Contents

1

Definitions and Theory

1.1 Topological Spaces

Definition 1. *A **topology** on a set X is a collection τ of subsets of X satisfying:*

1. *$\varnothing, X \in \tau$*
2. *An intersection of finite subcollections of τ is in τ*
3. *A union of any subcollection of τ is in τ*

*The ordered pair (X, τ) is called a **topological space**.*

Definition 2. *Let (X, τ) be a topological space. An **open subset** of X is a member of τ.*

Definition 3. *Let τ and σ be two topologies on a set X. We say that τ is **weaker** (or smaller, coarser) than σ if $T \subseteq \sigma$. In this case, σ is then said to be **stronger** (or larger, finer) than τ.*

Definition 4. *Let X be any set. The collection $\tau = P(X)$ is a topology on X and is called the **discrete topology** on X. Here (X, τ) is called the **discrete topological space**.*

Definition 5. *Let X be any set. The collection $\tau = \{\varnothing, X\}$ is called the **indiscrete topology** on X. Here (X, τ) is called the **indiscrete topology**.*

Definition 6. *Let X be any set. The collection $tau = \{A \subseteq X : X \setminus A \text{ is finite }\} \cup \{\varnothing\}$ is called the **co-finite topology**.*

Definition 7. *Let X be any set. The collection $tau = \{A \subseteq X : X \setminus A \text{ is countable }\} \cup \{\varnothing\}$ is called the **co-finite topology**.*

Definition 8. *A topology τ on a set X is said to be* ***metrizable*** *if there exists a metric d on X such that the topology τ_d generated by the metric d coincides with τ.*

Definition 9. *Two metrics defined on a set X are said to be* ***equivalent*** *if they generate the same topology. In other words, d_1 and d_2 are equivalent if the collection of open sets in (X, d_1) and (X, d_2) are the same.*

Definition 10. *The topology generated by the Euclidean metric on $\mathbb{R}^n$ is called the* ***usual topology*** *on $\mathbb{R}^n$. For $Y \subseteq \mathbb{R}^n$, the topology generated by the Euclidean metric is called the usual topology on Y.*

Definition 11. *By a* ***neighbourhood*** *of a point x in a topological space (X, τ), we mean an open set containing x.*

Definition 12. *A subset A of a topological space (X, τ) is said to be* ***closed*** *if $X \setminus A$ is open in X, that is $X \setminus A \in \tau$*

Theorem 1.

- *$\varnothing$ and X are closed in X*
- *An intersection of any collection of closed sets is closed in X*
- *A union of a finite collection of closed sets in X is closed in X*

Definition 13. *Let X be a topological space and $A \subseteq X$. A point $x \in X$ is said to be an* ***interior point*** *of A if there exists a neighbourhood U of X such that $U \subseteq A$ (or) if there exists an open set U in X such that $x \in U \subseteq A$.*
The set of all interior points of A is called the ***interior*** *of A and is denoted A°.*

Theorem 2. *Let (X, d) be a topological space and $S \subseteq X$. Then S° is the largest open set in X that is contained in S.*

Theorem 3. *Let X be a topological space and $S \subseteq X$ Then S is open if and only if $S = S^{\circ}$.*

1.2 Bases and Subbases

Definition 14. *Let X be a topological space. A collection β of open subsets of X is said to be a* ***basis*** *for the topology on X if for every open set U in X and $x \in U$, there exists a $B \in \beta$ such that $x \in B \subseteq U$.*
Members of β are called ***basis open sets*** *corresponding to basis β.*

Note 1. *For any topological space (X, τ), τ is a basis for τ.*

Note 2. *In $\mathbb{R}$ the set $\{(x-\varepsilon, x+\varepsilon) : x \in \mathbb{R}$ and $\varepsilon > 0\}$ is a basis for the usual topology.*

Note 3. *For $(\mathbb{R}, \tau)$, where τ is the discrete topology, $\beta = \{\{x\} : x \in \mathbb{R}\}$ is a basis for τ on $\mathbb{R}$.*

Note 4. *In $\mathbb{R}$ the set $\left\{\left(x - \frac{1}{n}, x + \frac{1}{n}\right) : x \in \mathbb{R} \text{ and } n \in \mathbb{N}\right\}$ is a basis for the usual topology.*

Note 5. *In $\mathbb{R}$ the set $\{(a, b) : a, b \in \mathbb{Q}$ and $a < b\}$ is a basis for the usual topology. This is a countable basis for $\mathbb{R}$ with the usual topology.*

Note 6. *Let (X, τ) be a metrizable space. Then $\beta = \{B(x, \varepsilon) : x \in X, \varepsilon > 0\}$ is a basis for τ.*

Definition 15. *Let (X, τ) be a topological space. A collection $S \subseteq \tau$ is said to be a **subbasis** for the topology τ if for every open set U in τ and $x \in U$, there exists a finite subcollection $\{S_1, S_2, ..., S_n\}$ in S such that $x \in \bigcap_{i=1}^{n} S_i \subseteq U$*
*Members of S are called **subbasis open sets** corresponding to S.*

Note 7. *For a topological space (X, τ) a collection S is a subbasis for the topology τ if and only if the collection of all finite intersections of members in S forms a basis for the topology τ.*

Theorem 4. *Let (X, τ) be a topological space and β be a collection of open sets in (X, τ).*
Then β is a basis for the topology τ on X if and only if every open set U in (X, τ) can be written as a union of members in β.

Theorem 5. *If X is a set, a basis for a topology on X is a collection β of subsets of X such that*

- *for every $x \in X$ there exists $B \subset \beta$ containing x.*
- *if $x \in B_1 \cap B_2$, for some $B_1, B_2 \in \beta$, then there exists $B_3 \in \beta$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

Definition 16. *The topology generated by the basis $\beta = \{[a, b) : a, b \in \mathbb{R}, a < b\}$ is known as the **lower limit topology** on $\mathbb{R}$.*
*The space $\mathbb{R}$ equipped with the lower limit topology is known as the **Sorgenfrey line** $\mathbb{R}_l$.*

Proposition 1.2.1. *The lower limit topology on $\mathbb{R}$ is strictly finer than the usual topology on $\mathbb{R}$*

Definition 17. *Let* $K = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. *Consider the basis* $\beta = \{(a,b) \setminus K : a,b \in \mathbb{Q}, a < b\} \cup \{(a,b) : a,b \in \mathbb{R}, a < b\}$. *The topology generated by the basis* β *is known as the* k***-topology*** *on* $\mathbb{R}$.

Proposition 1.2.2. *The* k*-topology on* $\mathbb{R}$ *is strictly finer than the usual topology on* $\mathbb{R}$.

Proposition 1.2.3. *The lower limit topology and the* k*-topology on* $\mathbb{R}$ *are incomparable.*

Definition 18. *Let* X *be a set. Let* S *be a collection of subsets of* X *whose union is in* X. *Any such collection is called a* ***subbasis*** *for a topology* X. *(TODO: Verify)*

Theorem 6. *Let* S *be a subbasis for a topology on* X. *Then* $\tau = \{U \subseteq X :$ *for every* $x \in U$ *there exists a finite number of members* $S_1, S_2, ..., S_n \in S$ *such that* $x \in S_1 \cap S_2 \cap ... \cap S_n \subseteq U\}$ *is the topology generated by the subbasis* S.

Theorem 7. *Let* β *be a basis for a topology on* X. *Then the topology generated by* β *is equal to the intersection of all topologies on* X *that contains* β.

Theorem 8. *If* β *is a basis for a topology on a set* X, *then the topology generated by* β *on* τ *is the smallest topology on* X *that contains* β.

Definition 19. *Let* X *be a set. A function* $f \colon \mathbb{N} \to X$ *is called a* ***sequence*** *in* X.

Definition 20. *Let* (X, Y) *be a topological space. A sequence* (X_n) *is said to* ***converge*** *to* $x \in X$ *if for every neighbourhood* U *of* x, *there exists* $n_0 \in \mathbb{N}$ *such that* $x_n \in U \ \forall n \geq n_0$

Definition 21. *A sequence* (x_n) *is said to be* ***eventually constant*** *if there exists* $n_0 \in \mathbb{N}$ *such that* $x_n = x_{n_0} \ \forall n \geq n_0$.

Theorem 9. *In any topological space, every eventually constant sequence is convergent.*

Definition 22. *Let* $f \colon \mathbb{N} \to X$ *be a sequence in* X. *Then for any strictly increasing function* $g \colon \mathbb{N} \to \mathbb{N}$, *the composition* $f \circ g \colon \mathbb{N} \to X$ *is called a* ***subsequence*** *of* f.

Theorem 10. *In a topological space* (X, τ), *every subsequence of a convergent sequence is convergent.*

Definition 23. *Let* $\mathcal{A}$ *be a non-empty set. A relation* $\leq$ *on a set* A *is called a* ***partial order relation*** *if the following condition holds for all* α, β, γ *in* A:

1. *reflexive:* $\alpha \leq \alpha$

2. *anti-symmetric:* $\alpha \leq \beta$ *and* $\beta \leq \alpha \implies \alpha = \beta$

3. *transitive:* $\alpha \leq \beta$ *and* $\beta \leq \gamma \implies \alpha \leq \gamma$

Definition 24. *A **directed set** J is a set with a partial order relation $\leq$ such that for each pair α and β of J, there exists a $Y \in J$ such that $\alpha \leq Y$ and $\beta \leq Y$.*

Definition 25. *A **net** in X is a function f from a directed set J to X.*

Note that every sequence is a net.

Definition 26. *Let J be a directed set with a partial order relation '$\leq$'. A subset K of J is said to be **cofinal** in J if for each $\alpha \in J$, there exists $\beta \in K$ such that $\alpha \leq \beta$.*

Proposition 1.2.4. *If $g \colon \mathbb{N} \to \mathbb{N}$ is a strictly increasing function then $g(\mathbb{N})$ is cofinal in $\mathbb{N}$.*

Theorem 11. *If J is a directed set and K is cofinal in J, then K is a directed set.*

Definition 27. *Let $f \colon J \to X$ be a net in X. If I is a directed set and $g \colon I \to J$ such that*

- $i < j \implies g(i) < g(j)\ \forall i, j \in I$
- $g(I)$ *is cofinal in* J

*Then $f \circ g \colon I \to X$ is called a **subnet** of f.*

Definition 28. *The net $(X_\alpha)_{\alpha \in J}$ is said to **converge** to a point $x \in X$ if for every neighbourhood U of x, there exists $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$.*

Definition 29. *Let (X, τ) be a topological space and $S \subseteq X$. A point $x \in X$ is called a **closure point** of S if for every neighbourhood U of x, we have $U \cap S \neq \varnothing$. THe set of all closure points of S is called the **closure** of A and is denoted $cl(A)$ or $\bar{A}$.*

Definition 30. *A point $x \in X$ is said to be a **limit point** of S if for every neighbourhood of x, we have $(U \cap S) \setminus \{x\} \neq \varnothing$.*
*The set of all limit points is called the **derived set** and is denoted S'*

Theorem 12. *Let X be a topological space and $S \subseteq X$. Then $\bar{S}$ is the smallest closed set in X that contains S.*

Proposition 1.2.5. *A subset of a topological space X is closed if and only if $S = \bar{S}$*

Theorem 13. *Let (X,τ) be a metrizable space and $S \subseteq X$. Then $x \in \bar{S}$ if and only if there exists a sequence $\langle x_n \rangle$ in S such that $x_n \to x$.*

Theorem 14. *Let X be a topological space and $S \subseteq X$. Then show that $x \in \bar{S}$ if and only if there exists a net $\langle x_\lambda \rangle$ in S such that $x_\lambda \to x$*

Theorem 15. *Let X be a topological space and $S \subseteq X$. Then S^{o} is the largest open set contained in S.*

Theorem 16. *Let X be a topological space and $S \subseteq X$. Then S is open if and only if $S = S^{\mathrm{o}}$.*

Definition 31. *Let (X,τ_1) and (Y,τ_2) be two topological spaces. The collection $\mathcal{B} = \tau_1 \times \tau_2 = \{u \times v : u \in \tau_1, v \in \tau_2\}$ is a basis for a topology on $X \times Y$. The topology generated by $\mathcal{B}$ is called the product topology on $X \times Y$.*

Theorem 17. *Let (X,τ) and (Y,σ) be two topological spaces. Let $\mathcal{B}$ be a basis for τ and $\mathcal{C}$ be a basis for σ. Then the collection $\mathcal{D} = \{U \times V : U \in \mathcal{B}, V \in \mathcal{C}\}$ is a basis for the product topology on $X \times Y$.*

2

Exercises

---

1. Let (X, d) be a metric space and τ_d be the collection of all open subsets of X. Show that τ_d is a topology on X *This is called the topology on X generated by d*

Source. Class, Aug 07

2. Let X be any set and $A \subseteq X$. Then show that $\tau = \{\varnothing, A, X\}$ is a topology on X.

Source. Class, Aug 07

3. Let X be any set. Let $\tau = \{A \subseteq X : X \setminus A \text{ is finite }\} \cup \{\varnothing\}$ Then show that τ is a topology on X.

Source. Class, Aug 07

4. Let X be any set and $\tau = \{A \subseteq X : X \setminus A \text{ is countable }\} \cup \{\varnothing\}$. Then show that τ is a topology on X.

Source. Class, Aug 07

5. Show that every discrete topological space is metrizable

Source. Class, Aug 07

6. Let X be any set with more than one element. Show that the topology $\tau = \{\varnothing, X\}$ is not metrizable.

Source. Class, Aug 07

7. Let $X = \mathbb{R}^n$.
For any $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$, define

1. $d_1(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$
2. $d_2(x, y) = \sum_{i=1}^{n} |x_i - y_i|$
3. $d_3(x, y) = \max\{|x_i - y_i| : 1 \leq i \leq n\}$

Show that d_1, d_2, d_3 all define the same topology on $\mathbb{R}^n$.

Source. Class, Aug 09

8. Let (X, τ) be a topological space and $A \subseteq X$. Suppose for each $x \in A$, there exists an open set U in X such that $x \in U \subseteq A$. Show that A is open in X.

Source. Problem Sheet 01, Q1

9. Let X be a set, and let $\{\tau_\alpha\}_{\alpha \in I}$ be a collection of topologies on X. Show that $\bigcap\limits_{\alpha \in I} \tau_\alpha$ is a topology on X.

Source. Problem Sheet 01, Q2

10. Show by an example that a union of two topologies on a set X may not be a topology.

Source. Problem Sheet 01, Q3

11.

1. Consider (X, τ) where τ is the discrete topology. Then show that the collection of closed sets in (X, τ) is the power set $P(X)$
2. Consider (X, τ) where τ is the co-finite topology. Then show that the collection of closed sets in (X, τ) is $\{A \subseteq X : A \text{ is finite }\} \cup \{X\}$.

Source. Class, Aug 11

12. Given a topological space (X, τ), show that:

1. $\varnothing$ and X are closed in X
2. An intersection of any collection of closed sets is closed in X
3. A union of a finite collection closed sets in X is closed in X

Source. Class, Aug 11

13. Consider $(\mathbb{R}, \tau)$ where τ is the usual topology on $\mathbb{R}$. Find interior ${}^{\circ}o_f A$ where:

1. $A = (0, 1)$
2. $A = \mathbb{Q}$

Source. Class, Aug 11

14. Consider $(\mathbb{R}, \tau)$ where τ is the discrete topology on $\mathbb{R}$. Find interior $\mathbb{Q}^{\circ}$

Source. Class, Aug 11

15. Consider $(\mathbb{R}, \tau)$ where τ is the indiscrete topology on $\mathbb{R}$. Find interior $\mathbb{Q}^{\circ}$

Source. Class, Aug 11

16. Consider $(\mathbb{R}, \tau)$ where τ is the co-finite topology on $\mathbb{R}$. Find interior ${}^{\circ}o_f A$ where $A = (0, 1)$.

Source. Class, Aug 11

17. Let (X, d) be a topological space and $S \subseteq X$. Then show that S° is the largest open set in X that is contained in S.

Source. Class, 11 Aug

18. Let X be a topological space and $S \subseteq X$. Then show that S is open if and only if $S = S^{o}$.

Source. Class, Aug 11

19. For subsets A and B and A_α of a topological space (X, τ) show that:

1. $A \subseteq B \implies A^{o} \subseteq B^{o}$
2. $(A \cap B)^{o} = A^{o} \cap B^{o}$
3. $A^{o} \cup B^{o} \subseteq (A \cup B)^{o}$
4. $(\cap A_\alpha)^{o} \subseteq \cap (A_\alpha)^{o}$

Source. Problem Sheet 01, Q6

20. Let τ be a topology on X consisting of four sets $\tau = \{X, \phi, A, B\}$, where A, B are non-empty, distinct proper subsets of X. What conditions must A and B satisfy?

Source. Schaum's P73, Q3

21. Let $f \colon X \to Y$ be a function from a non-empty set X into a topological space (Y, σ). Furthermore, let τ be the class of inverses of open subsets of Y: $\tau = \{f^{-1}(G) : G \in \sigma\}$ Show that τ is a topology on X.

Source. Schaums, P74, Q5

22. Let A be a subset of a topological space in X with the property that each point $p \in A$ belongs to an open set G_p contained in A. Then prove that A is open.

Source. Schaums, P74, Q8

23. Let τ be the class of subsets of $\mathbb{R}$ consisting of $\mathbb{R}$, $\varnothing$ and all open infinite intervals $E_a = (a, \infty)$ with $a \in \mathbb{R}$. Show that τ is a topology on $\mathbb{R}$.

Source. Schaum's P75, Q9

24. Let S be a subbasis for a topology on X. Show that the topology generated by S is equal to the intersection of all topologies on X that contains S.

Source. Problem Sheet 01, Q5

25. Show that a subset S of a topological space X is closed if and only if $S' \subseteq S$, that is S contains all its limit points.

Source. Problem Sheet 03, Q1

26. Let $X = \mathbb{R}$ with the co-finite topology. Show that $\mathbb{N}$ is a dense subset of $\mathbb{R}$. Is $\mathbb{N}$ dense in $\mathbb{R}$ with the usual topology?

Source. Problem Sheet 03, Q2

27. Find the closure of $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ in $\mathbb{R}, \mathbb{R}_l, \mathbb{R}_K$.

Source. Problem Sheet 03, Q3

28. Let $X = \{a, b, c, d\}$ Construct a non-discrete topology on X such that $\{a, b\}$ is both closed and open in X.

Source. Problem Sheet 03, Q4

29. Let $X = \{a, b, c, d\}$. Construct a non-discrete topology on X such that $\{a, b\}$ is neither closed nor open in X.

Source. Problem Sheet 03, Q5

30. Let A, B, A_α denote subsets of a topological space X. Show the following:

1. If $A \subseteq B$ then $\bar{A} \subseteq \bar{B}$
2. $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$
3. $\overline{A \cup B} = \bar{A} \cup \bar{B}$
4. $\overline{\cup A_\alpha} \subset \overline{A_\alpha}$

Also give examples where the equality in point 2. and 4. fail.

Source. Problem Sheet 03, Q6

31. Show that if U is open in X and A is closed in X, then $U \setminus A$ is open in X. What about $A \setminus U$?

Source. Problem Sheet 03, Q7

32. Let $X = \mathbb{R}$ and $\tau_1 = P(X)$ and $\tau_2 =$ usual topology .
Does $\beta = \{(a, b) : a, b \in \mathbb{R}, a < b\}$ form a basis for τ_1 and τ_2?

Source. Class, Aug 14

33. Let β be a basis for a topology on X. Show that the topology generated by β is equal to the intersection of all topologies on X that contains β.

Source. Problem Sheet 01, Q4

34. For $\mathbb{R}$ with the usual topology, is $S = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$ a basis? Is it a subbasis?

Source. Class, Aug 14

35. Let (X, τ) be a topological space and β be a collection of open sets in (X, τ). Show that β is a basis for the topology τ on X if and only if every open set U in (X, τ) can be written as a union of members in β.

Source. Class, Aug 14

36. If X is a set, then show that a basis for a topology on X is a collection β of subsets of X such that

- for every $x \in X$ there exists $B \subset \beta$ containing x.
- if $x \in B_1 \cap B_2$, for some $B_1, B_2 \in \beta$, then there exists $B_3 \in \beta$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Source. Class, Aug 16

37. Show that the topology generated by $\beta = \{(a, b) : a, b \in \mathbb{R}, a < b\}$ is the usual topology.

Source. Class, Aug 16

38. Show that $\beta = \{[a, b) : a, b \in \mathbb{R}, a < b\}$ is a basis for a topology on $\mathbb{R}$.

Source. Class, Aug 16

39. Show that the lower limit topology on $\mathbb{R}$ is strictly finer than the usual topology on $\mathbb{R}$.

Source. Class, Aug 16

40. Let $K = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Consider the collection $\beta = \{(a, b) \setminus K : a, b \in \mathbb{Q}, a < b\} \cup \{(a, b) : a, b \in \mathbb{R} a < b\}$. Show that the β is a basis for a topology on $\mathbb{R}$.

Source. Class, Aug 18

41. Show that the k-topology on $\mathbb{R}$ is strictly finer than the usual topology.

Source. Class, Aug 18

42. Show that the lower limit topology and the k topology on $\mathbb{R}$ are incomparable.

Source. Class, Aug 18

43. Let S be a subbasis for a topology on X. Then show that $\tau = \{U \subseteq X$: for every $x \in U$ there exists a finite number of members $S_1, S_2, ..., S_n \in S$ such that $x \in S_1 \cap S_2 \cap ... \cap S_n \subseteq U\}$ is a topology generated by the subbasis S.

Source. Class, Aug 18

44. Let β be a basis for a topology on X. Show that the topology generated by β is equal to the intersection of all topologies on X that contains β.

Source. Class, Aug 18

45. Show that the collection $\beta = \{(a, b) : a < b, a$ and $b \in \mathbb{Q}\}$ is a basis for a topology on $\mathbb{R}$ and the topology generated by the basis is the usual topology on $\mathbb{R}$.

Source. Problem Sheet 01, Q7

46. Show that the collection $\beta = \{[a, b) : a < b$ and $a, b \in \mathbb{Q}\}$ is a basis for a topology on $\mathbb{R}$ and the topology generated by this basis is different from the lower limit topology on $\mathbb{R}$.

Source. Problem Sheet 01, Q8

47. Show that if J is a directed set and K is cofinal in J, then K is a directed set.

Source. Class, Aug 23

48. Show that a subnet of a convergent net is convergent.

Source. Class, Aug 23

49. Let X be an infinite set and τ be the co-finite topology. Then find the closure of X (?)

Source. Class, Aug 23

50. Show that $\bar{S} = S' \cup S$.

Source. Class, Aug 23

51. Let (X, τ) be a metrizable space and $S \subseteq X$. Then show that $x \in \bar{S}$ if and only if there exists a sequence $\langle x_n \rangle$ in S such that $x_n \to x$.

Source. Class, Aug 28

52. Let $X = \mathbb{R}$ and τ be the co-countable topology. Consider $S = \mathbb{R} \setminus \{0\}$. Does there exist a sequence $\langle x_n \rangle$ in S that such that $x_n \to 0$?

Source. Class, Aug 28

53. Let X be a topological space and $S \subseteq X$. Then show that $x \in \bar{S}$ if and only if there exists a net $\langle x_\lambda \rangle$ in S such that $x_\lambda \to x$

Source. Class, Aug 28

54. Let β be a basis for the topology on X. Show that a net (x_λ) converges to x in X if for every $B \in \beta$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in B \; \forall n \geq n_0$.

Source. Problem Sheet 02, Q1

55. In the space $\mathbb{R}$ with the co-finite topology, to what point or points does the sequence $\frac{1}{n}$ converge?

Source. Problem Sheet 02, Q2

56. Show that a collection $\mathcal{A}$ of subsets of S that is closed under finite intersection, partially ordered by the reverse set inclusion (that is, for $A, B \in \mathcal{A}$, $A \leq B$ if $A \supseteq B$) is a directed set. Here for $A, B \in \mathcal{A}$, $A \cap B \in \mathcal{A}$ and $A \supseteq A \cap B$ and $B \supseteq A \cap B$.

Source. Problem Sheet 02, Q3

57. Show that the collection $\mathcal{F}$ of all closed subsets of a space X, partially ordered by the set inclusion is a directed set.

Source. Problem Sheet 02, Q4

58. Show that if a net f converges to x in a space X, so does any subnet of f.

Source. Problem Sheet 02, Q5

59. Show that a convergent sequence in a discrete topological space is eventually constant.

Source. Problem Sheet 02, Q6

60. Show that every net in a indiscrete topological space converges to every point of X.

Source. Problem Sheet 02, Q7

61. Let $X = \{a, b, c\}$. Construct a topology τ on X such that there exists a sequence in (X, τ) that converges to a and b both but does not converge to c.

Source. Problem Sheet 02, Q8

62. Let (X, d) be a metric space and (x_n) be a sequence in X. Then $x_n \to x$ if and only if for every open set U in X containing x, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U \; \forall n \geq n_0$.

Source. Class, Aug 21

63. Let $X = \mathbb{R}$ and τ be the indiscrete topology. Let $x_n = \frac{1}{n}$ be a sequence in (X, τ). Find if the sequence (x_n) converges, and the point to where it converges.

Source. Class, Aug 21

64. Let X be a discrete topological space. Show that every convergent sequence in X is eventually constant.

Source. Class, Aug 21

65. Let X and τ be the co-countable topological space. Show that every convergent sequence in X is eventually constant.

Source.

66. Show that in a topological space (X, τ), every subsequence of a convergent sequence is convergent.

Source. Class, Aug 21

67. Let S be any set. Consider $P(S)$ = the power set of S. Define $A \leq B$ if $A \subseteq B$. Verify that $\leq$ is a partial order relation. Verify the same when $A \leq B$ when $A \supseteq B$.

Source. Class, Aug 21

68. Let X be a topological space and $S \subseteq X$. Then S^{o} is the largest open set contained in S.

Source. Class, Sept 01

69. Let (X, τ_1) and (Y, τ_2) be two topological spaces. Show that the cartesian product $\mathcal{B} = \tau_1 \times \tau_2 = \{u \times v : u \in \tau_1, v \in \tau_2\}$ will not be a topology on $X \times Y$. Verify that $\mathcal{B}$ is a basis for the product topology on $X \times Y$.

Source. Class, Sept 01

70. Is the usual topology on $\mathbb{R}^2$ different from the product topology on $\mathbb{R} \times \mathbb{R}$.

Source. Class, Sept 01

71. Let (X, τ) and (Y, σ) be two topological spaces. Let $\mathcal{B}$ be a basis for τ and $\mathcal{C}$ be a basis for σ. Then show that the collection $\mathcal{D} = \{U \times V : U \in \mathcal{B}, V \in \mathcal{C}\}$ is a basis for the product topology on $X \times Y$.

Source. Class, Sept 01

72 ([Stub for projection maps and one theorem (before axiom of choice))**.**]

Source. Class, Sept 01