HW 3

(1) Matrix is orthogonally diagonalizable iff it is symmetric

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

(2) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (3) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

(4) If the sum $A+B$ remains symmetric, it must be orthogonally diagonalizable

$A = A^T$ $\quad A+B = A^T + B^T$

$B = B^T$ $\quad A+B = (A+B)^T$ Since $A+B$ is equal to $(A+B)^T$ (its transpose), $A+B$ is symmetric and therefore orthogonally diagonalizable

(5) a)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\lambda_1 = -1$ $\quad v_1 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

$\lambda_2 = 1$ $\quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_3 = 2$ $\quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Make columns of Q orthonormal:

$$||v_1|| = \sqrt{(-1)^2+(-1)^2+(2)^2} = \sqrt{6}$$

$$||v_2|| = \sqrt{(-1)^2+1^2+0^2} = \sqrt{2}$$

$$||v_3|| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$QDQ^T = \begin{bmatrix} -1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$\quad\quad\quad\quad\quad$ " v_1 $\quad$ " v_2 $\quad$ " v_3

b) $\lambda_1 = -1$

$$P_1 = v_1 v_1^T$$

$$P_1 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/6 & 1/6 & -2/6 \\ 1/6 & 1/6 & -2/6 \\ -2/6 & -2/6 & 4/6 \end{bmatrix}$$

$\lambda_2 = 1$

$$P_2 = v_2 v_2^T$$

$$P_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\lambda_3 = 2$

$$P_3 = v_3 v_3^T$$

$$P_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$A = -1P_1 + 1P_2 + 2P_3$

(6) a) $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ $\lambda_1 = \lambda_2 = 2$ $v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_3 = 5$ $v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Gram-Schmidt Orthogonalization

$v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{v_1 \cdot v_2}{v_1 \cdot v_1} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{v_3 \cdot v_2}{v_2 \cdot v_2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\|v_1\| = \sqrt{2}$

$\|v_2\| = \sqrt{6}$

$v_2 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ $\longrightarrow$

$\|v_3\| = \sqrt{3}$

$$⑥\quad QDQ^T = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

b) $\lambda_1 = 2$

$P_1 = v_1 v_1^T$

$$P_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\lambda_2 = 2$

$P_2 = v_2 v_2^T$

$$P_2 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/6 & 1/6 & -1/3 \\ 1/6 & 1/6 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

$\lambda_3 = 5$

$$P_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$A = 2P_1 + 2P_2 + 5P_3$$

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Orthogonal Basis $= \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix} \right\}$

a)

Orthonormal Basis $= \left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{8} \\ 1/\sqrt{8} \\ 2/\sqrt{8} \\ 1/\sqrt{8} \\ 1/\sqrt{8} \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ -1/2 \\ 1/2 \end{bmatrix} \right\}$

$$P = \underbrace{\begin{bmatrix} 1/2 & -1/\sqrt{8} & 1/2 \\ -1/2 & 1/\sqrt{8} & 1/2 \\ 0 & 2/\sqrt{8} & 0 \\ 1/2 & 1/\sqrt{8} & -1/2 \\ 1/2 & 1/\sqrt{8} & 1/2 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} 1/2 & -1/2 & 0 & 1/2 & 1/2 \\ -1/\sqrt{8} & 1/\sqrt{8} & 2/\sqrt{8} & 1/\sqrt{8} & 1/\sqrt{8} \\ 1/2 & 1/2 & 0 & -1/2 & 1/2 \end{bmatrix}}_{V^T}$$

$$P = \begin{bmatrix} 5/8 & -1/8 & -1/4 & -1/8 & 3/8 \\ -1/8 & 5/8 & 1/4 & -3/8 & 1/8 \\ -1/4 & 1/4 & 1/2 & 1/4 & 1/4 \\ -1/8 & -3/8 & 1/4 & 5/8 & 1/8 \\ 3/8 & 1/8 & 1/4 & 1/8 & 5/8 \end{bmatrix}$$

HW 3 cont'd

b) $\text{Proj}_A(v) = P\begin{bmatrix} -1 \\ 2 \\ 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}$