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Definitions and Theory

1.1 Topological Spaces

Definition 1. *A **topology** on a set X is a collection τ of subsets of X satisfying:*

1. *$\varnothing, X \in \tau$*
2. *An intersection of finite subcollections of τ is in τ*
3. *A union of any subcollection of τ is in τ*

*The ordered pair (X, τ) is called a **topological space**.*

Definition 2. *Let (X, τ) be a topological space. An **open subset** of X is a member of τ.*

Definition 3. *Let τ and σ be two topologies on a set X. We say that τ is **weaker** (or smaller, coarser) than σ if $T \subseteq \sigma$. In this case, σ is then said to be **stronger** (or larger, finer) than τ.*

Definition 4. *Let X be any set. The collection $\tau = P(X)$ is a topology on X and is called the **discrete topology** on X. Here (X, τ) is called the **discrete topological space**.*

Definition 5. *Let X be any set. The collection $\tau = \{\varnothing, X\}$ is called the **indiscrete topology** on X. Here (X, τ) is called the **indiscrete topology**.*

Definition 6. *Let X be any set. The collection*
*$tau = \{A \subseteq X : X \setminus A \text{ is finite }\} \cup \{\varnothing\}$ is called the **co-finite topology**.*

Definition 7. *Let X be any set. The collection*
*$tau = \{A \subseteq X : X \setminus A \text{ is countable }\} \cup \{\varnothing\}$ is called the **co-finite topology**.*

Definition 8. *A topology τ on a set X is said to be **metrizable** if there exists a metric d on X such that the topology τ_d generated by the metric d coincides with τ.*

Definition 9. *Two metrics defined on a set X are said to be **equivalent** if they generate the same topology. In other words, d_1 and d_2 are equivalent if the collection of open sets in (X, d_1) and (X, d_2) are the same.*

Definition 10. *The topology generated by the Euclidean metric on $\mathbb{R}^n$ is called the **usual topology** on $\mathbb{R}^n$. For $Y \subseteq \mathbb{R}^n$, the topology generated by the Euclidean metric is called the usual topology on Y.*

Definition 11. *By a **neighbourhood** of a point x in a topological space (X, τ), we mean an open set containing x.*

Definition 12. *A subset A of a topological space (X, τ) is said to be **closed** if $X \setminus A$ is open in X, that is $X \setminus A \in \tau$*

Proposition 1.1.1.

- *$\varnothing$ and X are closed in X*
- *An intersection of any collection of closed sets is closed in X*
- *A union of a finite collection of closed sets in X is closed in X*

Definition 13. *Let X be a topological space and $A \subseteq X$. A point $x \in X$ is said to be an **interior point** of A if there exists a neighbourhood U of X such that $U \subseteq A$*
or in other words, if there exists an open set U in X such that $x \in U \subseteq A$.
*The set of all interior points of A is called the **interior** of A and is denoted A°.*

Theorem 1. *Let (X, d) be a topological space and $S \subseteq X$. Then S° is the largest open set in X that is contained in S.*

Theorem 2. *Let X be a topological space and $S \subseteq X$ Then S is open if and only if $S = S^{\circ}$.*

1.2 Bases and Subbases

Definition 14. *Let X be a topological space. A collection β of open subsets of X is said to be a **basis** for the topology on X if for every open set U in X and $x \in U$, there exists a $B \in \beta$ such that $x \in B \subseteq U$.*
*Members of β are called **basis open sets** corresponding to basis β.*

Note 1. *For any topological space (X, τ), τ is a basis for τ.*

Note 2. *In* $\mathbb{R}$ *the set* $\{(x-\varepsilon, x+\varepsilon) : x \in \mathbb{R}$ *and* $\varepsilon > 0\}$ *is a basis for the usual topology.*

Note 3. *For* $(\mathbb{R}, \tau)$, *where* τ *is the discrete topology,* $\beta = \{\{x\} : x \in \mathbb{R}\}$ *is a basis for* τ *on* $\mathbb{R}$.

Note 4. *In* $\mathbb{R}$ *the set* $\left\{\left(x - \frac{1}{n}, x + \frac{1}{n}\right) : x \in \mathbb{R} \text{ and } n \in \mathbb{N}\right\}$ *is a basis for the usual topology.*

Note 5. *In* $\mathbb{R}$ *the set* $\{(a, b) : a, b \in \mathbb{Q}$ *and* $a < b\}$ *is a basis for the usual topology. This is a countable basis for* $\mathbb{R}$ *with the usual topology.*

Note 6. *Let* (X, τ) *be a metrizable space. Then* $\beta = \{B(x, \varepsilon) : x \in X, \varepsilon > 0\}$ *is a basis for* τ.

Definition 15. *Let* (X, τ) *be a topological space. A collection* $S \subseteq \tau$ *is said to be a* ***subbasis*** *for the topology* τ *if for every open set* U *in* τ *and* $x \in U$, *there exists a finite subcollection* $\{s_1, s_2, ..., s_n\}$ *in* S *such that* $x \in \bigcap_{i=1}^{n} S_i \subseteq U$
Members of S *are called* ***subbasis open sets*** *corresponding to* S.

Note 7. *For a topological space* (X, τ) *a collection* S *is a subbasis for the topology* τ *if and only if the collection of all finite intersections of members in* S *forms a basis for the topology* τ.

Theorem 3. *Let* (X, τ) *be a topological space and* β *be a collection of open sets in* (X, τ).
Then β *is a basis for the topology* τ *on* X *if and only if every open set* U *in* (X, τ) *can be written as a union of members in* β.

Theorem 4. *If* X *is a set, a basis for a topology on* X *is a collection* β *of subsets of* X *such that*

- *for every* $x \in X$ *there exists* $B \subset \beta$ *containing* x.
- *if* $x \in B_1 \cap B_2$, *for some* $B_1, B_2 \in \beta$, *then there exists* $B_3 \in \beta$ *such that* $x \in B_3 \subseteq B_1 \cap B_2$.

Definition 16. *The topology generated by the basis* $\beta = \{[a, b) : a, b \in \mathbb{R}, a < b\}$ *is known as the* ***lower limit topology*** *on* $\mathbb{R}$.
The space $\mathbb{R}$ *equipped with the lower limit topology is known as the* ***Sorgenfrey line*** $\mathbb{R}_l$.

Proposition 1.2.1. *The lower limit topology on* $\mathbb{R}$ *is strictly finer than the usual topology on* $\mathbb{R}$

Definition 17. *Let* $K = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. *Consider the basis* $\beta = \{(a, b) \setminus K : a, b \in r_a ti_o n_a l_s, a < b\} \cup \{(a, b) : a, b \in \mathbb{R} a < b\}$. *The topology generated by the basis* β *is known as the* ***k-topology*** *on* $\mathbb{R}$.

Proposition 1.2.2. *The* k*-topology on* $\mathbb{R}$ *is strictly finer than the usual topology on* $\mathbb{R}$.

Proposition 1.2.3. *The lower limit topology and the* k*-topology on* $\mathbb{R}$ *are incomparable.*

Definition 18. *Let* X *be a set. Let* S *be a collection of subsets of* X *whose union is in* X. *Any such collection is called a* ***subbasis*** *for a topology* X.

Theorem 5. *Let* S *be a subbasis for a topology on* X. *Then* $\tau = \{U \subseteq X : \text{for every } x \in U \text{ there exists a finite number of members } S_1, S_2, ..., S_n \in S \text{ such that } x \in S_1 \cap S_2 \cap ... \cap S_n \subseteq U\}$ *is the topology generated by the subbasis* S.

Theorem 6. *Let* β *be a basis for a topology on* X. *Then the topology generated by* β *is equal to the intersection of all topologies on* X *that contains* β.

Theorem 7. *If* β *is a basis for a topology on a set* X, *then the topology generated by* β *on* τ *is the smallest topology on* X *that contains* β.

[TODO: Notes from 21st and 22nd August]

Definition 19. *Let* J *be a directed set with a partial order relation '*$\leq$*'. A subset* K *of* J *is said to be* ***cofinal*** *in* J *if for each* $\alpha \in J$, *there exists* $\beta \in K$ *such that* $\alpha \leq \beta$.

Proposition 1.2.4. *If* $g \colon \mathbb{N} \to \mathbb{N}$ *is a strictly increasing function then* $g(\mathbb{N})$ *is cofinal in* $\mathbb{N}$.

Theorem 8. *If* J *is a directed set and* K *is cofinal in* J, *then* K *is a directed set.*

Definition 20. *Let* $f \colon J \to X$ *be a net in* X. *If* I *is a directed set and* $g \colon I \to J$ *such that*

- $i < j \implies g(i) < g(j) \ \forall i, j \in I$
- $g(I)$ *is cofinal in* J

Then $f \circ g \colon I \to X$ *is called a* ***subnet*** *of* f.

Definition 21. *The net* $(X_\alpha)_{\alpha \in J}$ *is said to* ***converge*** *to a point* $x \in X$ *if for every neighbourhood* U *of* x, *there exists* $\alpha_0 \in J$ *such that* $x_\alpha \in U$ *for all* $\alpha \geq \alpha_0$.

Definition 22. *Let (X, τ) be a topological space and $S \subseteq X$. A point $x \in X$ is called a* ***closure point*** *of S if for every neighbourhood U of x, we have $U \cap S \neq \varnothing$. THe set of all closure points of S is called the* ***closure*** *of A and is denoted $cl(A)$ or $\bar{A}$.*

Definition 23. *A point $x \in X$ is said to be a* ***limit point*** *of S if for every neighbourhood of x, we have $(U \cap S) \setminus \{x\} \neq \varnothing$.*
The set of all limit points is called the ***derived set*** *and is denoted S'*

Theorem 9. *Let X be a topological space and $S \subseteq X$. Then $\bar{S}$ is the smallest closed set in X that contains S.*

Proposition 1.2.5. *A subset of a topological space X is closed if and only if $S = \bar{S}$*

Theorem 10. *Let (X, τ) be a metrizable space and $S \subseteq X$. Then $x \in \bar{S}$ if and only if there exists a sequence $\langle x_n \rangle$ in S such that $x_n \to x$.*

Theorem 11. *Let X be a topological space and $S \subseteq X$. Then show that $x \in \bar{S}$ if and only if there exists a net $\langle x_\lambda \rangle$ in S such that $x_\lambda \to x$*

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Exercises

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1. Let (X, d) be a metric space and τ_d be the collection of all open subsets of X. Show that τ_d is a topology on X *This is called the topology on X generated by d*

Source. Class, Aug 07

2. Let X be any set and $A \subseteq X$. Then show that $\tau = \{\varnothing, A, X\}$ is a topology on X.

Source. Class, Aug 07

3. Let X be any set. Let $\tau = \{A \subseteq X : X \setminus A \text{ is finite }\} \cup \{\varnothing\}$ Then show that τ is a topology on X.

Source. Class, Aug 07

4. Let X be any set and $\tau = \{A \subseteq X : X \setminus A \text{ is countable }\} \cup \{\varnothing\}$. Then show that τ is a topology on X.

Source. Class, Aug 07

5. Show that every discrete topological space is metrizable

Source. Class, Aug 07

6. Let X be any set with more than one element. Show that the topology $\tau = \{\varnothing, X\}$ is not metrizable.

Source. Class, Aug 07

7. Let $X = \mathbb{R}^n$.
For any $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$, define

1. $d_1(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$
2. $d_2(x, y) = \sum_{i=1}^{n} |x_i - y_i|$
3. $d_3(x, y) = \max\{|x_i - y_i| : 1 \leq i \leq n\}$

Show that d_1, d_2, d_3 all define the same topology on $\mathbb{R}^n$.

Source. Class, Aug 09

8. Let (X, τ) be a topological space and $A \subseteq X$. Suppose for each $x \in A$, there exists an open set U in X such that $x \in U \subseteq A$. Show that A is open in X.

Source. Problem Sheet 01, Q1

9. Let X be a set, and let $\{\tau_\alpha\}_{\alpha \in I}$ be a collection of topologies on X. Show that $\bigcap_{\alpha \in I} \tau_\alpha$ is a topology on X.

Source. Problem Sheet 01, Q2

10. Show by an example that a union of two topologies on a set X may not be a topology.

Source. Problem Sheet 01, Q3

11. Consider (X, τ) where τ is the discrete topology. Then the collection of closed sets is the power set $P(X)$ which coincides with the collection open sets in (X, τ). Consider (X, τ) where τ is the co-finite topology. Then the collection of closed sets in (X, τ) is $\{A \in \tau : A \text{ is finite }\} \cup \{X\}$.

Source. Class, Aug 11

12. Given a topological space (X, τ), show that:

1. $\varnothing$ and X are closed in X
2. An intersection of any collection of closed sets is closed in X
3. A union of a finite collection closed sets in X is closed in X

Source. Class, Aug 11

13. Consider $(\mathbb{R}, \tau)$ where τ is the usual topology on $\mathbb{R}$. Find A° where:

1. $A = (0, 1)$
2. $A = \mathbb{Q}$

Source. Class, Aug 11

14. Consider $(\mathbb{R}, \tau)$ where τ is the discrete topology on $\mathbb{R}$. Find $\mathbb{Q}^{\circ}$

Source. Class, Aug 11

15. Consider $(\mathbb{R}, \tau)$ where τ is the indiscrete topology on $\mathbb{R}$. Find $\mathbb{Q}^{\circ}$

Source. Class, Aug 11

16. Consider $(\mathbb{R}, \tau)$ where τ is the co-finite topology on $\mathbb{R}$. Find A° where $A = (0, 1)$.

Source. Class, Aug 11

17. Let (X, d) be a topological space and $S \subseteq X$. Then show that S° is the largest open set in X that is contained in S.

Source. Class, 11 Aug

18. Let X be a topological space and $S \subseteq X$. Then show that S is open if and only if $S = S^{\circ}$.

Source. Class, Aug 11

19 (Stub for Problem Sheet 1, Q6)**.** and remaining from Problem Sheet 1 and 2

Source.

20. Let τ be a topology on X consisting of four sets $\tau = \{X, \phi, A, B\}$, where A, B are non-empty, distinct proper subsets of X. What conditions must A and B satisfy?

Source. Schaum's P73, Q3

21. Let $f\colon X \to Y$ be a function from a non-empty set X into a topological space (Y, σ). Furthermore, let τ be the class of inverses of open subsets of Y: $\tau = \{f^{-1}(G) : G \in \sigma\}$ Show that τ is a topology on X.

Source. Schaums, P74, Q5

22. Let A be a subset of a topological space in X with the property that each point $p \in A$ belongs to an open set G_p contained in A. Then prove that A is open.

Source. Schaums, P74, Q8

23. Let τ be the class of subsets of $\mathbb{R}$ consisting of $\mathbb{R}$, $\varnothing$ and all open infinite intervals $E_a = (a, \infty)$ with $a \in \mathbb{R}$. Show that τ is a topology on $\mathbb{R}$.

Source. Schaum's P75, Q9

24. Let $X = \mathbb{R}$ and $\tau_1 = P(X)$ and $\tau_2 =$ usual topology .
Does $\beta = \{(a, b) : a, b \in \mathbb{R}, a < b\}$ form a basis for τ_1 and τ_2?

Source. Class, Aug 14

25 (Stub for Problem 4, 7, 8 from Problem Sheet 01)**.**

Source. Problem Sheet 01

26. For $\mathbb{R}$ with the usual topology, is $S = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$ a basis? Is it a subbasis?

Source. Class, Aug 14

27. Let (X, τ) be a topological space and β be a collection of open sets in (X, τ). Show that β is a basis for the topology τ on X if and only if every open set U in (X, τ) can be written as a union of members in β.

Source. Class, Aug 14

28. If X is a set, then show that a basis for a topology on X is a collection β of subsets of X such that

- for every $x \in X$ there exists $B \subset \beta$ containing x.
- if $x \in B_1 \cap B_2$, for some $B_1, B_2 \in \beta$, then there exists $B_3 \in \beta$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Source. Class, Aug 16

29. Show that the topology generated by $\beta = \{(a, b) : a, b \in \mathbb{R}, a < b\}$ is the usual topology.

Source. Class, Aug 16

30. Show that $\beta = \{[a, b) : a, b \in \mathbb{R}, a < b\}$ is a basis for a topology on $\mathbb{R}$.

Source. Class, Aug 16

31. Show that the lower limit topology on $\mathbb{R}$ is strictly finer than the usual topology on $\mathbb{R}$.

Source. Class, Aug 16

32. Let $K = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Consider the collection $\beta = \{(a,b) \setminus K : a, b \in \mathbb{Q}, a < b\} \cup \{(a,b) : a, b \in \mathbb{R} a < b\}$. Show that the β is a basis for a topology on $\mathbb{R}$.

Source. Class, Aug 18

33. Show that the k-topology on $\mathbb{R}$ is strictly finer than the usual topology.

Source. Class, Aug 18

34. Show that the lower limit topology and the k topology on $\mathbb{R}$ are incomparable.

Source. Class, Aug 18

35. Let S be a subbasis for a topology on X. Then show that
$\tau = \{U \subseteq X$: for every $x \in U$ there exists a finite number of members $S_1, S_2, ..., S_n \in S$ such that $x \in S_1 \cap S_2 \cap ... \cap S_n \subseteq U\}$ is a topology generated by the subbasis S.

Source. Class, Aug 18

36. Let β be a basis for a topology on X. Show that the topology generated by β is equal to the intersection of all topologies on X that contains β.

Source. Class, Aug 18

37 (Stub for Problem 8 from Sheet 1)**.**

Source. Problem Sheet 01, Q8

38. Show that if J is a directed set and K is cofinal in J, then K is a directed set.

Source. Class, Aug 23

39. Show that a subnet of a convergent net is convergent.

Source. Class, Aug 23

40. Let X be an infinite set and τ be the co-finite topology. Then find the closure of X (?)

Source. Class, Aug 23

41. Show that $\bar{S} = S' \cup S$.

Source. Class, Aug 23

42. Let (X, τ) be a metrizable space and $S \subseteq X$. Then show that $x \in \bar{S}$ if and only if there exists a sequence $\langle x_n \rangle$ in S such that $x_n \to x$.

Source. Class, Aug 28

43. Let $X = \mathbb{R}$ and τ be the co-countable topology. Consider $S = \mathbb{R} \setminus \{0\}$. Does there exist a sequence $\langle x_n \rangle$ in S that such that $x_n \to 0$?

Source. Class, Aug 28

44. Let X be a topological space and $S \subseteq X$. Then show that $x \in \bar{S}$ if and only if there exists a net $\langle x_\lambda \rangle$ in S such that $x_\lambda \to x$

Source. Class, Aug 28