20MA008 - NUMERICAL COMPUTING

UNIT - I

Solution of algebraic and transcendental equations and Eigen value problems:

⇒ Solution to algebraic and transcendental equations : Bisection method - Iteration method - Method of false position - Newton Raphson method. Iteration method for Eigen-values : Power method - Jacobi's method.

BISECTION METHOD: [Bolzano's Method].

(Fundamental theorem):

If $f(x)$ is continuous in the interval (a,b) and if $f(a)$ and $f(b)$ are of opposite signs, then the equation, $F(x)=0$ will have atleast one real root between a and b.

Q Find the positive root of $x^3-x=1$ correct to four decimal places by bisection method.

Solution:

Given $x^3-x=1$

$x^3-x-1=0$.

$f(x)=0$

$\therefore f(x) = x^3-x-1$

now, $f(0) = 0^3-0-1 = -1 = -ve$.

$f(1) = (1)^3 - 1 - 1 = -1 = -ve.$

$f(2) = 2^3 - 2 - 1 = 5 = +ve$

The root lies between 1 and 2 because here f(1) and f(2) are of opposite signs.

let $x_0 = \frac{a+b}{2} = \frac{1+2}{2} = \frac{3}{2} = 1.5$

now $f(1.5) = (1.5)^3 - (1.5) - 1$

$= 0.8750$

$= +ve.$

f(1) and f(1.5) are of opposite signs.

The root lies between 1 and 1.5.

let $x_1 = \frac{1+1.5}{2} = \frac{2.5}{2} = 1.25$

now, $f(1.25) = (1.25)^3 - (1.5) - 1$

$= -0.2969$

$= -ve.$

f(1.25) and f(1.5) are of opposite signs.

The root lies between 1.25 and 1.5.

let $x_2 = \frac{1.25+1.5}{2} = \frac{2.75}{2} = 1.3750$

now, $f(1.3750) = (1.3750)^3 - 1.3750 - 1$

$= 0.2246$

$= +ve.$

$f(1.25)$ and $f(1.3750)$ are of opposite sign

The root lies between 1.25 and 1.3750.

$$\text{let } x_3 = \frac{1.3750 + 1.25}{2}$$

$$= \frac{2.6250}{2}$$

$$= 1.3125$$

$$\text{now } f(1.3125) = (1.3125)^3 - 1.3125 - 1$$

$$= -0.0515$$

$$= -ve.$$

$f(1.3125)$ and $f(1.3750)$ are of opposite signs.

The root lies between 1.3125 and 1.3750.

$$\text{let } x_4 = \frac{1.3125 + 1.3750}{2}$$

$$= \frac{2.6875}{2}$$

$$= 1.3438$$

$$\text{now } f(1.3438) = (1.3438)^3 - 1.3438 - 1$$

$$= 0.0828$$

$$= +ve.$$

$f(1.3125)$ and $f(1.3438)$ are of opposite signs

The root lies between 1.3125 and 1.3438.

$$\text{let } x_5 = \frac{1.3125 + 1.3438}{2}$$

$$= \frac{2.6563}{2}$$

$$= 1.3282$$

$f(1.3282) = (1.3282)^3 - 1.3282 - 1$

$= 0.0149$

$= +ve.$

The root lies between 1.3125 and 1.3282.

let $x_6 = \frac{1.3125 + 1.3282}{2}$

$= \frac{2.6407}{2}$

$= 1.3204$

$f(1.3204) = (1.3204)^3 - 1.3204 - 1$

$= -0.0183$

$= -ve$

The root lies between 1.3204 and 1.3282.

let $x_7 = \frac{1.3204 + 1.3282}{2}$

$= \frac{2.6486}{2}$

$= 1.3243$

$f(1.3243) = (1.3243)^3 - 1.3243 - 1$

$= -0.0018$

$= -ve.$

$\therefore$ The root lies between 1.3243 and 1.3282

let $x_8 = \frac{1.3243 + 1.3282}{2} = \frac{2.6525}{2}$

$$= 1.3263$$

$$f(1.3263) = (1.3263)^3 - (1.3263) - 1$$
$$= 0.0068$$
$$= +ve.$$

The root lies between 1.3243 and 1.3263.

let $x_9 = \dfrac{1.3243 + 1.3263}{2}$

$$= \frac{2.6506}{2}$$

$$= 1.3253$$

$$f(1.3253) = (1.3253)^3 - 1.3253 - 1$$
$$= 0.0025$$
$$= +ve.$$

The root lies between 1.3243 and 1.3253

let $x_{10} = \dfrac{1.3243 + 1.3253}{2}$

$$= \frac{2.6496}{2}$$

$$\boxed{x_{10} = 1.3248}$$

$$f(1.3248) = (1.3248)^3 - 1.3248 - 1$$
$$= 0.0003$$
$$= +ve.$$

The root lies between 1.3243 and 1.3248

let $x_{11} = \dfrac{1.3243 + 1.3248}{2}$

$$= \frac{2.6491}{2} = 1.3246$$

$f(1.3282) = (1.3282)^3 - 1.3282 - 1$

$= 0.0149$

$= +ve.$

The root lies between 1.3125 and 1.3282.

let $x_6 = \dfrac{1.3125 + 1.3282}{2}$

$= \dfrac{2.6407}{2}$

$= 1.3204$

$f(1.3204) = (1.3204)^3 - 1.3204 - 1$

$= -0.0183$

$= -ve$

The root lies between 1.3204 and 1.3282.

let $x_7 = \dfrac{1.3204 + 1.3282}{2}$

$= \dfrac{2.6486}{2}$

$= 1.3243$

$f(1.3243) = (1.3243)^3 - 1.3243 - 1$

$= -0.0018$

$= -ve.$

$\therefore$ The root lies between 1.3243 and 1.3282

let $x_8 = \dfrac{1.3243 + 1.3282}{2} = \dfrac{2.6525}{2}$

$= 1.3263$

$f(1.3263) = (1.3263)^3 - (1.3263) - 1$

$= 0.0068$

$= +ve.$

The root lies between 1.3243 and 1.3263.

let $x_9 = \dfrac{1.3243 + 1.3263}{2}$

$= \dfrac{2.6506}{2}$

$= 1.3253$

$f(1.3253) = (1.3253)^3 - 1.3253 - 1$

$= 0.0025$

$= +ve.$

The root lies between 1.3243 and 1.3253

let $x_{10} = \dfrac{1.3243 + 1.3253}{2}$

$= \dfrac{2.6496}{2}$

$\boxed{x_{10} = 1.3248}$

$f(1.3248) = (1.3248)^3 - 1.3248 - 1$

$= 0.0003$

$= +ve.$

The root lies between 1.3243 and 1.3248

let $x_{11} = \dfrac{1.3243 + 1.3248}{2}$

$= \dfrac{2.6491}{2} = 1.3246$

$$f(1.3246) = (1.3246)^3 - 1.3246 - 1$$
$$= -0.0005$$
$$= -ve.$$

The root lies between 1.3246 and 1.3248.

$$\text{Let } x_{12} = \frac{1.3246 + 1.3248}{2}$$
$$= \frac{2.6494}{2}$$
$$= 1.3247$$

$$f(1.3247) = (1.3247)^3 - 1.3247 - 1$$
$$= -0.0001$$
$$= -ve.$$

The root lies between 1.3247 and 1.3248

$$\text{Let } x_{13} = \frac{1.3247 + 1.3248}{2}$$
$$= \frac{2.6495}{2}$$

$$\boxed{x_{13} = 1.3248}$$

if $f(x) = 0.000$ then the x_n is the answer

$x_{10} = x_{13}$

Hence the approximate root is $x = 1.3248$

2) Assuming that a root of $x^3-9x+1=0$ lies in the interval $(2,4)$, find that root by Bisection method.

Solution:

$$x^3-9x+1=0.$$

$$f(x)=0$$

$$\therefore f(x)=x^3-9x+1.$$

now $f(2) = 2^3-9(2)+1$

$= 8-18+1$

$= -9$

$= -ve$

$f(4) = 4^3-9(4)+1$

$= 64-36+1$

$= 29$

$= +ve.$

$\therefore$ The root lies between 2 and 4.

let $x_0 = \frac{2+4}{2} = \frac{6}{2} = 3.$

$f(3) = 3^3-9(3)+1$

$= 27-27+1$

$= 1$

$= +ve.$

The root lies between 2 and 3.

let $x_1 = \frac{2+3}{2} = \frac{5}{2} = 2.5$

$f(2.5) = (2.5)^3-9(2.5)+1$

$= -5.8750$

$= -ve$

The root lies between 2.5 and 3.

$$\text{let } x_2 = \frac{2.5+3}{2} = \frac{5.5}{2} = 2.75$$

$$f(2.75) = (2.75)^3 - 9(2.75) + 1$$

$$= -2.9531$$

$$= -ve.$$

The root lies between 2.75 and 3.

$$\text{let } x_3 = \frac{2.75+3}{2} = \frac{5.75}{2} = 2.8750$$

$$f(2.8750) = (2.8750)^3 - 9(2.8750) + 1$$

$$= -1.1113$$

$$= -ve.$$

The root lies between 2.8750 and 3.

$$\text{let } x_4 = \frac{2.8750+3}{2} = \frac{5.8750}{2} = 2.9375$$

$$f(2.9375) = (2.9375)^3 - 9(2.9375) + 1$$

$$= -0.0901$$

$$= -ve.$$

The root lies between 2.9375 and 3

$$\text{let } x_5 = \frac{2.9375+3}{2} = \frac{5.9375}{2} = 2.9688$$

$$f(2.9688) = (2.9688)^3 - 9(2.9688) + 1$$

$$= 0.4479$$

$$= +ve$$

The root lies between 2.9375 and 2.9688

let $x_6 = \dfrac{2.9375 + 2.9688}{2}$

$= \dfrac{5.9063}{2}$

$= 2.9532$

$f(2.9532) = (2.9532)^3 - 9(2.9532) + 1$

$= 0.1759$

$= +ve$

The root lies between 2.9532 and 2.9375.

let $x_7 = \dfrac{2.9375 + 2.9532}{2} = \dfrac{5.8907}{2} = 2.9454$

$f(2.9454) = (2.9454)^3 - 9(2.9454) + 1$

$= 0.0426$

$= +ve$.

The root lies between 2.9454 and 2.9375

let $x_8 = \dfrac{2.9375 + 2.9454}{2} = \dfrac{5.8829}{2} = 2.9415$

$f(2.9415) = (2.9415)^3 - 9(2.9415) + 1$

$= -0.0237$

$= -ve$.

The root lies between 2.9415 and 2.9454

let $x_9 = \dfrac{2.9415 + 2.9454}{2} = \dfrac{5.8869}{2} = 2.9435$

$f(2.9435) = (2.9435)^3 - 9(2.9435) + 1$

$= 0.0103$

$= +ve$.

The root lies between 2.9415 and 2.9435

let $x_{10} = \frac{2.9415 + 2.9435}{2} = \frac{5.8850}{2} = 2.9425$

$f(2.9425) = (2.9425)^3 - 9(2.9425) + 1$
$= -0.0054$
$= -ve$

The root lies between 2.9425 and 2.9435

let $x_{11} = \frac{2.9425 + 2.9435}{2} = \frac{5.8860}{2} = 2.9430$

$f(2.9430) = (2.9430)^3 - 9(2.9430) + 1$
$= 0.0031$
$= +ve$

The root lies between 2.9425 and 2.9430

let $x_{12} = \frac{2.9425 + 2.9430}{2} = \frac{5.8855}{2} = 2.9428$

$f(2.9428) = (2.9428)^3 - 9(2.9428) + 1$
$= -0.0016$
$= -ve.$

The root lies between 2.9428 and 2.9430.

let $x_{13} = \frac{2.9428 + 2.9430}{2} = \frac{5.8858}{2} = 2.9429$

$f(2.9429) = (2.9429)^3 - 9(2.9429) + 1$
$= 0.0014$
$= +ve.$

The root lies between 2.9428 and 2.9429

let $x_{14} = \frac{2.9428 + 2.9429}{2} = \frac{5.8857}{2} = 2.9429$

Hence, the approximate root is $\boxed{x = 2.9429.}$

1717 2. Find the positive roots of $x - \cos x = 0$ by bisection method. Trigonometric - Rad.

Sol:

$$f(x) = x - \cos x$$

$$f(0) = 0 - \cos 0 = -1 = -ve$$

$$f(1) = 1 - \cos 1 = 0.4597 = +ve$$

The root lies between 0 and 1

let, $x_0 = \frac{0+1}{2} = \frac{1}{2} = 0.5$

$$f(0.5) = 0.5 - \cos(0.5) = -0.3776 = -ve$$

The root lies between 0.5 and 1

let $x_1 = \frac{0.5+1}{2} = \frac{1.5}{2} = 0.75$

$$f(0.75) = 0.75 - \cos(0.75) = 0.0183 = +ve$$

The root lies between 0.5 and 0.75

let $x_2 = \frac{0.75+0.5}{2} = \frac{1.25}{2} = 0.6250$.

$$f(0.6250) = 0.6250 - \cos(0.6250) = -0.1860 = -ve$$

The root lies between 0.6250 and 0.75.

let $x_3 = \frac{0.6250 + 0.75}{2} = \frac{1.3750}{2} = 0.6875$

$$f(0.6875) = 0.6875 - \cos(0.6875) = -0.0853 = -ve$$

The root lies between 0.6875 and 0.75

let $x_4 = \frac{0.6875 + 0.75}{2} = \frac{1.4375}{2} = 0.7188$

$$f(0.7188) = 0.7188 - \cos(0.7188) = -0.0338 = -ve$$

The root lies between 0.7188 and 0.75.

let $x_5 = \frac{0.7188 + 0.75}{2} = \frac{1.4688}{2} = 0.7344$

$f(0.7344) = 0.7344 - \cos(0.7344) = -0.0078.$

The root lies between 0.7344 and 0.75

$$\text{let } x_6 = \frac{0.7344 + 0.75}{2} = \frac{1.4844}{2} = 0.7422.$$

$f(0.7422) = 0.7422 - \cos(0.7422) = 0.0052 = +ve.$

The root lies between 0.7344 and 0.7422

$$\text{let } x_7 = \frac{0.7344 + 0.7422}{2} = \frac{1.4766}{2} = 0.7383$$

$f(0.7383) = 0.7383 - \cos(0.7383) = -0.0013 = -ve.$

The root lies between 0.7383 and 0.7422.

$$\text{let } x_8 = \frac{0.7383 + 0.7422}{2} = \frac{1.4805}{2} = 0.7403$$

$f(0.7403) = 0.7403 - \cos(0.7403) = 0.0020 = +ve.$

The root lies between 0.7383 and 0.7403

$$\text{let } x_9 = \frac{0.7383 + 0.7403}{2} = \frac{1.4786}{2} = 0.7393$$

$f(0.7393) = 0.7393 - \cos(0.7393) = 0.0004 = +ve.$

The root lies between 0.7393 and 0.7483

$$\text{let } x_{10} = \frac{0.7393 + 0.7483}{2} = \frac{1.4776}{2} = 0.7388$$

$f(0.7388) = 0.7388 - \cos(0.7388) = -0.0005 = -ve$

The root lies between 0.7393 and 0.7388.

$$\text{let } x_{11} = \frac{0.7388 + 0.7393}{2} = \frac{1.4781}{2} = 0.7391$$

$f(0.7391) = 0.7391 - \cos(0.7391) = 0.0000.$

Hence, the approximate root is 0.7391

4. Using bisection method find the real root of $x^3 - 4x + 9 = 0$.

Solution:

Let $f(x) = x^3 - 4x + 9$.

$f(0) = 0^3 - 4(0) + 9 = 9 = +ve$.

$f(1) = 1^3 - 4(1) + 9 = 6 = +ve$

$f(2) = 2^3 - 4(2) + 9 = 9 = +ve$

$f(3) = 3^3 - 4(3) + 9 = 24 = +ve$.

$f(4) = 4^3 - 4(4) + 9 = 57 = +ve$.

$f(5) = 5^3 - 4(5) + 9 = 114 = +ve$.

Let $f(-1) = (-1)^3 - 4(-1) + 9 = 12 = +ve$

$\therefore$ The root lies between -1 and 0.

Let $x_0 = \frac{-1+0}{2} = -0.5000$

$f(-0.5) = (-0.5)^3 - 4(-0.5) + 9 = 10.8750 = +ve$.

$\therefore$ The roots are -0.5 and 0.

Let $x_1 =$

The root lies between -2.7066 and -2.7065

Let $x_{14} = \frac{-2.7066 - 2.7065}{2} = -2.7066$.

$\therefore$ The approximate value of x is -2.7066.

4. using bisection method find the real root of

$x^3 - 4x + 9 = 0$.

Solution:

let $f(x) = x^3 - 4x + 9$.

$f(-1) = (-1)^3 - 4(-1) + 9 = 12 = +ve$.

$f(-2) = (-2)^3 - 4(-2) + 9 = 9 = +ve$.

$f(-3) = (-3)^3 - 4(-3) + 9 = -6 = -ve$.

$\therefore$ The roots lies between -2 and -3

let $x_0 = \dfrac{(-2)+(-3)}{2} = -2.5$

$f(-2.5) = (-2.5)^3 - 4(-2.5) + 9 = 3.3750 = +ve$.

The roots lies between -2.5 and -3

let $x_1 = \dfrac{(-2.5)+(-3)}{2} = -2.75$

$f(-2.75) = (-2.75)^3 - 4(-2.75) + 9 = \cancel{0.3170} = \cancel{+ve}$. -0.7969 $-ve$

The root lies between -2.75 and -3.

let $x_2 = \dfrac{(-2.75)+(-3)}{2} = -2.875$

$f(-2.875) = (-2.875)^3 - 4(-2.875) + 9 =$.

The root lies between -2.75 and -2.5

let $x_3 = \dfrac{-2.75 - 2.5}{2} = -2.6250$

$f(-2.6250) = (-2.6250)^3 - 4(-2.6250) + 9 = 1.4121 = +ve$

The root lies between -2.6250 and -2.75

let $x_4 = \dfrac{-2.6250 - 2.75}{2} = -2.6875$

$f(-2.6875) = (-2.6875)^3 - 4(-2.6875) + 9 = 0.3391 = +ve$

The root lies between -2.6875 and -2.75

Let $x_5 = \frac{-2.6875 - 2.75}{2} = -2.7188$

$f(-2.7188) = (-2.7188)^3 - 4(-2.7188) + 9 = -0.2018 = -ve$

The root lies between -2.7188 and -2.6875

Let $x_6 = \frac{-2.7188 - 2.6875}{2} = -2.7032$

$f(-2.7032) = (-2.7032)^3 - 4(-2.7032) + 9 = 0.0597 = +ve$

The root lies between -2.7032 and -2.7188

Let $x_7 = \frac{-2.7032 - 2.7188}{2} = -2.7110$

$f(-2.7110) = (-2.7110)^3 - 4(-2.7110) + 9 = -0.0806 = -ve$

The root lies between -2.7110 and -2.7032

Let $x_8 = \frac{-2.7110 - 2.7032}{2} = -2.7071$

$f(-2.7071) = (-2.7071)^3 - 4(-2.7071) + 9 = -0.0103 = -ve$

The root lies between -2.7071 and -2.7032

Let $x_9 = \frac{-2.7071 - 2.7032}{2} = -2.7052$

$f(-2.7052) = (-2.7052)^3 - 4(-2.7052) + 9 = 0.0239 = +ve$

The root lies between -2.7052 and -2.7071

Let $x_{10} = \frac{-2.7052 - 2.7071}{2} = -2.7062$

$f(-2.7062) = (-2.7062)^3 - 4(-2.7062) + 9 = 0.0059 = +ve$

The root lies between -2.7062 and -2.7071

Let $x_{11} = \frac{-2.7062 - 2.7071}{2} = -2.7067$, $f(-2.7067) = -0.0031$
$= -ve$

The root lies between -2.7067 and -2.7062

Let $x_{12} = \frac{-2.7067 - 2.7062}{2} = -2.7065$, $f(-2.7065) = 0.0005 = +ve$

The root lies between -2.7065 and -2.7067

Let $x_{13} = \frac{-2.7065 - 2.7067}{2} = -2.7066$, $f(-2.7066) = -0.0013 = -ve$

ITERATION METHOD.

condition for convergence of iteration method:

* [Let $f(x)=0$ be the given equation. It can be written as $x=\phi(x)$ and the condition for convergence is $|\phi'(x)|<1$.] 2M

Let I be the interval containing the root $x=\alpha$. If $|\phi'(x)|<1$ for all x in I, then the sequence of approximation $x_0, x_1, \dots x_n$ will converge to α, if the initial value x_0 is chosen in I.

Order of convergence of iteration method:

If $P\geq 1$ can be found such that $|e_{i+1}| \leq |e_i|^P . K$ where K is a positive constant for every i then P is called the order of convergence.

NOTE:

If $P=1$, the convergence is linear.

If $P=2$, it is quadratic.

1. solve $e^x-3x=0$ by method of iteration.

let $e^x-3x=0$

$\therefore f(x)=0$

$f(x) = e^x-3x$

$f(0) = e^0-3(0) = 1 = +ve$

$f(1) = e^1-3(1) = -0.2817 = -ve.$

$f(0)$ & $f(1)$ are of opposite signs.

$\therefore$ The root lies between 0 and 1.

Solve for α:

$$e^{x} - 3x = 0$$

$$x = \frac{e^{x}}{3}.$$

Let $x = \phi(x) = \frac{e^{x}}{3}$.

Solve for condition:

$\Rightarrow \phi'(x) = \frac{e^{x}}{3}$.

$\phi'(1) = \frac{e^{1}}{3} = \frac{2.7183}{3} = 0.9061 < 1$

$\phi'(0) = \frac{e^{0}}{3} = \frac{1}{3} = 0.3333 < 1.$

$\therefore |\phi'(x)| < 1$ satisfies.

NOTE: [Error]

let $x_0, x_1 \ldots x_n$ be the successive approximati of the root α of $f(x)=0$ then e_i be the error of the [illegible] is given by,

$e_i = x_i - \alpha$;

$e_{i+1} = x_{i+1} - \alpha$

Let $x_0 = 0.5$

$x_1 = \frac{e^{x_0}}{3} = \frac{e^{0.5}}{3} = \frac{1.6487}{3} = 0.5496.$

$x_2 = \frac{e^{x_1}}{3} = \frac{1.7326}{3} = 0.5775.$

$x_3 = \frac{e^{x_2}}{3} = \frac{e^{0.5775}}{3} = \frac{1.7816}{3} = 0.5939$

$x_4 = \frac{e^{x_3}}{3} = \frac{e^{0.5939}}{3} = 0.6037.$

$x_5 = \frac{e^{x_4}}{3} = \frac{e^{0.6037}}{3} = 0.6096$

$x_6 = \frac{e^{x_5}}{3} = \frac{e^{0.6096}}{3} = 0.6132.$

$x_7 = \frac{e^{x_6}}{3} = \frac{e^{0.6132}}{3} = 0.6154$

$x_8 = \frac{e^{x_7}}{3} = \frac{e^{0.6154}}{3} = 0.6168$

$$x_9 = \frac{e^{x_8}}{3} = \frac{e^{0.6168}}{3} = 0.6177$$

$$x_{10} = \frac{e^{x_9}}{3} = \frac{e^{0.6177}}{3} = 0.6182.$$

$$x_{11} = \frac{e^{10}}{3} = \frac{e^{0.6182}}{3} = 0.6185$$

$$x_{12} = \frac{e^{11}}{3} = \frac{e^{0.6185}}{3} = 0.6187$$

$$x_{13} = \frac{e^{12}}{3} = \frac{e^{0.6187}}{3} = 0.6188$$

$$x_{14} = \frac{e^{13}}{3} = \frac{e^{0.6188}}{3} = 0.6189$$

$$x_{15} = \frac{e^{14}}{3} = \frac{e^{0.6189}}{3} = 0.6190$$

$$x_{16} = \frac{e^{15}}{3} = \frac{e^{0.6190}}{3} = 0.6190$$

Hence, the approximate root is $x = 0.6190$.

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2. Solve $x^3 + x^2 - 1 = 0$ for the positive root by iteration method.

Sol:

Given, $x^3 + x^2 - 1 = 0$.

$$f(x) = x^3 + x^2 - 1$$

$$f(0) = -1 = -ve$$

$$f(1) = 1 = +ve$$

$\therefore$ The root lies between 0 and 1

we have, $x^3 + x^2 - 1 = 0$

$$x^3 + x^2 = 1$$

$$x^2(x+1) = 1$$

$$x^2 = \frac{1}{x+1}$$

$$x = \frac{1}{\sqrt{x+1}}$$

let $x = \phi(x) = \frac{1}{\sqrt{x+1}}$

$$\phi'(x) = \frac{d}{dx}\frac{1}{\sqrt{x+1}} = \frac{d}{dx}\frac{1}{(x+1)^{1/2}}$$

$$\Rightarrow \frac{d}{dx}(x+1)^{-1/2}$$

$$\because x^n = nx^{n-1}$$

$$= -1/2\,(x+1)^{-1/2-1}$$

$$= -1/2\,(x+1)^{-3/2}$$

$$\phi'(x) = -\frac{1}{2(x+1)^{3/2}}$$

Now, $|\phi(0)| = -\frac{1}{2(0+1)^{3/2}} = -1/2 < 1$

$|\phi(1)| = -\frac{1}{2(1+1)^{3/2}} = 0.1768 < 1$

$\therefore |\phi'(x)| < 1$ in $(0,1)$

Hence iteration method can be applied.

let $x_0 = 0.7$

$$x_1 = \frac{1}{\sqrt{x_0+1}} = \frac{1}{\sqrt{0.7+1}} = 0.7670$$

$$x_2 = \frac{1}{\sqrt{x_1+1}} = \frac{1}{\sqrt{0.7670+1}} = 0.7523$$

$$x_3 = \frac{1}{\sqrt{x_2+1}} = \frac{1}{\sqrt{0.7523+1}} = 0.7554$$

$$x_4 = \frac{1}{\sqrt{x_3+1}} = \frac{1}{\sqrt{0.7554+1}} = 0.7548.$$

$$x_5 = \frac{1}{\sqrt{x_4+1}} = \frac{1}{\sqrt{0.7548+1}} = 0.7549$$

$$x_6 = \frac{1}{\sqrt{x_5+1}} = \frac{1}{\sqrt{0.7549+1}} = 0.7549.$$

$\therefore$ The approximate root is $x = 0.7549$.

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3. Find the real root of the equation $\cos x = 3x-1$ Correct to four decimal places by iteration method.

Sol.

Given:

$$\cos x = 3x-1$$

$$\cos x - 3x + 1 = 0$$

$$f(x) = \cos x - 3x + 1.$$

$$f(0) = \cos(0) - 3(0) + 1 = 2 = +ve$$

$$f(1) = \cos(1) - 3(1) + 1 = -1.4597 = -ve.$$

$\therefore$ The root lies between 0 and 1.

Solve for x,

$$\cos x = 3x-1$$

$$\phi(x) \Rightarrow x = \frac{\cos x + 1}{3}$$

$$\phi'(x) = \frac{1}{3}\cdot(-\sin x)$$

$$|\phi'(x)| = |\phi'(0)| = \frac{1}{3}\sin 0 = 0 < 1$$

$$|\phi'(1)| = \frac{1}{3}\sin 1 = 0.2805 < 1$$

$\therefore |\phi'(x)| < 1$ satisfies in $(0,1)$

now, $x_0 = 0.3$

$$x_1 = \frac{\cos x_0 + 1}{3} = \frac{0.9553+1}{3} = 0.6518$$

$$x_2 = \frac{\cos(0.6518)+1}{3} = \frac{0.7950+1}{3} = 0.5983$$

$$x_3 = \frac{\cos(0.5983)+1}{3} = \frac{0.8263+1}{3} = 0.6088$$

$$x_4 = \frac{\cos(0.6088)+1}{3} = \frac{0.8203+1}{3} = 0.6068$$

$$x_5 = \frac{\cos(0.6068)+1}{3} = \frac{0.8215+1}{3} = 0.6072$$

$$x_6 = \frac{\cos(0.6072)+1}{3} = \frac{0.8212+1}{3} = 0.6071$$

$$x_7 = \frac{\cos(0.6071)+1}{3} = \frac{0.8213+1}{3} = 0.6071$$

$\therefore$ The approximate root is $x = 0.6071$.

---

4. Solve $x^3 = 2x+5$ for the positive root by iteration method.

Sol.

Given, $x^3 = 2x+5$

$x^3 - 2x - 5 = 0$

$f(x) = x^3 - 2x - 5$

$f(0) = 0^3 - 2(0) - 5 = -5 = -ve$

$f(1) = 1^3 - 2(1) - 5 = -6 = -ve$

$f(2) = 2^3 - 2(2) - 5 = -1 = -ve.$

$f(3) = 3^3 - 2(3) - 5 = 16 = +ve.$

$$\phi'(x) = \frac{d}{dx}\frac{5}{x^2-2}$$

$$= \frac{d}{dx} 5(x^2-2)^{-1} \qquad x^n = nx^{n-1}$$

$$= -5(x^2-2)^{-2} \cdot 2x$$

$$= \frac{-10x}{(x^2-2)^2}$$

$$|\phi'(2)| = \frac{+10(2)}{(4-2)^2} = \frac{20}{4}$$

Solve for x,

$$x^3 - 2x - 5 = 0$$

$$x^3 = 2x+5$$

$$\phi(x) = x = (2x+5)^{1/3}$$

$$\phi'(x) = \frac{1}{3}(2x+5)^{-2/3} \cdot 2$$

$$= \frac{2}{3(2x+5)^{2/3}}$$

$$|\phi'(2)| = \frac{2}{3(2(2)+5)^{2/3}} = \frac{2}{3(4+5)^{2/3}} = 0.1541 < 1$$

$$|\phi'(3)| = \frac{2}{3(2(3)+5)^{2/3}} = \frac{2}{3(6+5)^{2/3}} = 0.1348 < 1$$

$\therefore |\phi'(x)| < 1$ in $(2,3)$ satisfies.

Let $x_0 = 2.5$

$$x_1 = (2x_0+5)^{1/3} = (2(2.5)+5)^{1/3} = 2.1173.$$

$x_2 = (2(2.1179)+5)^{1/3} = 2.0981$

$x_3 = (2(2.0981)+5)^{1/3} = 2.0951$

$x_4 = (2(2.0951)+5)^{1/3} = 2.0946$

$x_5 = (2(2.0946)+5)^{1/3} = 2.0946$

$\therefore$ The approximate value is 2.0946.

---

8. Find the positive roots of $3x - \sqrt{1+\sin x} = 0$ by iterative method.

Soln:

Given $3x - \sqrt{1+\sin x} = 0$.

$$f(x) = 3x - \sqrt{1+\sin x} \qquad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

$$f(0) = 3(0) - \sqrt{1+\sin(0)} = -1 = -ve$$

$$f(1) = 3(1) - \sqrt{1+\sin(1)} = 1.6430 = +ve$$

The root lies between 0 and 1.

Solve for x,

$$3x - \sqrt{1+\sin x} = 0$$

$$3x = \sqrt{1+\sin x}$$

$$\phi(x) = x = \frac{\sqrt{1+\sin x}}{3}$$

$$\phi'(x) = \frac{1}{3} \cdot \frac{1}{2\sqrt{1+\sin x}} \cdot \cos x.$$

$$|\phi'(0)| = \frac{1}{3} \cdot \frac{\cos 0}{2\sqrt{1+\sin 0}} = 0.1667 < 1$$

$$|\phi'(1)| = \frac{0.5403}{3 \cdot 2\sqrt{1+\sin 1}} = 0.0664 < 1$$

$|\phi'(x)| < 1$ satisfies in $(0,1)$

0.3919

Let $x_0 = 0.7$

$$x_1 = \sqrt{\frac{1+\sin x_0}{3}} = \sqrt{\frac{1+\sin(0.7)}{3}} = 0.4274.$$

$$x_2 = \sqrt{\frac{1+\sin x_1}{3}} = \sqrt{\frac{1+\sin(0.4274)}{3}} = 0.3964$$

$$x_3 = \sqrt{\frac{1+\sin x_2}{3}} = \sqrt{\frac{1+\sin(0.3964)}{3}} = 0.3924$$

$$x_4 = \sqrt{\frac{1+\sin x_3}{2}} = \sqrt{\frac{1+\sin(0.3924)}{2}} = 0.3919$$

$$x_5 = \sqrt{\frac{1+\sin x_4}{2}} = \sqrt{\frac{1+\sin(0.3919)}{2}} = 0.3919.$$

$\therefore$ The approximate value is 0.3919.

REGULA FALSI METHOD

(or)

METHOD OF FALSE POSITION.

Formula : $x_1 = \dfrac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$ $(a<b)$

NOTE : The order of convergence in regula falsi method is 1.618.

1. Solve for a positive root of $x^3 - 4x + 1 = 0$. by regula falsi method.

Solution :

Given, $x^3 - 4x + 1 = 0$.

$f(x) = x^3 - 4x + 1$

$f(0) = 0^3 - 4(0) + 1 = +1 = +ve$ $(a<b)$

$f(1) = 1^3 - 4(1) + 1 = -2 = -ve$.

$\therefore$ The root lies between $f(0)$ and $f(1)$.

$f(0) = 1$ $f(1) = -2$.

Here $a = 0$, $b = 1$, $f(a) = 1$, $f(b) = -2$

Let $x_1 = \dfrac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$

$= \dfrac{0f(1) - 1f(0)}{-2-1} = \dfrac{0(-2) - 1(1)}{-2-1} = \dfrac{-1}{-3}$

$= 0.3333$

Now, $f(0.3333) = (0.3333)^3 - 4(0.3333) + 1$

$= -0.2962$

$= -ve$

The root lies between 0.3333 and 0.
$a = 0$ $b = 0.3333$ $f(a) = +1$ $f(b) = -0.2962$.

let $x_2 = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

$= \dfrac{0f(0.3333) - 0.3333(1)}{-0.2962 - 1}$

$= \dfrac{-0.3333}{-1.2962} = 0.2571$

now, $f(0.2571) = (0.2571)^3 - 4(0.2571) + 1$

$= -0.0114 = -ve$.

The root lies between 0 and 0.2571

$a = 0$ $b = 0.2571$ $f(a) = 1$ $f(b) = -0.0114$

let $x_3 = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

$= \dfrac{0(-0.0114) - 0.2571(1)}{-0.0114 - 1}$

$= \dfrac{-0.2571}{-1.0114} = 0.2542$

now, $f(0.2542) = (0.2542)^3 - 4(0.2542) + 1$

$= -0.0004 = -ve$.

The root lies between 0 and 0.2542

$a = 0$ $b = 0.2542$ $f(0) = 1$ ie $f(a)$ $f(0.2542) = -0.0004$ ie $f(b)$

let $x_4 = \dfrac{af(b) - bf(a)}{f(b) - f(a)} = \dfrac{0(-0.0004) - 0.2542(1)}{-0.0004 - 1}$

$= \dfrac{-0.2542}{-1.0004}$

$= 0.2541$

$$f(0.2541) = (0.2541)^3 - 4(0.2541) + 1$$

$$= 0.0000.$$

$\therefore$ The approximate value of x is 0.2541.

---

2. Find an approximate root of $x \log_{10} x - 1.2 = 0$ by regula falsi method.

Solution:

Given $x \log_{10} x - 1.2 = 0$.

Let $f(x) = x \log x - 1.2$.

$f(0) = -1.2 = -ve$

$f(1) = -1.2 = -ve$

$f(2) = -0.5979 = -ve$

$f(3) = 0.2314 = +ve$.

$\therefore$ The root lies between 2 and 3.

$a = 2,\ b = 3,\ \underset{f(a)}{f(2)} = -0.5979\quad \underset{f(b)}{f(3)} = 0.2314$

Let $x_1 = \dfrac{a f(b) - b f(a)}{f(b) - f(a)} = \dfrac{2(0.2314) - 3(-0.5979)}{0.2314 + 0.5979}$

$$= \frac{2.2565}{0.8293} = 2.7210$$

$$f(2.7210) = 2.7210 \log 2.7210 - 1.2$$

$$= -0.0171 = -ve.$$

The root lies between 2.7210 and 3.

$a = 2.7210$, $b = 3$ $f(a) = -0.0171$ $f(b) = 0.2314$

Let $x_2 = \dfrac{af(b) - bf(a)}{f(b) - f(a)} = \dfrac{2.7210(0.2314) - 3(-0.0171)}{0.2314 + 0.0171}$

$= \dfrac{0.6296 + 0.0513}{0.2485} = \dfrac{0.6809}{0.2485} = 2.7402$

$f(2.7402) = 2.7402 \log(2.7402) - 1.2 = -0.0004 = -ve.$

The root lies between 2.7402 and 3.

$a = 2.7402$. $b = 3$ $f(a) = -0.0004$ $f(b) = 0.2314$

Let $x_3 = \dfrac{af(b) - bf(a)}{f(b) - f(a)} = \dfrac{2.7402(0.2314) - 3(-0.0004)}{0.2314 - (-0.0004)}$

$= 2.7406.$

$f(2.7406) = 2.7406 \log(2.7406) - 1.2 = 0.0000$

$\therefore$ The approximate value of x is 2.7406.

---

3. Solve for a positive root of $x - \cos x = 0$ by the method of regula falsi.

Solution.

Let $f(x) = x - \cos x.$

$f(0) = 0 - \cos 0 = -1 = -ve$

$f(1) = 1 - \cos 1 = 0.4597 = +ve.$

$\therefore$ The root lies between 0 and 1.

$a = 0$ $b = 1$ $f(a) = -1$ $f(1) = 0.4597.$

Let $x_1 = \dfrac{af(b) - bf(a)}{f(b) - f(a)} = \dfrac{0(0.4597) - 1(-1)}{0.4597 - (-1)} = 0.6851$

$f(0.6851) = 0.6851 - \cos(0.6851) = -0.0893 = -ve.$

$\therefore$ The root lies between 0.6851 and 1

$a = 0.6851 \quad b = 1 \quad f(a) = -0.0893 \quad f(b) = 0.4597$

Let $x_2 = \dfrac{af(b) - bf(a)}{f(b) - f(a)} = \dfrac{0.6851(0.4597) - 1(-0.0893)}{0.4597 - (-0.0893)}$

$= 0.7363$

$f(0.7363) = 0.7363 - \cos(0.7363) = -0.0047 = -ve$

$\therefore$ The root lies between 0.7363 and 1

$a = 0.7363 \quad b = 1 \quad f(a) = -0.0047 \quad f(b) = 0.4597$

Let $x_3 = \dfrac{af(b) - bf(a)}{f(b) - f(a)} = \dfrac{0.7363(0.4597) - 1(-0.0047)}{0.4597 - (-0.0047)}$

$= 0.7390$

$f(0.7390) = 0.7390 - \cos 0.7390 = -0.0001 = -ve$

$\therefore$ The root lies between 0.7390 and 1

$a = 0.7390 \quad b = 1 \quad f(a) = -0.0001 \quad f(b) = 0.4597$

Let $x_4 = \dfrac{af(b) - b(f(a))}{f(b) - f(a)} = \dfrac{0.7390(0.4597) - 1(-0.0001)}{0.4597 - (-0.0001)}$

$= 0.7391$

$f(0.7391) = 0.7391 - \cos(0.7391) = 0.0000$

$\therefore$ The approximate value of x is 0.7391.

4. Find the positive root of $xe^x = 2$ by regula falsi method.

Solution:

Let $f(x) = xe^x - 2$.

$f(0) = 0(e^0) - 2 = -2 = -ve$

$f(1) = 1e^1 - 2 = 0.7183 = +ve$

$\therefore$ The root lies between 0 and 1.

$a = 0, \; b = 1 \quad f(a) = -2 \quad f(b) = 0.7183.$

Let $x_1 = \dfrac{af(b) - bf(a)}{f(b) - f(a)} = \dfrac{0(0.7183) - 1(-2)}{0.7183 - (-2)}$

$= 0.7358$

$f(0.7358) = 0.7358\, e^{0.7358} - 2 = -0.4643 = -ve$

$\therefore$ The root lies between 0.7358 and 1.

$a = 0.7358, \; b = 1, \; f(a) = -0.4643 \quad f(b) = 0.7183$

Let $x_2 = \dfrac{af(b) - bf(a)}{f(b) - f(a)} = \dfrac{0.7358(0.7183) - 1(-0.4643)}{0.7183 - (-0.4643)}$

$= 0.8395$

$f(0.8395) = 0.8395\, e^{0.8395} - 2 = -0.0564 = -ve$

$\therefore$ The root lies between 0.8395 and 1

$a = 0.8395, \; b = 1 \quad f(a) = -0.0564 \quad f(b) = 0.7183$

Let $x_3 = \dfrac{af(b) - bf(a)}{f(b) - f(a)} = \dfrac{0.8395(0.7183) - 1(-0.056}{0.7183 - (-0.0564)}$

$= 0.8512$

$f(0.8512) = 0.8512\, e^{0.8512} - 2 = -0.0061 = -ve$

$\therefore$ The root lies between 0.8512 and 1

$a = 0.8512 \quad b=1 \quad f(a) = -0.0061 \quad f(b) = 0.7183$

Let $x_4 = \dfrac{af(b) - bf(a)}{f(b) - f(a)} = \dfrac{0.8512(0.7183) - 1(-0.0061)}{0.7183 - (-0.0061)}$

$= 0.8525$

$f(0.8525) = 0.8525\, e^{0.8525} - 2 = -0.0005 = -ve$

$\therefore$ The root lies between 0.8525 and 1

$a = 0.8525 \quad b=1 \quad f(a) = -0.0005 \quad f(b) = 0.7183$

Let $x_5 = \dfrac{af(b) - bf(a)}{f(b) - f(a)} = \dfrac{0.8525(0.7183) - 1(-0.0005)}{0.7183 - (-0.0005)}$

$= 0.8526$

$f(0.8526) = 0.8526\, e^{0.8526} - 2 = 0.0000$

$\therefore$ The approximate value of x is 0.8526.

5. Solve the equation $3x + \sin x - e^x = 0$ by regula falsi method.

Solution:

Let $f(x) = 3x + \sin x - e^x$

$f(0) = 3(0) + \sin 0 - e^0 = -1 = -ve$

$f(1) = 3(1) + \sin 1 - e^1 = 1.1232 = +ve$

$\therefore$ The root lies between 0 and 1.

by regula falsi method,

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Here $a = 0$, $f(0) = -1$, $b = 1$, $f(1) = 1.1232$.

$$x_1 = \frac{0(1.1232) - 1(-1)}{1.1232 + 1} = 0.4710$$

$f(0.4710) = 3(0.4710) + \sin(0.4710) = 1.8668 = +ve$

$\therefore$ The root lies between 0.4710 and 0

$a = 0$, $f(a) = -1$, $b = 0.4710$, $f(0.4710) = 1.8668$

$$x_2 = \frac{0(1.8668) - (0.4710)(-1)}{1.8668 + 1} = 0.1643$$

$f(0.1643) = 3(0.1643) + \sin(0.1643)$

$f(0.4710) = 3(0.4710) + \sin(0.4710) - e^{0.4710} = 0.2652$

The root lies between 0 and 0.2652.

$a = 0$ $f(0) = -1$, $f(b) = 0.2652$, $b = 0.4710$.

$$x_2 = \frac{0(0.2652) - 0.4710(-1)}{0.2652 + 1} = \frac{0.4710}{1.2652} = 0.3723$$

$f(0.3723) = 3(0.3723) + \sin(0.3723) - e^{0.3723} = 0.0296$

The root lies between 0 and 0.0296.

$a = 0$, $b = 0.3723$, $f(0) = -1$, $f(0.3723) = 0.0296$

$$x_3 = \frac{0(0.0296) - 0.3723(-1)}{0.0296 + 1} = 0.2873.$$

$f(0.2873) = 3(0.2873) + \sin(0.2873) - e^{0.2873} = -0.1876$
$= -ve$

The root lies between 0.2873 and 0.3723

$a = 0.2873$, $b = 0.3723$, $f(a) = -0.1876$,

$f(b) = 0.0296$.

$$x_4 = \frac{0.2873(0.0296) - 0.3723(-0.1876)}{0.0296 + 0.1876}$$

$$= 0.3607.$$

$$f(0.3607) = 3(0.3607) + \sin(0.3607) - e^{0.3607} = 0.0007.$$

The root lies between 0.2873 and 0.3607.

$a = 0.2873$, $b = 0.3607$, $f(a) = -0.1876$, $f(b) = 0.0007$.

$$x_5 = \frac{0.2873(0.0007) - 0.3607(-0.1876)}{0.0007 + 0.1876}$$

$$= 0.3604$$

$$f(0.3604) = 3(0.3604) + \sin(0.3604) - e^{0.3604} = -0.0001$$

The root lies between 0.3604 and 0.3607

$a = 0.3604$, $f(a) = -0.0001$, $f(b) = 0.0007$, $b = 0.3607$

$$x_6 = \frac{0.3604(0.0007) - 0.3607(-0.0001)}{0.0007 + 0.0001}$$

$$= 0.3064$$

$\therefore$ The approximate value of x is 0.3064.

NEWTON RAPHSON METHOD

Given an approximate value of the root of an equation, a better and closer approximation to the root can be found by using an iterative process called Newton's method.

NOTE :

1. The iterative formula of newtons method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad , \quad n = 0, 1, 2, 3, \dots$$

2. Condition or criterion for the convergence in Newton Raphson method:

The sequence $x_1, x_2, x_3, \dots$ converges to the exact value if $\left| f(x) f''(x) \right| < \left[f'(x) \right]^2$

3. Order of convergence of Newton's method.

The order of convergence of Newton Raphson method is of order 2 and hence it is quadratic (nature)

4. This method is also called as method of Tangents.

1. Find the positive root of $f(x) = 2x^3 - 3x - 6 = 0$ by Newton's method. Correct to five decimal places.

solution:

Given, $f(x) = 2x^3 - 3x - 6$, $f'(x) = 6x^2 - 3$

let $f(0) = 2(0)^3 - 3(0) - 6 = -6 = -ve$

$f(1) = 2(1)^3 - 3(1) - 6 = -7 = -ve$

$f(2) = 2(2)^3 - 3(2) - 6 = 4 = +ve$

$\therefore$ The root lies between 1 and 2.

We know that,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

let, $n = 0$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

let $x_0 = 1.7$

$$x_1 = 1.7 - \frac{f(1.7)}{f'(1.7)}$$

$$= 1.7 - \frac{[2(1.7)^3 - 3(1.7) - 6]}{6[1.7]^2 - 3}$$

$$= 1.7 - \frac{(-1.27400)}{14.34000}$$

$$= \frac{24.37800 + 1.27400}{14.34000} = \frac{25.65200}{14.34000}$$

$x_1 = 1.78884$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.78884 - \frac{[2(1.78884)^3 - 3(1.78884) -}{6(1.78884)^2 - 3}$$

$$= 1.78884 - \frac{[0.08187]}{16.19969}$$

$$= \frac{28.97866 - 0.08187}{16.19969}$$

$$= \frac{28.89679}{16.19969} = 1.78379.$$

Let $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$= 1.78379 - \frac{[2(1.78379)^3 - 3(1.78379) - 6]}{[6(1.78379)^2 - 3]}$$

$$= 1.78379 - \frac{0.00034}{16.09144}$$

$$= \frac{28.70375 - 0.00034}{16.09144} = \frac{28.70341}{16.09144} = 1.78377$$

Let $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$

$$= 1.78377 - \frac{[2(1.78377)^3 - 3(1.78377) - 6]}{6(1.78377)^2 - 3}$$

$$= 1.78377 - \frac{0.00002}{16.09101}$$

$$= \frac{28.70267 - 0.00002}{16.09101} = 1.78377$$

$\therefore$ The approximate value of x is 1.78377

2. Find the real positive root of $3x - \cos x - 1 = 0$ by Newton's method.

Solution:

Given

$f(x) = 3x - \cos x - 1$ $\quad f'(x) = 3 + \sin x.$

$f(0) = -2 = -ve$

$f(1) = 3(1) - \cos(1) - 1 = 1.4597 = +ve.$

$\therefore$ The root lies between 0 and 1

by Newton's raphsons method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

let $n = 0, \quad x_0 = 0.7$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.7 - \frac{f(0.7)}{f'(0.7)}$$

$$= 0.7 - \frac{[3(0.7) - \cos(0.7) - 1]}{3 + \sin(0.7)}$$

$$= 0.7 - \frac{0.3352}{3.6442}$$

$$= \frac{2.5510 - 0.3352}{3.6442}$$

$$= \frac{2.2158}{3.6442}$$

$$x_1 = 0.6080.$$

Let $x = 2$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.6080 - \frac{[3(0.6080) - \cos(0.6080) - 1]}{3 + \sin(0.6080)}$$

$$= 0.6080 - \frac{0.0032}{3.5712}$$

$$= \frac{2.1713 - 0.0032}{3.5712}$$

$$= \frac{2.1681}{3.5712}$$

$$= 0.6071$$

Let $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$= 0.6071 - \frac{[3(0.6071) - \cos(0.6071) - 1]}{3 + \sin(0.6071)}$$

$$= 0.6071 - \frac{0.0000}{3.5705}$$

$$= \frac{2.1676 - 0.0000}{3.5705}$$

$$= 0.6071$$

Let $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$

$$= 0.6071 -$$

$\therefore$ The approximate value of x is 0.6071

3. Find an iterative formula to find $\sqrt{N}$ where N is a positive number.

Solution:

$$x = \sqrt{N}$$

i.e. $x^2 = N$

$\therefore x^2 - N = 0$

Let $f(x) = x^2 - N$ $\quad f'(x) = 2x$

by Newtons method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \rightarrow (1).$$

$$= x_n - \frac{(x_n^2 - N)}{2x_n}$$

$$= \frac{2x_n^2 - x_n^2 - N}{2x_n}$$

$x = \sqrt{N}$ $\quad x = \sqrt{N}$

$x^2 = N$ $\quad x^2 = N$

$$= \frac{x_n^2 + N}{2x_n}$$

$$\boxed{x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]}$$

4. Find the value of $\sqrt{5}$ by Newton's method.

Let $x = \sqrt{5}$

ie $x^2 = 5$

$f(x) = x^2 - 5$ $\quad f'(x) = 2x$.

Let $f(0) = 0^2 - 5 = -5 = -ve$

$f(1) = 1^2 - 5 = -4 = -ve$

$f(2) = 2^2 - 5 = -1 = -ve$

$f(3) = 3^2 - 5 = 4 = +ve$. $\quad \therefore$ the root lies between 2 and 3.

We know that,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

let $n=0$, $x_0 = 2.2$

$$x_1 = \frac{1}{2}\left(x_0 + \frac{N}{x_0}\right)$$

$$x_1 = \frac{1}{2}\left(2.2 + \frac{5}{2.2}\right)$$

$$= \frac{1}{2}\left(\frac{4.84+5}{2.2}\right)$$

$$= \frac{1}{2}\left(\frac{9.84}{2.2}\right)$$

$$= \frac{1}{2} \times 4.4727$$

$$x_1 = 2.2364$$

let $n=1$, $x_1 = 2.2364$

$$x_2 = \frac{1}{2}\left[x_1 + \frac{N}{x_1}\right]$$

$$= \frac{1}{2}\left[2.2364 + \frac{5}{2.2364}\right]$$

$$= 0.5\left(\frac{5.0015+5}{2.2364}\right)$$

$$= 0.5\left(\frac{10.0015}{2.2364}\right)$$

$$x_2 = 2.2361$$

let $n = 2$, $x_2 = 2.2361$

$$x_3 = \frac{1}{2}\left[2.2361 + \frac{5}{2.2361}\right]$$

$$= \frac{1}{2}\left[\frac{5.0001 + 5}{2.2361}\right]$$

$$= \frac{1}{2}\left[\frac{10.0001}{2.2361}\right]$$

$$= \frac{4.4721}{2}$$

$$x_3 = 2.2361$$

$\therefore$ The value of $\sqrt{5}$ is 2.2361.

---

5. Find an iterative formula to find the reciprocal of a given number N and hence find the value of $1/19$.

Let $x = \frac{1}{N}$

$$\Rightarrow N = \frac{1}{x}$$

Let $f(x) = \frac{1}{x} - N$. $\quad f'(x) = -\frac{1}{x^2}$

we know that,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\left(\frac{1}{x_n} - N\right)}{-\frac{1}{x_n^2}}$$

$$= x_n + x_n^2\left(\frac{1}{x_n} - N\right)$$

$$= x_n + x_n - N x_n^2$$

$$x_{n+1} = 2x_n - Nx_n^2$$

$$x_{n+1} = x_n(2 - Nx_n) \longrightarrow (1)$$

to find the value of $1/19$.

take $N = 19$, 0.0526

$f(x) = \frac{1}{x} - 19$ $\quad f'(x) = -\frac{1}{x^2}$

~~let f(x)~~ let $x_0 = 0.05$

$$x_1 = x_0(2 - Nx_0)$$

$$x_1 = 0.05(2 - 19(0.05))$$

$$x_1 = 0.0525$$

let $x_2 = x_1(2 - Nx_1)$

$$= 0.0525(2 - 19(0.0525))$$

$$x_2 = 0.0526$$

let $x_3 = x_2(2 - Nx_2)$

$$= 0.0526(2 - 19(0.0526))$$

$$x_3 = 0.0526.$$

$\therefore$ The value of $\frac{1}{19}$ is 0.0526.

Find the value of $\sqrt{12}$ by Newton's method.

solution:

let $x = \sqrt{12}$

ie, $x^2 = 12$.

$x^2 - 12 = 0$

$f(x) = x^2 - 12$.

$f(0) = -12 = -ve$

$f(1) = -11 = -ve$

$f(2) = -8 = -ve$

$f(3) = -3 = -ve$

$f(4) = 4 = +ve$.

$\therefore$ The root lies b/n 3 & 4

we know that,

$$x_{n+1} = \left[x_n + \frac{N}{x_n}\right]\frac{1}{2}$$

let $n=0$, $x_0 = 3.5$

$$x_1 = \left[x_0 + \frac{N}{x_0}\right]\frac{1}{2}$$

$$= \left[3.5 + \frac{12}{3.5}\right]\frac{1}{2}$$

$$x_1 = 3.4643$$

let $n=1$, $x_1 = 3.4643$.

$$x_2 = \left[x_1 + \frac{N}{x_1}\right]\frac{1}{2}$$

$$= \left[3.4643 + \frac{12}{3.4643}\right]\frac{1}{2}$$

$$x_2 = 3.4641$$

let $n=2$, $x_2 = 3.4641$

$$x_3 = \frac{1}{2}\left[x_2 + \frac{N}{x_2}\right] = \frac{1}{2}\left[3.4641 + \frac{12}{3.4641}\right]$$

$$x_3 = 3.4641$$

$\therefore$ The value of $\sqrt{12}$ is 3.4641

7. Find the value of $\frac{1}{31}$ by Newton's method.

Solution:

Let $x = \frac{1}{31}$

$f(x) = \frac{1}{x} - 31$

We know that,

$x_{n+1} = x_n (2 - N x_n)$

Let $n = 0$, $x_0 = 0.03$, $N = 31$

$x_{0+1} = x_0 (2 - N x_0)$

$x_1 = 0.03 (2 - 31(0.03))$

$x_1 = 0.0321$

Let $n = 1$, $x_1 = 0.0321$, $N = 31$

$x_{1+1} = x_1 (2 - N x_1)$

$x_2 = 0.0321 (2 - 31(0.0321))$

$x_2 = 0.0323$

Let $n = 2$, $x_2 = 0.0323$, $N = 31$.

$x_{2+1} = x_2 (2 - N x_2)$

$x_3 = 0.0323 (2 - 31(0.0323))$

$x_3 = 0.0323$

$\therefore$ The approximate value of $\frac{1}{31}$ is 0.0323.

8. Find the real positive root of $x - \cos x = 0$. by Newtons method.

Solution:

Let $f(x) = x - \cos x$ $\qquad f'(x) = 1 + \sin x$.

$f(0) = 0 - \cos 0 = -1 = -ve$

$f(1) = 1 - \cos 1 = 0.4597 = +ve$.

$\therefore$ The root lies between 0 and 1.

we know that,

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

Let $n=0$, $x_0 = 0.8$

$$x_{0+1} = x_0 - \frac{f(0)}{f'(0)}$$

$$= 0.8 - \frac{[0.8 - \cos(0.8)]}{1 + \sin(0.8)}$$

$$= 0.8 - \frac{0.1033}{1.7174}$$

$$x_1 = \frac{1.3739 - 0.1033}{1.7174} = \frac{1.2706}{1.7174} = 0.7398$$

Let $n=1$, $x_1 = 0.7398$

$$x_{1+1} = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.7398 - \frac{[0.7398 - \cos(0.7398)]}{1 + \sin(0.7398)}$$

$$= 0.7398 - \frac{0.0012}{1.6741}$$

$$= \frac{1.2385 - 0.0012}{1.6741}$$

$$= \frac{1.2373}{1.6741}$$

$$x_2 = 1.8355.$$

Let $n=2$, $x_2 = 1.8355.$

$$x_{2+1} = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 1.8355 - \frac{[1.8355 - \cos(1.8355)]}{1+\sin(1.8355)}$$

$$= 1.8355 - \frac{2.0971}{1.9652}$$

$$= \frac{3.6071 - 2.0971}{1.9652} = \frac{1.5100}{1.9652} = 0.7684$$

Let $n=3$, $x_3 = 0.7684$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.7684 - \frac{[0.7684 - \cos 0.7684]}{1+\sin(0.7684)}$$

$$= 0.7684 - \frac{0.0494}{1.6950}$$

$$= \frac{1.3024 - 0.0494}{1.6950}$$

$$= \frac{1.2530}{1.6950} = 0.7392$$

Let $n=4$, $x_4 = 0.7392$.

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 0.7392 - \frac{[0.7392 - \cos 0.7392]}{1+\sin 0.7392}$$

$$= 0.7392 - \frac{0.0002}{1.6737}$$

$$= \frac{1.2372 - 0.0002}{1.6737} = \frac{1.2370}{1.6737} = 0.7391$$

Let, $n = 5$, $x_5 = 0.7391$

$$x_6 = x_5 - \frac{f(x_5)}{f'(x_5)}$$

$$= 0.7391 - \frac{[0.7391 - \cos(0.7391)]}{1 + \sin(0.7391)}$$

$$= 0.7391 - \frac{0.0000}{1.6736}$$

$$= 0.7391$$

$\therefore$ The approximate value of x is 0.7391.

---

9. Find the real positive root of $x^3 = 6x - 4$ by newton raphson method.

Solution:

$f(x) = x^3 - 6x + 4$ $\qquad f'(x) = 3x^2 - 6$.

$f(0) = 0^3 - 6(0) + 4 = 4 = +ve$

$f(1) = 1^3 - 6(1) + 4 = -1 = -ve$

$\therefore$ The root lies between 0 and 1.

We know that,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let $n = 0$, $x_0 = 0.3$.

$$x_1 = 0.3 - \frac{[(0.3)^3 - 6(0.3) + 4]}{3(0.3)^2 - 6.}$$

$$= 0.3 - \left[\frac{2.2270)}{-5.7300}\right.$$

$$= \frac{1.7190 + 2.2270}{5.7300} = \frac{3.9460}{5.7300} = 0.6887$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.6887 - \frac{[(0.6887)^3 - 6(0.6887) + 4]}{3(0.6887)^2 - 6}$$

$$= 0.6887 - \frac{0.1945}{-4.5771} = \frac{3.1522 + 0.1945}{4.5771}$$

$$= \frac{3.3467}{4.5771} = 0.7312.$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.7312 - \frac{[(0.7312)^3 - 6(0.7312) + 4]}{3(0.7312)^2 - 6}$$

$$= 0.7312 - \frac{0.0037}{-4.3960}$$

$$= \frac{3.2144 + 0.0037}{4.3960.} = \frac{3.2181}{4.3960} = 0.7321$$

$$x_4 = x_3 - \frac{f(x)}{f'(x)} = 0.7321 - \frac{[0.7321^3 - 6(0.7321) + 4]}{3(0.7321)^2 - 6}$$

$$= 0.7321 - 0.0000$$

$$= 0.7321$$

$\therefore$ The approximate value of x is 0.7321

Iterative method for Eigen Values

10/6/23

POWER METHOD :

↳ Power method is used to determine numerically largest eigen vale and the corresponding eigen vector of a given matrix A.

↳ Let A be a $n \times n$ square matrix and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the distinct eigen values of A. so that $|\lambda_1| > |\lambda_2| > |\lambda_3| \cdots > |\lambda_n| \rightarrow ①$

↳ Let $V_1, V_2, \ldots, V_n$ be their corresponding eigen vectors ie $AV_i = \lambda V_i, i = 1, 2, \ldots, n \rightarrow ②$

↳ This method is applicable only if the vectors $V_1, V_2, \ldots V_n$ are linearly independent. This may be true even if the eigen values $\lambda_1, \lambda_2, \ldots \lambda_n$ are not distinct.

NOTE :

1. To find the numerically smallest eigen values of A, obtain the numerically greatest eigen value of A^{-1} and then take its reciprocal.

2. The eigen values of A-KI are $\lambda_i - K$. where λ_i are the eigen values of A.

3. Some of the eigen values of the matrix is equal to some of the diagonal elements of the matrix.

4. To find the numerically smallest eigen value of A, obtain the dominant eigen value λ_1 of A and then find $\boxed{B = A - \lambda_1 I}$ and find the

dominant eigen value of B then the smallest eigen value of A is equal to the dominant eigen value of $B + \lambda_1$.

① Find the dominant eigen value of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ by power method & hence find the other eigen values also.

Solution:

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and let the initial vector be $X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ now,

$$A X_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+2 \\ 0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$= 4 \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$= 4 X_2$$

$$A X_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5+2 \\ 1.5+4 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5 \\ 5.5 \end{bmatrix}$$

$$= 5.5 \begin{bmatrix} 0.4545 \\ 1 \end{bmatrix}$$

$$= 5.5 X_3$$

$$AX_3 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.4545 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4545+2 \\ 1.3635+4 \end{bmatrix}$$

$$= \begin{bmatrix} 2.4545 \\ 5.3635 \end{bmatrix}$$

$$= 5.3635 \begin{bmatrix} 0.4576 \\ 1 \end{bmatrix}$$

$$= 5.3635 \; X_4$$

$$AX_4 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.4576 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4576+2 \\ 1.3728+4 \end{bmatrix}$$

$$= \begin{bmatrix} 2.4576 \\ 5.3728 \end{bmatrix}$$

$$= 5.3728 \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix}$$

$$= 5.3728 \; X_5$$

$$AX_5 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4574+2 \\ 1.3722+4 \end{bmatrix}$$

$$= \begin{bmatrix} 2.4574 \\ 5.3722 \end{bmatrix}$$

$$= 5.3722 \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix}$$

$$= 5.3722 \; X_6$$

$$AX_6 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4574+2 \\ 1.3722+4 \end{bmatrix}$$

$$= \begin{bmatrix} 2.4574 \\ 5.3722 \end{bmatrix}$$

$$= 5.3722 \begin{bmatrix} 0.4574 \\ 1 \end{bmatrix}$$

Hence $\lambda_1 = 5.3722$ is the dominant eigen value & the corresponding eigen vector is $\begin{pmatrix} 0.4574 \\ 1 \end{pmatrix}$

By property we have,

$$\boxed{\lambda_1 + \lambda_2 = \text{trace of } A \; [\text{sum of diagonal elements}]}$$

$$\lambda_1 + \lambda_2 = 5 \qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1+4 = 5$$

$$5.3722 + \lambda_2 = 5$$

$$\lambda_2 = -0.3722.$$

Find the numerically largest eigen value of

$A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ and the corresponding eigen vector.

Solution:

Let $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be the initial eigen vector,

now, $$AX_1 = \underset{3\times 3}{\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} 25+0+0 \\ 1+0+0 \\ 2+0+0 \end{bmatrix} = \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix}$$

$$= 25 \begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} \text{ ie } 25\, X_2$$

$$AX_2 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = \begin{bmatrix} 25+0.04+0.16 \\ 1+0.12+0 \\ 2+0-0.32 \end{bmatrix} = \begin{bmatrix} 25.2 \\ 1.12 \\ 1.68 \end{bmatrix}$$

$$= 25.2 \begin{bmatrix} 1 \\ 0.044 \\ 0.066 \end{bmatrix} \text{ ie } 25.2\, X_3$$

$$AX_3 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.044 \\ 0.066 \end{bmatrix} = \begin{bmatrix} 25+0.044+0.132 \\ 1+0.132+0 \\ 2+0-0.264 \end{bmatrix}$$

$$= \begin{bmatrix} 25.176 \\ 1.132 \\ 1.736 \end{bmatrix} \Rightarrow 25.176 \begin{bmatrix} 1 \\ 0.0397 \\ 0.0689 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0397 \\ 0.0689 \end{bmatrix} = \begin{bmatrix} 25+0.0397+0.1378 \\ 1+0.1191+0 \\ 2+0-0.2756 \end{bmatrix}$$

$$= \begin{bmatrix} 25.1775 \\ 1.1191 \\ 1.7244 \end{bmatrix} \Rightarrow 25.1775 \begin{bmatrix} 1 \\ 0.0444 \\ 0.0685 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0444 \\ 0.0685 \end{bmatrix} = \begin{bmatrix} 25+0.0444+0.137 \\ 1+0.1332+0 \\ 2+0-0.274 \end{bmatrix}$$

$$= \begin{bmatrix} 25.1814 \\ 1.1332 \\ 1.726 \end{bmatrix} \Rightarrow 25.1814 \begin{bmatrix} 1 \\ 0.7508 \\ 0.0685 \end{bmatrix}$$

$$AX_6 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.7508 \\ 0.0685 \end{bmatrix} = \begin{bmatrix} 25+0.7508+0.137 \\ 1+2.2524+0 \\ 2+0-0.274 \end{bmatrix}$$

$$= 25.88$$

$$AX_3 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0444 \\ 0.0667 \end{bmatrix} = \begin{bmatrix} 25+0.0444+0.1334 \\ 1+0.1332+0 \\ 2+0-0.2668 \end{bmatrix}$$

$$= \begin{bmatrix} 25.1778 \\ 1.1332 \\ 1.7332 \end{bmatrix} = 25.1778 \begin{bmatrix} 1 \\ 0.0450 \\ 0.0688 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0450 \\ 0.0688 \end{bmatrix} = \begin{bmatrix} 25+0.0450+0.1376 \\ 1+0.135+0 \\ 2+0-0.2752 \end{bmatrix}$$

$$= \begin{bmatrix} 25.1826 \\ 1.135 \\ 1.7248 \end{bmatrix} = 25.1826 \begin{bmatrix} 1 \\ 0.0450 \\ 0.0685 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0450 \\ 0.0685 \end{bmatrix} = \begin{bmatrix} 25+0.8810+0.137 \\ 1+0.435+0 \\ 2+0-0.274 \end{bmatrix}$$

$$= \Big[$$

$$A X_5 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0450 \\ 0.0685 \end{bmatrix}$$

$$= \begin{bmatrix} 25 + 0.0450 + 0.137 \\ 1 + 0.135 + 0 \\ 2 + 0 - 0.274 \end{bmatrix}$$

$$= \begin{bmatrix} 25.182 \\ 1.135 \\ 1.726 \end{bmatrix} = 25.182 \begin{bmatrix} 1 \\ 0.0450 \\ 0.0685 \end{bmatrix}$$

$\therefore \lambda_1 = 25.182$ and the corresponding eigen vector is $\begin{pmatrix} 1 \\ 0.0450 \\ 0.0685 \end{pmatrix}$.

⟹ JACOBI METHOD FOR FINDING EIGEN VALUES.

By this method we can find all eigen values and eigen vectors of a real symmetric matrix.

Working rule:

i) Use, $\theta = \frac{1}{2} \tan^{-1} \left[\frac{2a_{12}}{a_{11} - a_{12}} \right]$ if $a_{11} \neq a_{22}$

$\theta = \frac{\pi}{4}$ if $a_{11} = a_{22}$ & $a_{12} > 0$

$\theta = -\frac{\pi}{4}$ if $a_{11} = a_{22}$ & $a_{12} < 0$.

ii) write the <u>rotation matrix</u>,

$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ using the value of θ.

iii) Get $B = R'AR$

iv) The diagonal elements of B are the eigen values.

v) The columns of R are the corresponding eigen vectors.

Trigonometric table:

	$0°[0]$	$30°\left[\frac{\pi}{6}\right]$	$45°\ \frac{\pi}{4}$	$60°\ \frac{\pi}{3}$	$90°\ \frac{\pi}{2}$
sin	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞

1. using Jacobi method, find eigen values of

$$A = \begin{array}{cc} a_{11} & a_{12} \\ \multicolumn{2}{c}{\begin{pmatrix} 6 & \sqrt{3} \\ \sqrt{3} & 4 \end{pmatrix}} \\ a_{21} & a_{22} \end{array}.$$

Solution:

$$\theta = \frac{1}{2} \tan^{-1}\left(\frac{2a_{12}}{a_{11} - a_{22}} \right).$$

$$\theta = \frac{1}{2} \tan^{-1}\left(\frac{2\sqrt{3}}{6-4} \right)$$

$$\theta = \frac{1}{2} \tan^{-1}\left(\frac{\cancel{2}\sqrt{3}}{\cancel{2}} \right)$$

$$\theta = \frac{1}{2} \tan^{-1}(\sqrt{3})$$

$$= \frac{1}{2}\left(\frac{\pi}{3}\right)$$

$$\boxed{\theta = \frac{\pi}{6}}.$$

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\pi/6 & -\sin\pi/6 \\ \sin\pi/6 & \cos\pi/6 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ +1/2 & \sqrt{3}/2 \end{pmatrix}$$

$$B = R'AR$$

$$= \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 6 & \sqrt{3} \\ \sqrt{3} & 4 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}.$$

$$= \begin{pmatrix} 3\sqrt{3} + \frac{\sqrt{3}}{2} & \frac{3}{2} + 2 \\ -3 + \frac{3}{2} & -\frac{\sqrt{3}}{2} + 2\sqrt{3} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{6\sqrt{3}+\sqrt{3}}{2} & \frac{3+4}{2} \\ \frac{-6+3}{2} & \frac{-\sqrt{3}+4\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

$$= \begin{pmatrix} 7\sqrt{3}/2 & 7/2 \\ -3/2 & \frac{3\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{21}{4}+\frac{7}{4} & \cancel{\frac{-7\sqrt{3}}{4}}+\cancel{\frac{7\sqrt{3}}{4}} \\ \cancel{\frac{3\sqrt{3}}{4}}+\cancel{\frac{3\sqrt{3}}{4}} & \frac{3}{4}+\frac{9}{4} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{28}{4} & 0 \\ 0 & \frac{12}{4} \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix}$$

Eigen values are $\lambda_1 = 7, \lambda_2 = 3$

Eigen vectors are $X_1 = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$, $X_2 = \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$

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2. Using Jacobi method, find eigen values of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$\theta = \frac{\pi}{4}$

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$B = R'AR = \begin{pmatrix} 1/\sqrt{2} & +1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 3/\sqrt{2} & 3/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 3/2 + 3/2 & -3/2 + 3/2 \\ -1/2 + 1/2 & 1/2 + 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

Eigen values are $\lambda_1 = 3$, $\lambda_2 = 1$

Eigen vectors are $X_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$; $X_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

3. Using Jacobi's method, find eigen values of $\begin{array}{cc} a_{11} & a_{12} \\ \multicolumn{2}{c}{\begin{pmatrix} 3 & 2\sqrt{3} \\ 2\sqrt{3} & 2 \end{pmatrix}} \\ a_{21} & a_{22} \end{array}$

Solution:

Here $a_{11} \neq a_{22}$. So,

$$\theta = \frac{1}{2} \tan^{-1}\left[\frac{2a_{12}}{a_{11} - a_{12}}\right]$$

$$= \frac{1}{2} \tan^{-1}\left[\frac{2(2\sqrt{3})}{3 - 2\sqrt{3}}\right].$$

$$= \frac{1}{2} \tan^{-1}\left[\frac{4\sqrt{3}}{3 - 2\sqrt{3}}\right]$$

$$= \frac{1}{2} \tan^{-1}\left[\frac{4\sqrt{3} \times (3 + 2\sqrt{3})}{(3 - 2\sqrt{3}) \times (3 + 2\sqrt{3})}\right]$$

$$= \frac{1}{2} \tan^{-1} \left[\frac{12\sqrt{3} + 8\sqrt{3}}{9 + 6\sqrt{3} - 6\sqrt{3} - 4\sqrt{3}} \right]$$

$$= \frac{1}{2} \tan^{-1} \left[\frac{20\sqrt{3}}{9 - 12} \right]$$

$$= \frac{1}{2} \tan^{-1} \left[\frac{20\sqrt{3}}{-3} \right]$$

2) Using Jacobi method, find the eigen values & eigen vectors of $\begin{array}{cc} a_{11} & a_{12} \\ \multicolumn{2}{c}{\begin{pmatrix} 7 & 6 \\ 6 & -2 \end{pmatrix}} \\ a_{21} & a_{22} \end{array}$ by Jacobi method.

Solution:

Here $a_{11} \neq a_{22}$

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{2a_{12}}{a_{11} - a_{12}} \right)$$

$$= \frac{1}{2} \tan^{-1} \left(\frac{2(6)}{7 - 6} \right)$$

$$= \frac{1}{2} \tan^{-1} \left(\frac{12}{1} \right)$$

$$= \frac{1}{2} \tan^{-1} 12.$$

$$= \frac{1}{2} \times 1.488.$$

$$\theta = 0.744$$

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos 0.744 & -\sin 0.744 \\ \sin 0.744 & \cos 0.744 \end{pmatrix}$$

$$= \begin{pmatrix} 0.736 & -0.677 \\ 0.677 & 0.736 \end{pmatrix}$$

Eigen vectors $X_1 = \begin{pmatrix} 0.736 \\ 0.677 \end{pmatrix}$ $\quad X_1 = \begin{pmatrix} -0.677 \\ 0.736 \end{pmatrix}$

$B = R'AR$

$$= \begin{pmatrix} 0.736 & 0.677 \\ -0.677 & 0.736 \end{pmatrix} \begin{pmatrix} 7 & 6 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} 0.736 & -0.677 \\ 0.677 & 0.736 \end{pmatrix}$$

$$= \begin{pmatrix} 5.152 + 4.062 & 4.416 - 1.354 \\ -4.739 + 4.416 & -4.062 + (-1.472 \end{pmatrix} \begin{pmatrix} 0.736 & -0.677 \\ 0.677 & 0.736 \end{pmatrix}$$

$$= \begin{pmatrix} 9.214 & 3.062 \\ -0.323 & -5.534 \end{pmatrix} \begin{pmatrix} 0.736 & -0.677 \\ 0.677 & 0.736 \end{pmatrix}$$

$$= \begin{pmatrix} 6.782 + 2.073 & -6.238 + 2.254 \\ -0.2378 - 3.747 & 0.219 - 4.073 \end{pmatrix}$$

$$B = \begin{pmatrix} 8.855 & -3.984 \\ -3.985 & -3.854 \end{pmatrix}$$

Eigen values $\lambda_1 = 8.855$ $\lambda_2 = -3.854$.

JACOBI ~~POWER~~ METHOD.

Use Jacobi method find Eigen values of $A = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}$.

Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$

Sol:

Here the largest off-diagonal element is

$a_{13} = a_{31} = 2$ and $a_{11} = 1$, $a_{33} = 1$

Hence take the rotation matrix,

$$B_1 = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

select θ so that $\cot 2\theta = \dfrac{a_{11} - a_{33}}{2a_{13}} = \dfrac{2-2}{4} = 0$

$2\theta = \dfrac{\pi}{2}$, $\theta = \dfrac{\pi}{4}$.

$$B_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$B_1 = B_1^{-1} A B_1 \Rightarrow B_1^{-1} A B_1$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 2\sqrt{2} & 3\sqrt{2} & 0 \\ 3 & 2 & -1 \end{bmatrix}.$$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

again annihilate the largest off-diagonal element

$a_{12} = a_{21} = 2$ in B_1 also, $a_{11} = 3$, $a_{22} = 3$

take,

$$R_2 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Select θ so that $\cot 2\theta = \dfrac{a_{11} - a_{22}}{2a_{12}} = \dfrac{3-3}{4} = 0$

$\theta = \dfrac{\pi}{4}$

$$R_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

now, $B_2 = R_2' B_1 R_2$.

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 5 & -1 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & -\sqrt{2} \end{bmatrix}$$

$$= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

after 2 rotations, A is reduced to the diagonal matrix B_2. Hence the eigen values of A are $5, 1, -1$

$$R = \cancel{B_1 B_2} \; S_1 S_2 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

[redacted]

$$= \begin{bmatrix} 1/2 & -1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 1/\sqrt{2} \end{bmatrix}$$

Hence the corresponding eigen vectors are $\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}$

$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Find the eigen values and eigen vectors of

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

sol:

Hence, the largest off diagonal element is

$a_{13} = 1$. Let us annihilate the element.

take $S_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

$$\tan 2\theta = \frac{2a_{13}}{a_{11}-a_{33}} = \frac{2}{2-2} = \infty \quad \therefore \theta = \pi/4$$

$$S_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$B_1 = S_1' A S_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & -1 \\ 0 & 2\sqrt{2} & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

though it is a 3x3 matrix, we are lucky to get the diagonal matrix in one rotation because

$a_{12} = 0, a_{23} = 0$.

$\therefore$ the eigen values of A are 3, 2, 1.

The corresponding eigen vectors of A are the columns of R_1 are $\begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}-1\\0\\1\end{pmatrix}$

Find eigen values and eigen vectors of $A = \begin{pmatrix}1 & 0 & 0\\0 & 3 & -1\\0 & -1 & 3\end{pmatrix}$

sol:

The element $|a_{23}| = 1$ is to be annihilated.

take $S_1 = \begin{bmatrix}1 & 0 & 0\\0 & \cos\theta & -\sin\theta\\0 & \sin\theta & \cos\theta\end{bmatrix}$

$$\tan 2\theta = \frac{2a_{23}}{a_{22}-a_{33}} = \frac{-2}{0} = \infty \quad \therefore \theta = \pi/4$$

$$\therefore S_1 = \begin{bmatrix}1 & 0 & 0\\0 & 1/\sqrt{2} & -1/\sqrt{2}\\0 & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$$

$$S_1 = S_1^{-1} A S_1$$

$$= \begin{bmatrix}1 & 0 & 0\\0 & 1/\sqrt{2} & 1/\sqrt{2}\\0 & -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 3 & -1\\0 & -1 & 3\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 1/\sqrt{2} & -1/\sqrt{2}\\0 & 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}\sqrt{2} & 0 & 0\\0 & 1 & 1\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 3 & -1\\0 & -1 & 3\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0 & 0\\0 & 1 & -1\\0 & 1 & 1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 2 & -4 \\ 0 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The eigen values of A are 1, 2, 4

The corresponding eigen vectors are columns of R_1

namely $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

24/8/23.

UNIT- 2.

SOLUTION OF SIMULTANEOUS, LINEAR ALGEBRAIC EQUATIONS.

Direct methods

For solving a system of algebraic equations, the two direct methods are

1. Gauss Elimination method.
2. Gauss Jordan method.

Gauss Elimination method:

Consider n linear equations with n unknowns ie

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned}\right\} \rightarrow (1)$$

The above system 1 can be written in matrix form as, $AX = B$

where $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$

$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

now we form the augmented matrix (A, B) as

$$(A,B) \sim \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right) \rightarrow ②$$

the above matrix ② can be reduced to upper triangular matrix by using elementary row options.

Then by using back substitution we can find the solution.

Gauss Jordan Method:

In the above equation system, the augmented matrix is converted / transferred to diagonal matrix [upper triangular & lower triangular]. then without back substitution, we can find the solution.

2m:

Compare Gauss elimination & Gauss Jordan method.

Gauss Elimination	Gauss Jordan.
* It is direct method.	* It is also a direct method.
* The coefficient matrix A is transformed to a upper triangular matrix.	* The coefficient matrix A is transformed to a diagonal matrix.
* By using back substitution we can find the solution.	* No need of back substitution, directly we can find the solution.

1. Solve the following system $2x+y=3$;

(4.) $7x-3y=4$ by Gauss elimination method.

Solution:

The given system of equations can be written in matrix form $AX=B$.

Here $A = \begin{pmatrix} 2 & 1 \\ 7 & -3 \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \end{pmatrix}$ $B = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

The augmented matrix is,

$$(A,B) = \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 7 & -3 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 0 & -13 & -13 \end{array}\right) \quad R_2 \to 2R_2 - 7R_1$$

by using back substitution method,

$$-13y = -13$$

$$\boxed{y=1}$$

Sub $y=1$, $2x+y=3$

$$2x+1=3$$

$$2x = 3-1$$

$$2x = 2$$

$$\boxed{x=1}$$

To Check,

Sub $x=1$, $y=1$ in $2x+y=3$ & $7x-3y=4$

$2(1)+1 = 3$ $\qquad 7(1)-3(1) = 4$

$3 = 3$ $\qquad 7-3=4$

$\qquad 4=4$

Solve the following system by Gauss Elimination method: $2x+3y-z=5$; $4x+4y-3z=3$; $2x-3y+2z=2$

Solution:

The given system of equations can be written in matrix from $AX=B$.

Here $A=\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 2 \end{pmatrix}$ $X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $B=\begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$

The augmented matrix is

$$(A,B)=\left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ 2 & -3 & 2 & 2 \end{array}\right)$$

$$\approx\left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & -6 & 3 & -3 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\approx\left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & 6 & 18 \end{array}\right) R_3 \to R_3 - 3R_2.$$

by using back substitution method,

$6z=18$

$\boxed{z=3}$

also, $-2y-z=-7$.

$-2y-3=-7$

$-2y=-4$

$\boxed{y=2}$

$2x+3y-z=5$

$2x+6-3=5$

$2x=2$

$\boxed{x=1}$

$\therefore$ The Solution is $x=1, y=2, z=3$.

(4) Solve the following system of equations by Gauss elimination method: $x + 2y + z = 3$, $2x + 3y + 3z = 10$, $3x - y + 2z = 13$.

Here, $AX = B$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad B = \begin{pmatrix} 3 \\ 10 \\ 13 \end{pmatrix}$$

The augmented matrix is,

$$(A,B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right)$$

$$\begin{array}{cccc} 2 & 3 & 3 & 10 \\ -2 & -4 & -2 & -6 \\ \hline 0 & -1 & 1 & 4 \end{array} \quad \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array} \quad \begin{array}{cccc} 3 & -1 & 2 & 13 \\ -3 & -6 & -3 & -9 \\ \hline 0 & -7 & -1 & 4 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right) R_3 \to R_3 - 7R_2 \quad \begin{array}{cccc} 0 & -7 & -1 & 4 \\ 0 & -7 & 7 & 28 \\ \hline 0 & 0 & -8 & -24 \end{array}$$

By backward substitution method,

$-8z = -24$

$\boxed{z = 3}$

$-y + z = 4$

$-y + 3 = 4$

$-y = 4 - 3$

$\boxed{y = -1}$

$x + 2y + z = 3.$

$x + 2(-1) + 3 = 3$

$x - 2 + 3 = 3$

$x + 1 = 3$

$\boxed{x = 2}$

$\therefore$ The solution is $x = 2$, $y = -1$, $z = 3$.

— x —

Solve by Gauss elimination method, $3x+4y+5z=18$

$2x-y+8z=13$, $5x-2y+7z=20$.

Solution:

The given system of equations is can be written in matrix form i.e. $AX=B$.

$$\therefore A=\begin{pmatrix}3 & 4 & 5\\ 2 & -1 & 8\\ 5 & -2 & 7\end{pmatrix} \quad X=\begin{pmatrix}x\\ y\\ z\end{pmatrix} \quad B=\begin{pmatrix}18\\ 13\\ 20\end{pmatrix}$$

The augmented matrix is,

$$(A,B)=\left(\begin{array}{ccc|c}3 & 4 & 5 & 18\\ 2 & -1 & 8 & 13\\ 5 & -2 & 7 & 20\end{array}\right)$$

$$\begin{array}{cccc}15 & -6 & 21 & 60\\ 15 & 20 & 25 & 90\\ \hline 0 & -26 & -4 & -30\end{array}$$

$$\sim\left(\begin{array}{ccc|c}3 & 4 & 5 & 18\\ 0 & -11 & 14 & 3\\ 0 & -26 & -4 & -30\end{array}\right) \quad \begin{array}{l}R_2\to 3R_2-2R_1\\ R_3\to 3R_3-5R_1\end{array}$$

$$\begin{array}{cccc}6 & -3 & 24 & 39\\ 6 & 8 & 10 & 36\\ \hline 0 & -11 & 14 & 3\end{array}$$

$$\sim\left(\begin{array}{ccc|c}3 & 4 & 5 & 18\\ 0 & -11 & 14 & 3\\ 0 & 0 & -408 & -408\end{array}\right) \quad R_3\to 11R_3-26R_2$$

$$\begin{array}{cccc}0 & -286 & -44 & -330\\ 0 & -286 & 364 & 78\\ \hline 0 & 0 & -408 & -408\end{array}$$

by backward substitution method,

$-408z=-408$

$\boxed{z=1}$

$-11y+14z=3$

$-11y+14(1)=3$

$-11y=3-14$

$-11y=-11$

$\boxed{y=1}$

$3x+4y+5z=18$

$3x+4+5=18$

$3x=18-9$

$3x=9$

$\boxed{x=3}$.

$\therefore$ The solution is $x=2$, $y=1$, $z=1$.

— x —

Solve by Gauss elimination method $2x+y+4z=12$

$8x-3y+2z=20$; $4x+11y-z=33$.

Sol) The given system of equations can be written in matrix form as $AX=B$.

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad B = \begin{pmatrix} 12 \\ 20 \\ 33 \end{pmatrix}$$

The augmented matrix,

$$(A,B) = \left(\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 33 \end{array}\right)$$

$$\begin{array}{cccc} 4 & 11 & -1 & 33 \\ -4 & -2 & -8 & -24 \\ \hline 0 & 9 & -9 & 9 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 9 & -9 & 9 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 \to 2R_1 \end{array}$$

$$\begin{array}{cccc} 8 & -3 & 2 & 20 \\ -8 & -4 & -16 & -48 \\ \hline 0 & -7 & 14 & -21 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & +2 & 4 \\ 0 & -1 & 1 & 1 \end{array}\right) \begin{array}{l} R_2 \to \frac{R_2}{-7} \\ R_3 \to \frac{R_3}{9} \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -3 & -3 \end{array}\right) R_3 \to R_3 - R_2$$

by backward substitution method,

$-3z = -3$

$\boxed{z=1}$

$y+2z=4$

$y+2=4$

$\boxed{y=2}$

$2x+y+4z=12$

$2x+2+4=12$

$2x+6=12$

$2x = 12-6$

$2x=6$

$\boxed{x=3}$

$\therefore$ The solution is $x=3$, $y=2$, $z=1$

Apply Gauss Jordan method to find the solution of the following system: $10x+y+z=12$; $2x+10y+z=13$; $x+y+5z=7$.

Sol

The given ~~elet~~ system of equations can be written in matrix form as $AX=B$.

$$A=\begin{pmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{pmatrix} \quad X=\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad B=\begin{pmatrix} 12 \\ 13 \\ 7 \end{pmatrix}$$

the augmented matrix is,

$$(A,B)=\left(\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{array}\right) \quad R_1 \leftrightarrow R_3$$

$$\sim\left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{array}\right) \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 10R_1 \end{array}$$

$$\sim\left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & -9 & -49 & -58 \end{array}\right) \quad R_2 \to R_2/8$$

$$\sim\left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & 0 & -59.125 & -59.125 \end{array}\right) \quad R_3 \to R_3 + 9R_2$$

$$\sim\left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad R_2 \to \frac{R_3}{-59.125}$$

$$\sim\left(\begin{array}{ccc|c} 1 & 0 & 6.125 & 7.125 \\ 0 & 1 & -1.125 & -1.125 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad R_1 \to R_1 - R_2$$

$$\sim\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad R_1 \to R_1 - 6.125R_3$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad R_2 \to R_2 + 1.125 R_3.$$

Hence, $x = 1,\ y = 1,\ z = 1$

Solve the following system of equations by Gauss Jordan method: $x + 2y + z = 3$; $2x + 3y + 3z = 10$
$3x - y + 2z = 13.$

Solution:

The system of equations can be written in matrix form as, $AX = B$.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad B = \begin{pmatrix} 3 \\ 10 \\ 13 \end{pmatrix}$$

The augmented matrix,

$$(A, B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right)$$

$$\begin{array}{cccc} 3 & -1 & 2 & 13 \\ -3 & -6 & -3 & -9 \\ \hline 0 & -7 & -1 & 4 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right) \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1. \end{array}$$

$$\begin{array}{cccc} 2 & 3 & 3 & 10 \\ -2 & -4 & -2 & -6 \\ \hline 0 & -1 & 1 & 4 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right) \quad R_3 \to R_3 - 7R_2$$

$$\begin{array}{cccc} 0 & -7 & -1 & 4 \\ 0 & 7 & -7 & -28 \\ \hline 0 & 0 & -8 & -24 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right) \quad R_1 \to R_1 + 2R_2$$

$$\begin{array}{cccc} 1 & 2 & 1 & 3 \\ 0 & -2 & 2 & 8 \\ \hline 1 & 0 & 3 & 11 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right) \quad \begin{array}{l} R_2 \to -R_2 \\ R_3 \to \frac{R_3}{-8} \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{l} R_1 \to R_1 - 3R_3 \\ R_2 \to R_2 + R_3 \end{array}$$

$R_1 - 3R_3$

$$\begin{array}{cccc} 1 & 0 & 3 & 11 \\ 0 & 0 & -3 & -9 \\ \hline 1 & 0 & 0 & 2 \end{array}$$

$$\begin{array}{cccc} 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 3 \\ \hline 0 & 1 & 0 & -1 \end{array}$$

from this,

$x = 2, \; y = -1, \; z = 3.$

Solve the following system of equations by Gauss Jordan method: $x - y + z = 1$; $-3x + 2y - 3z = -6$; $2x - 5y + 4z = 5$

solution:

The system of equations can be written in matrix form as, $AX = B$.

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -3 & 2 & -3 \\ 2 & -5 & 4 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ -6 \\ 5 \end{pmatrix}$$

The augmented matrix,

$$(A, B) = \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -3 & 2 & -3 & -6 \\ 2 & -5 & 4 & 5 \end{array}\right)$$

$$\begin{array}{cccc} 2 & -5 & 4 & 5 \\ -2 & 2 & -2 & -2 \\ \hline 0 & -3 & 2 & 3 \end{array}$$

$$\begin{array}{cccc} -3 & 2 & -3 & -6 \\ 3 & -3 & 3 & 3 \\ \hline 0 & -1 & 0 & -3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & -3 & 2 & 3 \end{array}\right) \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & -3 & 2 & 3 \end{array}\right) R_2 \to -R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & 0 \end{array}\right) R_3 \to R_3 - 3R_1$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 12 \end{array}\right) R_3 \to R_3 + 3R_2$$

$$\begin{array}{cccc} 0 & -3 & 2 & 3 \\ 0 & 3 & 0 & 9 \\ \hline 0 & 0 & 2 & 12 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right) R_3 \to \frac{R_3}{2}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right) R_1 \to R_1 + R_2$$

$$\begin{array}{cccc} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ \hline 1 & 0 & 1 & 4 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right) R_1 \to R_1 - R_3$$

$$\begin{array}{cccc} 1 & 0 & 1 & 4 \\ 0 & 0 & -1 & 6 \\ \hline 1 & 0 & 0 & -2 \end{array}$$

Hence, $x = -2$, $y = 3$, $z = 6$.

$x - y + z = 1$
$-2 - 3 + 6 = 1$
$- 5 + 6 = 1$
$1 = 1$ ✓

Solve by Gauss Jordan method: $10x + y + z = 12$; $x + 10y + z = 12$; $x + y + 10z = 12$.

The system of equations can be written in matrix form $AX = B$.

$$A = \begin{pmatrix} 10 & 1 & 1 \\ 1 & 10 & 1 \\ 1 & 1 & 10 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad B = \begin{pmatrix} 12 \\ 12 \\ 12 \end{pmatrix}$$

The augmented matrix

$$(A, B) = \left(\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 1 & 10 & 1 & 12 \\ 1 & 1 & 10 & 12 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 10 & 1 & 12 \\ 10 & 1 & 1 & 12 \\ 1 & 1 & 10 & 12 \end{array}\right) R_1 \leftrightarrow R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 10 & 1 & 12 \\ 0 & -99 & -9 & -108 \\ 0 & -9 & 9 & 0 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 10R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\begin{array}{cccc} 10 & 1 & 1 & 12 \\ -10 & -100 & -10 & -120 \\ \hline 0 & -99 & -9 & -108 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 10 & 1 & 12 \\ 0 & 11 & 1 & 12 \\ 0 & -1 & 1 & 0 \end{array}\right) \begin{array}{l} R_2 \to \frac{R_2}{-9} \\ R_3 \to \frac{R_3}{9} \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 10 & 1 & 12 \\ 0 & 11 & 1 & 12 \\ 0 & 0 & 12 & 12 \end{array}\right) R_3 \to 11R_3 + R_2$$

$$\begin{array}{cccc} 0 & -11 & 11 & 0 \\ 0 & 11 & 1 & 12 \\ \hline 0 & 0 & 12 & 12 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 10 & 1 & 12 \\ 0 & 11 & 1 & 12 \\ 0 & 0 & 1 & 1 \end{array}\right) R_3 \to \frac{R_3}{12}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 10 & 1 & 12 \\ 0 & 11 & 0 & 11 \\ 0 & 0 & 1 & 1 \end{array}\right) R_2 \to R_2 - R_3$$

$$\begin{array}{cccc} 0 & 11 & 1 & 12 \\ 0 & 0 & -1 & -1 \\ \hline 0 & 11 & 0 & 11 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 10 & 1 & 12 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) R_2 \to \frac{R_2}{11}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) R_1 \to R_1 - 10R_2$$

$$\begin{array}{cccc} 1 & 10 & 1 & 12 \\ 0 & -10 & 0 & -10 \\ \hline 1 & 0 & 1 & 2 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) R_1 \to R_1 - R_3$$

$$\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ \hline 1 & 0 & 0 & 1 \end{array}$$

Hence $x = 1, y = 1, z = 1$

$x + 10y + z = 12$

$1 + 10(1) + 1 = 12$

$1 + 10 + 1 = 12$

$12 = 12$

Q. Use Gauss elimination method solve

i) $11x+3y=17$; $2x+7y=16$

ii) $4x-3y=11$; $3x+2y=4$

i) $11x+3y=17$, $2x+7y=16$.

The system of equations can be written as, $AX=B$.

$$A=\begin{bmatrix}11 & 3\\ 2 & 7\end{bmatrix} \quad X=\begin{bmatrix}x\\ y\end{bmatrix} \quad B=\begin{bmatrix}17\\ 16\end{bmatrix}$$

The augmented matrix,

$$(A,B)=\left[\begin{array}{cc|c}11 & 3 & 17\\ 2 & 7 & 16\end{array}\right]$$

$$\sim\left[\begin{array}{cc|c}11 & 3 & 17\\ 0 & 71 & 142\end{array}\right] \quad R_2\to 11R_2-2R_1$$

$$\begin{array}{ccc}22 & 77 & 176\\ -22 & -6 & -34\\ \hline 0 & 71 & 142\end{array}$$

by backward substitution method,

$71y=142$

$\boxed{y=2}$

$11x+3y=17$

$11x+6=17$

$11x=11$

$\boxed{x=1}$

$\therefore$ The solution is $x=1$, $y=2$

ii) The system of equations can be written as, $AX=B$.

$$A=\begin{bmatrix}4 & -3\\ 3 & 2\end{bmatrix} \quad X=\begin{bmatrix}x\\ y\end{bmatrix} \quad B=\begin{bmatrix}11\\ 4\end{bmatrix}$$

The augmented matrix,

$$(A,B)=\left[\begin{array}{cc|c}4 & -3 & 11\\ 3 & 2 & 4\end{array}\right]$$

$$\sim\left[\begin{array}{cc|c}4 & -3 & 11\\ 0 & 17 & -17\end{array}\right] \quad R_2\to 4R_2-3R_1$$

$$\begin{array}{ccc}12 & 8 & 16\\ -12 & 9 & -33\\ \hline 0 & 17 & -17\end{array}$$

by backward substitution method,

$17y=-17$

$\boxed{y=-1}$

$4x-3y=11$

$4x+3=11$

$4x=11-3$

$\boxed{x=2}$

LUD method:

[This method is also called as decomposition method. In this method, the coefficient matrix A of the system $AX=B$, is decomposed or factorized into the product of a lower triangular matrix L & an upper triangular matrix U.] 1st part

2nd part

[
$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

i.e.
$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Equating corresponding coefficients we get nine equations in nine unknowns. From these 9 equations, we can solve for 3l's & 6u's.]

2nd part

[consider the system of equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \quad \rightarrow (1). \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

This system is equivalent $AX=B$ $\rightarrow (2)$.

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$LUX = B \quad \rightarrow (3)$$

Let $\quad UX = Y \quad \rightarrow (4)$

then $\quad LY = B \quad \rightarrow (5)$

i.e., $$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \rightarrow (6)$$

$y_1 = b_1, \quad l_{21} y_1 + y_2 = b_2, \quad l_{31} y_1 + l_{32} y_2 + y_3 = b_3$

by forward substitution, $y_1\ y_2\ y_3$ can be found out if L is known,

From (4)

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

i.e. $$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$$

$$u_{22}x_2 + u_{23}x_3 = y_2$$

$$u_{33}x_3 = y_3$$

From these x_1, x_2, x_3 can be solved by back substitution, since $y_1, y_2\ y_3$ are known if U is known. Now L & U can be found from $LU = A$

Solve the following system by LU Decomposition Method: $x+5y+z=14$; $2x+y+3z=13$; $3x+y+4z=17$.

The given system of equations can be written in matrix form as $AX=B$.

(ie) $$\begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 17 \end{pmatrix}$$

$$LU = A$$

ie $$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix}$$

ie. $u_{11} = 1$; $u_{12} = 5$; $u_{13} = 1$

$l_{21}u_{11} = 2$; $l_{21}u_{12} + u_{22} = 1$; $l_{21}u_{13} + u_{23} = 3$

$\Rightarrow \boxed{l_{21} = 2}$ $\quad 2(5) + u_{22} = 1$ $\quad 2(1) + u_{23} = 3$

$\boxed{u_{22} = -9}$ $\quad \boxed{u_{23} = 1}$

$l_{31}u_{11} = 3$; $l_{31}u_{12} + l_{32}u_{22} = 1$; $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 4$

$\boxed{l_{31} = 3}$

$3(5) + l_{32}(-9) = 1$

$15 - 9l_{32} = 1$

$-9l_{32} = 1 - 15$

$-9l_{32} = -14$

$l_{32} = \frac{14}{9}$

$3(1) + \frac{14}{9}(1) + u_{33} = 4$

$3 + \frac{14}{9} + u_{33} = 4$

$27 + 14 + 9u_{33} = 36$

$41 + 9u_{33} = 36$

$9u_{33} = 36 - 41$

$9u_{33} = -5$

$u_{33} = -5/9$

since we have $AX = B$, & $LU = A$

$\Rightarrow \quad LUX = B$

let $UX = Y$

$\therefore \quad LY = B$

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 17 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 17 \end{pmatrix}$$

$\boxed{y_1 = 14}$; $2y_1 + y_2 = 13$

$2(14) + y_2 = 13$

$28 + y_2 = 13$

$y_2 = 13 - 28$

$\boxed{y_2 = -15}$

; $3y_1 + \frac{14}{9} y_2 + y_3 = 17$

$27y_1 + 14y_2 + 9y_3 = 153$

$27(14) + 14(-15) + 9y_3 = 153$

$378 - 210 + 9y_3 = 153$

$168 + 9y_3 = 153$

$9y_3 = 153 - 168$

$y_3 = -\frac{15}{9}$

$\boxed{y_3 = -1.67}$

Now,

$UX = Y$

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\underset{3\times3}{\begin{pmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -\frac{5}{9} \end{pmatrix}} \underset{3\times1}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \begin{pmatrix} 14 \\ -15 \\ -1.67 \end{pmatrix}$$

$$\begin{pmatrix} x+5y+z \\ -9y+z \\ -\frac{5}{9}z \end{pmatrix} = \begin{pmatrix} 14 \\ -15 \\ -1.63 \end{pmatrix}$$

$x+5y+z=14$; $-9y+z=-15$; $-\frac{5}{9}z=-1.63$

$x+5(2)+3=14$ $\quad$ $-9y+(3)=-15$ $\quad$ $z=\frac{14.67}{5}$

$x+10+3=14$ $\quad$ $-9y=-15-3$

$\boxed{x=1}$ $\quad$ $-9y=-18$ $\quad$ $z=2.934.$

$\boxed{y=2}$ $\quad$ $\boxed{z=3}$

The solution is $x=1, y=2, z=3$.

solve ~~by~~ LUD method: $x+y+z=1; 4x+3y-z=6;$
$3x+5y+3z=4$

solution:

The given system of equations can be written in matrix form: $AX=B$.

Here:

$$A=\begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix} \quad X=\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad B=\begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}.$$

Now, $LU=A$

ie $$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}=\begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix}.$$

$u_{11}=1$; $u_{12}=1$; $u_{13}=1$;

$l_{21}u_{11}=4$; $l_{21}u_{12}+u_{22}=3$; $l_{21}u_{13}+u_{23}=-1$

$l_{21}(1)=4$ $\quad$ $4(1)+u_{22}=3$ $\quad$ $4(1)+u_{23}=-1$

$\boxed{l_{21}=4}$ $\quad$ $\boxed{u_{22}=-1}$ $\quad$ $u_{23}=-1-4$

$\boxed{u_{23}=-5}$

$l_{31} u_{11} = 3$; $l_{31} u_{12} + l_{32} u_{22} = 5$;

$l_{31}(1) = 3$ $\quad$ $3(1) + l_{32}(-1) = 5$

$\boxed{l_{31} = 3}$ $\quad$ $3 - l_{32} = 5$

$-l_{32} = 5-3$

$\boxed{l_{32} = -2}$

$l_{31} u_{13} + l_{32} u_{23} + u_{33} = 3$

$(3)(1) + (-2)(-5) + u_{33} = 3.$

$3 + 10 + u_{33} = 3$

$+13 + u_{33} = 3$

$\boxed{u_{33} = -10}$

since we have $LU = A$; $B = AX$

$\Rightarrow LUX = B.$

$\because UX = Y$

$LY = B$

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}$$

$y_1 = 1$; $4y_1 + y_2 = 6$; $3y_1 - 2y_2 + y_3 = 4$

$4 + y_2 = 6$ $\quad$ $3 - 2(2) + y_3 = 4$

$\boxed{y_2 = 2}$ $\quad$ $3 - 4 + y_3 = 4$

$-1 + y_3 = 4$

$\boxed{y_3 = 5}$

$\Rightarrow UX = Y$

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

$x + y + \cancel{y} = 1$; $-y - 5z = 2$; $-10z = 5$

$x + \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} = 1$ $\quad$ $-y - \frac{5}{2} = 2$ $\quad$ $\boxed{z = -1/2}$

$\boxed{x = 1}$ $\quad$ $-2y + 5 = 4$

$\boxed{y = +1/2}$

using LUD method solve the following system of equations: $5x - 2y + z = 4$; $7x + y - 5z = 8$; $3x + 7y + 4z = 10$.