Topology

Definition. *A **topology** on a set X is a collection τ of subsets of X satisfying:*

1. $\varnothing, X \in \tau$
2. *An intersection of finite subcollections of τ is in τ*
3. *A union of any subcollection of τ is in τ*

*The ordered pair (X, τ) is called a **topological space**.*

Definition. *Let (X, τ) be a topological space. An **open subset** of X is a member of τ.*

Definition. *Let τ and σ be two topologies on a set X. We say that τ is **weaker** (or smaller, coarser) than σ if $T \subseteq \sigma$. In this case, σ is then said to be **stronger** (or larger, finer) than τ.*

Definition. *Let X be any set. The collection $\tau = P(X)$ is a topology on X and is called the **discrete topology** on X. Here (X, τ) is called the **discrete topological space**.*

Definition. *Let X be any set. The collection $\tau = \{\varnothing, X\}$ is called the **indiscrete topology** on X. Here (X, τ) is called the **indiscrete topology**.*

Definition. *Let X be any set. The collection $tau = \{A \subseteq X : X \setminus A \text{ is finite }\} \cup \{\varnothing\}$ is called the **co-finite topology**.*

Definition. *Let X be any set. The collection $tau = \{A \subseteq X : X \setminus A \text{ is countable }\} \cup \{\varnothing\}$ is called the **co-countable topology**.*

Definition. *A topology τ on a set X is said to be **metrizable** if there exists a metric d on X such that the topology τ_d generated by the metric d coincides with τ.*

Definition. *Two metrics defined on a set X are said to be **equivalent** if they generate the same topology. In other words, d_1 and d_2 are equivalent if the collection of open sets in (X, d_1) and (X, d_2) are the same.*

Definition. *The topology generated by the Euclidean metric on $\mathbb{R}^n$ is called the **usual topology** on $\mathbb{R}^n$. For $Y \subseteq \mathbb{R}^n$, the topology generated by the Euclidean metric is called the usual topology on Y.*

Definition. *By a **neighbourhood** of a point x in a topological space (X, τ), we mean an open set containing x.*

Definition. *A subset A of a topological space (X, τ) is said to be **closed** if $X \setminus A$ is open in X, that is $X \setminus A \in \tau$*

Theorem.

- *$\varnothing$ and X are closed in X*
- *An intersection of any collection of closed sets is closed in X*
- *A union of a finite collection of closed sets in X is closed in X*

Definition. *Let X be a topological space. A collection β of open subsets of X is said to be a **basis** for the topology on X if for every open set U in X and $x \in U$, there exists a $B \in \beta$ such that $x \in B \subseteq U$.*
*Members of β are called **basis open sets** corresponding to basis β.*

Note. *For any topological space (X, τ), τ is a basis for τ.*

Note. *In $\mathbb{R}$ the set $\{(x - \varepsilon, x + \varepsilon) : x \in \mathbb{R} \text{ and } \varepsilon > 0\}$ is a basis for the usual topology.*

Note. *For $(\mathbb{R}, \tau)$, where τ is the discrete topology, $\beta = \{\{x\} : x \in \mathbb{R}\}$ is a basis for τ on $\mathbb{R}$.*

Note. *In $\mathbb{R}$ the set $\left\{\left(x - \frac{1}{n}, x + \frac{1}{n}\right) : x \in \mathbb{R} \text{ and } n \in \mathbb{N}\right\}$ is a basis for the usual topology.*

Note. *In $\mathbb{R}$ the set $\{(a, b) : a, b \in \mathbb{Q} \text{ and } a < b\}$ is a basis for the usual topology. This is a countable basis for $\mathbb{R}$ with the usual topology.*

Note. *Let* (X, τ) *be a metrizable space. Then* $\beta = \{B(x, \varepsilon) : x \in X, \varepsilon > 0\}$ *is a basis for* τ.

Definition. *Let* (X, τ) *be a topological space. A collection* $S \subseteq \tau$ *is said to be a* ***subbasis*** *for the topology* τ *if for every open set* U *in* τ *and* $x \in U$, *there exists a finite subcollection* $\{S_1, S_2, ..., S_n\}$ *in* S *such that* $x \in \bigcap_{i=1}^{n} S_i \subseteq U$
Members of S *are called* ***subbasis open sets*** *corresponding to* S.

Note. *For a topological space* (X, τ) *a collection* S *is a subbasis for the topology* τ *if and only if the collection of all finite intersections of members in* S *forms a basis for the topology* τ.

Theorem. *Let* (X, τ) *be a topological space and* β *be a collection of open sets in* (X, τ).
Then β *is a basis for the topology* τ *on* X *if and only if every open set* U *in* (X, τ) *can be written as a union of members in* β.

Theorem. *If* X *is a set, a basis for a topology on* X *is a collection* β *of subsets of* X *such that*

- *for every* $x \in X$ *there exists* $B \subset \beta$ *containing* x.
- *if* $x \in B_1 \cap B_2$, *for some* $B_1, B_2 \in \beta$, *then there exists* $B_3 \in \beta$ *such that* $x \in B_3 \subseteq B_1 \cap B_2$.

Definition. *The topology generated by the basis* $\beta = \{[a, b) : a, b \in \mathbb{R}, a < b\}$ *is known as the* ***lower limit topology*** *on* $\mathbb{R}$.
The space $\mathbb{R}$ *equipped with the lower limit topology is known as the* ***Sorgenfrey line*** $\mathbb{R}_l$.

Proposition. *The lower limit topology on* $\mathbb{R}$ *is strictly finer than the usual topology on* $\mathbb{R}$

Definition. *Let* $K = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. *Consider the basis*
$\beta = \{(a, b) \setminus K : a, b \in \mathbb{Q}, a < b\} \cup \{(a, b) : a, b \in \mathbb{R}, a < b\}$. *The topology generated by the basis* β *is known as the* k***-topology*** *on* $\mathbb{R}$.

Proposition. *The* k*-topology on* $\mathbb{R}$ *is strictly finer than the usual topology on* $\mathbb{R}$.

Proposition. *The lower limit topology and the* k*-topology on* $\mathbb{R}$ *are incomparable.*

Definition. *Let* X *be a set. Let* S *be a collection of subsets of* X *whose union equals* X. *Any such collection is called a* ***subbasis*** *for a topology* X.

Theorem. *Let S be a subbasis for a topology on X. Then* $\tau = \{U \subseteq X$: *for every* $x \in U$ *there exists a finite number of members* $S_1, S_2, ..., S_n \in S$ *such that* $x \in S_1 \cap S_2 \cap ... \cap S_n \subseteq U\}$ *is the topology generated by the subbasis* S.

Theorem. *Let β be a basis for a topology on X. Then the topology generated by β is equal to the intersection of all topologies on X that contains β.*

Theorem. *If β is a basis for a topology on a set X, then the topology generated by β on τ is the smallest topology on X that contains β.*

Definition. *Let X be a set. A function $f: \mathbb{N} \to X$ is called a* ***sequence*** *in X.*

Definition. *Let (X, Y) be a topological space. A sequence (X_n) is said to* ***converge*** *to $x \in X$ if for every neighbourhood U of x, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U \; \forall n \geq n_0$*

Definition. *A sequence (x_n) is said to be* ***eventually constant*** *if there exists $n_0 \in \mathbb{N}$ such that $x_n = x_{n_0} \; \forall n \geq n_0$.*

Theorem. *In any topological space, every eventually constant sequence is convergent.*

Definition. *Let $f: \mathbb{N} \to X$ be a sequence in X. Then for any strictly increasing function $g: \mathbb{N} \to \mathbb{N}$, the composition $f \circ g: \mathbb{N} \to X$ is called a* ***subsequence*** *of f.*

Theorem. *In a topological space (X, τ), every subsequence of a convergent sequence is convergent.*

Definition. *Let $\mathcal{A}$ be a non-empty set. A relation $\leq$ on a set A is called a* ***partial order relation*** *if the following condition holds for all α, β, γ in A:*

1. *reflexive:* $\alpha \leq \alpha$
2. *anti-symmetric:* $\alpha \leq \beta$ *and* $\beta \leq \alpha \implies \alpha = \beta$
3. *transitive:* $\alpha \leq \beta$ *and* $\beta \leq \gamma \implies \alpha \leq \gamma$

Definition. *A* ***directed set*** *J is a set with a partial order relation $\leq$ such that for each pair α and β of J, there exists a $Y \in J$ such that $\alpha \leq Y$ and $\beta \leq Y$.*

Definition. *A **net** in X is a function f from a directed set J to X.*

Note. *Every sequence is a net.*

Definition. *Let J be a directed set with a partial order relation '$\leq$'. A subset K of J is said to be **cofinal** in J if for each $\alpha \in J$, there exists $\beta \in K$ such that $\alpha \leq \beta$.*

Proposition. *If $g \colon \mathbb{N} \to \mathbb{N}$ is a strictly increasing function then $g(\mathbb{N})$ is cofinal in $\mathbb{N}$.*

Theorem. *If J is a directed set and K is cofinal in J, then K is a directed set.*

Definition. *Let $f \colon J \to X$ be a net in X. If I is a directed set and $g \colon I \to J$ such that*

- $i < j \implies g(i) < g(j) \ \forall i, j \in I$
- *$g(I)$ is cofinal in J*

*Then $f \circ g \colon I \to X$ is called a **subnet** of f.*

Definition. *The net $(X_\alpha)_{\alpha \in J}$ is said to **converge** to a point $x \in X$ if for every neighbourhood U of x, there exists $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$.*

Definition. *Let (X, τ) be a topological space and $S \subseteq X$. A point $x \in X$ is called a **closure point** of S if for every neighbourhood U of x, we have $U \cap S \neq \varnothing$.
The set of all closure points of S is called the **closure** of A and is denoted $cl(A)$ or $\bar{A}$.*

Definition. *A point $x \in X$ is said to be a **limit point** of S if for every neighbourhood of x, we have $(U \cap S) \setminus \{x\} \neq \varnothing$.
The set of all limit points is called the **derived set** and is denoted S'*

Theorem. *Let X be a topological space and $S \subseteq X$. Then $\bar{S}$ is the smallest closed set in X that contains S.*

Proposition. *A subset of a topological space X is closed if and only if $S = \bar{S}$*

Theorem. *Let (X, τ) be a metrizable space and $S \subseteq X$. Then $x \in \bar{S}$ if and only if there exists a sequence $\langle x_n \rangle$ in S such that $x_n \to x$.*

Theorem. *Let X be a topological space and $S \subseteq X$. Then show that $x \in \bar{S}$ if and only if there exists a net $\langle x_\lambda \rangle$ in S such that $x_\lambda \to x$*

Definition. *Let X be a topological space and $A \subseteq X$. A point $x \in X$ is said to be an* ***interior point*** *of A if there exists a neighbourhood U of X such that $U \subseteq A$ (or) if there exists an open set U in X such that $x \in U \subseteq A$.*
The set of all interior points of A is called the ***interior*** *of A and is denoted A^{o}.*

Theorem. *Let X be a topological space and $S \subseteq X$. Then S^{o} is the largest open set contained in S.*

Theorem. *Let X be a topological space and $S \subseteq X$. Then S is open if and only if $S = S^{\mathrm{o}}$.*

Definition. *Let (X, τ_1) and (Y, τ_2) be two topological spaces. The collection $\mathcal{B} = \tau_1 \times \tau_2 = \{u \times v : u \in \tau_1, v \in \tau_2\}$ is a basis for a topology on $X \times Y$. The topology generated by $\mathcal{B}$ is called the* ***product topology*** *on $X \times Y$.*

Theorem. *Let (X, τ) and (Y, σ) be two topological spaces. Let $\mathcal{B}$ be a basis for τ and $\mathcal{C}$ be a basis for σ. Then the collection $\mathcal{D} = \{U \times V : U \in \mathcal{B}, V \in \mathcal{C}\}$ is a basis for the product topology on $X \times Y$.*

Result. *The product topology on $\mathbb{R} \times \mathbb{R}$ coincides with the usual topology on $\mathbb{R}^2$*
