Lab 2 : Theory

2) Let $A = [3, 4, 2, 5]$

To find median of A :
Order elements : $[2, 3, 4, 5]$
Find midpoint : $\frac{3+4}{2} = 3.5$

i.e. ~~midd~~ middle element or midpoint of middle elements

sample mean of $A = \frac{14}{4} = 3.5$

Suppose A is now $[3, 4, 2, 5, x]$ where $x \geq 5$. Now consider

$$\frac{3+4+2+5+x}{5} \geq \text{median of } A = 3.5$$

Previous sample mean was 3.5. So, 3.5 is the smallest value of x such that the above holds

$\therefore x = 3.5$

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3) By definition, for any x :

$(\text{Bern}(p) + \text{Bern}(p))(x)$

$= +_*(\text{Bern}(p) \otimes \text{Bern}(p))(x)$

$= (\text{Bern}(p) \otimes \text{Bern}(p))(\{(x_1, x_2) \mid x_1 + x_2 = x, x_1, x_2 \in \{0, 1\}\})$

NOTE: $+ : \mathbb{R}^2 \to \mathbb{R}$, $(x_1, x_2) \mapsto x_1 + x_2$

Support of Bern(p) is $\{0, 1\}$, so, support of $\text{Bern}(p) \otimes \text{Bern}(p)$ is $\{0,1\} \times \{0,1\}$

$\therefore$ Support of $\text{Bern}(p) + \text{Bern}(p)$

$= \{x \mid \exists\, x_1, x_2 \in \{0, 1\}$ such that $x_1 + x_2 = x\}$

$= \{0, 1, 2\}$

Now, note that for any x :

$\text{Binom}(2, p)(x) = \binom{2}{x} p^x (1-p)^{2-x}$

Clearly, its support is also $\{0, 1, 2\}$.

$(\mathrm{Bern}(p) + \mathrm{Bern}(p))(0)$

$= (\mathrm{Bern}(p) \otimes \mathrm{Bern}(p))(\{(x_1, x_2) \mid x_1 + x_2 = 0, \; x_1, x_2 \in \{0, 1\}\})$

$= (\mathrm{Bern}(p) \otimes \mathrm{Bern}(p))(\{(0, 0)\})$

$= \mathrm{Bern}(p)(0)\, \mathrm{Bern}(p)(0)$

By definition of product measure

$= (1-p)^2$

Also, we have

$\mathrm{Binom}(2, p)(0) = \binom{2}{0} p^0 (1-p)^{2-0}$

$= (1-p)^2$

$\dots \; 1$

Similarly,

$(\mathrm{Bern}(p) + \mathrm{Bern}(p))(1)$

$= (\mathrm{Bern}(p) \otimes \mathrm{Bern}(p))(\{(0,1), (1,0)\})$

$= 2 \cdot \mathrm{Bern}(p)(0)\, \mathrm{Bern}(p)(1)$

$= 2(1-p)p = 2p(1-p)$

Also, we have

$\mathrm{Binom}(2, p)(1) = \binom{2}{1} p^1 (1-p)^{2-1}$

$= 2p(1-p)$

$\dots \; 2$

Lastly,

$(\mathrm{Bern}(p) + \mathrm{Bern}(p))(2)$

$= (\mathrm{Bern}(p) \otimes \mathrm{Bern}(p))(\{(1,1)\})$

$= \mathrm{Bern}(p)(1)\, \mathrm{Bern}(p)(1)$

$= p^2$

Also, we have

$\mathrm{Binom}(2, p)(2) = \binom{2}{2} p^2 (1-p)^{2-2}$

$= p^2$

$\dots \; 3$

①, ② & ③ $\Rightarrow$

$\mathrm{Bern}(p) + \mathrm{Bern}(p) = \mathrm{Binom}(2, p)$

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4) Let $g(x) = \frac{1}{2}x \quad \forall x$

Now we have that for any x,

$$\mathbb{P}^{dice}(x) = \begin{cases} \frac{1}{6} & \text{if } x \in \{1,2,3,4,5,6\} \\ 0 & \text{else} \end{cases}$$

Hence, using the definition of pushforward, we get

$$g_* \mathbb{P}^{dice}(x) = \frac{1}{2}\mathbb{P}^{dice}(x)$$

$$= \mathbb{P}^{dice}(\{x_0 \mid \tfrac{1}{2}x_0 = x\})$$

$$= \mathbb{P}^{dice}(2x)$$

$\therefore$

$$\frac{1}{2}\mathbb{P}^{dice}(x) = \begin{cases} \frac{1}{6} & \text{if } 2x \in \{1,2,3,4,5,6\} \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \frac{1}{6} & \text{if } x \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\} \\ 0 & \text{else} \end{cases}$$

This is the required PMF