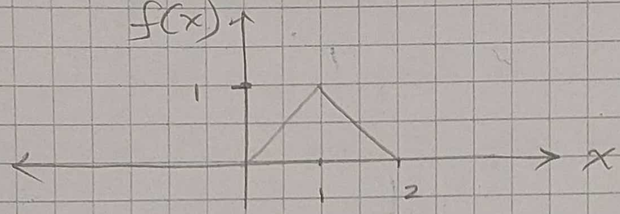
Lab 1 : Theory.

1)



Put CDF of the above as F.

If $t < 0$, clearly $F(t) = 0$

If ~~F(t~~ $t \in [0, 1]$,

$$F(t) = \int_{-\infty}^{t} x \, dx$$

$$= \left[\frac{x^2}{2} \right]_0^t = \frac{t^2}{2}$$

NOTE: By definition

$$F(t) = \int_{-\infty}^{t} f(x) \, dx$$

To elaborate,

$$F(t) = \int_{-\infty}^{t} f(x) dx = \int_{0}^{t} x \, dx$$

If $t \in [1, 2]$,

$$F(t) = \int_{-\infty}^{t} f(x) dx = \int_0^1 x \, dx + \int_1^t 2 - x \, dx$$

$$= \frac{1}{2} + \left[2x - \frac{x^2}{2} \right]_1^t$$

$$= \frac{1}{2} + \left(2t - \frac{t^2}{2} \right) - \left(2 - \frac{1}{2} \right)$$

$$= 2t - \frac{t^2}{2} - 1$$

$$\therefore F(t) = \begin{cases} 0 & \text{if } t < 0 \\ t^2/2 & \text{if } t \in [0, 1] \\ 2t - t^2/2 - 1 & \text{if } t \in [1, 2] \\ 1 & \text{else} \end{cases}$$

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2) Denote this distribution as $\mathbb{P}$.

Mean of $\mathbb{P}$ ~~$\mu(\mathbb{P}) = p$~~

$$= \mu(\mathbb{P}) = \sum_{x \in [1, 2]} x \, \mathbb{P}(x) = 1 \cdot (1 - p) + 2 \cdot p$$

$$= 1 + p$$

Variance of $\mathbb{P}$

$$= \operatorname{var}(\mathbb{P}) = \sum (x - \mu(\mathbb{P}))^2 \mathbb{P}(x)$$

$= (1-1-p)^2(1-p) + (2-1-p)^2(p)$

$\cancel{= (p^2 - p) + (p - p^2) = 0}$

$= p^2(1-p) + (1-p)^2 p$

$= p(1-p)(p + 1 - p)$

$= p(1-p)$

NOTE: The variance is the same as for a regular Bernoulli distribution; shifting the mean does not change the variance!

b) Mean of triangular distr. from (1)

$$= \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_0^1 x^2\,dx + \int_1^2 2x - x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[x^2 - \frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{3} + \left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)$$

$= 1$ (easily seen in the plot earlier)

c) Std. dev. of Uniform(a, b) is

$\sqrt{Var(Uniform(a,b)}$

$Var(Uniform(a,b)$

$$= \int_{-\infty}^{\infty} \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a}\,dx$$

$$= \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a}\,dx$$

$$= \left[\frac{\left(x - \frac{a+b}{2}\right)^3}{3}\right]_a^b \cdot \frac{1}{b-a}$$

$$= \left(\frac{(b-a)^3}{3} - \frac{(a-b)^3}{3}\right)\frac{1}{b-a}$$

$$= \frac{1}{3}\left((b-a)^2 + (b-a)^2\right)$$

$$= \frac{2(b-a)^2}{3}$$

$$\Rightarrow \text{std. dev.} = (b-a)\sqrt{\frac{2}{3}}$$

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3) ~~By~~ By definition of pushforward:

$$f_* \mathbb{P}(-\infty, t] = \mathbb{P}(\{x \mid x^2 \in (-\infty, t]\})$$

(where $\mathbb{P}$ ~~= Uniform(0, 1)~~) is the probability measure with density Uniform(0, 1)

Case: $t \in [0, 1]$

$$= \mathbb{P}((-\infty, |\sqrt{t}|])$$

$$= \int_{-\infty}^{|\sqrt{t}|} \text{Uniform}(0,1)(x)\, dx$$

$$= \int_{0}^{|\sqrt{t}|} 1 \, dx$$

$$= [x]_0^{|\sqrt{t}|} = |\sqrt{t}|$$

NOTE: PDF is not the same as probability. If something has prob. density as 1, does not mean it is certain to occur

$\therefore$ CDF of $f_* \mathbb{P}$ is $\sqrt{t}$ if $t \in [0, 1]$

consider only +ve root i.e. assume $\sqrt{t} = |\sqrt{t}|$

If $t > 1$, CDF $= 1$

if $t < 0$, CDF $= 0$

Hence

$$\text{CDF}_{f_* \mathbb{P}}(t) = \begin{cases} 0 & \text{if } t < 0 \\ \sqrt{t} & \text{if } t \in [0, 1] \\ 1 & \text{if } t > 1 \end{cases}$$

To get $\text{PDF}_{f_* \mathbb{P}}$, we must differentiate $\text{CDF}_{f_* \mathbb{P}}$: $\dfrac{d\, \text{CDF}_{f_* \mathbb{P}}}{t}$ is...

Case 1: $t < 0$

$= 0$

Case 2: $t \in [0, 1]$

$$= \frac{d\sqrt{t}}{dt} = \frac{1}{2}(t)^{-\frac{1}{2}} = \frac{1}{2\sqrt{t}}$$

Case 3: $t > 1$

$= 0$

$\therefore$ $\text{PDF}_{f_* \mathbb{P}}(x) = \begin{cases} \frac{1}{2\sqrt{t}} & \text{if } t \in [0,1] \\ 0 & \text{else} \end{cases}$

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4) PDF of Uniform $(0,1]$

$= \begin{cases} 1 & \text{if } x \in (0,1] \\ 0 & \text{else} \end{cases}$

Consider

$A_i = \left(\frac{1}{2^{i+1}}, \frac{1}{2^i}\right]$, $0 \leq i$

1st to show all A_i's are pairwise disjoint ...

Consider A_i & A_j, where ~~$i \neq j$~~ $i < j$

Let $t \in A_i$ & $t \in A_j$

$\Rightarrow t \in \left(\frac{1}{2^{i+1}}, \frac{1}{2^i}\right]$ & $t \in \left(\frac{1}{2^{j+1}}, \frac{1}{2^j}\right]$

$\Rightarrow t < \min\left(\frac{1}{2^{i+1}}, \frac{1}{2^{j+1}}\right) = \frac{1}{2^{j+1}}$

& $t \geq \max\left(\frac{1}{2^i}, \frac{1}{2^j}\right) = \frac{1}{2^i}$

$\therefore t \in \left(\frac{1}{2^{j+1}}, \frac{1}{2^i}\right]$

But $A_i \subset \left(\frac{1}{2^{j+1}}, \frac{1}{2^i}\right]$ &

$A_j \subset \left(\frac{1}{2^{j+1}}, \frac{1}{2^i}\right]$ & $A_i \neq A_j$

$\Rightarrow A_i$ & A_j

$\Rightarrow \frac{1}{2^{j+1}} < \frac{1}{2^j} \leq \frac{1}{2^{i+1}} < \frac{1}{2^i}$

$\Rightarrow$ There exist elements in A_i not in A_j and vice versa, since their ranges are not equal if $i \neq j$.

Consider A_i.

$A_i \cup A_{i+1} = \left(\frac{1}{2^{i+1}}, \frac{1}{2^i}\right] \cup \left(\frac{1}{2^{i+2}}, \frac{1}{2^{i+1}}\right]$

$= \left(\frac{1}{2^{i+2}}, \frac{1}{2^i}\right]$

$$\Rightarrow A_i \cup A_{i+1} \cup A_{i+2}$$

$$= \left(\frac{1}{2^{i+2}}, \frac{1}{2^i}\right] \cup \left(\frac{1}{2^{i+3}}, \frac{1}{2^{i+2}}\right]$$

$$= \left(\frac{1}{2^{i+3}}, \frac{1}{2^i}\right]$$

In general,

$$\bigcup_{i=0}^{n} A_i = \left(\frac{1}{2^{n+1}}, \frac{1}{2^0}\right] = \left(\frac{1}{2^{n+1}}, 1\right]$$

$$\therefore \lim_{n \to \infty} \bigcup_{i=0}^{n} A_i \cancel{=} = \lim_{n \to \infty} \left(\frac{1}{2^{n+1}}, 1\right]$$

$$= (0, 1]$$

$$\Rightarrow \bigcup_{i=0}^{\infty} A_i = (0, 1]$$