Question-1

Integer-Program

$$\begin{aligned} \min_x \quad & \sum_e c_e x_e \\ \text{s.t.} \quad & \sum_{e \in P_i} x_e \geq 1 \\ & x_e \in \{0,1\} \end{aligned}$$

Dual of Relaxed problem

$$\begin{aligned} \max_y \quad & \sum_{i=1}^{k} y_i \\ \text{s.t.} \quad & \sum_{i: e \in P_i} y_i \leq c_e \qquad \forall e \in E \\ & y_i \geq 0 \end{aligned}$$

Algorithm 1 Primal-Dual Algorithm for Multi-Cut problem

procedure MULTICUT($T = (V, E), r \in V, c_e \geq 0\ \forall e, (s_i, t_i)\ i = 1 \ldots n$)
  $F \leftarrow \varnothing$
  while F is not a multi-cut **do** ▷
    i be the index of the unseparated pair (s_i, t_i) having highest $depth(lct(s_i, t_i))$
    Increase y_i such till edge e becomes tight
    $F \leftarrow F \cup \{e\}$
  end while
end procedure

In reverse delete step, go through the edges in the reverse order in which they are addded to F. Delete an edge e if $F - \{e\}$ is a feasible multi-cut. Return F finally.

Theorem 1.

$$cost(F) = \sum_{e \in F} c_e \leq 2 * OPT \tag{1}$$

Proof. Let y be the dual feasible solution given by the algorithm. Hence, $\sum_{i=1}^{k} y_i \leq OPT$. We also have

$$\begin{aligned} cost(F) &= \sum_{e \in F} c_e \\ &= \sum_{e \in F} \sum_{i: e \in P_i} y_i && \text{(Since, an edge is added only when tight)} \\ &= \sum_{i=1}^{k} y_i |F \cap P_i| \end{aligned}$$

Claim 1. $y_i > 0 \Rightarrow |F \cap P_i| \leq 2$

Suppose there is an i such that $y_i > 0$ and $|F \cap P_i| > 2$. $y_i > 0$ implies that there is an edge $e \in F$ added to disconnect s_i, t_i. Let u be the least common ancestor of s_i and t_i. As there are more than 2 selected edges on the path (s_i, t_i), we can without loss of generality assume that the path between s_i and u has atleast 2 edges in F. Let e_1 and e_2 be the edges in F on the path $s_i \to u$ and e_1 be the edge nearest to the root r. As the edges are added in the decreasing order of depth of least common ancestors of unconnected pairs, we have e_2 added before e_1. Thus, while deleting, we see e_1 before e_2 and removing e_2 still disconnects s_i, t_i as e_1 which is already in F disconnects them. Hence, we get a contradiction.

From the claim above, we have $y_i |F \cap P_i| \leq 2y_i$. Hence, the $cost(F) = \sum_{i=1}^{k} y_i |F \cap P_i| \leq \sum_{i=1}^{k} 2y_i \leq 2OPT$. □

Question-4

Define $A \cdot X = \sum_{i,j} a_{ij} x_{ij}$.
Primal SDP

$$\begin{aligned} \max_X \quad & \sum_{i<j} w_{ij}(1-x_{ij})/2 \\ \text{s.t.} \quad & x_{ii} = 1 \quad \forall i \\ & X \succeq 0 \end{aligned}$$

Dual

$$\begin{aligned} \min_\gamma \quad & \frac{1}{2}\sum_{i<j} w_{ij} + \frac{1}{4}\sum_i \gamma_i \\ \text{s.t.} \quad & W + diag(\gamma) \succeq 0 \end{aligned}$$

We have W is a symmetric matrix with $w_{ii} = 0$. To show weak duality, we need to show that given $X \succeq 0$, $x_{ii} = 1\ \forall i$, $W + diag(\gamma) \succeq 0$, we have

$$\frac{1}{2}\sum_{i<j} w_{ij}(1-x_{ij}) \le \frac{1}{2}\sum_{i<j} w_{ij} + \frac{1}{4}\sum_i \gamma_i$$

Lemma 2. *If X, Y are positive semidefinite matrices, then $X \cdot Y \ge 0$*

Proof. Given matrices X, Y we have $X \cdot Y = \text{tr}(X^T Y)$. As X, Y are p.s.ds, we can write $X = LL^T$ and $Y = MM^T$. Hence, $\text{tr}(X^T Y) = \text{tr}(LL^T MM^T) = \text{tr}(L^T MM^T L) = \text{tr}(L^T M (L^T M)^T) = ||L^T M||_F^2 \ge 0$. Second equality follows from the fact that $\text{tr}(AB) = \text{tr}(BA)$. □

Proof.

$$\begin{aligned} & \frac{1}{2}\sum_{i<j} w_{ij}(1-x_{ij}) \le \frac{1}{2}\sum_{i<j} w_{ij} + \frac{1}{4}\sum_i \gamma_i \\ \Leftrightarrow \quad & -\frac{1}{2}\sum_{i<j} w_{ij}x_{ij} \le \frac{1}{4}\sum_i \gamma_i \\ \Leftrightarrow \quad & -\frac{1}{4}\sum_{i\neq j} w_{ij}x_{ij} \le \frac{1}{4}\sum_i \gamma_i && \text{(Since, } x_{ij} = x_{ji} \ \&\ w_{ij} = w_{ji}) \\ \Leftrightarrow \quad & -\frac{1}{4}\sum_{i,j} w_{ij}x_{ij} \le \frac{1}{4}\sum_i \gamma_i && \text{(Since, } w_{ii} = 0) \\ \Leftrightarrow \quad & 0 \le \sum_{i,j} w_{ij}x_{ij} + \sum_i \gamma_i \\ \Leftrightarrow \quad & 0 \le \sum_{i,j} w_{ij}x_{ij} + \sum_i \gamma_i x_{ii} && \text{(Since, } x_{ii} = 1) \\ \Leftrightarrow \quad & 0 \le (W + diag(\gamma)) \cdot X && \text{(Dot Product of p.s.ds is } \ge 0) \end{aligned}$$

□