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Problem Set 2 #2-5

(1) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Upper triangular matrix, $\therefore$ the eigenvalues are the diagonal entries. Since all the entries are the same, there is only one distinct eigenvalue, $\lambda = 1$.

(2) a) $Av = \lambda v$

$$A \cdot Av = A \cdot \lambda v \rightarrow A^2 v = A \cdot \lambda v \rightarrow$$

$$A \cdot A^2 v = A \cdot A \cdot \lambda v \rightarrow A^3 v = A^2 \cdot \lambda v \rightarrow A^3 v = (A^2 v)\lambda \rightarrow$$

$$A^3 v = (A \cdot \lambda v)\lambda \rightarrow A^3 v = (Av)\lambda^2 \rightarrow A^3 v = \lambda v \cdot \lambda^2 =$$

$$A^3 v = \lambda^3 v$$

The associated eigenvalue of A^3 w/ eigenvector v is λ^3.

c) $Av = \lambda v$ Given: $7A$ $(7A)v \rightarrow 7(Av)$

$\rightarrow 7(\lambda v)$

$\rightarrow (7\lambda)v$

$\therefore (7A)v = (7\lambda)v$

Therefore, v is an eigenvector of $7A$, with associated eigenvalue 7λ.

(2)

b) Given transformed matrix:

$$A + 2I$$

given eigenvalue λ and eigenvector v

$Av = \lambda v$

some eigenvalue

$(A+2I)v = \lambda_2 v$

$Av + 2Iv = \lambda_2 v$

$\lambda v + 2Iv = \lambda_2 v \rightarrow \lambda v + 2Iv = (A+2I)v$

$\rightarrow \lambda v + 2v = (A+2I)v$

$\rightarrow (\lambda + 2)v = (A+2I)v$

$A+2I$ has eigenvector v w/ associated eigenvalue $\lambda + 2$

Assume v is an eigenvector for eigen values λ and λ_2 $\rightarrow$ v is an eigenvector for eigenvalues λ and $\lambda_2 = \lambda + 2$.

(3) a) $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

$\lambda_1 = \lambda_2 = 2$ $\qquad$ $\lambda_3 = 5$

$A - 2I = 0$ $\qquad$ $A - 5I = 0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - x_3 = 0$

$x_1 = x_3$

$x_2 - x_3 = 0$

$x_2 = x_3$

$x_1 + x_2 + x_3 = 0$

$x_1 = -x_2 - x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\text{Eigenvector}_{\lambda=5} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\text{Eigenvectors}_{\lambda=2} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$A = SDS^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

b) $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $\qquad$ $\lambda_1 = \lambda_2 = 0$ $\qquad$ $\lambda_3 = 1$

$A - 0I = 0$ $\qquad$ $A - 1I$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & -1.5 & 0 \\ 0 & 1 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The matrix is not diagonalizable, as the set of eigenvectors only contains 2, when A is a 3×3 matrix, requiring 3 linearly independent eigenvectors associated with 3 eigenvalues in order to be diagonalized.

c) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$\lambda_1 = -1$ $A + 1I = 0 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0.5 & 0 \\ 0 & 1 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$V_1 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$\lambda_2 = 1$ $A - 1I = 0 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 + x_2 = 0$

$x_1 = -x_2$

$x_3 = 0$

$$V_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda_3 = 2$ $A - 2I = 0 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

HW cont'd

$A = SDS^{-1} \rightarrow \begin{bmatrix} -1/2 & -1 & 1 \\ -1/2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1/3 & -1/3 & 2/3 \\ -1/2 & 0.5 & 0 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$

(4) full set of linearly independent eigenvectors? ✓ $\left\{ \overset{v_1}{\begin{bmatrix} 2 \\ -5 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} -1 \\ 3 \end{bmatrix}} \right\}$

Associated Eigenvalues? ✓ $\lambda_1 = 1$, $\lambda_2 = 3$

Original matrix, A, can be diagonalized and therefore written in the format:

$$A = SDS^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

$S^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}^{-1}$

$= \frac{1}{6-5} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2 & -3 \\ -5 & 9 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -9 & -4 \\ 30 & 13 \end{bmatrix} = A}$$

(5) $A = \begin{bmatrix} 13 & 18 \\ -6 & -8 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 13-\lambda & 18 \\ -6 & -8-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (13-\lambda)(-8-\lambda) + 108 = 0$

$-104 - 13\lambda + 8\lambda + \lambda^2 + 108 = 0$

$\lambda^2 - 5\lambda + 4 = 0$

$(\lambda - 4)(\lambda - 1) = 0$

$\lambda = \{4, 1\}$

$\lambda_1 = 4 \qquad A - 4I = 0$

$$\left[\begin{array}{cc|c} 9 & 18 & 0 \\ -6 & -12 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\lambda_2 = 1 \qquad A - 1I = 0$

$(B^2)^{1/2} = (A)^{1/2}$

$B = A^{1/2}$

$$\left[\begin{array}{cc|c} 12 & 18 & 0 \\ -6 & -9 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 2 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$v_2 = \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$$

$A = SDS^{-1}$

$$A = \begin{bmatrix} -2 & -3/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 2 & 4 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} -2 & -3/2 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{-2 + 3/2} \begin{bmatrix} 1 & 3/2 \\ -1 & -2 \end{bmatrix} = -2 \begin{bmatrix} 1 & 3/2 \\ -1 & -2 \end{bmatrix}$$

$$B = A^{1/2} = \begin{bmatrix} -2 & -3/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}^{1/2} \begin{bmatrix} -2 & -3 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -3/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \pm 2 & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 2 & 4 \end{bmatrix} \qquad = \begin{bmatrix} -2 & -3 \\ 2 & 4 \end{bmatrix}$$

$$SD_1S^{-1} = \begin{bmatrix} -2 & -3/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 2 & 4 \end{bmatrix} =$$

$$= \begin{bmatrix} -4 & -3/2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix}$$

HW 1 cont'd

$$SD_2S^{-1} = \begin{bmatrix} -2 & -3/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 3/2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 11 & 18 \\ -6 & -10 \end{bmatrix}$$

$$SD_3S^{-1} = \begin{bmatrix} -2 & -3/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3/2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -11 & -18 \\ 6 & 10 \end{bmatrix}$$

$$SD_4S^{-1} = \begin{bmatrix} -2 & -3/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3/2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -5 & -6 \\ 2 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix}, \begin{bmatrix} 11 & 18 \\ -6 & -10 \end{bmatrix}, \begin{bmatrix} -11 & -18 \\ 6 & 10 \end{bmatrix}, \begin{bmatrix} -5 & -6 \\ 2 & 2 \end{bmatrix}$$