(8) a) $v_1 = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix}$ $\quad u_1 = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix}$ $\quad u_2 = \begin{bmatrix} 2 \\ 9 \\ 8 \\ 2 \end{bmatrix}$

$$v_2 = \begin{bmatrix} 2 \\ 9 \\ 8 \\ 2 \end{bmatrix} - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/17 \\ 9/17 \\ -8/17 \\ -2/17 \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \\ -8 \\ -2 \end{bmatrix}$$

$\|v_1\| = \sqrt{34}$

$$\text{Basis } W = \left\{ \frac{1}{\sqrt{34}}\begin{bmatrix}1\\4\\4\\1\end{bmatrix}, \frac{1}{\sqrt{153}}\begin{bmatrix}-2\\9\\-8\\-2\end{bmatrix} \right\}$$

$\|v_2\| = \sqrt{153}$

b) $$W^{\perp} = \text{span}\left\{ \underset{v_1}{\begin{bmatrix}-4\\0\\1\\0\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}} \right\}$$

Gram-Schmidt orthogonalization

$$v_1 = \begin{bmatrix}-4\\0\\1\\0\end{bmatrix}$$

$$v_2 = \begin{bmatrix}-1\\0\\0\\1\end{bmatrix} - \frac{4}{17}\begin{bmatrix}-4\\0\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\\-4\\17\end{bmatrix}$$

$\|v_1\| = \sqrt{17}$

$\|v_2\| = \sqrt{306}$

$$\text{Orthonormal Basis} = \left\{ \frac{1}{\sqrt{17}}\begin{bmatrix}-4\\0\\1\\0\end{bmatrix}, \frac{1}{\sqrt{306}}\begin{bmatrix}-1\\0\\-4\\17\end{bmatrix} \right\}$$

c) Projection matrix W:

$$P_W = \begin{bmatrix} 1/\sqrt{34} & -2/\sqrt{153} \\ 4/\sqrt{34} & 9/\sqrt{153} \\ 4/\sqrt{34} & -8/\sqrt{153} \\ 1/\sqrt{34} & -2/\sqrt{153} \end{bmatrix} \begin{bmatrix} 1/\sqrt{34} & 4/\sqrt{34} & 4/\sqrt{34} & 1/\sqrt{34} \\ -2/\sqrt{153} & 9/\sqrt{153} & -8/\sqrt{153} & -2/\sqrt{153} \end{bmatrix} =$$

$$\begin{bmatrix} 1/18 & 0 & 2/9 & 1/18 \\ 0 & 1 & 0 & 0 \\ 2/9 & 0 & 8/9 & 2/9 \\ 1/18 & 0 & 2/9 & 1/18 \end{bmatrix}$$

Projection matrix $W^{\perp}$:

$$P_{W^{\perp}} = \begin{bmatrix} -4/\sqrt{17} & -1/\sqrt{306} \\ 0 & 0 \\ 1/\sqrt{17} & -4/\sqrt{306} \\ 0 & 17/\sqrt{306} \end{bmatrix} \begin{bmatrix} -4/\sqrt{17} & 0 & 1/\sqrt{17} & 0 \\ -1/\sqrt{306} & 0 & -4/\sqrt{306} & 17/\sqrt{306} \end{bmatrix} =$$

$$P_{W^\perp} = \begin{bmatrix} 17/18 & 0 & -2/9 & -1/18 \\ 0 & 0 & 0 & 0 \\ -2/9 & 0 & 1/9 & -2/9 \\ -1/18 & 0 & -2/9 & 17/18 \end{bmatrix}$$

$$x = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix} \qquad x = u + v \quad : u$$

$$u = P_W x = \begin{bmatrix} 5/6 \\ 1 \\ 10/3 \\ 5/6 \end{bmatrix}$$

$$v = P_{W^\perp} x = x - u = \begin{bmatrix} 13/6 \\ 0 \\ -4/3 \\ 19/6 \end{bmatrix}$$

(9) $A = QDQ^T$ Symmetric matrix: $A = A^T$

$A^T = (QDQ^T)^T$

$A^T = QDQ^T$ $\quad * D = D^T$

Since $A = A^T = QDQ^T$, A is orthogonally diagonalizable and therefore symmetric.

(10) a)

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} a-\lambda & b \\ b & c-\lambda \end{bmatrix}$$

$A = QDQ^T$ $\qquad \det(A) = (a-\lambda)(c-\lambda) - b^2 = 0$

$ac - a\lambda - c\lambda + \lambda^2 - b^2 = 0$

$\lambda^2 + (-a-c)\lambda + (ac - b^2) = 0 \longrightarrow$

HW 3 Cont'd

(10) a)

$$\lambda = \frac{-(-a-c) \pm \sqrt{(-a-c)^2 - 4(1)(ac-b^2)}}{2(1)}$$

$$\lambda = \frac{a+c \pm \sqrt{a^2+2ac+c^2-4ac+4b^2}}{2}$$

$$\lambda = \frac{a+c \pm \sqrt{a^2+4b^2+c^2-2ac}}{2}$$

$$a^2+4b^2+c^2-2ac \geq 0$$
$$(a^2-2ac+c^2)+4b^2 \geq 0$$
$$(a^2-ac-ac+c^2)+4b^2 \geq 0$$
$$(a(a-c)-c(a-c))+4b^2 \geq 0$$
$$(a-c)^2+4b^2 \geq 0$$

The sum of squares is always a positive value, therefore if the discriminant is always positive, then the root is always real. →

$$\lambda_1 = \frac{a+c+\sqrt{(a-c)^2+4b^2}}{2} \qquad \lambda_2 = \frac{a+c-\sqrt{(a-c)^2+4b^2}}{2}$$

b) Matrix A has two, distinct, real eigenvalues, meaning each eigenvalue has a corresponding eigenvector. This means A can be diagonalized and S, the matrix containing eigenvectors, can be converted to an orthogonal matrix Q and further establish the form $A = QDQ^T$

(11) a) Each row of $P_i = Q_i Q_i^T$ is a multiple of each other,

$$Q_i Q_i^T = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_n \end{bmatrix} = \begin{bmatrix} c_1c_1 & c_1c_2 & c_1c_3 & \cdots & c_1c_n \\ c_2c_1 & c_2c_2 & c_2c_3 & \cdots & c_2c_n \\ & & \vdots & & \end{bmatrix} \begin{array}{l} \leftarrow \text{scalar } c_1 \\ \leftarrow \text{scalar } c_2 \\ \vdots \end{array}$$

Therefore, there is only one linearly independent row, → rank = 1

HW 3

b) $P_i Q_j = 0$; $i \neq j$

$Q_i Q_i^T Q_j \rightarrow$; $Q_i^T Q_j = Q_i \cdot Q_j = 0$

$Q_i (Q_i \cdot Q_j) \rightarrow$

$Q_i (0) = 0$

c) $P_i P_j = 0$; $i \neq j$

$Q_i Q_i^T Q_j Q_j^T \rightarrow$

$Q_i (Q^T Q_j) Q_j^T \rightarrow$

$Q_i (0) Q_j^T = 0$

d) $P_i Q_i = Q_i Q_i^T Q_i \rightarrow$

$Q_i (Q_i \cdot Q_i)$

$Q_i (1) = Q_i$

(12) $\bar{v}^T(Av) = (\bar{v}^T A)v = (A^T \bar{v})^T v$

$= \bar{v}^T(\lambda v)$ $\qquad = (-A\bar{v})^T v$

$= \lambda \bar{v}^T v$ $\qquad = (-\bar{\lambda}\bar{v})^T v$

$= -\bar{\lambda}\bar{v}^T v$

$\lambda \bar{v}^T v = -\bar{\lambda}\bar{v}^T v$

$\lambda \bar{v}^T v + \bar{\lambda}\bar{v}^T v = 0$

$\bar{v}^T v(\lambda + \bar{\lambda}) = 0$

Let

$\lambda = c + di$ be an eigenvalue of A

w/ corresponding eigenvector $v = u + iw$

$v = u + iw$

$0 + di = \lambda$

$\lambda = di$, where d = real #

Case 1

$\lambda + \bar{\lambda} = 0$

$(c + di) + (c - di) = 0$

$c + \cancel{di} + c - \cancel{di} = 0$

$2c = 0$

$c = 0$

Case 2

$\bar{v}^T v = 0$

$\bar{v} \cdot v = 0$

$(u - iw)(u + iw) = 0$

$u \cdot u + \cancel{u \cdot iw - iw \cdot u} + w \cdot w = 0$

$\|u\|^2 + \|w\|^2 = 0$

only happens when v and w are 0, however $v=0$ is not possible because v is an eigenvector.