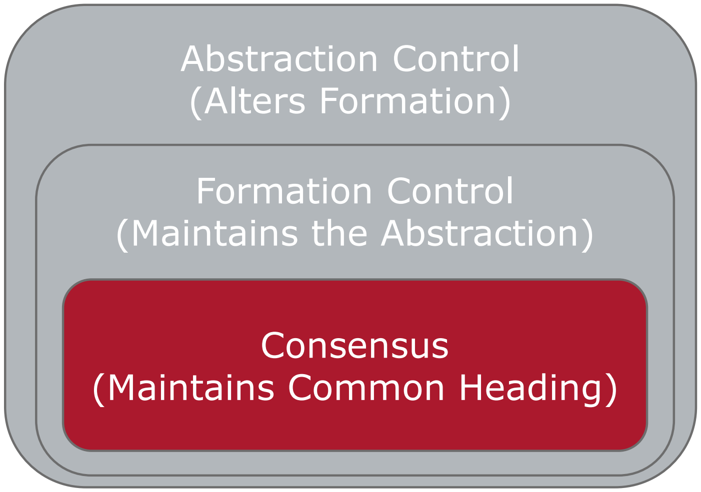
- Lowest layer
  - Consensus on heading with neighbors
  - Effectively creates common arbitrary reference frame
- Middle layer
  - Formation control
  - Keeps agents contained to abstraction region
- Highest layer
  - Controls the abstraction
  - Updates the goal of the abstraction layer



Lowest Layer – Consensus

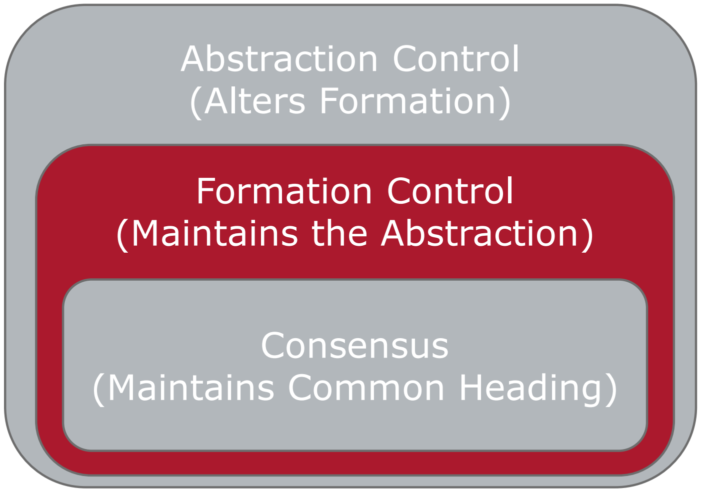
- If agents can observe each others' headings, each agent updates their heading according to

$$\dot{\theta}_i = \frac{1}{\beta}\sum_j (\theta_j - \theta_i)$$



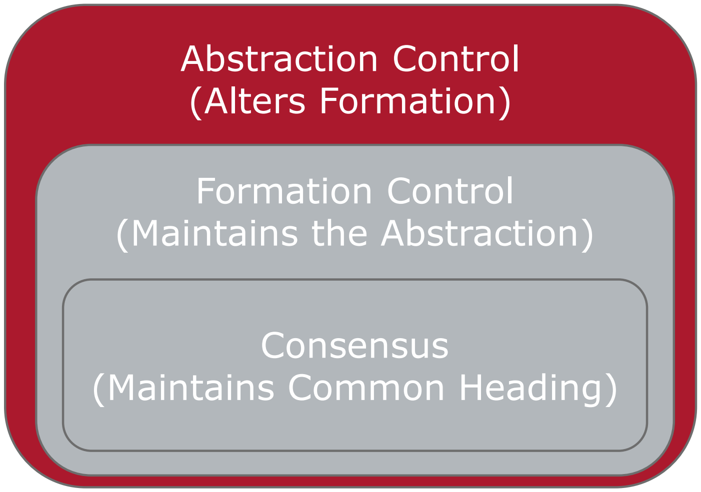
Middle Layer – Formation Control

- No prescribed distance vector d
- Instead, assume initial group inter-agent vectors are desired separation
- Formation control keeps agents in same relative positions

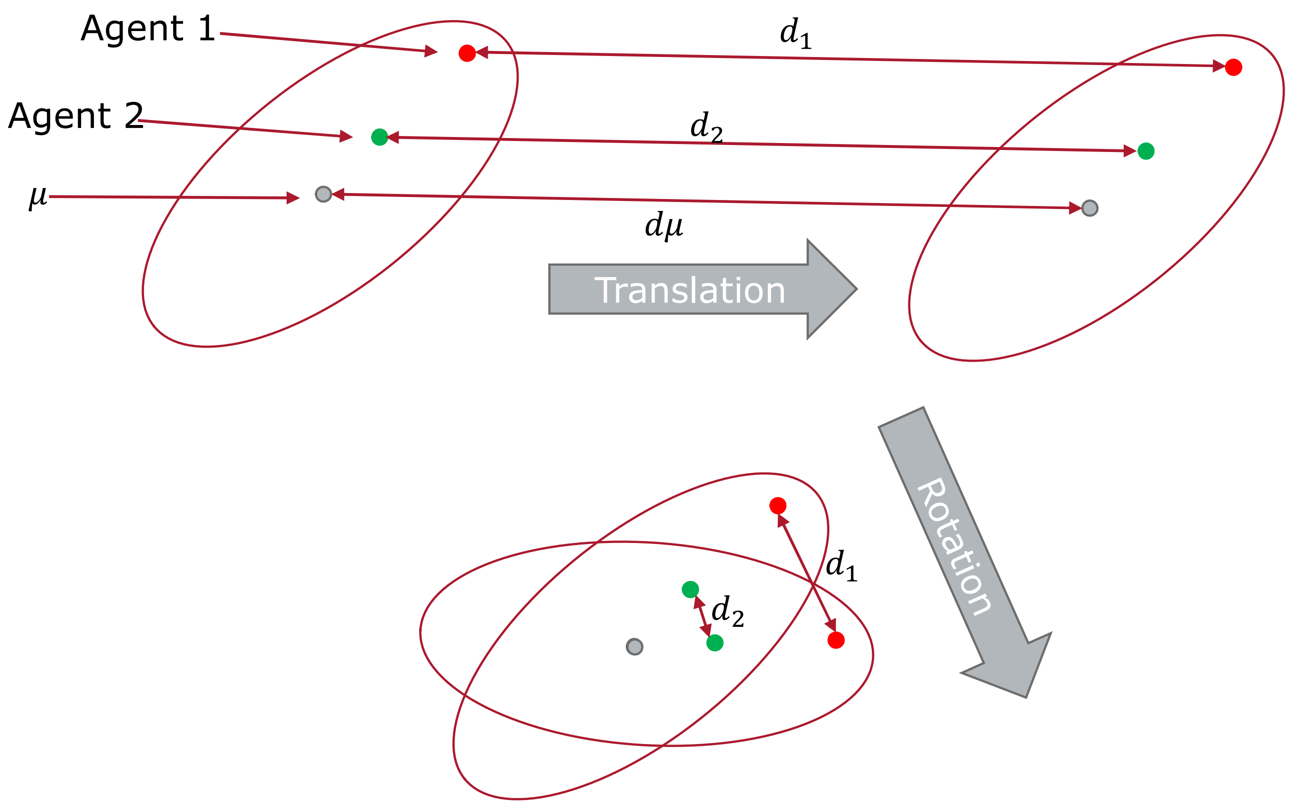


Highest Layer – Abstraction Control

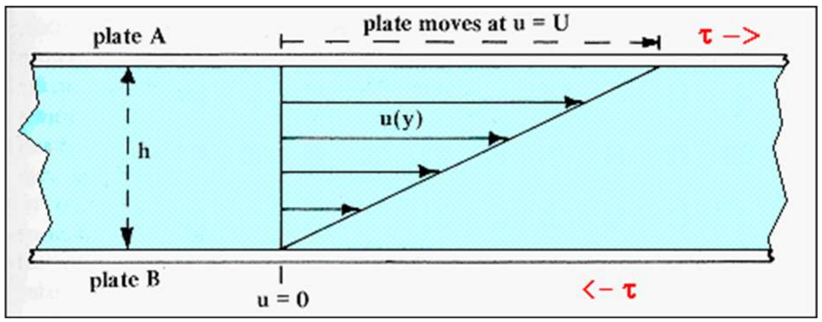
- Four operations at abstraction level
  - Translation – trivial as we saw last time
    - $\nu = \dot{\mu}$
  - Scaling – multiply distances by α as we saw last time
    - $\nu = \alpha$
  - Rotation – requires global knowledge – not possible

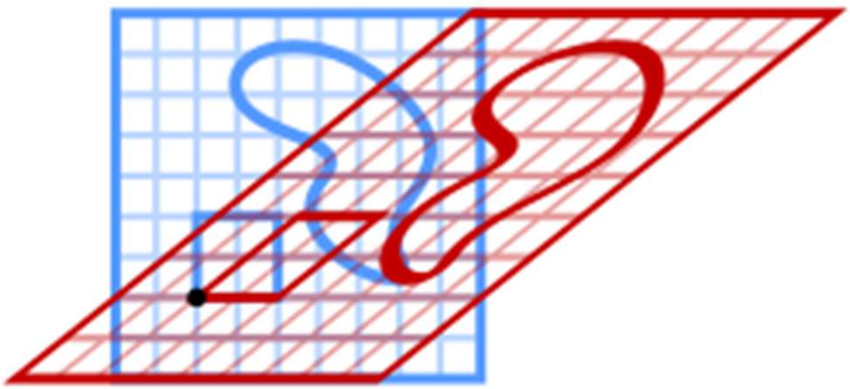


Agents and Distance from μ



Shear





Shear

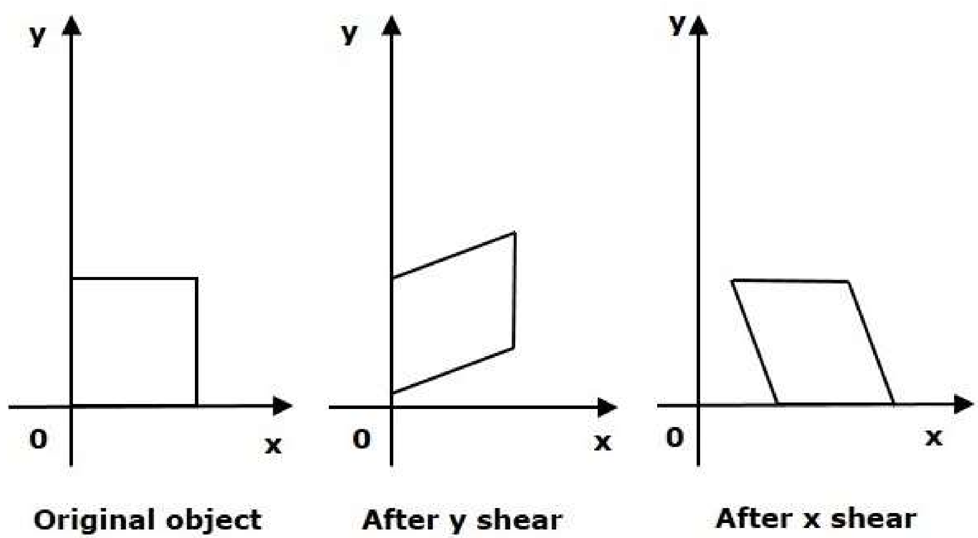
- **Shear** is a linear operation that transforms one component of a point in proportion to another value (e.g., x in proportion to y)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

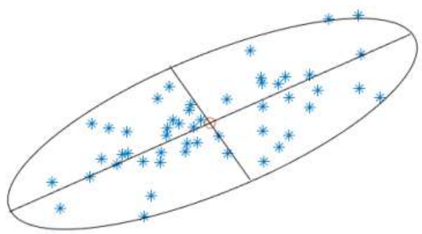
$$x' = x + hy$$
$$y' = y$$

- This example is an $\boldsymbol{x}$**-shear**, but can do the same for y

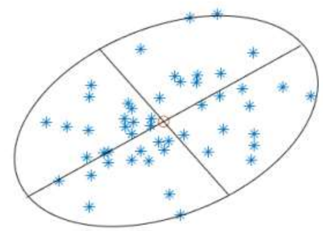
Shear



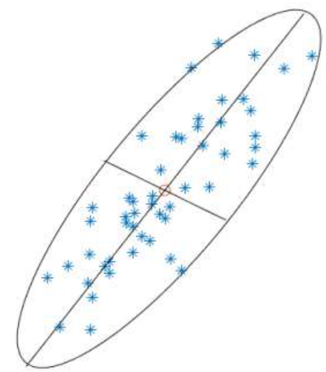
Shear



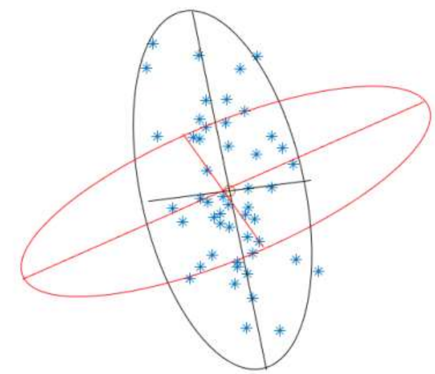
Original Position



Horizontal Shear



Vertical Shear



Final Position after one
more horizontal shear

Shear as Rotation

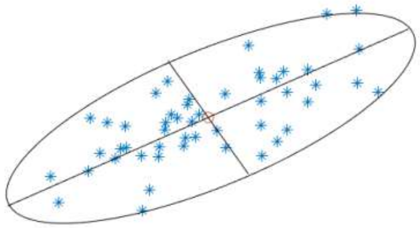
- Rotation by any angle $0^\circ \leq \psi \leq 90^\circ$ can be decomposed from rotation matrix R to a series of three shears

$$R = \begin{bmatrix} 1 & -\tan(\frac{\psi}{2}) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \sin(\psi) & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan(\frac{\psi}{2}) \\ 0 & 1 \end{bmatrix}$$

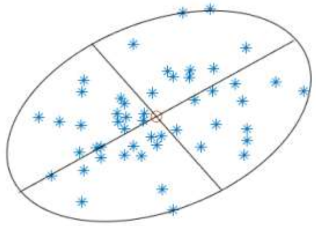
- This operates on vectors in $\mathbb{R}^2$

- We can use it on the entries of d

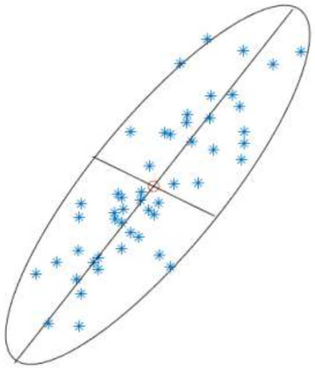
Shear



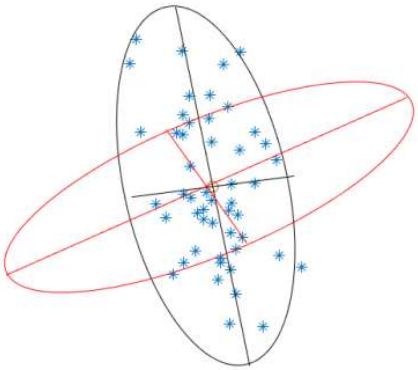
Original Position



Horizontal Shear

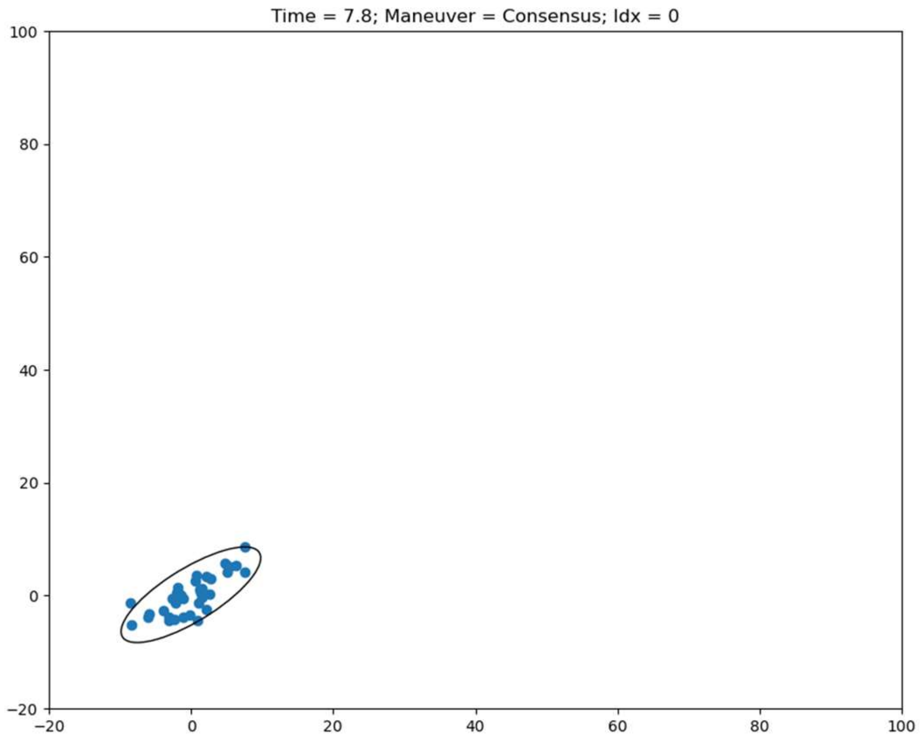


Vertical Shear



Final Position after one more horizontal shear

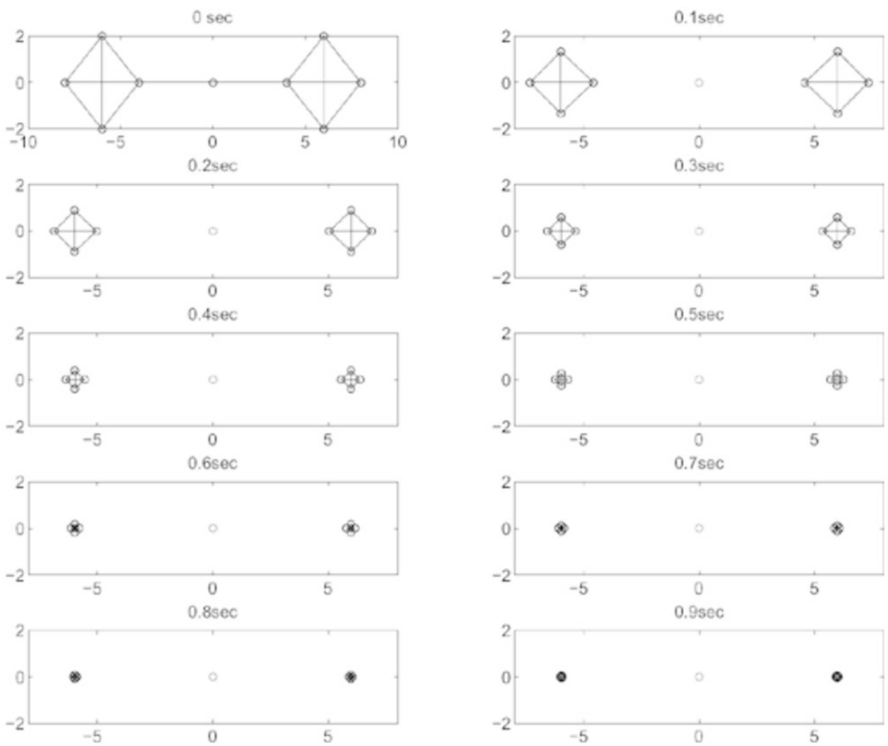
Results



Time = 7.8; Maneuver = Consensus; Idx = 0
100
80
60
40
20
0
−20
−20
0
20
40
60
80
100

An aside about weighting and connectivity

The Problem



When does this happen?

- For distance-based edges $\{v_i, v_j\} \in E$ iff $\|x_i - x_j\| \leq \Delta$
- If comms are lost beyond distance δ, we want edges to be at most $\delta - \epsilon$
- Can we define a controller such that
- If all edges are $< \delta - \epsilon$, we can guarantee $\|x_i - x_j\| \leq \delta$

The Solution

- Define ϵ-interior of δ-constrained graph realization as
$$D_{G,\delta}^{\epsilon} = \{x \in \mathbb{R}^{pn} \mid \|l_{ij}\| \leq \delta - \epsilon \; \forall\{v_i, v_j\} \in E\}$$

- Edge tension $\mathcal{V}_{ij}(\delta, x) = \begin{cases} \frac{\|l_{ij}(x)\|^2}{\delta - \|l_{ij}(x)\|} & if \; \{v_i, v_j\} \in E \\ 0 & otherwise \end{cases}$

- Gradient: $\frac{\partial \mathcal{V}_{ij}(\delta,x)}{\partial x_i} = \begin{cases} \frac{2\delta - \|l_{ij}(x)\|}{(\delta - \|l_{ij}(x)\|)^2}(x_i - x_j) & if \; \{v_i, v_j\} \in E \\ 0 & otherwise \end{cases}$

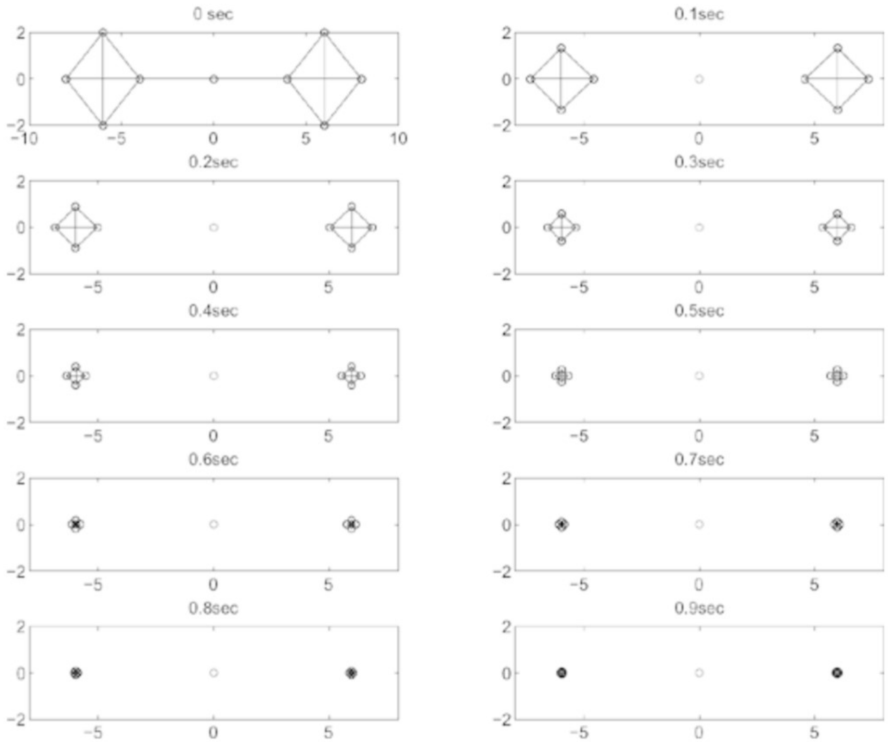
Total Energy

- Total Energy is defined as

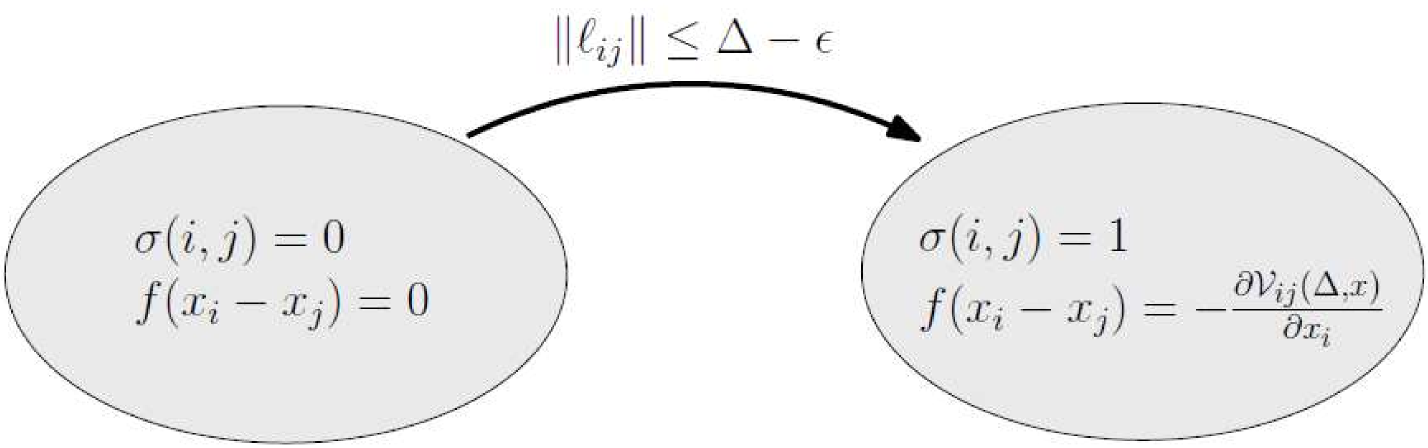
$$\mathcal{V}(\delta, x) = \frac{1}{2}\sum_{i}^{n}\sum_{j}^{n}\mathcal{V}_{ij}(\delta, x)$$

- Gradient of total energy is non-increasing!!

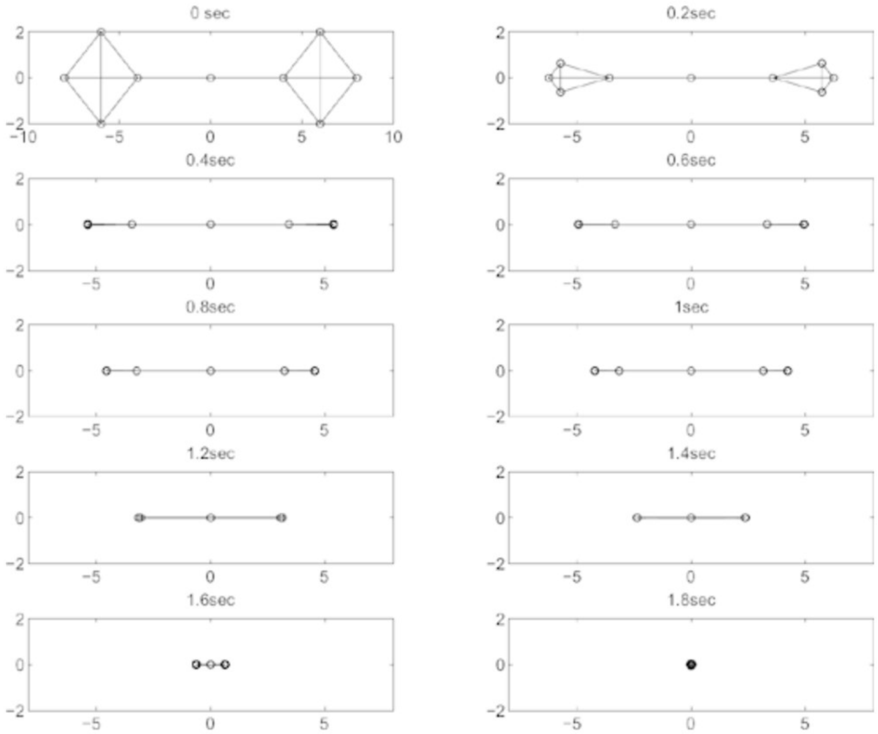
But what’s going on here?



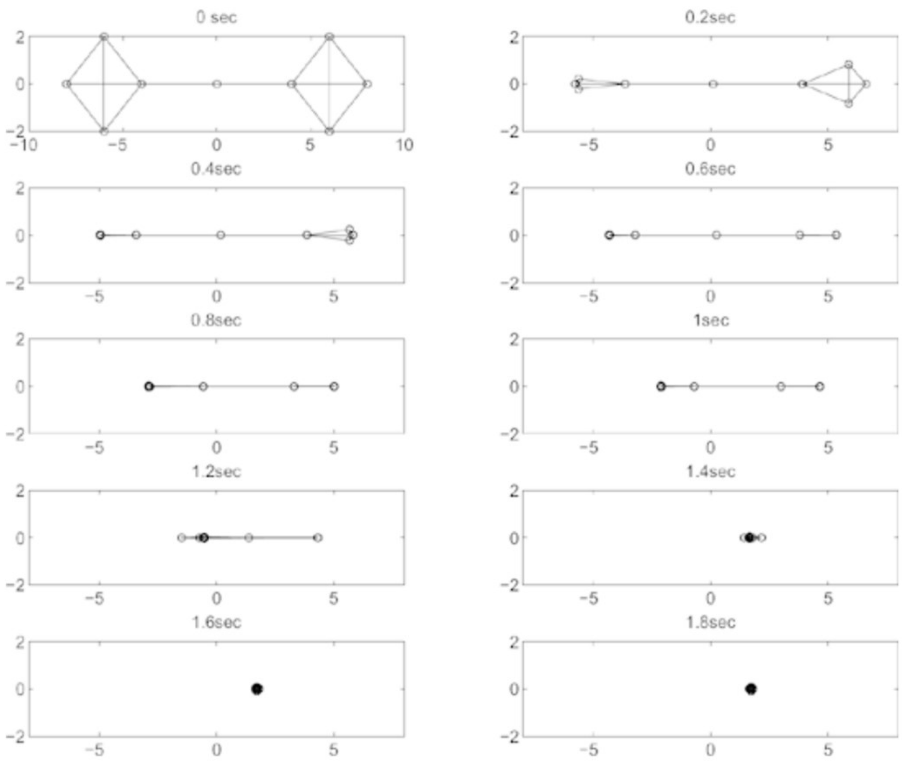
The Real Solution



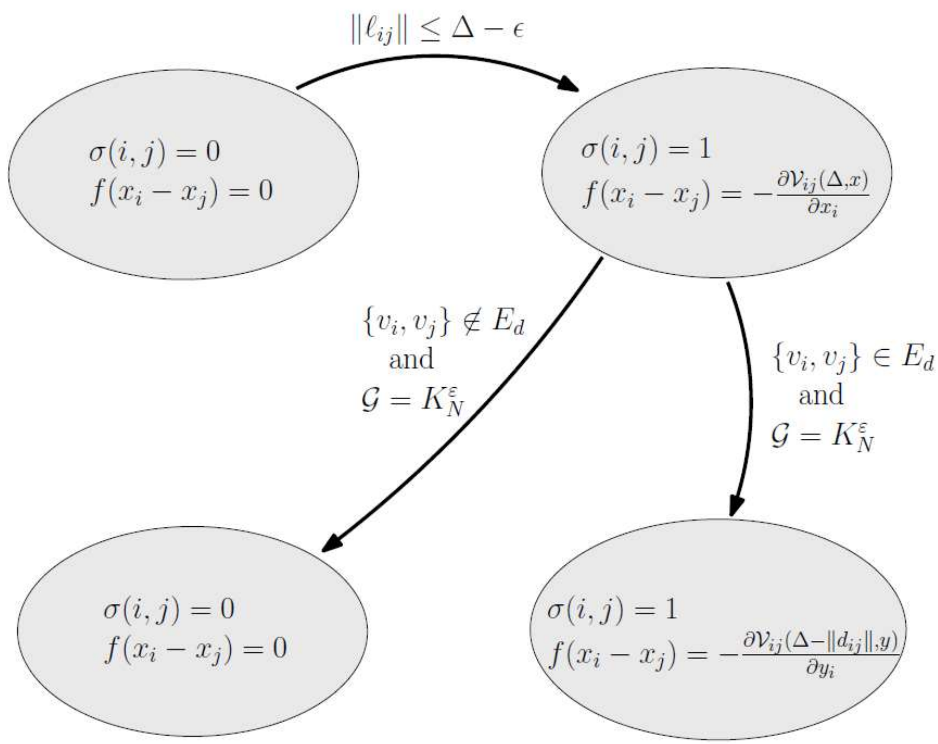
Rendezvous



Weighted Rendezvous

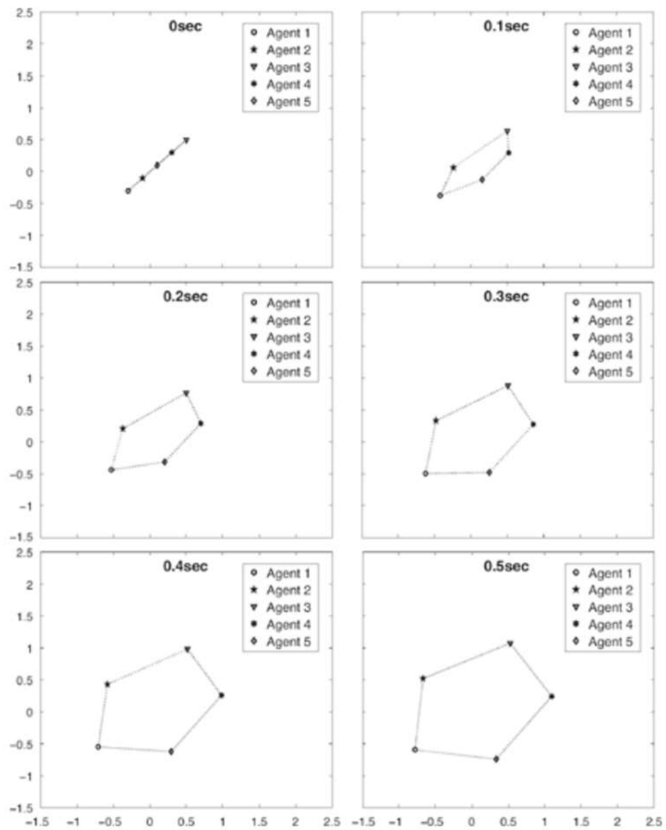


Formations



$\|\ell_{ij}\| \leq \Delta - \epsilon$
$\sigma(i,j) = 0$
$f(x_i - x_j) = 0$
$\sigma(i,j) = 1$
$f(x_i - x_j) = -\frac{\partial \mathcal{V}_{ij}(\Delta, x)}{\partial x_i}$
$\{v_i, v_j\} \notin E_d$
and
$\mathcal{G} = K_N^{\varepsilon}$
$\{v_i, v_j\} \in E_d$
and
$\mathcal{G} = K_N^{\varepsilon}$
$\sigma(i,j) = 0$
$f(x_i - x_j) = 0$
$\sigma(i,j) = 1$
$f(x_i - x_j) = -\frac{\partial \mathcal{V}_{ij}(\Delta - \|d_{ij}\|, y)}{\partial y_i}$

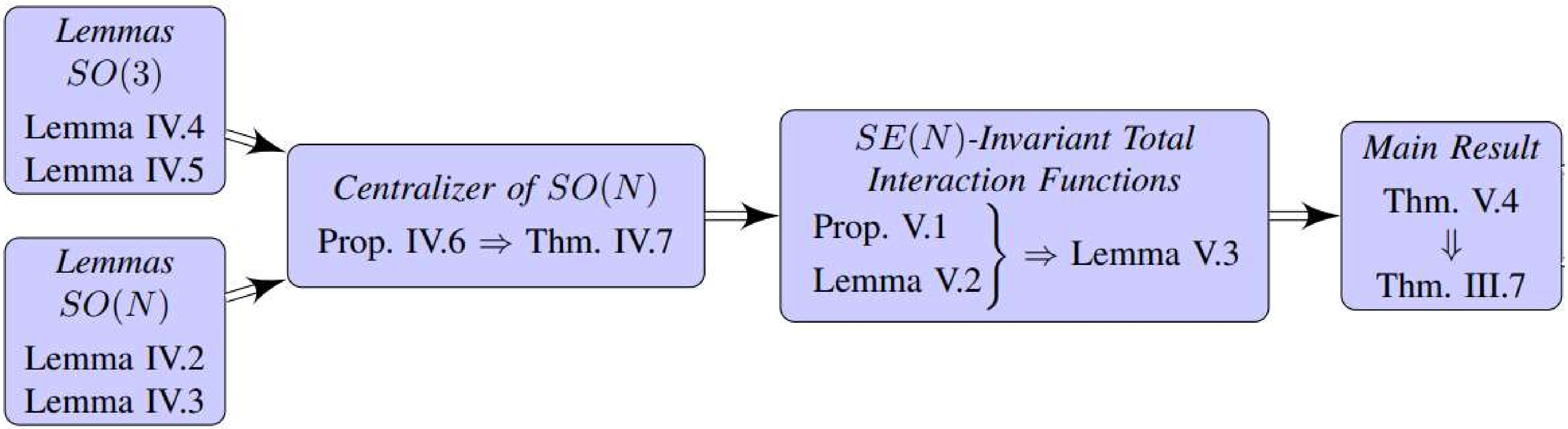
Formations


0sec
0.1sec
0.2sec
0.3sec
0.4sec
0.5sec
Agent 1
Agent 2
Agent 3
Agent 4
Agent 5

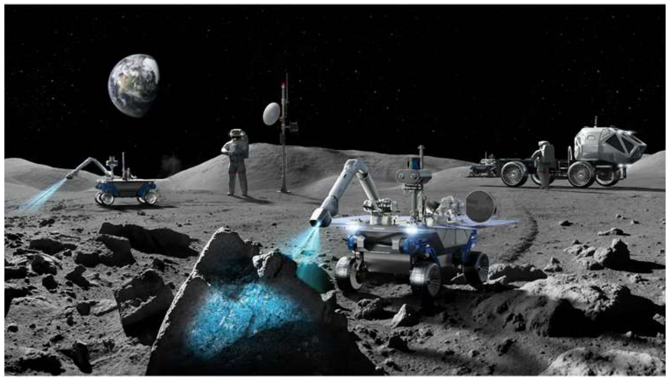
Pairwise Interactions and $SE(N)$-invariance

Source

- Vasile, Schwager, and Belta, “Translational and Rotational Invariance in Networked Dynamical Systems.” In *IEEE Trans. On Control of Network Systems*, 2018

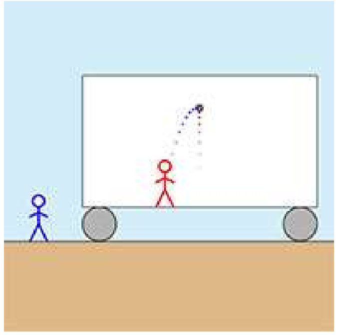


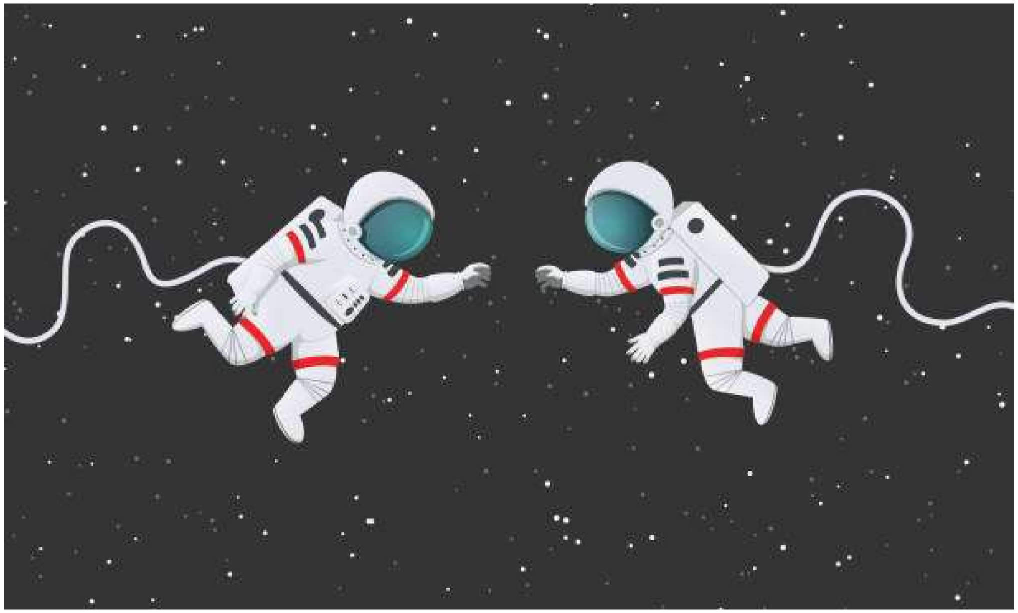
Robot Teams with Only Local Information



How can we design an effective controller when there is no global reference frame?

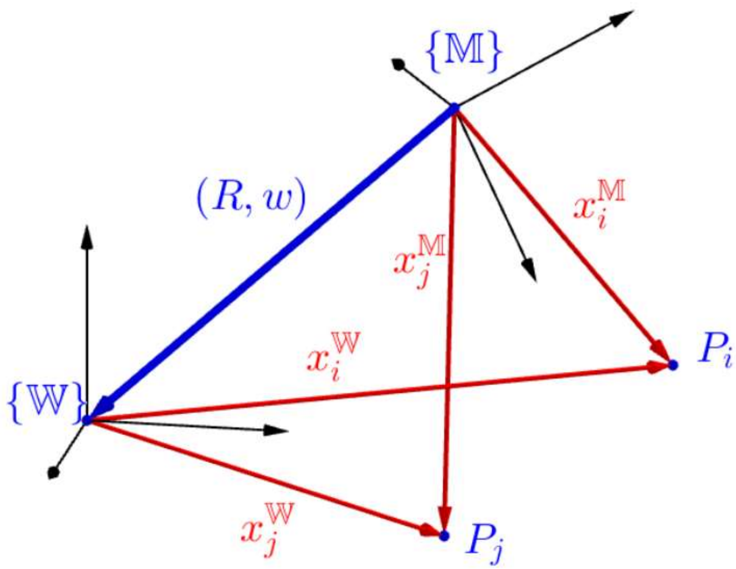
Inspiration – Physics





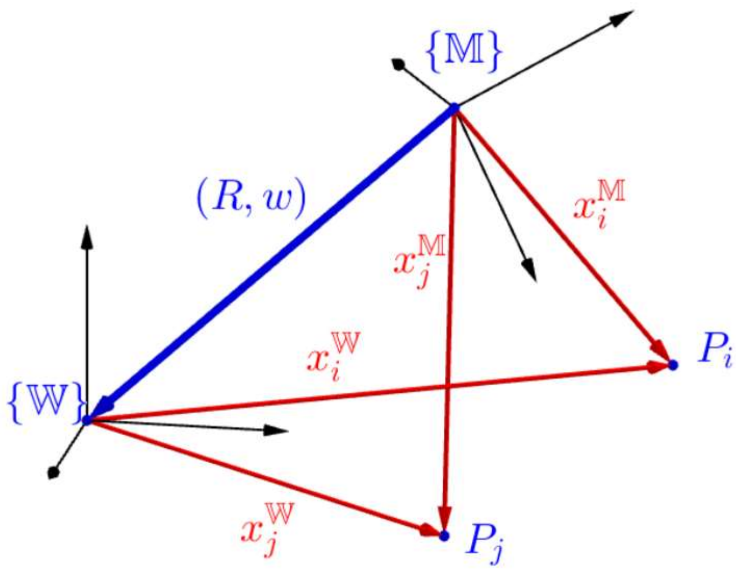
The laws of physics are the same in all inertial reference frames

Reference Frames



- $x_i^{\mathbb{W}}$- state of agent i in frame $\mathbb{W}$
- $x_i^{\mathbb{M}}$- state of agent i in frame $\mathbb{M}$
- Frame $\mathbb{M}$ related to $\mathbb{W}$ by pair (R, w)
  - $x_i^{\mathbb{W}} = R x_i^{\mathbb{M}} + w$
  - $v_i^{\mathbb{W}} = R v_i^{\mathbb{M}}$

Reference Frames



- Consider pairwise interaction between agents i and j
  - $v_i^{\mathbb{W}} = f_{ij}\left(x_i^{\mathbb{W}}, x_j^{\mathbb{W}}\right)$
  - $v_i^{\mathbb{M}} = f_{ij}\left(x_i^{\mathbb{M}}, x_j^{\mathbb{M}}\right)$
- Substituting:
  - $Rv_i^{\mathbb{M}} = f_{ij}\left(Rx_i^{\mathbb{M}} + w, Rx_j^{\mathbb{M}} + w\right)$
  - $Rv_i^{\mathbb{M}} = Rf_{ij}\left(x_i^{\mathbb{M}}, x_j^{\mathbb{M}}\right)$
- Combining:
  - $Rf_{ij}\left(x_i^{\mathbb{M}}, x_j^{\mathbb{M}}\right) = f_{ij}\left(Rx_i^{\mathbb{M}} + w, Rx_j^{\mathbb{M}} + w\right)$

The Special Euclidean Group

- $SE(N)$ is the **Special Euclidean group** in N-dimensions
  - The set of all possible rotations and displacements in $\mathbb{R}^N$

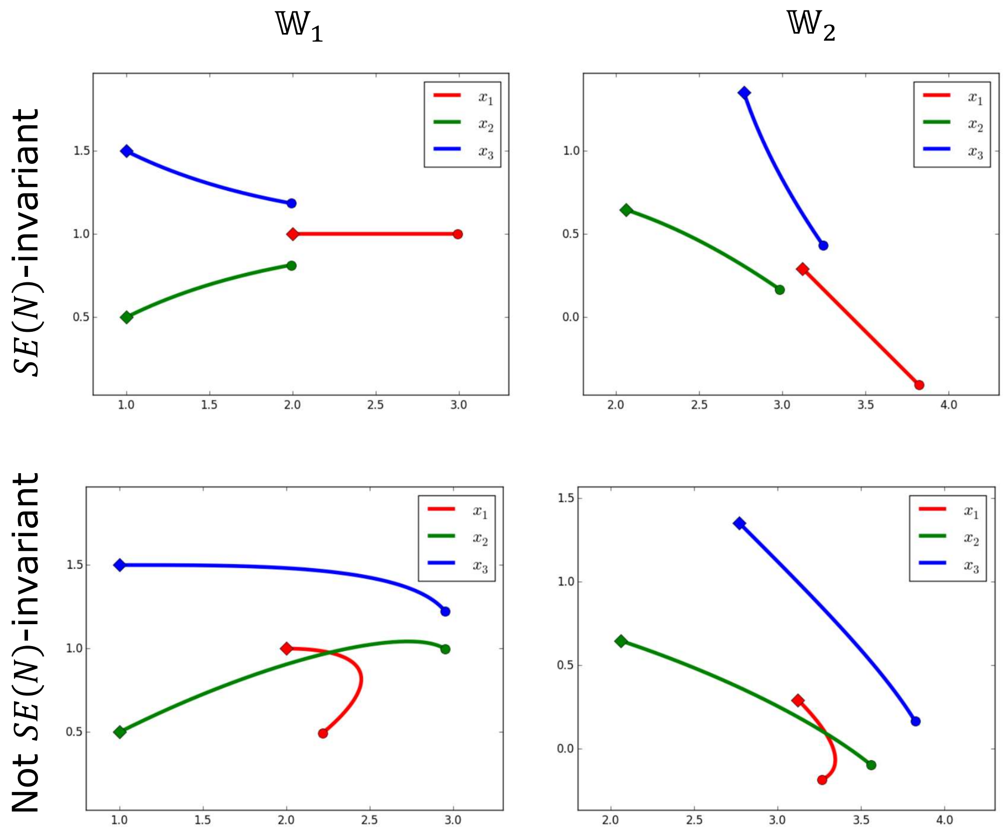
- A function $f: \mathbb{R}^N \times \cdots \times \mathbb{R}^N \to \mathbb{R}^N$ is $\boldsymbol{SE(N)}$**-invariant** if for all $R \in SO(N)$ and all $w \in \mathbb{R}^N$

$$Rf(x_1, \ldots, x_p) = f(Rx_1 + w, \ldots, Rx_p + w)$$

- This is the property we would like for reference-frame invariant control policies!

Example

- Consider two frames, $\mathbb{W}_1$ and $\mathbb{W}_2$
- Related by a clockwise rotation of $\pi/4$ and a translation of $[1,1]^T$



Pairwise Interaction System

- A **pairwise interaction system** is a double (G, F), where
  - G is a graph
  - $F = \left\{ f_{ij} \mid f_{ij}\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N, (i,j) \in E(G) \right\}$

- Dynamics of agent

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} f_{ij}\left(x_i, x_j\right)$$

- Call this the **total interaction** on agent i

$$S_i(x_1, \dots, x_n) = \sum_{j \in \mathcal{N}_i} f_{ij}\left(x_i, x_j\right)$$

Pairwise Interaction System

- A pairwise interaction system (G, F) is $SE(N)$-invariant if, for all $i \in V(G)$, the total interaction function S_i are $SE(N)$-invariant

- Note: this does not necessarily mean the *individual* f_{ij} functions are $SE(N)$-invariant. It is their *sum* for an agent.

Quasi-Linearity

- A function $f: \mathbb{R}^N \to \mathbb{R}^N$ is called **quasi-linear** if there is a function $k: \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that
$$f(x) = k(\|x\|)x$$

For all $x \in \mathbb{R}^N$

Quasi-Linear Pairwise Interactions

- A pairwise interaction system (G, F) is quasi-linear if the total interaction function of each agent is a sum of quasi-linear functions

$$S_i = \sum_{j \in \mathcal{N}_i} k_{ij}\left(\left\|x_j - x_i\right\|\right)(x_j - x_i)$$

- $k_{ij}: \mathbb{R}_{\geq 0} \to \mathbb{R}$
- $N \geq 3$

Main Result

- A pairwise interaction system (G, F) is $SE(N)$-invariant if and only if it is quasi-linear
  - If we have a pairwise interaction system, we can check for invariance
  - If we are designing one, we have a design criterion
  - Does not imply any result about stability
  - Does not impose restrictions on G or F

Only for pairwise systems

- $S_1(x_1, x_2, x_3) = \|x_2 - x_1\|(x_3 - x_2)$

What about stability?

- Can prove (using Lyapunov) that if (G, F)
  - (G, F) is $SE(N)$-invariant
  - G is strongly connected
  - (G, F) is balanced: $\underbrace{\sum_{j \in \mathcal{N}_i} f_{ij}(x_i, x_j)}_{\text{Outgoing}} + \underbrace{\sum_{j \in \mathcal{N}_i} f_{ji}(x_j, x_i)}_{\text{Incoming}} = 0$
  - For all $(i, j) \in E$, $x_i \neq x_j$: $\left(x_j - x_i\right)^T f_{ij}(x_i, x_j) > 0$

- Then the consensus set is globally asymptotically stable

Let’s look at some examples

$$S_i = \sum_{j \in \mathcal{N}_i} k_{ij}(\|x_j - x_i\|)(x_j - x_i)$$

- Consensus:

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i)$$

- Formation:

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} \left(\|x_i - x_j\|^2 - d_{ij}\right)(x_i - x_j)$$

Let's look at some examples

$$S_i = \sum_{j \in \mathcal{N}_i} k_{ij}(\|x_j - x_i\|)(x_j - x_i)$$

- Navigation functions:

$$\dot{x}_i = -\alpha \nabla_{x_i} \left(\frac{\gamma_i(x)}{\left(\gamma_i(x)^k + \beta_i(x)\right)^{\frac{1}{k}}} \right)$$

- Hamiltonian physics:

$$\ddot{x}_i = \frac{1}{m_i} \sum_{j=1, j \neq i}^{n} \frac{G m_i m_j}{\|x_i - x_j\|^3} (x_j - x_i)$$

Let’s look at some examples

$$S_i = \sum_{j \in \mathcal{N}_i} k_{ij}(\|x_j - x_i\|)(x_j - x_i)$$

- Edge potentials:

$$\frac{\partial \mathcal{V}_{ij}(\delta, x)}{\partial x_i} = \begin{cases} \dfrac{2\delta - \|l_{ij}(x)\|}{\left(\delta - \|l_{ij}(x)\|\right)^2}\left(x_i - x_j\right) & if\ \{v_i, v_j\} \in E \\ 0 & otherwise \end{cases}$$

Not Just Consensus

- Consider complete graph with 3 agents

$$f_{ij}(x_i, x_j) = \begin{cases} x_j & (i,j) \in \{(1,2),(2,3),(3,1)\} \\ -x_j & otherwise \end{cases}$$

- Pairwise interactions: quasi-linear?

Not Just Consensus

- Consider complete graph with 3 agents

$$f_{ij}(x_i, x_j) = \begin{cases} x_j & (i,j) \in \{(1,2),(2,3),(3,1)\} \\ -x_j & otherwise \end{cases}$$

- SE(N)-invariant?

Total Interaction Function

- But $SE(N)$-invariance $\Leftrightarrow$ quasi-linear?

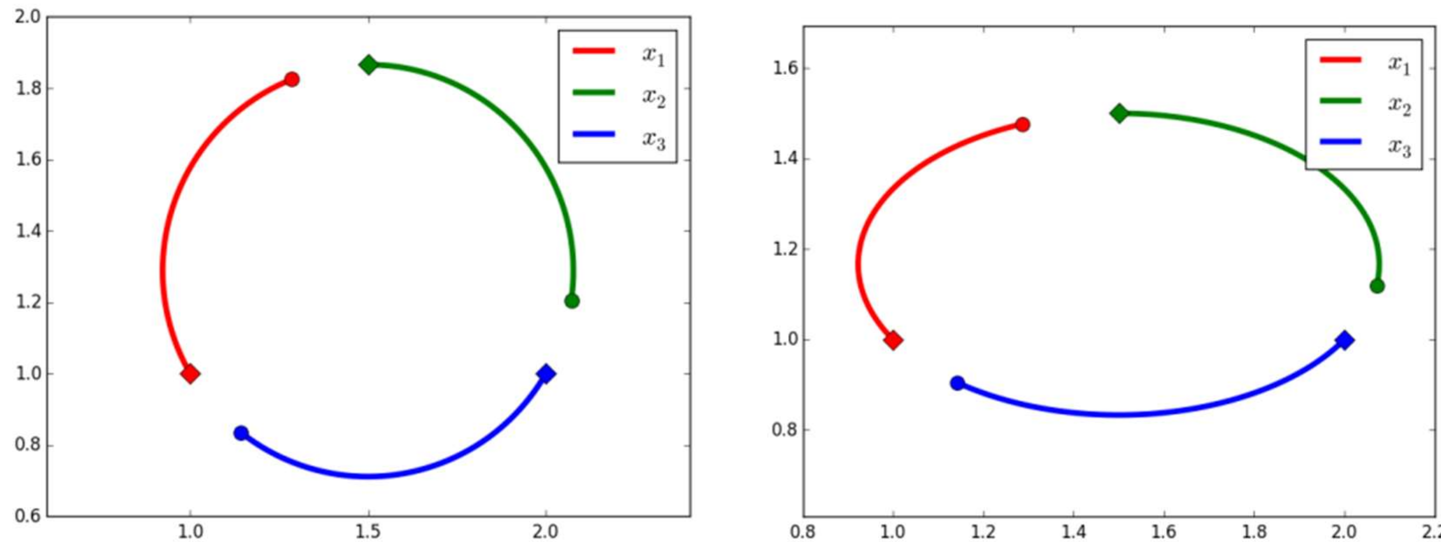
Not Just Consensus

- Consider complete graph with 3 agents

$$f_{ij}(x_i, x_j) = \begin{cases} x_j & (i,j) \in \{(1,2),(2,3),(3,1) \\ -x_j & otherwise \end{cases}$$

- Total interaction: quasi-linear?

Result: Periodic Orbits



(a) $x_1^T(0) = [1, 1], x_2^T(0) = [\frac{3}{2}, 1 + \frac{\sqrt{3}}{2}]$, $x_3^T(0) = [2, 1]$

(b) $x_1^T(0) = [1, 1]$, $x_2^T(0) = [\frac{3}{2}, \frac{3}{2}]$, $x_3^T(0) = [2, 1]$

Where are we now

Recap

- Consensus and nearest-neighbor rules
  - Average consensus for undirected, connected, intermittently connected
    - Requirements: comms/sensing, synchronous updates (in what we discussed, at least)
    - Advantages: scalable, decentralized, local
  - Formations: consensus-like approach
    - Requirements: same as consensus + global information
    - Advantages: same as consensus, but global information/awareness required
  - Leader-follower control: same as consensus
- Centralized formation control:
  - Aggregated control
    - Requirements: centralization, localization
    - Advantages: very little comms (one to many), scalable (size-agnostic)

Recap

- Distributed Formation Control:
  - Requirements: comms/sensing, synchronous updates (in what we discussed, at least)
  - Advantages: scalable, decentralized, local
  - Disadvantages: shearing instead of rotation
- Connectivity maintenance in nearest-neighbor formations
  - Hybrid approach
  - Local information, local reference frames, distributed
- $SE(N)$-invariance
  - Iff quasi-linear
  - Design requirement/validation criteria to ensure local reference frame control

Overview

- Distributed algorithms and control
- Sensing and estimation
- Communication and information sharing
- Cooperative decision making

Distributed Algorithm with Control

Distributed Algorithm

- A **distributed algorithm** DA for a network S consists of the sets
  - $\mathbb{A}$: a set containing the **alphabet**, including the `null` symbol
  - $W^{[i]}, i \in I$: the **processor state sets**
  - $W_0^{[i]} \subseteq W^{[i]}, i \in I$: the **allowable initial values**

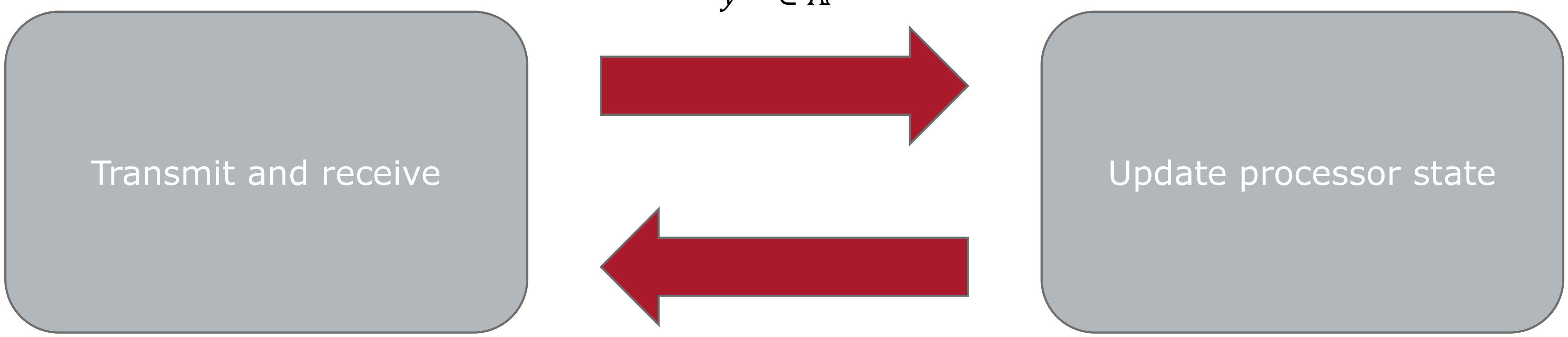
- It also has the maps
  - $msg^{[i]}: W^{[i]} \times I \to \mathbb{A}, i \in I$: the **message-generation functions**
  - $stf^{[i]}: W^{[i]} \times \mathbb{A}^n \to W^{[i]}, i \in I$: the **state-transition functions**

Communication on a Network

State: $w^{[i]} \in W^{[i]}$

Update state:
$w^{[i]} \leftarrow stf^{[i]}(w^{[i]}, y^{[i]})$

Receive messages:
$y^{[i]} \in \mathbb{A}^n$



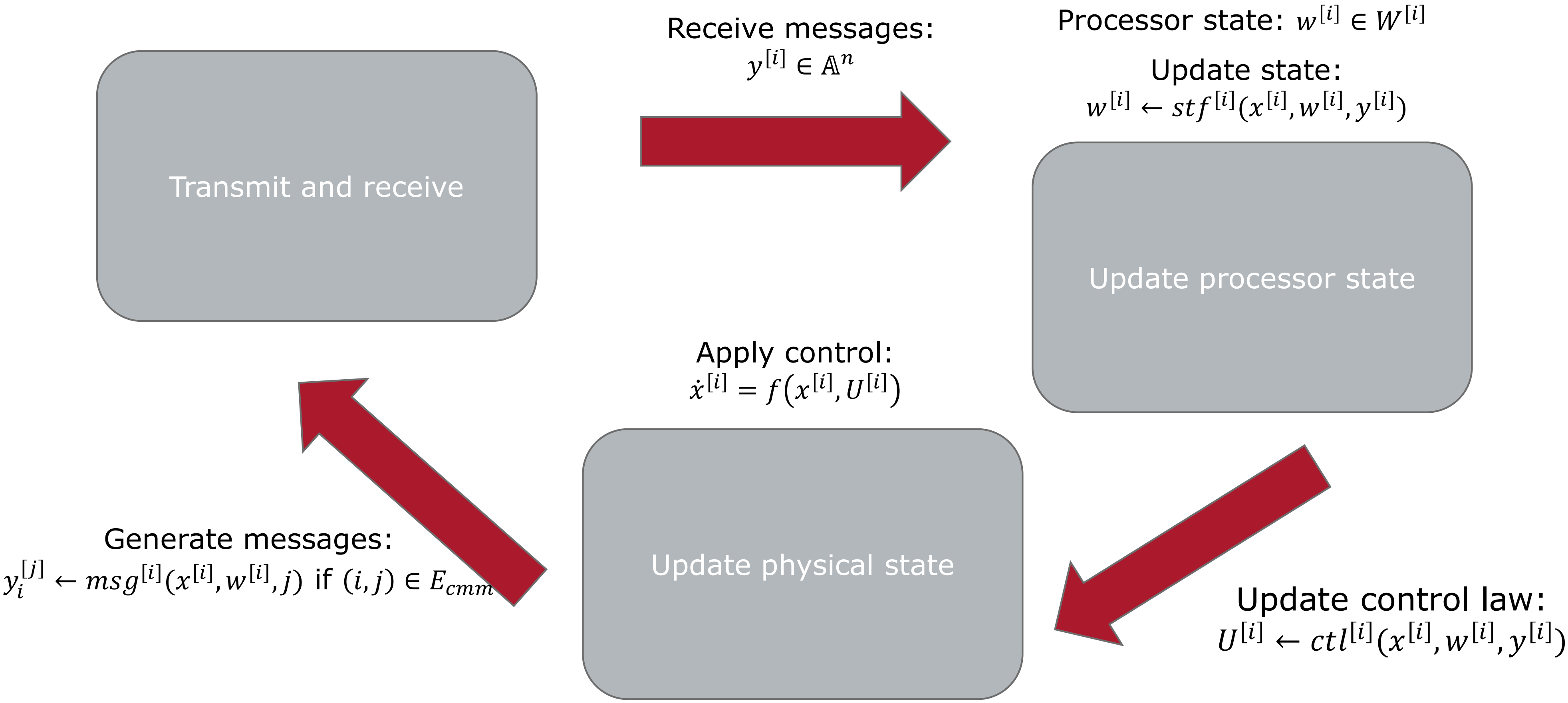
Generate messages:
$y_i^{[j]} \leftarrow msg^{[i]}(w^{[i]}, j)$ if $(i, j) \in E_{cmm}$

Control and Communication Law

- A **control and communication law** CC for a robotic network S consists of the sets
  - $\mathbb{A}$: a set containing the **alphabet**, including the `null` symbol
  - $W^{[i]}, i \in I$: the **processor state sets**
  - $W_0^{[i]} \subseteq W^{[i]}, i \in I$: the **allowable initial values**

- It also has the maps
  - $msg^{[i]}: X^{[i]} \times W^{[i]} \times I \to \mathbb{A}, i \in I$: the **message-generation functions**
  - $stf^{[i]}: X^{[i]} \times W^{[i]} \times \mathbb{A}^n \to W^{[i]}, i \in I$: the **state-transition functions**
  - $ctl^{[i]}: X^{[i]} \times X^{[i]} \times W^{[i]} \times \mathbb{A}^n \to U^{[i]}, i \in I$: the **(motion) control functions**

Communication and Control on a Network



Wrap Up

Recap

- Closing out consensus and formation control
  - Combining reference frame invariance with abstraction-based control
  - Generalizing the results for reference-frame invariance
- Next time: starting sensing, deployment, and coverage