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1

Definitions and Theory

1.1 Topological Spaces

Definition 1. A **topology** on a set X is a collection τ of subsets of X satisfying:

1. $\emptyset, X \in \tau$
2. An intersection of finite subcollections of τ is in τ
3. A union of any subcollection of τ is in τ

The ordered pair (X, τ) is called a **topological space**.

Definition 2. Let (X, τ) be a topological space. An **open subset** of X is a member of τ .

Definition 3. Let τ and σ be two topologies on a set X . We say that τ is **weaker** (or smaller, coarser) than σ if $\tau \subseteq \sigma$. In this case, σ is then said to be **stronger** (or larger, finer) than τ .

Definition 4. Let X be any set. The collection $\tau = P(X)$ is a topology on X and is called the **discrete topology** on X . Here (X, τ) is called the **discrete topological space**.

Definition 5. Let X be any set. The collection $\tau = \{\emptyset, X\}$ is called the **indiscrete topology** on X . Here (X, τ) is called the **indiscrete topology**.

Definition 6. Let X be any set. The collection $\tau = \{A \subseteq X : X \setminus A \text{ is finite}\} \cup \{\emptyset\}$ is called the **co-finite topology**.

Definition 7. Let X be any set. The collection $\tau = \{A \subseteq X : X \setminus A \text{ is countable}\} \cup \{\emptyset\}$ is called the **co-finite topology**.

Definition 8. A topology τ on a set X is said to be **metrizable** if there exists a metric d on X such that the topology τ_d generated by the metric d coincides with τ .

Definition 9. Two metrics defined on a set X are said to be **equivalent** if they generate the same topology. In other words, d_1 and d_2 are equivalent if the collection of open sets in (X, d_1) and (X, d_2) are the same.

Definition 10. The topology generated by the Euclidean metric on \mathbb{R}^n is called the **usual topology** on \mathbb{R}^n . For $Y \subseteq \mathbb{R}^n$, the topology generated by the Euclidean metric is called the usual topology on Y .

Definition 11. By a **neighbourhood** of a point x in a topological space (X, τ) , we mean an open set containing x .

Definition 12. A subset A of a topological space (X, τ) is said to be **closed** if $X \setminus A$ is open in X , that is $X \setminus A \in \tau$

Proposition 1.1.1.

- \emptyset and X are closed in X
- An intersection of any collection of closed sets is closed in X
- A union of a finite collection of closed sets in X is closed in X

Definition 13. Let X be a topological space and $A \subseteq X$. A point $x \in X$ is said to be an **interior point** of A if there exists a neighbourhood U of x such that $U \subseteq A$

or in other words, if there exists an open set U in X such that $x \in U \subseteq A$.

The set of all interior points of A is called the **interior** of A and is denoted ${}^\circ o_f A$.

Theorem 1. Let (X, d) be a topological space and $S \subseteq X$. Then ${}^\circ o_f S$ is the largest open set in X that is contained in S .

Theorem 2. Let X be a topological space and $S \subseteq X$. Then S is open if and only if $S = {}^\circ o_f S$.

1.2 Bases and Subbases

Definition 14. Let X be a topological space. A collection β of open subsets of X is said to be a **basis** for the topology on X if for every open set U in X and $x \in U$, there exists a $B \in \beta$ such that $x \in B \subseteq U$.

Members of β are called **basis open sets** corresponding to basis β .

Note 1. For any topological space (X, τ) , τ is a basis for τ .

Note 2. In \mathbb{R} the set $\{(x - \varepsilon, x + \varepsilon) : x \in \mathbb{R} \text{ and } \varepsilon > 0\}$ is a basis for the usual topology.

Note 3. For (\mathbb{R}, τ) , where τ is the discrete topology, $\beta = \{\{x\} : x \in \mathbb{R}\}$ is a basis for τ on \mathbb{R} .

Note 4. In \mathbb{R} the set $\{(x - \frac{1}{n}, x + \frac{1}{n}) : x \in \mathbb{R} \text{ and } n \in \mathbb{N}\}$ is a basis for the usual topology.

Note 5. In \mathbb{R} the set $\{(a, b) : a, b \in \mathbb{Q} \text{ and } a < b\}$ is a basis for the usual topology. This is a countable basis for \mathbb{R} with the usual topology.

Note 6. Let (X, τ) be a metrizable space. Then $\beta = \{B(x, \varepsilon) : x \in X, \varepsilon > 0\}$ is a basis for τ .

Definition 15. Let (X, τ) be a topological space. A collection $S \subseteq \tau$ is said to be a **subbasis** for the topology τ if for every open set U in τ and $x \in U$, there exists a finite subcollection $\{s_1, s_2, \dots, s_n\}$ in S such that $x \in \bigcap_{i=1}^n s_i \subseteq U$.

Members of S are called **subbasis open sets** corresponding to S .

Note 7. For a topological space (X, τ) a collection S is a subbasis for the topology τ if and only if the collection of all finite intersections of members in S forms a basis for the topology τ .

Theorem 3. Let (X, τ) be a topological space and β be a collection of open sets in (X, τ) .

Then β is a basis for the topology τ on X if and only if every open set U in (X, τ) can be written as a union of members in β .

Theorem 4. If X is a set, a basis for a topology on X is a collection β of subsets of X such that

- for every $x \in X$ there exists $B \in \beta$ containing x .
- if $x \in B_1 \cap B_2$, for some $B_1, B_2 \in \beta$, then there exists $B_3 \in \beta$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Definition 16. The topology generated by the basis $\beta = \{[a, b) : a, b \in \mathbb{R}, a < b\}$ is known as the **lower limit topology** on \mathbb{R} .

The space \mathbb{R} equipped with the lower limit topology is known as the **Sorgenfrey line** \mathbb{R}_l .

Proposition 1.2.1. The lower limit topology on \mathbb{R} is strictly finer than the usual topology on \mathbb{R} .

Definition 17. Let $K = \{\frac{1}{n} : n \in \mathbb{N}\}$. Consider the basis $\beta = \{(a, b) \setminus K : a, b \in \text{rationals}, a < b\} \cup \{(a, b) : a, b \in \mathbb{R}, a < b\}$. The topology generated by the basis β is known as the **k-topology** on \mathbb{R} .

Proposition 1.2.2. The k-topology on \mathbb{R} is strictly finer than the usual topology on \mathbb{R} .

Proposition 1.2.3. The lower limit topology and the k-topology on \mathbb{R} are incomparable.

Definition 18. Let X be a set. Let S be a collection of subsets of X whose union is in X . Any such collection is called a **subbasis** for a topology X .

Theorem 5. Let S be a subbasis for a topology on X . Then $\tau = \{U \subseteq X : \text{for every } x \in U \text{ there exists a finite number of members } S_1, S_2, \dots, S_n \in S \text{ such that } x \in S_1 \cap S_2 \cap \dots \cap S_n \subseteq U\}$ is the topology generated by the subbasis S .

Theorem 6. Let β be a basis for a topology on X . Then the topology generated by β is equal to the intersection of all topologies on X that contains β .

Theorem 7. If β is a basis for a topology on a set X , then the topology generated by β on τ is the smallest topology on X that contains β .

Definition 19. Let X be a set. A function $f: \mathbb{N} \rightarrow X$ is called a **sequence** in X .

Definition 20. Let (X, Y) be a topological space. A sequence (X_n) is said to **converge** to $x \in X$ if for every neighbourhood U of x , there exists $n_0 \in \mathbb{N}$ such that $x_n \in U \forall n \geq n_0$

Definition 21. A sequence (x_n) is said to be **eventually constant** if there exists $n_0 \in \mathbb{N}$ such that $x_n = x_{n_0} \forall n \geq n_0$.

Theorem 8. In any topological space, every eventually constant sequence is convergent.

Definition 22. Let $f: \mathbb{N} \rightarrow X$ be a sequence in X . Then for any strictly increasing function $g: \mathbb{N} \rightarrow \mathbb{N}$, the composition $f \circ g: \mathbb{N} \rightarrow X$ is called a **subsequence** of f .

Theorem 9. In a topological space (X, τ) , every subsequence of a convergent sequence is convergent.

Definition 23. Let A be a non-empty set. A relation \leq on a set A is called a **partial order relation** if the following condition holds for all α, β, γ in A :

1. reflexive: $\alpha \leq \alpha$
2. anti-symmetric: $\alpha \leq \beta$ and $\beta \leq \alpha \implies \alpha = \beta$
3. transitive: $\alpha \leq \beta$ and $\beta \leq \gamma \implies \alpha \leq \gamma$

Definition 24. A **directed set** J is a set with a partial order relation \leq such that for each pair α and β of J , there exists a $\gamma \in J$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

Definition 25. A **net** in X is a function f from a directed set J to X .

Note that every sequence is a net.

Definition 26. Let J be a directed set with a partial order relation ' \leq '. A subset K of J is said to be **cofinal** in J if for each $\alpha \in J$, there exists $\beta \in K$ such that $\alpha \leq \beta$.

Proposition 1.2.4. If $g: \mathbb{N} \rightarrow \mathbb{N}$ is a strictly increasing function then $g(\mathbb{N})$ is cofinal in \mathbb{N} .

Theorem 10. If J is a directed set and K is cofinal in J , then K is a directed set.

Definition 27. Let $f: J \rightarrow X$ be a net in X . If I is a directed set and $g: I \rightarrow J$ such that

- $i < j \implies g(i) < g(j) \quad \forall i, j \in I$
- $g(I)$ is cofinal in J

Then $f \circ g: I \rightarrow X$ is called a **subnet** of f .

Definition 28. The net $(x_\alpha)_{\alpha \in J}$ is said to **converge** to a point $x \in X$ if for every neighbourhood U of x , there exists $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$.

Definition 29. Let (X, τ) be a topological space and $S \subseteq X$. A point $x \in X$ is called a **closure point** of S if for every neighbourhood U of x , we have $U \cap S \neq \emptyset$. The set of all closure points of S is called the **closure** of S and is denoted $\text{cl}(S)$ or \bar{S} .

Definition 30. A point $x \in X$ is said to be a **limit point** of S if for every neighbourhood U of x , we have $(U \cap S) \setminus \{x\} \neq \emptyset$.

The set of all limit points is called the **derived set** and is denoted S' .

Theorem 11. Let X be a topological space and $S \subseteq X$. Then \bar{S} is the smallest closed set in X that contains S .

Proposition 1.2.5. A subset of a topological space X is closed if and only if $S = \bar{S}$.

Theorem 12. *Let (X, τ) be a metrizable space and $S \subseteq X$. Then $x \in \bar{S}$ if and only if there exists a sequence $\langle x_n \rangle$ in S such that $x_n \rightarrow x$.*

Theorem 13. *Let X be a topological space and $S \subseteq X$. Then show that $x \in \bar{S}$ if and only if there exists a net $\langle x_\lambda \rangle$ in S such that $x_\lambda \rightarrow x$.*

Theorem 14. *Let X be a topological space and $S \subseteq X$. Then S° is the largest open set contained in S .*

Theorem 15. *Let X be a topological space and $S \subseteq X$. Then S is open if and only if $S = S^\circ$.*

Definition 31. *Let (X, τ_1) and (Y, τ_2) be two topological spaces. The collection $\mathcal{B} = \tau_1 \times \tau_2 = \{u \times v : u \in \tau_1, v \in \tau_2\}$ is a basis for a topology on $X \times Y$. The topology generated by \mathcal{B} is called the product topology on $X \times Y$.*

Theorem 16. *Let (X, τ) and (Y, σ) be two topological spaces. Let \mathcal{B} be a basis for τ and \mathcal{C} be a basis for σ . Then the collection $\mathcal{D} = \{U \times V : U \in \mathcal{B}, V \in \mathcal{C}\}$ is a basis for the product topology on $X \times Y$.*

2

Exercises

1. Let (X, d) be a metric space and τ_d be the collection of all open subsets of X . Show that τ_d is a topology on X . *This is called the topology on X generated by d*

Source. Class, Aug 07

2. Let X be any set and $A \subseteq X$. Then show that $\tau = \{\emptyset, A, X\}$ is a topology on X .

Source. Class, Aug 07

3. Let X be any set. Let $\tau = \{A \subseteq X : X \setminus A \text{ is finite}\} \cup \{\emptyset\}$. Then show that τ is a topology on X .

Source. Class, Aug 07

4. Let X be any set and $\tau = \{A \subseteq X : X \setminus A \text{ is countable}\} \cup \{\emptyset\}$. Then show that τ is a topology on X .

Source. Class, Aug 07

5. Show that every discrete topological space is metrizable

Source. Class, Aug 07

6. Let X be any set with more than one element. Show that the topology $\tau = \{\emptyset, X\}$ is not metrizable.

Source. Class, Aug 07

7. Let $X = \mathbb{R}^n$.

For any $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, define

1. $d_1(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$

2. $d_2(x, y) = \sum_{i=1}^n |x_i - y_i|$

3. $d_3(x, y) = \max\{|x_i - y_i| : 1 \leq i \leq n\}$

Show that d_1, d_2, d_3 all define the same topology on \mathbb{R}^n .

Source. Class, Aug 09

8. Let (X, τ) be a topological space and $A \subseteq X$. Suppose for each $x \in A$, there exists an open set U in X such that $x \in U \subseteq A$. Show that A is open in X .

Source. Problem Sheet 01, Q1

9. Let X be a set, and let $\{\tau_\alpha\}_{\alpha \in I}$ be a collection of topologies on X . Show that $\bigcap_{\alpha \in I} \tau_\alpha$ is a topology on X .

Source. Problem Sheet 01, Q2

10. Show by an example that a union of two topologies on a set X may not be a topology.

Source. Problem Sheet 01, Q3

11. Consider (X, τ) where τ is the discrete topology. Then the collection of closed sets is the power set $P(X)$ which coincides with the collection open sets in (X, τ) . Consider (X, τ) where τ is the co-finite topology. Then the collection of closed sets in (X, τ) is $\{A \in \tau : A \text{ is finite}\} \cup \{X\}$.

Source. Class, Aug 11

12. Given a topological space (X, τ) , show that:

1. \emptyset and X are closed in X
2. An intersection of any collection of closed sets is closed in X
3. A union of a finite collection closed sets in X is closed in X

Source. Class, Aug 11

13. Consider (\mathbb{R}, τ) where τ is the usual topology on \mathbb{R} . Find ${}^{\circ}o_f A$ where:

1. $A = (0, 1)$
2. $A = \mathbb{Q}$

Source. Class, Aug 11

14. Consider (\mathbb{R}, τ) where τ is the discrete topology on \mathbb{R} . Find \mathbb{Q}°

Source. Class, Aug 11

15. Consider (\mathbb{R}, τ) where τ is the indiscrete topology on \mathbb{R} . Find \mathbb{Q}°

Source. Class, Aug 11

16. Consider (\mathbb{R}, τ) where τ is the co-finite topology on \mathbb{R} . Find ${}^{\circ}o_f A$ where $A = (0, 1)$.

Source. Class, Aug 11

17. Let (X, d) be a topological space and $S \subseteq X$. Then show that S° is the largest open set in X that is contained in S .

Source. Class, 11 Aug

18. Let X be a topological space and $S \subseteq X$. Then show that S is open if and only if $S = S^{\circ}$.

Source. Class, Aug 11

19. For subsets A and B and A_α of a topological space (X, τ) show that:

1. $A \subseteq B \implies A^\circ \subseteq B^\circ$
2. $(A \cap B)^\circ = A^\circ \cap B^\circ$
3. $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$
4. $(\cap A_\alpha)^\circ \subseteq \cap (A_\alpha)^\circ$

Source. Problem Sheet 01, Q6

20. Let τ be a topology on X consisting of four sets $\tau = \{X, \phi, A, B\}$, where A, B are non-empty, distinct proper subsets of X . What conditions must A and B satisfy?

Source. Schaum's P73, Q3

21. Let $f: X \rightarrow Y$ be a function from a non-empty set X into a topological space (Y, σ) . Furthermore, let τ be the class of inverses of open subsets of Y : $\tau = \{f^{-1}(G) : G \in \sigma\}$ Show that τ is a topology on X .

Source. Schaums, P74, Q5

22. Let A be a subset of a topological space in X with the property that each point $p \in A$ belongs to an open set G_p contained in A . Then prove that A is open.

Source. Schaums, P74, Q8

23. Let τ be the class of subsets of \mathbb{R} consisting of \mathbb{R} , \emptyset and all open infinite intervals $E_a = (a, \infty)$ with $a \in \mathbb{R}$. Show that τ is a topology on \mathbb{R} .

Source. Schaum's P75, Q9

24. Let S be a subbasis for a topology on X . Show that the topology generated by S is equal to the intersection of all topologies on X that contains S .

Source. Problem Sheet 01, Q5

25. Show that a subset S of a topological space X is closed if and only if $S' \subseteq S$, that is S contains all its limit points.

Source. Problem Sheet 03, Q1

26. Let $X = \mathbb{R}$ with the co-finite topology. Show that \mathbb{N} is a dense subset of \mathbb{R} . Is \mathbb{N} dense in \mathbb{R} with the usual topology?

Source. Problem Sheet 03, Q2

27. Find the closure of $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ in $\mathbb{R}, \mathbb{R}_l, \mathbb{R}_K$.

Source. Problem Sheet 03, Q3

28. Let $X = \{a, b, c, d\}$ Construct a non-discrete topology on X such that $\{a, b\}$ is both closed and open in X .

Source. Problem Sheet 03, Q4

29. Let $X = \{a, b, c, d\}$. Construct a non-discrete topology on X such that $\{a, b\}$ is neither closed nor open in X .

Source. Problem Sheet 03, Q5

30. Let A, B, A_α denote subsets of a topological space X . Show the following:

1. If $A \subseteq B$ then $\bar{A} \subseteq \bar{B}$
2. $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$
3. $\overline{A \cup B} = \bar{A} \cup \bar{B}$
4. $\overline{\cup A_\alpha} \subset \overline{A_\alpha}$

Also give examples where the equality in point 2. and 4. fail.

Source. Problem Sheet 03, Q6

31. Show that if U is open in X and A is closed in X , then $U \setminus A$ is open in X . What about $\bar{A} \setminus U$?

Source. Problem Sheet 03, Q7

32. Let $X = \mathbb{R}$ and $\tau_1 = P(X)$ and $\tau_2 =$ usual topology. Does $\beta = \{(a, b) : a, b \in \mathbb{R}, a < b\}$ form a basis for τ_1 and τ_2 ?

Source. Class, Aug 14

33. Let β be a basis for a topology on X . Show that the topology generated by β is equal to the intersection of all topologies on X that contains β .

Source. Problem Sheet 01, Q4

34. For \mathbb{R} with the usual topology, is $S = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$ a basis? Is it a subbasis?

Source. Class, Aug 14

35. Let (X, τ) be a topological space and β be a collection of open sets in (X, τ) . Show that β is a basis for the topology τ on X if and only if every open set U in (X, τ) can be written as a union of members in β .

Source. Class, Aug 14

36. If X is a set, then show that a basis for a topology on X is a collection β of subsets of X such that

- for every $x \in X$ there exists $B \in \beta$ containing x .
- if $x \in B_1 \cap B_2$, for some $B_1, B_2 \in \beta$, then there exists $B_3 \in \beta$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Source. Class, Aug 16

37. Show that the topology generated by $\beta = \{(a, b) : a, b \in \mathbb{R}, a < b\}$ is the usual topology.

Source. Class, Aug 16

38. Show that $\beta = \{[a, b) : a, b \in \mathbb{R}, a < b\}$ is a basis for a topology on \mathbb{R} .

Source. Class, Aug 16

39. Show that the lower limit topology on \mathbb{R} is strictly finer than the usual topology on \mathbb{R} .

Source. Class, Aug 16

40. Let $K = \{\frac{1}{n} : n \in \mathbb{N}\}$. Consider the collection $\beta = \{(a, b) \setminus K : a, b \in \mathbb{Q}, a < b\} \cup \{(a, b) : a, b \in \mathbb{R}, a < b\}$. Show that the β is a basis for a topology on \mathbb{R} .

Source. Class, Aug 18

41. Show that the k -topology on \mathbb{R} is strictly finer than the usual topology.

Source. Class, Aug 18

42. Show that the lower limit topology and the k topology on \mathbb{R} are incomparable.

Source. Class, Aug 18

43. Let S be a subbasis for a topology on X . Then show that $\tau = \{U \subseteq X : \text{for every } x \in U \text{ there exists a finite number of members } S_1, S_2, \dots, S_n \in S \text{ such that } x \in S_1 \cap S_2 \cap \dots \cap S_n \subseteq U\}$ is a topology generated by the subbasis S .

Source. Class, Aug 18

44. Let β be a basis for a topology on X . Show that the topology generated by β is equal to the intersection of all topologies on X that contains β .

Source. Class, Aug 18

45. Show that the collection $\beta = \{(a, b) : a < b, a \text{ and } b \in \mathbb{Q}\}$ is a basis for a topology on \mathbb{R} and the topology generated by the basis is the usual topology on \mathbb{R} .

Source. Problem Sheet 01, Q7

46. Show that the collection $\beta = \{(a, b) : a < b \text{ and } a, b \in \mathbb{Q}\}$ is a basis for a topology on \mathbb{R} and the topology generated by this basis is different from the lower limit topology on \mathbb{R} .

Source. Problem Sheet 01, Q8

47. Show that if J is a directed set and K is cofinal in J , then K is a directed set.

Source. Class, Aug 23

48. Show that a subnet of a convergent net is convergent.

Source. Class, Aug 23

49. Let X be an infinite set and τ be the co-finite topology. Then find the closure of X (?)

Source. Class, Aug 23

50. Show that $\bar{S} = S' \cup S$.

Source. Class, Aug 23

51. Let (X, τ) be a metrizable space and $S \subseteq X$. Then show that $x \in \bar{S}$ if and only if there exists a sequence $\langle x_n \rangle$ in S such that $x_n \rightarrow x$.

Source. Class, Aug 28

52. Let $X = \mathbb{R}$ and τ be the co-countable topology. Consider $S = \mathbb{R} \setminus \{0\}$. Does there exist a sequence $\langle x_n \rangle$ in S such that $x_n \rightarrow 0$?

Source. Class, Aug 28

53. Let X be a topological space and $S \subseteq X$. Then show that $x \in \bar{S}$ if and only if there exists a net $\langle x_\lambda \rangle$ in S such that $x_\lambda \rightarrow x$

Source. Class, Aug 28

54. Let β be a basis for the topology on X . Show that a net (x_λ) converges to x in X if for every $B \in \beta$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in B \forall n \geq n_0$.

Source. Problem Sheet 02, Q1

55. In the space \mathbb{R} with the co-finite topology, to what point or points does the sequence $\frac{1}{n}$ converge?

Source. Problem Sheet 02, Q2

56. Show that a collection \mathcal{A} of subsets of S that is closed under finite intersection, partially ordered by the reverse set inclusion (that is, for $A, B \in \mathcal{A}$, $A \leq B$ if $A \supseteq B$) is a directed set. Here for $A, B \in \mathcal{A}$, $A \cap B \in \mathcal{A}$ and $A \supseteq A \cap B$ and $B \supseteq A \cap B$.

Source. Problem Sheet 02, Q3

57. Show that the collection \mathcal{F} of all closed subsets of a space X , partially ordered by the set inclusion is a directed set.

Source. Problem Sheet 02, Q4

58. Show that if a net f converges to x in a space X , so does any subnet of f .

Source. Problem Sheet 02, Q5

59. Show that a convergent sequence in a discrete topological space is eventually constant.

Source. Problem Sheet 02, Q6

60. Show that every net in a indiscrete topological space converges to every point of X .

Source. Problem Sheet 02, Q7

61. Let $X = \{a, b, c\}$. Construct a topology τ on X such that there exists a sequence in (X, τ) that converges to a and b both but does not converge to c .

Source. Problem Sheet 02, Q8

62. Let (X, d) be a metric space and (x_n) be a sequence in X . Then $x_n \rightarrow x$ if and only if for every open set U in X containing x , there exists $n_0 \in \mathbb{N}$ such that $x_n \in U \forall n \geq n_0$.

Source. Class, Aug 21

63. Let $X = \mathbb{R}$ and τ be the indiscrete topology. Let $x_n = \frac{1}{n}$ be a sequence in (X, τ) . Find if the sequence (x_n) converges, and the point to where it converges.

Source. Class, Aug 21

64. Let X be a discrete topological space. Show that every convergent sequence in X is eventually constant.

Source. Class, Aug 21

65. Let X and τ be the co-countable topological space. Show that every convergent sequence in X is eventually constant.

Source.

66. Show that in a topological space (X, τ) , every subsequence of a convergent sequence is convergent.

Source. Class, Aug 21

67. Let S be any set. Consider $P(S)$ = the power set of S . Define $A \leq B$ if $A \subseteq B$. Verify that \leq is a partial order relation. Verify the same when $A \leq B$ when $A \supseteq B$.

Source. Class, Aug 21

68. Let X be a topological space and $S \subseteq X$. Then S° is the largest open set contained in S .

Source. Class, Sept 01

69. Let (X, τ_1) and (Y, τ_2) be two topological spaces. Show that the cartesian product $\mathcal{B} = \tau_1 \times \tau_2 = \{u \times v : u \in \tau_1, v \in \tau_2\}$ will not be a topology on $X \times Y$. Verify that \mathcal{B} is a basis for the product topology on $X \times Y$.

Source. Class, Sept 01

70. Is the usual topology on \mathbb{R}^2 different from the product topology on $\mathbb{R} \times \mathbb{R}$.

Source. Class, Sept 01

71. Let (X, τ) and (Y, σ) be two topological spaces. Let \mathcal{B} be a basis for τ and \mathcal{C} be a basis for σ . Then show that the collection $\mathcal{D} = \{U \times V : U \in \mathcal{B}, V \in \mathcal{C}\}$ is a basis for the product topology on $X \times Y$.

Source. Class, Sept 01

72 ([Stub for projection maps and one theorem (before axiom of choice)).]

Source. Class, Sept 01