Discre**te Mathematics**

**Semigroups an*d* Groups**

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**Prop*erties* o*f Bi*nary O*perations***

**There are many properties of the binary operati**ons which are asi. follows: 1. Closure Property*:*

Consider a non-empty set A and a binary operation \* on A, then **A closed under the o*p*eration \***

if a . bea, where á and b are elements of A 1982

S*EMIGROUP A*N*D GROUP*S

9:1 Binary Operations

Consider a non-empty set 'A' and a function 'f srch that f: A A → A is called a binary operation on 'A'. If \* is a binary operation on A, then it may be written as a = b.

A bin**ary operation can be denoted by any of the s*y*mbols** +,-, \*, ,9,0;v, n etc. The value of the binar**y operation is** de:zcied by placing the operator between the two ope**rands.**

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For example: The operation of addition is a binary operation on the set of natural number. i.c. a + b.

Ex. : Consider the set. A = {1, 2, 3) and a binary o**peration \* on** the set A defined by a \* b = 2a + 26: Represent operation \* as a table on A. Solation : The table of the operation is shown in below:

a = 1, b=1=a\*b = 2a + 2b = 2(1)+2(1)= 4. a = 1, .b=25 2a +2b = 2(1)+2(1) = 2+4 = 621

mi 2. 3. :

*14 6 3* 2*16*1:8 10

**For example** : The speration of addition on the set of integet.isla **closed operation.** . Ex. : Consider the se:A= (*-1,*0,1). Determine whether A' is close*d* under (i) addition (ü):nultiplication. **Solution** : Given th it: Set A = *(*-1,0, 1} (i) The sum **of the elements is**

(-1)+(-1)=-2. and

1+1=2 does not belong to . Hence A is not **closed under addit**ion. ; (ii) The multiplicatio**n of every two elemen**ts of the set are :

-1\* 0=0 -1+1=-, -1\* -1=1 0\* -1=0 : 0 + 1 = 0 , 0.\* )=0

1\*-=-i 1 + 0 = 0. 1\*1=1:1 Since, cach multiplication belongs to A... **E*l*onna** A is closed undei multiplication.

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**2. Associative Property:** . . .

Consider a non-empty set A and a binary operation \* on A, then the operation \* on A is associative, if for every a, b, CEA,... we have (a + b) \* C=a\* (b\* c).

Ex.: Consider the binary operation \* on V, UIC stuur owww.. defined by a - ib=a? +62 V a, b EQ. : Determine whether \* is commutative. **Solution : Let us assume some e**lements 2, 6EQ, then by definition

a\*b = a? + b2 = 6\* a Hence, \* is commutative.

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Ex.: Consider the binary operation \* on Q, the set of rational number defined by a.\* b = 36 V a, b EQ **Determine** whether \* is (i) associative (ii) commutative. Solution : (i) Let 2 DEQ, then we have

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Ex. : Consider the binary operation on Q, the set of rational number defined by a \* b=a+b-ab W'a, b EQ.... **Determine whether \* is associative. Solution : Let us assume somc element**s 2.b,c EQ, . thien by definition . .,

(a + b) \* c=(a+b-ab) \*C

= (a + b-ab)+c-(a+b-ab)c

· = a +1b-ab+c-ca - bc + abc

=a+b+c-ab-ac-bc+abc -- () Safnilərly, we have

a \* (6 \* C) =a +b+c-ab-ac- bc+abc ------ (ii) f rom equation (i) and ii)

*Th*erefore, (a + b) \* ¢=a\* (6 \*C) **Hence***,* \* is associative.

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**Hence, \* is commutative.** (ii). Let a, b, c EQ, then definition, we have

(a\*\*b) \* c= (0) - c

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Sim*ilarly*

*.*

**Commutativ**e Property:.

Consider a non wimply set A and a binary operation \* on A, then *the op*eration \* in A is commutative, .. *if f*or every a, b eA. **we have** a + 1 = ht: ::

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**Se*migrou*ps and Groups**

**Discrete Mathematics**

**Therefore,**

a " (b\* c) = 2 \* (b + c) Hence, \* is associative.

4. Identity :

**Consider a non-empty set** A and a binary operation \* on A, then the operation **has on identity property if there exists an element l**e in A such that

a. e (right identity) = e. a (left identity).

= a Va ea

Similarly, . a\*e=a, a el . and = a, e=2 -----(2) From (1) and *(2*) for e = 2; we have

e\* a = a \*e='a.= 2 **The*ref*ore, *2 is* the *i*dentity element for \*.** 5. · Inverse: ....

**Consider a non-empty set A and a b**inary operation \* on A. iren operation \* has the inverse property if for each a EA; there exists **an ele**ment 'b' in *'*A' such that ....

**a \* b *(*right inverse) = b \*'a*. (*Left inverse)**

**e** where b is called an invere *of '*a'

Theorem : Prove that e; = ewhere ei is a right identity and . e' is a left identity of a binary operation. . Proof: We know that -es is a right identity... Hence, ' e\* e; = ; Also, we know that e" is a left identity. Hence, e'\* ej = ; Thus, **we can say that** if 'e' is a right identity of a binary operation. then 'e' is also a left identity or there is no left identity.

**6. Idempotent :** . Consider ä non-empty set A and a binary operation \* on A. *A* and a bir siy operation \* on A; then the operation \* has the idemo . property,. ..... :

**if for each a EA, we have**

\*\*.a - a Va eA]

Ex. : Consider the binary operation \* on I,, the set of positive int**egers def**ined loy .

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Determine the id:niity for the binary operation \* , if-exists. Solution : Let us a:l!ne that 'e' be a positive integer number, then

e\* a = i). it and = a, -=? ------ (1)

*. 7.* Distributivity :

. Consider a non-empty set A and two binary operations operation,+ on A, then the operation \* distributes over + if for every a, b, c ea, w**e have**

a wcb + c) = (a + b)+(a + c) (Left distributivity) | and (b+c) \* 4 = *(*b + a*)*+(\* a) Right distributivity]

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6. Cancellation :

:::.. ! Iconsi**der a non-empty set A and a binary operation \* on A, then *t*he operation has the cancellation property,**

folievery a, b, c ea, we have \*\*\*

a b =a\* > b=c (Left cancellatiou] ande:5 á = c\* a soc"(Right cancellation]

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So, as canu - "w --- **Finally, we mu*s*t show that**, *f*or a*ll* a and b in A, .. . a.\* b = a žo *(*the ***gr*eates***t lo*wer bound o*f* a and b with *r*espect 10 31 We have

a\*bra \* (b + b)

1 = (a \* b) \* 6 *.* So, a + bis b.

In a similar way, we can show that a + bsa So a \* b is a lower bound for a and b. Now,

if csa and csb, then

c=cra and crc\* b [by definition] Thus, c= (©\* a) \* 6

=c\* (a + b) So, csa\* b. This show that a \* b is t**he greatest l**ower bound of a ard b.

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Ex. vietbe a binary operation on a set A and suppose that \* satisfies the following properties for any a, b and c in A.

1) a = a\* a ; Idempotent propery (2) a \* b.= h \* a

Commutative property (3) 4 \* (b\* c)=(a\*b) \*c A**ssociative property** Define a relation s on A by a s b if and only if a = a \* b. Show.chat (A, 5) is a. poset, and for all a, b in A, GLB (a, b) = a\*b Solution : We must show that s is r**eflexive, antisymmetri**c and ***p*ansitive.** Since : a = a \* a, usa for all a in A and s is reflexive. Now! suppose that a s b and b s a. *S*h*e*n, by detinition :

ja = a \* b = b maab

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is commutative and

Ex.: Determine whether the binary operation whether it is associative on the ser. (i) On 2\* , 'where a\*b is a + b + 2 **Solution :** (a) Let a; b €2+, then we have

a\* b = a +b+? = b \* a Hence, \* is commutative.

*mişoa =b*

*..*

be

Trus sis antisynimetric., :: If a's b and bsc, then

\*

a\* *(*b\*ic)

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EZT, then by: detinition, we have

**\**b)***

**c**

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(a\*b) \* c = (a +b+2) \* C

= 2.0+is.c+?.6

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Se*mig*roup**s and *Gr*oups**

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Discrete Mathematics

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a» (b\* c) = (a \* b) \* C. . . := (a +b+*2)* \* C

=C+b.co+2.c .: From (1) and (2)

(a - b) - c=a - (b + c) Hence it is associative. (ii) On 2, where a + b is abi Solution : (a) Let a, b EZ, then we have

8- b= ab = b \* & Hence, is commutative.

(ii) Let a, b ER, then by defination, we have

(a - b) \* c = (axl61) \* C

= ac x 161 C ------ (1*)* Similarly,

a \* (b + c) = 2 \* (b + c)

= (a + b) \* C = (a + [b]) \* c

= ac \* 161 C---- *(2)* :: From (1) and (2) ...

(a \* b) \* C=a\* (b + *c)* Hence \* is associative. *(*iv) On R, where a \* b is the minimum of a and b. Solution : (a) Let a, b ER, then we have

· a + b = min {, b} = b \*a . .

**Hence, \* is commutative.** (a) Let a, b ER, then by defination, we have

(a + b) \* C = min. {a, b} \* C ----- (i) Similarly,

a \* (b + c) =a \* (b + c)

= (a +-6) \* C :.- = mín.{a, b} \* C 1. From (i) and (ii) we have : . (a\*b).\* c=a\*(b\* c)

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(1)

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(b) Lei a, hez, then we have (a + b) \* c = (ab) \* C

= abc Similarly,

a \* (b + c)= a \* (b + c)

:=: (a + b)\* c

= (ab) \* C

"=abc : From (1) and (*2) He*rce\* is associative.

.... (2*)*

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On R, where a \* b is a x.16*1*

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**Se*m*igroups an*d Grou*p*s***

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(3) b \* c = c + b (by commutative) From given : b \* c = a.

so c\* b = a

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x: : Fill in the following table, so that the binary operation \*, is : coinmutative

i la b c

ab b c i' a 1 :

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a b c bc ba *:*

C la a

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Solution : Given that

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Since, the entries in the table ar**e s*y*mmetric with respect t**o main diagonal. .: + is commutative.

TO . Ic: b

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Srom given :

a\* a a \*a= b\* a=c b\*b2b

not Semigroup w116dim

Let S be a non-enıpty set and be a binary operation on S. The algebric system (S, \*) is called a senrigrou*p* if the operation is: (1) The operation - is a closed operatior. on set A. (ü) Tie operation is an associative operation. or (S, \*) is a semigroup if for any x, y, z ES,

(x + y) + z = x + (y + 2) |

h\* c= a

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C\*a = a

co c= c We have to find a \* b = ?, Q\* C-?, C\*b = *?* We know that, : (1) a "b-b\* a (by commutative)

From given : bna'da

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Free Semigroup :

f. is an associative binary operation and (A , •) is a Senugorun. The semigroup ta') is called free Semigroup generated by A. ?

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Ex.: Consideran algebric system (S, +) where $ =(1,1,1,1,1.9...' the set of all positive odd integers and is a binary operation means : multiplication. Determine whether (S. \*) is a semigroup.

*(*2) C16\* in Lby commutative) *F*ioni given : Cha = c

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a \* c = a

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Solution : closure Property : The operation \* is a closed operation because multiplication of two. Hve cdd integers is a tve odd number. i.e. 1 - 3 = 3, 3 \* 5 = 15, *5*\* *7* = 35 ---- . . **Associative Property :** The operation is associative since we have

(a + b) \* c = a + (b + c) y a, b, c. Since, the algebric system is closed and associative. Hence, it is a semigroup.

Semigrou**ps and Groups**

**Erampreet Fosemiocup** Ex. : \_Let *(*A, \*) b*e s*emigroup. *S*how that for a, t \*\*C-C\*a and b \*C=C \*b, th*en (*a \* b) \* C=C \* Solution : Given that lei (A, \**)* b**e *s*emigroup**

a\* c=c\* & and b\*c=c\*b. Take LHS, we have, . .

(a + b) \* c = a\* (b\*c) 1\*\* is a associative] = a\* *(*c\* b)

[: b\*C=C \* b] = (a + c) \* b

[:\* \* is associative] = *(*C\* a) \* b . *:* (ai c= c \* a] > c\* (a + b)

. [:\* is associati*v*e] : „which is equal to RHS... Hence; (amb).. CEC\* (a + b37.;.

Ex.: Consider the algebric system ({0,1)), where is a multiplication operation. Determine whether ({0, 1}; \*) is a semigroup. Solution : Closure Property : Since. 0 + 0 = 0 ;

0.1 = 0; 1 + 0 = 0;

1 \* 1 = 1; Hence, The operation \* is a closed one on the given set. Associative Property : The operation - is a**ssociative** Since we have

*(*a \* b \* c = a + b + c) V a, b, c. *"* ... olanhain critem is closed and associ**ative.** .

Subsemigroup

Let (S, \*) be **a semigroup and** TSS, if the set T is closed under the op**eration \* then (T, \*) is said t**o be subsemigroups of (S, \*).

. : **For example : Consider a şemigrou**p A, +), where N is the set of all natural numbers and + is an addition operation. The algebric system (E, +) is a subsemigroup of (N, +), where E is set of all +ve even integer.

:: Product of Semigroup Theorem : If (S,, \*) and (S2, +) are semigroups then (S, RS26*2)* is a semigroup, w**hère - is defined by** .. (S, S4) \* (S1, S2)=(S \* Ss. Sa \* S).

*A*SS. b. Cel cini. 02

**Associativity of** Ler a, b, c es, xsz

i a. (b + c) = (2,. 22) - ((b, bz) (9,,:C2*))*

=(a,. 82) \* (b... 67, 62.2 C2) = (2,0, (6,6,6,), 12.2 (by .? Cz) =(a,.;b,)\*,, (az .2 b2*)* \*Cy) = (a,., bj, 82.2 bx) \* (C1, C2): = ((ay, ay) \* (6,bz)) \* (c.cz)

= (a \* b) \* C Since, \* is closed and associ**ative.** Hence, S, \*S**is a semigroup.**

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**There exists an i*d*enuity element in sec N w.ric, the opera*von* - The element 0 is an identity element w.r.t. the operation +.** Since, the operation + is a closed, associative and **There exists a**n identity. Hence, 'the algebric systèni (N, +) is a nonrid.

**Theorem** : Let (S. Ə) be a given semigroup there exists ai. homomorphism 8:5 5s, where *1$*$, \*) is a semigroup of function from S to under the operation o*r (*left) composition. Proof : (i) For any element à es

Left y(a) = f(a) where ta es: and fa is defined by **fa(b)= a +** b *f*or an*y* bes

g(a - b) = fr.o (ii) Now f*oo*d *(C*) = (a + b) + c

=a \* (b + c)

= (fa o fb) (c) .: 8(a b) = f,.

= fm ofi = g(a).g(b)

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این مدت Monoid و

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**(Let us consider an algebric syste**m (M, \*), where is a binary operation on A.) Then the system (M. \*) is said to be a monoid if it

**satisfies** the following prope**rius:** *10* The op**eration is a closure operation on set A.**

***0 T*he operation is an associative opereation. *w)! T*here exists an identity element w. r. I. the operation .**

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**Ex.: Consider an algebric sys**tem (N. +), where the sel N = {0, 1, 2, 3 ---} the set of natural numbers and + is an addition, ***oper*at*ion. Deter*mine whether *(*N. +) is a nionoid.** isolation : *O Clo*sure Properly: *T*he operation + is *cl*osed. ***Sinc*e sum o*f* two naliiral no. is a nutral nunber.** *2* Asso*ciative Proper*ty *: T*he operation + is an associ**ative property.** Since we have (a+b))+ c = 11+(b+c) V a, b, cEN : S

(iii) Lasi step.show that .: S$. is a hoinonorphism of ($. \*;' into (S$, \*) corresponding to element il e S, the function la is completly determined from the entries in the row corresponds to 'a'm' the composition table (S, \*). Since, f(a)=g(a), e*v*ery row of such a table determines the image under thč homomorphism g.

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Discs!f Mathematics

***Sem*i*gro*ups *a*nd Gr*o*ups**

Submonoid

Let (M. \*, e) be a monoid and TSM if the set T is closed **under the cperation \* and e** eT, then (T, \*, e) is said to be - submonoid of (M, \*, e). It satisfy the following prop**erties**

(1) it is closed under the operation (ü) There exists an identity element e ET.

Theorem : For any commutative monoid M, \*), the set of idempoteni element of M forms a submonoid. Proof: Since, the identity element e e M is a idempotent, e es, where S is the set : f ideinpotents of Mi' .2 2.5ES, so that a . a = a and b b = b NOH (? - b) = (a - b) = (a - b) = (b + a) [:*(*M ) is comm.)

= a \* (

bb) a = a + b\*a = a\* a\*b

= a + b .. a \*beS and (S, \*) is a subimonoid.

Asomorphic :

The semigroups (S; \*) and (T, \*') are isomorphic and w*e* write ]

S=T. To show that **two se*m*ig**roup *(*S, \*)an*d (*T, \**')* are isomorphic. we are the followi**ng procedure.** (1) Define a function f:ST with DOM (7) = S (2) Show that 'f is one-to-one (3) Show that it is onto : *(*4) Show that *f(*a + b)= *f(a*)*.\*?* f*(*b).

Imp w116 Ex.: Let T be the set of all even integer. Show that the semigroups : 1 (2, +) and (T, +) are isomorphic.. Solution : Let T be the set of all e**ven integer.** To show : Semigroups (2, +) and (T; +) a**re isomorphic** Proof:

i : Step *(1)*: We defi**ne the function: f:2 +** Step (2): We now show that fis one-to-one as follows Suppose that f(ar) = f(az) : Then 2a, = 222 So; a, = az. j . . Hence, fis one-to-one. . Step (3): Now show thdef.is onto Suppose that b is any even integer. - Then a = b*/2. E*Z and f(a) = f(b*/2)* = 2(6*/*2) = b . So; f'is onto: Step (4): We have i

: f(a+b) = 2(a + b) = 2a +26

5 yi Isomorphisın an**d Homomorphism**

Isomorphism :

Lai (S, ~) and 'T, - *'*) be two seinigroup. A function f: 5+1 is called an iso*m*or*p*h*ism from (*S, \*) *1*0 *(T*, \*') if it is a one-to-one correspondence from S 10 T and if

*f(a* - b) = f*(*a*) \*' f(*b)

**etheoren** : . Let (S, \*) and (T, \*') be monoids with identities Lilac an**d es respectively. Let f:s → T be an isomorphism, then**

**Homomorphism**

Let (S, *)* and (T, -*'*) be w**o s*em*igro*u*ps*.* An everywhere defined fun**ction fis + T is called a h**omomorphism fro**m (S, ~) and (T, \*').

;; :

f(a + b) = f'(a) \*' f(b)

Proof: Let 6 be any element of T. ; Since, fis onto, there is an element a in S Such that f(a)= b then

a = a \* e b= f(a) ::

= f(a \* e)

· = f(a) \*' f(e)

= b \* fle) ------ (1) Similarly, since

for all. 'a and b in S. . If fis also onto. W*e* say that Tis a homomorphic image of S.

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=f(e) = f(a

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b = (e) \*' 6 : ------- (2)

From (1) and (2) Thus, for any bet

:: b= b*x' f(e) = f(*e)\*' which means that f(e) is an identity for T, and *e*' is identity for T

f(e) = e" **H*e*nce pro**ved. . .

**Theorem : Let f be a homomorhism frors a semig**roup (S. \*) to a semigroup (T, \*'). It s'is a subsemigroup of (s, \*). then f(S') = {t ET|1=f(s) for some s ES}

The image of S'under f, is a subsemigroup of (T, \**'*). :: Proof: If t, and iz are any elements of f*(S*')

Then, there exist s, and sq in S' with

t;= f(s,) and 17 = f(sz) Then, t, \*' tą = f(31) \*' f(s*)*

= f(s, \* $;)

= f(sz) Where, 'S; = S, \* 'S: ES' . Hence, t, \*' lz ells') . Thus, f(S') is closed under the operation \*'. Since, the associative property hold in T, it holds in f(S'). So, f(S') is a subsemigroup of (T, \*').

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Semig*r*aph*ı***s *& G*rapb*x***

Ex. : Let R\* be the set of all positive real numbers, show that the

Theorem : If 't is a homomorphism from a commutative semigroup

function f:R\* →R defined by *f*(x) = In x is an isomorphism of

(S, \*) onto a semigroup (T, \*'), then (T, \*') is also commutative.

the semigroup (R\*, x) to **the semigroup** .(R, +), 'wh*e*re X and

Proof: I fis homo.phism from a commutajive semigroup (S, \*) onto

**are ordinary multiplication an*d*** a*ddi*tion*, respectively.*

semigroup (T, \**'*)

**Solution : Le**t x, y ER\*, .

(5') = {t € T/t ef*(s);* fo*r s*ome ses'} ....... (1)

In(x \* y) = In(x) + In(y)

Leli, and , are'any elements of t.

So, In is a homomorphism.

Then, there exist s, and s2 in S with

Suppose XER.

t; = f(s,) anii 12 = f(sz) (from equation (1)

Then, et ER\* and In(e\*\**)*= x

So, In is onto R\*..

Therefore, i, o tz = f(s, ) \*' f(sz) ..

Suppose In(x) = In(y)

= f(s, \* $*7)*

Then, e!M(x) = (%) and x = y

= f(s. \* 5,)

Hence, In is one to one

= f(s*z)* \*' f(s*)*

.. and an isomorphism between (R\*, \*) and (R, +*)*.

=\*' . . .'.

9.5 Co**ngruence Relation**

Hence,' (T, - ') is also commutative.

**An equivalence relation R on the semigr**oup *(*S, \*) is called a

Ex. : Let (S,, \*;), (92, \*2) and (s3, \*3) be semigroups and congruence relation if fis, -+S2 and 8: S2 + 5z be homomorphisms. Prove that gof.

aRa' and bRb' = (a + b) R (a'\*

is a homomo**rphism from s, to sz**

Ex.:. Consider the semigroup *(*2, +) **and the equivalen**ce relationR

on Z defined by aRb if and orly if a = b (mod 2*)*

Solution : Let x, y es

Solution : Remember that if.a's b (mod 2), then 21.2 - 0.

*(g*of)(x \*, y) = g(f(x \*, y))

We now show that th**is relation is a congruence re**lation as follows*:*

a = b (mod 2)

= g(f(x) \*, f*(y)) .*

arid ced (mod 2).

Then 2 divide a ieb

-- $(f(x)) \*, g(f(y*))*

arid 2 divide ö d

*-loof* (x)\*, (2*0f) (Y)*

1..' So, a - b = 2m

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Carollary 1 : Let R b**e a congruence rel**ation on the monoid (S. \*); **If we define the opera**tion in SIR by (a) (b) = (a - *b)*. then (SIR, ) is a monoid.

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tov.ba, wakat*i* aki

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Adding, we have

(a+b)+ic-d) = 2m +2R ! Wish (a + c)-(b+d)= 2(m+n)

So, a+csb+d (mod 2). Hence, the relation is congruence relation : **Thcorei**n : Let R be a co**ngruence relation on the sem**igroup (S, \*).

Consider the relation from S*/R*XS/R I S*/*R in which the ordered TER: S pair.((a), (b)) is, forja and.b in S, related to (a + b).

la 0 is a function from S/RXS*/*R 10 SIR; and as usual we denote .

([a)(a)) by (a) [b]. Thus, [a] 8 [b] = (a + b) :

*SI*R, ) is a scmigroup. Proof : Suppose that

i(la). [b]) = ([a'); [b']). Then aRa and bro' **So we must have**

a\* b. Ra' b'! .. Since, R is a congruence relation.

Thus, (a ..b] = [a' - 6') ::Thai.is ® is a function . ***-*This means that is a binary.operation on *S/*R.** Next, we must verify that is an **associateive operation.** We have *[a] ®* ([b*]* → [c]) = [a] - [b+c] ... ? *= (*a + *(*b + c)]

= [(a + b) \* c] by **associative p**ropery of - in s - 120 b] © (C)

= {{a) o [h]) % (c) Hence. S:R is a semigroup.

***T*heorem: Let R be a congruence relation on a semig**roup :S.) and let (S*/*R, 0) be the corresponding quotient semigroup, then the function fr:S+*S/*R defined by ff*(*2)=[2] **is an onto homomorphsim called natural homomorphism. Proof**: If (a) E*S/*R then fr(a)= [2] So, friis on onto function. **If a and b are elem**ents of S, then fq(a + b) = (a + b)= [a] [b]

= {p(a) Ir(6) So, fx is a homomorphism. **Fundamental Homomorphism Theorem Theorem : Let f:S\* T be a homomorphism** o: the semigroup (S, \*) onto the semigroup (T, \*'). Let Rbe ihe relation on S čefined by arb if and only if f(a) = f(b), for a and b in S. The: (a) R is a congruence relacion. (b) (T, \*') an**d the quouent semigr**oup (S*/*R, ) are isomorphic. ; Pr*o*of: (a) We show that R is an equival**ence relation.** First, ana for every a ES **Since aRb a**nd brc , then .

f(a)= f(b) and f(b) = f(c) So f(a) = f*(*c) and aRc. Hence, R is an equivalence relation. Now, suppose that aRa, and 'Rb. the*n*

fla) = f(a,) and f(0) = f(b)

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Semigraphs & Graphs

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Discrete Ma**thematics**

Finally, f([a]) [b]) = f((a + b) .

=f*(*a\* b) . =f(a) \*' f(b)

= F ([a]) \*' ] ([b]). **Hence, f is an isomorphi*sm.****.*

***N*atural H*om*omorph*ism***

It follows from the defination of lie and F that Fofa = f': Since, *(*Fofa*)(*a) = f (fr (a)*)*

::F([a])

=f(a)

*.*

Multiplying in T iv. obrain

(a) i(b) = f(a,) f(b,). Sinc-, fis a homo.morphism, this last equation can be rewritten as

f(a - b) = f(a, b;) Hence, (a - b) R (a, \* 6,*)* and R is a co**ngruence relation.** (0) New co**nsider the relation † f**rom *S*/R to T **defir.ed as follows:**

F = {([a), f(a))| (a) E*S/*R} **Show that f is a function** Suppose thai [2] = [a')', then aRa', So. () = fi*a*'), which implies that is a function.. We may now write

T *: S/*R +T, where F ([a]) = f(a) for (aj e*S/*R Now show that s is one to one. Suppose that ([a]) = F ([a']) Then fla) = f(a*')* So, aRa', which implies that (a)= [a') . Hence s is one to one. Nuw, 'we show that F is onto

Described by the di**agram shown in fig.** Here te is the natu**ral homomorphism.**

*en aRa:*

Ex.: Let (S, \*) and (T, \*') be co**mmutative semig**roup. Show SxT is a also a co**mmutative semigr**oup. Solution : Let (S, \*)and (T, \*) be co**mmutative** semigroup; **To show : SxT is a co.nmutative semigroup.** Let (s), (,), (S2, 12). €9xT .

(S1, 1,) #." (S2..tz*)* = *(*s, \* $2; 1; \* 12) So, " is a binary operation :

· Coiisider,

(si, 11) \*\*:(5x. (a) \*" (sz, 1))

= (s. 17) \*" (sz \* Sg. 17 \* 13) .

Thus, ($x T. \*") is a semigroup.

9*.*6 Group

**Con*s*ider a*n* algebric s*y*st*em (*G -), where is a binar**y o*peration* . on G Then the system *(*G \*) is said to be a group if it sati*s*fies

followi**ng properties :** (1) The operation \* is a closed operation. . . i.e. a, b eG > a, b eG (2) The operation \* is an associative operation.

i.e: a, b, c eG = a\* (b\* c) = (a \* b) \*c (3) There exists an identity **element w..t. the operation \***

j.e: a\* e = a \* x = x *(4*) For every. a EG the**re exists an e**lement a 'EG such that i

a-'\*.a = a\* a-' = e

**"o*n,***

1:

1.

= (sz \* $1, tz \*',

Esq.f)\*" ($i ty) Hence, " is c**ommutative**. .. Ex. Let (S, \*) and (T, \* n be semigroups. Show that the function

-SXT →S defined by f(s, t) = Sii: a homomorphism of the Semigroup S: solution : Let (S, ) and (T, \*') be 'a semigroups. Let != (s, 2,). (sz, Iz) eSXT . Then, f((s; '; 11) \*" (S2, t*z))*

= f(s, \* 52, , \*' tz) : = 5,.\* sz .

= f(s, t) • F(S), 17) fisa homomorphism.

***T*h*e*orem :: Is** (S, \*) and (T, \*') **are semigr**oups, then (SxT, \*"*)* **is a semig**roup, where" is defined by (sjá by:) \*" *f($2*, 12*)* = (s, \* 52, t, \*'(z) Proof: Let \*" is a **binary operațion, because bot**h \* and \*"are. Consider, ($j. 4) \*" ($2; *;]* \*" (szi t*z)) (s*. 1,)\*" (($2612) «" (S3, 13))

*= (*s. 1,) \**(*S2 \* Szo tz \*13) , = (s, \* *(*$, \* S3): 1, \*' (tz \*' 13)) = ((s, \* $2) \* 53, (t; \*' tz) \*'13)

= ((s). 1)" (Sz. 12) 4" ($3, 13) **Thus, " is associat**ive. .

Ex. : Consider an algebric system *(*Q, \*), where Q is the set of rational number and \* is a binary operation defined by

a\* b=a+b-ab Ya, ben Determine whether (Q, \*) is a group. Solution : Closure Property : Since the element a, b EQ for every a, b e Q. **Hence, the set Q is closed under the operation'\*. Associative.Property: Let us assume a**, b, c EQ, then We have

(a + b) \* c = (a + b-ab) \* C

= (a + b - ab) +C-(a+b-ab) s = 8 +6-ab+C-4C - DC +abc = a + b + c - ab - ac – bc + abc

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Discrete M**athematics**

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**Semigraphs & *G*raphs** o r wet 13

Teorem : Show that the identity eleinent in a group is unique. Or *T*h*e id*entity of group G is unique. **Proof: *L*et G be a group with two ident**ity elements:e and *e!* Since, e eG and e' is an identity. We have e'e = e.e're ----- (i) Als*o, e*' EG and e is an identity. We have e'e=e.e'

-- *(ii)* F*r*om (i) and (ii)

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Similarly, a \* (b. c) = a.+b+c-ab- ac – bc +abc Therefore, (a \* 6).\* c=a\* (b + c) .: \* is associative. Identity : Lei e is an identity element. Then we have

amera Vae a + e-ae =a o*r* e-ae = 0

e*l*:-a)= 0 or e=0 . . Similarly, e \* a = a ta en Therefore, for e=0 We have a \* e= e \* a = a Thus a is the identity **clement.** Inverse : Let us a**ssume an element a en** Ler a-l is an inverse *o*f a, where a-'EQ. Then we have

\* a -= 0

*[I*dentity? i i-a-'-aa-' = 0. ut i 'Li-it) = -a or a 'a

**He**nce, identity in a group is unique.

w wlis o **Theorem : Show that inverse of an eleme**nt 'a' in the group is unique. OR Let G be a group; then every a eG has unique inverse in G. : Proof : Let u**s assume t**hat a E G..-be an élerrieni. Suppose that a,-' and az-' be tw**o inverse element** of a. , Then w**e have**

2,.-'a = 2.,-!= ---- (1) .. . and a .-'a = a.az-'=e .......-(2) Now, a, a,-!, ,!ęc then by associative law

9,-'(a.az-') = (a, -\*.a).a.""

2,-*'*(e) = (e).az-' [by equation (1) an*d* (2)] ? 2,-' = a; -'

.

· Hence, the inverse of rn element is unique for each i E G.

Ilcorém : Shout thilt (a-')' = a for all i Ci where! group and 1" :: in inverse of a. Droof. Inca be a group with identity elenient c *..*

=0. if a ra

a a-1

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Now. -

1.

*T*herefore, ev**ery element has inverse** such that a

*j*...sinfor all the propertil - of a group.

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:1. Theorem : Prove the lecture

SH Now

a"'.= . . (:-) = (0,1"')'.c

= [(a+'; \*\*-\*-"). a) "{say a :sociative }

i.e. ab = ac boc v a, b, c*eG (L*eft cancellation law) **Pro**of :: Given that ab = ac To show b= c Then, we have

b=eb

= (a 'a)

...

..

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= a-'(ab)

= a-'(ac) (: (ab = ac)]

= (a-ta) c

= ec

19 BCA-'} = n Wa; d-dieg is hul teoreni : ; Show that (ab)-' = 6-'a-'\_ for all a, b eo ,

Proof: Let G be group with identity element e. 6 ! Let a, b € Gand : 4-6-'G El We have to prove that ab is inverse.of 6-18-!...

işo." (ab) (6" : "') (6-4a-') (ab)

Now, take

LHS = (ab) (6-10-23..

= [(ab)b“']a-' . . : By associative property] : = [a(6b+1)] a-' [:: By associative property] = (ae) a- . . 1: By Identity property]

[::But Inverse property]

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.....

Hence, ab = ac > b=C

**Theorem** : Prove the right cancellation law i.e. ba = ca = b = c V a, b, cEG (Right cancellation law) Pro*o*f : Give that bar.ca To show b = c Then, we have

b = be

= 6 (aa-'*)* = (ba) a = (ca) a-'(: (ba = ca)] - c (aa') = ce : aa-?=e]

=a a-.

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SWEE*T*

Similarly, RHS = (6-4a-') (ab)

1 : =[b'(a-'a) b]

=[b+c] b]

[::By associative]

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Hence proved. :

Hence, ba = ca = b = c.

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Discrete Mathem**atics**

**Semi*gr*aphs & Grapos** *V*OTT6Impo i*k* : L*e*i G is the set of a*ll* non-z**ero rea*l* num**bers and is a binär operation defined by

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ab **2**

AN

c*e*lcuk

a\* b *-* a*b* a\**b* =

***a n*on-zero**

Ex: Lei (Go) be a group. Show that (G o) is an Abelian group if and only it (a - b)2 ='a? ob? for all a and 6 in G. Solution : 'We know that

(ao b) = (a ob) ciaob) [:: o is a associative]

= a (boa)ob Now Let G is an Abelian group

= a'o (aob) ob

= {a0a)o(bob) Hence, (ar)) = a oba a, beG

Thus, the group G is abelian.if and only if

(206) = a-ob? a, beG. Subgroup

Consider a group (G \*). Also let SCG, then (S. \*) is called a subgroup if it satisfies following conditions. (i) The operation \* is closed operation S. . (2*)* The operation \* is a associative operation. (3) As 'e' is an identity eleme*n*t belong to *fg*. It must belong to the

set S i.e. *T*he indeiry element o*f (*G, \*) must belong to (S, »). H) For every elemel!" a ES, a“ also belong to S. : US Abelian Group

Let us consider, ar. algebric systern *(*G \*), where is a binary operation on G Then the system (G \*) is said to be an abelian group ti: saisties all the properies of the group plus an additional following

Show that *(*G **\*) is an abelian group.** Solution : If a, b are element in G the 50 is an To show : (G \**)* is an abelian group. Closure property: The set *G* is c*l*osed under the op**eration \*.** Șince, a + b =\* is a real number. Hence, belongs to G . **Associative property :** The operation \* is associative. Let a, b, c EG, then We have ·

: (a + b) \*c=()

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Similarly, a " (15\* C

16 **abc**

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oir üs4 8. Similarly, a d a . is.

AC à or e = 4 :: : : Thus, the identity elemen in G is a *I*nver*s*c: Suppose that. a E G. If a-'EQ is an inverse of a, then a\* -' = 4. Therefore, a =4 or a '16 Similarly, a-'\* a=4

"aa **Therefore, 34** or a''\_ Thus, the inverse of element 'a' in G is . Commutative: The operation \*on G is commutative. Since, and = 25 = 6\* a Thus, the algebraic s*y*stem *(*G \*) is closed, associative, identity element, inverse and commutative. Hence, the system (G, \*) is an abelian group. . h eorem :: Let G be a group each element, a in G has only one inverse of G Proof: Let a'and a" be inverse of a. Then. a'(aa") = a'c = a' and (aa) a“ = ea" za" Hence, by associativity a' = a"

Hence proved.

**i*dn*eyitw**. ww. - - -

***ta) T*he equation ax = b has a unique solut**ion in G. (b) *T*he equation ya = b has a linugie solution in G. Proof: . (a) *T*he element x = a-'b is a solution of the equation

ax = b Since, ala-'b) = (aa-')b = eb = b Suppose now that x, and x, are two solutions of the equation

i ax = b Then, ax, = b and ax, = b Hence, ax, = ax, It implies that X1 = xạ........ .

Hence proved. (b) The element y = a -'b is a solution of the equation va = b Since, (a-'b)a = (ba-')a

= b[a'a)

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= be

= b Suppose now that y, and y, are iwo solution of the equation ya = by Then, y, a = b and yqa = b Hence, y,a'= y2a. It implies that I = y;]

... Hence proved.

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**Se*m*i*graph*s & Gra*ph*s**

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**Discrete Mathematics**

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9.9\_Normal Subgroupdet

fr. 11134 Consider a group (G \*) and subgroup (H, :-) of the group, then the (H?. \*) is called a normal subgroup if for any a E G we have

H = Ha ie if it is a normal subgroup, then both the left and right co**sets of** Hin G are equal. **Theorem** : A non-empty sub-set H of the group G is a subgroup of Gif and only if

(i) ,, DEH = a.beH

(ii) a EH = 'EH Proof : Le: H be a sub-set of G and G be a group with operation of G. Part-1: Let H be a subgroup of G. = li satisfied all the group **properties.**

a. berabeH 16h10 is the closure property. Part - II : (i) 2. b CH = a.beH

(ii) a eH = a-'CH To pro*v*e H is subgroup of G i.e. H satisfied all group prop**erties.** (i) Clo**sure property f**rom (i)

a. bEH = RobeH (ii) Associativ**e property.** Let a, b, c EH then a. b. ceH' (HCG) Bur Gis a **group.**

C...*cr*i. {by associative}

*(*iii) Existan*ce of id*entity element. Let a eH inverse of *7*-1EH and a oa ' eH by (*i)* But aoa-leG *7: HC*G)

goa-'-e eG saoa-'=C EG (:: *H*CG) **se is the identity element in H.** (iv) Existance of inverse by *(ü) .*

a EH = a-'CH .: H is subgroup of G Theorem : I*f* H and K are subgroup of G:show that H*N*K is a subgroup of G. **Proof**: H and K be the subgroup of G. Let å beHNK > a, b eH and a, b ek

ĐaobeH and aob ek(Hard K be the subgrou)

> RobeHOK . 'sa; BEHNK = a0bEHNK ---- (1*)* Now, : . Let a EHNK = a pH and a ek :

a'el and a-' eK :.

...a-'ėlink From (1) and (2) :: HNK is a subgroup of G **Theorem: Nis normal subgr**oup of Ġ if and only if yñ \*:: Proof': Part-1: Let N be a normal ohyroup of G;

:

Primultiplying by ge! and post-multiplying by g in equation (i)

E-1

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Theorem : Show that ine CosIW.. .. ***i*s a normal sub**group of G*. OR If H* and K are two no*r*mal subgroup o*f g*roup G then p*ro*ve 1.12: HNK is normal. *Proof : Let H* and K be the normal subgroup of *G c*hen Hand K are subgroup of G > HNK is a subgroup of G . We show that H*NK* is a normal subgroup *of* G. Let xdHNK xeH and xEK

: > gxg-'EH and gxg-EK :: H and K be the normal subgroup of G.

= gxg-EHOK XEHNK = gxy-'EHOK THNK > H*N*K is a normal subgroup of G

I

(8-'8)N(8"') = g'-'N 1: by associative} -- cNe = g"! Ng .

N= "Ng . , (i) From (i) and (ii)

SN -'=N Vg EO Part-: Lct Ny=N. We show that N is normal subgroup of

Ny-SEN VYEO.

Niis a normal subgroup of G. **Tbleorem** : The subgroup N of G is a normal subgroup of G if and only if cach left coset gN in G is a right coset of Ng in G OR Nis a normal subgroup of Gʻif and only if gN = Ng Proof: Part-1: Let N be a normal subgroup of G then yng-'=N Post multiplying above eqn. by & .

. 1 Ng" = Nga

(GN) (8-'s) = Ng {:: by associative} I gN) = Ng {: g"g = e*}* 1 2 BN = N*g (* g €G

Part II : Let gN = Ng B N)!-! = (Ny) g

N :' = N(!!-') : by associative? BNS" = Ne f ise" =c}

.

**Theorem** : A subset S'of G is a subgroup of *(*G \*) if and only if for any pair of element a, bes. ab-'eS. . Proof: By defination of subgroup *I*f S is a subgroup of G and G is a subgroup then S is also subgroup. :: S is a subgroup of G 7

Å, DES = abes and a les a. bes sabes and b'eS

a, b'eS > a.bu'es Hence proved.

0.1

Ex.: Let G be the group of real number under addition and Ini be the under multiplication, let 1: =) ( 01401110D [?? 11:17 Show that f is an is a morphism of G onto O'.

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ARKIKLIAI K**ITA LEMANLARININ**

**468*;***

Discrete Mathe**matics**

***Semigrap*hs & *G*ra*phs*** Pr*oof:* (a) Let x = *f(*e) then

\*\*\* x = *f(*e*) \*'. f(*e)

**= *f(*e\* *e)*** = f*(*e*)*

**TALDEA**

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Solution : If *f*(a) =r{b) So tha: e" = ed

> a = b Thus iis one-to-one Ji CEG'.then log CeG

f(log C)= ellos () - C Su fis onto . f(a + b) = 2\*\*6=e""= *f*(a).*f*(b) Hence, fis an isomorphism.

v wl13 **Theorem: Show that every su**bgroup H of an abelian group G is. il normal. Or Show that if G is an abelian group then every subgroup

of G is a nor**mal subgroup.** Proof : Let H be a subgroup of an abelian group of G Lei h = H and geG

ghg' = (gh)g

= (hg)*g-'* 1 : But Commutative property) = híge-') 1 :: But Associative property] = he

:: But Inverse property] =r!" = h > ghg'EH Y DEG, heH :: Every subgroup abelian of G is a normal subgroup. *Theorem:* Let *(*G \**) a*nd *(*G'#') de two groups, and let f: G+G*'.* be a homomor*p*hism fr*om* ( to G'*.* (a) li e is the identity in G and e' is the identity in G*',* then f(e) = '' (b) If a *b c* then *fl*a-*'*) = f(a))-' *loo*d If H is a subgroup of G then *f(H) = {f(*h)/h eh}

So, X\* X= x Multiplying both sides by x-! on *ri*ght we obtain

x=x\*x\*x+'=; \*'x'= Thus, f*(e)*=ę'. (b) a \*a-*-.*... So, f(a+a-) = *f(*e) :

=e' by : 'art (a*)* or f(*a) \*'* f(a-') = e' . Since, fis a homomorhism . Similarly, f(a+') #' f(a)= Hence, f(a-') =*(f*(a))

**EVME*LE***

(c*)* If H, and H, are any element of f(H), then these exist h, and ha in H with

· Hy = f(hj), and H, = f(h) Then. H, -' *H;* = fch.) with

= f(h, \*H)

= *f*h): where he = h. \* ha es' .

P

1. Hence, 9, \*hj ef(H)

Thus' f(H) is closed order the operation \*'. since the associative property hold in-T, it holds in f(H*)*

f(H) is subgroup.. Ex. : Let' (S, \*) be a commutative semigroup. Show that it xr x= x any \* y = y; then. (\* \* y) + (x \* y) = x + y. Solution : Take LHS = (x \* y) + (\* \* y*)* .

(x + y) + (y + x) 1: (S, \*) is a commutative semigroup] .\* \*y\*y\*x. 1: y + y = y)

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X # \* \*yi

: commutative semigroup) . \*y

..[:x\*:; = x] ence (\* \* y) \* (x \* y)= X \* y.

Case I': x.y = x Now assume

y.y = y.(\*.x)

= (y.x) x 1:' is associative] = *(*x•y).x

1:. x.y = y.x]

= X X

[: x+y = x3 . = *y* Case II : x.y = y

lis=!) Now assume

y.*y* =(x:x).y .[: associative]

= x•(x•*y)* [:: xuy=y].... = X:*Y*

=y **Hence proved. Ex: Determine whether a semig**roup with more than one idenpolent element can be group. **Solution : Le**t *(*A, \*) be a gemigroup with two idempotent element a and b. Then we have

a\* a = a ..... (i) .! b\*b = b ---- (i) Now, assume that A is a group with identit: elements. Then, a \* e = a. and b \* e = b. From (i) and (d), we have

. a \*a = ane and b + b = b \* e

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**Sinib**

Ex.: Let: 64\*. y}, .) be a semigroup, where x.x=y show that · (i) x.y=y.x (ii) y.y=y

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**...**

**\***\*?:

(1) To show that \*•y=y. X We know that .

XXX = \*•\*:x [: 8.x = y)

X.!= y.x .. (ii) To show that. y.y=*y* We know that the set (x, y) is closed under the operation. Therefore, we have two options

xy=x, x•y=y

**Semi*g*raphs & Gra*ph*s**

**Discrete Mathematics**

*T*hus, w**e have** I a *e*GoG, *and b 6G*, G, > ablegi*ng He*nce*, Ging*, is a subgroup o*f* G. (ii) *It i*s not always neces*s*ary *th*at 'G, UG*, i*s a subgroup o*f G.*

By law of left cancellation, we get

a = e = b = which is a contradiction to a = b Hence *(*A, \*) can not be group. Ex.: Let G, and G, be subgroup of group G. (i) Show that Gin G2 is also a subgroup of G.. (ii) Is GyG2 alwa*y*s a subgroup of G? Solution : (i? Let G, and G. be two subgroup of G. 11:en we have *0*.62 = [:*: I*dentity element is common to both G, and G2] To show that : GnG, is a subgroup. We shail have to prove that

a eG, Gand begin G > begin Let us assume

a eG, G2 > a EG, and a eG2 ani begin Gz = 'beG, and beG Bu: we know that

G, and G are subgroups. *T*herefore, a EG, and beG, > as'EG, ------ *(*1*)* anc a G2 an*d* begi = ab-'eG2 ------ (ii)

Ex: Let G be a finite group and *H* be a sub*g*roup of *G.* For a *EG defi*ne aH.- *(ahh el}* (a) Show that aH] = 1H/ (o) Show that for every pair of elenients, a, b eG either ..

aH = bH or aH and bhurc *di*sjoint. *So*lution': (a) To show that | H| = |H*|!.* Let H = {hi, biz, hz --- hn*}* be the 'n' elements of H Then, aH = {ah,, ah, ah, --- ah, } But, we know that

ah; = ah; or hi = hi. **Hence, the 'n' elements in aH are disj**oint. ***T*heref**ore, \ H1 = HI...

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(b) Let us assume that (i\* H)n(b\* H), is non-empty. Also, Let ce(a\* H)n(6\* H): Then, we have

c c**a \* H = a + Hac + *H*.** Also, ceb\* H = b\*t: =c+ H :... **There*f*ore**, a \*H=B\*H\* Since the cosets from a partition of G. .

A

walkenbené Let R be a congruences relation on the group (G S M o lin thd semigroup (DIR, ) i**s a group, where the operation ©**

lielined on G*/*R by (a) • b) = (a \* b).. .. INSI : Since a group is a minoid, we know that-G/R is a monoid.

i vaneed to show that each element of u/R has an inverse.

Lei (a) EG*/*R, thien [a"'JEG*/*R

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Hence, (G*/*R. Ə) is a group. .

Carollary 1: : K) If Ris a congruence relation on a group G then the function

Itis : 6 →G/R: given by t: (a) = (a) is a group homomorphism. (b) If f: 6+ (j' is a homomorphism from the group (G \*) onto

the group (Ci\*") and R is the relation defined on G by a Rb|

if and only it f(a) = f(b), for a and b. in G. then 10). 'R is a congruence relaliun. . 10) The function.f: G*/*R+Gʻ, given by f([a]) = f'(a) is an

isomorphism from the group *(*G/R, 0) onto the group *(*G', \* 7.1 Ex. :'Let G bc the symmetric group S,. The subset H = {f,, 8,} is a subgroup of G. Computer all the distinct left cosets of H in G. Solution : If a € Hi then aH = H Thus, f; H = ,H=H Also,' f H=(.. S}:.

fxH = {f;. £j} : %,

H iss. f;} = t;H

VH= 183. ;} = 1;H !! The distinci left cosets of H in G are H, fxH and IjH. .

Ex. : Show that if G is an abelian group, then every sub*gr*oup CIG **is a norm**al subgroup*.* **So*l*u*ti*on** : Le*t* H be a subgroup o*f* G and Let a eG and H. EH Then ha = ah So, Ha = ah which emplies that H is a normal subgroup of G. ***T*heorem** : Let R be a co**ngruence relation on a grou**p G and let H = {e}, the equivalence class containing the identity. Then H is a hormal subgroup of G and *f*or each a eG, ia] = 2H = Ha ***Pr*o*of*: Let a and b be any elements in G since R is an eq**uivalence relation bela) if and only if (b) = (a). Also G*/*R is a group. . Therefore, [b] = [a] if and only if

(e) = (a '] [b]

= (a - b] Thus, be [a] if and only if

H=[e] = [a-' b] 1. That is, be [a] if and only if

a-beH or beah This prove that

[a] = ah for *e*very a EG We can show Similarly that b [a] if and only if

H = [e]

= [b] (a)"" = [ba)"

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Discrete Mathematics

**S*emigr*a*phs* & G*raphs***

This is equivalent to the **statement (a) = Ha**

Thụs, R is an equi**valence relati**on on G.

*· Next* we show tha*t R is* a co**ngruence relat*i***on on C. ilus, (a)= aH = Ha and H is normal.

Supp*ose that a*rb *and cRd.* Note : Quotient group G/R consists of all the left coset*s of N*-(e). |

Then, a-'b*eN and c*o*ld en* The operation in *G*/R is given by

Since *N* i*s n*o*rm*al, 1 (AN) (B*N)* = [a] [b] = [ab] = abN and

. The function fR: G

*Nd = d*N G*/*R, defined by fr(a)= aN, **is a homomorphism f**rom G onto G*/R.*

*T*hat is, *f*or any n, en, n,d = *!nz f*ür *s*onie n, *EN* For this reasion, we will often write G*/R a*s *G/N. .* . Since a-'*b'en,* we have .. *T*heorem : Let N be a normal subgroup of a group G and let R be a -lbd = dn*g* ***f*or *s*ome ng *EN*:** the following relation on G:

*T*hen, (ac)-'b*d* = *(c*-'a-'*)* (b*d*) R5 if and only if 2-'beN

i =c-'a-'6) (*d*)

Klicka (a) Ris a congruence relation on G.

*= (c*-!d) na *EN*

(0) N is the equivalence class (e) :elative to R

S*o,* acRbd*. . . ;.* **where** e is the identity of G*.*

**Hence, R is a congruence relation on G.** *Proof:* L*e*t a € Q then aRa Since, a 'a=e EN

*(*b) Suppose that XEN

So, R is r**e*fl*exiive.**

*T*hen, x-le=x-Y*EN N*exi. suppose that aRb, so that a-beN.

Since, N is a subgroup.

*T*hen (a - b)-t = -'EN

**So xRe and therefore x € (e)**

So, Ra.

Thus N sl*e*j

Hence, R is symmeuic.

Conversely, : *Fina*lly, su*ppose t*hat aRb and bRc.

If: x € (e), then xRe

*T*hen a baN and b-'ce*N*

So, xoc sx"'*sN T*hell. *(*-5) *(*5-4C) =a 'c *en*

Then, sen and [e] S N

So. *arc.*

where

· Hence, N = (e)

\*

· Then x = ah, heH and fash) = x Thus, fa is onto Since, it is everywhere defined as well, fa. is a one-to-one correspondence between H and aH. Hence, 1H1 = |ah!

Note : ; il. Korneli! Let 't be a ho**momorphism form a gr**oup (G, -) onto.

a group! *(*G'*,* \*'); and ... Lel the Kernel of f,

ker (7), be defiend by wijl ker(t)={a ea|f'(a) = e'}

Then, (a). Ker (1) is a normal subgroup of G. (b): The quotient group G/Ker (1) is isomorphic to G'. Since, if R is the congruence relation on G. given by

aRb if and only if f(a) = f(b) Then, it is easy to show that ker (f) = {e] .

Ex.: Let G be a group and Ha subgroup of G. Let S be the set of all left cosets of H in G and let T be tre set of all right cosets of Hin G. Prove that function f:s + T defined by f(aH) = Haris one-to-one and onto. Solution : Suppose f(aH) = f(bH) Then Ha-' = Ho-and a-' = 16-4. LEH. Hence, a = bh-'EbH So, aH CbH This means fis one-to-one. If Hc is a right coset of H Then f(c-'H) = Hc So, f is also onto.

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:Ex: : Consider the homomorphism f from Z onto Zin.

Defined loy 1/11!) =|"), where r is the remainder, when m is divided : 1.-by 11. Find ker (1). Li

Solution : An integer m in z belongs to ker (t) if and only.if

l(m) = [0] That is, if and only il m is a multiple of n. Hence, ker (n) = n2.. .

. :Let it be : subgroup of a group G. Prove that every left coset .! aH of H has the same number of clements as H by showing that the fiinction fa: H+H defined by fa(h)= ah, for het, is one to one and ontu. . solution : Suppose !fa(hi) = fa(hz) Then :ah, = ah, and a-'(ath, ) = a " (aha) Hence, 1, = hy and fa is one to one.

Ex: Let f be a homomorphism from a groups G, onto a group G. and suppoe that G, is abelian. Show that ker (f) contains all elements of G, of the form aba-'6', where a and b are arbitrary in C Solution : Consider

f(aba-16-')= f(a) *f*(b) f(a -');(6-')

= f(a) f(a "') t(b) f(b-') = f(a*) (*f(a))' f(b) (f(b))"'

zeie

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1.

Det x E aH

Hence, {aba-16-1|a, b in G,} s. ker (1)

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**Discrete Mathematics**

***Semigraphs* & *Graphs***

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Ex. : Let H be a subgroup of the finite group G and suppose that there are only twoteft co*s*et*s* of H in G. Prove that H is a normal.

H*enc*e, *8*,82-'*eker (f)*

subgrol:p of G.

*T*hus, 6182-' = *e an*d si = 82 : Solution : Let a £H.

*H*ence*, f*is one-to-one.

The left coseis of H are H and aH. The righ: cosers are H and Ha.

***EXERCI*SE**

HoaH = Hn Ha = *{* }

- Imp w/16 .

and HuaH=HU Ha

*Qu*e *1 :* Consi*de*r the binary operation \* an*d* Q*,* the *s*et of rational

number *def*in*ed* by Thus a3 = Ha Since, a eH=H=H

a + b = 25 V a, beQ

We have

Dete*rm*ine whether \* *i*ś (i) Associative ; (*ii)* C*om*m**utative*. :*** XH = Hx XEG H is a normal subgroup of G.

Quc 2 : Prove that e; = em', where e; is a *right i*de*nti*ty an*d chiis*

a *left i*de*nt*ity o**f a binary operațio**n. ii Ex.: Lei f: 0 - G' be a group ho**momorphism. Prove that f is** one-10-one if ind only if ker (*1*) = *{*e}.

Que 3 : Let (A, **+) b*e semi*g**roup. Show: t*hat f*o*r* a, b, *c i*n Ar*if*

2 + c = c + a and b + c = c + b then (a #6) \* c = c + (a - b). S*ol*ution : Suppose 1: G→ G' is one-to-one. Les seker (1).

Quic *4 : If* (S,, \*) and *(*S,, \*) are semigroup then (S: XS, \**)* i*s-;.*

Theo i vore' *= tie*)

a *semi*group whe*ll* - is *defi*ned by i

: (S,'; S,*'*).\* (S,", SX"*)* = (S,'\* S,", S,'\* S;"*)* Thus x = e and ker *(*) = {e} . *Co*nersci*;*

*..* Que 5: Let ($, \*) **be a given semigroup there e*xi*sts a homomorph*is*in** suppose k*er (A) = {e*}

8:5+ s, where (S?, \*) is a semigroup o*f* function *f*rom S. to

If tro, ) = tig:)

under the operation *of l*eft composition.

Th*e*n, f(g, €2-') = *f(*g) 1(87-*')*

Quc 6 : For any commutative monoid *(*M, \**)* the set of idenipotent

**eleme**nt of M *f*or MS **a submonoid.** = *f(g)) (f(82))*'

Qu*bt* Let7. be the set of all even inte*g*ers. Show that, the = *f(g,) (F*(*8,3)*

**semigr**oups (2, +) and (T; +**) are isomorphic.** *= f(g) (f(g.))*"'

*S*uc *8 : L*et (S, \*) and (T, \*') be monoid with identities é andic?...