HW 4 9-12

(9) a)

Full SVD = $\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{65} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{13} & 3/\sqrt{13} \\ -3/\sqrt{13} & 2/\sqrt{13} \end{bmatrix}$

Reduced SVD = $\begin{bmatrix} 2\sqrt{5} \\ 1\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{65} \end{bmatrix} \begin{bmatrix} 2/\sqrt{13} & 3/\sqrt{13} \end{bmatrix}$

Rank One Decomposition $= \sqrt{65} \begin{bmatrix} 2\sqrt{5} \\ 1\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{13} & 3/\sqrt{13} \end{bmatrix}$

b) Reduced SVD = $\underbrace{\begin{bmatrix} 1/\sqrt{7} & 0 \\ 1/\sqrt{7} & 0 \\ -1/\sqrt{7} & 2/\sqrt{5} \\ 2/\sqrt{7} & 1/\sqrt{5} \end{bmatrix}}_{\tilde{U}}^{m\times r} \underbrace{\begin{bmatrix} \sqrt{35} & 0 \\ 0 & 5 \end{bmatrix}}_{D}^{r\times r} \underbrace{\begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}}_{\tilde{V}^T}^{r\times n}$

Rank one Decomposition $= \sqrt{35} \begin{bmatrix} 1/\sqrt{7} \\ 1/\sqrt{7} \\ -1/\sqrt{7} \\ 2/\sqrt{7} \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

c) Rank One Decomp, $= \sqrt{54} \begin{bmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} -1/3 & -2/3 & 2/3 \end{bmatrix}$

Reduced SVD = $\begin{bmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{54} \end{bmatrix} \begin{bmatrix} -1/3 & -2/3 & 2/3 \end{bmatrix}$

(10) a) $M = AA^T$ $\quad M = M^T$ $\quad$ * $A = A^T \rightarrow$ Symmetric Matrix

$AA^T = (AA^T)^T$

$AA^T = (A^T)^T A^T$

$AA^T = AA^T$

$A = U\Sigma V^T$

b) $M = AA^T$

$M = U\Sigma V^T (U\Sigma V^T)^T$ *V is an orthogonal matrix, $\therefore V^T V = I$

$M = U\Sigma V^T V \Sigma^T U^T$

$M = U \Sigma I \Sigma^T U^T$

$M = U \Sigma \Sigma^T U^T$

$M = U \Sigma^2 U^T$

- U is an orthogonal matrix
- Σ^2 is a diagonal matrix containing eigenvalues
- U^T is the transpose (or inverse) of U, since $U^{-1} = U^T$

$B = A^T A$

$B = (U\Sigma V^T)^T (U \Sigma V^T)$

$B = V\Sigma^T U^T U \Sigma V^T$ *$UU^T = I$

$B = V \Sigma^T I \Sigma V^T$

$B = V \Sigma^T \Sigma V^T$

$B = V \Sigma^2 V^T$

- V is an orthogonal matrix
- Σ^2 is a diagonal matrix containing eigenvalues
- V^T is the transpose (or inverse) of V, since $V^{-1} = V^T$

11) a) The rank of A is equal to the number of non-zero singular values in Σ

$\therefore$ Rank$(A) = 2$

b) Take rank$(A) = 2$ columns from U to get orthonormal basis for Col(A)

Orthonormal Basis for Col(A) = $\left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \right\}$

c) $$\begin{array}{ccc} v_1 & v_2 & v_3 = v_1 + v_2 \\ \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, & \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, & \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{array}$$

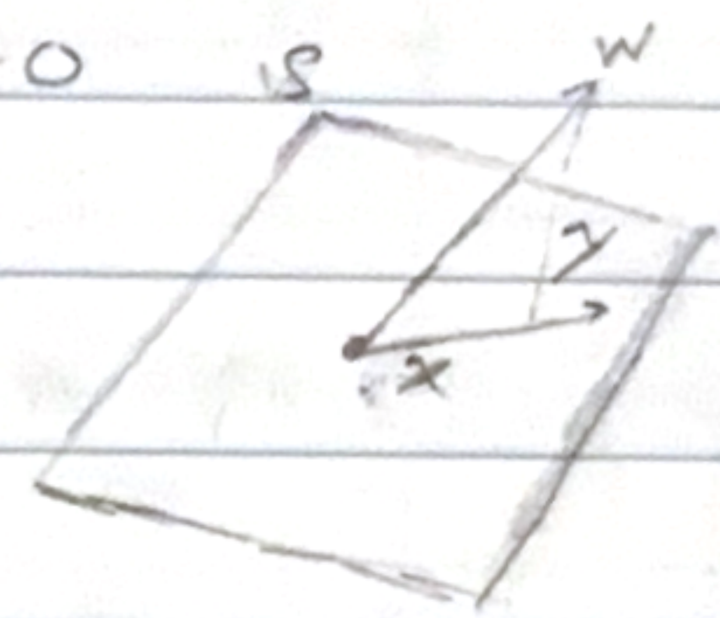
d) $P = U_r U_r^T$ $\quad r = \text{rank}(A) = 2$

$$\underset{4\times 2}{\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}} \underset{2\times 4}{\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}} = \underset{4\times 4}{\begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}} =$$ Projection mat onto Col(A

e) $w = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$

$w = x + y$; $x \in \text{Col}(A)$ $\quad y \cdot x = 0$

$w = \text{proj}_S w + (w - \text{proj}_S w)$



$$S = \text{Col}(A) = \left\{ \underset{u_1}{\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}} \right\}$$

$$x = \text{proj}_S w = \left(\frac{w \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{w \cdot u_2}{u_2 \cdot u_2}\right) u_2 = \begin{bmatrix} 2 \\ 1/2 \\ 1/2 \\ 2 \end{bmatrix}$$

$$y = w - \text{proj}_S w = \begin{bmatrix} 1 \\ -3/2 \\ 3/2 \\ -1 \end{bmatrix} \qquad x \cdot y = \begin{bmatrix} 2 \\ 1/2 \\ 1/2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3/2 \\ 3/2 \\ -1 \end{bmatrix} = 0$$

$$\boxed{x = \begin{bmatrix} 2 \\ 1/2 \\ 1/2 \\ 2 \end{bmatrix} \quad y = \begin{bmatrix} 1 \\ -3/2 \\ 3/2 \\ -1 \end{bmatrix}}$$

* Orthogonal but not orthonormal basis *

(f) $y \in Nul(A^T)$

(g) $Nul(A) = \left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right\}$

(h) $Row(A) = \left\{ \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}, \begin{bmatrix} 4/3 \\ 1/3 \\ 1/3 \end{bmatrix} \right\}$

$*A = A^T$

(12) $\sigma_i = |\lambda_i|$ $\quad * \sigma_i = \|Av_i\|$

v_i = right singular vector of A^TA, so a unit vector

$\|Av_i\| = |\lambda_i|$

$\|\lambda_i v_i\| \overset{?}{=} |\lambda_i|$

$|\lambda_i| \|v_i\| = |\lambda_i|$

$|\lambda_i| \cdot 1 = |\lambda_i|$

$|\lambda_i| = |\lambda_i|$

Since σ_i is an eigenvalue of A^TA, however $A^TA \to AA = A^2$, because A is symmetric, the associated right singular vector of σ_i (v_i) is also an eigenvector for λ_i, therefore $\lambda_i v_i$ is logical.

(13) b) $* \det(A) = \lambda_1 \lambda_2 \lambda_3 \lambda_4 \ldots \lambda_n$

$\det(A^TA) = \sigma_1^2 \sigma_2^2 \sigma_3^2 \sigma_4^2 \ldots \sigma_n^2$

$\det(A^T)\det(A) =$

$(\det(A))^2 = (|\lambda_1||\lambda_2||\lambda_3||\lambda_4| \ldots |\lambda_n|)^2$

$\sigma_i = \sqrt{|\lambda_i|}$, because $\lambda_i \geq 0$

$(\sigma_i)^2 = |\lambda_i|$

$\det(A^TA) = (\det(A))^2$

$\sqrt{\sigma_1^2 \sigma_2^2 \sigma_3^2 \ldots \sigma_n^2} = \sqrt{(|\lambda_1 \lambda_2 \lambda_3 \lambda_4 \ldots \lambda_n)^2}$

$\sigma_1 \sigma_2 \sigma_3 \sigma_4 \ldots \sigma_n = |\lambda_1 \lambda_2 \lambda_3 \lambda_4 \ldots \lambda_n|$

The eigenvalues of A^TA must be ≥ 0, or, rewritten, $|\lambda_i|$. The product of eigenvalues is the determinant of a matrix, so since $\lambda_i = \sigma_i^2$, $\det(A^TA) = \sigma_1^2 \sigma_2^2 \sigma_3^2 \ldots \sigma_n^2$.

The identity of $\det(A^TA)$ is $(\det(A))^2 = (|\lambda_1||\lambda_2||\lambda_3||\lambda_4| \ldots |\lambda_n|)^2$

c) If A is positive definite $\rightarrow$ A is symmetric

Since A is symmetric, $\sigma_i = |\lambda_i|$, therefore the diagonal entries of Σ, are simply $\begin{bmatrix} |\lambda_1| & & & 0 \\ & |\lambda_2| & & \\ 0 & & |\lambda_3| \ddots & \\ & & & |\lambda_n| \end{bmatrix}$, which also represents

D, a diagonal matrix w/ entries corresponding to the eigenvalues of matrix A. Since matrix A is symmetric, the left and right singular vectors are equal, so $U^T = V^T$, or $U = V$, which are both orthogonal matrices. Q is an orthogonal matrix, w/ orthonormal columns corresponding to the eigenvalue entries in D. Therefore, the SVD is equivalent to the orthogonal diagonalization of A.

$$U \Sigma U^T = Q D Q^T$$