Question-1

Integer-Program

$$\begin{aligned} \min_x \quad & \sum_e c_e x_e \\ \text{s.t.} \quad & \sum_{e \in P_i} x_e \geq 1 \\ & x_e \in \{0,1\} \end{aligned}$$

Dual of Relaxed problem

$$\begin{aligned} \max_y \quad & \sum_{i=1}^{k} y_i \\ \text{s.t.} \quad & \sum_{i: e \in P_i} y_i \leq c_e \qquad \forall e \in E \\ & y_i \geq 0 \end{aligned}$$

Algorithm 1 Primal-Dual Algorithm for Multi-Cut problem

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procedure MULTICUT(T = (V, E), r ∈ V, c_e ≥ 0 ∀e, (s_i, t_i) i = 1 ... n)
    F ← ∅
    while F is not a multi-cut do                                          ▷
        i be the index of the unseparated pair (s_i, t_i) having highest depth(lct(s_i, t_i))
        Increase y_i such till edge e becomes tight
        F ← F ∪ {e}
    end while
end procedure
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In reverse delete step, go through the edges in the reverse order in which they are addded to F. Delete an edge e if $F - \{e\}$ is a feasible multi-cut. Return F finally.

Theorem 1.

$$cost(F) = \sum_{e \in F} c_e \leq 2 * OPT \tag{1}$$

Proof. Let y be the dual feasible solution given by the algorithm. Hence, $\sum_{i=1}^{k} y_i \leq OPT$. We also have

$$\begin{aligned} cost(F) &= \sum_{e \in F} c_e \\ &= \sum_{e \in F} \sum_{i: e \in P_i} y_i && \text{(Since, an edge is added only when tight)} \\ &= \sum_{i=1}^{k} y_i |F \cap P_i| \end{aligned}$$

Claim 1. $y_i > 0 \Rightarrow |F \cap P_i| \leq 2$

Proof. Let $a \rightsquigarrow b$ denote the set of edges in the path from a to b. Suppose there is an i such that $y_i > 0$ and $|F \cap P_i| > 2$. Let u be the lowest common ancestor of s_i and t_i. Let e be the edge which became tight by increasing y_i.(Note: e might have been deleted from F in deletion step). As $|F \cap P_i| > 2$, we can, without loss of generality assume that $|F \cap (s_i \rightsquigarrow u)| \geq 2$. Let e_1, e_2 be two edges in $|F \cap (s_i \rightsquigarrow u)|$ such that e_1 is closer to r that e_2. Let pair (s_l, t_l) caused the addition of e_1 and $(s_{l'}, t_{l'})$ caused the addition of e_2. We claim that our reverse deletion step would have deleted the edge e_2 and hence obtain a contradiction. Given that $y_i > 0$, we can conclude that $depth(lct(s_i, t_i)) > depth(lct(s_l, t_l))$ and $depth(lct(s_i, t_i)) > depth(lct(s_{l'}, t_{l'}))$. Otherwise, edges e_1 or e_2 would have been added to F earlier that e and y_i couldn't have been raised. We can also conclude that all the pairs for which e_2 is the earliest separator that has been added to F have their lct depth lower that $depth(lct(s_i, t_i))$.Again, otherwise, e_2 would have been added before e and hence y_i couldn't have been raised. This implies that all those pairs for which e_2 is the earliest separator that has been added to F have e_1 in the path between the nodes. It is also easy to see that e_1 has been added to F after e_2. Thus, in reverse delete step we observe e_2 after e_1 and hence we will remove e_2 from F as e_1 separates all the pairs which require e_2. Hence having e_1 and e_2 both in F is a contradiction. Which gives $y_i > 0 \Rightarrow |F \cap P_i| \leq 2$. □

From the claim above, we have $y_i|F \cap P_i| \leq 2y_i$. Hence, the $cost(F) = \sum_{i=1}^{k} y_i|F \cap P_i| \leq \sum_{i=1}^{k} 2y_i \leq 2OPT$. □

Question-4

Define $A \cdot X = \sum_{i,j} a_{ij}x_{ij}$.
Primal SDP

$$\begin{aligned} \max_{X} \quad & \sum_{i<j} w_{ij}(1 - x_{ij})/2 \\ \text{s.t.} \quad & x_{ii} = 1 \quad \forall i \\ & X \succeq 0 \end{aligned}$$

Dual

$$\begin{aligned} \min_{\gamma} \quad & \frac{1}{2}\sum_{i<j} w_{ij} + \frac{1}{4}\sum_{i} \gamma_i \\ \text{s.t.} \quad & W + diag(\gamma) \succeq 0 \end{aligned}$$

We have W is a symmetric matrix with $w_{ii} = 0$. To show weak duality, we need to show that given $X \succeq 0$, $x_{ii} = 1\ \forall i$, $W + diag(\gamma) \succeq 0$, we have

$$\frac{1}{2}\sum_{i<j} w_{ij}(1 - x_{ij}) \leq \frac{1}{2}\sum_{i<j} w_{ij} + \frac{1}{4}\sum_{i} \gamma_i$$

Lemma 2. *If X, Y are positive semidefinite matrices, then $X \cdot Y \geq 0$*

Proof. Given matrices X, Y we have $X \cdot Y = \text{tr}(X^T Y)$. As X, Y are p.s.ds, we can write $X = LL^T$ and $Y = MM^T$. Hence, $\text{tr}(X^T Y) = \text{tr}(LL^T MM^T) = \text{tr}(L^T MM^T L) = \text{tr}(L^T M(L^T M)^T) = ||L^T M||_F^2 \geq 0$. Second equality follows from the fact that $\text{tr}(AB) = \text{tr}(BA)$. □

Proof.

$$
\begin{aligned}
& \frac{1}{2}\sum_{i<j} w_{ij}(1 - x_{ij}) \le \frac{1}{2}\sum_{i<j} w_{ij} + \frac{1}{4}\sum_i \gamma_i \\
\Leftrightarrow \quad & -\frac{1}{2}\sum_{i<j} w_{ij}x_{ij} \le \frac{1}{4}\sum_i \gamma_i \\
\Leftrightarrow \quad & -\frac{1}{4}\sum_{i\neq j} w_{ij}x_{ij} \le \frac{1}{4}\sum_i \gamma_i && \text{(Since, } x_{ij} = x_{ji} \text{ \& } w_{ij} = w_{ji}\text{)} \\
\Leftrightarrow \quad & -\frac{1}{4}\sum_{i,j} w_{ij}x_{ij} \le \frac{1}{4}\sum_i \gamma_i && \text{(Since, } w_{ii} = 0\text{)} \\
\Leftrightarrow \quad & 0 \le \sum_{i,j} w_{ij}x_{ij} + \sum_i \gamma_i \\
\Leftrightarrow \quad & 0 \le \sum_{i,j} w_{ij}x_{ij} + \sum_i \gamma_i x_{ii} && \text{(Since, } x_{ii} = 1\text{)} \\
\Leftrightarrow \quad & 0 \le (W + diag(\gamma)) \cdot X
\end{aligned}
$$

Given that the last inequality is true as both matrices are p.s.ds, we can follow the bi-implications backward and obtain what is required. □