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Definitions and Theory

1.1 Topological Spaces

Definition 1. A **topology** on a set X is a collection τ of subsets of X satisfying:

1. $\emptyset, X \in \tau$
2. An intersection of finite subcollections of τ is in τ
3. A union of any subcollection of τ is in τ

The ordered pair (X, τ) is called a **topological space**.

Definition 2. Let (X, τ) be a topological space. An **open subset** of X is a member of τ .

Definition 3. Let τ and σ be two topologies on a set X . We say that τ is **weaker** (or smaller, coarser) than σ if $\tau \subseteq \sigma$. In this case, σ is then said to be **stronger** (or larger, finer) than τ .

Definition 4. Let X be any set. The collection $\tau = P(X)$ is a topology on X and is called the **discrete topology** on X . Here (X, τ) is called the **discrete topological space**.

Definition 5. Let X be any set. The collection $\tau = \{\emptyset, X\}$ is called the **indiscrete topology** on X . Here (X, τ) is called the **indiscrete topology**.

Definition 6. Let X be any set. The collection $\tau = \{A \subseteq X : X \setminus A \text{ is finite}\} \cup \{\emptyset\}$ is called the **co-finite topology**.

Definition 7. Let X be any set. The collection $\tau = \{A \subseteq X : X \setminus A \text{ is countable}\} \cup \{\emptyset\}$ is called the **co-finite topology**.

Definition 8. A topology τ on a set X is said to be **metrizable** if there exists a metric d on X such that the topology τ_d generated by the metric d coincides with τ .

Definition 9. Two metrics defined on a set X are said to be **equivalent** if they generate the same topology. In other words, d_1 and d_2 are equivalent if the collection of open sets in (X, d_1) and (X, d_2) are the same.

Definition 10. The topology generated by the Euclidean metric on \mathbb{R}^n is called the **usual topology** on \mathbb{R}^n . For $Y \subseteq \mathbb{R}^n$, the topology generated by the Euclidean metric is called the usual topology on Y .

Definition 11. By a **neighbourhood** of a point x in a topological space (X, τ) , we mean an open set containing x .

Definition 12. A subset A of a topological space (X, τ) is said to be **closed** if $X \setminus A$ is open in X , that is $X \setminus A \in \tau$

Proposition 1.1.1.

- \emptyset and X are closed in X
- An intersection of any collection of closed sets is closed in X
- A union of a finite collection of closed sets in X is closed in X

Definition 13. Let X be a topological space and $A \subseteq X$. A point $x \in X$ is said to be an **interior point** of A if there exists a neighbourhood U of x such that $U \subseteq A$

or in other words, if there exists an open set U in X such that $x \in U \subseteq A$.

The set of all interior points of A is called the **interior** of A and is denoted A° .

Theorem 1. Let (X, d) be a topological space and $S \subseteq X$. Then S° is the largest open set in X that is contained in S .

Theorem 2. Let X be a topological space and $S \subseteq X$. Then S is open if and only if $S = S^\circ$.

1.2 Bases and Subbases

Definition 14. Let X be a topological space. A collection β of open subsets of X is said to be a **basis** for the topology on X if for every open set U in X and $x \in U$, there exists a $B \in \beta$ such that $x \in B \subseteq U$.

Members of β are called **basis open sets** corresponding to basis β .

Note 1. For any topological space (X, τ) , τ is a basis for τ .

Note 2. In \mathbb{R} the set $\{(x - \varepsilon, x + \varepsilon) : x \in \mathbb{R} \text{ and } \varepsilon > 0\}$ is a basis for the usual topology.

Note 3. For (\mathbb{R}, τ) , where τ is the discrete topology, $\beta = \{\{x\} : x \in \mathbb{R}\}$ is a basis for τ on \mathbb{R} .

Note 4. In \mathbb{R} the set $\{(x - \frac{1}{n}, x + \frac{1}{n}) : x \in \mathbb{R} \text{ and } n \in \mathbb{N}\}$ is a basis for the usual topology.

Note 5. In \mathbb{R} the set $\{(a, b) : a, b \in \mathbb{Q} \text{ and } a < b\}$ is a basis for the usual topology. This is a countable basis for \mathbb{R} with the usual topology.

Note 6. Let (X, τ) be a metrizable space. Then $\beta = \{B(x, \varepsilon) : x \in X, \varepsilon > 0\}$ is a basis for τ .

Definition 15. Let (X, τ) be a topological space. A collection $S \subseteq \tau$ is said to be a **subbasis** for the topology τ if for every open set U in τ and $x \in U$, there exists a finite subcollection $\{s_1, s_2, \dots, s_n\}$ in S such that $x \in \bigcap_{i=1}^n S_i \subseteq U$. Members of S are called **subbasis open sets** corresponding to S .

Note 7. For a topological space (X, τ) a collection S is a subbasis for the topology τ if and only if the collection of all finite intersections of members in S forms a basis for the topology τ .

Theorem 3. Let (X, τ) be a topological space and β be a collection of open sets in (X, τ) .

Then β is a basis for the topology τ on X if and only if every open set U in (X, τ) can be written as a union of members in β .

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Exercises

1. Let (X, d) be a metric space and τ_d be the collection of all open subsets of X . Show that τ_d is a topology on X . *This is called the topology on X generated by d*

Source. Class, Aug 07

2. Let X be any set and $A \subseteq X$. Then show that $\tau = \{\emptyset, A, X\}$ is a topology on X .

Source. Class, Aug 07

3. Let X be any set. Let $\tau = \{A \subseteq X : X \setminus A \text{ is finite}\} \cup \{\emptyset\}$. Then show that τ is a topology on X .

Source. Class, Aug 07

4. Let X be any set and $\tau = \{A \subseteq X : X \setminus A \text{ is countable}\} \cup \{\emptyset\}$. Then show that τ is a topology on X .

Source. Class, Aug 07

5. Show that every discrete topological space is metrizable

Source. Class, Aug 07

6. Let X be any set with more than one element. Show that the topology $\tau = \{\emptyset, X\}$ is not metrizable.

Source. Class, Aug 07

7. Let $X = \mathbb{R}^n$.

For any $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, define

1. $d_1(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$

2. $d_2(x, y) = \sum_{i=1}^n |x_i - y_i|$

3. $d_3(x, y) = \max\{|x_i - y_i| : 1 \leq i \leq n\}$

Show that d_1, d_2, d_3 all define the same topology on \mathbb{R}^n .

Source. Class, Aug 09

8. Let (X, τ) be a topological space and $A \subseteq X$. Suppose for each $x \in A$, there exists an open set U in X such that $x \in U \subseteq A$. Show that A is open in X .

Source. Problem Sheet 01, Q1

9. Let X be a set, and let $\{\tau_\alpha\}_{\alpha \in I}$ be a collection of topologies on X . Show that $\bigcap_{\alpha \in I} \tau_\alpha$ is a topology on X .

Source. Problem Sheet 01, Q2

10. Show by an example that a union of two topologies on a set X may not be a topology.

Source. Problem Sheet 01, Q3

11. Consider (X, τ) where τ is the discrete topology. Then the collection of closed sets is the power set $P(X)$ which coincides with the collection open sets in (X, τ) . Consider (X, τ) where τ is the co-finite topology. Then the collection of closed sets in (X, τ) is $\{A \in \tau : A \text{ is finite}\} \cup \{X\}$.

Source. Class, Aug 11

12. Given a topological space (X, τ) , show that:

1. \emptyset and X are closed in X
2. An intersection of any collection of closed sets is closed in X
3. A union of a finite collection closed sets in X is closed in X

Source. Class, Aug 11

13. Consider (\mathbb{R}, τ) where τ is the usual topology on \mathbb{R} . Find A° where:

1. $A = (0, 1)$
2. $A = \mathbb{Q}$

Source. Class, Aug 11

14. Consider (\mathbb{R}, τ) where τ is the discrete topology on \mathbb{R} . Find \mathbb{Q}°

Source. Class, Aug 11

15. Consider (\mathbb{R}, τ) where τ is the indiscrete topology on \mathbb{R} . Find \mathbb{Q}°

Source. Class, Aug 11

16. Consider (\mathbb{R}, τ) where τ is the co-finite topology on \mathbb{R} . Find A° where $A = (0, 1)$.

Source. Class, Aug 11

17. Let (X, d) be a topological space and $S \subseteq X$. Then show that S° is the largest open set in X that is contained in S .

Source. Class, 11 Aug

18. Let X be a topological space and $S \subseteq X$. Then show that S is open if and only if $S = S^\circ$.

Source. Class, Aug 11

19 (Stub for Problem Sheet 1, Q6). and remaining from Problem Sheet 1 and 2
Source.

20. Let τ be a topology on X consisting of four sets $\tau = \{X, \phi, A, B\}$, where A, B are non-empty, distinct proper subsets of X . What conditions must A and B satisfy?

Source. Schaum's P73, Q3

21. Let $f: X \rightarrow Y$ be a function from a non-empty set X into a topological space (Y, σ) . Furthermore, let τ be the class of inverses of open subsets of Y : $\tau = \{f^{-1}(G) : G \in \sigma\}$ Show that τ is a topology on X .

Source. Schaums, P74, Q5

22. Let A be a subset of a topological space in X with the property that each point $p \in A$ belongs to an open set G_p contained in A . Then prove that A is open.

Source. Schaums, P74, Q8

23. Let τ be the class of subsets of \mathbb{R} consisting of \mathbb{R} , \emptyset and all open infinite intervals $E_a = (a, \infty)$ with $a \in \mathbb{R}$. Show that τ is a topology on \mathbb{R} .

Source. Schaum's P75, Q9

24. Let $X = \mathbb{R}$ and $\tau_1 = P(X)$ and $\tau_2 =$ usual topology .

Does $\beta = \{(a, b) : a, b \in \mathbb{R}, a < b\}$ form a basis for τ_1 and τ_2 ?

Source. Class, Aug 14

25 (Stub for Problem 4, 7, 8 from Problem Sheet 01).

Source. Problem Sheet 01

26. For \mathbb{R} with the usual topology, is $S = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$ a basis? Is it a subbasis?

Source. Class, Aug 14

27. Let (X, τ) be a topological space and β be a collection of open sets in (X, τ) . Show that β is a basis for the topology τ on X if and only if every open set U in (X, τ) can be written as a union of members in β .

Source. Class, Aug 14