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1

Definitions and Theory

Definition 1. The **expectation** of a random variable X having values x_1, x_2, \dots, x_n is defined as $E(X) = x_1P(X = x_1) + x_2P(X = x_2) + \dots + x_nP(X = x_n) = \sum_{j=1}^n x_jP(X = x_j) = x_1f(x_1) + x_2f(x_2) + \dots + x_nf(x_n) = \sum_{j=1}^n x_jf(x_j)$ where f is the distribution of X .

Theorem 1.0.1. The expectation has the following properties:

1. $E(cX) = cE(X)$ where c is any constant
2. If X and Y are any random variables then $E(X + Y) = E(X) + E(Y)$
3. If X and Y are independent random variables, then $E(XY) = E(X)E(Y)$

Definition 2. The **variance** is defined as $\text{Var}(X) = \sigma^2 = E[(X - \mu)^2]$. The **standard deviation** is defined as $\sigma = \sqrt{\text{Var}(X)}$.

Theorem 1.0.2. The variance has the following properties:

1. $\sigma^2 = E[(X - \mu)^2] = E(X^2) - \mu^2 = E(X^2) - [E(X)]^2$ where $\mu = E(X)$
2. If c is any constant, $\text{Var}(cX) = c^2 \text{Var}(X)$
3. The quantity $E[(X - a)^2]$ is a minimum where $a = \mu = E(X)$
4. If X and Y are independent random variables

Definition 3 (Moments). The r -th **moment** of a random variable X about the mean μ is defined as $\mu_r = E[(X - \mu)^r]$ where $r = 0, 1, 2, \dots$. The r -th moment of X about the origin is defined as $\mu'_r = E(X^r)$.

It follows that $\mu_0 = 0$, $\mu_1 = 1$, $\mu_2 = \sigma^2$

Definition 4 (Moment Generating Function). *Let X be a random variable defined on (Ω, \mathcal{F}, P) . The function $M(t) = E[e^{tX}]$ is called the **moment generating function** (MGF) of the random variable X if the expectation on the right side exists in some neighbourhood of the origin. If the expectation on the right side does not exist in any neighbourhood of the origin, then we say the MGF does not exist.*

The r -th derivative of the moment generating function is the r -th moment about the origin μ'_r .

Theorem 1.0.3. *The moment generating function has the following properties:*

1. *For any constants a and b , the mgf of the random variable $aX + b$ is given by $M_{aX+b}(t) = e^{bt}M_X(at)$*
2. *If X and Y are independent random variables having moment generating functions $M_X(t)$ and $M_Y(t)$ respectively, then $M_{X+Y}(t) = M_X(t)M_Y(t)$*
3. **Uniqueness Theorem** *Suppose that X and Y are random variables having moment generating functions $M_X(t)$ and $M_Y(t)$ respectively. Then X and Y have the same probability distribution if and only if $M_X(t) = M_Y(t)$ identically.*

We can either sample *with replacement* or *without replacement*. A finite population sampled with replacement can be considered infinite. Sampling from a very large finite population can similarly be considered as sampling from an infinite population.

To properly choose the sample, we can make sure that every member of the population has an equal chance of being in the sample. Normally, since the sample size is much smaller than the population size, sampling without replacement will give practically the same results as sampling with replacement.

For a sample of size n from a population which we assume has distribution $f(x)$, we can choose members of the population at random, each selection corresponding to a random variable X_1, X_2, \dots, X_n with corresponding values x_1, x_2, \dots, x_n . In case we are assuming sampling without replacement, X_1, X_2, \dots, X_n will be independent and identically distributed random variables with probability distribution $f(x)$.

Definition 5 (Random Sample). *Let X be a random variable with a distribution f , and let X_1, X_2, \dots, X_n be iid random variables with the common distribution f . Then the collection X_1, X_2, \dots, X_n is called a **random sample** of size n from the population f .*

Since X_1, X_2, \dots, X_n are iid, the joint distribution of the random sample is $f(x_1, x_2, \dots, x_n) = f(x_1)f(x_2)\dots f(x_n)$.

Any quantity obtained from a sample for the purpose of estimating a population parameter is called a sample statistic, or briefly statistic. Mathematically, a sample statistic for a sample of size n can be defined as a function of the random variables X_1, X_2, \dots, X_n as $T(X_1, X_2, \dots, X_n)$. This itself is a random variable whose values can be represented as $T(x_1, x_2, \dots, x_n)$.

Definition 6 (Statistic). *Let X_1, X_2, \dots, X_n be a random sample of size n from the population whose distribution is $f(x|\theta)$ (the distribution f with unknown parameter θ). Let $T(x_1, x_2, \dots, x_n)$ be a real-valued or vector-valued function whose domain includes the range of (X_1, X_2, \dots, X_n) . Then the random variable or random vector $Y = T(X_1, \dots, X_n)$ is called a **statistic** provided that T is not a function of any unknown parameter θ .*

For example consider $X \approx N(\mu, \sigma^2)$ where μ is known but σ is unknown. Then $\frac{\sum_{i=1}^n X_i}{\sigma^2}$ is not a statistic but $\frac{\sum_{i=1}^n X_i}{\mu^2}$ is a statistic.

Two common statistics are the sample mean and sample variance.

Definition 7 (Sample Mean, Sample Variance, Sample Standard Deviation). *The **sample mean** is the arithmetic average of the values in the random sample. It is denoted by $\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$.*

*The **sample variance** is the statistic defined by $S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n - 1}$*

*The **sample standard deviation** is the statistic defined by $S = \sqrt{S^2}$.*

Definition 8 (Unbiased Estimator). *Let X_1, X_2, \dots, X_n be a random sample from a population $f(x|\theta)$. We say that a statistic $T(X_1, X_2, \dots, X_n)$ is an **unbiased estimator** of the parameter θ if $E_T = \theta$ for all possible values of θ .*

Theorem 1.0.4. *Let X_1, X_2, \dots, X_n be a random sample from a population with mean μ and variance $\sigma^2 < \infty$. Then:*

1. $E(\bar{X}) = \mu$
2. $Var(\bar{X}) = \frac{\sigma^2}{n}$
3. $E(S^2) = \sigma^2$

From the above theorem we see that the sample mean \bar{X} is an unbiased estimator of the population mean μ and the sample variance S^2 is an unbiased estimator of the population variance σ^2 . (The reason we included $1/n - 1$ in the definition of the sample variance was to make it an unbiased estimator)

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Exercises

1.

- (i) Let X be a standard normal variable $N(0, 1)$. Find the moment generating function of X .
- (ii) Let Y be a standard normal variable $N(\mu, \sigma^2)$. Find the moment generating function of Y .

ans.

$$M_X(t) = e^{\frac{1}{2}t^2}$$

$$M_Y(t) = e^{\mu t} e^{\frac{\sigma^2 t^2}{2}}$$

Source. Class, Lec 04

2. Let X have the PMF $P(X = k) = \frac{6}{\pi^2 k^2}$ where $k = 1, 2, \dots$. Show that the MGF of X does not exist.

Source. Class, Lec 04

3. Let $X \approx \exp(\lambda)$ with pdf $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$. Then show that $E[e^{tX}]$ exists for all $t < \lambda$.

Source. Class, Lec 04

4. Let X_1, X_2, \dots, X_n be a random sample of n identical circuit boards whose times until failure are thought to follow an exponential(β) population. Find the joint distribution of the sample. What is the probability that all the boards last more than 2 years?

ans.

$$f(x_1, x_2, \dots, x_n) = f(x_1)f(x_2)\dots f(x_n) = \prod_{i=1}^n \frac{1}{\beta} \exp\left(\frac{-x_i}{\beta}\right)$$

$$P(X_1 > 2, X_2 > 2, \dots, X_n > 2) = \exp\left(\frac{-2n}{\beta}\right)$$

Source. Class, Lec 02

5. If sample X_1, X_2, \dots, X_n are drawn from a finite population without replacement, then show that the random variables X_1, X_2, \dots, X_n are not mutually independent but that they are identically distributed.

Source. Class, Lec 02

6. Let $X \approx \text{Bernoulli}(p)$ where p is possibly unknown. Suppose that five independent observations on X are 0, 1, 1, 1, 0. Then find the sample mean, sample variance and sample standard deviation.

ans.

Mean $\bar{x} = 0.6$

Variance $s^2 = 0.3$

Standard Deviation $s = 0.55$

Source. Class, Lec 03

7. Let X_1, X_2, \dots, X_n be a random sample from a population with mean μ and variance $\sigma^2 < \infty$. Then show that:

1. $E(\bar{X}) = \mu$

2. $\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$

3. $E(S^2) = \sigma^2$

Source. Class, Lec 03

8. Color blindness appears in 1% of the people in a certain population. How large must a sample be if the probability of its containing a color-blind person is to be 0.95 or more? (Assume that the population is large enough to be considered infinite, so that sampling can be considered to be with replacement.)

ans.

Sample size greater than 299

Source. Class, Lec 03

9. Suppose that five observations on normal population are $-0.864, 0.561, 2.355, 0.582, -0.774$. Compute sample variance.

ans.

1.648

Source. Class, Lec 03

10. For a sample of size 5 that results in 7, 9, 1, 6, 2, find the sample mean and variance.

ans.

Sample Mean $\bar{x} = 6$ Sample Variance $S^2 = 2.5$

Source. Schaum's Example 5.5

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