# Topology

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**Definition**
A **topology** on a set $X$ is a collection $\tau$ of subsets of $X$ satisfying:

1. $\varnothing, X \in \tau$

2. An intersection of finite subcollections of $\tau$ is in $\tau$

3. A union of any subcollection of $\tau$ is in $\tau$

The ordered pair $(X, \tau)$ is called a **topological space**.

**Definition**
Let $(X, \tau)$ be a topological space. An **open subset** of $X$ is a member of $\tau$.

**Definition**
Let $\tau$ and $\sigma$ be two topologies on a set $X$. We say that $\tau$ is **weaker** (or smaller, coarser) than $\sigma$ if $T \subseteq \sigma$. In this case, $\sigma$ is then said to be **stronger** (or larger, finer) than $\tau$.

**Definition**
Let $X$ be any set. The collection $\tau = P(X)$ is a topology on $X$ and is called the **discrete topology** on $X$. Here $(X, \tau)$ is called the **discrete topological space**.

**Definition**
Let $X$ be any set. The collection $\tau = \{\varnothing, X\}$ is called the **indiscrete topology** on $X$. Here $(X, \tau)$ is called the **indiscrete topology**.

**Definition**
Let $X$ be any set. The collection $\tau = \{A \subseteq X : X \setminus A \text{ is finite }\} \cup \{\varnothing\}$ is called the **co-finite topology**.

**Definition**
Let $X$ be any set. The collection $\tau = \{A \subseteq X : X \setminus A \text{ is countable }\} \cup \{\varnothing\}$ is called the **co-countable topology**.

**Definition**
A topology $\tau$ on a set $X$ is said to be **metrizable** if there exists a metric $d$ on $X$ such that the topology $\tau_d$ generated by the metric $d$ coincides with $\tau$.

**Definition**
Two metrics defined on a set $X$ are said to be **equivalent** if they generate the same topology. In other words, $d_1$ and $d_2$ are equivalent if the collection of open sets in $(X, d_1)$ and $(X, d_2)$ are the same.

**Definition**
The topology generated by the Euclidean metric on $\mathbb{R}^n$ is called the **usual topology** on $\mathbb{R}^n$. For $Y \subseteq \mathbb{R}^n$, the topology generated by the Euclidean metric is called the usual topology on $Y$.

**Definition**
By a **neighbourhood** of a point $x$ in a topological space $(X, \tau)$, we mean an open set containing $x$.

**Definition**
A subset $A$ of a topological space $(X, \tau)$ is said to be **closed** if $X \setminus A$ is open in $X$, that is $X \setminus A \in \tau$

**Theorem**

- $\varnothing$ and $X$ are closed in $X$
- An intersection of any collection of closed sets is closed in $X$
- A union of a *finite* collection of closed sets in $X$ is closed in $X$

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**Definition**
Let $X$ be a topological space. A collection $\beta$ of open subsets of $X$ is said to be a **basis** for the topology on $X$ if for every open set $U$ in $X$ and $x \in U$, there exists a $B \in \beta$ such that $x \in B \subseteq U$.
Members of $\beta$ are called **basis open sets** corresponding to basis $\beta$.

**Note**
For any topological space $(X, \tau)$, $\tau$ is a basis for $\tau$.

**Note**
In $\mathbb{R}$ the set $\{(x - \varepsilon, x + \varepsilon) : x \in \mathbb{R} \text{ and } \varepsilon > 0\}$ is a basis for the usual topology.

**Note**
For $(\mathbb{R}, \tau)$, where $\tau$ is the discrete topology, $\beta = \{\{x\} : x \in \mathbb{R}\}$ is a basis for $\tau$ on $\mathbb{R}$.

**Note**
In $\mathbb{R}$ the set $\left\{\left(x - \frac{1}{n}, x + \frac{1}{n}\right) : x \in \mathbb{R} \text{ and } n \in \mathbb{N}\right\}$ is a basis for the usual topology.

**Note**
In $\mathbb{R}$ the set $\{(a, b) : a, b \in \mathbb{Q} \text{ and } a < b\}$ is a basis for the usual topology. This is a countable basis for $\mathbb{R}$ with the usual topology.

**Note**
Let $(X, \tau)$ be a metrizable space. Then $\beta = \{B(x, \varepsilon) : x \in X, \varepsilon > 0\}$ is a basis for $\tau$.

**Definition**
Let $(X, \tau)$ be a topological space. A collection $S \subseteq \tau$ is said to be a **subbasis** for the topology $\tau$ if for every open set $U$ in $\tau$ and $x \in U$, there exists a finite subcollection $\{S_1, S_2, ..., S_n\}$ in $S$ such that $x \in \bigcap_{i=1}^{n} S_i \subseteq U$
Members of $S$ are called **subbasis open sets** corresponding to $S$.

**Note**
For a topological space $(X, \tau)$ a collection $S$ is a subbasis for the topology $\tau$ if and only if the collection of all finite intersections of members in $S$ forms a basis for the topology $\tau$.

**Theorem**
Let $(X, \tau)$ be a topological space and $\beta$ be a collection of open sets in $(X, \tau)$. Then $\beta$ is a basis for the topology $\tau$ on $X$ if and only if every open set $U$ in $(X, \tau)$ can be written as a union of members in $\beta$.

**Theorem**
If $X$ is a set, a basis for a topology on $X$ is a collection $\beta$ of subsets of $X$ such that

- for every $x \in X$ there exists $B \subset \beta$ containing $x$.
- if $x \in B_1 \cap B_2$, for some $B_1, B_2 \in \beta$, then there exists $B_3 \in \beta$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

**Definition**
The topology generated by the basis $\beta = \{[a, b) : a, b \in \mathbb{R}, a < b\}$ is known as the **lower limit topology** on $\mathbb{R}$.
The space $\mathbb{R}$ equipped with the lower limit topology is known as the **Sorgenfrey line** $\mathbb{R}_l$.

**Proposition**
The lower limit topology on $\mathbb{R}$ is strictly finer than the usual topology on $\mathbb{R}$

**Definition**
Let $K = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Consider the basis
$\beta = \{(a, b) \setminus K : a, b \in \mathbb{Q}, a < b\} \cup \{(a, b) : a, b \in \mathbb{R}, a < b\}$. The topology generated by the basis $\beta$ is known as the **$k$-topology** on $\mathbb{R}$.

**Proposition**
The $k$-topology on $\mathbb{R}$ is strictly finer than the usual topology on $\mathbb{R}$.

**Proposition**
The lower limit topology and the $k$-topology on $\mathbb{R}$ are incomparable.

**Definition**
Let $X$ be a set. Let $S$ be a collection of subsets of $X$ whose union equals $X$. Any such collection is called a **subbasis** for a topology $X$.

**Theorem**
Let $S$ be a subbasis for a topology on $X$. Then
$\tau = \{U \subseteq X :$ for every $x \in U$ there exists a finite number of members $S_1, S_2, ..., S_n \in S$ such that $x \in S_1 \cap S_2 \cap ... \cap S_n \subseteq U\}$ is the topology generated by the subbasis $S$.

**Theorem**
Let $\beta$ be a basis for a topology on $X$. Then the topology generated by $\beta$ is equal to the intersection of all topologies on $X$ that contains $\beta$.

**Theorem**
If $\beta$ is a basis for a topology on a set $X$, then the topology generated by $\beta$ on $\tau$ is the smallest topology on $X$ that contains $\beta$.

**Definition**
Let $X$ be a set. A function $f\colon \mathbb{N} \to X$ is called a **sequence** in $X$.

**Definition**
Let $(X, Y)$ be a topological space. A sequence $(X_n)$ is said to **converge** to $x \in X$ if for every neighbourhood $U$ of $x$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ $\forall n \geq n_0$

**Definition**
A sequence $(x_n)$ is said to be **eventually constant** if there exists $n_0 \in \mathbb{N}$ such that $x_n = x_{n_0}$ $\forall n \geq n_0$.

**Theorem**
In any topological space, every eventually constant sequence is convergent.

**Definition**
Let $f\colon \mathbb{N} \to X$ be a sequence in $X$. Then for any strictly increasing function $g\colon \mathbb{N} \to \mathbb{N}$, the composition $f \circ g\colon \mathbb{N} \to X$ is called a **subsequence** of $f$.

**Theorem**
In a topological space $(X, \tau)$, every subsequence of a convergent sequence is convergent.

**Definition**
Let $\mathcal{A}$ be a non-empty set. A relation $\leq$ on a set $A$ is called a **partial order relation** if the following condition holds for all $\alpha, \beta, \gamma$ in $A$:

1. reflexive: $\alpha \leq \alpha$
2. anti-symmetric: $\alpha \leq \beta$ and $\beta \leq \alpha \implies \alpha = \beta$
3. transitive: $\alpha \leq \beta$ and $\beta \leq \gamma \implies \alpha \leq \gamma$

**Definition**
A **directed set** $J$ is a set with a partial order relation $\leq$ such that for each pair $\alpha$ and $\beta$ of $J$, there exists a $Y \in J$ such that $\alpha \leq Y$ and $\beta \leq Y$.

**Definition**
A **net** in $X$ is a function $f$ from a directed set $J$ to $X$.

**Note**
Every sequence is a net.

**Definition**
Let $J$ be a directed set with a partial order relation '$\leq$'. A subset $K$ of $J$ is said to be **cofinal** in $J$ if for each $\alpha \in J$, there exists $\beta \in K$ such that $\alpha \leq \beta$.

**Proposition**
If $g\colon \mathbb{N} \to \mathbb{N}$ is a strictly increasing function then $g(\mathbb{N})$ is cofinal in $\mathbb{N}$.

**Theorem**
If $J$ is a directed set and $K$ is cofinal in $J$, then $K$ is a directed set.

**Definition**
Let $f\colon J \to X$ be a net in $X$. If $I$ is a directed set and $g\colon I \to J$ such that

- $i < j \implies g(i) < g(j)\ \forall i, j \in I$
- $g(I)$ is cofinal in $J$

Then $f \circ g\colon I \to X$ is called a **subnet** of $f$.

**Definition**
The net $(X_\alpha)_{\alpha \in J}$ is said to **converge** to a point $x \in X$ if for every neighbourhood $U$ of $x$, there exists $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$.

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**Definition**
Let $(X, \tau)$ be a topological space and $S \subseteq X$. A point $x \in X$ is called a **closure point** of $S$ if for every neighbourhood $U$ of $x$, we have $U \cap S \neq \varnothing$.
The set of all closure points of $S$ is called the **closure** of $A$ and is denoted $\mathrm{cl}(A)$ or $\bar{A}$.

**Definition**
A point $x \in X$ is said to be a **limit point** of $S$ if for every neighbourhood of $x$, we have $(U \cap S) \setminus \{x\} \neq \varnothing$.
The set of all limit points is called the **derived set** and is denoted $S'$

**Theorem**
Let $X$ be a topological space and $S \subseteq X$. Then $\bar{S}$ is the smallest closed set in $X$ that contains $S$.

**Proposition**
A subset of a topological space $X$ is closed if and only if $S = \bar{S}$

**Theorem**
Let $(X, \tau)$ be a metrizable space and $S \subseteq X$. Then $x \in \bar{S}$ if and only if there exists a sequence $\langle x_n \rangle$ in $S$ such that $x_n \to x$.

**Theorem**
Let $X$ be a topological space and $S \subseteq X$. Then show that $x \in \bar{S}$ if and only if there exists a net $\langle x_\lambda \rangle$ in $S$ such that $x_\lambda \to x$

**Definition**
Let $X$ be a topological space and $A \subseteq X$. A point $x \in X$ is said to be an **interior point** of $A$ if there exists a neighbourhood $U$ of $X$ such that $U \subseteq A$ (or) if there exists an open set $U$ in $X$ such that $x \in U \subseteq A$.
The set of all interior points of $A$ is called the **interior** of $A$ and is denoted $A^{\text{o}}$.

**Theorem**
Let $X$ be a topological space and $S \subseteq X$. Then $S^{\text{o}}$ is the largest open set contained in $S$.

**Theorem**
Let $X$ be a topological space and $S \subseteq X$. Then $S$ is open if and only if $S = S^{\text{o}}$.

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**Definition**
Let $(X, \tau_1)$ and $(Y, \tau_2)$ be two topological spaces. The collection $\mathcal{B} = \tau_1 \times \tau_2 = \{u \times v : u \in \tau_1, v \in \tau_2\}$ is a basis for a topology on $X \times Y$. The topology generated by $\mathcal{B}$ is called the **product topology** on $X \times Y$.

**Theorem**
Let $(X, \tau)$ and $(Y, \sigma)$ be two topological spaces. Let $\mathcal{B}$ be a basis for $\tau$ and $\mathcal{C}$ be a basis for $\sigma$. Then the collection $\mathcal{D} = \{U \times V : U \in \mathcal{B}, V \in \mathcal{C}\}$ is a basis for the product topology on $X \times Y$.

**Result**
The product topology on $\mathbb{R} \times \mathbb{R}$ coincides with the usual topology on $\mathbb{R}^2$

**Note**
Missed quotient topology, check.

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**Definition**
Let $(X, \tau)$ be a topological space and $Y \subseteq X$.
The collection $\tau_Y$ is a topology on $Y$ called the **subspace topology** and with this topology, $Y$ is called a **subspace** of $X$.

**Theorem**
Let $Y$ be a subspace of $X$. Then a subset $U$ of $Y$ is open in $Y$ iff $U = V \cap Y$ for some open set $V$ in $X$.

**Theorem**
If $\mathscr{B}$ is a basis for the topology on $X$, then the collection $\mathscr{B}_Y = \{B \cap Y | B \in \mathscr{B}\}$ is a basis for the subspace topology on $Y$.

**Theorem**
Let $Y$ be an open subspace of $X$. Then a subset $U$ of $Y$ is open in $Y$ iff $U$ is open in $X$.

**Theorem**
Let $Y$ be a closed subspace of a topological space $X$. Then a subset $C$ of $Y$ is closed in $Y$ iff it is closed in $X$.

**Theorem**
Let $X$ be a topological space and $Y$ be a subspace of $X$.
Let $C \subseteq Y$. Then $C$ is closed in $Y$ iff there is a closed set $D$ in $X$ such that $Y \cap D = C$

**Theorem**
Let $X$ be a topological space and $Y$ be a closed subspace of $X$. Let $C \subseteq Y$. Then $C$ is closed in $Y$ iff $C$ is closed in $X$.

**Theorem**
If $A$ is a subspace of $X$ and $B$ is a subspace of $Y$, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

**Theorem**
Let $X$ be a topological space and $Y$ be a subspace of $X$. Let $A \subseteq Y$. Then $\mathrm{cl}_X(A) \cap Y = \mathrm{cl}_Y(A)$

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**Definition**
A function $f : (X, \tau) \to (Y, \sigma)$ is said to be **continuous** at $x_0 \in X$ if for every neighbourhood $U$ of $f(x_0)$, there exists a neighbourhood $V$ of $x_0$ such that $f(V) \subseteq U$.
The function $f$ is said to be **continuous** if it is continuous at each point of $X$.

**Theorem**
Let $f : (X, \tau) \to (Y, \sigma)$ be a function from a topological space $(X, \tau)$ to a topological space $(Y, \sigma)$. Then the following are equivalent:

- $f$ is continuous on $X$
- $f^{-1}(U)$ is an open subset of $X$ whenever $U$ is open in $X$
- $f(\overline{A}) = \overline{f(A)}$ for every $A \subseteq X$
- $f^{-1}(C)$ is a closed subset of $X$ whenever $C$ is closed in $Y$
- The net $x_\lambda \to x$ in $X \implies f(x_\lambda) \to f(x)$ in $Y$

**Theorem**
The composition of two continuous functions is continuous.

**Theorem**
The restriction of a continuous function on a topological space to a subspace is continuous.

**Theorem**
Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces and $Z = X \times Y$. For a net $(z_\lambda) = (x_\lambda, y_\lambda)$ in $Z$, $z_\lambda \to z \in Z \iff x_\lambda \to x \in X$ and $y_\lambda \to y \in Y$.

**Definition**
Let $X$ and $Y$ be topological spaces and let $f : X \to Y$ be a bijection. If both the function $f$ and the inverse function $f^{-1} : Y \to X$ are continuous, then $f$ is called a **homeomorphism**.

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**Definition**
Let $\{X_\alpha\}_{\alpha \in I}$ be an indexed family of topological spaces. For the product space $X = \prod\limits_{\alpha \in I} X_\alpha$, we take the basis $\mathscr{B} = \{\prod\limits_{\alpha \in I} U_\alpha | U_\alpha \text{ is open in } X_\alpha\}$. The topology generated by this basis is called the **box topology**.

**Note**
The above is a direct generalization of the earlier product topology defined on product of two spaces.

**Theorem**
A subset $W$ of $X$ is open in $X$ with the box topology iff for every $x = (x_\alpha)_{\alpha\in I}$, there exists an open set $U_\alpha$ in $X_\alpha$ such that $(x_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} U_\alpha \subseteq W$

**Definition**
Let $\{X_\alpha\}_{\alpha\in I}$ be an indexed family of topological spaces. For the product space $X = \prod_{\alpha\in I} X_\alpha$, we take the basis
$\mathscr{C} = \{\prod_{\alpha\in I} U_\alpha | U_\alpha$ is open in $X, U_\alpha = X_\alpha$ for all but finitely many $\alpha \in I\}$.
The topology generated by this basis is called the **product topology** on $X$.

**Note**
Unless mentioned otherwise, we usually consider the product topology on the product space $X$.

**Theorem**
The product topology on $X$ is weaker than the box topology on $X$.

**Note**
When the index $I$ is finite, the product topology and the box topology coincide.

**Definition**
Let $\{X_\alpha\}_{\alpha\in I}$ be an indexed family of topological spaces and $X = \prod_{\alpha\in I} X_\alpha$ be the product space.
For every $\beta \in I$, the $\beta$**-th projection map** $\Pi_\beta : X \to X_\beta$ is defined as
$\Pi_\beta((x_\alpha)_{\alpha\in I}) = x_\beta$

**Theorem**
Let $\{X_\alpha\}_{\alpha\in I}$ be an indexed family of topological spaces and $X = \prod_{\alpha\in I} X_\alpha$ be the product space.
The collection $\mathscr{S} = \{\Pi_\beta^{-1}(U_\beta) | U_\beta$ is open in $X_\beta, \beta \in I\}$is a subbasis for the product topology on $X$.

**Theorem**
Let $\{X_\alpha\}_{\alpha\in I}$ be an indexed family of topological spaces and $X = \prod_{\alpha\in I} X_\alpha$ be the product space equipped with the product topology.
Then a function $f : A \to X$ is continuous iff $f_\alpha = \Pi_\alpha \circ f : A \to X_\alpha$ is continuous for each $\alpha \in I$.

**Theorem**
The topologies on $\mathbb{R}^n$ induced by the euclidean metric and the square norm are the same as the product topology on $\mathbb{R}^n$.

**Definition**
Consider the standard bounded metric on $\mathbb{R}$ defined as $\overline{d} = \min\{|a-b|, 1\}$. Given an index set $I$ and points $x = (x_\alpha)_{\alpha \in I}$ and $y = (y_\alpha)_{\alpha \in I}$, the metric $\rho$ on $\mathbb{R}^I$ defined as $\rho(x, y) = \sup\{\overline{d}(x_\alpha, y_\alpha) | \alpha \in I\}$ is known as the **uniform metric** on $\mathbb{R}^I$.
The topology generated by the uniform metric is called the **uniform topology** on $\mathbb{R}^I$.

**Theorem**
The uniform topology on $\mathbb{R}^I$ is weaker than the box topology and stronger than the product topology. All these topologies are different if $I$ is infinite.

**Definition**
Let $X$ and $Y$ be topological spaces and let $p : X \to Y$ be a surjective map.
The map $p$ is said to be a **quotient map** provided a subset $U$ of $Y$ is open in $Y$ iff $p^{-1}(U)$ is open in $X$.

**Note**
We can replace 'open' in the above definition with 'closed'.

**Note**
Every quotient map is a continuous map.

**Definition**
A map $f : X \to Y$ is said to be **open** if for each open set $U$ in $X$, $f(U)$ is open in $Y$. A map $f : X \to Y$ is said to be **closed** if for each closed set $C$ in $X$, $f(C)$ is closed in $Y$.

**Note**
A surjective continuous map which is either open or closed is a quotient map.
Note, there are quotient maps which are neither open nor closed.

**Definition**
If $X$ is a topological space and $A$ is any set, and if $p : X \to A$ is a surjective map, then there exists exactly one topology $\tau$ on $A$ relative to which $p$ is a quotient map. This topology is known as the **quotient topology** induced by $p$.

**Definition**
Let $X$ be a topological space and $\sim$ be an equivalence relation on $X$.
Define a map $p : X \to \dfrac{X}{\sim}$ as $p(x) = [x]$ where $[x]$ is the equivalence class of $x$ under $\sim$.

Let $\tau$ be the quotient topology induced by $p$ on $\dfrac{X}{\sim}$. Then the space $(\dfrac{X}{\sim}, \tau)$ is called the **quotient space** of $X$.