Contents

1

Definitions and Theory

1.1 Topological Spaces

Definition 1. *A **topology** on a set X is a collection τ of subsets of X satisfying:*

1. *$\varnothing, X \in \tau$*
2. *An intersection of finite subcollections of τ is in τ*
3. *A union of any subcollection of τ is in τ*

*The ordered pair (X, τ) is called a **topological space**.*

Definition 2. *Let (X, τ) be a topological space. An **open subset** of X is a member of τ.*

Definition 3. *Let τ and σ be two topologies on a set X. We say that τ is **weaker** (or smaller, coarser) than σ if $T \subseteq \sigma$. In this case, σ is then said to be **stronger** (or larger, finer) than τ.*

Definition 4. *Let X be any set. The collection $\tau = P(X)$ is a topology on X and is called the **discrete topology** on X. Here (X, τ) is called the **discrete topological space**.*

Definition 5. *Let X be any set. The collection $\tau = \{\varnothing, X\}$ is called the **indiscrete topology** on X. Here (X, τ) is called the **indiscrete topology**.*

Definition 6. *A topology τ on a set X is said to be **metrizable** if there exists a metric d on X such that the topology τ_d generated by the metric d coincides with τ.*

Definition 7. *Two metrics defined on a set X are said to be* ***equivalent*** *if they generate the same topology. In other words, d_1 and d_2 are equivalent if the collection of open sets in (X, d_1) and (X, d_2) are the same.*

Definition 8. *The topology generated by the Euclidean metric on $\mathbb{R}^n$ is called the* ***usual topology*** *on $\mathbb{R}^n$. For $Y \subseteq \mathbb{R}^n$, the topology generated by the Euclidean metric is called the usual topology on Y.*

Definition 9. *By a* ***neighbourhood*** *of a point x in a topological space (X, τ), we mean an open set containing x.*

Definition 10. *A subset A of a topological space (X, τ) is said to be* ***closed*** *if $X \setminus A$ is open in X, that is $X \setminus A \in \tau$*

Definition 11. *Let X be a topological space and $A \subseteq X$. A point $x \in X$ is said to be an* ***interior point*** *of A if there exists a neighbourhood U of X such that $U \subseteq A$, or in other words, if there exists an open set U in X such that $x \in U \subseteq A$. The set of all interior points of A is called the* ***interior*** *of A and is denoted A°.*

2

Exercises

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1. Let (X, d) be a metric space and τ_d be the collection of all open subsets of X. Show that τ_d is a topology on X *This is called the topology on X generated by d*

Source. Class, Aug 07

2. Let X be any set and $A \subseteq X$. Then show that $\tau = \{\varnothing, A, X\}$ is a topology on X.

Source. Class, Aug 07

3. Let X be any set. Let $\tau = \{A \subseteq X : X \setminus A \text{ is finite }\} \cup \{\varnothing\}$ Then show that τ is a topology on X. *This is called the co-finite topology or finite-complement topology*

Source. Class, Aug 07

4. Let X be any set and $\tau = \{A \subseteq X : X \setminus A \text{ is countable }\} \cup \{\varnothing\}$. Then show that τ is a topology on X. *This is called the co-countable topology on X*

Source. Class, Aug 07

5. Show that every discrete topological space is metrizable

Source. Class, Aug 07

6. Let X be any set with more than one element. Show that the topology $\tau = \{\varnothing, X\}$ is not metrizable.

Source. Class, Aug 07

7. Let $X = \mathbb{R}^n$.
For any $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$, define

1. $d_1(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$
2. $d_2(x, y) = \sum_{i=1}^{n} |x_i - y_i|$
3. $d_3(x, y) = \max\{|x_i - y_i| : 1 \leq i \leq n\}$

Show that d_1, d_2, d_3 all define the same topology on $\mathbb{R}^n$.

Source. Class, Aug 09

8. Let (X, τ) be a topological space and $A \subseteq X$. Suppose for each $x \in A$, there exists an open set U in X such that $x \in U \subseteq A$. Show that A is open in X.

Source. Problem Sheet 01, Q1

9. Let X be a set, and let $\{\tau_\alpha\}_{\alpha \in I}$ be a collection of topologies on X. Show that $\bigcap_{\alpha \in I} \tau_\alpha$ is a topology on X.

Source. Problem Sheet 01, Q2

10. Show by an example that a union of two topologies on a set X may not be a topology.

Source. Problem Sheet 01, Q3

11. Consider (X, τ) where τ is the discrete topology. Then the collection of closed sets is the power set $P(X)$ which coincides with the collection open sets in (X, τ). Consider (X, τ) where τ is the co-finite topology. Then the collection of closed sets in (X, τ) is $\{A \in \tau : A \text{ is finite }\} \cup \{X\}$.

Source. Class, Aug 11

12. Given a topological space (X, τ), show that:

1. $\varnothing$ and X are closed in X
2. An intersection of any collection of closed sets is closed in X
3. A union of a finite collection closed sets in X is closed in X

Source. Class, Aug 11

13. Consider $(\mathbb{R}, \tau)$ where τ is the usual topology on $\mathbb{R}$. Find A^o where:

1. $A = (0, 1)$
2. $A = \mathbb{Q}$

Source. Class, Aug 11

14. Consider $(\mathbb{R}, \tau)$ where τ is the discrete topology on $\mathbb{R}$. Find $\mathbb{Q}^o$

Source. Class, Aug 11

15. Consider $(\mathbb{R}, \tau)$ where τ is the indiscrete topology on $\mathbb{R}$. Find $\mathbb{Q}^{\mathrm{o}}$

Source. Class, Aug 11

16. Consider $(\mathbb{R}, \tau)$ where τ is the co-finite topology on $\mathbb{R}$. Find A^{o} where $A = (0, 1)$.

Source. Class, Aug 11

17. Let (X, d) be a topological space and $S \subseteq X$. Then show that S^{o} is the largest open set in X that is contained in S.

Source. Class, 11 Aug

18. Let X be a topological space and $S \subseteq X$. Then show that S is open if and only if $S = S^{\mathrm{o}}$.

Source. Class, Aug 11

19. Let τ be a topology on X consisting of four sets $\tau = \{X, \phi, A, B\}$, where A, B are non-empty, distinct proper subsets of X. What conditions must A and B satisfy?

Source. Schaum's P73, Q3

20. Let $f\colon X \to Y$ be a function from a non-empty set X into a topological space (Y, σ). Furthermore, let τ be the class of inverses of open subsets of Y: $\tau = \{f^{-1}(G) : G \in \sigma\}$ Show that τ is a topology on X.

Source. Schaums, P74, Q5

21. Let A be a subset of a topological space in X with the property that each point $p \in A$ belongs to an open set G_p contained in A. Then prove that A is open.

Source. Schaums, P74, Q8

22. Let τ be the class of subsets of $\mathbb{R}$ consisting of $\mathbb{R}$, $\varnothing$ and all open infinite intervals $E_a = (a, \infty)$ with $a \in \mathbb{R}$. Show that τ is a topology on $\mathbb{R}$.

Source. Schaum's P75, Q9