CROUTS METHOD.

This is also a direct method. Here also we decompose the coefficient matrix A as LU where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$

and $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$.

Then proceed in the previous method (LU decomposition method).

1) Solve the following system by using crout's method.

$2x+3y+z=-1$, $5x+y+z=9$, $3x+2y+4z=11$.

Soln

The given system can be written in matrix form as $AX=B$, where $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$B = \begin{bmatrix} -1 \\ 9 \\ 11 \end{bmatrix}$

Let $LU = A$

(1,2,)

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

$l_{11} = 2$; $l_{11} u_{12} = 3$

$2 u_{12} = 3$

$u_{12} = 3/2$

$l_{11} u_{13} = 1$

$2 u_{13} = 1$

$u_{13} = 1/2$

$l_{21} = 5$; $l_{21} u_{12} + l_{22} = 1$

$5\left(\frac{3}{2}\right) + l_{22} = 1$

$l_{22} = \frac{-13}{2}$

$l_{21} u_{13} + l_{22} u_{23} = 1$

$5\left(\frac{1}{2}\right) + \left(\frac{-13}{2}\right) u_{23} = 1$

$\frac{-13}{\cancel{2}} u_{23} = -\frac{3}{\cancel{2}}$

$u_{23} = \frac{3}{13}$

$l_{31} = 3$; $l_{31} u_{12} + l_{32} = 2$

$$3\left(\frac{3}{2}\right) + l_{32} = 2$$

$$l_{32} = 2 - \frac{9}{2}$$

$$l_{32} = \frac{-5}{2}$$

$$l_{31} u_{13} + l_{32} u_{23} + l_{33} = 4$$

$$3\left(\frac{1}{2}\right) + \left(\frac{-5}{2}\right)\left(\frac{3}{13}\right) + l_{33} = 4$$

$$l_{33} = 4 + \frac{15}{26} -$$

$$l_{33} = \frac{104 + 15 - 3}{26}$$

$$l_{33} = \frac{80}{26}$$

W.K.T $AX = B$

So $LUX = B$

$LY = B$ where $UX = Y$

$$\begin{bmatrix} 2 & 0 & 0 \\ 5 & -13/2 & 0 \\ 3 & -5/2 & 80/26 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 11 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2y_1 \\ 5y_1 - \frac{13}{2} y_2 \\ 3y_1 - \frac{5}{2} y_2 + \frac{80}{26} y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 11 \end{bmatrix}$$

$$\Rightarrow 2y_1 = -1$$

$$y_1 = -1/2$$

$$\Rightarrow 5\left(-\frac{1}{2}\right) - \frac{13}{2} y_2 = 9$$

$$\frac{-5}{2} - \frac{13}{2} y_2 = 9$$

$$-13/2\, y_2 = 9 + \frac{5}{2}$$

$$\frac{-13}{2} y_2 = 23/2$$

$y_2 = -23/13$

$\Rightarrow 3y_1 - \frac{5}{2} y_2 + \frac{80}{26} y_3 = 11$

$3\left(\frac{-1}{2}\right) - \frac{5}{2}\left(\frac{-23}{13}\right) + \frac{80}{26} y_3 = 11$

$\frac{-39}{26} + \frac{115}{26} + \frac{80}{26} y_3 = 11$

$\frac{80}{26} y_3 = 11 - \frac{76}{26}$

$\frac{80}{\cancel{26}} y_3 = \frac{210}{\cancel{26}}$

$y_3 = \frac{21\cancel{0}}{8\cancel{0}}$

$y_3 = 21/8$

$$\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cancel{-1/2}\ x \\ \cancel{-23/13}\ y \\ \cancel{21/8}\ z \end{bmatrix} = \begin{bmatrix} -1/2 \\ -23/13 \\ 21/8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 3/13 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1/2 \\ -23/13 \\ 21/8 \end{bmatrix}$$

$x + \frac{3}{2} y + \frac{1}{2} z = -\frac{1}{2}$

$y + \frac{3}{13} z = -\frac{23}{13}$

$\Rightarrow z = 21/8$

$\Rightarrow y + \frac{3}{13}\left(\frac{21}{8}\right) = -\frac{23}{13}$

$y = -\frac{23}{13} - \frac{3 \times 21}{13(8)}$

$\cancel{x = \frac{95}{52}}$ $\quad y = \frac{-247}{104} \;\; \frac{\cancel{-86}}{} = -2.375$

$$x + \frac{3}{2}\left(\frac{95}{52}\right) + \frac{1}{2}\left(\frac{21}{8}\right) = \frac{-1}{2}$$

$$x + \frac{285}{104} + \frac{21}{16} = \frac{-1}{2}$$

$$x = \frac{-1}{2} - \frac{21}{16} - \frac{285}{104}$$

$$x = -\frac{29}{16} - \frac{285}{104}$$

$$x = -\frac{\cancel{3784} \; \cancel{7576} \; 1894 \; 947}{\cancel{1664} \; \cancel{832} \; \cancel{416} \; 208}$$

$$x + \frac{3}{2}(-2.375) + \frac{1}{2}\left(\frac{21}{8}\right) = -\frac{1}{2}$$

$$x + (-3.5625) + 1.3125 = -0.5$$

$$\boxed{x = 1.75}$$

ITERATIVE METHODS (INDIRECT METHODS):

2m

Iterative methods can be applied for the system of equns $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$ only if $|a_1| > |b_1| + |c_1|$; $|b_2| > |a_2| + |c_2|$; $|c_3| > |a_3| + |b_3|$ (i.e.,)

We can apply iterative method only when the system of equns is diagonally dominant.

There are 2 iterative methods

i) Gauss Jacobi method &

ii) Gauss Seidal method

GAUSS JACOBI METHOD:

The iteration scheme is given by $\cancel{x^{(r+1)} = \frac{1}{a_1}}$

$$x^{(r+1)} = \frac{1}{a_1}\left[d_1 - b_1(y)^r - c_1(z)^r\right]$$

$$y^{(r+1)} = \frac{1}{b_2}\left[d_2 - a_2(x)^r - c_2(z)^r\right]$$

$$z^{(r+1)} = \frac{1}{c_3}\left[d_3 - a_3(x)^r - b_3(y)^r\right]$$

Taking $x^0 = y^0 = z^0 = 0$, we can find the successive value of x, y, z.

1) Solve the following system by Gauss Jacobi method.

$10x - 5y - 2z = 3$, $4x - 10y + 3z = -3$, $x + 6y + 10z = -3$

Soln

Since the given system of equations are diagonally dominant, iteration method can be applied.

Solving for x, y & z we have,

$$x = \frac{1}{10}\left[3 + 5y + 2z\right]$$

$$y = \frac{1}{-10}\left[-3 - 4x - 3z\right]$$

$$z = \frac{1}{10}\left[-3 - x - 6y\right]$$

1st Iteration:

Taking $x = y = z = 0$

$$x^{(1)} = \frac{1}{10}(3) = 0.3$$

$$y^{(1)} = \frac{1}{10}(3) = 0.3$$

$$(z)^{(1)} = \frac{1}{10}(-3) = -0.3$$

2nd Iteration

~~Taking~~ $x^{(2)} = \frac{1}{10}\left[3 + 5(0.3) + 2(-0.3)\right] = 0.39$

$$y^{(2)} = \frac{1}{-10}\left[-3 - 4(0.3) - 3(-0.3)\right] = 0.33$$

$$z^{(2)} = \frac{1}{10}\left[-3 - (0.3) - 6(0.3)\right] = -0.51$$

3rd Iteration

$$x^{(3)} = \frac{1}{10}\left[3+5(0.33)+2(-0.51)\right] = 0.363$$

$$y^{(3)} = \frac{1}{-10}\left[-3-4(0.39)-3(-0.51)\right] = 0.303$$

$$z^{(3)} = \frac{1}{10}\left[-3-0.39-6(0.33)\right] = -0.537$$

4th Iteration

$$x^{(4)} = \frac{1}{10}\left[3+5(0.303)+2(-0.537)\right] = 0.3441$$

$$y^{(4)} = \cancel{\frac{1}{10}[}\ \frac{1}{-10}\left[-3-4(0.363)-3(-0.537)\right] = 0.2841$$

$$z^{(4)} = \frac{1}{10}\left[-3-0.363-6(0.303)\right] = -0.5181$$

5th Iteration

$$x^{(5)} = \frac{1}{10}\left[3+5(0.2841)+2(-0.5181)\right] = 0.3384$$

$$y^{(5)} = \frac{1}{-10}\left[-3-4(0.3441)-3(-0.5181)\right] = 0.2822$$

$$z^{(5)} = \frac{1}{10}\left[-3-0.3441-6(0.2841)\right] = -0.5049$$

6th Iteration

$$x^{(6)} = \frac{1}{10}\left[3+5(0.2822)+2(-0.5049)\right] = 0.3401$$

$$y^{(6)} = \frac{1}{-10}\left[-3-4(0.3384)-3(-0.5049)\right] = 0.2839$$

$$z^{(6)} = \frac{1}{10}\left[-3-0.3384-6(0.2822)\right] = 0.5032$$

7th Iteration

$x^{(7)} = 0.3413$

$y^{(7)} = 0.2851$

$z^{(7)} = -0.5043$

8th Iteration

$x^{(8)} = 0.3416$

$y^{(8)} = 0.2852$

$z^{(8)} = -0.5051$

9th Iteration

$x^{(9)} = 0.3416$

$y^{(9)} = 0.2851$

$z^{(9)} = -0.5058$

Hence the solution is $x = 0.342$, $y = 0.285$, $z = -0.505$ [Correct to 3 decimal places]

2) $8x - 3y + 2z = 20$, $4x + 11y - z = 33$, $6x + 3y + 12z = 35$

3) $27x + 6y - z = 85$, $6x + 15y + 2z = 72$, $x + y + 54z = 110$

Soln

2) Since the given system of equations are diagonally dominant, iteration method can be applied.

Solving for x, y, z we have

$$x = \frac{1}{8}\left[20 + 3y - 2z\right]$$

$$y = \frac{1}{11}\left[33 + z - 4x\right]$$

$$z = \frac{1}{12}\left[35 - 6x - 3y\right]$$

1st Iteration

Taking $x=y=z=0$

$x^{(1)} = \frac{1}{8}[20] = 2.5$

$y^{(1)} = \frac{1}{11}[33] = 3$

$z^{(1)} = \frac{1}{12}[35] = 2.9167$

2nd Iteration

$x^{(2)} = \frac{1}{8}[20+3(3)-2(2.9167)]$

$= 2.8959$

$y^{(2)} = \frac{1}{11}[33+(2.9167)-4(2.5)]$

$= 2.3561$

$z^{(3)} = \frac{1}{12}[35-6(2.5)-3(3)]$

$= 0.9167$

3rd Iteration

$x^{(3)} = \frac{1}{8}[20+3(2.356)-2(0.917)] = 3.154$

$y^{(3)} = \frac{1}{11}[33-4(2.896)+0.9167] = 2.030$

$z^{(3)} = \frac{1}{12}[35-6(0.896)-3(2.356)] = 0.879$

4th Iteration

$x^{(4)} = 3.041$

$y^{(4)} = 1.933$

$z^{(4)} = 0.832$

5th Iteration

$x^{(5)} = 3.016$

$y^{(5)} = 1.969$

$z^{(5)} = 0.912$

6th Iteration

$x^{(6)} = 3.010$

$y^{(6)} = 1.986$

$z^{(6)} = 0.916$

7th Iteration

$x^{(7)} = 3.015$

$y^{(7)} = 1.988$

$z^{(7)} = 0.915$

8th Iteration

$x^{(8)} = 3.016$

$y^{(8)} = 1.986$

$z^{(8)} = 0.912$

9th Iteration

$x^{(9)} = 3.016$

$y^{(9)} = 1.986$

$z^{(9)} = 0.912$

3) The given eqns are diagonally dominant.

Given,

$$x = \frac{1}{27}[85 - 6y + 2] \rightarrow (1)$$

$$y = \frac{1}{15}[72 - 6x - 2z] \rightarrow (2)$$

$$z = \frac{1}{54}[110 - x - y] \rightarrow (3)$$

1st Iteration: $x = y = z = 0$

$x^{(1)} = \frac{1}{27}[85] = \cancel{21.027}\ 3.148$

$y^{(1)} = \cancel{2.067}\ 4.800$

$z^{(1)} = \cancel{2.019}\ 2.037$

2nd Iteration

$x^{(2)} = 3.134$

$y^{(2)} = 4.783$

$z^{(2)} = 2.035$

3rd Iteration

$x^{(3)} = 2.161$

$y^{(3)} = 3.275$

$z^{(3)} = 1.890$

4th Iteration

$x^{(4)} = 2.490$

$y^{(4)} = 3.684$

$z^{(4)} = 1.936$

5th Iteration

$x^{(5)} = 2.401$

$y^{(5)} = 3.546$

$z^{(5)} = 1.923$

6th Iteration:

$x^{(6)} = 2.429$

$y^{(6)} = 3.601$

$z^{(6)} = 1.977$

7th Iteration

$x^{(7)} = 2.419$

$y^{(7)} = 3.583$

$z^{(7)} = 1.975$

8th Iteration

$x^{(8)} = 2.423$

$y^{(8)} = 3.587$ $z^{(8)} = 1.976$

~~9th Iteration~~

$\cancel{x^{(9)} =}$

9th Iteration

$x^{(9)} = 2.422$

$y^{(9)} = 3.585$

$z^{(9)} = 1.976$

10th Iteration

$x^{(10)} = 2.422$

$y^{(10)} = 3.586$

$z^{(10)} = 1.876$

Hence the values of $x = 2.422, y = 3.586, z = 1.976$

2nd Iteration

$x^{(2)} = 2.157$

$y^{(2)} = 3.269$

$z^{(2)} = 1.890$

3rd Iteration

$x^{(3)} = 2.352$

$y^{(3)} = 3.685$

$z^{(3)} = 1.937$

4th Iteration

$x^{(4)} = 2.401$

$y^{(4)} = 3.601$

$z^{(4)} = 1.925$

5th Iteration

$x^{(5)} = 2.419$

$y^{(5)} = 3.583$

$z^{(5)} = 1.926$

6th Iteration

$x^{(6)} = 2.423$

$y^{(6)} = 3.576$

$z^{(6)} = 1.926$

7th Iteration

$x^{(7)} = 2.425$

$y^{(7)} = 3.574$

$z^{(7)} = 1.926$

8th Iteration

$x^{(8)} = 2.425$

$y^{(8)} = 3.573$

$z^{(8)} = 1.926$

∴ The solution of x is 2.425
y is 3.573
z is 1.926

Correct to three decimal places

GAUSS SEIDAL METHOD:

The iteration scheme is given by

$$x^{(r+1)} = \frac{1}{a_1}\left[d_1 - b_1 y^{(r)} - c_1 z^{(r)}\right]$$

$$y^{(r+1)} = \frac{1}{b_1}\left[d_2 - a_2 x^{(r+1)} - c_2 z^{(r)}\right]$$

$$z^{(r+1)} = \frac{1}{c_3}\left[d_3 - a_3 x^{(r+1)} - b_3 y^{(r+1)}\right]$$

→ At each step, we use latest available values of x, y, z.

→ Initially we take $y^{(0)} = z^{(0)} = 0$ & find $x^{(1)}$.

→ we use $x^{(1)}, z^{(0)}$ to find $y^{(1)}$ and use $x^{(1)}$ and $y^{(1)}$ to find $z^{(1)}$ and so on.

<u>Note</u>:-

* The rate of convergence in Gauss seidal method will be more rapid than in Gauss Jacobi method
* The rate of convergence of Gauss seidal method is <u>roughly twice</u> that of Gauss Jacobi method.

1) Solve the following system : $28x + 4y - z = 32$, $x + 3y - 10z = 24$, $2x + 17y + 4z = 35$ by seidal method.

<u>Soln</u>:-

Since the diagonal elements are not ~~dominant~~ diagonally dominant. We rearrange the equations as follows.

$$28x + 4y - z = 32$$

$$2x + 17y + 4z = 35$$

$$x + 3y - 10z = 24$$

Hence solving for x, y, z, we have

$$x = \frac{1}{28}\left[32 - 4y + z\right]$$

$$y = \frac{1}{17}\left[35 - 2x - 4z\right]$$

$$z = \frac{1}{10}\left[24 - x - 3y\right].$$

1st Iteration

$$x^{(1)} = \frac{1}{28}\left[32 - 4(0) + 2(0)\right] = 1.1429$$

$$y^{(1)} = \frac{1}{17}\left[35 - 2(1.1429) - 4(0)\right] = 1.9244$$

$$z^{(1)} = \frac{1}{10}\left[24 - 1.1429 - 3(1.9244)\right] = 1.7084$$

2nd Iteration

$$x^{(2)} = \frac{1}{28}\left[32 - 4(1.9244) + 1.7084\right] = 0.9290$$

$$y^{(2)} = \frac{1}{17}\left[35 - 2(0.9290) - 4(1.7084)\right] = 1.5476$$

$$z^{(2)} = \frac{1}{10}\left[24 - 0.9290 - 3(1.5476)\right] = 1.8428$$

3rd Iteration

$$x^{(3)} = \frac{1}{28}\left[32 - 4(1.5476) + 1.8428\right] = 0.9876$$

$$y^{(3)} = \frac{1}{17}\left[35 - 2(0.9876) - 4 \times 1.8428\right] = 1.5090$$

$$z^{(3)} = \frac{1}{10}\left[24 - 0.9876 - 3(1.5090)\right] = 1.8485$$

4th Iteration

$$x^{(4)} = \frac{1}{28}\left[32 - 4(1.5090) + 1.8485\right] = 0.9933$$

$$y^{(4)} = \frac{1}{17}\left[35 - 2(0.9933) - 4(\cancel{1.8485})\right] = \cancel{1.5869}\ 1.507$$

$$z^{(4)} = \frac{1}{10}\left[24 - 0.9933 - 3(\cancel{1.5869}\ 1.5070)\right] = 1.8486$$

5th Iteration

$$x^{(5)} = \frac{1}{28}\left[32-4(\cancel{1.5869})+1.8246\right] =$$

$$x^{(5)} = \frac{1}{28}\left[32-4(1.5070)+1.8486\right] = 0.9933$$

$$y^{(5)} = \frac{1}{17}\left[35-2(0.9933)-4(1.8485)\right] = 1.5070$$

$$z^{(5)} = \frac{1}{10}\left[24-0.9933-3(1.5070)\right] = 1.8486$$

6th Iteration

$$x^{(6)} = 0.9933$$

$$y^{(6)} = 1.5070$$

$$z^{(6)} = 1.8486$$

2) Solve the following system of eqns by Gauss seidal method: $x+y+54z=110$; $27x+6y-z=85$; $6x+15y+2z=72$

Soln

Since it's not diagonally dominant, we rearrange the eqns as follows

$$27x+6y-z=85$$

$$6x+15y+2z=72$$

$$x+y+54z=110$$

Hence solving x, y, z we get

$$x = \frac{1}{27}\left[85-6y+z\right]$$

$$y = \frac{1}{15}\left[72-6x-2z\right]$$

$$z = \frac{1}{54}\left[110-x-y\right]$$

1st Iteration

$$x = \frac{1}{27}[85] = 3.1481$$

$$y = \frac{1}{15}[72 - 6(3.1481)] = 3.5408$$

$$z = \frac{1}{54}[110 - 3.1481 - 3.5408] = 1.9132$$

2nd Iteration

$$x = \frac{1}{27}[85 - 6(3.5408) + 1.9132] = 2.4322$$

$$y = \frac{1}{15}[72 - 6(2.4322) - 2(1.9132)] = 3.5720$$

$$z = \frac{1}{54}[110 - 2.4322 - 3.5720] = 1.9258$$

3rd Iteration

$$x = \frac{1}{27}(85 - 6(3.5720) + 1.9258) = 2.4257$$

$$y = \frac{1}{15}[72 - 6(2.4257) - 2(1.9258)] = 3.57$$

$$z = \frac{1}{54}[110 - 2.4257 - 3.5729] = 1.9260$$

4th Iteration

$$x = \frac{1}{27}[85 - 6(3.5729) + 1.9260] = 2.4255$$

$$y = \frac{1}{15}[72 - 6(2.4255) - 2(1.9260)] = 3.573$$

$$z = \frac{1}{54}[110 - 2.4255 - 3.5730] = 1.9260$$

5th Iteration:

$$x = 2.4255$$

$y = 3.5730$

$z = 1.9260$

$\therefore$ Approximate values of $x = 2.4255, y = 3.5730$
$z = 1.9260$

2 Marks :

1) when gauss elimination method fails?

* This method fails if the element in the top of the 1st column is zero.
* we can rectify this by interchanging the rows of the matrix

2. State the sufficient condition for Gauss Jacobi & Gauss Seidal method to converge.

* The coefficient matrix should be diagonally dominant.

3) Compare Gauss Jacobi & Gauss Seidal method

Gauss Jacobi	Gauss Seidal
* Indirect method	* Indirect method
* Rate of convergence is slow	* Rate of convergence is fast (roughly twice that of Jacobi)
* Coefficient should be diagonally dominant	* Coefficient should be diagonally dominant

Direct | Indirect

* Gauss elimination & Gauss Jordan are direct methods | * Gauss Jacobi & Gauss are indirect methods

* We get exact solution. | * We get approximate. so

* It's simple & takes less time. | * It takes more time

3)

Gauss Jacobi

1) Eqns: $5x - 2y + z = -4$, $x + 6y - 2z = -1$, $3x + y + 5z = 13$

Soln

Since the eqns are diagonally dominant iteration method can be applied

Solving for x, y, z we have

$$x = \frac{1}{5}[-4 - z + 2y]$$

$$y = \frac{1}{6}[-1 + 2z - x]$$

$$z = \frac{1}{5}[13 - 3x - y]$$

1st Iteration : $x = y = z = 0$

$x = -0.8$

$y = -0.1666$

$z = 2.6$

2nd Iteration

$x = \cancel{-1.1555}\ -1.3866$

$y = 0.8333$

$z = 3.1133$

Iteration

$x = \cancel{-1.3866}\ -1.0893$

$y = 1.3226$

$z = 3.2653$

4th Iteration

$x = -0.9240$

$y = 1.1033$

$z = 2.9891$

5th Iteration

$x = -0.9565$

$y = 0.9837$

$z = 2.9532$

6th Iteration

$x = -0.9972$

$y = 0.9772$

$z = 2.9772$

7th Iteration

$x = -1.0048$

$y = 0.9919$

$z = 3.0029$

8th Iteration

$x = -1.0039$

$y = 1.0017$

$z = 3.0044$

9th Iteration

$x = -1.0002$

$y = 1.0021$

$z = 2.9998$

10th Iteration

$x = -0.9991$

$y = 0.9999$

$z = 2.9997$

11th Iteration

$x = -0.9999$

$y = 0.9996$

$z = 2.9995$

UNIT-3

INTERPOLATION

→ The process of finding intermediate values of the function from the given set of table values.

$\boxed{2m}$ → For equal intervals of x, we use newton's forward and backward interpolation formula.

$\boxed{2m}$ → We use newton's forward interpolation formula to interpolate the value which is nearer to the beginning of the table.

$\boxed{2m}$ → We use newton's backward interpolation formula to interpolate the value which is nearer to the end of the table.

NEWTON'S FORWARD INTERPOLATION FORMULA:

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots\cdots$$

where $u = \dfrac{x - x_0}{h}$

NEWTON'S BACKWARD INTERPOLATION FORMULA:

$$y(x) = y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \cdots\cdots$$

where $v = \dfrac{x - x_n}{h}$

h - common diff in x value

Note:

1. The interpolating polynomial for n datas will be of degree n-1.

Q1) Find the values of y at $x=21$ and $x=28$.

x :	20	23	26	29
y :	0.3420	0.3907	0.4384	0.4848

Soln

Difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	0.3420			
		0.0487		
23	0.3907		-0.0010	
		0.0477		-0.0003
26	0.4384		-0.0013	
		0.0464		
29	0.4848			

$h=3$

To find y at $x=21$, we use NFIF. Since x is nearer to begng.

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \dots$$

where $u = \frac{x-x_0}{h} = \frac{21-20}{3} = 0.3333$

$$y(21) = 0.3420 + 0.3333(0.0487) + \frac{0.3333(0.3333-1)}{2!} \times (-0.0010) + \frac{(0.333)(0.3333-1)(0.3333-2)}{3!} \times (-0.0003)$$

$y(21) = 0.3583$

To find y at $x = 28$, we use NBIF

$$v = \frac{28-29}{3} = -0.3333$$

$$y(28) = y_n + v\Delta y_n + \frac{v(v+1)}{2!}\Delta^2 y_n + \frac{v(v+1)(v+2)}{3!}\Delta^3 y_n$$

$$= 0.4848 + (-0.3333)(0.0464) + \frac{(-0.3333)(-0.3333+1)}{2}(-0.0013) + \frac{(-0.3333)(-0.3333+1)(-0.3333+2)}{6}(-0.0003)$$

$$y(28) = 0.4695$$

2) The following data are taken from the steam table.

Temp °C	: 140	150	160	170	180
Pressure kgf/cm^2	: 3.685	4.854	6.302	8.076	10.225

Find the pressure at $t = 142°$ and $t = 175°$

Soln

Difference table

Temp (x)	Pressure (y)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
140	3.685				
		1.1690			
150	4.854		0.2790		
		1.4480		0.0470	
160	6.302		0.3260		0.0020
		1.7740		0.0490	
170	8.076		0.3750		
		2.1490			
180	10.225				

$$u = \frac{142-140}{10} = \frac{2}{10} = 0.2000$$

To find pressure $t = 142$ we use NFIF

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!} \times \Delta^4 y_0$$

$$y(142) = 3.685 + (0.2)(1.169) + \frac{(0.2)(0.2-1)}{2}(0.279) + \frac{(0.2)(0.2-1)(0.2-2)}{6} \times (0.047)$$

$$+ \frac{(0.2)(0.2-1)(0.2-2)(0.2-3)}{24} \times (0.002)$$

$$y(142) = 3.8987$$

$$v = \frac{175-180}{10} = \frac{-5}{10} = -0.5$$

To find pressure $t = 175$ we use NBIF

$$y(x) = y_n + v\Delta y_n + \frac{v(v+1)}{2!}\Delta^2 y_n + \frac{v(v+1)(v+2)}{3!}\Delta^3 y_n + \frac{v(v+1)(v+2)(v+3)}{4!}\Delta^4 y_n$$

$$\cancel{= (-0.5)(2.1490) + (-0.5)(2.1490) + \frac{(-0.5)(-0.5+1)}{2}}$$

$$= (10.225) + (-0.5)(2.1490) + \frac{(-0.5)(-0.5+1)}{2}(0.3750)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)}{6}(0.0490) + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{24} \times (0.002)$$

$$y(175) = 9.1005$$

3) The population of a town is as follows. Estimate population increase during the period 1946 to 1976

Year x	Population y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1941	20					
		4				
1951	24		1			
		5		1		
1961	29		2		0	
		7		1		-9
1971	36		3		-9	
		10		-8		
1981	46		-5			
		5				
1991	51					

To find y at $x=1946$, use forward interpolation formula

$$y(x) = y_0 + u\Delta y + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0$$

$$u = \frac{x-x_0}{h} = \frac{1946-1941}{10} = \frac{5}{10} = \frac{1}{2}$$

$$y(1946) = 20 + \left(\frac{1}{2}\right)4 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}(1) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{6}$$

$$+ \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{24}(0) +$$

$$\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-4\right)}{120}(-1)$$

$$y(1946) = 21.69$$

$$v = \frac{x-x_n}{h} = \frac{1976-1991}{10} = \frac{-3}{2}$$

$$y(1976) = 51 + \left(\frac{-3}{2}\right)5 + \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}+1\right)}{2}(-5) + \left[\frac{\left(\frac{-3}{2}\right)\left(-\frac{3}{2}+1\right)\left(-\frac{3}{2}+2\right)}{}\right.$$

$$+\left[\frac{-3/2\left(\frac{-3}{2}+1\right)\left(-3/2+2\right)\left(\frac{-3}{2}+3\right)}{24}\times(-9)\right]+$$

$$\left[\frac{-3/2\left(\frac{-3}{2}+1\right)\left(\frac{-3}{2}+2\right)\left(\frac{-3}{2}+3\right)\left(\frac{-3}{2}+4\right)}{120}(-9)\right]$$

(1976) $= 40.8086$

$\therefore$ Increase in population during the period

$= 40.8086 - 21.69 = 19.1186$ lakhs

4) From the following data find θ at $x=43$ & $x=84$ also express θ in terms of x.

x	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$
40	184				
		20			
50	204		2		
		22		0	
60	226		2		0
		24		0	
70	250		2		
		26		0	
80	276		2		
		28			
90	304				

To find θ at $x=43$, here $u = \frac{x-x_0}{h} = \frac{43-40}{10} = 0.3$

$\theta(43) = 184 + (0.3)(20) + \frac{(0.3)(0.3-1)}{2}(2)$

$= 184 + 60 - 0.21$

$\theta(43) = 189.79$

To find θ at $x=84$, hence $u = \frac{x - x_n}{h} = \frac{84-90}{10} =$

$$\theta\,(x=84) = 304 + (-0.6)(28) + \frac{(-0.6)(-0.6-1)}{2}(2)$$

$$= 286.96e.$$

$$\theta = \theta_0 + u\,\Delta\theta_0 + \frac{u(u-1)}{2!}\Delta^2\theta_0 +$$

$$= 184 + u(20) + \frac{u(u-1)}{2}(2) \quad \text{where } u = \frac{x-4}{10}$$

$$= 184 + 20\frac{(x-40)}{10} + \frac{(x-40)(x-40-10)}{100}$$

$$= 184 + 2x - 80 + \frac{1}{100}(x^2 - 90x + 2000)$$

$$= 0.01x^2 - 0.9x + 20 - 80 + 184 + 2x$$

$$\theta = 0.01x^2 + 1.1x + 124$$

5) Find a polynomial of degree four which takes the values:

x:	2	4	6	8	10
y:	0	0	1	0	0

Soln

Difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	0				
		0			
4	0		1		
		+1		-3	
6	1		-2		6
		-1		3	
8	0		1		
		0			
10	0				

Let us find polynomial using Newton's forward Interpolation formula

$$u = \frac{x - x_0}{h} = \frac{x-2}{2}$$

$$y(x) = y_0 + \Delta y_0 u + \frac{u(u-1)}{2} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \cdots$$

$$= 0 + u \times 0 + \frac{\left(\frac{x-2}{2}\right)\left(\frac{x-4}{2}\right)}{2}(1) + \frac{\left(\frac{x-2}{2}\right)\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)(-3)}{6}$$

$$+ \frac{\left(\frac{x-2}{2}\right)\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)\left(\frac{x-8}{2}\right)}{24}(6)$$

$$= \frac{(x-2)(x-4)}{8}\left[1 - \frac{1}{2}(x-6) + \frac{1}{8}(x-6)(x-8)\right]$$

$$= \frac{1}{64}(x-2)(x-4)\left[8 - 4x + 24 + x^2 - 14x + 48\right]$$

$$y(x) = \frac{1}{64}\left[x^4 - 24x^3 + 196x^2 - 624x + 640\right]$$

UNEQUAL INTERVALS:

For unequal intervals of x, we use the following two methods.

1. Lagrange's interpolation formula
2. Newton's divided difference formula

LAGRANGE'S INTERPOLATION FORMULA:

$$y(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} \times y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} \times y_1 + \frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)\ldots(x_2-x_n)} \times y_2 + \ldots$$

Problems

1) Using lagrange's interpolation formula, find $y(10)$ from the following table.

x :	5	6	9	11
y :	12	13	14	16

Soln

From the given datas, we have

$x_0 = 5$; $x_1 = 6$; $x_2 = 9$; $x_3 = 11$

$y_0 = 12$; $y_1 = 13$; $y_2 = 14$; $y_3 = 16$

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$y(x) = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)} \times 13 +$$

$$\frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)} \times 16$$

$$y(10) = \frac{4 \times 1 \times -1}{-1 \times -4 \times -6} \times 12 + \frac{5 \times 1 \times -1}{1 \times -3 \times -5} \times 13 + \frac{5 \times 4 \times -1}{4 \times 3 \times -2} \times 14$$

$$+ \frac{5 \times 4 \times 1}{6 \times 5 \times 2} \times 16$$

$$= 2 - 4.33 + 11.667 + 5.33$$

$$y(10) = 14.67$$

2) Use Lagrange's formula to fit a polynomial to the following data

x	-1	0	2	3
y	-8	3	1	12

Hence find $y(x=1)$

Soln

From the given datas, we have

$x_0 = -1$, $x_1 = 0$, $x_2 = 2$, $x_3 = 3$

$y_0 = -8$, $y_1 = 3$, $y_2 = 1$, $y_3 = 12$

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times$$

$$= \frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)} \times -8 + \frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)} \times 3$$

$$+ \frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)} \times 1 + \frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)} \times 12$$

$$= \frac{(x^2-2x)(x-3)}{-1 \times -3 \times -4} \times -8 + \frac{(x^2+x-2x-2)(x-3) \times 3}{1(-2)(-3)}$$

$$+ \frac{(x+1)(x^2-3x)}{3(2)(-1)} \times 1 + \frac{(x+1)(x^2-2x)}{4(3)(1)} \times 12$$

$$= \frac{8(x^3-2x^2-3x^2+6x)}{12} + \frac{3(x^3-3x^2-x^2+3x-2x+6)}{6}$$

$$+ \frac{x^3+x^2-3x^2-3x}{-6} + \frac{12(x^3-2x^2+x^2-2x)}{12}$$

$$= \frac{8x^3-16x^2-24x^2+48x}{12} + \frac{3x^3-9x^2-3x^2+9x-6x+12+18}{6}$$

$$+ \frac{x^3-2x^2-3x}{-6} + \frac{12x^3-12x^2-24x}{12}$$

$$= \frac{8x^3-16x^2-24x^2+48x+6x^3-18x^2-6x^2+18x-12x+24 -2x^3+4x^2+6x+12x^3-12x^2-24x}{12}$$

$$= \cancel{8x^3 - 4x} \quad \frac{24x^3 - \overset{72}{\cancel{60}}x^2 + \overset{36x}{\cancel{48x}} + \cancel{8}\,36}{12}$$

$$y(x) = \cancel{2x^3 - 6x^2 + 4x} + 3 \quad 2x^3 - 6x^2 + 3x$$

$$y(1) = 2(1) - 6(1) + 3(1) + 3$$

$$= \cancel{2 - 6 + 4} \quad 2 - 6 + 3 + 3$$

$$y(1) = \cancel{6}\ \cancel{4}\ 2$$

3) Find the parabola of the form $y = ax^2 + bx + c$ passing through the points $(0,0)$, $(1,1)$ and $(2,20)$

Soln

From the given points we have

$x_0 = 0 \quad x_1 = 1 \quad x_2 = 2$

$y_0 = 0 \quad y_1 = 1 \quad y_2 = 20.$

$$y(x) = \frac{(x - x_1)(x - x_2)\cancel{(x - x_3)}}{(x_0 - x_1)(x_0 - x_2)\cancel{(x_0 -}} \times y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \times y_1$$

$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \times y_2$$

$$= \frac{(x-1)(x-2)}{(0-1)(0-2)} \times 0 + \frac{(x-0)(x-2)}{(1-0)(1-2)} \times 1 +$$

$$\frac{(x-0)(x-1)}{(2-0)(2-1)} \times 20$$

$$\frac{\quad}{-1} \qquad \frac{(x-x)}{8} \times \frac{10}{20}$$

$y(x) = -x^2+2x+10x^2-10x$

$y(x) = 9x^2-8x$

3.625

Using Langrange's formula of interpolation find y(9.5).

x :	7	8	9	10
y :	3	1	1	9

Soln

From ~~the~~ given datas, we have

$x_0 = 7 \quad x_1 = 8 \quad x_2 = 9 \quad x_3 = 10$

$y_0 = 3 \quad y_1 = 1 \quad y_2 = 1 \quad y_3 = 9$

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$= \frac{(9.5-8)(9.5-9)(9.5-10) \times 3}{(7-8)(7-9)(7-10)} + \frac{(9.5-7)(9.5-9)(9.5-10) \times 1}{(8-7)(8-9)(8-10)}$$

$$+ \frac{(9.5-7)(9.5-8)(9.5-10)}{(9-7)(9-8)(9-10)} \times 1 + \frac{(9.5-7)(9.5-8)(9.5-9)}{(10-7)(10-8)(10-9)} \times 9$$

$$= \frac{1.5 \times 0.5 \times -0.5 \times 3}{-1 \times -2 \times -3} + \frac{(2.5)(0.5)(-0.5)}{1 \times -1 \times -2}$$

$$+ \frac{(2.5)(1.5)(-0.5)}{2(1)(-1)} + \frac{(2.5)(1.5)(0.5)}{3\times2\times1}\times 9$$

$$y(9.5) = 8.625$$

NEWTON'S DIVIDED DIFFERENCE FORMULA:

The first divided difference of $f(x)$ for two arguments x_0, x_1 is defined as $\frac{f(x_1)-f(x_0)}{x_1-x_0}$ and is denoted by $f(x_0, x_1)$ or $\underset{x_1}{\Delta} f(x_0)$.

The second divided difference of $f(x)$ for three arguments x_0, x_1, x_2 is denoted by $f(x_0, x_1, x_2)$ and is defined as $f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2)$$

$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3) + \cdots$$

1) Form the divided difference ~~formula~~ table for $f(x) = x^3 + x + 2$ for the arguments 1, 3, 6, 11

Soln

Difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
1	4	$28/(3-1) = 14$	$\frac{64-14}{6-1} = 10$
3	32	$192/(6-3) = 64$	$\frac{224-64}{11-3} = 20$
6	224	$1120/(11-6) = 224$	
11	1344		

$\Delta^3 f(x)$

$$\frac{20-10}{11-1} = 1$$

2) Find the polynomial from the following data.

x	0	1	3	4
y	1	4	40	85

Soln

Difference table

x	$y=f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	$f(x_0)$ $\boxed{1}$	$f(x_0,x_1)$ $\frac{4-1}{1-0}=\boxed{3}$	$f(x_0,x_1,x_2)$ $\frac{18-3}{3-0}=\boxed{5}$	$f(x_0,x_1,x_2,x_3)$ $\frac{9-5}{4-0}$
1	4	$\frac{40-4}{3-1}=18$	$\frac{45-18}{4-1}=\frac{27}{3}=9$	
3	40	$\frac{85-40}{4-3}=45$		
4	85			

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)$$
$$+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)+\dots$$

$$=1+(x-0)(3)+(x-0)(x-1)(5)+$$
$$(x-0)(x-1)(x-3)(1)$$

$$=1+3x+(x^2-x)5+(x^2-x)(x-3)$$

$$=1+3x+5x^2-5x+x^3-3x^2-x^2+3x$$

$$f(x)=x^3+x^2+x+1$$

3) From the following table, find $f(x)$ and hence find $f(6)$ using newton's divided difference formula

x	1	2	7	8
$f(x)=y$	1	5	5	4

Soln

Difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1			
		$\frac{5-1}{2-1}=4$		
2	5		$\frac{0-4}{7-1}=\frac{-4}{6}$	
		$\frac{5-5}{7-2}=0$		$\frac{\frac{-1}{6}+\frac{4}{6}}{8-1}=\frac{1/2}{7}=\frac{1}{14}$
7	5		$\frac{-1-0}{6}=\frac{-1}{6}$	
		$\frac{4-5}{1}=-1$		
8	4			

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)$$
$$+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)+\ldots$$

$$f(x)=1+(x-1)(4)+(x-1)(x-2)(-4/6)+(x-1)(x-2)(x-7)(1/14)$$

$$f(x)=1+4x-4+(x^2-3x+2)(-2/3)+(x^2-3x+2)(x-7)(1/14)$$

$$=1+4x-4+\frac{-2}{3}x^2+\frac{6}{3}x-\frac{4}{3}+\left(\frac{x^3}{14}-\frac{3x^2}{14}+\frac{2x}{14}-\frac{7x^2}{14}+\frac{21x}{14}-\frac{14}{14}\right)$$

$$=\cancel{1}+4x-4-0.66x^2+2x-1.33+(0.07x^3-0.21x^2+0.14x-0.5x^2+1.5x-1)$$

$$f(x)=0.07x^3-1.37x^2+7.64x-5.33$$

$$f(6)=6.31$$

Problem:

1) Apply Gauss Forward central difference formula & estimate $f(32)$ from the following table

x	25	30	35	40
$y=f(x)$	0.2707	0.3027	0.3386	0.3794

$\binom{n}{r} = {}^nC_r$

Soln

Let $u = \frac{x-x_0}{h}$

$= \frac{32-30}{5}$

$u = 0.4$

x	u	y	Δy	$\Delta^2 y$	$\Delta^3 y$
25	-1	0.2707			
			0.0320	$\Delta^2 y_{-1}$ $\boxed{0.0039}$	
30	0	y_0 $\boxed{0.3027}$			$\Delta^3 y_{-1}$ $\boxed{0.0010}$
			Δy_0 $\boxed{0.0359}$	0.0049	
35	1	0.3386			
			0.0408		
40	2	0.3794			

By Gauss Forward formula, we have

$$y(x) = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1} \dots$$

$$y(x) = 0.3027 + \binom{0.4}{1} 0.0359 + \binom{0.4}{2}(0.0039) + \binom{1.4}{3} 0.0010$$

$\binom{n}{r} = {}^nC_r$

$$= 0.3027 + (0.4)\, 0.03959 + \frac{(0.4)(0.4-1)}{2}(0.0039) + \frac{(1.4)(1.4-1)(1.4-2)}{3!}$$

$y(x) = 0.31653$

1) Apply Gauss Forward formula to obtain $f(x)$ at $x = 3.5$ from the table below.

x	2	3	4	5
$f(x)$	2.626	3.454	4.784	6.986

Soln

Let $u = \frac{x - x_0}{h}$

$= \frac{3.5 - 3}{1}$

$u = 0.5$

x	u	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2	-1	2.626			
			0.828		
3	0	y_0 $\boxed{3.454}$		$\Delta^2 y_{-1}$ $\boxed{0.502}$	
			Δy_0 $\boxed{1.330}$		$\Delta^3 y_{-1}$ 0.370
4	1	4.784		0.872	
			2.202		
5	2	6.986			

By Gauss forward formula, we have

$$y(x) = y_0 + \binom{u}{1} \Delta y_0 + \binom{u}{2} \Delta^2 y_{-1} + \binom{u+1}{3} \Delta^3 y_{-1} + \ldots$$

$$y(x) = 3.454 + \binom{0.5}{1} 1.330 + \binom{0.5}{2} 0.502 + \binom{1.5}{3} 0.37$$

$$= 3.454 + 0.665 + \frac{0.5(0.5-1)}{2}(0.502) + \frac{(1.5)(1.5-1)(1.5-2)}{3!} \times 0.37$$

$$= 3.454 + 0.665 - 0.06275 - 0.023125$$

$$y(x) = 4.0331$$

8) Use Gauss forward Interpolation formula to get y_{16} given

x	5	10	15	20	25
y	26.782	19.951	14.001	8.762	4.163

Soln

$$u = \frac{x - x_0}{h}$$

$$= \frac{16-15}{5} = 0.2$$

x	u	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
5	-2	26.782				
			-6.831			
10	-1	19.951		0.881		
			-5.95		-0.17	
15	0	$\boxed{14.001}$ (y_0)		$\boxed{0.711}$ ($\Delta^2 y_{-1}$)		$\boxed{0.099}$ ($\Delta^4 y_{-2}$)
			$\boxed{-5.239}$ (Δy_0)		$\boxed{-0.071}$ ($\Delta^3 y_{-1}$)	
20	1	8.762		0.64		
			-4.599			
25	2	4.163				

By Using Gauss forward formula, we have

$$y(x) = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1} + \binom{u+1}{4}\Delta^4 y_{-2}$$

$$= (14.001) + \binom{0.2}{1}(-5.239) + \binom{0.2}{2}(0.711) + \binom{1.2}{3}(-0.071) + \binom{1.2}{4}(0.099)$$

$$= 14.001 - 1.0478 + \frac{(0.2)(0.2-1)}{2}(0.711) + \frac{(1.2)(1.2-1)(0.2-1)}{3!}(-0.071) + \frac{(1.2)(1.2-1)(1.2-2)(1.2-3)}{4!}\times(0.099)$$

$$= 14.001 - 1.0478 - 0.05688 + 0.002272 + 0.014256 = 12.9128$$

Gauss Backward central difference formula

$$y(x) = y_0 + \binom{u}{1}\Delta y_{-1} + \binom{u+1}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-2} + \binom{u+2}{4}\Delta^4 y_{-2} + \dots$$

1) Using Gauss Backward Interpolation formula find the population for the year 1936

Year x :	1901	1911	1921	1931	1941	1951
Population in thousand } y :	12	15	20	27	39	52

Soln

Let $u = \dfrac{x - x_0}{h}$

$= \dfrac{1936 - 1941}{10}$

$u = -5/10 = -0.5$

x	u	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1901	-4	12					
			3				
1911	-3	15		2			
			5		0		
1921	-2	20		2		3	
			7		3		-10
1931	-1	27		5		-7	
			$\boxed{12}$		$\boxed{-4}$		
1941	0	$\boxed{39}$		$\boxed{1}$			
			13				
1951	1	52					

By using ~~Ba~~ Gauss Backward formula, we have

$$y(x) = y_0 + \binom{u}{1}\Delta y_{-1} + \binom{u+1}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y$$

$$+ \binom{u+2}{4}\Delta^4 y_{-2} + \ldots$$

$$y(1936) = 39 + (-0.5)12 + \binom{0.5}{2}1 + \binom{0.5}{3}(-4)$$

$$= 39 - 6 + \frac{(0.5)(0.5-1)}{2} + \frac{(0.5)(0.5-1)(0.5-\ldots}{6}$$

$$= 33 - 0.125 + 0.25$$

$$= \cancel{33}\ 32.625$$

2 Marks

1) If $f(x) = 1/x$ then find the divided differences $f(a,b,c)$

Soln

Given $f(x) = 1/x$

$$f(a) = 1/a \; ; \; f(b) = 1/b$$

Now, $$f(a,b) = \frac{f(b) - f(a)}{b-a}$$

$$= \frac{1/b - 1/a}{b-a} = \frac{a-b}{ab(b-a)} = \frac{-(b-\ldots}{ab}$$

$$f(a,b) = \frac{-1}{ab}$$

Now $f(a,b,c) = \dfrac{f(b,c) - f(a,b)}{c-a}$

$$= \frac{\frac{-1}{bc} + \frac{1}{ab}}{c-a} = \frac{\frac{\cancel{-ab+bc}\; -a+c}{abc}}{c-a}$$

$$f(a,b,c) = \frac{1}{abc}$$

2) If $f(x) = 1/x^2$, then find the divided differences of $f(a,b)$ and $f(a,b,c)$

Soln

Given $f(x) = 1/x^2$

$f(a) = 1/a^2$; $f(b) = 1/b^2$; $f(c) = 1/c^2$

$$f(a,b) = \frac{f(b) - f(a)}{b-a}$$

$$= \frac{1/b^2 - 1/a^2}{b-a} = \frac{\frac{a^2-b^2}{(ab)^2}}{(b-a)}$$

$$= \frac{\frac{(a+b)(a-b)}{ab \times ab}}{b-a}$$

$$f(a,b) = -\frac{(a+b)}{(ab)^2}$$

$$f(a,b,c) = \frac{f(b,c) - f(a,b)}{c-a} = \frac{-\frac{(b+c)}{(bc)^2} + \frac{(a+b)}{(ab)^2}}{c-a}$$

$$= \frac{-a^2b - a^2c + c^2a + c^2b}{(abc)^2(c-a)}$$