Lab 4 : Theory

1) a) All probability masses are non-negative & add up to 1

$\Rightarrow \mathbb{P}$ is a probability measure.

b) If $\mathbb{P}$ can be written as a product measure, then

$$\mathbb{P}(x,y) = \mathbb{P}(x)\,\mathbb{P}(y) \quad \forall (x,y) \in \{1,2,3\}^2$$

Hence, $\mathbb{P}(3,2) = 0$

$\Rightarrow \mathbb{P}(3)\,\mathbb{P}(2) = 0$

$\Rightarrow \mathbb{P}(3) = 0$ or $\mathbb{P}(2) = 0$ or both

If $\mathbb{P}(3) = 0$, then $\mathbb{P}(3,1) = 0$ but it is not so.

If $\mathbb{P}(2) = 0$, then $\mathbb{P}(2,1) = 0$ but it is not so

$\therefore$ Writing $\mathbb{P}$ as a product measure leads to a contradiction w.r.t. the given definition of $\mathbb{P}$

$\Rightarrow \mathbb{P}$ cannot be written as a product measure.

c) The marginals of $\mathbb{P}$ are

$\pi_{1*}\mathbb{P}$, where $\pi_1 : \{1,2,3\}^2 \to \{1,2,3\},\ (x,y) \mapsto x$

$\pi_{2*}\mathbb{P}$, where $\pi_2 : \{1,2,3\}^2 \to \{1,2,3\},\ (x,y) \mapsto y$

$\cancel{\neq}\ \pi_{1*}\mathbb{P}(1)\ \cancel{= \mathbb{P}(\{x}$

$= \mathbb{P}(\{(x,y) \mid \pi_1(x,y) = 1,\ \forall (x,y) \in \{1,2,3\}^2\})$

$\boxed{\text{Definition of pushforward leads to this}}$

$= \mathbb{P}(\{(1,1),(1,2),(1,3)\})$

$= 1/10 + 2/10 + 1/10 = 4/10$

Similarly, $\pi_{1*}\mathbb{P}(2) = 4/10$

$\pi_{1*}\mathbb{P}(3) = 2/10$

Likewise,

$\pi_{2*}\mathbb{P}(1)$

$= \mathbb{P}(\{(x,y) \mid \pi_2(x,y) = 1, \forall (x,y) \in \{1,2,3\}^2\})$

$\cancel{= \mathbb{P}(\{(2,1),(2,2),(2,3)\})}$

$= \mathbb{P}(\{(2,1),(3,1),(1,1)\})$

$= 3/10$

Similarly, $\pi_{2*}\mathbb{P}(2) = 3/10$

$\pi_{2*}\mathbb{P}(3) = 4/10$

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d) Mean of 1st marginal

$\mu_1 = 1 \cdot \frac{4}{10} + 2 \cdot \frac{4}{10} + 3 \cdot \frac{2}{10}$

$= 1.8$

Mean of 2nd marginal

$\mu_2 = 1 \cdot \frac{3}{10} + 2 \cdot \frac{3}{10} + \cancel{4}\, 3 \cdot \frac{4}{10}$

$= 2.1$

$\therefore Cov(x,y) \cancel{= \sum_{x\in\{1,2,3\}} \sum_{y \in \{1,}}$

$= \sum_{x\in\{1,2,3\}} \sum_{y\in\{1,2,3\}} (x-1.8)(y-2.1)\mathbb{P}(x,y)$

(not ~~gonna~~ going to calculate this...)

Variance of 1st marginal

$\cancel{Var} = \sigma_1^2 = (1-1.8)^2 \cdot \frac{4}{10} + (2-1.8)^2 \cdot \frac{4}{10}$

$+ (3-1.8)^2 \cdot \frac{2}{10}$

$$\sigma_2^{\,2} = (1-2.1)^2 \cdot \frac{3}{10} + (2-2.1)^2 \cdot \frac{3}{10} + (3-2.1)^2 \cdot \frac{4}{10}$$

$$\operatorname{Corr}(x,y) = \frac{\operatorname{Cov}(x,y)}{\sigma_1 \, \sigma_2}$$