CROUT'S METHOD.

This is also a direct method. Here also we decompose the coefficient matrix A as LU

where $L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$ and $U = \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$

Then proceed in the previous method (LUD)

1. Solve the following system by using crout's method $2x+3y+z=-1$; $5x+y+z=9$; $3x+2y+4z=11$.

Solution:

The given system can be written in matrix form as $AX=B$.

Here, $A = \begin{pmatrix} 2 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 4 \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $B = \begin{pmatrix} -1 \\ 9 \\ 11 \end{pmatrix}$.

Let $LU = A$

$$\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 4 \end{pmatrix}.$$

$l_{11} = 2$; $l_{11}u_{12} = 3$; $l_{11}u_{13} = 1$

$(2)u_{12} = 3$; $2u_{13} = 1$

$u_{12} = \frac{3}{2}$; $u_{13} = 1/2$

$l_{21} = 5$; $l_{21}u_{12} + l_{22} = 1$; $l_{21}u_{13} + l_{22}u_{23} = 1$

$(5)\left(\frac{3}{2}\right) + l_{22} = 1$

$15 + 2l_{22} = 2$

$2l_{22} = 2-15$

$l_{22} = -\frac{13}{2}$

$(5)\left(\frac{1}{2}\right) + \left(-\frac{13}{2}\right)u_{23} = 1$

$\frac{5}{2} - \frac{13u_{23}}{2} = 1$

$\cancel{15 - 26u_{23} = 2}$

$-26u_{23} = 6-15$

$5 - 13u_{23} = 2 \Rightarrow -13u_{23} = 2-5$

$-13u_{23} = -3 \qquad u_{23} = \frac{3}{13}$

$l_{31} = 3$; $l_{31}u_{12} = 2$

$(l_{31})u_{12} + l_{32} = 2$

$3\left(\frac{3}{2}\right) + l_{32} = 2$

$9 + 2l_{32} = 4$

$2l_{32} = 4-9$

$2l_{32} = -5$

$l_{32} = \frac{-5}{2}$

; $l_{31}u_{13} + l_{32}u_{23} + l_{33} = 4$

$(3)\left(\frac{1}{2}\right) + l_{32}\left(\frac{3}{13}\right) + l_{33} = 4$

$\frac{3}{2} - \frac{5}{2}\left(\frac{3}{13}\right) + l_{33} = 4$

$\frac{3}{2} - \frac{15}{26} + l_{33} = 4$

$\frac{39}{26} - \frac{15}{26} + l_{33} = 4$

$\frac{24}{26} + l_{33} = 4$

$24 + 26l_{33} = 104$

$26l_{33} = 104 - 24$

$26l_{33} = 80$

$l_{33} = \frac{\cancel{80}^{40}}{\cancel{26}_{13}}$

$l_{33} = \frac{40}{13}$

now, $LU = A$; $B = AX$

$LUX = B$

$LY = B \quad \because UX = Y$

now, $LY = B$

$$\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \\ 11 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 5 & -\frac{13}{2} & 0 \\ 3 & -\frac{5}{2} & \frac{40}{13} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \\ 11 \end{pmatrix}$$

$2y_1 = -1$

$\boxed{y_1 = -1/2}$

; $5y_1 - \frac{13}{2}y_2 = 9$

$5\left(-\frac{1}{2}\right) - \frac{13}{2}y_2 = 9$

$-5 - 13y_2 = 18$

$-13y_2 = 18 + 5$

$-13y_2 = 23$

$\boxed{y_2 = -\frac{23}{13}}$

$$3y_1 - \frac{5}{2}y_2 + \frac{40}{13}y_3 = 11$$

$$3\left(-\frac{1}{2}\right) - \frac{5}{2}\left(-\frac{23}{13}\right) + \frac{40}{13}y_3 = 11$$

$$-\frac{3}{2} + \frac{115}{26} + \frac{40}{13}y_3 = 11$$

$$-\frac{39}{26} + \frac{115}{26} + \frac{80}{26}y_3 = 11$$

$$-39 + 115 + 80y_3 = 11 \times 26$$

$$76 + 80y_3 = 286$$

$$80y_3 = 286 - 76$$

$$80y_3 = 210$$

$$y_3 = \frac{21\cancel{0}}{8\cancel{0}}$$

$$\boxed{y_3 = \frac{21}{8}}$$

$$Ux = y$$

$$\begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 3/13 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1/2 \\ -23/13 \\ 21/8 \end{pmatrix}$$

$z = \frac{21}{8}$; $y + \frac{3}{13}z = -\frac{23}{13}$

(or)

$z = 2.6250$

$13y + 3z = -23$

$13y + 3\left(\frac{21}{8}\right) = -23$

$104y + 63 = -184$

$104y = -184 - 63$

$104y = -247$

$y = -2.3750$

$$x + \frac{3}{2}\left(\frac{-247}{104}\right) + \frac{1}{2}\left(\frac{21}{8}\right) = -\frac{1}{2}$$

$$x + \frac{(-741)}{208} + \frac{21}{16} = -\frac{1}{2}$$

$$208x - 741 + 273 = -104$$

$$208x - 468 = -104$$

$$208x = -104 + 468$$

$$208x = 364$$

$$x = 1.7500$$

The solution is $x = 1.7500$; $y = -2.3750$, $z = 2.6250$.

Iterative methods (Indirect Method).

Iterative methods can be applied for the system of equation.

$a_1x_1 + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$

Only if $|a_1| > |b_1| + |c_1|$, $|b_2| > |a_2| + |c_2|$, $|c_3| > |a_3| + |b_3|$ (ie)

we can apply iterative method only when the system of equations are dynamically dominant.

There are two iterative methods:

(i) Gauss Jacobi method.

(ii) Gauss Seidal method.

Gauss Jacobi Method:

The iteration scheme is given by

$$x^{r+1} = \frac{1}{a_1}\left[d_1 - b_1(y)^r - c_1(z)^r\right]$$

$$y^{r+1} = \frac{1}{b_2}\left[d_2 - a_2(x)^r - c_2(z)^r\right]$$

$$z^{r+1} = \frac{1}{c_3}\left[d_3 - a_3(x)^r - b_3(y)^r\right].$$

Taking $x^0 = y^0 = z^0 = 0$, we can find the successive value of x, y, z.

1. Solve the following system of equations by Gauss Jacobi method.

$10x - 5y - 2z = 3$, $4x - 10y + 3z = -3$, $x + 6y + 10z = -3$.

Soln

Since the given system of equations are dynamically dominant, iteration method can be applied.

Solving for x, y and z we have,

$$x = \frac{1}{10}(3 + 5y + 2z)$$

$$y = -\frac{1}{10}(-3 - 4x - 3z)$$

$$z = \frac{1}{10}(-3 - x - 6y)$$

1st iteration:

take $x = y = z = 0$.

$$x^1 = \frac{1}{10}(3 + 5(0) + 2(0)) = \frac{3}{10}$$

$$y^1 = \frac{-1}{10}(-3 - 4(0) - 3(0)) = \frac{3}{10}$$

$$z^1 = \frac{1}{10}(-3 - (0) - 6(0)) = -\frac{3}{10}$$

2nd iteration:

$$x^2 = \frac{1}{10}\left(3 + 5\left(\frac{3}{10}\right) + 2\left(\frac{-3}{10}\right)\right)$$

$$= \frac{1}{10}\left(\frac{30 + 15 - 6}{10}\right)$$

$$x^2 = \frac{39}{100} = 0.39.$$

$$y^2 = -\frac{1}{10}\left(-3 - 4\left(\frac{3}{10}\right) - 3\left(\frac{-3}{10}\right)\right)$$

$$y^2 = 0.33$$

$$z^2 = \frac{1}{10}(-3 - 0.3 - 6(0.3))$$

$$= -0.51$$

3rd iteration:

$$x^3 = \frac{1}{10}(3 + 5(0.33) + 2(-0.51))$$

$$= 0.363$$

$y^3 = -\frac{1}{10}(-3 - 4(0.39) - 3(-0.51)) = 0.303$

$z^3 = \frac{1}{10}(-3 - 0.39 - 6(0.33)) = -0.537$

4th iteration:

$x^4 = \frac{1}{10}(3 + 5(0.303) + 2(-0.537)) = 0.3441$

$y^4 = -\frac{1}{10}(-3 - 4(0.363) - 3(0.537)) = 0.2841$

$z^4 = \frac{1}{10}(-3 - 0.363 - 6(0.303)) = -0.5181$

5th iteration:

$x^5 = \frac{1}{10}(3 + 5(0.2841) + 2(-0.5181)) = 0.3384$

$y^5 = -\frac{1}{10}(-3 - 4(0.3441) - 3(-0.5181)) = 0.2822$

$z^5 = \frac{1}{10}(-3 - 0.3441 - 6(0.2841)) = -0.5049.$

6th iteration:

$x^6 = 0.3401$

$y^6 = 0.2839$

$z^6 = 0.5032$

7th iteration:

$x^7 = 0.3413$

$y^7 = 0.2851$

$z^7 = 0.5043$

8th iteration:

$x^8 = 0.3416$

$y^8 = 0.2852$

$z^8 = -0.5051$

9th iteration:

$x^9 = 0.3416$

$y^9 = 0.2852$

$z^9 = -0.5051$

Hence the solution is $x = 0.342$,

$y = 0.285$ and $z = -0.505$)

(Corrected to three decimal places)

Gauss Jacobi Method.

2. $8x - 3y + 2z = 20$, $4x + 11y - z = 33$, $6x + 3y + 12z = 35$.

<u>sol</u>: Since the given system of equations are diagonally dominant. So iteration method can be applied.

solving $x, y, z \rightarrow$

$$x = \frac{1}{8}[20 + 3y - 2z]$$

$$y = \frac{1}{11}[33 - 4x + z]$$

$$z = \frac{1}{12}[35 - 6x - 3y]$$

1st iteration: $x = y = z = 0$

$$x^1 = \frac{1}{8}[20 + 3(0) - 2(0)] = 2.500$$

$$y^1 = \frac{1}{11}[33 - 4(0) + (0)] = 3.000$$

$$z^1 = \frac{1}{12}[35 - 6(0) - 3(0)] = 2.917$$

2nd iteration:

$$x^2 = \frac{1}{8}[20 + 3(3) - 2(2.917)] = 2.896.$$

$$y^2 = \frac{1}{11}[33 - 4(2.5) + 2.917] = 2.356$$

$$z^2 = \frac{1}{12}[35 - 6(2.5) - 3(3)] = 0.917$$

3rd iteration:

$$x^3 = \frac{1}{8}[20 + 3(2.356) - 2(0.917)] = 3.154.$$

$$y^3 = \frac{1}{11}[33 - 4(2.896) + 0.917] = 2.030$$

$$z^3 = \frac{1}{12}[35 - 6(2.896) - 3(2.356)] = 0.880$$

4^{th} Iteration:

$$x^4 = \frac{1}{8}[20 + 3(2.030) - 2[0.880]) = 3.041$$

$$y^4 = \frac{1}{11}[33 - 4(3.154) + 0.880] = 1.933$$

$$z^4 = \frac{1}{12}[35 - 6(3.154) - 3(2.030)] = 0.832$$

5^{th} Iteration:

$$x^5 = \frac{1}{8}[20 + 3(1.933) - 2(0.832)] = 3.017.$$

$$y^5 = \frac{1}{11}[33 - 4(3.041) + 0.832] = 1.970.$$

$$z^5 = \frac{1}{12}[35 - 6(3.041) - 3(1.933)] = 0.913$$

6^{th} Iteration:

$$x^6 = \frac{1}{8}[20 + 3(1.970) - 2(0.913)] = 1.533.$$

$$y^6 = \frac{1}{11}[33 - 4(3.017) + 0.913] = 1.986.$$

$$z^6 = \frac{1}{12}[35 - 6(3.017) - 3(1.970)] = 0.916$$

7^{th} Iteration:

$$x^7 = \frac{1}{8}[20 + 3(1.986) - 2(0.916)] = 3.016$$

$$y^7 = \frac{1}{11}[33 - 4(1.533) + 0.916] = 2.526$$

$$z^7 = \frac{1}{12}[35 - 6(1.533) - 3(1.986)] = 1.654$$

8^{th} Iteration:

$$x^8 = \frac{1}{8}[20 + 3(2.526) - 2(1.654)] = 3.034$$

$$y^8 = \frac{1}{11}[33 - 4(3.016) + 1.654] = 2.054.$$

$$z^8 = \frac{1}{12}[35 - 6(3.016) - 3(2.526)] = 0.777$$

9^{th} Iteration:

$$x^9 = \frac{1}{8}[20 + 3(2.054) - 2(0.777)] = 3.076.$$

$$y^9 = \frac{1}{11}[33 - 4(3.034) + 0.777] = 1.967.$$

$$z^9 = \frac{1}{12}[35 - 6(3.034) - 3(2.054)] = 0.886$$

10^{th} iteration:

$x^{10} = \frac{1}{8}[20 + 3(1.967) - 2(0.886)] = 3.016$

$y^{10} = \frac{1}{11}[33 - 4(3.076) + 0.886] = 1.962$

$z^{10} = \frac{1}{12}[35 - 6(3.076) - 3(1.967)] = 0.887.$

11^{th} iteration:

$x^{11} = \frac{1}{8}[20 + 3(1.962) - 2(0.887)] = 3.014.$

$y^{11} = \frac{1}{11}[33 - 4(3.016) + 0.887] = 1.984$

$z^{11} = \frac{1}{12}[35 - 6(3.016) - 3(1.962)] = 0.918$

12^{th} iteration:

$x^{12} = \frac{1}{8}[20 + 3(\overset{1.984}{\cancel{3.014}}) - 2(\overset{0.918}{\cancel{1.984}})] = \cancel{3.134}\ 3.015$

$y^{12} = \frac{1}{11}[33 - 4(3.014) + 0.918] = 1.987$

$z^{12} = \frac{1}{12}[35 - 6(3.014) - 3(\overset{1.984}{\cancel{0.918}})] = \cancel{1.180}\ 0.914$

13^{th} iteration:

$x^{13} = \frac{1}{8}[20 + 3(\overset{(1.987)}{\cancel{3.134}}) - 2(\overset{(0.914)}{\cancel{1.180}})] = 3.017$

$y^{13} = \frac{1}{11}[33 - 4(3.015) + 0.914] = 1.987$

$z^{13} = \frac{1}{12}[35 - 6(3.015) - 3(1.987)] = 0.912$

14^{th} iteration:

$x^{14} = \frac{1}{8}[20 + 3(1.987) - 2(0.912)] = 3.017$

$y^{14} = \frac{1}{11}[33 - 4(3.017) + 0.912] = 1.986$

$z^{14} = \frac{1}{12}[35 - 6(3.017) - 3(1.987)] = 0.911$

Hence the value of $x = 3.017$, $y = 1.986$, $z = 0.911$ correct to three decimal places.

3. $27x + 6y - z = 85$, $6x + 15y + 2z = 72$, $x + y + 54z = 110$.

Sol:

The given equations are diagonally dominant so iteration method can be ~~equa~~applied.

Solving $x, y, z \rightarrow$

$$x = \frac{1}{27}[85 - 6y + z]$$

$$y = \frac{1}{15}[72 - 6x - 2z]$$

$$z = \frac{1}{54}[110 - x - y]$$

1st iteration: $x = y = z = 0$

$$x^1 = \frac{1}{27}[85 - 6(0) + (0)] = 3.148$$

$$y^1 = \frac{1}{15}[72 - 6(0) - 2(0)] = 4.800$$

$$z^1 = \frac{1}{54}[110 - 0 - 0] = 2.037$$

2nd iteration:

$$x^2 = \frac{1}{27}[85 - 6(4.8) + (2.037)] = 2.157$$

$$y^2 = \frac{1}{15}[72 - 6(3.148) - 2.037] = 3.405$$

$$z^2 = \frac{1}{54}[110 - 3.148 - 4.8] = 1.890$$

3rd iteration:

$$x^3 = \frac{1}{27}[85 - 6(3.405) + 1.890] = 2.461$$

$$y^3 = \frac{1}{15}[72 - 6(2.157) - 1.890] = 3.811$$

$$z^3 = \frac{1}{54}[110 - 2.157 - 3.405] = 1.934$$

4th iteration:

$$x^4 = 1/27[85 - 6(3.811) + 1.934] = 2.373$$

$$y^4 = 1/15[72 - 6(2.461) - 1.934] = 3.687$$

$$z^4 = 1/54[110 - 2.461 - 3.811] = 1.921$$

5^{th} iteration:

$x^5 = \frac{1}{27}[85 - 6(\overset{3.687}{\cancel{3.811}}) + 1.921] =$

$y^5 = \frac{1}{15}[72 - 6(2.373) - 1.921]$

$z^5 = \frac{1}{54}[110 -$

2^{nd} iteration:

$x^2 = \frac{1}{27}[85 - 6(4.8) + 2.037] = 2.157.$

$y^2 = \frac{1}{15}[72 - 6(3.148) - 2(2.037)] = 3.269.$

$z^2 = \frac{1}{54}[110 - 3.148 - 4.8] = 1.890$

3^{rd} iteration:

$x^3 = \frac{1}{27}[85 - 6(3.269) + 1.890] = 2.352.$

$y^3 = \frac{1}{15}[72 - 6(2.157) - 2(1.890)] = 3.685.$

$z^3 = \frac{1}{54}[110 - 2.157 - 3.269] = 1.937$

4^{th} iteration:

$x^4 = \frac{1}{27}[85 - 6(3.685) + 1.937] = 2.401$

$y^4 = \frac{1}{15}[72 - 6(2.352) - 2(1.937)] = 3.601$

$z^4 = \frac{1}{54}[110 - 2.352 - 3.685] = 1.925$

5^{th} iteration:

$x^5 = 1/27\,[85 - 6(3.601) + 1.925] = 2.419$

$y^5 = 1/15\,[72 - 6(2.401) - 2(1.925)] = 3.583$

$z^5 = 1/54\,[110 - 2.401 - 3.601] = 1.926.$

6^{th} iteration:

$x^6 = 1/27\,(85 - 6(3.583) + 1.926) = 2.423.$

$y^6 = 1/15\,(72 - 6(2.419) - 2(1.926)) = 3.576$

$z^6 = 1/54\,(110 - 2.419 - 3.583) = 1.926$

7th iteration:

$x^7 = \frac{1}{27}[85 - 6(3.576) + 1.926] = 2.425$

$y^7 = \frac{1}{15}[72 - 6(2.423) - 2(1.926)] = 3.574$

$z^7 = \frac{1}{54}[110 - 2.423 - 3.576] = 1.926$

8th iteration:

$x^8 = \frac{1}{27}[85 - 6(3.574) + (1.926)] = 2.425$

$y^8 = \frac{1}{15}[72 - 6(2.425) - 2(1.926)] = 3.573$

$z^8 = \frac{1}{54}[110 - 2.425 - 3.574] = 1.926$

$\therefore$ The solution of x is 2.425, y is 3.573, z is 1.926. correct to three decimal places.

— x —

Gauss Seidal Method.

The iteration scheme is given by,

$$x^{(r+1)} = \frac{1}{a_1}[d_1 - b_1 y^{(r)} - c_1 z^{(r)}],$$

$$y^{(r+1)} = \frac{1}{b_2}[d_2 - a_2 x^{(r+1)} - c_2 z^{(r)}],$$

$$z^{(r+1)} = \frac{1}{c_3}[d_3 - a_3 x^{(r+1)} - b_3 y^{(r+1)}]$$ at each step we use the latest available values of x, y & z

Initially we take $y^{(0)} = z^{(0)} = 0$ and find $x^{(1)}$. We use $x^{(1)}, z^{(0)}$ to find $y^{(1)}$ and we use $x^{(1)}, y^{(1)}$ to find $z^{(1)}$ and so on.

Note :

The rate of convergence in gauss seidal method will be more rapid than in Gauss Jacobi method.

The rate of convergence of gauss seidel method is roughly twice that of Gauss Jacobi method.

1. Solve the following system by Gauss Seidal Method

$28x + 4y - z = 32, \quad x + 3y + 10z = 24, \quad 2x + 17y + 4z = 35.$

Sol :

Since, the diagonal elements are not dominant, we rearrange the equation as follows,

$$28x + 4y - z = 32 \Rightarrow x = \frac{1}{28}[32 - 4y + z]$$

$$2x + 17y + 4z = 35 \Rightarrow y = \frac{1}{17}[35 - 2x - 4z]$$

$$x + 3y + 10z = 24 \Rightarrow z = \frac{1}{10}[24 - x - 3y]$$

1st Iteration:

$x^{(1)} = \frac{1}{28}[32 - 4(0) + (0)] = 1.143$

$y^{(1)} = \frac{1}{17}[35 - 2(1.143) - 4(0)] = 1.924$

$z^{(1)} = \frac{1}{10}[24 - 1.143 - 3(1.924)] = 1.709$

2nd Iteration:

$x^{(2)} = \frac{1}{28}[32 - 4(1.924) + 1.709] = 0.929$

$\cancel{y^{(2)} = \frac{1}{17}[35 - 2(1.143) - 4(1.709)] =}$

$\cancel{z^{(2)} = \frac{1}{10}[24 - 1.143 - 3(1.921}$

$y^{(2)} = \frac{1}{17}[35 - 2(0.929) - 4(1.709)] = 1.547$

$z^{(2)} = \frac{1}{10}[24 - 0.929 - 3(1.547)] = 1.843$

3rd Iteration:

$x^{(3)} = \frac{1}{28}[32 - 4(1.547) + 1.843] = 0.988$

$y^{(3)} = \frac{1}{17}[35 - 2(0.988) - 4(1.843)] = 1.509$

$z^{(3)} = \frac{1}{10}[24 - 0.988 - 3(1.509)] = 1.849$

4th Iteration:

$x^{(4)} = \frac{1}{28}[32 - 4(1.509) + 1.849] = 0.993$

$y^{(4)} = \frac{1}{17}[35 - 2(0.993) - 4(1.849)] = 1.507$

$z^{(4)} = \frac{1}{10}[24 - 0.993 - 3(1.507)] = 1.849$

5th iteration:

$x^{(5)} = \frac{1}{28}[32 - 4(1.507) + 1.849] = 0.994$

$y^{(5)} = \frac{1}{17}[35 - 2(0.994) - 4(1.849)] = 1.507$

$z^{(5)} = \frac{1}{10}[24 - 0.994 - 3(1.507)] = 1.849$

6^{th} Iteration:

$$x^{(6)} = \frac{1}{28}\left[32 - 4(1.507) + 1.849\right] = 0.994$$

$$y^{(6)} = \frac{1}{17}\left[35 - 2(0.994) - 4(1.849)\right] = 1.507$$

$$z^{(6)} = \frac{1}{10}\left[24 - 0.994 - 3(1.507)\right] = 1.849.$$

The solution of x is 0.994, y is 1.507, z is 1.849.

(2) Solve the equations $x+y+54z = 110$, $27x+6y-z = 85$, $6x+15y+2z = 72$.

Sol:

Since the diagonal elements are not diagonally dominant, we rearrange the equations as follows:

Let

$27x+6y-z = 85 \rightarrow (1) \quad \Rightarrow x = \frac{1}{27}\left[85 - 6y + z\right]$

$6x+15y+2z = 72 \rightarrow (2) \quad \Rightarrow y = \frac{1}{15}\left[72 - 6x - 2z\right]$

$x+y+54z = 110 \rightarrow (3) \quad \Rightarrow z = \frac{1}{54}\left[110 - x - y\right]$

1^{st} Iteration:

$$x^{(1)} = \frac{1}{27}\left[85 - 6(0) + (0)\right] = 3.148$$

$$y^{(1)} = \frac{1}{15}\left[72 - 6(3.148) - 2(0)\right] = 3.541$$

$$z^{(1)} = \frac{1}{54}\left[110 - 3.148 - 3.541\right] = 1.913$$

2^{nd} iteration:

$$x^{(2)} = \frac{1}{27}\left[85 - 6(3.541) + 1.913\right] = 2.432$$

$$y^{(2)} = \frac{1}{15}\left[72 - 6(2.432) - 2(1.913)\right] = 3.572$$

$$z^{(2)} = \frac{1}{54}\left[110 - 2.432 - 3.572\right] = 1.926$$

3rd iteration:

$x^{(3)} = \frac{1}{27}[85 - 6(3.572) + 1.926] = 2.426$

$y^{(3)} = \frac{1}{15}[72 - 6(2.426) - 2(1.926)] = 3.573$

$z^{(3)} = \frac{1}{54}[110 - 2.426 - 3.573] = 1.926$

4th iteration

$x^{(4)} = \frac{1}{27}[85 - 6(3.573) + 2(1.926)] = 2.211$

$y^{(4)} = \frac{1}{15}[72 - 6(2.211) - 2(1.926)] = 3.659$

$z^{(4)} = \frac{1}{54}[110 - 2.211 - 3.659] = 1.928$

5th iteration:

$x^{(5)} = \frac{1}{27}[85 - 6(3.659) - 2(1.928)] = 2.192$

$y^{(5)} = \frac{1}{15}[72 - 6(2.192) - 2(1.928)] = 3.666$

$z^{(5)} = \frac{1}{54}[110 - 2.192 - 3.666] = 1.929$

6th iteration:

$x^{(6)} = \frac{1}{27}[85 - 6(3.666) - 2(1.929)] = 2.191$

$y^{(6)} = \frac{1}{15}[72 - 6(2.191) - 2(1.929)] = 3.666$

$z^{(6)} = \frac{1}{54}[110 - 2.191 - 3.666] = 1.929$

7th iteration

$x^{(7)} = \frac{1}{27}[85 - 6(3.666) - 2(1.929)] = 2.191$

$y^{(7)} = \frac{1}{15}[72 - 6(2.191) - 2(1.929)] = 3.666$

$z^{(7)} = \frac{1}{54}[110 - 2.191 - 3.666] = 1.929$

$\therefore$ The solution of x is 2.191, y is 3.666, z is 1.929

4th iteration:

$$x^{(4)} = \frac{1}{27}[85 - 6(3.573) + 1.926] = 2.425$$

$$y^{(4)} = \frac{1}{15}[72 - 6(2.425) - 2(1.926)] = 3.573.$$

$$z^{(4)} = \frac{1}{54}[110 - 2.425 - 3.573] = 1.926.$$

5th iteration:

$$x^{(5)} = \frac{1}{27}[85 - 6(3.573) + 2(1.926)] = \cancel{2.811}\ 2.425$$

$$y^{(5)} = \frac{1}{15}[72 - 6(2.425) - 2(1.926)] = 3.573.$$

$$z^{(5)} = \frac{1}{54}[110 - 2.425 - 3.573] = 1.926.$$

6th iteration:

$$x^{(6)} = \frac{1}{27}[85 - 6(3.573) + 1.926] = 2.425$$

$$y^{(6)} = \frac{1}{15}[72 - 6(2.425) - 2(1.926)] = 3.573.$$

$$z^{(6)} = \frac{1}{54}[110 - 2.425 - 3.573] = 1.926.$$

The solution of x is 2.425, y is 3.573, z is 1.926.

2 Marks:

1. When Gauss Elimination Method fails?

This method fails if the element in the top of first column is 0, we can rectify this by interchanging the rows of the matrix.

2. State the sufficient condition for Gauss Jacobi and Gauss Seidal method to convergence?

29. The coefficient matrix should be diagonally dominant.

3.

Gauss Jacobi Method	Gauss Seidel Method.
* It is a indirect method.	* It is also an indirect method.
* Rate of convergence is slow	* Rate of convergence is fast (roughly twice that of jacobi)
* coefficient matrix should be diagonally dominant.	* coefficient matrix should be diagonally dominant.

4.

Direct Method.	Indirect Method.
* Gauss elimination & Jordan are direct method.	* Gauss Jacobi and seidel are indirect method.
* We get exact solution.	* We get approximate solution
* Its simple & takes less time.	* It takes more time to converge.

— x —

UNIT IV - Interpolation

Definition

2M The process of finding intermediate value of the function from the given set of tabular values.

For equal intervals of x, we use Newton's forward and Newton's backward interpolation formula.

2M We use Newton's forward interpolation formula to interpolate the value which is nearer to the beginning of the table.

2M We use Newton's backward interpolation formula to interpolate the value which is nearer to the end of the table.

Newton's forward interpolation formula:

$$y(x) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots$$

where $u = \frac{x - x_0}{h}$ h - difference in x value

Newton's backward interpolation formula:

$$y(x) = y_n + v.\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \cdots \quad \text{where } v = \frac{x - x_n}{h}$$

NOTE:

The interpolating polynomial for n data will be of degree n-1

1. Find the values of y at $x = 21$, and $x = 28$ from the following data.

x :	20	23	26	29
y :	0.3420	0.3907	0.4284	0.4848

Difference table:

x	y	Δy	$\Delta^2 y\ [\Delta(\Delta y)]$	$\Delta^3 y\ [\Delta(\Delta(\Delta y))]$ (or) $[\Delta(\Delta^2 y)]$
20	0.3420			
		0.0487		
			−0.0010	
23	0.3907			−0.0003
		0.0477		
			−0.0013	
26	0.4284			
		0.0464		
29	0.4848			

To find y at $x = 21$, we use Newton Forward interpolation formula,

$$y(x) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots$$

$$u = \frac{x - x_0}{h} = \frac{21-20}{3} = 0.3333$$

$$y(21) = 0.3420 + 0.3333(0.0487) + \frac{(0.3333)(0.3333-1)}{2}(-0.0010) + \frac{(0.3333)(0.3333-1)(0.3333-2)}{6}(-0.0003)$$

$y(21) = 0.3583$

To find y at $x = 28$, we use Newton backward interpolation formula, $y(x) = y_n + v\,\nabla y_n + \frac{v(v+1)}{2!}\nabla y_n + \cdots$

$$v = \frac{28-29}{3} = -0.3333$$

$$y(28) = 0.4848 + (-0.3333)(0.0464) + \frac{(-0.3333)(-0.3333+1)}{2}(-0.0013) + \frac{(-0.3333)(-0.3333+1)(-0.3333+2)}{6}(-0.0003)$$

$$y(28) = 0.4895.$$

— x —

2. The following data are taken from the steam table.

temp °c	pressure kgf/cm²	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
140	3.685				
		1.1690			
150	4.854		0.2790		
		1.4480		0.0470	
160	6.302		0.3260		0.0020
		1.7740		0.0490	
170	8.076		0.3750		
		2.1490			
180	10.225				

find the pressure at temperature $t = 142^\circ$, $t = 175^\circ$

To find y at $x = 142^\circ$, we use Newton Forward Interpolation formula,

$$y(x) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots$$

here $u = \frac{x - x_0}{h} = \frac{142-140}{10} = 0.2$

$$y(142) = 3.685 + 0.2(1.1690) + \frac{0.2(0.2-1)}{2}(0.2790) + \frac{0.2(0.2-1)(0.2-2)}{6}(0.047) + \frac{0.2(0.2-1)(0.2-2)(0.2-3)}{24}(0.002)$$

$$y(142) = 3.8987$$

To find y at $x = 175$, use Newton backward interpolation formula,

$$y(x) = y_n + v \Delta y_n + \frac{v(v+1)}{2!} \nabla^2 y_n + \cdots$$

$$v = \frac{x - x_n}{h} = \frac{175 - 180}{10} = -0.5$$

$$y(175) = 10.225 + (-0.5)(2.149) + \frac{(-0.5)(-0.5+1)}{2} (0.375)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)}{6} (0.049) + \left[\frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{24} \times (0.002) \right]$$

$$y(175) = 9.1005.$$

— x —

3) The population of a town is as follows. Estimate the population increase during the period 1946 to 1976.

year x	population in lakhs y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1941	20					
		4				
1951	24		1			
		5		1		
1961	29		2		0	
		7		1		-9
1971	36		3		-9	
		10		-8		
1981	46		-5			
		5				
1991	51					

To find y at $x = 1946$, use forward interpolation formula,

$$y(x) = y_0 + u \Delta y + \frac{u(u-1)}{2} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{6} \Delta^3 y_0 + \cdots$$

$$u = \frac{x - x_0}{h} = \frac{1946 - 1941}{10} = \frac{5}{10} = \frac{1}{2}.$$

$$y(1946) = 20 + \left(\frac{1}{2}\right)4 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}(1) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{6}(1)$$

$$+ \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{24}(0) +$$

$$\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-4\right)}{120}(-1)$$

$$y(1946) = 21.69.$$

$$v = \frac{x - x_n}{h} = \frac{1976 - 1991}{10} = \frac{-3}{2}.$$

$$y(1976) = 51 + \left(\frac{-3}{2}\right)5 + \frac{\left(\frac{-3}{2}\right)\left(\frac{-3}{2}+1\right)}{2}(-5) + \left[\frac{\left(\frac{-3}{2}\right)\left(\frac{-3}{2}+1\right)\left(\frac{-3}{2}+2\right)}{6}\right.$$

$$\left.(-8)\right] + \left[\frac{\left(\frac{-3}{2}\right)\left(\frac{-3}{2}+1\right)\left(\frac{-3}{2}+2\right)\left(\frac{-3}{2}+3\right)}{24}(-9)\right] +$$

$$\left[\frac{\left(\frac{-3}{2}\right)\left(\frac{-3}{2}+1\right)\left(\frac{-3}{2}+2\right)\left(\frac{-3}{2}+3\right)\left(\frac{-3}{2}+4\right)}{120}(-9)\right]$$

$$y(1976) = 40.80859375.$$

$\therefore$ Increase in population during the period

$$= 40.809 - 21.69 = 19.119 \text{ lakhs}.$$

4) From the following data find θ at $x = 43$ & $x = 84$ also express θ in terms of x.

x	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$	$\Delta^5\theta$
40	184					
		20				
50	204		2			
		22		0		
60	226		2		0	
		24		0		0
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

To find θ at $x = 43$, here $u = \frac{x - x_0}{h} = \frac{43-40}{10} = \frac{3}{10} = 0.3$

$$\theta(x = 43) = 184 + (0.3)(20) + \frac{0.3(0.3-1)}{2}(2)$$

$$= 184 + 60 - 0.21$$

$$\theta(43) = 189.79$$

To find θ at $x = 84$, hence $v = \frac{x - x_n}{h} = \frac{84-90}{10} = -0.6$

$$\theta(x = 84) = 304 + (-0.6)(28) + \frac{(-0.6)(-0.6-1)}{2}(2)$$

$$= 286.96$$

$$\theta = \theta_0 + u\Delta\theta_0 + \frac{u(u-1)}{2!}\Delta^2\theta_0 + \cdots$$

$$= 184 + u(20) + \frac{u(u-1)}{2}(2) \quad \text{where } u = \frac{x-40}{10}$$

$$= 184 + \frac{20(x-40)}{10} + \frac{(x-40)(x-40-10)}{100}$$

$$= 184 + 2x - 80 + \frac{1}{100}[x^2 - 90x + 2000]$$

$$= 0.01x^2 - 0.9x + 20 - 80 + 184 + 2x$$

$\theta = 0.01x^2 + 1.1x + 124.$

5. Find a polynomial of degree four which takes the values:

x :	2	4	6	8	10
y :	0	0	1	0	0

Let us form difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	0				
		0			
4	0		1		
		1		-3	
6	1		-2		6
		-1		3	
8	0		1		
		0			
10	0				

Let us find polynomial using Newton's forward interpolation formula,

$$u = \frac{x - x_0}{h} = \frac{x-2}{2}.$$

$$y(x) = y_0 + \Delta y_0 u + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 0 + u \times 0 + \frac{\left(\frac{x-2}{2}\right)\left(\frac{x-4}{2}\right)}{2}(1) + \frac{\left(\frac{x-2}{2}\right)\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)}{6}(-3)$$

$$+ \frac{\left(\frac{x-2}{2}\right)\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)\left(\frac{x-8}{2}\right)}{24}(6)$$

$$= \frac{(x-2)(x-4)}{8}\left[1 - \frac{1}{2}(x-6) + \frac{1}{8}(x-6)(x-8)\right]$$

$$= \frac{1}{64}(x-2)(x-4)\left[8-4x+24+x^2-14x+4x\right]$$

$$= \frac{1}{64}(x-2)(x-4)(x-8)(x-10)$$

$$= \frac{1}{64}\left[x^4-24x^3+196x^2-624x+640\right]$$

For unequal intervals of x, we use the following two methods.

1. Lagrange's interpolation formula
2. Newton's divided difference formula.

Lagrange's Interpolation formula:

$$y(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} \times y_0 +$$

$$\frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} \times y_1 + \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)\cdots(x_2-x_n)} \times y_2$$

$+\cdots$

$x : x_0 \; x_1 \; x_2 \cdots x_n \qquad y : y_0 \; y_1 \; y_2 \cdots y_n$

1. Using Lagrange's interpolation formula, find $y(10)$ from the following table:

$x : 5 \quad 6 \quad 9 \quad 11 \qquad y : 12 \quad 13 \quad 14 \quad 16$

From the given data, we have

$x_0 = 5; \quad x_1 = 6; \quad x_2 = 9; \quad x_3 = 11$

$y_0 = 12; \quad y_1 = 13; \quad y_2 = 14; \quad y_3 = 16.$

$$y(x) = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)} \times 13 +$$

$$\frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)} \times 16$$

when $x = 10$,

$$y(10) = \frac{4(1)(-1)}{(-1)(-4)(-6)} \times 12 + \frac{5(1)(-1)}{1(-3)(-5)} \times 13 + \frac{5(4)(-1)}{4(3)(-2)} \times 14$$

$$+ \frac{5(4)(1)}{6(5)(2)} \times 16$$

$$y(10) = \frac{-4}{-24}(12) + \frac{-5}{15} \times 13 + \frac{-20}{-24}(14) + \frac{20}{60} \times 16$$

$$= 2\left(-\frac{1}{3}\right)(13) + \frac{5}{3} \times 7 + \frac{1}{3} \times 16$$

$$= 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3}$$

$$= \frac{6 - 13 + 35 + 16}{3}$$

$$y(10) = 14.6667$$

2. Use Lagrange's formula to fit a polynomial to the following data:

$x: -1 \quad 0 \quad 2 \quad 3$; $y: -8 \quad 3 \quad 1 \quad 12$ and hence find $y(x=1)$

Sol:

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$y(x) = \frac{(x-0)(x-2)(x-3)}{(-1)(-1-2)(-1-3)} \times (-8) + \frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)} \times 3 +$$

$$\frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2+3)} \times 1 + \frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)} \times 12.$$

$$y(x) = \frac{8}{12} x(x^2-5x+6) + \frac{3}{6}(x+1)(x^2-5x+6) - \frac{1}{6} x(x^2-2x-3) + \frac{12}{12} x(x^2-x-2)$$

$$= \frac{2}{3}(x^3-5x^2+6x) + \frac{1}{2}(x^3-5x^2+6x+x^2-5x+6) - \frac{1}{6}(x^3-2x^2-3x) + (x^3-x^2-2x)$$

$$= \frac{2}{3}(x^3-5x^2+6x) + \frac{1}{2}(x^3-4x^2+x+6) - \frac{1}{6}(x^3-2x^2-3x) + (x^3-x^2-2x)$$

$$= \frac{4x^3-20x^2+24x+3x^3-12x^2+3x+18-x^3+2x^2+3x+6x^3-6x^2-12x}{6}$$

$$= \frac{1}{6}[12x^3-36x^2+18x+18]$$

$$= 2x^3-6x^2+3x+3$$

$y(x=1) = 2.$

(3) Find the parabola of the form $y = ax^2+bx+c$ passing through the points $(0,0)$, $(1,1)$ and $(2,20)$.

Sol.

From the given points we have $x_0 = 0, x_1 = 1, x_2 = 2$

$y_0 = 0, y_1 = 1, y_2 = 20$

By Lagrange's interpolation formula we have,

$$y(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \times y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \times y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \times y_2$$

$$= \frac{(x-1)(x-2)}{(0-1)(0-2)} \times 0 + \frac{(x-0)(x-2)}{(1-0)(1-2)} \times 1 + \frac{(x-0)(x-1)}{(2-0)(2-1)} \times 20$$

$$= -(x^2-2x) + 10(x^2-x) = -x^2+2x+10x^2-10x$$

$y = 9x^2-8x.$

(8) using lagranges formula of interpolation find $y(9.5)$ given:

$x : 7 \; 8 \; 9 \; 10, \quad y : 3 \; 1 \; 1 \; 9.$

sol

From the given data, $x_0 = 7, x_1 = 8, x_2 = 9, x_3 = 10.$

$y_0 = 3, y_1 = 1, y_2 = 1, y_3 = 9.$

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$= \frac{(x-8)(x-9)(x-10)}{(7-8)(7-9)(7-10)} \times 3 + \frac{(x-7)(x-9)(x-10)}{(8-7)(8-9)(8-10)} \times 1$$

$$+ \frac{(x-7)(x-8)(x-10)}{(9-7)(9-8)(9-10)} \times 1 + \frac{(x-7)(x-8)(x-9)}{(10-7)(10-8)(10-9)} \times 9.$$

$$y(9.5) = \frac{(9.5-8)(9.5-9)(9.5-10)}{-6} \times 3 + \frac{(9.5-7)(9.5-9)(9.5-10)}{2}$$

$$+ \frac{(9.5-7)(9.5-8)(9.5-10)}{-2} + \frac{(9.5-7)(9.5-8)(9.5-9)}{6} \times 9$$

$$= \frac{1.5 \times 0.5 \times (-0.5)}{-2} + \frac{2.5 \times 0.5 \times (-0.5)}{2} + \frac{2.5 \times 1.5 \times (-0.5)}{-2}$$

$$+ \frac{2.5 \times 1.5 \times 0.5}{2} \times 3$$

$= 0.1875 - 0.3125 + 0.9375 + 2.8125$

$= 3.625$

$y(9.5) = 3.625$

Newton's divided difference formula: (unequal)

The first divided difference of $f(x)$ is for the arguments x_0, x_1 is defined as $\frac{f(x_1)-f(x_0)}{x_1-x_0}$ and it's denoted by $f(x_0, x_1)$ or $\underset{x_1}{\Delta} f(x_0)$ Δ - divided diff

The second divided difference of $f(x)$ for 3 arguments is denoted by $f(x_0, x_1, x_2)$ & is defined as,

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

Formula:

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3) + \ldots$$

1. From the following table. Form divided difference table of $f(x) = x^3 + x + 2$. for the arguments 1, 3, 6, 11

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	4			
		$\frac{32-4}{3-1} = 14$		
3	32		$\frac{64-14}{6-1} = 10$	
		$\frac{224-32}{6-3} = 64$		$\frac{20-10}{11-1} = 1$
6	224		$\frac{224-64}{11-3} = 20$	
		$\frac{1344-224}{11-6} = 224$		
11	1344			

2. Find polynomial for the given data:

x : 0 1 3 4 , y : 1 4 40 85 .

x	$y=f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0 0	1 $f(x_0)$			
		$\frac{4-1}{1-0}=3$ $f(x_0,x_1)$		
x_1 1	4		$f(x_0,x_1,x_2)$ $\frac{18-3}{3-0}=5$	
		$\frac{40-4}{3-1}=18$		$f(x_0,x_1,x_2,x_3)$ $\frac{9-5}{4-0}=1$
x_2 3	40		$\frac{45-18}{4-1}=9$	
		$\frac{85-40}{4-3}=45$		
x_3 4	85			

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3)$$
$$= 1 + (x-0)(3) + (x-0)(x-1)5 + (x-0)(x-1)(x-3)(1)$$
$$= 1 + 3x + 5(x^2-x) + x(x^2-4x+3)$$
$$= 1 + 3x + 5x^2 - 5x + x^3 - 4x^2 + 3x$$
$$f(x) = x^3 + x^2 + x + 1$$

(3) From the following table find $f(x)$, find $f(6)$ using Newton divided diff formula

(X)

$x: 1\ 2\ 7\ 8$, $f(x) = 1\ 5\ 5\ 4$.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1 x_0	1 $f(x_0)$			
		$f(x_0,x_1)$ $\frac{5-1}{2-1}=4$	$f(x_0,x_1,x_2)$	$f(x_0,x_1,x_2,x_3)$
2 x_1	5		$\frac{0-4}{7-1}=-0.6667$	$\frac{+0.1667+0.6667}{8-1}$
		$\frac{5-5}{7-2}=0$		
7 x_2	5		$\frac{-1-0}{8-2}=-0.1667$	$=0.0714$
		$\frac{4-5}{8-7}=-1$		
8 x_3	4			

$$f(x) = 1 + (x-1)(4) + (x-1)(x-2)(-0.6667) +$$
$$(x-1)(x-2)(x-7)(0.0714)$$
$$= 1 + 4x - 4 + x^2 - 3x + 2\,(-0.6667) + (x-7)(x^2-3x+2)$$
$$(0.0714)$$
$$= 1 + 24 - 4 + 36 - 18 + 2(-0.6667) + (-1)(36-18+2)$$
$$(0.0714)$$

$f(6) = 6.238.$

Using Newton's divided diff formula. find the values of $f(2)$, $f(8)$ and $f(15)$ given the following table.

x:	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
4	48					
		$\frac{100-48}{5-4}=52$	$\frac{97-52}{7-4}=15$			
5	100			$\frac{21-15}{10-4}=1$		
		$\frac{294-100}{7-5}=97$				
7	294		$\frac{202-97}{10-5}=21$		0	
		$\frac{900-294}{10-7}=202$		$\frac{27-21}{11-5}=1$		0
10	900		$\frac{310-202}{11-7}=27$		0	
		$\frac{1210-900}{11-10}=310$				
11	1210			$\frac{33-27}{13-7}=1$		
		$\frac{2028-1210}{13-11}=409$	$\frac{409-310}{13-10}=33$			
13	2028					

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+$$
$$(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)+\dots$$

$$f(x)=48+(x-4)52+(x-4)(x-5)15+(x-4)(x-5)(x-7)1$$

$$f(2)=48+(2-4)(52)+(2-4)(2-5)15+(2-4)(2-5)(2-7)=4$$

$$f(8)=48+(8-4)52+(8-4)(8-5)15+(8-4)(8-5)(8-7)=448$$

$$f(15)=48+(15-4)52+(15-4)(15-5)15+(15-4)(15-5)(15-7)$$
$$=3150$$

Central difference formula: [Interpolation formula]

Gauss Forward Interpolation Formula: (for equal intervals)

$$y(x) = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1} + \binom{u+1}{4}\Delta^4 y_{-2} + \binom{u+2}{5}\Delta^5 y_{-2} + \cdots$$

where $\binom{u}{r} = \dfrac{u(u-1)(u-2)\ldots(u-r+1)}{r!}$

1. Apply Gauss forward central diff formula & estimate f(32) from the following table.

x :	25	30	35	40
$y=f(x)$:	0.2707	0.3027	0.3386	0.3794

Let $u = \dfrac{x-x_0}{h} = \dfrac{32-30}{5} = 0.4$

above central (-ve)
central value - 0
below central (+ve)

x	u	y	Δy	$\Delta^2 y$	$\Delta^3 y$
25	-1	0.2707			
			0.032		
30	0	$\boxed{0.3027}$ y_0		$\boxed{0.0039}$	
			$\boxed{0.0359}$		$\boxed{0.0010}$
35	1	0.3386		0.0049	
			0.0408		
40	2	0.3794			

For central value, because Δy_{-1} value must be (+ve)

By Gauss forward Interpolation formula,

$$y(x) = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1} + \binom{u+1}{4}\Delta^4 y_{-2} + \cdots$$

$$= (0.3027) + \binom{0.4}{1}(0.0359) + \binom{0.4}{2}(0.0039) +$$

$\binom{n}{r} = {}^nC_r$

$$\binom{0.4+1}{3}(0.0010) + \cdots$$

$$= (0.3027) + {}^{0.4}C_1(0.0359) + {}^{0.4}C_2(0.0039) +$$

$${}^{1.4}C_3(0.0010) + \ldots$$

$$= 0.3027 + 0.4(0.0359) + \frac{(0.4)(0.4-1)}{2!}(0.0029)$$

$$+ \frac{(1.4)(1.4-1)(1.4-2)}{3!}(0.0010)$$

$$= 0.3027 + 0.0144 \mp 0.0005 - 0.0001$$

$y(32) = 0.31653.$

2. Apply Gauss forward formula to obtain $f(x)$ at $x = 3.5$ from the table below.

x	2	3	4	5
$f(x)$	2.626	3.454	4.784	6.986

Let $u = \frac{x - x_0}{h}$ for x_0 take the central value

$$= \frac{3.5 - 3}{1}$$

$\boxed{u = 0.5}$ let $y = f(x)$

x	u	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2	-1	2.626			
			0.8280		
3	0	y_0 $\boxed{3.454}$		$\Delta^2 y_{-1}$ $\boxed{0.502}$	
			Δy_0 $\boxed{1.3300}$		$\Delta^3 y_{-1}$ $\boxed{0.370}$
4	1	4.784		0.872	
			2.2020		
5	2	6.986			

we know that,

Gauss forward interpolation formula,

$$y(x) = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1}$$

$$+ \binom{u+1}{4}\Delta^4 y_{-2} + \cdots$$

$$y(3.5) = 3.454 + \binom{0.5}{1}(1.330) + \binom{0.5}{2}(0.502) + \binom{0.5+1}{3}(0.370) + \cdots$$

$$= 3.454 + {}^{0.5}C_1\,(1.330) + {}^{0.5}C_2\,(0.502) + {}^{1.5}C_3\,(0.370) + \cdots$$

$$= 3.454 + 0.5(1.330) + \frac{0.5(0.5-1)}{2!}(0.502)$$

$$+ \frac{(1.5)(1.5-1)(1.5-2)}{3!}(0.370) + \cdots$$

$$= 3.454 + 0.665 - 0.063 - 0.023$$

$$y(3.5) = 4.033.$$

3. use Gauss forward interpolation formula. to get y_{16} given

x :	5	10	15	20	25
y :	26.782	19.951	14.001	8.762	4.163.

let $u = \frac{x - x_0}{h} = \frac{16-15}{5} = \frac{1}{5} = 0.2$

x	u	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
5	-2	26.782				
			-6.831			
10	-1	19.951		0.881 ~~12.781~~		
			-5.950		-0.170 ~~23.970~~	
15	0	$\boxed{14.001}$ y_0		$\boxed{0.711}$ ~~11.189~~		$\boxed{0.099}$ $\Delta^4 y_{-2}$
			$\boxed{-5.239}$ Δy_0		$\boxed{-0.071}$ $\Delta^3 y_{-1}$	
20	1	8.762		0.640 ~~9.838~~		
			-4.599			
25	2	4.163				

we know that,

Gauss forward interpolation formula.

$$y(x) = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1} + \binom{u+1}{4}\Delta^4 y_{-2}$$

$$y(16) = 14.001 + \binom{0.2}{1}(-5.239) + \binom{0.2}{2}(0.711)$$

$$+ \binom{0.2+1}{3}(-0.071) + \binom{0.2+1}{4}(0.099)$$

$$= 14.001 - 1.048 + {}^{0.2}C_2(0.711) + {}^{1.2}C_3(-0.071)$$

$$+ {}^{1.2}C_4(0.099)$$

$$= 14.001 - 1.048 - 0.057 + 0.002 + 0.001$$

$$y(16) = 12.9$$

Gauss Backward Central difference formula:

$$y(x) = y_0 + \binom{u}{1}\Delta y_{-1} + \binom{u+1}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-2}$$

$$+ \binom{u+2}{4}\Delta^4 y_{-2} + \binom{u+2}{5}\Delta^5 y_{-3} + \cdots$$

Ex ① using gauss backward interpolation formula find the population for the year 1936 given.

year (x) :	1901	1911	1921	1931	1941	1951
population (in thousand) (y) :	12	15	20	27	39	52

let $u = \frac{x - x_0}{h} = \frac{1936 - 1941}{10} = -0.5$

x	u	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1901	-4	12					
			3				
1911	-3	15		2			
			5		0		
1921	-2	20		2		3	-10
			7		3		
1931	-1	27		5		-7	
			Δy_{-1} $\boxed{12}$		$\boxed{-4}$ $\Delta^3 y_{-2}$		
1941	0	y_0 $\boxed{39}$		$\Delta^2 y_{-1}$ $\boxed{1}$			
			13				
1951	1	52					

we know that,

Gauss backward interpolation formula,

$$y(x) = y_0 + \binom{u}{1}\Delta y_{-1} + \binom{u+1}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-2} + \binom{u+2}{4}\Delta^4 y_{-2}$$

$$+ \binom{u+2}{5}\Delta^5 y_{-3}$$

$$= 39 + \binom{-0.5}{1} 12 + \binom{-0.5+1}{2}(1) + \binom{-0.5+1}{3}(-4)$$

$$= 39 + (-0.5)_{C_1}(12) + (+0.5)_{C_2}(1) + (+0.5)_{C_3}(-4)$$

$$= 39 + (-0.5)(12) + \frac{(0.5)(0.5-1)}{2!}(1) + \frac{(0.5)(0.5-1)(0.5-2)}{3!}(-4)$$

$$= 39 - 6 + (-0.125) - 0.250$$

$$y(1936) = 32.625$$

(7) Newton's divide and difference table:

Q. If $f(x) = \frac{1}{x}$ then find the divided difference of $f(a,b,c)$.

Sol:

$$f(x) = \frac{1}{x}.$$

$$f(a) = \frac{1}{a}, \quad f(b) = \frac{1}{b}, \quad f(c) = \frac{1}{c}.$$

now, $$f(a,b) = \frac{f(b) - f(a)}{b-a}$$

$$= \frac{\frac{1}{b} - \frac{1}{a}}{b-a}$$

$$= \frac{\frac{a-b}{ab}}{b-a}$$

$$= -\frac{\cancel{(b-a)}}{ab} \times \frac{1}{\cancel{(b-a)}}$$

$$f(a,b) = -\frac{1}{ab}$$

we know that,

$$f(a,b,c) = \frac{f(b,c) - f(a,b)}{c-a}$$

$$= \frac{-\frac{1}{bc} - \left(-\frac{1}{ab}\right)}{c-a}$$

$$= \frac{-\frac{1}{bc} + \frac{1}{ab}}{c-a}$$

$$= \frac{-\frac{(a+c)}{abc}}{c-a}$$

$$= -\frac{\cancel{(c-a)}}{abc} \times \frac{1}{\cancel{c-a}} = -\frac{1}{abc}.$$

(2) If $f(x) = \frac{1}{x^2}$ then find the divided differences of $f(a,b,c)$

Sol:

$$f(x) = \frac{1}{x^2}$$

let $f(a) = \frac{1}{a^2}$, $f(b) = \frac{1}{b^2}$

now $f(a,b) = \frac{f(b) - f(a)}{b-a}$

$$= \frac{\frac{1}{b^2} - \frac{1}{a^2}}{b-a}$$

$$= \frac{a^2-b^2}{a^2b^2} \times \frac{1}{b-a}$$

$$= \frac{(a+b)(a-b)}{a^2b^2} \times \frac{1}{b-a}$$

$$= \frac{(a+b)[-\cancel{(b-a)}]}{a^2b^2} \times \frac{1}{\cancel{b-a}}$$

$$f(a,b) = \frac{-(a+b)}{a^2b^2} =$$

we know that,

$$f(a,b,c) = \frac{f(b,c) - f(a,b)}{c-a}$$

$$= \frac{\frac{-(b+c)}{b^2c^2} + \frac{(a+b)}{a^2b^2}}{c-a}$$

$$= \frac{c^2(a+b) - a^2(b+c)}{a^2b^2c^2} \times \frac{1}{c-a}$$

$$= \frac{c^2a + c^2b - a^2b - a^2c}{a^2b^2c^2} \times \frac{1}{c-a}$$

$$= \frac{ac(c-a) + b(c^2-a^2)}{a^2b^2c^2} \times \frac{1}{c-a}$$

$$= \frac{ac(c-a) + b[(c+a)(c-a)]}{a^2b^2c^2} \times \frac{1}{c-a}$$

$$= \frac{ac(c-a) + b[(c+a)(c-a)]}{a^2b^2c^2} \times \frac{1}{c-a}$$

$$= \frac{\cancel{(c-a)}\,[ac + bc + ba]}{a^2b^2c^2} \times \frac{1}{\cancel{c-a}}$$

$$f(a,b,c) = \frac{ac + ab + bc}{a^2b^2c^2}$$