ECS764P –
Applied Statistics

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Question 1

a) $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

must be true for f if it is a probability density function

b) $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{1} (ax-2)^2\,dx$$

$$= \int_0^1 (a^2x^2 + 4 - 4ax)\,dx$$

$$= \left[\frac{a^2x^3}{3} + 4x - \frac{4ax^2}{2} \right]_0^1$$

$$= \frac{a^2}{3} + 4 - 2a$$

* Put the above as = 1

$$\Rightarrow \frac{a^2}{3} + 4 - 2a = 1$$

$$\cancel{\Rightarrow a^2 + -6a + 12 = 1}$$

$$\cancel{\Rightarrow a = 3 + 1.73205\,i}$$

$$\cancel{\text{or } a = 3 - 1.73205\,i}$$

$$\Rightarrow a^2 + 12 \cancel{x} - 6a = 3$$

$$\Rightarrow a^2 - 6a + 9 = 0$$

$$\Rightarrow a - 3a - 3a + 9 = 0$$

$$\Rightarrow a(a-3) - 3(a-3) = 0$$

$$\Rightarrow \underline{\underline{a = 3}}$$

c) $\mathbb{P}([-1, \frac{1}{2}])$

$$= \int_{-1}^{1/2} (3x-2)^2 \, dx$$

$$= \int_{0}^{1/2} (3x-2)^2 \, dx \quad \left(\begin{array}{l}\text{since}\\ \text{support is}\\ [0,1]\end{array}\right)$$

$$= \left[\frac{3^2 x^3}{3} + 4x - \frac{4 \cdot 3x^2}{2}\right]_0^{1/2}$$

$$= \left[3x^3 + 4x - 6x^2\right]_0^{1/2}$$

$$= \frac{3}{8} + \frac{4}{2} - \frac{6}{4}$$

$$= \underline{\underline{\frac{7}{8}}}$$

d) Let F be the CDF.

$$F(t) = \int_{-\infty}^{t} f(x) \, dx$$

$$= \int_{-\infty}^{t} (3x-2)^2 \, dx$$

if $t \in [0, 1]$.

$$F(t) = 3t^3 + 4t - 6t^2$$

if $t < 0$ ~~or $t > 1$~~,

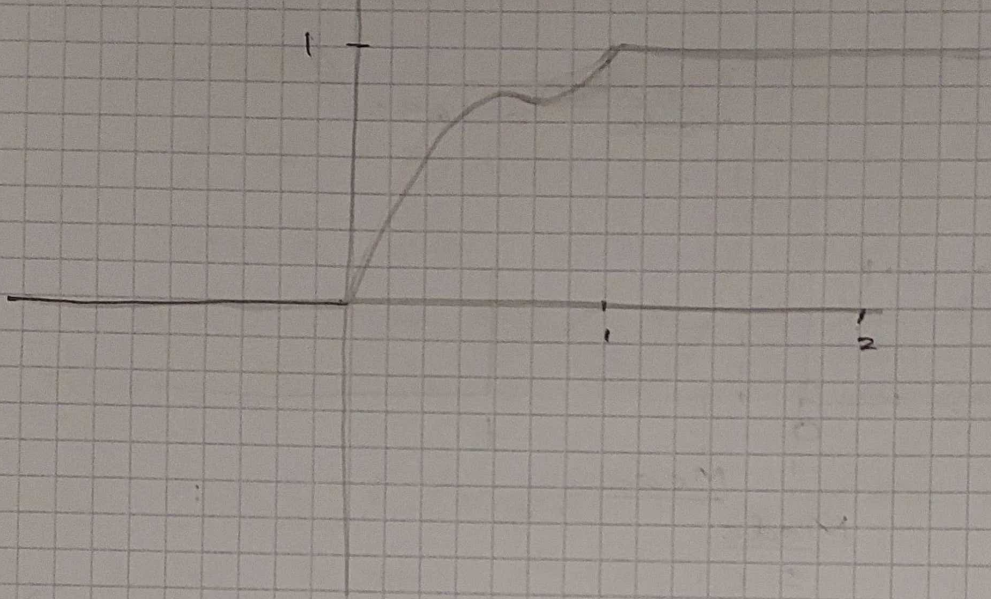
$$F(t) = 0$$

if $t > 1$,

$$F(t) = 1$$

Hence,

$$F(t) = \begin{cases} 3t^3 + 4t - 6t^2 & \text{if } t \in [0,1] \\ 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases}$$



e) Mean $= \mu(\mathbb{P})$

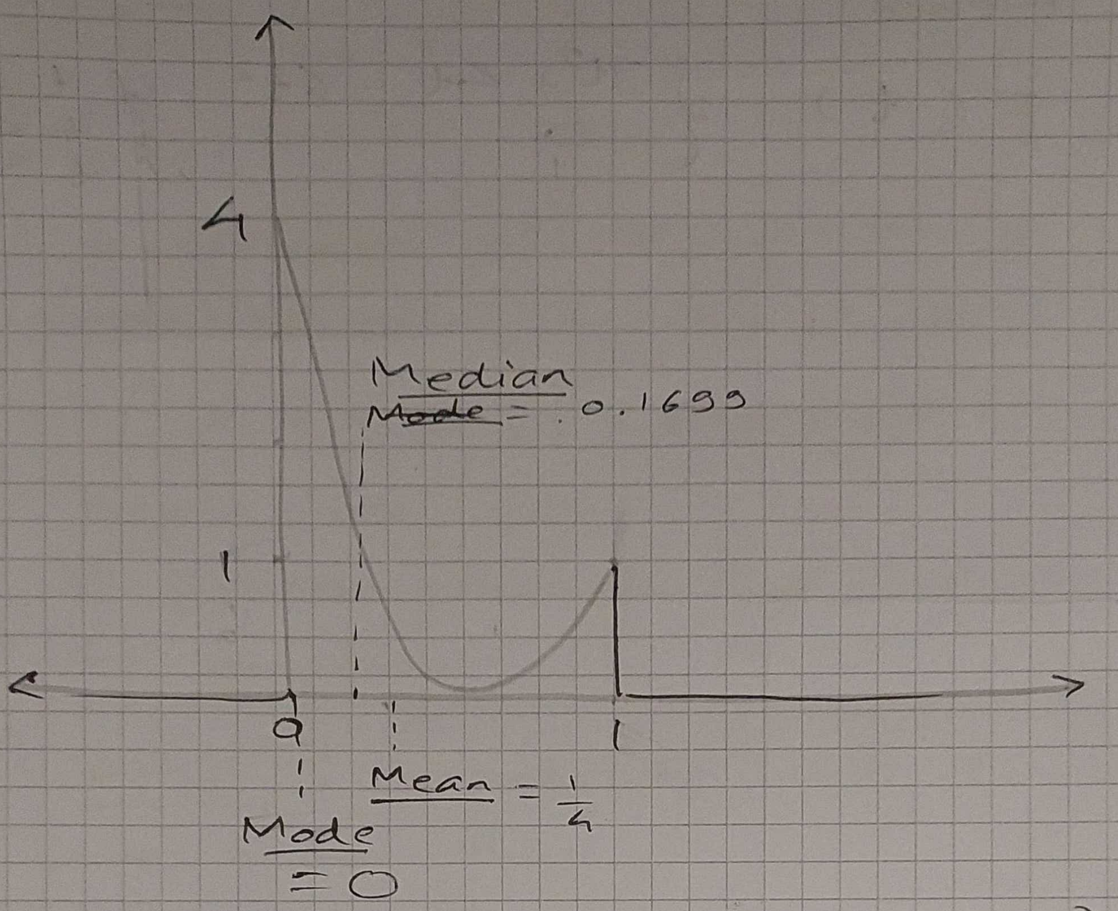
$$= \int_0^1 (3x-2)^2 \cdot x \, dx$$

$$= \int_0^1 (9x^3 + 4x - 12x^2) \, dx$$

$$= \left[\frac{9}{4}x^4 + \frac{4x^2}{2} - \frac{12x^3}{3} \right]_0^1$$

$$= \frac{9}{4} + 2 - 4$$

$$= \cancel{4} \quad \frac{1}{4}$$



Mode < Mean. (Mode = 0)

$\Rightarrow$ Positive skewness

~~Mod~~

I would not choose mode as the measure of centrality since the ~~data is h~~ points are highly skewed, thus the mode would not reflect an accurate central tendency.

Question 2

Part a

i) Support of $\mathbb{P} + \mathbb{P} + \mathbb{P}$ is exactly all possible sums

$$\{ x_1 + x_2 + x_3 \mid x_1, x_2, x_3 \in \{-1, 1\} \}$$

Hence, we get

$\{-3, \cancel{-2}, -1, 1, 3\}$ as the support

ii) PMF (Probability mass function)

We know that

$$\mathbb{P} + \mathbb{P} + \mathbb{P}$$
$$= +_* (\mathbb{P} \otimes \mathbb{P} \otimes \mathbb{P})$$

where $+ : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ s.t.

$$(x_1, x_2, x_3) \to x_1 + x_2 + x_3$$

Support was obtained as

$$\{-3, -1, 1, 3\}$$

$$\mathbb{P} + \mathbb{P} + \mathbb{P} (-3)$$
$$= +_* (\mathbb{P} \otimes \mathbb{P} \otimes \mathbb{P}) (-3)$$
$$= (\mathbb{P} \otimes \mathbb{P} \otimes \mathbb{P}) (\{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = -3, \; x_1, x_2, x_3 \in \{-1, 1\}\})$$

By definition of pushforward

$$= (\mathbb{P} \otimes \mathbb{P} \otimes \mathbb{P}) ((-1, -1, -1))$$
$$= \mathbb{P}(-1) \; \mathbb{P}(-1) \; \mathbb{P}(-1)$$

(by definition of product measure)

$$= \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{64}$$

Similarly we obtain

$$(\mathbb{P} + \mathbb{P} + \mathbb{P})(-1)$$

$$= (\mathbb{P} \otimes \mathbb{P} \otimes \mathbb{P})(\{(1,-1,-1), (-1,1,-1), (-1,-1,1)\})$$

$$= \mathbb{P}(1)\,\mathbb{P}(-1)\,\mathbb{P}(-1) + \mathbb{P}(-1)\,\mathbb{P}(1)\,\mathbb{P}(-1) + \mathbb{P}(-1)\,\mathbb{P}(-1)\,\mathbb{P}(1)$$

$$= 3 \cdot \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{4}$$

$$= \frac{9}{64}$$

$$(\mathbb{P} + \mathbb{P} + \mathbb{P})(1)$$

$$= (\mathbb{P} \otimes \mathbb{P} \otimes \mathbb{P})(\{(-1,1,1), (1,-1,1), (1,1,-1)\})$$

$$= 3 \cdot \mathbb{P}(-1)\,\mathbb{P}(1)\,\mathbb{P}(1)$$

$$= 3 \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{64}$$

Hence, $(\mathbb{P} + \mathbb{P} + \mathbb{P})(3)$

$$= 1 - \frac{27}{64} - \frac{9}{64} - \frac{1}{64}$$

(since probability measure of the support must be 1)

$$= \frac{27}{64}$$

$$\left(\frac{\mathbb{P}+\mathbb{P}+\mathbb{P}}{3}\right)\left(-\frac{1}{3}\right) = (\mathbb{P}+\mathbb{P}+\mathbb{P})(-1)$$
$$= \frac{9}{64}$$

$$\left(\frac{\mathbb{P}+\mathbb{P}+\mathbb{P}}{3}\right)\left(\frac{1}{3}\right) = (\mathbb{P}+\mathbb{P}+\mathbb{P})(1)$$
$$= \frac{27}{64}$$

$$\left(\frac{\mathbb{P}+\mathbb{P}+\mathbb{P}}{3}\right)(1) = (\mathbb{P}+\mathbb{P}+\mathbb{P})(3)$$
$$= \frac{27}{64}$$

iii) Let $\overline{\mathbb{P}}_{25}$ model the distribution of the average step size of a walk with 25 steps (i.e. sample size $= n = 25$).

CLT states:

$$\frac{\sqrt{n}\,(\overline{\mathbb{P}}_{n} - \mu(\mathbb{P}))}{\sqrt{Var(\mathbb{P})}} \approx Norm(0,1)$$

as

($\overline{\mathbb{P}}_n$ = distribution of the average step size of n random walk steps)

$$\Rightarrow \overline{\mathbb{P}}_n \approx Norm\left(\mu(\mathbb{P}), \frac{Var(\mathbb{P})}{n}\right)$$

Now, we have that

$$\mu(\mathbb{P}) = (-1)\frac{1}{4} + 1 \cdot \frac{3}{4} = \frac{1}{2}$$

$$Var(\mathbb{P}) = \left(-1-\frac{1}{2}\right)^2\frac{1}{4} + \left(1-\frac{1}{2}\right)^2\frac{3}{4}$$
$$= \frac{9}{16} + \frac{3}{16} = \frac{3}{4}$$

$n = 25$

iii) This random walk is ~~distributed~~ modelled by $\mathbb{P} + \mathbb{P} + \mathbb{P}$.

Mean of $\mathbb{P} + \mathbb{P} + \mathbb{P}$

$$= \sum_{x \in \{-3,-1,1,3\}} x \cdot (\mathbb{P} + \mathbb{P} + \mathbb{P})(x)$$

$$= (-3) \cdot \frac{1}{64}$$
$$+ (-1) \cdot \frac{9}{64}$$
$$+ (1) \cdot \frac{27}{64}$$
$$+ (3) \cdot \frac{27}{64}$$
$$= 1.96875$$

Hence, mean position after 3 steps $= 1.96875$

Part b

i) Support of $\frac{\mathbb{P} + \mathbb{P} + \mathbb{P}}{3}$

$$= \frac{1}{3} \cdot \text{support of } \mathbb{P} + \mathbb{P} + \mathbb{P}$$

$$= \left\{-1, -\frac{1}{3}, \frac{1}{3}, 1\right\}$$

ii) ~~PMF of~~ Note that

$$\frac{\mathbb{P} + \mathbb{P} + \mathbb{P}}{3} = g_* (\mathbb{P} + \mathbb{P} + \mathbb{P})$$

where $g : \mathbb{R} \to \mathbb{R}, \ x \mapsto \frac{x}{3}$

Hence, the PMF of $\frac{\mathbb{P} + \mathbb{P} + \mathbb{P}}{3}$ is

~~$\mathbb{R}(X=1)$~~

$$\left(\frac{\mathbb{P} + \mathbb{P} + \mathbb{P}}{3}\right)(-1) = (\mathbb{P} + \mathbb{P} + \mathbb{P})(-3) = \frac{1}{64}$$

$\therefore \overline{\mathbb{P}}_{25} \approx \text{Norm}\left(\frac{1}{2}, \frac{3/4}{25}\right)$

Probability of average step size being negative is then

$\text{Norm}\left(\frac{1}{2}, \frac{3/4}{25}\right)((-\infty, 0))$

$= 0.0019462$

$= 0.19462$ % probability.

Part c

Weak law of large numbers:

$$\lim_{N \to \infty} \overline{\mathbb{P}}_N([\mu(\mathbb{P}) - \varepsilon, \mu(\mathbb{P}) + \varepsilon]) = 1 \quad (\text{for any } \varepsilon > 0)$$

Hence, from this, (i) & (v) follow.

Part d.

~~Cauchy dist~~

Cauchy distribution.

Reason: the probability measure under this distribution has no defined mean.

Question 3

a) MLE (maximurnum likelihood estimator) is an estimator for a parameter ($\hat{v}$ in this case) which maximises the probability of observing a given set of observations under the assumption that they are sampled from the given distribution.

$$MLE(\hat{v}) = \underset{\hat{v}}{\operatorname{argmax}} \left(\prod_{i=1}^{n} f_{\hat{v}}(x_i) \right)$$

The joint probability of observing each observation x_i $(i = 1, 2 \ldots n)$ must be maximised by optimising $\hat{v}$.

Difficulty: To find the maximum of the above function (i.e. joint probability), we must partially differentiate it with respect to $\hat{v}$, then find $\hat{v}$ for which this partial derivative is zero.

However, differentiation of a product is complex.

To solve this, we transform

the product into sum ~~using~~ using logarithm (which preserves the maxima). Hence,

$$MLE(\hat{v}) = \underset{\hat{v}}{\operatorname{argmax}} \sum_{i=1}^{n} \log f_{\hat{v}}(x_i)$$

(log ⇒ natural log)

(NOTE: $\log \prod f_{\hat{v}}(x_i) = \sum \log f_{\hat{v}}(x_i)$)

$\hat{v}$ is the computed by solving

$$\frac{\delta \sum_{i=1}^{n} \log f_{\hat{v}}(x_i)}{\delta v} = 0$$

s.t. $$\frac{\delta^2 \sum_{i=1}^{n} \log f_{\hat{v}}(x_i)}{\delta v^2} < 0$$

↳ to ensure it is maximum not minimum.

b) $MLE(\hat{v})$ is calculated by solving:

$$\frac{\delta \sum_{i=1}^{n} \log f_{\hat{v}} f(x_i)}{\delta v} = 0$$

$$\Rightarrow \frac{\delta \sum_{i=1}^{n} \log(2\hat{v}x_i e^{-\hat{v}x_i^2})}{\delta \hat{v}} = 0 \quad \ldots (1)$$

NOTE: log here is natural log (base e)

Now, note:

$$\log(2\hat{v}x_i e^{\hat{v}x_i^2})$$

$$= \log 2 + \log \hat{v} + \log x_i + \log e^{-\hat{v}x_i^2}$$

$$= \log 2 + \log \hat{v} + \log x_i - \hat{v}x_i^2$$

∴ (1) becomes

$\cancel{\sum_{i=1}^{n} -x_i^2 \neq 0}$

$\cancel{\Rightarrow}$

$$\Rightarrow \sum_{i=1}^{n} \frac{\delta\left(\log 2 + \log\hat{v} + \log x_i - \hat{v}x_i^2\right)}{\delta \hat{v}} = 0$$

$$\Rightarrow \sum_{i=1}^{n}\left(\frac{1}{\hat{v}} - x_i^2\right) = 0$$

$$\Rightarrow \frac{n}{\hat{v}} - \sum_{i=1}^{n} x_i^2 = 0$$

$$\Rightarrow \hat{v} = \frac{n}{\sum_{i=1}^{n} x_i^2}$$

c) ~~The law states:~~

$$\cancel{\mu(g * \mathbb{P}) = \int g(x) f(x)\,dx}$$

~~Here,~~ $\cancel{g(x) = x^2}$

$$\therefore \cancel{\mu(\mu_2') = \int x^2 f(x)\,dx}$$

$$\cancel{= \int \cancel{2x^2}\, x^2\, 2vx e^{-vx^2}\,dx}$$

$$\cancel{= \int 2v x^3 e^{-vx^2}\,dx}$$

$$\cancel{= 2v \int x^3 e^{-vx^2}\,dx}$$

~~Now,~~

$$\cancel{\frac{1}{\hat{v}} = \frac{n}{\sum x_i}} \quad \cancel{\frac{\sum_{i=1}^{n} x_i^2}{n}}$$

$$\cancel{\sim\ g * (\mathbb{P} \otimes \mathbb{R} \ldots \mathbb{P})}$$

~~where~~ $\cancel{g: \mathbb{R}^n \to \mathbb{R},\ (x_1, x_2 \ldots x_n) \mapsto \frac{\sum x_i^2}{n}}$

(done after exam)

Background for question 3c

<u>Change of variable formula</u>

Let X_1 & X_2 be 2 sets, & let f be a bijective function $f: X_1 \to X_2$.

Let g be some real valued function.

~~Also, define: (given $\mathbb{P}$ is the probability measure)~~

~~• p_1 as the density of distr. $\mathbb{P}(X_1)$~~

~~• p_2 as the density of distr. $\mathbb{P}(X_2)$~~

~~Then, the foll~~

Let $\mathbb{P}$ be ~~a~~ the probability measure on ~~X_2~~. X_1. Then, the probability measure on X_2 is defined by the pushforward measure $f_* \mathbb{P}$. Define:

- p_1 as the density of $\mathbb{P}$
- p_2 as the density of $f_* \mathbb{P}$

Then, the following holds:

$$\int g(x)\, p_2(x)\,dx = \int g(f(x))\, p_1(x)\,dx$$

$\boxed{\text{Change of variable formula}}$

Now, put $g(x) = x$.

$$\Rightarrow \int x\, p_2(x)\,dx = \int f(x)\, p_1(x)\,dx$$

But $\int x\, p_2(x)\,dx = \mu(f_* \mathbb{P})$

(by definition of mean)

$\Rightarrow \mu(f_* \mathbb{P}) = \int f(x)\, p_1(x)\, dx$

↳ $\boxed{\text{Law of the unconscious statistician (LUS)}}$

Question 3c : (done after exam)

NOTE: There's a change in notation:

$f \Rightarrow$ the density of $\mathbb{P}$

Hence, to restate LUS:

$\mu(g_* \mathbb{P}) = \int g(x) f(x)\, dx$

(g does not refer to anything previously used, it is just an arbitrary function)

Now, we are given:

$\mu_2' = \int x^2 f(x)\, dx$

↳ 2nd raw moment

~~This matches~~ (i.e. the RHS) The RHS matches the RHS of LUS as formulated, with $g(x) = x^2$.

Hence, $\mu(g_* \mathbb{P}) = \int x^2 f(x)\, dx$ where $g(x) = x^2$ ... (1)

Now, consider the MLE of $\hat{\lambda}$, i.e.

$\hat{\lambda} = n / \sum_{i=1}^{n} x_i^2$

$\Rightarrow \frac{1}{\hat{\lambda}} = \frac{\sum_{i=1}^{n} x_i^2}{n}$

Each x_i ~~x_i^2~~ is an independent sample from $\mathbb{P}$, which means each x_i^2 is an independent sample of $g_* \mathbb{P}$ (where $g(x) = x^2$ as defined)

$\Rightarrow \frac{1}{\hat{v}} = \frac{\sum x_i^2}{n}$ follows the distribution

$$\frac{1}{n} \underbrace{(g_* \mathbb{P}) \otimes (g_* \mathbb{P}) \cdots (g_* \mathbb{P})}_{n \text{ times}}$$

$$= \cancel{g_* \mathbb{P}} \ (\overline{g_* \mathbb{P}})_n$$

Now, we know that for any ~~arbitrart~~ arbitrary distribution K, with mean μ,

$\mu(\overline{K_n}) = \mu(K)$ (i.e. sample mean is unbiased estimator of mean)

$$\Rightarrow \mu((\overline{g_* \mathbb{P}})_n) = \mu(g_* \mathbb{P})$$

$$= \int x^2 f(x)\,dx \quad (\text{from } ①)$$

$$= \mu_2'$$

$\Rightarrow \frac{1}{\hat{v}}$ is an unbiased estimator of μ_2'.

Question 4

Part a

i) We estimate β_1 as

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

where

$x_i \in x$, $y_i \in y$

$\bar{x}$ = sample mean of x

$\bar{y}$ = sample mean of y

$n = 12$ (sample size)

We get:

$\bar{x} = 8$

$\bar{y} = -4.5$

Upon computation,

$\hat{\beta}_1 = -2.95238$

ii) We estimate β_0 as

$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$.

Using previously obtained values, we get

$\hat{\beta}_0 = 19.1190476$

iii) In the case where we model the joint distribution of two variables X & Y (from which we have sampled $x \in X^{12}$ & $y \in Y^{12}$) as a linearly dependent distribution where Y is a linear function of X, i.e.

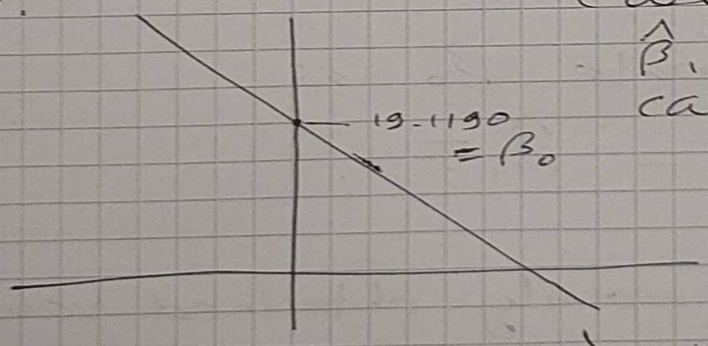
$$Y = \beta_1 X + \beta_0$$

Here, (which is a linear regression model), we have that

β_1 = slope of the regression

β_0 = Y-axis intercept of the regression

<u>ex.</u>



(using $\hat{\beta}_1$ & $\hat{\beta}_0$ values calculated before)

slope
$= -2.95238$
$= \beta_1$

$\Rightarrow$ p-value = 0.380

NOTE: t-test is 2 sided, so we compare as follows with the confidence level α:

$$0.380 > \frac{1-\alpha}{2} \quad (\alpha = 95\%)$$

i.e., $0.380 > \frac{0.05}{2}$

Hence, we fail to reject the null hypothesis that the samples ~~as~~ were drawn from the joint distribution $\mathbb{P}(X \times Y)$ wherein

$$Y = \beta_1 X + \beta_0 + \varepsilon$$

(with the given β_1 value that was tested & the assumed β_0 value)

ii) Similarly as (i), we do student's t-test for β_0 assuming $\beta_1 = \cancel{0}\ -3$.

Test statistic for β_0:

$$\frac{(\hat{\beta}_0 - \beta_0)\sqrt{n(n-2)\sum_{i=1}^{n}(x_i - \bar{x})^2}}{\sqrt{\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2 \sum_{i=1}^{n} x_i^2}} \sim t(n-2)$$

Test statistic = $\cancel{8}$ -0.6275

$\Rightarrow$ p-value = $0.27218875 > \frac{1-\alpha}{2}$

Hence, we ~~can~~ fail to reject the ~~null~~ null hypothesis

iii) A joint distribution is a probability measure applied to a product space.

In our case, the above model does define a joint distribution over the product space

$$\{ (x, \beta_0 + \beta_1 x + \varepsilon) \mid x \in X, \varepsilon \sim Norm(0, \sigma) \}$$

$\beta_0 + \beta_1 x + \varepsilon$ is also normally distributed, & x ~~is~~ may be considered uniformly distributed. (since ε is ~~& x is~~ normally distributed and x is a uniform random variable).

Thus, the joint distribution is also normal, hence a valid probability distribution

iv) β_1 is the rate of change in the target variable y with a unit change in the independent variable x. We got

$\hat{\beta}_1 = -2.95238 < 0$

$\Rightarrow$ There is inverse relation between x & y

$\Rightarrow$ ρ_{xy} (i.e. correlation coefficient between x & y) is less than 0

i.e. $\rho_{x,y} < 0$

~~Part c~~ (done after exam)

We ~~a~~ have a joint density function

$f(x,y) = \mathbb{1}_{[0,1]}(x)\,\mathbb{1}_{[x-\frac{1}{2},x+\frac{1}{2}]}(y)$

Consider: what does marginal density mean? Formally, it is the density of the pushforward $\pi_i * \mathbb{P}$, where $\mathbb{P}$ is the joint distribution (defined by f in our case), & π_i is the projection of the ith coordinate, i.e.

$\pi_i : \mathbb{R}^n \to \mathbb{R}, \quad (x_1, x_2 \dots x_n) \mapsto x_i$

(i.e. takes a tuple from the joint distr. & ~~*~~ returns the ith coordinate).

In our case,

$\pi_1 : \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto x$

$\pi_2 : \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto y$

i) Consider $\pi_1 * \mathbb{P}$. This defines the distribution of x based on the ~~distribution~~ ~~of each value~~ joint occurrence of each x with every possible y.

Hence, density of $\pi_1 * \mathbb{P}$ (call it f_1) for each value of x is the ~~*~~ density of that value taken with every possible y.

$\therefore$ ~~$\pi_1 * \mathbb{P}$~~ $f_1(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

(i.e. take x as constant).

Now, note that if $x < 0$ or $x > 0$, $f(x,y) = 0$. Hence,

$f_1(x) = 0$ if $x < 0$ or $x > 1$

if $x \in [0,1]$,

$$f_1(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} f(x,y)\,dy$$ (since $f(x,y)$ is 0 for the range beyond)

$$= \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \mathbb{1}_{[x-\frac{1}{2},\, x+\frac{1}{2}]}(y)\,dy$$

$$= \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} 1\,dy$$ (since $\mathbb{1}_{[x-\frac{1}{2},\, x+\frac{1}{2}]}(y) = 1$ $\forall\, y \in [x-\frac{1}{2}, x+\frac{1}{2}]$)

$$= \cancel{[\;x\;]}$$

$$= [\,y\,]_{x-\frac{1}{2}}^{x+\frac{1}{2}}$$

$$= x + \frac{1}{2} - x + \frac{1}{2}$$

$$= 1$$

$$\therefore f_1(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{if } x < 0 \text{ or } x > 1 \end{cases}$$

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ii) Similarly,

$$f_2 \cancel{(x)} (y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$\boxed{\text{density of } \pi_2 * \mathbb{P}}$

$$= \int_0^1 f(x,y)\,dx$$ (since $\mathbb{1}_{[0,1]}(x) = 0$ $\forall\, x \notin [0,1]$.

If $y > 1 + \frac{1}{2} = \frac{3}{2}$, then

$y \notin [x - \frac{1}{2}, x + \frac{1}{2}]$ $\forall\, x \in [0,1]$

Similarly, if $y < 0 - \frac{1}{2} = -\frac{1}{2}$,

$y \notin [x - \frac{1}{2}, x + \frac{1}{2}]$ $\forall\, x \in [0,1]$

$\therefore f_2(y) = 0$ if $y > \frac{3}{2}$ or $y < -\frac{1}{2}$

If $y \in [-\frac{1}{2}, \frac{3}{2}]$,

$$f_2(y) = \int_0^1 1 \, \cancel{dy} \, dx$$

$$= [x]_0^1 = 1$$

$$\therefore f_2(y) = \begin{cases} 1 & \text{if } y \in [-\frac{1}{2}, \frac{3}{2}] \\ 0 & \text{if } y < -\frac{1}{2} \text{ or } y > \frac{3}{2} \end{cases}$$

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iii) Mean of $\pi_1 * \mathbb{P}$ is

$$\mu(\pi_1 * \mathbb{P}) = \int_{-\infty}^{\infty} x f_1(x) \, dx$$

$$= \int_0^1 x f_1(x) dx = \int_0^1 x \, dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

Mean of $\pi_2 * \mathbb{P}$ is

$$\mu(\pi_2 * \mathbb{P}) = \int_{-\infty}^{\infty} y f_2(y) \, dy$$

$$= \int_{-1/2}^{3/2} y f_2(y) dy = \int_{-1/2}^{3/2} y \, dy = \left[\frac{y^2}{2}\right]_0^1$$

$$= \frac{9}{8} - \frac{1}{8} = 1$$

$$\text{Cov}(\mathbb{P}) = \iint (x - \tfrac{1}{2})(y - 1) f(x, y) \, dx \, dy$$