(9)

$$\operatorname{col}(A) = \left\{ \begin{array}{c} u_1 \\ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \end{array}, \begin{array}{c} u_2 \\ \begin{bmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{bmatrix} \end{array}, \begin{array}{c} u_3 \\ \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix} \end{array} \right\}$$

Gram-Schmidt orthogonalization

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{bmatrix} - \left(\frac{16}{4}\right)\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix} - \left(\frac{14}{4}\right)\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{12}{8}\right)\begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix} \right\}$$

HW 2 cont'd

(10) $v_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}$

Orthogonal basis: - Linearly independent set of vectors

- Set of vectors must span the entire vector space

- Each pair of vectors in the basis must be orthogonal

$V_1 \cdot V_2 = 0$

$$\overset{V_1}{\begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}} \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0 \qquad \overset{V_2}{\begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}} \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

$2x + y + 0 - w = 0$ $\qquad$ $x + 0 + 3z + 2w = 0$

$2x + y + 0 - w = 0$

$x + 0 + 3z + 2w = 0$

$\rightarrow \left[\begin{array}{cccc|c} 2 & 1 & 0 & -1 & 0 \\ 1 & 0 & 3 & 2 & 0 \end{array}\right]$

$\begin{bmatrix} -3x_3 - 2x_4 \\ 6x_3 + 5x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} -3 \\ 6 \\ 1 \\ 0 \end{bmatrix}}_{V_3} + x_4 \underbrace{\begin{bmatrix} -2 \\ 5 \\ 0 \\ 1 \end{bmatrix}}_{V_4}$

$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 0 \\ 0 & 1 & -6 & -5 & 0 \end{array}\right]$

x_3 free, x_4 free

$x_1 = -3x_3 - 2x_4$

$x_2 = 6x_3 + 5x_4$

Check: $V_1 \cdot V_3 = 0$ $\quad V_2 \cdot V_3 = 0$

$V_1 \cdot V_4 = 0$ $\quad V_2 \cdot V_4 = 0$

(11) $x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad$ Basis $B = \left\{ \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$ $\qquad \begin{bmatrix} 2 \\ 9 \\ 8 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$

$1x + 4y + 4z + w = 0$ $\qquad 2x + 9y + 8z + 2w = 0$

$x + 4y + 4z + w = 0$

$2x + 9y + 8z + 2w = 0$

$\rightarrow \left[\begin{array}{cccc|c} 1 & 4 & 4 & 1 & 0 \\ 2 & 9 & 8 & 2 & 0 \end{array}\right]$

$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] \leftarrow \left[\begin{array}{cccc|c} 1 & 4 & 4 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right]$

x_3 free $\quad x_4$ free

$x_1 = -4x_3 - x_4$

$x_2 = 0$

- orthogonal rows/columns
- Orthonormal columns
- Columns orthogonal to eachother (orthogonal basis)
- $Q^T Q = I = Q^{-1} Q \rightarrow \det(Q) \neq 0,\ Q^{-1} = Q^T$
- upper triangular

(12) $Q = n \times n$ orthogonal matrix

$$\overset{\text{upper triangular}}{Q}\ \overset{\text{lower triangular}}{Q^T} = I$$

$$\begin{bmatrix} * & * & * & * & \cdots \\ 0 & * & * & * & \\ 0 & 0 & * & * & \\ 0 & 0 & 0 & * & \\ & \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} * & 0 & 0 & 0 & \cdots \\ * & * & 0 & 0 & \\ * & * & * & 0 & \\ * & * & * & * & \\ & \vdots & & & \ddots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ & & & 1 & \\ & \vdots & & & \ddots \end{bmatrix}$$

$Q_1 \cdot Q_1 = 1$

All the elements in Q_1 must be squared, and their sum must equal 1. This is only possible if only one of Q_1's elements is 1 or -1 and the rest of the elements are 0 ($(1)^2 = (-1)^2 = 1$).

$$\begin{bmatrix} q_1 \cdot q_1 & 0 & 0 & \\ 0 & q_2 \cdot q_2 & 0 & \cdots \\ 0 & 0 & q_3 \cdot q_3 & \\ & \vdots & & \ddots \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 0 & 1 \end{bmatrix},$$

Since Q is upper triangular, this 1 or -1 must exist on the main diagonal. In the end Q shapes out to be an $n \times n$ diagonal matrix w/ 1 or -1 diagonal entries.

(13) a) $QQ^T = I \rightarrow \det(QQ^T) = \det(I) \Rightarrow \det(QQ^T) = 1 \rightarrow$

$*\det(A) = \det(A^T)$ $\quad \det(Q)\det(Q^T) = 1 \rightarrow \det(Q)^2 = 1 \rightarrow \boxed{\det Q = \pm 1}$

b) $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, columns $\overset{\shortparallel}{v_1}$, $\overset{\shortparallel}{v_2}$

- $\|q_1\| = 1$
- $\|q_2\| = 1$

$\operatorname{Det} Q = (1)(1) = 1$

- $q_1 \cdot q_2 = 0$

- $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ - Basis

HW 2 cont'd

(13) c) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = Q$

d) $Q^T = Q^{-1}$ $\quad \det(A^T) = \det(A)$

$QQ^T = I$ $\quad \det(Q) = \det(Q^T) = -1$

Let

$x = \det(I+Q) = \det((I+Q)^T)$

$Q^T(I+Q) = (I+Q)^T$

$\det(Q^T(I+Q)) = \det((I+Q)^T)$

$\det(Q^T)\det(I+Q) = \det((I+Q)^T)$

$-1\, x = x$

$-x = x$

$+x \quad +x$

$2x = 0$

$x = 0 \quad \therefore \det(I+Q) = 0$

and not invertible

(14) $n \times n$ matrix A satisfies $A^T = -A \rightarrow -(A^T) = A$

$* \; -(A^T) = (-A)^T$

a) $\|x\|^2 = x^T x$

$(A+I)x = 0$

$x^T(A+I)x = 0$

$x^T A x + x^T I x = 0$

$-x^T A x + x^T x = 0$

$-x^T A x = -x^T x$

$x^T A x = x^T x$

$A(x^T x) = x^T x$

$x^T A x = (x^T A^T x)^T$

$= (x^T(-A)x)^T$

$x^T A x = -x^T A^T x$

$\|x\| = 0$

$\uparrow$

This can only be possible if $\|x\|^2 = 0$, further implying x must be the zero vector. This implies $(A+I)x = 0$ only has the trivial sol'n, $x = 0$. This further means $A+I$ is invertible.

$* A+I$ is invertible

Orthogonal matrix: $UU^T = I$

b) $M = (I-A)(I+A)^{-1}$

$M^T = ((I-A)(I+A)^{-1})^T$

$M^T = ((I+A)^{-1})^T (I-A)^T$

$M^T = ((I+A)^T)^{-1} (I-A)^T$

$M^T = (I+A^T)^{-1}(I-A^T)$

$M^T = (I-A)^{-1}(I+A)$

Show $MM^T = I$: $* A^T = -A$

$$
\begin{aligned}
MM^T &= ((I-A)(I+A)^{-1})((I-A)^{-1}(I+A)) \\
&= (I-A)(I+A)^{-1}(I-A)^{-1}(I+A) \\
&= ((I-A)(I-A)^{-1})((I+A)^{-1}(I+A)) \\
&= I\,I \\
&= I
\end{aligned}
$$

Since $MM^T = I$, we can conclude M is an orthogonal matrix