Graph Theory and Dyson Swarms:

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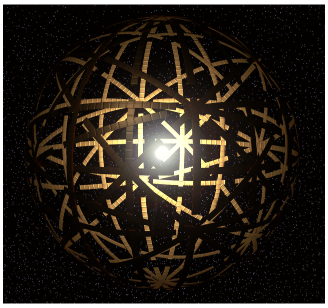
1 Abstract:

1.1 What are Dyson Swarms?



As a civilization advances, the main factor preventing its future growth is the ever important problem of energy production. To mitigate these issues, it has been theorized that a civilization that has reached a sufficient level of technological advancement and spread in their stellar system must decide to build a mega structure that can provide more than enough energy for the multi-planet civilization. This mega structure is known as a Dyson sphere, or more feasibly, a Dyson swarm. A Dyson sphere is a sphere of solar energy collectors that encloses the sun, while a Dyson swarm is a swarm of small (on the order of current day spacecrafts) solar energy collection spacecraft that orbit the sun in a controlled manner. Due to physical constraints caused by the need to survive the immense tidal forces a sphere enclosing the sun would have to endure, a Dyson swarm is a much more viable option and thus will be the focus of this project. In this project, we suppose that we are working with a multi-planet civilization who has built a Dyson Swarm. Dyson Swarms by nature are complicated and dynamic; power must be collected from the sun by the orbiting spacecrafts and then sent to some collection stations (which collection station will depend on the position of the spacecraft in its orbit relative to the different stations) and from said stations will be beamed to one of the occupied planets to satisfy their energy needs. It is important to note that the transmission of power is not perfectly efficient.

1.2 The Aims of the Project:

The energy network of a Dyson swarm is a directed, acyclic graph, which is a familiar concept from our study. We will use this perspective to discuss properties of the graph along with developing a framework for discussing how the connectivity of the graph changes with the relative motion between the orbits of planets, satellites and collection stations along with other special considerations such as the interference of space junk, downage of collection nodes, etc. As a function of the changing connectivity we may perform a max flow/min path optimization to mitigate the aforementioned losses in this system. This may be formulated as a MILP problem with time dependent inputs and binary choice variables. The goal of this project is ultimately threefold: contextualize the problem of optimizing the output of a Dyson Swarm to potentially many bodies that orbit

the sun with graph theoretical formalism, discuss the dynamics of the system with respect to this perspective and prove some interesting results about it, and formulate a MILP optimization.

2 The Swarm as a Network:

2.1 Nodes and Connectivity:

For this section we will be discussing the structure of the network that represents our Dyson swarm. The network's basis is in energy flow, and thus each node will be some physical entity that receives/transmits power, and each directed edge will represent the flow of energy (bounded for each edge by some weight) from one node to another. Further, in this section we will be disregarding the dynamics of the graph, instead focusing on describing the graph instantaneously at some arbitrary point in time. In the network there will be 4 types of nodes and we will be discussing them in order of energy flow:

2.1.1 Sun:

In the Dyson swarm system, the sun is the source of all energy, and thus in our network the sun will be the only source node in our network. It's properties are that it has a $\delta_-(N_s) = 0$ by virtue of being a source. Further it has a $\delta_+(N_s) = \infty$; this is a valid approximation as compared to the number of Dyson collectors a civilization can make, both the surface area and the energy output of the sun is much to large to consider limitation on the suns out-degree.

2.1.2 Collectors:

In the Dyson swarm system, collectors are the main devices that swarm around and collect energy directly from the sun. Thus there will be many of these nodes (for our model there will be n_c nodes). It's properties are that it has an $\delta_-(N_c) = 1$ as each collector has only one in-connection, that being the sun. Further it has a $\delta_+(N_c) = 1$ as each collector can only connect to one node in the next layer, the distributor.

2.1.3 Distributors:

In the Dyson swarm system, distributors are the intermediary devices that collect energy from many collectors and then distribute them either to other distributors OR directly to a consumer node. Thus there will be many of these nodes (for our model there will be n_d nodes) and further there will be many layers of distributors depending on the distance of the consumer from the sun (for our model there will be y_d layers). It's properties are that it has an $\delta_-(N_d) = D_n$ as each distributor can connect to many collectors/distributors depending on the distance of each distributor from other collectors/distributors. Further it has a $\delta_+(N_d) = 1$ as each distributor can only connect to one node in the next layer by construction of the problem.

2.1.4 Consumers:

In the Dyson swarm system, the consumers are the sinks of all energy, and thus in our network the consumers will be the sinks node in our network. Further, since the consumer nodes represent the many planets/moons/colonies an interplanetary species would inhabit, there will be many consumer nodes. There properties are that it has a $\delta_{-}(N_s) = \infty$ as we are assuming each consumer can set up an arbitrary number of receivers on there colony. Further it has a $\delta_{+}(N_s) = 0$ by virtue of it being a sink.

2.2 Properties:

Clearly, this system is acyclic and directed so it admits a topological sort which corresponds to distance from the sun. It is intuitive to consider this network the classes of nodes discussed above under this sort. Discussion of the node classes and network size/location can be done in terms of layers.

The swarm, which consists of the collector and distributor nodes takes the form of a forest of oriented trees where the leaves are collection nodes that propagate towards roots, which represent the final layer of distribution nodes. It is through these objects that we can continue our discussion of more interesting properties, namely dynamics and optimization in the following sections. A sample network is shown below – note that ideally, the reduction of layer size would be logarithmic, maximizing the in degree of each distributor.

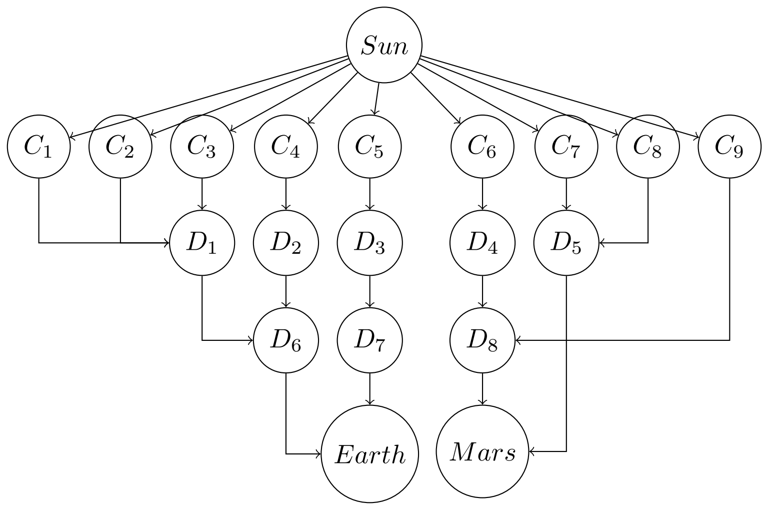


Figure 1: Sample Network

3 The Dynamics of the Network:

3.1 Why is the network Dynamic?

Given the network established above, it may seem that this is all one needs to describe the Dyson Swarm system. However this is not true for one important fact, *nothing* in a system governed by gravity is static. All collectors and distributors orbit the sun in *different* orbits. All collectors have the same orbital period but the same is not true for distributors, who by virtue of being different distances from the sun have different periods from each other. Further, despite the fact that collectors have the same period, by construction of the swarm, they all orbit on different orbital planes. Even more worryingly, the different consumers the distributors attempt to connect to are *also* orbiting the sun at different periods from each other, all the distributors and the collectors. This is not to include the revolution of each of these bodies and the true non-perfect chaotic orbits all objects follow which we will not being dealing with in this project. Since these objects are orbiting differently, the distance between each is constantly changing and thus the ability for each node to send power to one another is constantly changing. To handle this we must develop some framework for dynamics in a graph.

3.2 The Dynamic Graph Framework:

The basis of a "dynamic graph" like the one discussed above is an incidence function which is determined by conditions at time t. In directed cyclic network, incidence can be equivalently thought of as a choice map which takes as input a node and outputs a set of nodes to which it is incident. In the case where a node is excluded at some time, the choice may be represented as the empty set.

We can simplify this choice as follows: Consider a "possibility" graph in which there is an arc from node u to v if v is a valid choice. In a dynamic network, time-dependent incidence is a stochastic choice of a spanning sub-graph of the possibility graph. It is important to note that although the conditions which determine choice may be continuously defined, the choice is made on a discrete interval (stochastically) that can be described by some characteristic time-scale parameter, t_{char}, s.t a choice is made every t_{char} interval. This is to say:

$$\Phi(t, t_{char}) = \Phi_{(t \ \% \ t_{char})} \tag{1}$$

given that Φ_n is defined $\forall n$.

3.3 The Specifics:

Given this framework, we can now define what this means in the context of our problem. The possibilities graph will be a graph where the sun is connected to all collector nodes, the collector nodes are connected to all distributors, and all distributors are connected to all other distributors and to all consumers. Next, the choice of spanning sub graph is made based on the distance between all man-made nodes to each other given some characteristic range of each device, and given constraints on power demand and efficiency requirements. The characteristic time-scale of choice in this problem depends on the difficulty, both in terms of energy and time, for a node to switch connections. However, in general the characteristic range should be significantly shorter than the orbital period of the collectors.

4 How to Choose: Mixed-Integer Linear Modeling:

Now that we have defined the dynamics graph framework, both in general and for this specific situation, the next question is, since this is an engineered system, where we have freedom to change the graph at all times for some purpose, how should we choose the connectivity of the graph at any point of time? The important consideration of *why* we would want to change the connectivity over time is the fact that all nodes in the graph are constantly orbiting and thus changing distances from each other. Further, it is important that an optimal route of power flow is chosen as an excessively long route or a route with too many connections results in a massive loss of power efficiency of the system. Given this, the choice of connectivity can be formulated as an MILP that can be solved every t_{char} to redefine the graph for each t_{char} interval that minimizes power-loss while defining connectivity. Such an MILP can be formulated as such:

Parameters:

L_T = The percent loss of power for each connection.

L_D = The percent loss of power for each mile of distance.

d_i = The distance of edge i for each edge in the possibilities graph at the current time.

$PD_{consumer}$ = The demand of consumer

P_i = The maximum power carried by edge i

M = An arbitrarily large number

Decision Variables:

$e_i \triangleq$ Whether or not there edge i in the possibilities graph (with N edges) is in the current graph.

$p_i \triangleq$ Power distributed through edge i.

Objective Function and Constraints:

$$\min \ L_T \cdot \sum_{i=0}^{N} e_i + L_D \cdot \sum_{i=0}^{N} d_i e_i \tag{1}$$

s.t.

$$\begin{aligned}
\sum_{i \in e^-_{collectors}} e_i = \sum_{i \in e^+_{collectors}} e_i &\leq 1 \\
\sum_{i \in e^-_{distributors}} e_i &\leq D_n \\
\sum_{i \in e^+_{distributors}} e_i &\leq 1 \\
p_i &\leq M \cdot e_i \\
\sum_{i \in e^+_v} p_i = \sum_{i \in e^-_v} p_i \quad &\forall v \in G \\
\sum_{i \in e^-_{consumer}} p_i = PD_{consumer} \quad &\forall \ consumers \\
p_i \leq P_i \quad &\forall i \\
p_i \geq 0 \quad &\forall i \\
e_i \in \mathbb{B} \quad &\forall i \\
p_i \in \mathbb{R} \quad &\forall i
\end{aligned} \tag{2}$$

With edge choice variables (choosing edges from the possibility graph), the objective is a minimization of power loss given the number of edges and the distance parameter of each edge. It is expect that through solar system tracking, the distance parameter of edge would be updated every t_{char}. Next, the first three constraints insure that the collectors and distributors have the correct in and our degrees. The fourth constraint then insures the two optimization variables are consistent, while the fifth insures that the amount of power entering any node is the same as the power exiting. The sixth insures that power demand is meet for all consumers, and the seventh insures that an edge is not overloaded with power.

4.1 Complexity and Interesting Dynamics

In envisioning a functional swarm, the size of this system is considerably large. Since the branch and bound solution is typically exponential, computation will be a concern for large systems with many decision variables which relies on continuous optimizations.

Aside from the size of the system, it is interesting to discuss the cyclic nature of the system from a complexity perspective. The position of orbiting bodies and thereby the optimization conditions/choices are periodic. The length of these periods depends on the orbits of the satellites — this placement can be optimized to minimize the period of the system thereby minimizing the computational cost. A short period is preferable in the sense that the ”initial cost” of time that it takes to be able to reuse optimizations is short.

Though this seems to imply that computation is not necessary to repeat after the first period of the system, orbits have chaotic influences. However, it is interesting that in this system there are definite ”resonances” in the optimization conditions of the graph. If understood, these resonances are an opportunity to introduce an element of learning to the optimization. This should be left as a topic for further study as it has applications to a breadth of dynamic systems that have a similar property of ”resonance”.

4.2 The Greedy Approach

As mentioned above, the complexity of the system is potentially prohibitive. In this case it is important to discuss a lighter optimization strategy in a greedy approach.

One approach that shows promise is to make connectivity choices one topological layer at a time. This looks like beginning with a set of collectors assigned to a single consumer and making one degree of connectivity choices at a time. Once a layer is determined, the remaining ones can be iterative chosen. In this way the choice subtree of the possibility graph can be determined. The drawback of this approach is that it neglects larger ”synergies” of the system between orbital layers ie. it is not always the case that a globally optimal solution connects nodes in adjacent orbits. This touches on the important distinction between the topological layers of the choice subtree and the layers of the orbit – they dont always agree.

An alternative approach is to find a path to consumer for each collector satellite one at a time. This takes into account the aforementioned synergies but requires us to manage the indegree constraint for distributors, D_i, in some other way (or to naively disregard this constraint altogether).

Whichever of the two greedy solutions is better is a function of the dimensions of the graph (shorter graphs may benefit more from the first approach while narrower swarms with less collectors may benefit more from the second). In selection between the two it is important to also weigh the relative effects of distance and touch points in power loss using the given technology.

5 Treatment of Resonance Dynamics and Optimization

5.1 Motivation

Recall that we formalize a choice at a given time t as a subtree of a possibility graph. Ideally, this choice changes to accommodate the continuously changing optimization parameters, which are periodic. To do leverage the periodicity of the parameters to lighten computational load we must create some tools to discuss periodic dynamics of a system of multiple oscillators.

5.2 Formalising Resonance

5.2.1 Claim:

The decomposition of a system of multiple oscillators admits a partial ordering and tree representation of resonance. This has implications on optimization.

5.2.2 The Proof and Discussion:

We start by defining a set R_{sys} the set of oscillators in a system, which is in fact a finite commutative group. The operation of addition is the superposition of the time-parametrized waveforms that represent the oscillators. Identity in this group is a flat line. The generating set is the set of simple oscillators in the system which we denote $\{r_i\}$ (note that here we assume these simple oscillators are independent of each other, a reasonable assumption in our case where masses of satellites are relatively small in mass). The group is then populated by complex oscillators that are the result of the superposition of simple oscillators.

In general, these oscillators can represent the dynamics of multiple elements (in our case satellites). Each oscillator has a periodicity and the crests and troughs of complex oscillators characterize the resonance of the simple oscillators that form them. For this reason we will call the elements of the group "resonance classes." For all resonance classes we define a deformation function $D(m)$ which outputs the set of simple oscillators that compose m. $D(m)$ is a bijection from R to the power set of $\{r_i\}$ and so R inherits the partial ordering of this power set. An example of the containment graph is shown below where $R = \{a, b, c\}$.

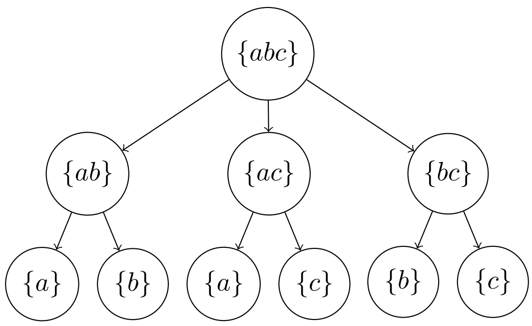


Figure 2: Resonance Network

Since the optimization parameters continuously determine choice via MILP, we can use the resonance network of parameters to discuss the cyclic property of choice. It is important to note that though choice is defined for all time t, the choice itself is discrete.

5.3 Relation to Optimization:

5.3.1 Comparison and Optimization

Given this structure, and given the fact that all elements in a set in a given node in the resonance tree represent time dependent functions, each node in a tree essentially has time dependent value. The relevant value in our orbital example for the node $\{a\}$ is the function $\theta_a(t)$ that represents where in the orbit a given satellite is, ranging from $0 - 2\pi$. Nodes with two or more elements will have the value of the taken from the addition of the sub wave-forms. However in order to find important properties (in this case the property of distance between two nodes) we can reassign there value to be $|\theta_a(t) - \theta_b(t)|$ for the $\{a, b\}$ node or more generally:

$$\{a, b, c, \dots\} \longrightarrow \frac{1}{N} \cdot \sum_{j}^{N} \sum_{i}^{N} |\theta_i(t) - \theta_j(t)| \tag{1}$$

for more than two to represent distance between 2 or more satellites. Over time, this distance will have maximal and minimal values, we are interested in the minimal values as these represent when the nodes are closest to each other and thus this is when resonance occurs. In relation to optimization, each time a node $\{a, b, c, \dots\}$ reaches a minimal value a specific portion of the graph involving the satellites in $\{a, b, c, \dots\}$ will be optimal in our formulated MILP.

5.3.2 Graphical Representation:

Since each minimal value of $\{a, b, c, \dots\}$ will represent a a specific portion of the graph involving the satellites in $\{a, b, c, \dots\}$ we can define the entire graph as a set of these minimal values composed of just one of each satellite in the energy network. For example an optimal energy network of a swarm with just satellites a,b and c, can be represented as $\{\{ab\}_m\{c\}_m\}$ at a time when $|\theta_a(t) - \theta_b(t)|$ is minimal and while this is not true for any other combination of a,b and c. With this we can then define a directed graph where each node represents a total optimal graph described in this set of minimal values and where the edges and direction represent the next optimal graph in time. Since this is periodic, we except to graph to form a directed cycle. This is the goal of this section as now we can simply compute the optimal graph that form these nodes *once* and reuse the optimal connectivity every cycle.

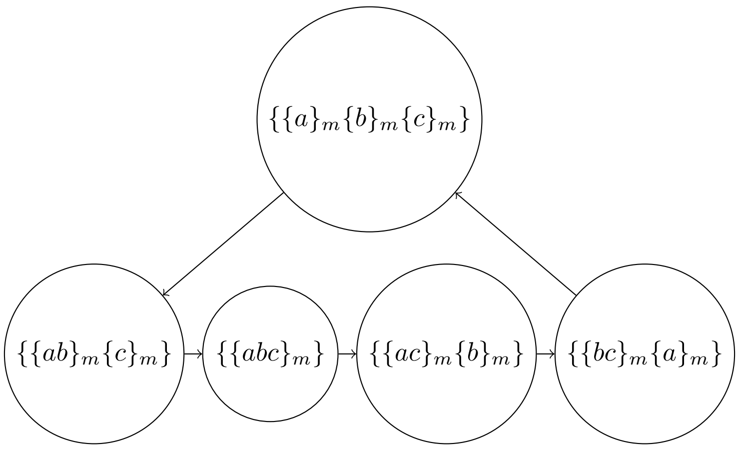


Figure 3: Time Graph

6 Work Cited:

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