HW 2 1-8

(1) Suppose $u, v \in \mathbb{R}^n$

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

$$\left(\sqrt{(u+v)\cdot(u+v)}\right)^2 + \left(\sqrt{(u-v)\cdot(u-v)}\right)^2 = 2\left(\sqrt{u\cdot u}\right)^2 + 2\left(\sqrt{v\cdot v}\right)^2$$

$$(u+v)\cdot(u+v) + u-v \cdot u-v = 2(u\cdot u) + 2(v\cdot v)$$

$$u\cdot u + \cancel{u\cdot v} + \cancel{v\cdot u} + v\cdot v + u\cdot u - \cancel{u\cdot v} - \cancel{v\cdot u} + v\cdot v = 2(u\cdot u) + 2(v\cdot v)$$

$$2(u\cdot u) + 2(v\cdot v) = 2(u\cdot u) + 2(v\cdot v) = 2\|u\|^2 + 2\|v\|^2$$

(2) Suppose $u\cdot v = 0$, $u, v \in \mathbb{R}^n$

$$\|u\| = \|v\| = 1 \rightarrow \sqrt{u\cdot u} = \sqrt{v\cdot v} = 1 \Leftarrow \sqrt{1} \quad \therefore u\cdot u = 1$$

$$v\cdot v = 1$$

$$\|u-v\| = \sqrt{2}$$

$$\sqrt{(u-v)\cdot(u-v)} \rightarrow \sqrt{u\cdot u - \cancel{u\cdot v} - \cancel{v\cdot u} + v\cdot v} = \sqrt{2}$$

$$\sqrt{u\cdot u + v\cdot v} = \sqrt{2}$$

$$\sqrt{1+1} = \sqrt{2}$$

$$\sqrt{2} = \sqrt{2}$$

Aside

$$(\sqrt{x})^2 = (1)^2$$

$$x = 1$$

(3) $u, v, y \in \mathbb{R}^n$

$y \cdot u = 0$

$y \cdot v = 0$

$$y \cdot (c_1 u + c_2 v) = 0$$
$$y \cdot c_1 u + y \cdot c_2 v = 0$$
$$c_1 (y \cdot u) + c_2 (y \cdot v) = 0$$
$$c_1 (0) + c_2 (0) = 0$$
$$0 = 0 \checkmark$$

$\therefore$ y is orthogonal to $c_1 u + c_2 v$ for any scalars c_1 and c_2

(4) Let U and V be $n \times n$ orthogonal matrices:

$U^T U = I$, $U^T = U^{-1}$

$V^T V = I$, $V^T = V^{-1}$

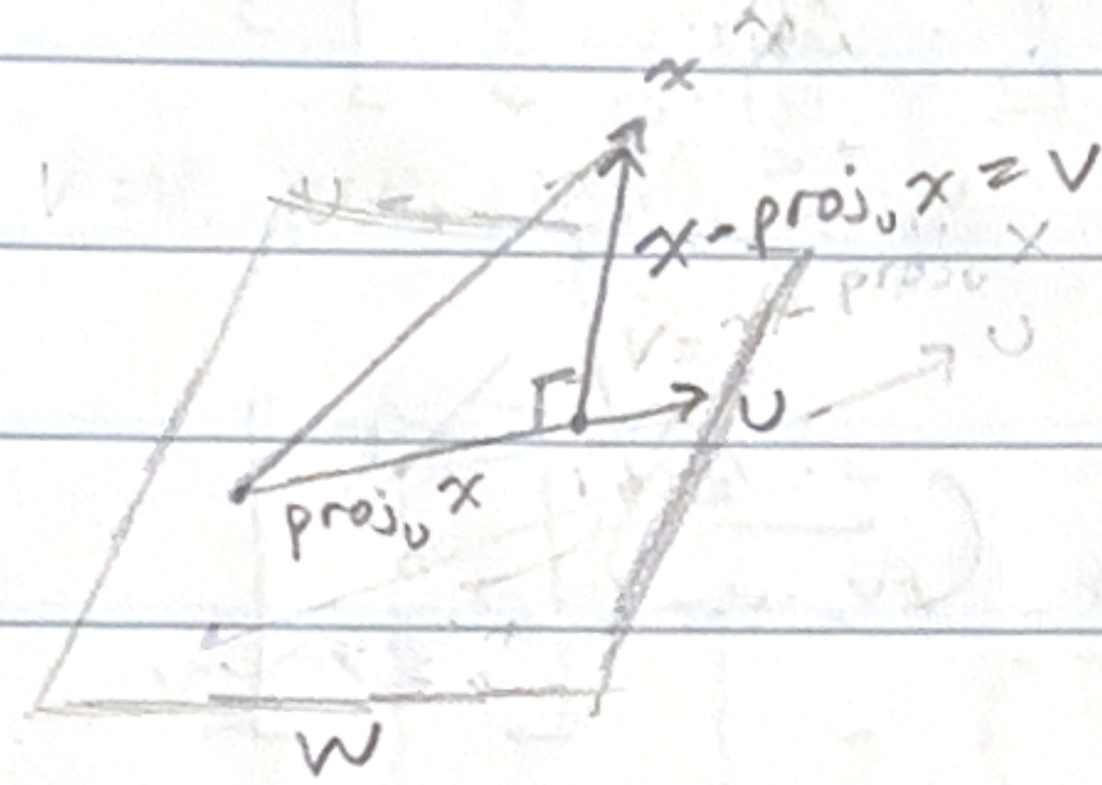
Prove UV is an orthogonal matrix:

$$(UV)^T (UV) = I$$
$$V^T (U^T U) V = I$$
$$V^T I V = I$$
$$V^T V = I$$
$$I = I$$

(5)

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}, \quad x = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

$*$ $\left\{ \underset{u_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}}, \underset{u_3}{\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}} \right\}$ is an orthogonal set



$(x = u + v = \text{proj}_u x + (x - \text{proj}_u x)$

$$u = \text{proj}_u x = \left(\frac{x \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{x \cdot u_2}{u_2 \cdot u_2}\right) u_2 + \left(\frac{x \cdot u_3}{u_3 \cdot u_3}\right) u_3$$

$$= \left(\frac{1}{3}\right)\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \left(\frac{14}{3}\right)\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \left(\frac{-5}{3}\right)\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

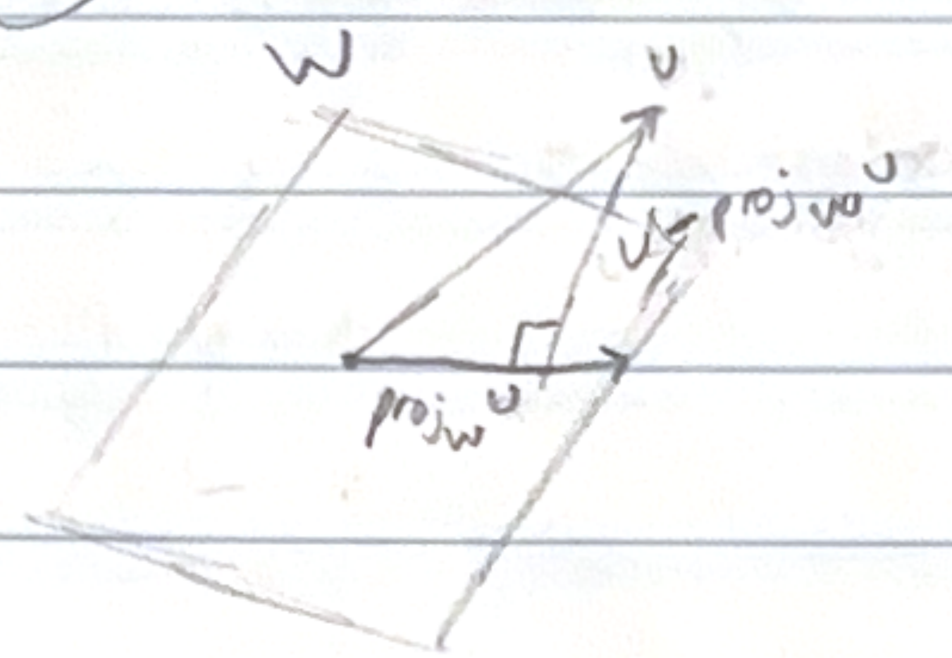
$$= \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ -1/3 \end{bmatrix} + \begin{bmatrix} 14/3 \\ 0 \\ 14/3 \\ 14/3 \end{bmatrix} + \begin{bmatrix} 0 \\ 5/3 \\ -5/3 \\ 5/3 \end{bmatrix}$$

$$u = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

$$V = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

$$u \cdot v = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix} = 0$$

(6)



$$u = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$$

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} \right\}$$

v_1 v_2

$$\text{proj}_W u = \left(\frac{u \cdot v_1}{v_1 \cdot v_1}\right) \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + \left(\frac{u \cdot v_2}{v_2 \cdot v_2}\right) \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

$$= \left(\frac{30}{10}\right) \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + \left(\frac{26}{26}\right) \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -6 \\ -3 \\ 6 \end{bmatrix} + \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\boxed{\text{proj}_W u = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}}$$

(7) $\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix} \right\}$ (v_1, v_2) $\quad v_1 \cdot v_2 = 0, \quad u = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix}$

*Orthogonal basis → lin. ind. set

$$W = \text{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix} \right\}$$

$$\text{proj}_W u = \left(\frac{u \cdot v_1}{v_1 \cdot v_1}\right) v_1 + \left(\frac{u \cdot v_2}{v_2 \cdot v_2}\right) v_2$$

$$= \left(\frac{7}{14}\right) \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix} + (0) \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix} \rightarrow$$

$$\boxed{\operatorname{proj}_W u = \begin{bmatrix} 1 \\ 0 \\ -1/2 \\ -3/2 \end{bmatrix}}$$

(8) $u_1 = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$ $u_2 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$ u_1 and u_2 are <u>not</u> orthogonal

$v_1 = u_1 = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$ * Gram-Schmidt orthogonalization process *

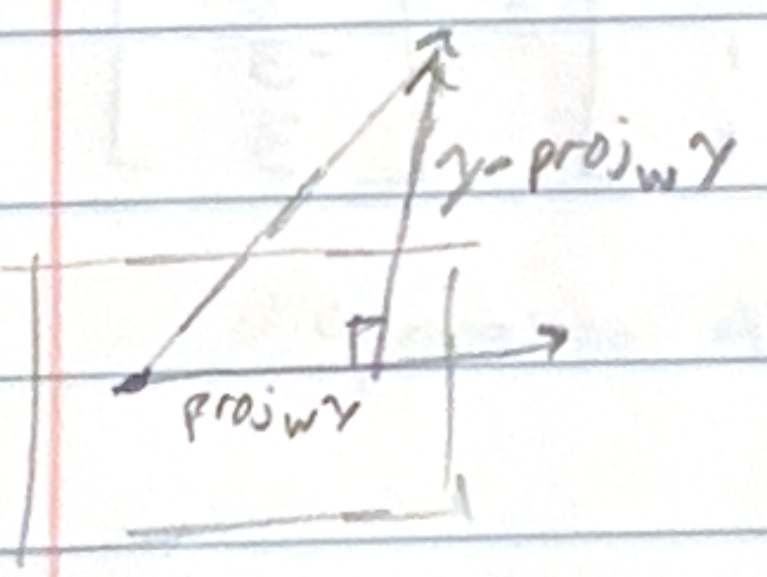
$$v_2 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} - \left(\frac{u_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$$

$$= \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} - \left(\frac{20}{35}\right) \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -12/7 \\ 20/7 \\ 4/7 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -9/7 \\ -6/7 \\ 3/7 \end{bmatrix} \cdot 7 = \begin{bmatrix} -9 \\ -6 \\ 3 \end{bmatrix}$$

$$W = \operatorname{Span}\left\{ \underset{v_1}{\begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} -9 \\ -6 \\ 3 \end{bmatrix}} \right\} \qquad y = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix}$$



$$\operatorname{proj}_W y = \left(\frac{y \cdot v_1}{v_1 \cdot v_1}\right) v_1 + \left(\frac{y \cdot v_2}{v_2 \cdot v_2}\right) v_2$$

$$= \left(\frac{-55}{35}\right) \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix} + \left(\frac{24}{126}\right) \begin{bmatrix} -9 \\ -6 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 33/7 \\ -55/7 \\ -11/7 \end{bmatrix} + \begin{bmatrix} -12/7 \\ -8/7 \\ 4/7 \end{bmatrix}$$

$$\operatorname{proj}_W y = \begin{bmatrix} 3 \\ -9/7 \\ -1 \end{bmatrix}$$

→

$\|y - \text{proj}_W y\|$

$$\left\| \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ -9 \\ -1 \end{bmatrix} \right\|$$

$$\left\| \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} \right\| = \sqrt{2^2+0^2+6^2} = \sqrt{40} = \boxed{2\sqrt{10} = \text{distance}}$$