Topology

Definition
A **topology** on a set X is a collection τ of subsets of X satisfying:

1. $\varnothing, X \in \tau$
2. An intersection of finite subcollections of τ is in τ
3. A union of any subcollection of τ is in τ

The ordered pair (X, τ) is called a **topological space**.

Definition
Let (X, τ) be a topological space. An **open subset** of X is a member of τ.

Definition
Let τ and σ be two topologies on a set X. We say that τ is **weaker** (or smaller, coarser) than σ if $T \subseteq \sigma$. In this case, σ is then said to be **stronger** (or larger, finer) than τ.

Definition
Let X be any set. The collection $\tau = P(X)$ is a topology on X and is called the **discrete topology** on X. Here (X, τ) is called the **discrete topological space**.

Definition
Let X be any set. The collection $\tau = \{\varnothing, X\}$ is called the **indiscrete topology** on X. Here (X, τ) is called the **indiscrete topology**.

Definition
Let X be any set. The collection $\tau = \{A \subseteq X : X \setminus A \text{ is finite }\} \cup \{\varnothing\}$ is called the **co-finite topology**.

Definition
Let X be any set. The collection $\tau = \{A \subseteq X : X \setminus A \text{ is countable }\} \cup \{\varnothing\}$ is called the **co-countable topology**.

Definition
A topology τ on a set X is said to be **metrizable** if there exists a metric d on X such that the topology τ_d generated by the metric d coincides with τ.

Definition
Two metrics defined on a set X are said to be **equivalent** if they generate the same topology. In other words, d_1 and d_2 are equivalent if the collection of open sets in $\left(X, d_1\right)$ and $\left(X, d_2\right)$ are the same.

Definition
The topology generated by the Euclidean metric on $\mathbb{R}^n$ is called the **usual topology** on $\mathbb{R}^n$. For $Y \subseteq \mathbb{R}^n$, the topology generated by the Euclidean metric is called the usual topology on Y.

Definition
By a **neighbourhood** of a point x in a topological space (X, τ), we mean an open set containing x.

Definition
A subset A of a topological space (X, τ) is said to be **closed** if $X \setminus A$ is open in X, that is $X \setminus A \in \tau$

Theorem

- $\varnothing$ and X are closed in X
- An intersection of any collection of closed sets is closed in X
- A union of a *finite* collection of closed sets in X is closed in X

Definition
Let X be a topological space. A collection β of open subsets of X is said to be a **basis** for the topology on X if for every open set U in X and $x \in U$, there exists a $B \in \beta$ such that $x \in B \subseteq U$.
Members of β are called **basis open sets** corresponding to basis β.

Note
For any topological space (X, τ), τ is a basis for τ.

Note
In $\mathbb{R}$ the set $\{(x-\varepsilon, x+\varepsilon) : x \in \mathbb{R} \text{ and } \varepsilon > 0\}$ is a basis for the usual topology.

Note
For $(\mathbb{R}, \tau)$, where τ is the discrete topology, $\beta = \{\{x\} : x \in \mathbb{R}\}$ is a basis for τ on $\mathbb{R}$.

Note
In $\mathbb{R}$ the set $\left\{ \left(x - \frac{1}{n}, x + \frac{1}{n} \right) : x \in \mathbb{R} \text{ and } n \in \mathbb{N} \right\}$ is a basis for the usual topology.

Note
In $\mathbb{R}$ the set $\{(a, b) : a, b \in \mathbb{Q} \text{ and } a < b\}$ is a basis for the usual topology. This is a countable basis for $\mathbb{R}$ with the usual topology.

Note
Let (X, τ) be a metrizable space. Then $\beta = \{B(x, \varepsilon) : x \in X, \varepsilon > 0\}$ is a basis for τ.

Definition
Let (X, τ) be a topological space. A collection $S \subseteq \tau$ is said to be a **subbasis** for the topology τ if for every open set U in τ and $x \in U$, there exists a finite subcollection $\{S_1, S_2, ..., S_n\}$ in S such that $x \in \bigcap_{i=1}^{n} S_i \subseteq U$
Members of S are called **subbasis open sets** corresponding to S.

Note
For a topological space (X, τ) a collection S is a subbasis for the topology τ if and only if the collection of all finite intersections of members in S forms a basis for the topology τ.

Theorem
Let (X, τ) be a topological space and β be a collection of open sets in (X, τ).
Then β is a basis for the topology τ on X if and only if every open set U in (X, τ) can be written as a union of members in β.

Theorem
If X is a set, a basis for a topology on X is a collection β of subsets of X such that

- for every $x \in X$ there exists $B \subset \beta$ containing x.
- if $x \in B_1 \cap B_2$, for some $B_1, B_2 \in \beta$, then there exists $B_3 \in \beta$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Definition
The topology generated by the basis $\beta = \{[a, b) : a, b \in \mathbb{R}, a < b\}$ is known as the **lower limit topology** on $\mathbb{R}$.
The space $\mathbb{R}$ equipped with the lower limit topology is known as the **Sorgenfrey line** $\mathbb{R}_l$.

Proposition
The lower limit topology on $\mathbb{R}$ is strictly finer than the usual topology on $\mathbb{R}$

Definition
Let $K = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. Consider the basis
$\beta = \{(a, b) \setminus K : a, b \in \mathbb{Q}, a < b\} \cup \{(a, b) : a, b \in \mathbb{R}, a < b\}$. The topology generated by the basis β is known as the **k-topology** on $\mathbb{R}$.

Proposition
The k-topology on $\mathbb{R}$ is strictly finer than the usual topology on $\mathbb{R}$.

Proposition
The lower limit topology and the k-topology on $\mathbb{R}$ are incomparable.

Definition
Let X be a set. Let S be a collection of subsets of X whose union equals X. Any such collection is called a **subbasis** for a topology X.

Theorem
Let S be a subbasis for a topology on X. Then
$\tau = \{U \subseteq X :$ for every $x \in U$ there exists a finite number of members $S_1, S_2, ..., S_n \in S$ such that $x \in S_1 \cap S_2 \cap ... \cap S_n \subseteq U\}$ is the topology generated by the subbasis S.

Theorem
Let β be a basis for a topology on X. Then the topology generated by β is equal to the intersection of all topologies on X that contains β.

Theorem
If β is a basis for a topology on a set X, then the topology generated by β on τ is the smallest topology on X that contains β.

Definition
Let (X, τ) be a topological space. A collection $\mathscr{B}_x$ of neighbourhoods of x is said to be a **basis at x** if for every neighbourhood U of x, there exists $B \in \mathscr{B}_x$ such that $x \in B \subseteq U$.

Definition
Let X be a set. A function $f : \mathbb{N} \to X$ is called a **sequence** in X.

Definition
Let (X, Y) be a topological space. A sequence $\left(X_n\right)$ is said to **converge** to $x \in X$ if for every neighbourhood U of x, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U \ \forall n \geq n_0$

Definition
A sequence $\left(x_n\right)$ is said to be **eventually constant** if there exists $n_0 \in \mathbb{N}$ such that $x_n = x_{n_0} \ \forall n \geq n_0$.

Theorem
In any topological space, every eventually constant sequence is convergent.

Definition
Let $f : \mathbb{N} \to X$ be a sequence in X. Then for any strictly increasing function $g : \mathbb{N} \to \mathbb{N}$, the composition $f \circ g : \mathbb{N} \to X$ is called a **subsequence** of f.

Theorem
In a topological space (X, τ), every subsequence of a convergent sequence is convergent.

Definition
Let $\mathcal{A}$ be a non-empty set. A relation $\leq$ on a set A is called a **partial order relation** if the following condition holds for all α, β, γ in A:

1. reflexive: $\alpha \leq \alpha$
2. anti-symmetric: $\alpha \leq \beta$ and $\beta \leq \alpha \implies \alpha = \beta$
3. transitive: $\alpha \leq \beta$ and $\beta \leq \gamma \implies \alpha \leq \gamma$

Definition
A **directed set** J is a set with a partial order relation $\leq$ such that for each pair α and β of J, there exists a $Y \in J$ such that $\alpha \leq Y$ and $\beta \leq Y$.

Definition
A **net** in X is a function f from a directed set J to X.

Note
Every sequence is a net.

Definition
Let J be a directed set with a partial order relation '$\leq$'. A subset K of J is said to be **cofinal** in J if for each $\alpha \in J$, there exists $\beta \in K$ such that $\alpha \leq \beta$.

Proposition
If $g : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function then $g(\mathbb{N})$ is cofinal in $\mathbb{N}$.

Theorem
If J is a directed set and K is cofinal in J, then K is a directed set.

Definition
Let $f : J \to X$ be a net in X. If I is a directed set and $g : I \to J$ such that

- $i < j \implies g(i) < g(j) \ \forall i, j \in I$
- $g(I)$ is cofinal in J

Then $f \circ g : I \to X$ is called a **subnet** of f.

Definition
The net $\left(X_\alpha\right)_{\alpha \in J}$ is said to **converge** to a point $x \in X$ if for every neighbourhood U of x, there exists $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$.

Definition
Let (X, τ) be a topological space and $S \subseteq X$. A point $x \in X$ is called a **closure point** of S if for every neighbourhood U of x, we have $U \cap S \neq \varnothing$.
The set of all closure points of S is called the **closure** of A and is denoted $\mathrm{cl}(A)$ or $\bar{A}$.

Definition
A point $x \in X$ is said to be a **limit point** of S if for every neighbourhood of x, we have $(U \cap S) \setminus \{x\} \neq \varnothing$.
The set of all limit points is called the **derived set** and is denoted S'

Theorem
Let X be a topological space and $S \subseteq X$. Then $\bar{S}$ is the smallest closed set in X that contains S.

Proposition
A subset of a topological space X is closed if and only if $S = \bar{S}$

Theorem
Let (X, τ) be a metrizable space and $S \subseteq X$. Then $x \in \bar{S}$ if and only if there exists a sequence $\langle x_n \rangle$ in S such that $x_n \to x$.

Theorem
Let X be a topological space and $S \subseteq X$. Then show that $x \in \bar{S}$ if and only if there exists a net $\langle x_\lambda \rangle$ in S such that $x_\lambda \to x$

Definition
Let X be a topological space and $A \subseteq X$. A point $x \in X$ is said to be an **interior point** of A if there exists a neighbourhood U of X such that $U \subseteq A$ (or) if there exists an open set U in X such that $x \in U \subseteq A$.
The set of all interior points of A is called the **interior** of A and is denoted A°.

Theorem
Let X be a topological space and $S \subseteq X$. Then S° is the largest open set contained in S.

Theorem
Let X be a topological space and $S \subseteq X$. Then S is open if and only if $S = S^{\circ}$.

Definition
Let (X, τ_1) and (Y, τ_2) be two topological spaces. The collection
$\mathcal{B} = \tau_1 \times \tau_2 = \{u \times v : u \in \tau_1, v \in \tau_2\}$ is a basis for a topology on $X \times Y$. The topology generated by $\mathcal{B}$ is called the **product topology** on $X \times Y$.

Theorem
Let (X, τ) and (Y, σ) be two topological spaces. Let $\mathcal{B}$ be a basis for τ and $\mathcal{C}$ be a basis for σ. Then the collection $\mathcal{D} = \{U \times V : U \in \mathcal{B}, V \in \mathcal{C}\}$ is a basis for the product topology on $X \times Y$.

Result
The product topology on $\mathbb{R} \times \mathbb{R}$ coincides with the usual topology on $\mathbb{R}^2$

Note
Missed quotient topology, check.

Definition
Let (X, τ) be a topological space and $Y \subseteq X$.
The collection τ_Y is a topology on Y called the **subspace topology** and with this topology, Y is called a **subspace** of X.

Theorem
Let Y be a subspace of X. Then a subset U of Y is open in Y iff $U = V \cap Y$ for some open set V in X.

Theorem
If $\mathscr{B}$ is a basis for the topology on X, then the collection $\mathscr{B}_Y = \{B \cap Y | B \in \mathscr{B}\}$ is a basis for the subspace topology on Y.

Theorem
Let Y be an open subspace of X. Then a subset U of Y is open in Y iff U is open in X.

Theorem
Let Y be a closed subspace of a topological space X. Then a subset C of Y is closed in Y iff it is closed in X.

Theorem
Let X be a topological space and Y be a subspace of X.
Let $C \subseteq Y$. Then C is closed in Y iff there is a closed set D in X such that $Y \cap D = C$

Theorem
Let X be a topological space and Y be a closed subspace of X. Let $C \subseteq Y$. Then C is closed in Y iff C is closed in X.

Theorem
If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Theorem
Let X be a topological space and Y be a subspace of X. Let $A \subseteq Y$. Then
$\mathrm{cl}_X(A) \cap Y = \mathrm{cl}_Y(A)$

Definition
A function $f : (X, \tau) \to (Y, \sigma)$ is said to be **continuous** at $x_0 \in X$ if for every neighbourhood U of $f(x_0)$, there exists a neighbourhood V of x_0 such that $f(V) \subseteq U$. The function f is said to be **continuous** if it is continuous at each point of X.

Theorem
Let $f : (X, \tau) \to (Y, \sigma)$ be a function from a topological space (X, τ) to a topological space (Y, σ). Then the following are equivalent:

- f is continuous on X
- $f^{-1}(U)$ is an open subset of X whenever U is open in X

- $f(\overline{A}) = \overline{f(A)}$ for every $A \subseteq X$
- $f^{-1}(C)$ is a closed subset of X whenever C is closed in Y
- The net $x_\lambda \to x$ in $X \implies f(x_\lambda) \to f(x)$ in Y

Theorem
The composition of two continuous functions is continuous.

Theorem
The restriction of a continuous function on a topological space to a subspace is continuous.

Theorem
Let (X, τ) and (Y, σ) be topological spaces and $Z = X \times Y$. For a net $(z_\lambda) = (x_\lambda, y_\lambda)$ in Z, $z_\lambda \to z \in Z \iff x_\lambda \to x \in X$ and $y_\lambda \to y \in Y$.

Definition
Let X and Y be topological spaces and let $f : X \to Y$ be a bijection. If both the function f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a **homeomorphism**.

Definition
Let $\{X_\alpha\}_{\alpha \in I}$ be an indexed family of topological spaces. For the product space $X = \prod_{\alpha \in I} X_\alpha$, we take the basis $\mathscr{B} = \{\prod_{\alpha \in I} U_\alpha | U_\alpha \text{ is open in } X_\alpha\}$. The topology generated by this basis is called the **box topology**.

Note
The above is a direct generalization of the earlier product topology defined on product of two spaces.

Theorem
A subset W of X is open in X with the box topology iff for every $x = (x_\alpha)_{\alpha \in I}$, there exists an open set U_α in X_α such that $(x_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} U_\alpha \subseteq W$

Definition
Let $\{X_\alpha\}_{\alpha \in I}$ be an indexed family of topological spaces. For the product space $X = \prod_{\alpha \in I} X_\alpha$, we take the basis
$\mathscr{C} = \{\prod_{\alpha \in I} U_\alpha | U_\alpha \text{ is open in } X, U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \in I\}$.
The topology generated by this basis is called the **product topology** on X.

Note
Unless mentioned otherwise, we usually consider the product topology on the product space X.

Theorem
The product topology on X is weaker than the box topology on X.

Note
When the index I is finite, the product topology and the box topology coincide.

Definition
Let $\{X_\alpha\}_{\alpha\in I}$ be an indexed family of topological spaces and $X = \prod_{\alpha\in I} X_\alpha$ be the product space.
For every $\beta \in I$, the **β-th projection map** $\Pi_\beta : X \to X_\beta$ is defined as $\Pi_\beta((x_\alpha)_{\alpha\in I}) = x_\beta$

Theorem
Let $\{X_\alpha\}_{\alpha\in I}$ be an indexed family of topological spaces and $X = \prod_{\alpha\in I} X_\alpha$ be the product space.
The collection $\mathcal{S} = \{\Pi_\beta^{-1}(U_\beta) | U_\beta \text{ is open in } X_\beta, \beta \in I\}$is a subbasis for the product topology on X.

Theorem
Let $\{X_\alpha\}_{\alpha\in I}$ be an indexed family of topological spaces and $X = \prod_{\alpha\in I} X_\alpha$ be the product space equipped with the product topology.
Then a function $f : A \to X$ is continuous iff $f_\alpha = \Pi_\alpha \circ f : A \to X_\alpha$ is continuous for each $\alpha \in I$.

Theorem
The topologies on $\mathbb{R}^n$ induced by the euclidean metric and the square norm are the same as the product topology on $\mathbb{R}^n$.

Definition
Consider the standard bounded metric on $\mathbb{R}$ defined as $\overline{d} = \min\{|a-b|, 1\}$. Given an index set I and points $x = (x_\alpha)_{\alpha\in I}$ and $y = (y_\alpha)_{\alpha\in I}$, the metric ρ on $\mathbb{R}^I$ defined as $\rho(x, y) = \sup\{\overline{d}(x_\alpha, y_\alpha) | \alpha \in I\}$ is known as the **uniform metric** on $\mathbb{R}^I$.
The topology generated by the uniform metric is called the **uniform topology** on $\mathbb{R}^I$.

Theorem
The uniform topology on $\mathbb{R}^I$ is weaker than the box topology and stronger than the product topology. All these topologies are different if I is infinite.

Definition
Let X and Y be topological spaces and let $p : X \to Y$ be a surjective map.
The map p is said to be a **quotient map** provided a subset U of Y is open in Y iff $p^{-1}(U)$ is open in X.

Note
We can replace 'open' in the above definition with 'closed'.

Note
Every quotient map is a continuous map.

Definition
A map $f : X \to Y$ is said to be **open** if for each open set U in X, $f(U)$ is open in Y. A map $f : X \to Y$ is said to be **closed** if for each closed set C in X, $f(C)$ is closed in Y.

Note
A surjective continuous map which is either open or closed is a quotient map. Note, there are quotient maps which are neither open nor closed.

Definition
If X is a topological space and A is any set, and if $p : X \to A$ is a surjective map, then there exists exactly one topology τ on A relative to which p is a quotient map. This topology is known as the **quotient topology** induced by p.

Definition
Let X be a topological space and $\sim$ be an equivalence relation on X.
Define a map $p : X \to \dfrac{X}{\sim}$ as $p(x) = [x]$ where $[x]$ is the equivalence class of x under $\sim$.
Let τ be the quotient topology induced by p on $\dfrac{X}{\sim}$. Then the space $(\dfrac{X}{\sim}, \tau)$ is called the **quotient space** of X.

Definition
A subset A of a topological space is said to be **dense** if the closure of A in X is equal to X.

Definition
A topological space (X, τ) is said to be **seperable** if it has a countable dense subset.

Note
The discrete topology on X is seperable iff X is countable.

Note
A subspace of a seperable space may not be seperable.

Theorem
A continuous image of a seperable space is seperable.

Definition
A topological space X is said to be **second countable** if it has a countable basis.

Note
The discrete topology on X is second countable iff X is countable.

Theorem
Every second countable space is seperable.

Theorem
Any subspace of a second countable space is second countable.

Theorem
A seperable metrizable space is second countable.

Theorem
A countable product of second countable spaces is second countable.

Definition
A topological space (X, τ) is said to be **first countable** if it has a countable basis at each of its points.

Theorem
Every metrizable space is first countable.

Theorem
Every second countable space is first countable.

Theorem
Any subspace of a first countable space is first countable.

Theorem
Let $f : (X, \tau) \to (X, \sigma)$ be a function from a first countable space X to any topological space Y.
Let $x \in X$. Then f is continuous at x iff whenever $x_n \to x$ in X implies $f(x_n) \to f(x)$ in Y.

Definition
A topological space X is called a $\boldsymbol{T_1}$**-space** if for every pair of distinct points x and y in X, there exists neighbourhoods U of x and V of y such that $x \notin V$ and $y \notin U$.

Theorem
A topological space X is T_1 iff every finite set of X is closed.

Theorem
Let X be a T_1 space and $A \subseteq X$. Then x is a limit point of A iff every neighbourhood of x intersects A infinitely many times.

Definition
A topological space is called a $\boldsymbol{T_2}$**-space** (or **Hausdroff**) if for every pair of distinct points $x, y \in X$, there exists two disjoint open sets U and V such that $x \in U$ and $y \in V$.

Note
Every metrizable space is T_2.

Theorem
Let X be any set and τ_1 and τ_2 be two topologies on X. If $\tau_1 \subseteq \tau_2$ and (X, τ_1) is Hausdroff, then (X, τ_2) is Hausdroff.

Theorem
Let (X, τ) be a Hausdroff space. Then any net in X has a unique limit.

Theorem
Every subspace of a Hausdroff space is Hausdroff.

Theorem
Any arbitrary product of Hausdroff spaces is Hausdroff.

Theorem
Let $f : X \to Y$ be a continuous function from a topological space X to a Hausdroff space Y. Let D be a dense set in X and $y_0 \in Y$.
Then $f(x) = y_0$ for all $x \in D$ implies $f(x) = y_0$ for all $x \in X$

Theorem
Let $f, g : X \to \mathbb{R}$ be two continuous functions from a topological space X to $\mathbb{R}$.
Let D be a dense set in X. Then $f(x) = g(x)$ for all $x \in D$ implies $f(x) = g(x)$ for all $x \in X$.

Definition
A T_1 space X is called a $\boldsymbol{T_3}$**-space** (or **regular**) if for every pair of a point x and a closed set A not containing x, there exists two disjoint open sets U and V such that $x \in U$ and $A \subseteq V$.

Theorem
Every T_3 space is T_2.

Theorem
Let X be a T_1 space. Then X is regular iff given a point x and a neighbourhood U of x, there exists a neighbourhood V of x such that $x \in V \subseteq \overline{V} \subseteq U$.

Theorem
A subspace of a regular space is regular.

Theorem
A product of regular spaces is regular.

Theorem (Urysohn Metrization Theorem)
Every second countable regular space is metrizable.

Definition
A T_1 space X is said to be a ***T*₄-space** (or **normal**) if for every two disjoint closed sets A and B in X, there exists two disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Definition
Every T_4 space is a T_3 space.

Note
Product of normal spaces need not be normal.

Theorem
Every metrizable space is normal.

Theorem
Every second countable regular space is normal.

Theorem
A closed subspace of a normal space is normal.

Theorem (Urysohn's Lemma)
Let X be a normal space and A and B be two disjoint closed subsets of X. Then there exists a continuous map $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.

Definition
A T_1 space X is called a **$T_{3\frac{1}{2}}$-space** (or **completely regular**) if for every point x_0 and a closed set A not containing x_0, there exists a continuous function $f : X \to [0, 1]$ with $f(x_0) = 1$ and $f(x) = 0$ for all $x \in A$.

Note
Every normal space is completely regular.

Note
Every completely regular space is regular.

Theorem
A subspace of a completely regular space is completely regular.

Theorem
An arbitrary product of completely regular spaces is completely regular.

Definition
Let X be a topological space. A collection $\mathscr{A} = \{A_\alpha\}_{\alpha \in I}$ of open subsets of X is said to be an **open cover** of X if $X = \bigcup_{\alpha \in I} A_\alpha$.

By a **finite subcover** of X, we mean a finite subcollection of $\mathscr{A}$ that itself is a cover of A.

Definition
A topological space X is said to be **compact** if everyopen cover of X has a finite subcover.