HW 2

(6) A and B are diagonalizable:

$A = SD_1S^{-1}$

$B = SD_2S^{-1}$

$AB = BA$

$(SD_1S^{-1})(SD_2S^{-1}) = (SD_2S^{-1})(SD_1S^{-1})$

$SD_1D_2S^{-1} = SD_2D_1S^{-1}$ ✓

Given D_1 and D_2 are ~~mxn~~ diagonal matrices, they commute.

(7) $\underset{A}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{A^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$ $\det(A) = 1 \neq 0$

$\lambda = 1;$ $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Matrix A is invertible because it's columns are linearly independent and its determinant is $\neq 0$. The set of eigenvectors is not linearly independent therefore it fails to be diagonalizable.

(8)

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

$\det(A) = 0$

$\lambda = 1$, $A - 1I = 0 \rightarrow$ $\begin{array}{c} x_2 \text{ free} \\ \left[\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 0 & -1 & 5 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right] \end{array}$ $x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda = 6$, $A - 6I = 0 \rightarrow$ $\left[\begin{array}{ccc|c} -5 & 1 & 4 & 0 \\ 0 & -6 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_3 \begin{bmatrix} \frac{29}{30} \\ 5/6 \\ 1 \end{bmatrix}$

x_3 free

$-6x_2 + 5x_3 = 0$ $\qquad -5x_1 + \frac{5}{6}x_3 + 4x_3 = 0$

$x_2 = \frac{5}{6}x_3$ $\qquad \frac{-1}{5}(-5x_1) = \left(-\frac{29}{6}x_3\right)\frac{-1}{5}$

$x_1 = \frac{29}{30}x_3$

$\lambda = 0$, $A - 0I = 0 \rightarrow$ $\left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right] \rightarrow$ $x_1 = -x_2$

~~x_2~~

$x_3 = 0$

x_2 free

$x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Eigenvectors = $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 29/30 \\ 5/6 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\lambda = 1 \qquad \lambda = 6 \qquad \lambda = 0$

The matrix is diagonalizable because it has a linearly independent set of eigenvectors w/ respective eigenvalues. Matrix A fails to be invertible, as its columns are linearly dependent and $\det(A) = 0$

(9) Nilpotent matrix $\rightarrow A^k = 0$

a)

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ 1 & -1 & 2 \end{bmatrix}^k = 0 \; ; \; k = 2$$

$$A^2 = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b) Given any strictly upper or lower triangular matrix, where the entries on the main diagonal are zero, the matrix is always nilpotent. Therefore, since the main diagonal entries are 0, the only possible eigenvalue for the matrix is 0.

Given A^k is a nilpotent, $A^k = 0$: $A^k v = \lambda^k v$

$0v = \lambda^k v$

v is a non-zero vector, therefore

$\lambda^k = 0 \rightarrow \lambda = 0$

c) The approximate eigenvalues calculated by MATlab:

$$\lambda = \lambda_2 = \lambda_3 = 0$$

10)

$\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$ are n eigenvalues obtained from an $n \times n$ matrix. Assuming n linearly independent eigenvectors:

$v_1, v_2, \dots, v_n \neq 0$

$$Av_1 = \lambda_1 v_1 \rightarrow Av_1 = 0v_1 \rightarrow Av_1 = 0$$

$$Av_2 = \lambda_2 v_2 \rightarrow Av_2 = 0v_2 \rightarrow Av_2 = 0$$

$$\vdots$$

Since,

$\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$

$$Av_n = \lambda_n v_n \rightarrow Av_n = 0v_n \rightarrow Av_n = 0$$

The non-trivial solutions of $(A - 0I)v = 0$ will determine eigenvectors $v_1, v_2, \dots, v_n$. Given a non-zero, nilpotent $n \times n$ matrix A w/ lin. dependent columns, the set of eigenvectors will strictly always be less than n, as there will always be one column w/ a pivot.

Ex) non-zero nilpotent $\rightarrow$ $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & & & & & \end{bmatrix}$

$\rightarrow n-1$ frees $= n-1$ eigenvectors

Therefore a set of n eigenvectors corresponding w/ n eigenvalues is not possible $\rightarrow$ a non-zero nilpotent matrix is not diagonalizable.

(11) $A^2 = A \rightarrow$ Idempotent

a) Assuming A is idempotent,

$A(Av) = A(\lambda v)$

$A^2 v = A\lambda v$

$Av = (Av)\lambda$

$Av = (\lambda v)\lambda$

$\lambda v = \lambda^2 v \rightarrow \lambda = \lambda^2 \rightarrow \lambda = \{0, 1\}$

b) Assuming A is idempotent, $A^2 = A$,

$M = I - A \rightarrow (M)^2 = (I - A)^2$,

$\rightarrow M^2 = I^2 - AI - AI + A^2$

$\rightarrow M^2 = I - A - A + A^2$

$\rightarrow M^2 = I - A$

$\rightarrow M^2 = M$

c) $A^2 = A$, $A \neq I$;

$F \rightarrow F = T$

Contradiction: If A is invertible, idempotent matrix, then A is the identity matrix

$A^2 = A$

$A^{-1}A^2 = A^{-1}A$

$(A^{-1}A)A = I$

$IA = I \rightarrow A = I$

$\therefore A$ an idempotent matrix that is not the identity matrix is not invertible.

(12) a) ~~2~~ Rank = dimension of column space (lin. ind. columns in the matrix)

Since A contains $\lambda = 0$, it is non-invertible: $\det(A) = 0 \cdot 1 \cdot 2 = 0 = 0$, therefore lin. dep. columns. This means that A does not have a full rank and $\text{Rank}(A) < 3$. It is known that given 3 distinct eigenvalues, there must be 3 distinct eigenvectors. Therefore we must obtain an eigenvector corresponding to each eigenvalue.

Solving $(A - 0I)\vec{v} = \vec{0}$, matrix A, we <u>must</u> obtain only one free column to obtain an eigenvector for $\lambda = 0$ and fulfill 3 distinct eigenvalues = 3 eigenvectors

b) $\det(B^T B) \rightarrow \det(B^T) = \det(B) \rightarrow \det(B) = 0 \cdot 1 \cdot 2$

$\det(B) = 0$

$\downarrow$

$\det(B^T) = 0$

$\downarrow$

$\therefore \det(B^T B) = 0$

$\downarrow$

$\det(B^T) \cdot \det(B)$

$\downarrow$

$\det(B^T) \cdot 0 = 0$

$\downarrow$

$\therefore \det(B^T B) = 0$

HW 1 cont'd

(12) c) Given $\det(B^T B) \to \det(B^T) = \det(B) = 0$

Given $tr(B) = tr(B^T) = 0 + 1 + 2 = 3$

$\det(B) = 0 \cdot 1 \cdot 2 = 0$ $\quad$ $tr(B) = 1 + 2 + 0 = 3$

$\det(B^T) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 0$ $\quad$ $tr(B^T) = \lambda_1 + \lambda_2 + \lambda_3 = 3$

$\therefore \lambda_1, \lambda_2, \lambda_3 = \{(0,0,3), (0,1,2), \ldots\}$

$\therefore$ There is not information provided, as though the determinant and trace of B and B^T are equal, there are more than 1 possible triplets of eigenvalues.

d) $(B+I)^{-1}v = \lambda v$

Given transformed matrix $(B+I)$, the matrix will contain the same eigenvectors as B, however, the eigenvalues will be incremented by 1

$\lambda_1 = 0+1 = 1,\ \lambda_2 = 1+1 = 2,\ \lambda_3 = 2+1 = 3$

Given the matrix $B+I$ has 3 distinct eigenvectors corresponding w/ 3 distinct eigenvalues (w/o 0), the columns of the matrix are linearly independent and therefore invertible. Applying the inverse to $B+I$, the associated eigenvalues become the reciprical:

$\lambda_1 = 1,\ \lambda_2 = \frac{1}{2},\ \lambda_3 = \frac{1}{3}$