1.19. Gauss' Divergence Theorem

According to this theorem, *the volume integral of divergence of a vector field* $\vec{A}$ *over a volume* V *is equal to the surface integral of that vector field* $\vec{A}$ *taken over the surface* S *which encloses that volume* V. *i.e.,*

$$\iiint_V (\text{div}\,\vec{A})\,dV = \iint_S \vec{A}\cdot\vec{da} \qquad ...(1.71)$$

This is the Gauss' divergence theorem.

Alternative method : Consider a volume V enclosed by a surface S (Fig. 1.33). This volume can be divided into small elements of volumes V_1, V_2, ... which are enclosed by the elementary surfaces S_1, S_2, ... respectively. By definition, the flux of a vector field $\vec{A}$ diverging out of the i^{th} element is given as :

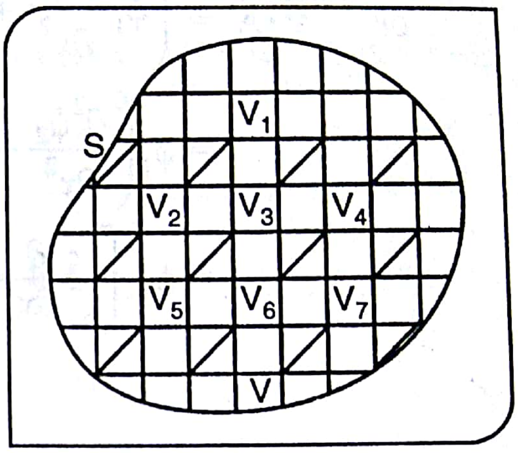

Fig. 1.33. Gauss' divergence theorem

$$(\text{div } \vec{A})_i = \frac{\iint\limits_{S_i} \vec{A}\cdot\vec{da}}{V_i}$$

$$\text{or} \quad (\text{div } \vec{A})_i \, V_i = \iint\limits_{S_i} \vec{A}\cdot\vec{da}$$

If we add the quantity $(\text{div}\vec{A})_i V_i$ on the left side of above eqn. for each element V_1, V_2, ..., we get

$$\sum (\text{div}\vec{A})_i V_i = \iiint\limits_V (\text{div}\vec{A})\, dV$$

Similarly, if we add the quantity $\iint\limits_{S_i} \vec{A}\cdot\vec{da}$ on the right side of above eqn. for each of the elementary surfaces S_1, S_2 ...., we get the terms only on the outer surface S because the value of $\vec{A}\cdot\vec{da}$ for each of the inner surface get cancelled due to equal and opposite terms for the surfaces being common for the two elementary volumes. Thus this sum comes out to be

$$\iint\limits_{S_i} \vec{A}\cdot\vec{da} = \iint\limits_S \vec{A}\cdot\vec{da}$$

Thus

$$\iiint\limits_V (\text{div}\vec{A})\, dV = \iint\limits_S \vec{A}\cdot\vec{da}$$

This is the Gauss' divergence theorem.

field.

1.22. Stokes' Theorem

According to this theorem, *the line integral of a vector field* $\vec{A}$ *along the boundary of a closed curve* C *is equal to the surface integral of Curl of that vector field when the surface integration is done over a surface* S *enclosed by the boundary* C, *i.e.,*

$$\oint_C \vec{A}\cdot\vec{dl} = \iint_S \text{Curl}\,\vec{A}\cdot\vec{da} \quad \text{or} \quad \oint_C \vec{A}\cdot\vec{dl} = \iint_S (\vec{\nabla}\times\vec{A})\cdot\vec{da} \qquad ...(1.79)$$

Obviously, this theorem is used to change the surface integral into the line integral or to change the line integral into the surface integral.

Proof **:** Consider a vector $\vec{A}$ which is the function of position. We are to find the line integral $\oint_C \vec{A}\cdot\vec{dl}$ along the boundary of a closed curve C shown in Fig. 1.35 (a).

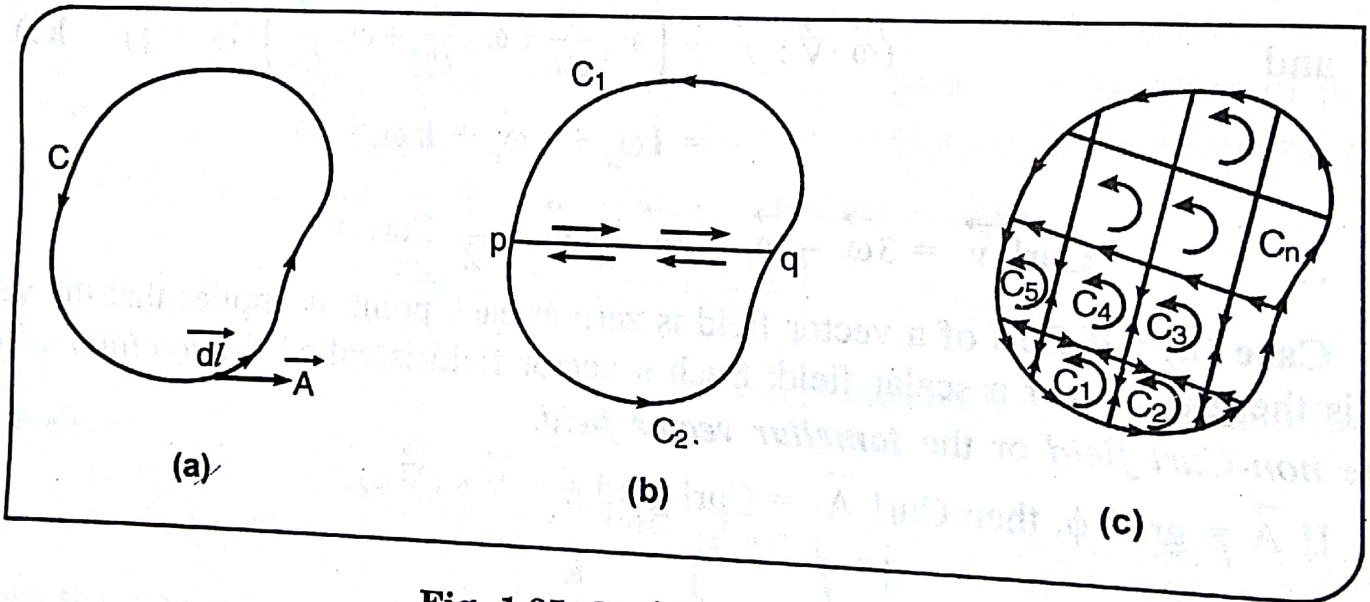

Fig. 1.35. Stokes' theorem

If we divide the area enclosed by the curve C in two parts by a line *pq*, we get the two closed curves C_1 and C_2 [Fig 1.35 (b)]. The line integral of vector $\vec{A}$ along the boundary of curve C will be equal to the sum of line integral of $\vec{A}$ along C_1pqC_1 and C_2qpC_2 (taken anti-clockwise), since the line integral along *pq* for the curve C_1 is cancelled by the line integral along *qp* for the curve C_2, *i.e.,*

$$\oint_C \vec{A}\cdot\vec{dl} = \oint_{C_1} \vec{A}\cdot\vec{dl} + \oint_{C_2} \vec{A}\cdot\vec{dl}$$

If we divide the area enclosed by the curve C in small elements of area da_1, da_2, da_3 ..... by the curves C_1, C_2, C_3, ...as shown in Fig. 1.35 (c), then the sum of line integrals along the boundary of these curves C_1, C_2.... (taken anti-clockwise) will be equal to the line integral along the boundary C. *i.e.*,

$$\oint_C \vec{A} \cdot \vec{dl} = \sum \oint_{C_n} \vec{A} \cdot \vec{dl}$$

But by the definition of Curl, we have Curl $\vec{A} \cdot \vec{da}_n = \oint_{C_n} \vec{A} \cdot \vec{dl}$

$$\therefore \qquad \oint_C \vec{A} \cdot \vec{dl} = \sum \text{Curl}\,\vec{A} \cdot \vec{da}_n = \iint_S \text{Curl}\,\vec{A} \cdot \vec{da}$$

This is the Stokes' theorem. It may be mentioned here that it is not necessary that the boundary or the surface enclosed by it should be plane. The surface can be a curved surface.

It is also evident from this theorem that if Curl $\vec{A} = 0$ at each point, the line integral of the vector field $\vec{A}$ is also zero.

(2) Otto Engine or Petrol Engine

This engine was invented by Dr. Otto in 1876. In each cycle of the engine there are four strokes, so it is also called the four-stroke engine.

the rotating cams C_1 and C_2. The springs ...

Working—Fig. 1.19 shows the working of Otto engine, in which only the main part of the engine (cylinder fitted with piston) is shown. I and E are the inlet and exit valves respectively and S is the spark plug. In each cycle, the following four strokes are repeated in subsequent order :

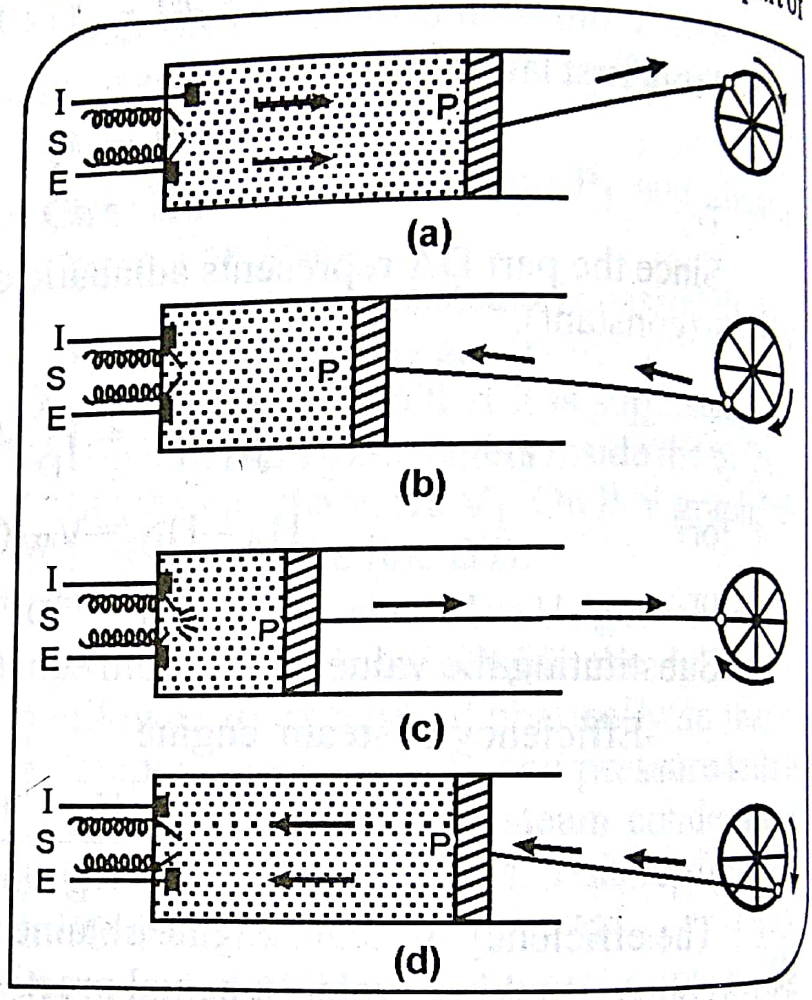

Fig. 1.19. Strokes of Otto engine

(1) First stroke (suction stroke)—In this stroke [Fig. 1.19 (a)], the inlet valve I opens and the petrol vapours mixed with air (98% air, 2% petrol vapours) enter the cylinder from the carburettor. The piston P moves out of the cylinder. The inlet valve closes when sufficient amount of petrol vapours enter in the cylinder.

(2) Second stroke (compression stroke)—In this stroke [Fig. 1.19(b)], both valves I and E remain closed and as the wheel rotates, the piston moves in the cylinder. As a result, the air-petrol mixture gets compressed adiabatically to $\frac{1}{5}$ th of its original volume. Due to this, the temperature of petrol vapour rises to about 600°C.

(3) **Third stroke (working stroke)**—In this stroke [Fig. 1.19(c)], the combustion of petrol vapours takes place due to the spark produced at the plug at about 600°C, and the temperature of vapour increases at a constant volume to about 2000°C. During the stroke, there is almost no movement of the piston. Now the compressed gas at such a high temperature expands adiabatically due to which the piston moves out of the cylinder with a large velocity and the wheel of the vehicle rotates rapidly. In this operation, the pressure and temperature of air falls to a good extent. As the piston reaches the end of the cylinder, the exit valve E opens.

(4) **Fourth stroke (exit stroke)**—In this stroke [Fig. 1.19(d)], the gas of the cylinder at a high pressure and high temperature escapes out of the exit valve E, till the pressure of gas inside the cylinder becomes equal to the atmospheric pressure. In this process, heat is rejected out of the cylinder and now the piston moves in the cylinder so that all the exhaust petrol vapours escape out of the cylinder. After this stroke, the exit valve E automatically closes and the inlet valve I opens to start the first stroke. Thus the cycle is repeated continuously.

called ...

Efficiency—Let us assume that 1 mole of air inside the cylinder completes the cy
ABCDA. In each cycle, it absorbs heat Q_1 at a constant volume in part BC and rejects h
Q_2 at a constant volume in part DA. Let temperature of air at A, B, C and D be T_a, T_b, T_c
T_d respectively and C_V be the molar specific heat of air at constant volume.

Heat absorbed in part BC $= Q_1 = C_V (T_c - T_b)$

Heat rejected in part DA $= Q_2 = C_V (T_a - T_d)$

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$$\therefore \quad \text{Efficiency of engine } \eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{C_V (T_a - T_d)}{C_V (T_c - T_b)}$$

$$= 1 - \frac{(T_a - T_d)}{(T_c - T_b)} \qquad \ldots(1.59)$$

If V_1 and V_2 be the volumes of the mixture at A and B respectively, then in adiabatic compression AB

$$T_a V_1^{\gamma-1} = T_b V_2^{\gamma-1}$$

Similarly, V_2 and V_1 are the volumes at C and D respectively, therefore, in adiabatic expansion CD

$$T_c V_2^{\gamma-1} = T_d V_1^{\gamma-1}$$

or

$$T_d V_1^{\gamma-1} = T_c V_2^{\gamma-1}$$

From the above two relations, $(T_b - T_c)\, V_2^{\gamma-1} = (T_a - T_d)\, V_1^{\gamma-1}$

or

$$\frac{(T_a - T_d)}{(T_b - T_c)} = \left(\frac{V_2}{V_1}\right)^{\gamma-1} = \left(\frac{1}{\rho}\right)^{\gamma-1} \qquad \ldots(1.60)$$

where $\rho = \dfrac{V_1}{V_2}$ is called the compression ratio.

$\therefore$ From eqns. (1.59) and (1.60),

$$\eta = 1 - \left(\frac{1}{\rho}\right)^{\gamma-1} \qquad \ldots(1.61)$$

In an actual engine the value of ρ (*i.e.*, V_1/V_2) can not exceed more than 5, otherwise the temperature in adiabatic compression will increase to such a value that explosion may occur before the spark. Thus if $\rho = 5$ and $\gamma = 1{\cdot}4$,

$$\eta = 1 - \left(\frac{1}{5}\right)^{1\cdot4-1} = 1 - \left(\frac{1}{5}\right)^{0\cdot4} = 0{\cdot}52 \text{ or } (52\%)$$

Thus the efficiency of a petrol engine can be 52% in ideal conditions. In practice, the efficiency is less than 52% because of loss in energy due to turbulence in air and friction of piston etc.

(3) Diesel Engine

This engine was invented by a German Engineer Diesel in 1900. The construction and working of diesel engine is similar to that of petrol engine. The only difference is that spark plug is replaced by a valve called the oil valve. The diesel oil enters the cylinder through this valve.

(3) Diesel Engine

This engine was invented by a German Engineer Diesel in 1900. The construction and working of diesel engine is similar to that of petrol engine. The only difference is that spark plug is replaced by a valve called the oil valve. The diesel oil enters the cylinder through this valve.

Fig. 1.21 represents the working of a diesel engine. It consists of a cylinder fitted with a piston. There are three valves in the cylinder—air inlet valve I, oil inlet valve I' and exit valve E. These valves open and close automatically in the following four strokes, in one cycle.

(1) First stroke (suction stroke)—In this stroke [Fig. 1.21(a)], the air inlet valve I opens while the oil inlet valve I' and the exit valve E remain closed. Piston moves out of the cylinder and the air enters the cylinder at the atmospheric pressure through the valve I.

(2) Second stroke (compression stroke)—In this stroke [Fig. 1.21(b)], all the three valves remain closed and the piston moves inwards inside the cylinder due to which the air inside the cylinder gets compressed adiabatically and the volume of air becomes $\frac{1}{17}$th of its initial volume with an increase in temperature to about 1000°C. The oil inlet valve I' opens at the end of this stroke.

(3) Third stroke (working stroke)—In this stroke [Fig. 1.21 (c)], the piston moves outwards of the cylinder. At this instant, the spray of diesel oil enters the cylinder through the valve I'. The oil gets ignited at 1000°C temperature. To control the quantity of oil, the increase in temperature of air is so controlled that the air coming out of cylinder expands at a constant pressure.

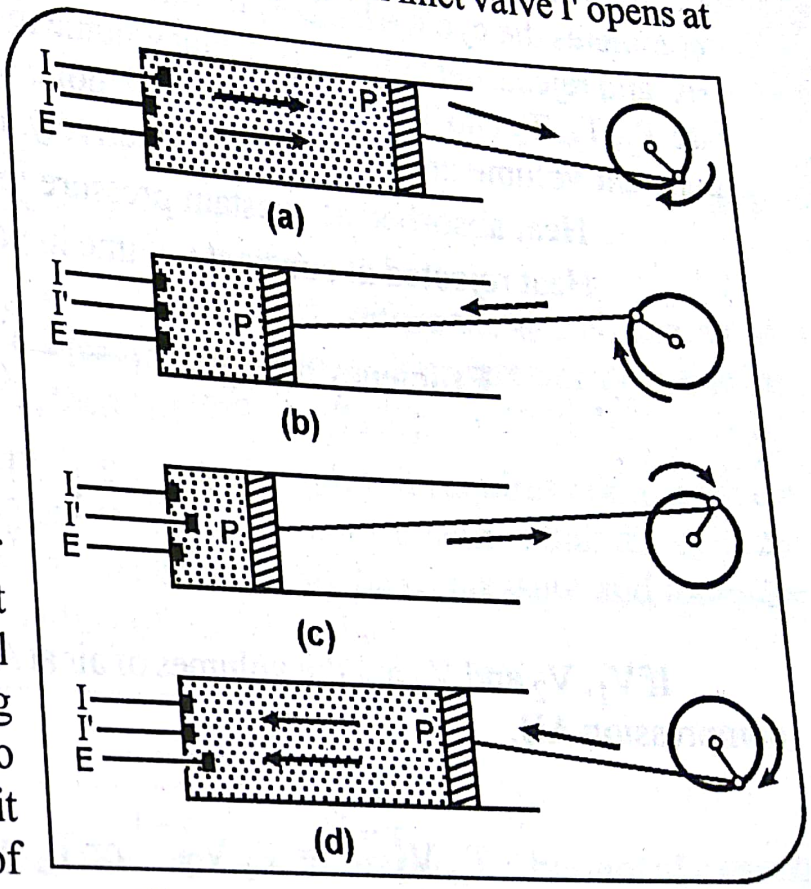
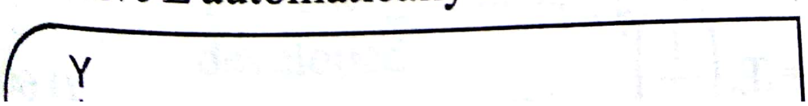


Fig. 1.21. Strokes of diesel engine

As a result, the temperature of air becomes 2000°C. At this instant, the oil inlet valve I' closes and thus the incoming of oil in the cylinder stops. Now in the remaining stroke, the air expands adiabatically due to which its temperature falls. Now the exit valve E opens due to which the pressure of air decreases to the atmospheric pressure. In this process, the heat is rejected from the cylinder.

(4) Fourth stroke (exit stroke)—In this stroke [Fig. 1.21(d)], the piston enters in the cylinder so that exhaust gases escape out of exit valve E. On completion of this stroke, the exit valve E automatically closes and the air inlet valve I opens to start the first stroke and the cycle repeats.



Diesel cycle—The P-V diagram o