HW 4 1-8

(3) A is positive semidefinite $\rightarrow$ $x^T A x \geq 0$

B is positive semidefinite $\rightarrow$ $x^T B x \geq 0$

$x^T (A+B) x \geq 0$

$x^T A x + x^T B x \geq 0$ Since $x^T A x \geq 0$ and $x^T B x \geq 0$,

$x^T A x + x^T B x \geq 0$

(4) Suppose $x^T A x$ is negative definite :

Choose v to be a unit eigenvector corresponding to λ

$$v^T A v < 0 \Rightarrow v^T(\lambda v) < 0 \Rightarrow \lambda(v^T v) < 0 \Rightarrow \lambda < 0$$

Suppose all eigenvalues of A satisfy $\lambda < 0$

Suppose $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the eigenvalues of A

w/ corresponding eigenvectors $v_1, v_2, v_3, \ldots, v_n$ (assume orthonormal set)

Let x be an arbitrary vector in $\mathbb{R}^n$

Since $v_1, v_2, v_3, \ldots, v_n$ is a basis for $\mathbb{R}^n$, there exist constants $c_1, c_2, c_3, \ldots, c_n$

such that $x = c_1 v_1 + c_2 v_2 + c_3 v_3 + \ldots + c_n v_n$

$$Ax = c_1 A v_1 + c_2 A v_2 + c_3 A v_3 + \ldots + c_n A v_n$$

$$Ax = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + c_3 \lambda_3 v_3 + \ldots + c_n \lambda_n v_n \rightarrow$$

$x^TAx = (c_1 v_1^T + c_2 v_2^T + c_3 v_3^T + \dots + c_n v_n^T)(c_1\lambda_1 v_1 + c_2\lambda_2 + v_2 + \dots + c_n \lambda_n v_n)$

$x^TAx = c_1^2\lambda_1 + c_2^2\lambda_2 + c_3^2\lambda_3 + \dots + c_n^2\lambda_n < 0$

The values $c_1^2\lambda_1, c_2^2\lambda_2, \dots, c_n^2\lambda_n$ are all negative, therefore their sum will also be negative

(5) a) $A = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$ $B = A^TA$

$B = \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$

$B - \lambda I = \begin{bmatrix} 20-\lambda & 30 \\ 30 & 45-\lambda \end{bmatrix} \leftarrow B = \begin{bmatrix} 20 & 30 \\ 30 & 45 \end{bmatrix}$ $* B^T = B$

$(20-\lambda)(45-\lambda) - 900 = 0$

$900 - 20\lambda - 45\lambda + \lambda^2 - 900 = 0$

$\lambda^2 - 65\lambda = 0$

$\lambda(\lambda - 65) = 0$

$\lambda = \{0, 65\}$

$\sigma_1 = 0$ $\Sigma = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{65} \end{bmatrix}$

$\sigma_2 = \sqrt{65}$

$\lambda = 0 \rightarrow B - 0I = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \rightarrow 2x_1 + 3x_2 = 0 \rightarrow x_1 = -\frac{3}{2}x_2 \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} -\frac{3}{2}x_2 \\ x_2 \end{bmatrix}$

$\lambda =$

$v_1 = \frac{1}{\sqrt{13}}\begin{bmatrix} -3 \\ 2 \end{bmatrix} \leftarrow v_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$\lambda = 65 \rightarrow B - 65I = \begin{bmatrix} -45 & 30 \\ 30 & -20 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix} \rightarrow$

$3x_1 - 2x_2 = 0$

$x_1 = \frac{2x_2}{3}$

$V_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \rightarrow \frac{1}{\sqrt{13}} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$V^T = \begin{bmatrix} 2/\sqrt{13} & 3/\sqrt{13} \\ -3/\sqrt{13} & 2/\sqrt{13} \end{bmatrix}$

$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{13}} \cdot \frac{1}{\sqrt{65}} \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{13\sqrt{5}} \begin{bmatrix} 26 \\ 13 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

$U = \begin{bmatrix} 2\sqrt{5} & x \\ 1/\sqrt{5} & y \end{bmatrix}$

$2\sqrt{5} \cdot x + 1/\sqrt{5} \cdot y = 0$

$x = 1$

$y = -2 \rightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$

$U = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$

SVD: $\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{65} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{13} & 3/\sqrt{13} \\ -3/\sqrt{13} & 2/\sqrt{13} \end{bmatrix}$

b)

$A = \begin{bmatrix} -1 & 2 \\ -1 & 2 \\ 5 & 0 \\ 0 & 5 \end{bmatrix}$

$B = A^T A = \begin{bmatrix} -1 & -1 & 5 & 0 \\ 2 & 2 & 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -1 & 2 \\ 5 & 0 \\ 0 & 5 \end{bmatrix} \rightarrow$

$$B = \begin{bmatrix} 27 & -4 \\ -4 & 33 \end{bmatrix} \rightarrow B - \lambda I = \begin{bmatrix} 27-\lambda & -4 \\ -4 & 33-\lambda \end{bmatrix}$$

$$(27-\lambda)(33-\lambda) - 16 = 0$$

$$891 - 60\lambda + \lambda^2 - 16 = 0$$

$$\lambda^2 - 60\lambda + 875 = 0$$

$$\lambda^2 + (-60)\lambda + 875 = 0$$

$$\lambda = \frac{60 \pm \sqrt{(60)^2 - 4(1)(875)}}{2}$$

$$\lambda = \frac{60 \pm \sqrt{100}}{2}$$

$$\lambda_1 = \frac{60+10}{2} \qquad \lambda_2 = \frac{60-10}{2}$$

$$\lambda_1 = 35 \qquad \lambda_2 = 25$$

$$\sigma_1 = \sqrt{35} \qquad \sigma_2 = 5$$

$$\Sigma = \begin{bmatrix} \sqrt{35} & 0 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\lambda_1 = 35 \qquad B - 35I \rightarrow \begin{bmatrix} -8 & -4 \\ -4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \qquad 2x_1 + x_2 = 0$$

$$x_1 = \frac{-1}{2} x_2$$

$$v_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \frac{1}{\sqrt{5}}$$

$$\lambda_2 = 25 \qquad B - 25I \rightarrow \begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \qquad x_1 - 2x_2 = 0$$

$$x_1 = 2x_2$$

$$v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \frac{1}{\sqrt{5}}$$

$$V^T = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{35}} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & 2 \\ -1 & 2 \\ 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{5\sqrt{7}} \begin{bmatrix} 5 \\ 5 \\ -5 \\ 10 \end{bmatrix} \rightarrow \frac{1}{\sqrt{7}} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{5} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & 2 \\ -1 & 2 \\ 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{5\sqrt{5}} \begin{bmatrix} 0 \\ 0 \\ 10 \\ 5 \end{bmatrix} \rightarrow \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$U = \frac{1}{\sqrt{35}} \begin{bmatrix} 1 & 0 & x_1 & x_2 \\ 1 & 0 & y_1 & y_2 \\ -1 & 2 & z_1 & z_2 \\ 2 & 1 & w_1 & w_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix} [x_1 \ \ y_1 \ \ z_1 \ \ w_1] =$$

$$x_1 + y_1 - z_1 + 2w_1 = 0$$

$$\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} [x_1 \ y_1 \ z_1 \ w_1] =$$

$$2z_1 + w_1 = 0$$

$$x_1 + y_1 - z_1 + 2w_1 = 0$$

$$2z_1 + w_1 = 0$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 \end{array}\right]$$

y free, w free

$$z = -\frac{1}{2} w$$

$$x = -y - \frac{5}{2} w$$

$$y \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + w \underbrace{\begin{bmatrix} -5 \\ 0 \\ -1 \\ 2 \end{bmatrix}}_{v_2}$$

Gram Schmidt orthogonalization

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -5 \\ 0 \\ -1 \\ 2 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -5/2 \\ -5/2 \\ -1 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} -5 \\ -5 \\ -2 \\ 4 \end{bmatrix}$$

$$SVD = \begin{bmatrix} 1/\sqrt{7} & 0 & -1/\sqrt{2} & -5/\sqrt{70} \\ 1/\sqrt{7} & 0 & -1/\sqrt{2} & -5/\sqrt{70} \\ -1/\sqrt{7} & 2/\sqrt{5} & 0 & -2/\sqrt{20} \\ 2/\sqrt{7} & 1/\sqrt{5} & 0 & 4/\sqrt{20} \end{bmatrix} \begin{bmatrix} \sqrt{35} & 0 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

c)

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -1 & -2 & 2 \end{bmatrix} \qquad B = A^TA = \begin{bmatrix} 6 & 12 & -12 \\ 12 & 24 & -24 \\ -12 & -24 & 24 \end{bmatrix}$$

$\lambda_1 = 54$ $\qquad \sigma_1 = \sqrt{54}$

$\lambda_2 = \lambda_3 = 0$ $\qquad \sigma_2 = 0$

$\sigma_3 = 0$

$$\lambda_1 = 54 \qquad B - 54I \Rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -1/2\, x_3$

$x_2 = -x_3$

$$v_1 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \frac{1}{3}$$

$$\lambda_2 = 0 \qquad B - 0I \rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -2x_2 + 2x_3 \qquad x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$V^T = \begin{bmatrix} -1/3 & -2/3 & 2/3 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{5}}$$

$$v_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \frac{1}{\sqrt{5}}$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{3} \cdot \frac{1}{\sqrt{54}} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

$$= \frac{1}{9\sqrt{6}} \begin{bmatrix} -9 \\ -18 \\ 9 \end{bmatrix} \rightarrow \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

$$U = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 & x_1 & x_2 \\ -2 & y_1 & y_2 \\ 1 & z_1 & z_2 \end{bmatrix}$$

$$-1x_1 - 2y_1 + z_1 = 0$$

$$V_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$-1x_2 - 2y_2 + z_2 = 0$$

$$V_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Gram-Schmidt Orthogonalization

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \qquad V_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{11}\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -14/11 \\ 8/11 \\ 2/11 \end{bmatrix} \cdot \frac{11}{2} =$$

$$\left\| \begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix} \right\| = \sqrt{66}$$

$$\text{svd} = \begin{bmatrix} -1/\sqrt{6} & 1/\sqrt{11} & -7/\sqrt{66} \\ -2/\sqrt{6} & 1/\sqrt{11} & 4/\sqrt{66} \\ 1/\sqrt{6} & 3/\sqrt{11} & 1/\sqrt{66} \end{bmatrix}\begin{bmatrix} \sqrt{54} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1/3 & -2/3 & 2/3 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \end{bmatrix}$$

6) $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -3 & -1 \end{bmatrix}$

1) $B = A^TA = \begin{bmatrix} 10 & 0 & 5 \\ 0 & 10 & 5 \\ 5 & 5 & 5 \end{bmatrix}$

$\lambda_1 = 15 \qquad \sigma_1 = \sqrt{15}$

$\lambda_2 = 10 \qquad \sigma_2 = \sqrt{10}$

$\lambda_3 = 0 \qquad \sigma_3 = 0$

$\lambda_1 = 15$ $\quad B - 15I \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = x_3$ $\quad x_2 = x_3$ $\quad v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{3}}$

$\lambda_2 = 10$ $\quad B - 10I \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = -x_2$ $\quad x_3 = 0$ $\quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}}$

$\lambda_3 = 0$ $\quad B - 0I \rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = -\frac{1}{2} x_3$ $\quad x_2 = -\frac{1}{2} x_3$ $\quad v_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \frac{1}{\sqrt{6}}$

$$u_1 = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{15}} \begin{bmatrix} 3 & 1 & 2 \\ 1 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3\sqrt{5}} \begin{bmatrix} 6 \\ -3 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & 1 & 2 \\ 1 & -3 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2\sqrt{5}} \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\text{SVD} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ -1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{15} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}$$

2) $A^T = \begin{bmatrix} 3 & 1 \\ 1 & -3 \\ 2 & -1 \end{bmatrix}$ $\quad B = (A^T)^T (A^T) = \begin{bmatrix} 14 & -2 \\ -2 & 11 \end{bmatrix}$

$\lambda_1 = 15 \quad \lambda_2 = 10$

$\sigma_1 = \sqrt{15} \quad \sigma_2 = \sqrt{10}$

$\lambda_1 = 15 \rightarrow B - 15I \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \rightarrow x_1 = -2x_2$

$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \frac{1}{\sqrt{5}}$

$\lambda_2 = 10 \quad B - 10I \rightarrow \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \rightarrow x_1 = \frac{1}{2}x_2 \quad v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \frac{1}{\sqrt{5}}$

$u_1 = \frac{1}{\sqrt{15}} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 3 & 1 \\ 1 & -3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \frac{1}{5\sqrt{3}} \begin{bmatrix} -5 \\ -5 \\ -5 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$

$u_2 = \frac{1}{\sqrt{10}} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 3 & 1 \\ 1 & -3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{5\sqrt{2}} \begin{bmatrix} 5 \\ -5 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$-x - y - z = 0$
$x - y + 0 = 0$
$\rightarrow \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1/2 & 0 \end{array}\right]$

$x_1 = -1/2\, x_3$

$x_2 = -1/2\, x_3$

$u_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \frac{1}{\sqrt{6}}$

$$SVD = \left(\begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{15} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \right)^T \rightarrow$$

$(U \Sigma V^T)^T \rightarrow (V^T)^T \Sigma^T U^T$

$$SVD = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{15} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}$$

(7) A is an $m \times n$ matrix $\underset{m \times m}{U}$ $\underset{m \times m}{M} = \underset{m \times n}{A} \underset{n \times m}{A^T}$

$M = A A^T$

$M = (U \Sigma V^T)(U \Sigma V^T)^T$

$M = U \Sigma V^T V \Sigma^T U^T$

$M = U \Sigma I \Sigma^T U^T$

$M = U \Sigma \Sigma^T U^T$

$M = U \Sigma^2 U^T$ U is an orthogonal matrix, therefore $U^T = U^{-1}$

Σ^2 is a diagonal matrix w/ eigenvalue entries

M is shown to be diagonalized in the form SDS^{-1}, where $S = U$, an orthogonal matrix, $D = \Sigma^2$, and $S^{-1} = U^T$. This implies the columns of U are the eigenvectors of M, w/ corresponding eigenvalues in Σ^2.

(8) * $Q^T Q = I$

$B_1 = A^T A$

$B_2 = (QA)^T (QA) = A^T Q^T Q A = A^T I A = A^T A$

$\Big\}\ B_1 = B_2$

Since both B matrices are the same, their eigenvalues and singular values are equal.