$$= \frac{ac(c-a) + b[(c+a)(c-a)]}{a^2b^2c^2} \times \frac{1}{c-a}$$

$$= \frac{\cancel{(c-a)}\,[ac + bc + ba]}{a^2b^2c^2} \times \frac{1}{\cancel{c-a}}$$

$$f(a,b,c) = \frac{ac + ab + bc}{a^2b^2c^2}$$

12/10/23

UNIT-IV - NUMERICAL DIFFERENTIATION AND INTEGRATION.

Numerical differentiation is the process of computing the value of the derivative $\frac{dy}{dx}$ for some particular value of x from the given data.

Derivatives from difference table:

1) Newtons forward difference formula:

i) $$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \cdots\right]$$

ii) $$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + \cdots\right]$$

iii) $$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \left(\frac{6u^2+18u+11}{12}\right)\Delta^4 y_0 + \cdots\right]$$

iv) $$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - 1\Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right]$$

2) Newtons Backward difference formula:

i) $$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left[\nabla y_n + \left(\frac{2v+1}{2}\right)\nabla^2 y_n + \left(\frac{3v^2+6v+2}{6}\right)\nabla^3 y_n + \ldots\right]$$

ii) $$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \ldots\right]$$

iii) $$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\nabla^2 y_n + (v+1)\nabla^3 y_n + \left(\frac{6v^2+18v+11}{12}\right)\nabla^4 y_n + \ldots\right]$$

iv) $$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \ldots\right].$$

① Find the first two derivatives of $x^{1/3}$ at $x=50$ and $x=56$ given the table below.

x:	50	51	52	53	54	55	56
$y = x^{1/3}$:	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

Sol:

difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
50	3.6840			
		0.0244		
51	3.7084		-0.0003	
		0.0241		0
52	3.7325		-0.0003	
		0.0238		0
53	3.7563		-0.0003	
		0.0235		0
54	3.7798		-0.0003	
		0.0232		0
55	3.8030		-0.0003	
		0.0229		
56	3.8259			

To find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 50$ we use

newtons forward difference formula.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{\;-1}{2}\right)\Delta^2 y_0 + \dots\right].$$

$u = \frac{x - x_0}{h}$

$\because$ h = difference in x value ie 1

$u = \frac{50-50}{1} = 0$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{1}\left[0.0244 - \frac{1}{2}(-0.0003) + \frac{1}{3}(0)\right]$$

$$= 0.0244 + \frac{0.0003}{2}$$

$$= 0.0246$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \dots\right]$$

$$= \frac{1}{1}\left[-0.0003 - 0\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = -0.0003$$

To find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 56$, we use newton's

backward difference formula.

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \dots\right]$$

$$= \frac{1}{1}\left[0.0229 + \frac{1}{2}(-0.0003) + \frac{1}{3}(0) + \dots\right]$$

$$= 0.0229 - \frac{0.0003}{2}$$

$$\left(\frac{dy}{dx}\right)_{x=x_n} = 0.0228$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \dots\right].$$

$$= \frac{1}{1}\left[-0.0003 - 0\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = -0.0003.$$

— x —

2) Find the value of $\sec 31^\circ$ using the following table.

θ(in degrees) :	31°	32°	33°	34°
$\tan\theta$:	0.6008	0.6249	0.6494	0.6745

$\because \frac{d}{d\theta}\tan\theta = \sec^2\theta$

Let θ be x and $\tan\theta$ be y, form the difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
31	0.6008			
		0.0241		
32	0.6249		0.0004	
		0.0245		0.0002.
33	0.6494		0.0006	
		0.0251		
34	0.6745			

we have, newtons forward difference formula.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{1}{3}\Delta^3 y_0 + \dots\right]$$

$1^\circ = \frac{\pi}{180}$

$$\sec^2 31^\circ = \frac{1}{1^\circ}\left[0.0241 - \frac{1}{2}(0.0004) + \frac{1}{3}(0.0002)\right]$$

$$= \frac{1}{\pi/180}\left[0.0241 - \frac{1}{2}(0.0004) + \frac{1}{3}(0.0002)\right]$$

$$= \frac{180}{\pi}\left[0.0240\right]$$

$\sec^2 31^\circ = 1.3736$

$\sec 31^\circ = 1.17$

(2) (x) A rod is rotating in a plane. The following table gives the angle θ (in radians) through which the rod has turned for various values of time t (seconds). Calculate the angular velocity & angular acceleration of the rod at, $t = 0.6$ seconds

$V = \frac{d\theta}{dt}$

$A = \frac{dV}{dt}$ i.e. $\frac{d}{dt}\left(\frac{d\theta}{dt}\right)$

$A = \frac{d^2\theta}{dt^2}$

t :	0	0.2	0.4	0.6	0.8	1.0
θ :	0	0.12	0.49	1.12	2.02	3.20

sol:

Difference table.

t	θ	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	0				
		0.12			
0.2	0.12		0.25		
		0.37		0.01	
0.4	0.49		0.26		0
		0.63		0.01	
0.6	1.12		0.27		0
		0.90		0.01	
0.8	2.02		0.28		
		1.18			
1.0	3.20				

$h = 0.2$

Since $t = 0.6$ is towards the end, we use newtons backward difference formula.

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \left(\frac{2v+1}{2}\right)\nabla^2 y_n + \left(\frac{3v^2+6v+2}{6}\right)\nabla^3 y_n + \dots\right]$$

$$v = \frac{x - x_n}{h} = \frac{0.6 - 1.0}{0.2} = -2.$$

Angular Velocity $= \frac{d\theta}{dt} = \frac{dy}{dx}$.

$$\therefore \frac{d\theta}{dt} = \frac{1}{0.2}\left[1.18 + \left(\frac{2(-2)+1}{2}\right)0.28 + \left(\frac{3(2)^2-6(2)+2}{6}\right)(0.01) + \dots\right]$$

$$= \frac{1}{0.2}\left[1.18 + \left(\frac{-3}{2}\right)0.28 + \left(\frac{12-12+2}{6}\right)(0.01) + \dots\right]$$

$$= \frac{1}{0.2}\left[1.18 - \frac{0.84}{2} + \frac{0.01}{3}\right]$$

$$\therefore \frac{d\theta}{dt} = 3.8167 \text{ rad/sec}$$

Angular acceleration $= \frac{d^2\theta}{dt^2} = \frac{d^2y}{dx^2}$.

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\nabla^2 y_n + (v+1)\nabla^3 y_n + \dots\right]$$

$$= \frac{1}{(0.2)^2}\left[0.28 + (-2+1)(0.01)\right]$$

$$\frac{d^2\theta}{dt^2} = 6.75 \text{ rad/sec}^2$$

4. The population of a certain town is given below. Find the rate of growth of the population in 1931, 1941, 1961 and 1971.

Year (x) :	1931	1941	1951	1961	1971
population in thousands (y) :	40.62	60.80	79.95	103.56	132.65

Soln:

difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1931	40.62				
		20.18			
1941	60.80		-1.03		
		19.15		5.49	
					-4.47
1951	79.95		4.46		
		23.61		1.02	
1961	103.56		5.48		
		29.09			
1971	132.65				

$h = 10.$ $\quad u = \dfrac{x - x_0}{h} =$

$x = 1931,$

Newton's forward distance formula,

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + \cdots\right].$$

$$= \frac{1}{10}\left[20.18 - \frac{1}{2}(-1.03) + \frac{1}{3}(5.49)\right]$$

$$= 0.0044.$$

$x = 1941$

Newton's forward distance formula,

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \cdots\right)$$

$$u = \frac{x - x_0}{h} = \frac{1941 - 1931}{10} = \frac{10}{10} = 1$$

$$= \frac{1}{10}\left[20.18 + \left(\frac{2-1}{2}\right)(-1.03) + \left(\frac{3-6+2}{6}\right)(5.49)\right]$$

$$\left(\frac{dy}{dx}\right) = 1.8750$$

$x = 1961$

Newton's backward distance formula,

$$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left(\nabla y_n + \left(\frac{2v+1}{2}\right)\nabla^2 y_n + \left(\frac{3v^2+6v+2}{6}\right)\nabla^3 y_n + \dots\right)$$

$$v = \frac{x - x_n}{h} = \frac{1961 - 1971}{10} = \frac{-10}{10} = -1$$

Derivatives using divided difference formula.
[unequal intervals]

(1). Find $y'(6)$ from the following data

x :	0	2	3	4	7	9
y :	4	26	58	112	466	922.

soln:

Since x values are of unequal intervals, we use newton's divided difference formula.

Difference table.

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	$\boxed{4}$ $f(x_0)$				
		$\frac{26-4}{2-0} = 11$ $f(x_0,x_1)$			
2	26		$\frac{32-11}{3-0} = 7$ $f(x_0,x_1,x_2)$		
		$\frac{58-26}{3-2} = 32$		$\frac{11-7}{4-0} = 1$ $f(x_0,x_1,x_2,x_3)$	
3	58		$\frac{54-32}{4-2} = 11$		0
		$\frac{112-58}{4-3} = 54$		$\frac{16-11}{7-2} = 1$	
4	112		$\frac{118-54}{7-3} = 16$		0
		$\frac{466-112}{7-4} = 118$		$\frac{22-16}{9-3} = 1$	
7	466.		$\frac{228-118}{9-4} = 22$		
		$\frac{922-466}{9-7} = 228$			
9	922				

$$y = f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2).$$
$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3) + \cdots$$

$$= 4 + (x-0)\,11 + (x-0)(x-2)\,7 + (x-0)(x-2)(x-3)\,1$$

$$= 4 + 11x + 7x^2 - 14x + x(x^2-5x+6)$$

$$y = f(x) = 4 + 11x + 7x^2 - 14x + x^3 - 5x^2 + 6x$$

$$f(x) = x^3 + 2x^2 + 3x + 4 \qquad \frac{d}{dx} x^n = nx^{n-1}$$

$$y'(x) = 3x^2 + 4x + 3$$

$$\therefore\ y'(6) = 3(6)^2 + 4(6) + 3$$

$$y'(6) = 135.$$

— x —

2) A curve passes through the points $(0, 18)$, $(1, 10)$, $(3, -18)$ and $(6, 90)$. Find the slope of the curve at $x = 2$.

Soln:

Difference table.

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
0	$\boxed{18}$ $f(x_0)$			
		$f(x_0, x_1)$ $\frac{10-18}{1-0} = -8$		
1	10		$f(x_0, x_1, x_2)$ $\frac{-14-(-8)}{3-0} = -2$	
		$\frac{-18-10}{3-1} = -14$		$f(x_0, x_1, x_2, x_3)$ $\frac{10-(-2)}{6-0} = 2$
3	-18		$\frac{36-(-14)}{6-1} = 10$	
		$\frac{90-(-18)}{6-3} = 36$		
6	90			

$$y = f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$

$$y = f(x) = 18 + (x - 0)(-8) + (x - 0)(x - 1)(-2)$$
$$+ (x - 0)(x - 1)(x - 3)(2)$$

$$= 18 - 8x - 2x^2 + 2x + 2x(x^2 - 4x + 3)$$
$$= 18 - 8x - 2x^2 + 2x + 2x^3 - 8x^2 + 6x$$

$$y = f(x) = 2x^3 - 10x^2 + 18$$

$$\frac{dy}{dx} = y'(x) = 6x^2 - 20x$$

↑
slope

$$y'(2) = 6(2)^2 - 20(2)$$

$$\boxed{\left(\frac{dy}{dx}\right)_{x=2} = -16}$$

(3). Using divided difference table find $f'(x)$ which takes the values 1, 4, 40, 85 as $x = 0, 1, 3, 4$ and hence find $f'(2)$.

soln:

difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
	$f(x_0)$	$f(x_0, x_1)$	$f(x_0, x_1, x_2)$	$f(x_0, x_1, x_2, x_3)$
0	$\boxed{1}$			
		$\frac{4-1}{1-0} = 3$	$\frac{18-3}{3-0} = 5$	$\frac{9-5}{4-0} = 1$
1	4			
		$\frac{40-4}{3-1} = 18$	$\frac{45-18}{4-1} = 9$	
3	40			
		$\frac{85-40}{4-3} = 45$		
4	85			

$$y = f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$

$$y = f(x) = 1 + (x-0)(3) + (x-0)(x-1)5 + (x-0)(x-1)(x-3)1$$

$$= 1 + 3x + 5x^2 - 5 + x(x^2 - 4x + 3)$$

$$= 1 + 3x + 5x^2 - 5x + x^3 - 4x^2 + 3x$$

$$y = f(x) = x^3 + x^2 + 1x - 4$$

$$y'(x) = 3x^2 + 2x + 1$$

$$y'(2) = 3(2)^2 + 2(2) + 1$$

$$= 12 + 4 + 1$$

$$\boxed{y'(2) = 17}$$

— x —

Numerical Integration.

I. Trapezoidal Rule:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\dots)\right].$$

Note:

↳ The order of error in trapezoidal rule is h^2.

II. Simpson's 1/3 Rule:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+y_6+\dots) + 4(y_1+y_3+\dots)\right]$$

Note:

↳ The order of error in simpson's 1/3 rule is h^4.

↳ This method can be applied when the no. of intervals is even.

III. Simpson's 3/8 Rule:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\dots) + 2(y_3+y_6+y_9+\dots)\right].$$

Note:

↳ This method can be applied when n is multiple of 3.

1. Evaluate $\int_{-3}^{3} x^4 \, dx$ by using trapezoidal rule and simpson's rule verify your result by actual integration.

let $f(x) = x^4$.

let interval, $n = b - a$ → number of intervals.

where $b \rightarrow$ upper limit, $a \rightarrow$ lower limit.

$\therefore n = +3-(-3) = 6$

$\because h = \frac{b-a}{n} = \frac{3-(-3)}{6} = \frac{6}{6} = 1$

h → difference in x value

x:	-3	-2	-1	0	1	2	3
$y=f(x)$ $=x^4$	81	16	1	0	1	16	81
	y_0	y_1	y_2	y_3	y_4	y_5	y_n

I. Trapezoid rule:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\dots)\right].$$

$$\int_{-3}^{3} x^4\,dx = \frac{1}{2}\left[(81+81) + 2(16+1+0+1+16)\right]$$

$$= \frac{1}{2}\left[162 + 2(34)\right]$$

$$= 115.$$

II Simpson's 1/3 Rule:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+\dots) + 4(y_1+y_3+\dots)\right].$$

$$\int_{-3}^{3} x^4\,dx = \frac{1}{3}\left[(81+81) + 2(1+1) + 4(16+0+16)\right]$$

$$= 98$$

iii. Simpson's 3/8 Rule:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\dots)+2(y_3+y_6+\dots)\right]$$

$$\int_{-3}^{3} x^4\,dx = \frac{3}{8}\left[(81+81)+3(16+1+1+16)+2(0)\right]$$

$$= 99.$$

$\int x^n dx = \frac{x^{n+1}}{n+1}$

By actual integration,

$$\int_{-3}^{3} x^4\,dx = \left(\frac{x^5}{5}\right)_{-3}^{3} = \frac{(3)^5}{5} - \frac{(-3)^5}{5} = 97.2$$

∴ The Simpson's rule is approximately equal to the actual integration.

2. Evaluate $I = \int_0^6 \frac{1}{1+x}\,dx$ using i) Trapezoidal Rule ii) Simpson's Rule iii) Also check by direct integration.

Let

$$f(x) = \frac{1}{1+x}.$$

Let $n = b-a = 6-0 = 6$

$$h = \frac{b-a}{n} = \frac{6}{6} = 1$$

x	0	1	2	3	4	5	6
$y=f(x) = \frac{1}{1+x}$	1	0.5	0.3333	0.25	0.2	0.1667	0.1429.

I) Trapezoidal rule:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\dots)\right]$$

$$= \frac{1}{2}\left[(1+0.1429)+2(0.5+0.3333+0.25+0.2+0.1667)\right]$$

$$= 2.02145.$$

II. Simpson's 1/3 rule:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+y_6+y_8+\dots)+4(y_1+y_3+\dots)\right]$$

$$\int_0^6 \frac{1}{1+x}\,dx = \frac{1}{3}\left[(1+0.1429)+2(\cancel{0.5}+0.3333+0.2+\cancel{0.1667}) + 4(0.5+0.25+0.1667)\right]$$

$$= 1.95877$$

III simpson's 3/8 rule:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\dots)+2(y_3+y_6+\dots)\right]$$

$$\int_0^6 \frac{1}{1+x}\,dx = \frac{3}{8}\left[(1+0.1429)+3(0.5+0.3333+0.2+0.1667) + 2(0.25)\right]$$

$$= 1.9661$$

By actual integration,

$$\int_0^6 \frac{1}{1+x}\,dx = \left[\log(1+x)\right]_0^6$$

$\log_e =$ ln

$$= \log_e 7 - \log_e 1$$

$$= 1.9459$$

∴ Simpson's rule is approximately equal to actual integration.

3. Evaluate: $\int_0^1 \frac{dx}{1+x^2}$ using (i) Trapezoidal rule with $h=0.2$. Hence obtain an approximate value of π.

Solution:

Let $f(x) = \frac{1}{1+x^2}$

no. of intervals

Given, $h = 0.2$ i.e. $n = 5$

x:	0	0.2	0.4	0.6	0.8	1
$y=f(x) = \frac{1}{1+x^2}$	1	0.9615	0.8620	0.7353	0.6097	0.5
	y_0	y_1	y_2	y_3	y_4	y_n

No. of intervals is odd so we cannot use simpson's rule

Trapezoidal rule:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\ldots)\right].$$

$$= \frac{0.2}{2}\left[(1+0.5)+2(0.9615+0.8620+0.7353+0.6097)\right]$$

$$= 0.7837.$$

By actual integration,

$$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^1$$

$$= \tan^{-1}1 - \tan^{-1}0$$

$$= 45^\circ - 0^\circ$$

$$= \frac{\pi}{4} - 0$$

$$= \pi/4$$

$$= 0.7837$$

$\therefore \frac{\pi}{4} = 0.7837$ $\quad 45^\circ \times \frac{\pi}{180} = \frac{\pi}{4}$

$\pi = 3.14$

5. By dividing the range into 10 equal parts, evaluate $\int_0^{\pi} \sin x \, dx$ by using trapezoidal rule and simpsons rule. Verify your answer with actual integration.

Soln:

Given $n=10$,

$$h = \frac{b-a}{n} = \frac{\pi - 0}{10} = \frac{\pi}{10}.$$

$$\therefore h = \frac{\pi}{10}.$$

Let $f(x) = \sin x$.

x	: 0	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$	$5\pi/10$	$6\pi/10$	$7\pi/10$	$8\pi/10$	$9\pi/10$	π
$y = f(x)$	: 0	0.3090	0.5878	0.8090	0.9511	1	0.9511	0.8090	0.5878	0.3090	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

i) Trapezoidal rule:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots)\right].$$

$$\int_0^{\pi} \sin x\,dx = \frac{\pi}{20}\Big[(0+0) + 2\big[0.3090 + 0.5878 + 0.8090 + 0.9511 + 1 + 0.9511 + 0.8090 + 0.5878 + 0.3090\big]\Big]$$

$$= 1.9835$$

Simpsons 1/3 rule:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+y_6+\dots)+4(y_1+y_3+\dots)\right]$$

$$\int_0^{\pi} \sin x\,dx = \frac{\pi}{30}\Big[(0+0)+2[0.5878+0.9511+0.9511+0.5878]$$

$$+4[0.8090+0.8090+1+0.8090$$

$$+0.3090]\Big]$$

$$= 2.0001$$

Simpson's 3/8 rule:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\Big[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\dots)+$$

$$2(y_3+y_6+y_9+\dots)\Big]$$

$$\int_0^{\pi} \sin x\,dx = \frac{3\pi}{80}\Big[(0+0)+3[0.3090+0.5878+0.9511+1+0.8090$$

$$+0.5878]+2[0.8090+0.9511+0.3090]\Big]$$

$$= \cancel{1.6343}\ 1.9877$$

By actual integration,

$$\int_0^{\pi} \sin x\,dx = (-\cos x)_0^{\pi}$$

$$= -\cos\pi + \cos 0$$

$$= 2.$$

Simpson's rule is approximately equal to simpson's 1/3 rule.

1. When does simpson's rule give exact result?

Simpson's rule will give exact result if the entire curve $y = f(x)$ is itself a parabola

2. What are the errors in trapezoidal & simpsons rule?

Error in trapezoidal rule is $|E| < \frac{(b-a)h^2}{12} m$.

Error in simpson's rule is $|E| < \frac{(b-a)h^4}{180} m$.

— × —

DOUBLE INTEGRALS

Trapezoidal Rule:

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \frac{hk}{4}\Big[\text{sum of the values of } f \text{ at four corners} + 2(\text{sum of the values of } f \text{ at the remaining nodes on the boundary}) + 4(\text{sum of the values of } f \text{ at the interior nodes})\Big].$$

Simpson's 1/3 rule:

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \frac{hk}{9}\Big[\text{sum of the values of } f \text{ at four corners} + \{2(\text{sum of the values of } f \text{ at odd positions}) + 4(\text{sum of the values of } f \text{ at even positions}) \text{ on the boundary except the corners}\} + \{4(\text{sum of the values of } f \text{ at odd positions}) + 8(\text{sum}$$

of the values of f at even positions) on the odd rows of the matrix except boundaries } + { 8 (sum of the values of f at odd positions) + 16 (sum of the values of f at even positions) on the even rows of the matrix except boundaries }].

where $h = \Delta x$ and $k = \Delta y$.

1. Evaluate $\int_0^1 \int_0^1 e^{x+y}\, dx\, dy$ using trapezoidal & Simpson's rule.

Soln

Let $f(x,y) = e^{x+y}$.

Let $h = k = 0.5$

y \ x	0	0.5	1
0	1	1.6487	2.7183
0.5	1.6487	2.7183	4.4817
1	2.7183	4.4817	7.3891

i) Trapezoidal Rule:

$$\int_0^1 \int_0^1 e^{x+y}\, dx\, dy = \left[\frac{(0.5)(0.5)}{4}\right]\Big[(1 + 2.7183 + 2.7183 + 7.3891)$$

$$+ 2(1.6487 + 4.4817 + 4.4817 + 1.6487)$$

$$+ 4(2.7183)\Big]$$

$$= 0.0625[13.8257 + 24.5216 + 10.8732]$$

$= 3.0763$

Simpson's 1/3 rule:

$$\int_0^1\int_0^1 e^{x+y}\,dx\,dy = \frac{(0.5)(0.5)}{9}\Big[(1+2.7183+2.7183+7.3891)$$

$$+\{2(0)+4(1.6487+1.6487+4.4817+4.4817)\}+\{4(0)+8(0)\}+\{8(0)+16(2.7183)\}\Big].$$

$$= 0.0278[13.8257+\{0+49.0432\}+\{0+0\}+\{0+43.4928\}]$$

$$= 0.0278[13.8257+49.0432+43.4928]$$

$$= 2.9569.$$

2) Evaluate $\int_0^{\pi/2}\int_0^{\pi/2} \sin(x+y)\,dx\,dy$ using trapezoidal rule & simpsons rule.

divide the range of x and y into two equal parts. let $h = k = \frac{\pi}{4}$ ie $\left[\frac{\text{upper limit} - \text{lower limit}}{2}\right]$

we have, $f(x,y) = \sin(x+y)$ $\left[\frac{\frac{\pi}{2}-0}{2}\right] \Rightarrow \frac{\pi}{4}$.

Table: $\sin(x+y)$

$y \backslash x$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
0	0	0.7071	1
$\frac{\pi}{4}$	0.7071	1	0.7071
$\frac{\pi}{2}$	1	0.7071	0

i) Trapezoidal Rule:

$$\int_0^{\pi/2}\int_0^{\pi/2} \sin(x+y)\, dx = \frac{\left(\frac{\pi}{4}\right)\left(\frac{\pi}{4}\right)}{4}\left[(0+1+1+0) + 2(0.7071+0.7071+0.7071+0.7071) + 4(1)\right]$$

$$= 0.1542\,[\,2 + 5.6568 + 4\,]$$

$$= 0.1542\,(11.6568)$$

$$= 1.7975.$$

ii) Simpson's 1/3 rule:

$$\int_0^{\pi/2}\int_0^{\pi/2} \sin(x+y)\, dx = \frac{\left(\frac{\pi}{4}\right)\left(\frac{\pi}{4}\right)}{9}\left[(0+1+1+0) + \{2(0) + 4(0.7071+0.7071+0.7071+0.7071)\} + \{4(0) + 8(0)\} + \{8(0) + 16(1)\}\right]$$

$$= 2.0091$$

3) Evaluate $\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\, dx\, dy$ using trapezoidal rule & simpson's rule.

Soln

Divide the range of x and y into 4 equal parts. Let $h = k = 0.1$

i.e. $\left[\frac{\text{upper limit} - \text{lower limit}}{4}\right]$

h = difference in x

$= \frac{1.4 - 1}{4} = 0.1$

k = difference in y

$= \frac{2.4 - 2}{4} = 0.1$

We have $f(x, y) = \frac{1}{xy}$.

$y \backslash x$	2	2.1	2.2	2.3	2.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3968	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

i) Trapezoidal rule:

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \frac{hk}{4}\Big[\text{sum of the values of } f \text{ at four corners} + 2(\text{sum of the values of } f \text{ at the remaining nodes on the boundary}) + 4(\text{sum of the values of } f \text{ at the interior nodes})\Big]$$

$$\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\,dx\,dy = \frac{(0.1)(0.1)}{4}\Big[(0.5+0.4167+0.3571+0.2976)$$

$$+2(0.4762+0.4545+0.4348+0.3788+0.3472+0.3205+0.3106+0.3247+0.3401+0.3846+0.4167+0.4545)$$

$$+4(0.4329+0.4132+0.3953+0.3968+0.3788+0.3623+0.3663+0.3497+0.3344)\Big]$$

$$= 0.0025\big[1.5714+2(4.6432)+4(3.4297)\big]$$

$$= 0.0025\big[1.5714+9.2864+13.7188\big]$$

$= 0.0025(24.5766)$

$= 0.0614$

ii) Simpson's 1/3 rule:

$$\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\,dx\,dy = \frac{(0.1)(0.1)}{9}\Big[(0.5+0.4167+0.3571+0.2976)$$

$$+\Big\{2(0.4545+0.3472+0.3247+0.4167)+$$

$$4(0.4762)\Big\} + \Big\{8(0.4348+0.3788+0.3205$$

$$+0.3106+0.3401+0.3846+0.4545)\Big\}+$$

$$+0.3623\Big\}+$$

$$\Big\{4(0.3788)+8(0.3968+8\Big\{8(0.4329+$$

$$0.3953+0.3663+0.3344)+16(0.4132+$$

$$0.3497)\Big\}\Big]$$

0.0613

$$\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\,dx\,dy = \frac{(0.1)(0.1)}{9}\Big[(0.5+0.4167+0.2976+0.3571)$$

$$+\Big\{2(0.4545+0.3472+0.3247+0.4167)+$$

$$4(0.4762+0.4348+0.3788+0.3205+$$

$$0.3106+0.3401+0.3846+0.4545)\Big\}$$

$$+\Big\{4(0.3788)+8(0.3968+0.3623)\Big\}+$$

$$\Big\{8(0.4132+0.3497)+16(0.4329+0.3953$$

$$+0.3663+0.3344)\Big\}\Big]$$

$$\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\,dx\,dy = 0.0613$$

By actual Integration:

$$\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\, dx\, dy = \int_1^{1.4}\int_2^{2.4} \frac{1}{x}\,\frac{1}{y}\, dx\, dy$$

$$= (\log x)_2^{2.4}\ (\log y)_1^{1.4}$$

$$= (\log 2.4 - \log 2)\ (\log 1.4 - \log 1)$$

$$= \log\left(\frac{2.4}{2}\right) \cdot \log(1.4)$$

$$= \log(1.2)\ \log(1.4).$$

see $\ln(1.2)$, $\ln(1.4)$

$$= 0.0613$$

4) Evaluate $\int_1^{1.2}\int_1^{1.4} \frac{1}{x+y}\, dx\, dy$ by using trapezoidal rule and simpson's rule.

Soln:

Let us divide the range into four equal parts ie $n=4$

$$h = \frac{b-a}{n} = \frac{1.4-1}{4} = \frac{0.4}{4} = 0.1$$

$$k = \frac{b-a}{n} = \frac{1.2-1}{4} = \frac{0.2}{4} = 0.05$$

we have $f(x,y) = \frac{1}{x+y}$

y \ x	1	1.1	1.2	1.3	1.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.05	0.4878	0.4651	0.4444	0.4255	0.4082
1.10	0.4762	0.4545	0.4348	0.4167	0.4000
1.15	0.4651	0.4444	0.4255	0.4082	0.3922
1.2	0.4545	0.4348	0.4167	0.4000	0.3846

i) Trapezoidal Rule:

$$\int_1^{1.2}\int_1^{1.4} \frac{1}{x+y}\,dx\,dy = \frac{(0.1)(0.05)}{4}\Big[(0.5+0.4167+0.3846+0.4545)$$

$$+2(0.4762+0.4545+0.4348+0.4082+0.4+0.3922+0.4000+0.4167+0.4348+0.4651+0.4762+0.4878)+4(0.4651+0.4444+0.4255+0.4545+0.4348+0.4167+0.4444+0.4255+0.4082)\Big]$$

$$= 0.0013\left[1.7558+2(5.2465)+4(3.9191)\right]$$

$$= 0.0013(27.9252)$$

$$= 0.0363$$

Simpsons 1/3 rule:

$$\int_1^{1.2}\int_1^{1.4} \frac{1}{x+y}\,dx\,dy = \frac{(0.1)(0.05)}{9}\Big[(0.5+0.4167+0.3846+0.4545)$$

$$+\{2(0.4545+$$

UNIT - V NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS.

Taylor's series method:

consider first order differential equation of the form $\frac{dy}{dx} = f(x,y)$ with initial condition, $y(x_0) = y_0$.

Then,

$$y_1 = y(x_1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \dots$$

$$y_2 = y(x_2) = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv} + \dots$$

Proceeding likewise we get,

$$y_{n+1} = y(x_{n+1})$$

$$= y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \frac{h^4}{4!} y_n^{iv} + \dots \rightarrow ①$$

① is called general taylor series formula.

Truncation error of Taylor series method is $O(h^{n+1})$.

Problem:

①. Solve $\frac{dy}{dx} = x+y$, given $y(1) = 0$ & get $y(1.1)$ & $y(1.2)$ by Taylor's series.

Given equation is $\frac{dy}{dx} = x+y \rightarrow ①$.

ie $y' = x+y$.

$y'' = 1+y'$; $y''' = y''$; $y^{iv} = y'''$

$\Rightarrow$ Given, $x_0 = 1$, $y_0 = 0$, $h = 0.1$ (difference in x) $h = x_1 - x_0$

$x_1 = 1.1$ $y_1 = ?$

$x_2 = 1.2$ $y_2 = ?$

now,

$y_0' = x_0 + y_0 = 1 + 0 = 1$

$y_0'' = 1 + y_0' = 1 + 1 = 2.$

$y_0''' = y'' = 2.$

$y_0^{iv} = y''' = 2.$

By Taylors series, we have.

$$y_1 = y(x_1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \dots$$

$$y(1.1) = 0 + 0.1(1) + \frac{(0.1)^2}{\cancel{2}}(\cancel{2}) + \frac{(0.1)^3}{\cancel{6}\,3}(\cancel{2}) + \frac{(0.1)^4}{\cancel{24}\,12}(\cancel{2})$$

$$= 0.1 + 0.01 + 0.0003 + 0.0021$$

$$y(1.1) = 0.1124.$$

$$y_2 = y(x_2) = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv} + \dots$$

now,

$y_1' = x_1 + y_1 = 1.1 + 0.1124 = 1.2124$

$y_1'' = 1 + y_1' = 1 + 1.2124 = 2.2124.$

$y_1''' = y_1'' = 2.2124, \quad y_1^{iv} = 2.2124.$

$$y_2 = y(1.2) = 1.2124 + \frac{0.1}{1!}(1.2124) + \frac{(0.1)^2}{2}(2.2124) + \frac{(0.1)^3}{6}(2.2124) + \frac{(0.1)^4}{24}(2.2124)$$

$$= 1.2124 + 0.1212 + 0.0111 + 0.0004 + 0$$

$$y(1.2) = 1.3451$$

$\therefore \; y_1 = 0.1124 \quad y_2 = 1.3451$

— x —

② Using Taylor series method, find y at $x = 0.1(0.1)0.4$, given $\frac{dy}{dx} = x^2 - y$, $y(0) = 1$ correct to 4 decimal places.

Let $f(x,y) = x^2 - y$. Let $x_0 = 0$; $y_0 = 1$

i.e) $y' = x^2 - y$, $y'' = 2x - y'$, $y''' = 2 - y''$, $y^{iv} = -y'''$

Now,

$y_0' = x_0^2 - y_0 = -1$

$y_0'' = 0 - y_0' = +1$

$y_0''' = 2 - y_0'' = 2 - 1 = 1$

$y_0^{iv} = 0 - y_0''' = -1$

By Taylor's series, we have.

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y^{iv} + \dots$$

$$y(x_1) = y_1 = 1 + 0.1(-1) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(1) + \frac{(0.1)^4}{24}(-1) + \dots$$

$\therefore y_1 = 0.9052$

i.e $\boxed{y(0.1) = 0.9052} \Rightarrow [y(x_1) = y_1]$

Now,

$y_1' = x_1^2 - y_1 = (0.1)^2 - (0.9052) = -0.8952$.

$y_0'' = 2x - y_1' = 2(0.1) + 0.8952 = 1.0952$.

$y_0''' = 2 - y'' = 2 - 1.0952 = 0.9048$.

By Taylor series,

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \dots$$

$$= 0.9052 + \frac{(0.1)}{1}(-0.8952) + \frac{(0.1)^2}{2}(1.0952) + \frac{(0.1)^3}{6}(0.9048)$$

$y_2 = y(x_2) = 0.8213$

$y(0.2) = 0.8213 \Rightarrow y(x_2) = y_2$

Similarly

$y(0.3) = 0.7492$

$y(0.4) = 0.6897$

— x —

(5) using Taylor series method find (correct to 4 decimal places) the value of $y(0.1)$ given $\frac{dy}{dx} = x^2+y^2$ and $y(0) = 1$

$y(x_1)$

$y(x_0) = y_0$

Given:

$$\frac{dy}{dx} = x^2+y^2, \quad y(0) = 1, \quad y(0.1) = ?$$

let $y' = x^2+y^2$

$$y'' = 2x + \underbrace{2yy'}_{uv} \qquad (uv)' = uv' + u'v$$

$$y''' = 2(1) + 2[yy'' + y'y'] = 2 + 2yy'' + 2y'^2$$

Given, $y(0) = 1$

let $x_0 = 0$, $y_0 = 1$, $h = 0.1$ ie $h = x_1 - x_0 = 0.1 - 0 = 0.1$

now,

$$y_0' = x_0^2 + y_0^2 = 1$$

$$y_0'' = 2x_0 + 2y_0y_0' = 2(0) + 2(1)(1) = 2$$

$$y_0''' = 2 + 2y_0y_0'' + 2y_0'^2 = 2 + 4 + 2 = 8$$

$y_0^{iv} = 2[yy''' + y'y''] + 4yy$

By Taylor's series method we have.

$$y(x_1) = y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$$

$$= 1 + \frac{0.1}{1}(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(8) + \cdots$$

$y(0.1) = y_1 = 1.1113$ — x —

(7). Using Taylor's series method, find the value of $y(0.1)$ given $\frac{dy}{dx} - 2y = 3e^x$, $y(0) = 0$

(7). Using Taylor's series find y at $x = 0.1, 0.2$ correct to 3 decimal places given $\frac{dy}{dx} - 2y = 3e^x$, $y(0) = 0$.

$y' = 3e^x + 2y$
$y'' = 3e^x + 2y'$
$y''' = 3e^x + 2y''$
$y'''' = 3e^x + 2y'''$

Given:

$$\frac{dy}{dx} = 3e^x + 2y$$

$$y' = 3e^x + 2y$$

$$y'' = 3e^x + 2y'$$

$$y''' = 3e^x + 2y''$$

$$y^{iv} = 3e^x + 2y'''$$

$y(0) = 0$ ie $x_0 = 0$, $y_0 = 0$, $h = x_1 - x_0$
$x_1 = 0.1$, $y_1 = ?$ $\quad = 0.1 - 0$
$x_2 = 0.2$, $y_2 = ?$ $\quad \boxed{h = 0.1}$

now

$$y_0' = 3e^{x_0} + 2y_0 = 2$$

$$y_0'' = 3e^{x_0} + 2y_0' = 9$$

$$y_0''' = 3e^{x_0} + 2y_0'' = 21$$

$$y_0^{iv} = 3e^{x_0} + 2y_0''' = 45.$$

By Taylor's series method,

$$y(x_1) = y_1 = y_0 + \frac{h^1}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \dots$$

$$= 0 + \frac{0.1}{1}(3) + \frac{(0.1)^2}{2}(9) + \frac{(0.1)^3}{6}(21) + \frac{(0.1)^4}{24}(45)$$

$= 0.3 + 0.0450 + 0.0035 + 0.0002.$

$$\boxed{y_1 = 0.3487}$$

now,

$y_1' = 3e^{x_1} + 2y_1 = 4.0129$

$y_1'' = 3e^{x_1} + 2y_1' = 11.3413$

$y_1''' = 3e^{x_1} + 2y_1'' = 25.9981$

$y_1^{iv} = 3e^{x_1} + 2y_1''' = 55.3117$

By Taylors series method,

$$y(x_2) = y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv} + \cdots$$

$$y(0.2) = y_2 = 0.3487 + 0.1(4.0129) + \frac{(0.1)^2}{2}(11.3413)$$

$$+ \frac{(0.1)^3}{6}(25.9981) + \frac{(0.1)^4}{24}(55.3117)$$

$$= 0.3487 + 0.4013 + 0.0567 + 0.0043 + 0.0002$$

$$\boxed{y(0.2) = y_2 = 0.8112}$$

$\therefore y(0.1) = 0.3487.$

$\Rightarrow y(0.2) = 0.8112.$

— v —

Euler's method.

To solve $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$

we use $\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$ where $n = 0, 1, 2, \ldots$

This formula is called as Euler's algorithm.

problem:

① Given $y' = -y$, $y(0) = 1$ determine the values of y at $x = \underset{\text{Start}}{0.01}\,(\underset{\text{Increment}}{\underline{0.01}})\,\underset{\text{End}}{0.04}$ by Euler's method.

h

Soln Given, $y(0) = 1$

Here $x_0 = 0$, $y_0 = 1$, $h = 0.01$, $x_1 = 0.01$, $x_2 = 0.02$, $x_3 = 0.03$, $x_4 = 0.04$.

We have to find y_1, y_2, y_3, y_4

by Euler's method, we have.

$$y_{n+1} = y_n + hf(x_n, y_n) \longrightarrow ①.$$

sub $n=0$ in ①,

$$y_1 = y_0 + hf(x_0, y_0)$$
$$= 1 + 0.01(-y_0)$$
$$= 1 + 0.01(-1)$$
$$\boxed{y_1 = 0.99}$$

sub $n=1$ in ①

$$y_2 = y_1 + hf(x_1, y_1)$$
$$= 0.99 + (0.01)(-y_1)$$
$$= 0.99 + (0.01)(-0.99)$$
$$\boxed{y_2 = 0.9801}$$

sub $n=2$ in ①,

$$y_3 = y_2 + hf(x_2, y_2)$$
$$= 0.9801 + (0.01)(-y_2)$$
$$= 0.9801 + (0.01)(-0.9801)$$
$$\boxed{y_3 = 0.9703}$$

sub $n=3$ in ①,

$$y_4 = y_3 + hf(x_3, y_3)$$
$$= 0.9703 + (0.01)(-y_3)$$
$$= 0.9703 + (0.01)(-0.9703)$$
$$\boxed{y_4 = 0.9606}$$

(2) Using Euler's method, solve numerically the eq

$y' = x + y$, $y(0) = 1$ for $x = 0(0.2)(1.0)$

$y(x_0) = y_0$

Soln

$$y' = x + y$$

$\therefore f(x, y) = x + y$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$

Let $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1.0$.

Need to find y_1, y_2, y_3, y_4, y_5.

We know that,

$$y_{n+1} = y_n + hf(x_n, y_n) \rightarrow (1).$$

Sub $n=0$ in (1)

$$y_1 = y_0 + hf(x_0, y_0)$$
$$= 1 + 0.2[x_0 + y_0]$$
$$= 1 + 0.2[0+1]$$
$$y_1 = 1.2$$

Sub $n=1$ in (1)

$$y_2 = y_1 + hf(x_1, y_1)$$
$$= 1.2 + (0.2)[x_1 + y_1]$$
$$= 1.2 + (0.2)[0.2 + 1.2]$$
$$y_2 = 1.48$$

Sub $n=2$ in (1)

$$y_3 = y_2 + hf(x_2, y_2)$$
$$= 1.48 + (0.2)[x_2 + y_2]$$
$$= 1.48 + 0.2[0.4 + 1.48]$$
$$y_3 = 1.856$$

Sub $n=3$ in (1)

$$y_4 = y_3 + hf(x_3, y_3)$$
$$= 1.856 + (0.2)[x_3 + y_3]$$
$$= 1.856 + 0.2[0.6 + 1.856]$$
$$y_4 = 2.3472.$$

Sub $n=4$ in (1).

$$y_5 = y_4 + hf(x_4, y_4)$$
$$= 2.3472 + (0.2)[x_4 + y_4]$$
$$= 2.3472 + 0.2[0.8 + 2.3472]$$
$$y_5 = 2.9766.$$

(3) using Euler's method find the solution of the initial value problem $\frac{dy}{dx} = \log(x+y)$, $y(0)=2$ at $x=0.2$ by assuming $h=0.2$

soln

let $f(x,y) = \log(x+y)$, $x_0=0$, $y_0=2$, $h=0.2$

let $x_1 = 0.2$, $y_1 = ?$

we know that,

$$y_{n+1} = y_n + hf(x_n, y_n)$$

sub $n=0$, $y_1 = y_0 + hf(x_0, y_0)$

$$= 2 + (0.2)[\log(x_0+y_0)]$$

$$= 2 + 0.2[\log(0+2)]$$

$$= 2 + 0.2[\log 2]$$

$$= 2 + 0.2(0.3010)$$

$$y_1 = 2.0602.$$

(4) Using Euler's method find $y(0.4)$ given $y' = xy$, $y(0)=1$.

Let $y' = xy$, $x_0 = 0$, $y_0 = 1$, $h = \overset{x_1}{0.4} - \overset{x_0}{0} = 0.4$

$x_1 = 0.4$, $y_1 = ?$

By Eulers formula we've,

$$y_{n+1} = y_n + hf(x_n, y_n) \rightarrow (1)$$

sub $n=0$ in (1)

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + 0.4(x_0 \times y_0)$$

$$= 1 + 0.4(0)$$

$$y_1 = 1.0$$

Modified Euler's Method.

$$y_{n+1} = y_n + h\left[f\left(x_n + h/2,\ y_n + \frac{h}{2} f(x_n, y_n)\right)\right]$$

1) compute y at $x = 0.25$ by modified euler's method. Given $y' = 2xy$, $y(0) = 1$.

soln

Let $f(x,y) = 2xy$

Here $x_0 = 0$, $y_0 = 1$, let $h = \overset{x_1 - x_0}{0.25 - 0} = 0.25$

let $x_1 = 0.25$, $y_1 = ?$

using modified Euler's method,

$$y_{n+1} = y_n + h\left[f\left(x_n + \frac{h}{2},\ y_n + \frac{h}{2} f(x_n, y_n)\right)\right]$$

sub $n = 0$,

$$y_1 = y_0 + h\left[f\left(x_0 + \frac{h}{2},\ y_0 + \frac{h}{2} f(x_0, y_0)\right)\right]$$

$$= 1 + 0.25\left[f\left(0 + \frac{0.25}{2},\ 1 + \frac{0.25}{2} f(0,1)\right)\right]$$

$$= 1 + 0.25\left[f\left(\frac{0.25}{2}, 1\right)\right]$$

$$= 1 + 0.25\left[2 \times 0.125 \times 1\right]$$

$$\therefore \boxed{y_1 = 1.0625.}$$

— x —

2) using Modified Euler Method find y at $x = 0.2$

Given $\frac{dy}{dx} = y - x^2$ and $y(0) = 1$.

soln

Let $f(x,y) = y - x^2$, $\boxed{h = 0.2 - 0 = 0.2}$

Let $x_0 = 0,\ y_0 = 1,\ x_1 = 0.2,\ y_1 = ?$

using modified Euler's method,

$$y_{n+1} = y_n + h\left[f\left(x_n + h/2,\ y_n + h/2\, f(x_n, y_n)\right)\right]$$

Sub $n = 0$,

$$y_1 = y_0 + h\left[f\left(x_0 + \frac{h}{2},\ y_0 + \frac{h}{2} f(x_0, y_0)\right)\right]$$

$$= 1 + 0.2\left[f\left(0 + \frac{0.2}{2},\ 1 + \frac{0.2}{2}(y_0 - x_0^2)\right)\right]$$

$$= 1 + 0.2\left[f\left(\frac{0.2}{2},\ 1 + 0.1\right)\right]$$

$$= 1 + 0.2\left[f(0.1, 1.1)\right]$$

$$= 1 + 0.2\left[1.1 - (0.1)^2\right]$$

$$y_1 = 1.2180.$$

– x – x –

3) using modified Euler's method find $y(0.1)$ Given

$\frac{dy}{dx} = y^2 + x^2$ and $y(0) = 1$

soln:

Let $f(x,y) = y^2 + x^2$.

Let $x_0 = 0$ $y_0 = 1$, $h = x_1 - x_0 = 0.1$

let $x_1 = 0.1$, $y_1 = ?$

By Modified Eulers method,

$$y_{n+1} = y_n + h\left[f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)\right]$$

sub $n=0$,

$$y_1 = y_0 + h\left[f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right)\right]$$

$$= 1 + 0.1\left[f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}(y_0^2 + x_0^2)\right)\right]$$

$$= 1 + 0.1\left[f\left(\frac{0.1}{2}, 1 + \frac{0.1}{2}(1^2 + 0^2)\right)\right]$$

$$= 1 + 0.1\left[f\left(\frac{0.1}{2}, 1.05\right)\right]$$

$$= 1 + 0.1\left[(1.05)^2 + (0.05)^2\right]$$

$$= 1 + 0.1\left[1.1025 + 0.0025\right]$$

$$= 1.1105$$

—x—

4) Using Modified Eulers method, solve $\frac{dy}{dx} = 1 - y$.

Given $y(0) = 0$, and find $y(0.1)$ & $y(0.2)$

soln:

let $f(x, y) = 1 - y$.

let $x_0 = 0$, $y_0 = 0$, $x_1 = 0.1$, $y_1 = ?$

$x_2 = 0.2$, $y_2 = ?$

$h = x_1 - x_0$

$= 0.1 - 0$

$\boxed{h = 0.1}$

using modified eulers method,

$$y_{n+1} = y_n + h\left[f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)\right]$$

sub $n=0$,

$$y_1 = y_0 + h\left[f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right)\right]$$

$$= 0 + 0.1\left[f\left(0+\frac{0.1}{2}, 0+\frac{0.1}{2}(1-y_0)\right)\right]$$

$$= 0 + 0.1\left[f\left(\frac{0.1}{2}, \frac{0.1}{2}(1-0)\right)\right]$$

$$= 0.1\left[f\left(\frac{0.1}{2}, \frac{0.1}{2}\right)\right]$$

$$= 0.1\left[1-\frac{0.1}{2}\right]$$

$$= 0.1[1-0.05].$$

$$\boxed{y_1 = 0.095}$$

sub $n=1$,

$$y_2 = y_1 + h\left[f\left(x_1+\frac{h}{2}, y_1+\frac{h}{2}f(x_1,y_1)\right)\right]$$

$$= 0.095 + 0.1\left[f\left(0.1+\frac{0.1}{2}, 0.095+\frac{0.1}{2}[1-y_1]\right)\right]$$

$$= 0.095 + 0.1\left[f\left(0.15, 0.095+0.05(0.9050)\right)\right]$$

$$= 0.095 + 0.1\left[f(0.15, 0.14)\right]$$

$$= 0.095 + 0.1[1-0.14].$$

$$\boxed{y_2 = 0.1810}$$

– x –

Fourth order Runge-Kutta Method - [RK method].

The fourth order RK algorithm is given by

$K_1 = hf(x, y)$

$K_2 = hf\left(x + \frac{h}{2}, y + K_1/2\right)$ $\qquad \frac{K_1}{2}$

$K_3 = hf\left(x + h/2, y + K_2/2\right)$

$K_4 = hf\left(x + h, y + K_3\right)$

$$\Delta y = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$y(x+h) = \Delta y + y(x).$$

1) Apply IVth order RK method to find $y(0.2)$. Given,

$$\frac{dy}{dx} = x+y, \quad y(0) = 1.$$

Soln

let $f(x, y) = x+y$.

let $x_0 = 0$, $y_0 = 1$, $x_1 = 0.2$, $y_1 = ?$ $\qquad h = x_1 - x_0$
$= 0.2 - 0$
$\boxed{h = 0.2}$

By RK method,

$K_1 = hf(x, y)$
$= 0.2(x_0 + y_0)$
$= 0.2(0+1)$

$\boxed{k_1 = 0.2}$

$K_2 = hf\left(x + h/2, y + \frac{K_1}{2}\right)$
$= 0.2\left[f\left(x_0 + \frac{0.2}{2}, y_0 + \frac{0.2}{2}\right)\right]$

$$= 0.2 f\left[0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right]$$

$$= 0.2 f\left[0.1, 1.1\right]$$

$$= 0.2 \left[0.1 + 1.1\right]$$

$$= 0.2 \left[1.2\right]$$

$$\boxed{k_2 = 0.24}$$

$$k_3 = hf\left(x + \frac{h}{2}, y + \frac{k_2}{2}\right)$$

$$= 0.2 f\left[x_0 + \frac{0.2}{2}, y_0 + \frac{0.24}{2}\right]$$

$$= 0.2 f\left[0 + \frac{0.2}{2}, 1 + \frac{0.24}{2}\right)$$

$$= 0.2 f\left[0.1, 1.12\right]$$

$$= 0.2 \left[0.1 + 1.12\right]$$

$$\boxed{k_3 = 0.2440}$$

$$k_4 = hf\left(x + h, y + k_3\right)$$

$$= 0.2 f\left(x_0 + 0.2, y_0 + 0.2440\right)$$

$$= 0.2 f\left(0 + 0.2, 1 + 0.2440\right)$$

$$= 0.2 f\left(0.2, 1.2440\right)$$

$$= 0.2 \left[0.2 + 1.2440\right]$$

$$= 0.2 \left[1.4440\right]$$

$$\boxed{k_4 = 0.2888}$$

$\therefore \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}[0.2 + 2(0.24) + 2(0.2440) + 0.288]$

$= \frac{1}{6}[1.4560]$

$\boxed{\Delta y = 0.2427}$

$y_1 = y(0.2) = y_0 + \Delta y$

$y_1 = 1 + 0.2427$

$\boxed{y_1 = 1.2427}$

2) Using RK method of IV th order find $y(0.8)$ correct to four decimal places. If $y' = y - x^2$ and $y(0.6) = 1.7379$

Soln

let $f(x,y) = y - x^2$.

let $x_0 = 0.6$, $y_0 = 1.7379$, $x_1 = 0.8$, $y_1 = ?$

$h = x_1 - x_0 = 0.8 - 0.6 = 0.2$

By RK method,

$k_1 = hf(x,y)$

$= 0.2(y_0 - x_0^2)$

$= 0.2(1.7379 - (0.6)^2)$

$= 0.2(1.3779)$

$k_1 = 0.2756$.

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.2 f\left(0.6 + \frac{0.2}{2}, 1.7379 + \frac{0.2756}{2}\right)$

$= 0.2 f(0.7000, 1.8757)$

$= 0.2(1.8757 - (0.7)^2) = 0.2771$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= 0.2 f\left(0.6 + \frac{0.2}{2}, 1.7379 + \frac{0.2771}{2}\right)$

$= 0.2 f\left(0.7000, 1.8765\right)$

$= 0.2\left(1.8765 - (0.7)^2\right)$

$k_3 = 0.2773$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.2 f(0.6 + 0.2, 1.7379 + 0.2773)$

$= 0.2 f(0.8000, 2.0152)$

$= 0.2\left(2.0152 - (0.8)^2\right)$

$k_4 = 0.2750$

$\Delta y = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$= \frac{1}{6}\left[0.2756 + 2(0.2771) + 2(0.2773) + 0.275\right.$

$\Delta y = 0.2766$

$y(x_0 + h) = \Delta y + y(x_0)$

$y(x_1) = y_0 + \Delta y$

$y(0.8) = 1.7379 + 0.2766$

$= 20145.$

2) using $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ Given $y(0)=1$ at $x=0.2, 0.4$,

Soln

Given $x_0 = 0,\ y_0 = 1,\ x_1 = 0.2,\ x_2 = 0.4$

$y_1 = ?\ ,\ y_2 = ?$

$$f(x,y) = \frac{y^2-x^2}{y^2+x^2}$$

$h = x_1 - x_0$

$= 0.2 - 0$

$\boxed{h = 0.2}$

By RK method,

$K_1 = hf(x,y) \Rightarrow hf(x_0, y_0)$

$$= 0.2\left[\frac{y_0^2 - x_0^2}{y_0^2 + x_0^2}\right]$$

$$= 0.2\left[\frac{1^2 - 0^2}{1^2 + 0^2}\right]$$

$K_1 = 0.2$

$$K_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_1}{2}\right)$$

$$= 0.2\, f\left(0 + \frac{0.2}{2},\ 1 + \frac{0.2}{2}\right)$$

$$= 0.2\, f\left(\overset{x}{0.1},\ \overset{y}{1.1}\right)$$

$$= 0.2\left[\frac{(1.1)^2 - (0.1)^2}{(1.1)^2 + (0.1)^2}\right]$$

$$= 0.2\left[\frac{1.2}{1.220}\right]$$

$K_2 = 0.1967$

$$K_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2}\right)$$

$$= 0.2 . f\left(0 + \frac{0.2}{2},\ 1 + \frac{0.1967}{2}\right)$$

$= 0.2\, f(0.1, 1.0984)$

$$= 0.2\left[\frac{(1.0984)^2 - (0.1)^2}{(1.0984)^2 + (0.1)^2}\right] = 0.2\left[\frac{1.1965}{1.2165}\right]$$

$$\cancel{= 0.2(0.9457)}$$

$$K_3 = 0.1967$$

$$K_4 = hf(x_0 + h, y_0 + K_3)$$

$$= 0.2 f\left[0 + 0.2,\ 1 + 0.1967\right]$$

$$= 0.2\left[0.2,\ 1.1967\right]$$

$$= 0.2\left[\frac{(1.1967)^2 - (0.2)^2}{(1.1967)^2 + (0.2)^2}\right]$$

$$= 0.2\left[0.9457\right]$$

$$K_4 = 0.1891.$$

$$\Delta y = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$= \frac{1}{6}\left[0.2 + 2(0.1967) + 2(0.1967) + 0.1891\right]$$

$$\Delta y = 0.1960$$

$$y_1 = y(0.2) = \Delta y + y_0$$

$$= 0.1960 + 1$$

$$= 1.1961$$

By R.K Method,

$$K_1 = hf(x, y) \Rightarrow hf(x_1, y_1)$$

$$= 0.2\left[\frac{(1.1961)^2 - (0.2)^2}{(1.1961)^2 + (0.2)^2}\right]$$

$$k_1 = 0.9456 \times 0.2 = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}\right)$$

$$= 0.2 f\left[0.2 + \frac{0.2}{2},\ 1.1961 + \frac{0.1891}{2}\right]$$

$$= 0.2 f\left[0.3,\ \overset{1.2907}{\cancel{1.6689}}\right].$$

$$= 0.2\left[\cancel{\frac{(1.6689)^2 - (0.3)^2}{(1.6689)^2 + (0.3)^2}}\right] \cdot \left[\frac{(1.2907)^2 - (0.3)^2}{(1.2907)^2 + (0.3)^2}\right]$$

$$= \cancel{0.1875}.\ 0.1795$$

$$k_3 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2}\right)$$

$$= 0.2 f\left[0.2 + \frac{0.2}{2},\ 1.1961 + \frac{0.1795}{2}\right].$$

$$= 0.2 f\left[0.3,\ 1.2859\right]$$

$$= 0.2\left[\frac{(1.2859)^2 - (0.3)^2}{(1.2859)^2 + (0.3)^2}\right] = 0.1794$$

$$k_4 = hf(x_1 + h,\ y_1 + k_3)$$

$$= 0.2 f(0.2 + 0.2,\ 1.1961 + 0.1794)$$

$$= 0.2 f(0.4,\ 1.3755)$$

$$= 0.2\left[\frac{(1.3755)^2 - (0.4)^2}{(1.3755)^2 + (0.4)^2}\right].$$

$$= 0.2\left[\frac{1.7320}{2.0520}\right] = 0.1688$$

$$\Delta y = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right] = 0.9181$$

$$y_2 = y(0.4) = \Delta y + y_1 = 0.9181 + 1.1961 = 2.1142$$

— x —

Multi-Step Method. [Predictor-Corrector Method]

In single step methods, we use only the information from the last step computed. But in multi-step methods like Milne's predictor corrector and adam's predictor-corrector method, first we predict a value by using one formula and then correct that value by using another formula.

NOTE:

To use this method, we should know four initial values of y namely y_{n-3}, y_{n-2}, y_{n-1} and y_n to calculate y_{n+1}.

MILNE'S PREDICTOR-CORRECTOR METHOD:

To solve $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$

we use the predictor formula,

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2f_{n-2} - f_{n-1} + 2f_n\right]$$

and the corrector formula is,

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[f_{n-1} + 4f_n + f_{n+1}\right].$$

Q ① Find $y(2)$ if $y(x)$ is the solution of $\frac{dy}{dx} = \frac{1}{2}(x+y)$ given $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$ and $y(1.5) = 4.968$. by using Milne's method.

soln

Let $f(x,y) = \frac{1}{2}(x+y)$ and $x_0 = 0,\ y_0 = 2,$

$x_1 = 0.5,\ y_1 = 2.636,\ x_2 = 1,\ y_2 = 3.595,\ x_3 = 1.5,$

$y_3 = 4.968,\quad y_4 = ?,\ x_4 = 2.$

$h \Rightarrow$ difference in x value

ie $h = x_1 - x_0$

$= 0.5 - 0$

$\boxed{h = 0.5}$

By Milne's Predictor formula,

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3}\left[2f_{n-2} - f_{n-1} + 2f_n\right]$$

sub $n=3$,

$$y_{4,P} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

now,

$f_1 = \frac{1}{2}(x_1+y_1) = 1.5680,$

$f_2 = \frac{1}{2}(x_2+y_2) = 2.2975$

$f_3 = \frac{1}{2}(x_3+y_3) = 3.2340$

$$\Rightarrow y_{4,P} = 2 + \frac{4(0.5)}{3}\left[2(1.5680) - 2.2975 + 2(3.2340)\right]$$

$= 2 + 0.6667[7.3065]$

$= 2 + 4.8712$

$\boxed{y_{4,P} = 6.8712}$

$y(2) = 6.8712$

By Milne's - corrector formula,

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[f_{n-1} + 4f_n + f_{n+1}\right].$$

sub, $n=3$.

$$y_{4,c} = y_2 + \frac{0.5}{3}\left[f_2 + 4f_3 + f_4\right].$$

now, $f_4 = \frac{1}{2}(x_4+y_4)$

$= \frac{1}{2}[2+6.8712]$

$$\boxed{f_4 = 4.4351}$$

$y_{4,c} = 3.595 + 0.1667[2.2975 + 4(3.2340) + 4.4351]$

$= 3.595 + 0.1667(19.6686)$

$= 3.595 + 3.2788$

$$\boxed{y_{4,c} = 6.8738}$$

2) Given $\frac{dy}{dx} = \frac{1}{2}(1+x^2)y^2$ and $y(0)=1$, $y(0.1)=1.06$, $y(0.2)=1.12$, $y(0.3)=1.21$ and find $y(0.4)$ by Milne's method.

Soln

let $f(x,y) = \frac{1}{2}(1+x^2)y^2$

let $x_0 = 0$, $y_0 = 1$,

$x_1 = 0.1$, $y_1 = 1.06$,

$x_2 = 0.2$, $y_2 = 1.12$,

$x_3 = 0.3$, $y_3 = 1.21$,

$x_4 = 0.4$, $y_4 = ?$

$h = x_1 - x_0$

$= 0.1 - 0$

$$\boxed{h = 0.1}$$

using milne's predictors formula,

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2f_{n-2} - f_{n-1} + 2f_n\right].$$

put $n=3$,

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right].$$

now,

$$f_1 = \frac{1}{2}(1+x_1^2)y_1^2 = 0.5674$$

$$f_2 = \frac{1}{2}(1+x_2^2)y_2^2 = 0.6523$$

$$f_3 = \frac{1}{2}(1+x_3^2)y_3^2 = 0.7979.$$

$$\Rightarrow y_{4,p} = 1 + \frac{4(0.1)}{3}\left[2(0.5674) - 0.6523 + 2(0.7979)\right]$$

$$= 1 + 0.1333\left[2.0783\right]$$

$$= 1 + 0.2770$$

$$y_{4,p} = 1.2770.$$

$$\boxed{y(0.4) = 1.2770}$$

using milne's corrector formula,

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[f_{n-1} + 4f_n + f_{n+1}\right]$$

sub, $n=3$,

$$y_{4,c} = y_2 + \frac{0.1}{3}\left[f_2 + 4f_3 + f_4\right]$$

$$f_4 = \frac{1}{2}(1+x_4^2)y_4^2 = 0.9458.$$

$$y_{4,c} = 1.12 + 0.0333\left[0.6523 + 4(0.7979) + 0.9458\right]$$

$$= 1.12 + 0.0333\left[4.7897\right]$$

$$= 1.12 + 0.1595$$

$$\boxed{y_{4,c} = 1.2795}$$

3) Using Milne's method find $y(4.4)$. Given $5xy' + y^2 - 2 = 0$. Given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$ and $y(4.3) = 1.0143$.

Soln

$$5xy' + y^2 - 2 = 0$$

$$\Rightarrow \quad 5xy' = 2 - y^2$$

$$y' = \frac{2-y^2}{5x}.$$

$$f(x,y) = \frac{dy}{dx} = \frac{2-y^2}{5x}.$$

Let, $x_0 = 4$, $y_0 = 1$

$x_1 = 4.1$, $y_1 = 1.0049$,

$x_2 = 4.2$, $y_2 = 1.0097$.

$x_3 = 4.3$, $y_3 = 1.0143$

$x_4 = 4.4$, $y_4 = ?$

$h = x_1 - x_0$

$= 4.1 - 4$

$\boxed{h = 0.1}$

using Milne's predictor method,

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2f_{n-2} - f_{n-1} + 2f_n\right]$$

Sub $n = 3$,

$$y_{4,p} = y_0 + \frac{4(0.1)}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$f_1 = \frac{2 - y_1^2}{5x_1} = 0.0483$$

$$f_2 = \frac{2 - y_2^2}{5x_2} = 0.0467$$

$$f_3 = \frac{2 - y_3^2}{5x_3} = 0.0452$$

$$\Rightarrow \quad y_{4,p} = 1 + 0.1333\left[2(0.0483) - 0.0467 + 2(0.0452)\right]$$

$$= 1 + 0.1333\left[0.1403\right]$$

$$= 1 + 0.0187$$

$$y_{4,p} = 1.0187.$$

$$\boxed{y(4.4) = 1.0187}$$

by using Milne's corrector formula,

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[f_{n-1} + 4f_n + f_{n+1}\right]$$

$n=3$

$$y_{4,c} = y_2 + \frac{0.1}{3}\left[f_2 + 4f_3 + f_4\right]$$

$$f_4 = \frac{2 - y_4^2}{5x_4} = 0.0437$$

$$y_{4,c} = 1.0097 + 0.0333\left[0.0467 + 4(0.0452) + 0.0437\right]$$

$$= 1.0097 + 0.0333\left[0.2712\right]$$

$$= 1.0097 + 0.0090$$

$$\boxed{y_{4,c} = 1.0187}$$

— x —

ADAM'S PREDICTOR CORRECTOR METHOD

Adam's predictor method:

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right].$$

Adam's Corrector Method:

$$y_{n+1,c} = y_n + \frac{h}{24}\left[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right].$$

① Using Adam's method find $y(0.4)$ given $\frac{dy}{dx} = \frac{1}{2}xy$, $y(0)=1$, $y(0.1)=1.01$, $y(0.2)=1.022$, $y(0.3)=1.023$.

soln

Let $f(x,y) = \frac{1}{2}xy$.

$x_0 = 0, \quad y_0 = 1$

$y_1 = 1.01, \quad x_1 = 0.1$

$x_2 = 0.2, \quad y_2 = 1.022$

$x_3 = 0.3, \quad y_3 = 1.023$

$y_4 = ?, \quad x_4 = 0.4$

$h = x_1 - x_0$
$= 0.1 - 0$
$\boxed{h = 0.1}$.

using Adam's Predictor method,

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right].$$

sub $n=3$,

$$y_{4,p} = y_3 + \frac{0.1}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right]$$

$f_1 = \frac{1}{2}x_1y_1 = 0.0505, \quad f_2 = \frac{1}{2}x_2y_2 = 0.1022$

$f_3 = \frac{1}{2} x_3 y_3 = 0.1535$, $f_0 = \frac{1}{2} x_0 y_0 = 0$

$$\Rightarrow y_{4,P} = 1.023 + 0.0042\left[55(0.1535) - 59(0.1022) + 37(0.0505) - 9(0)\right]$$

$$= 1.023 + 0.0042[4.2812]$$

$$= 1.023 + 0.0180$$

$$y_{4,P} = 1.0410$$

$$\boxed{y(0.4) = 1.0410}$$

using adam's corrector method,

$$y_{n+1,C} = y_n + \frac{h}{24}\left[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right]$$

sub $n = 3$,

$$y_{4,C} = y_3 + \frac{0.1}{24}\left[9f_4 + 19f_3 - 5f_2 + f_1\right]$$

$$f_4 = \frac{1}{2} x_4 y_4 = 0.2082$$

$$y_{4,C} = 1.023 + 0.0042\left[9(0.2082) + 19(0.1535) - 5(0.1022) + 0.0505\right]$$

$$= 1.023 + 0.0042[4.3298]$$

$$= 1.023 + 0.0182$$

$$\boxed{y_{4,C} = y(0.4) = 1.0412}$$

— x —

3) Solve to get $y(2)$ given $\frac{dy}{dx} = \frac{1}{2}(x+y)$, $y(0)=2$, $y(0.5) = 2.636$, $y(1) = 3.595$, $y(1.5) = 4.968$ by adam's method.

<u>soln.</u>

$$\text{Let } f(x,y) = \frac{1}{2}(x+y)$$

$x_0 = 0$, $y_0 = 2$,

$x_1 = 0.5$, $y_1 = 2.636$,

$x_2 = 1$, $y_2 = 3.595$,

$x_3 = 1.5$, $y_3 = 4.968$.

$h = x_1 - x_0$

$= 0.5 - 0$

$\boxed{h = 0.5}$

using Adam's predictor method,

$$y_{n+1,P} = y_n + \frac{h}{24}\left[55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right]$$

sub $n=3$,

$$y_{4,P} = y_3 + \frac{0.5}{24}\left[55 f_3 - 59 f_2 + 37 f_1 - 9 f_0\right]$$

$f_0(x_0, y_0) = \frac{1}{2}(x_0 + y_0) = 1$

$f_1 = \frac{1}{2}(x_1 + y_1) = 1.568$

$f_2 = \frac{1}{2}(x_2 + y_2) = 2.298$

$f_3 = \frac{1}{2}(x_3 + y_3) = 3.234$.

$$y_{4,P} = 4.968 + 0.021\left[55(3.234) - 59(2.298) + 37(1.568) - 9(1)\right]$$

$$y_{4,P} = 6.885$$

using adam's corrector method,

$$y_{n+1,c} = y_n + \frac{h}{24}\left[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right]$$

Sub $n=3$,

$$y_{4,c} = y_3 + \frac{0.5}{24}\left[9f_4 + 19f_3 - 5f_2 + f_1\right].$$

$$f_4 = \frac{1}{2}(x_4 + y_4) = 4.443.$$

$$y_{4,c} = 4.968 + 0.021\left[9(4.443) + 19(3.234) - 5(2.298) + 1.568\right].$$

$$y_{4,c} = 6.890$$

② Given $y' = 1+y^2$, $y(0)=0$, $y(0.2)=0.2027$, $y(0.4)=0.4228$, $y(0.6)=0.6841$, Find $y(0.8)$ by Adams method

Sol.

Let $\boxed{f(x,y) = 1+y^2.}$ $f_1(x_1, y_1) = 1+y_1^2$

$x_0 = 0,\quad y_0 = 0$

$x_1 = 0.2,\quad y_1 = 0.2027$

$x_2 = 0.4,\quad y_2 = 0.4228$

$x_3 = 0.6,\quad y_3 = 0.6841$

$h = x_1 - x_0$

$= 0.2 - 0$

$\boxed{h = 0.2}$

By ~~Milne's~~ Adam's Predictor method,

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right]$$

Sub $n=3$

$$y_{4,p} = y_3 + \frac{0.2}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right]$$

$f_0 = 1+y_0^2 = 1$, $f_1 = 1+y_1^2 = 1.0411$

$f_2 = 1+y_2^2 = 1.1788$ $f_3 = 1+y_3^2 = 1.4680$

$f(x_4, y_4) = 1 + y_4^2$

$$y_{4,P} = 0.6841 + 0.0083[55(1.4680) - 59(1.1788) + 37(1.0411) - 9(1)]$$

$$y_{4,P} = 1.0234.$$

using Adams corrector method,

$$y_{n+1,C} = y_n + \frac{h}{24}[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}]$$

sub $n=3$,

$$y_{4,C} = y_3 + \frac{0.2}{24}[9f_4 + 19f_3 - 5f_2 + f_1]$$

$$f_4 = 1 + y_4^2 = 1 + (1.0234)^2 = 2.0473.$$

$$y_{4,C} = 0.6841 + \frac{0.2}{24}[9(2.0473) + 19(1.4680) - 5(1.1788) + 1.0411]$$

$$y_{4,C} = 1.0296.$$

compare single step and multi step method.

single step method	multi step method
↳ Taylors series method, Eulers method, modified Euler's method, RK method are single step methods.	↳ Milne's and adams predictor corrector method are multi step method.

↳ We use only the information from the last step computed.

↳ We use one formula to predict the values and we use another formula to correct that method.

State the disadvantage of Taylors series method over RK method. / advantage of RK over Taylors series.

Why RK method is preferred to Taylor's series method.

compare RK method & Taylors

↳ RK method do not require prior calculati-on of higher derivatives of $y(x)$ as the Taylors method does.

↳ Since the differential eqns using in applications often complecated, the calculation of derivatives may be difficult.

How many prior value are required in multi step method? Four (4)