HW5 1-5

(1) $SVD = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ -1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{15} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}$

By Theorem 2, A is a 3×2 matrix w/ Rank(A) = 2, ∴

$$A^+ = (A^TA)^{-1}A^T$$

$$A^+ = \begin{bmatrix} 7/30 & 1/30 & 2/15 \\ 2/15 & -4/15 & -1/15 \end{bmatrix}$$

(2) By Theorem 5 A is a 3×2 matrix w/ Rank(A) = 2, ∴

$$\bar{x} = A^+ b$$

$$\bar{x} = \begin{bmatrix} 7/30 & 1/30 & 2/15 \\ 2/15 & -4/15 & -1/15 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} -1/30 \\ -11/15 \end{bmatrix}$$

(3) 5a) Reduced $SVD = \underbrace{\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}}_{\substack{\tilde{U} \\ m\times r}} \underbrace{\begin{bmatrix} \sqrt{65} \end{bmatrix}}_{\substack{\Sigma \\ r\times r}} \underbrace{\begin{bmatrix} 2/\sqrt{13} & 3/\sqrt{13} \end{bmatrix}}_{\substack{\tilde{V}^T \\ r\times n}}$

$$A^+ = \begin{bmatrix} 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix} \begin{bmatrix} 1/\sqrt{65} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 4/65 & 2/65 \\ 6/65 & 3/65 \end{bmatrix}$$

5c) Reduced SVD = $\begin{bmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{54} \end{bmatrix} \begin{bmatrix} -1/3 & -2/3 & 2/3 \end{bmatrix}$

$$A^+ = \underbrace{\begin{bmatrix} -1/3 \\ -2/3 \\ 2/3 \end{bmatrix}}_{\tilde{U}} \underbrace{\begin{bmatrix} 1/\sqrt{54} \end{bmatrix}}_{\Sigma^{-1}} \underbrace{\begin{bmatrix} -1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}}_{\tilde{V}^T} = \begin{bmatrix} 1/54 & 2/54 & -1/54 \\ 2/54 & 4/54 & -2/54 \\ -2/54 & -4/54 & 2/54 \end{bmatrix}$$

4 a) $A = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$ $b = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

Rank(A) = 1

Since Rank(A) = 1 $\neq$ n=2, By Theorem 6:

$$\bar{x} = A^+ b$$

$$\bar{x} = \begin{bmatrix} 4/65 & 2/65 \\ 6/65 & 3/65 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} 8/65 \\ 12/65 \end{bmatrix}$$

b) $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -1 & -2 & 2 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

$\bar{x} = A^{+} b = \begin{bmatrix} 1/18 \\ 1/9 \\ -1/9 \end{bmatrix}$

Both A matrices do not contain full rank, therefore each contains a free-variable which means the null-space is non-trivial, meaning there are infinitely many solutions

(5) a) Σ contains 3 non-zero singular values, therefore the rank of 3x3 matrix A is $\text{Rank}(A) = 3$. Since A has full rank, it must be invertible

b)

$$A^{+} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

c) If $AA^{-1} = I$ and $A^{+} = A^{-1}$, then $AA^{+} = I$

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$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1/4 & -1/2 & 1/4 \\ -1/2 & 0 & 1/2 \\ -1/4 & -1/2 & -1/4 \end{bmatrix} = I$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$