HW 5 6-8

(6) a) ① $AA^+ = (AA^+)^T$

Reduced SVD of $A = \tilde{U} D \tilde{V}^T$

$A^+ = \tilde{V} D^{-1} \tilde{U}^T$

$$AA^+ = \tilde{U} D \tilde{V}^T \tilde{V} D^{-1} \tilde{U}^T$$
$$= \tilde{U} D I D^{-1} \tilde{U}^T$$
$$= \tilde{U} D D^{-1} \tilde{U}^T$$
$$= \tilde{U} I \tilde{U}^T$$
$$= \tilde{U} \tilde{U}^T = I$$

$$(AA^+)^T = (A^+)^T A^T$$
$$= (\tilde{V} D^{-1} \tilde{U}^T)^T (\tilde{U} D \tilde{V}^T)^T$$
$$= \tilde{U} D^{-1} \tilde{V}^T \tilde{V} D \tilde{U}^T$$
$$= \tilde{U} D^{-1} I D \tilde{U}^T$$
$$= \tilde{U} D^{-1} D \tilde{U}^T$$
$$= \tilde{U} I \tilde{U}^T$$
$$= \tilde{U} \tilde{U}^T = I$$

② $A^+A = (A^+A)^T$

$$A^+A = \tilde{V} D^{-1} \tilde{U}^T \tilde{U} D \tilde{V}^T$$
$$= \tilde{V} D^{-1} I D \tilde{V}^T$$
$$= \tilde{V} D^{-1} D \tilde{V}^T$$
$$= \tilde{V} I \tilde{V}^T$$
$$= \tilde{V} \tilde{V}^T = I$$

$$(A^+A)^T = A^T (A^+)^T$$
$$= (\tilde{U} D \tilde{V}^T)^T (\tilde{V} D^{-1} \tilde{U}^T)^T$$
$$= \tilde{V} D \tilde{U}^T U D^{-1} \tilde{V}^T$$
$$= \tilde{V} D I D^{-1} \tilde{V}^T$$
$$= \tilde{V} D D^{-1} \tilde{V}^T$$
$$= \tilde{V} I \tilde{V}^T$$
$$= \tilde{V} \tilde{V}^T = I$$

b) $AA^+ = (AA^+)^2$

$$I = AA^+ AA^+$$
$$I = A I A^+$$
$$I = AA^+$$
$$I = I$$

$A^+A = (A^+A)^2$

$$I = A^+ A A^+ A$$
$$I = A^+ I A$$
$$I = A^+ A$$
$$I = I$$

c) $(AA^+)A = A$ or $A(A^+A) = A$

$IA = A$ $\qquad$ $AI = A$

$A = A$ $\qquad$ $A = A$

$(A^+A)A^+ = A^+$ or $A^+(AA^+) = A^+$

$IA^+ = A^+$ $\qquad$ $A^+I = A^+$

$A^+ = A^+$ $\qquad$ $A^+ = A^+$

d) $\text{Row}(A) = \{v_1, v_2, \dots, v_r\}$

The row space of A is spanned by the columns of $\tilde{V}$, therefore the projection matrix onto the row space of A is:

$$P_{\text{Row}(A)} = \tilde{V}\tilde{V}^T$$

Similarly, $A^+A = \tilde{V}\tilde{V}^T$, A^+A is idempotent (property of a projection matrix), and $A^+A = (A^+A)^T$, which ensures the orthogonal projection.

e) $A^+ u_i = \frac{1}{\sigma_i} v_i$

$$\underset{n\times r}{\tilde{V}}\,\underset{r\times r}{D^{-1}}\,\underset{r\times m}{\tilde{U}^T}\,\underset{m\times 1}{u_i} = \frac{1}{\sigma_i} v_i$$

$$\tilde{V} D^{-1} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$\tilde{V} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1/\sigma_i \\ \vdots \\ 0 \end{bmatrix}$$

$$\underset{n\times r}{\begin{bmatrix} v_1 & v_2 & \cdots & v_i & \cdots & v_r \\ \downarrow & \downarrow & & \downarrow & & \downarrow \end{bmatrix}} \underset{r\times 1}{\begin{bmatrix} 0 \\ 0 \\ 1/\sigma_i \\ \vdots \\ 0 \end{bmatrix}} \rightarrow 0 + 0 + \frac{1}{\sigma_i} v_i + 0 + \dots 0 = \boxed{\frac{1}{\sigma_i} v_i}$$

HW 5 cont'd

7 a)

$$A^{+} = \underbrace{\begin{bmatrix} -780/1351 & -985/1393 \\ -780/1351 & 985/1393 \\ -780/1351 & 0 \end{bmatrix}}_{\tilde{V}} \underbrace{\begin{bmatrix} 195/1351 & 0 \\ 0 & 1189/3363 \end{bmatrix}}_{D^{-1}} \underbrace{\begin{bmatrix} -1/2 & -1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix}}_{\tilde{U}^{T}}$$

$$A^{+} = \begin{bmatrix} -1/12 & 1/6 & -1/12 & 1/6 \\ 1/6 & -1/12 & 1/6 & -1/12 \\ 1/24 & 1/24 & 1/24 & 1/24 \end{bmatrix}$$

b) $\bar{x} = A^{+} b$ c) d)

$$\bar{x} = \begin{bmatrix} -5/6 \\ 11/12 \\ 1/24 \end{bmatrix}$$

8 a) $A = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$ $\quad A^{+} = \left(\begin{bmatrix} -1 & 3 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} \right)^{-1} \begin{bmatrix} -1 & 3 & -2 \end{bmatrix}$

$$A^{+} = \begin{bmatrix} -1/14 & 3/14 & -1/7 \end{bmatrix}$$

Since A is 3×1 matrix w/ rank 1, we can apply theorem 2.
$* m \times n$ $\quad * n$

b) If $A = \begin{bmatrix} 3 & 1 & -2 \end{bmatrix}$, a 1×3 matrix w/ rank 1, this matrix does not have full rank (columns = pivots, $3 \neq 1$)

Since A^T has ~~full~~ full rank:

$$A^+ = \underset{1\times1}{(A^TA)^{-1}}\ \underset{3\times T}{A^T}$$

$$\downarrow$$

$$*(A^T)^+ = (A^+)^T * \quad \left((A^T)^+\right)^T = \left((AA^T)^{-1}A\right)^T$$

$$\downarrow$$

$$A^+ = A^T(AA^T)^{-1}$$

$$A^+ = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \left(\begin{bmatrix} 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \right)^{-1}$$

$$A^+ = \begin{bmatrix} 3/14 \\ 1/14 \\ -1/7 \end{bmatrix}$$

c)

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix} \qquad \text{Rank}(A) = 2$$

2×3

$$A^+ = A^T(AA^T)^{-1}$$

$$A^+ = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix} \right)^{-1}$$

$$A^+ = \begin{bmatrix} 1/9 & 0 \\ -5/18 & 1/2 \\ 13/18 & -1/2 \end{bmatrix}$$

9 a) $Ax = b$ has a solution $\longleftrightarrow$ $(AA^+)b = b$

1) $Ax = b$ has a solution, meaning $\exists x_0$, such that $Ax_0 = b$

$(AA^+)b = b \rightarrow (AA^+A)x_0 = b$ $* AA^+A = A$

$\rightarrow Ax_0 = b$ $\therefore$ $(AA^+)b = b$

2) $(AA^+)b = b$ is true $* A$ is $m \times n$

$* A^+ = \tilde{V} \tilde{D}^{-1} \tilde{U}^T$ is $n \times m$

Since, A^+A is the orthogonal projection of $\mathbb{R}^n$ onto Row(A), AA^+ is the orthogonal projection of $\mathbb{R}^m$ onto Col(A) and $b \in Col(A)$

Therefore, given $A = [a_1 \ a_2 \ a_3 \ \dots \ a_n]$ and $x_0 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, there must exist a linear combination using the linearly ind. columns of A to obtain b. This affirms $Ax = b$ has a solution.

b) $A = [1 \ \ 2]$ $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$(AB)^+ = \Big(\underset{1\times 2}{[1 \ \ 2]}\underset{2\times 1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}\Big)^+ = ([1])^+ = [1]$$

HW 5 cont'd

$$A^+ = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix} \qquad B^+ = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$B^+A^+ = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 1/5 \end{bmatrix}$$

$1 \times 2 \qquad 2 \times 1$

$\therefore$, $(AB)^+ \neq B^+A^+$