# Error Propagation

# Outline

Error types

Numeric imprecision in formulas

$\mu$ and $\sigma$ in a linear case

$\mu$ and $\sigma$ in a non-linear case

(Physical) Errors in digital cameras
- Parallax
- What is a "point" and how is it mapped?

# Error Types: Input, Method, Roundoff, Truncation, Modelling, Machine, and Human Errors

## Input errors

Given numbers are not machine numbers (e.g. $\sqrt{2}$)

## Method errors

Accumulated roundoff errors per calculation

## Truncation errors

Systematic errors when stopping an approximation too early

## Modelling errors

Too strong idealizations

## Machine + Human errors

Hardware errors, programming errors

# Some Rules for Reporting Measurements

## Rule for stating uncertainties

Experimental uncertainties should **almost always** be rounded to one significant figure

$$x_{\text{measured}} = x_{\text{best}} \pm \Delta x$$

$x_{\text{best}}$ = best estimate of $x$
$\Delta x$ = uncertainty of measurement error

## Rule for stating answers

The **last significant figure** in any stated answer should usually be **of the same order of magnitude** (in the same decimal position) as the **uncertainty**

## Fractional uncertainty

$$\frac{\Delta x}{|x_{\text{best}}|}$$

## Approximate correspondence between significant figures and fractional uncertainties

| Number of significant figures | Corresponding fractional uncertainty is between | or roughly |
|---|---|---|
| 1 | 10% and 100% | 50% |
| 2 | 1% and 10% | 5% |
| 3 | 0.1% and 1% | 0.5% |

# Uncertainty in Experiments

## Counting experiment

The uncertainty in any counted number of random events - as an estimate of the true average number - is the square root of the counted number $v$:

$$\text{average number of events in time } T = v \pm \sqrt{v}$$

## Example: $14$ births in two weeks

$$\text{average number of births in a two-week period} = 14 \pm 4$$

# Uncertainties in Sums, Differences, Products, and Quotients

## Sums and differences

If $q = x + \ldots z - (u + \ldots + w)$, then

$$\Delta q \begin{cases} = \sqrt{(\Delta x)^2 + (\Delta y)^2 + \ldots (\Delta w)^2}, & \text{if independent and random} \\ \leq \Delta x + \Delta y + \ldots + \Delta w, & \text{always} \end{cases}$$

## Products and Quotients

If $q = \frac{x \cdot y \cdot \ldots \cdot z}{u \cdot v \cdot \ldots \cdot w}$, then

$$\frac{\Delta q}{|q|} \begin{cases} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2 + \ldots + \left(\frac{\Delta w}{w}\right)^2}, & \text{if independent and random} \\ \leq \frac{\Delta x}{|x|} + \frac{\Delta y}{|y|} + \ldots + \frac{\Delta w}{|w|}, & \text{always} \end{cases}$$

# Uncertainties: Special Cases

**$q = Bx$, $B$ is exactly known**

$$\Delta q = |B| \Delta x$$

**$q$ a function of one variable, i.e. $q = q(x)$**

$$\Delta q = \left| \frac{dq}{dx} \right| \Delta x$$

**$q$ is a power, i.e. $q = x^n$**

$$\frac{\Delta q}{|q|} = |n| \frac{\Delta x}{|x|}$$

# Differential Error Analysis

Let $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} = \phi(x), \mathbf{y} \in \mathbb{R}^n$

If $\Delta_x$ is the vector of absolute data errors in $x$ and

$$\mathrm{Jac}(\phi) = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial x_1} & \cdots & \frac{\partial \phi_n}{\partial x_m} \end{pmatrix} \in \mathbb{R}^{n \times m}$$

then it holds that the absolute output error is given by (to first order)

$$\Delta_y = \mathrm{Jac}(\phi)\Delta_x$$

if we calculate it in the absence of round-off errors

# Curvature Radius Example

$$R(v_r, v_l) = \frac{d}{2}\frac{v_r + v_l}{v_r - v_l}$$

$$\frac{\partial R}{\partial v_r} = \frac{d}{2}\left[\frac{(v_r - v_l) - (v_r + v_l)}{(v_r - v_l)^2}\right] = -d\frac{v_l}{(v_r - v_l)^2}$$

$$\frac{\partial R}{\partial v_l} = \frac{d}{2}\left[\frac{(v_r - v_l) + (v_r + v_l)}{(v_r - v_l)^2}\right] = -d\frac{v_r}{(v_r - v_l)^2}$$

$$\Delta R = -d\left[\frac{v_l}{(v_r - v_l)^2}\Delta v_r + \frac{v_r}{(v_r - v_l)^2}\Delta v_l\right]$$

Note: When $v_l \approx v_r$, $R$ is very imprecise; in other words, the input errors are grossly amplified when $v_l$ and $v_r$ are close

# Linear Case: How Expectations Map

In the linear case, $\mathbf{y}$ is a linear map of $\mathbf{x}$

$$\mathbf{y} = F(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$$

$$\begin{aligned} E[\mathbf{y}] &= E[A\mathbf{x} + \mathbf{b}] \\ &= \int\int\int \dots \int (A\mathbf{x} + \mathbf{b}) p(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \end{aligned}$$

Taking component $j$, we have

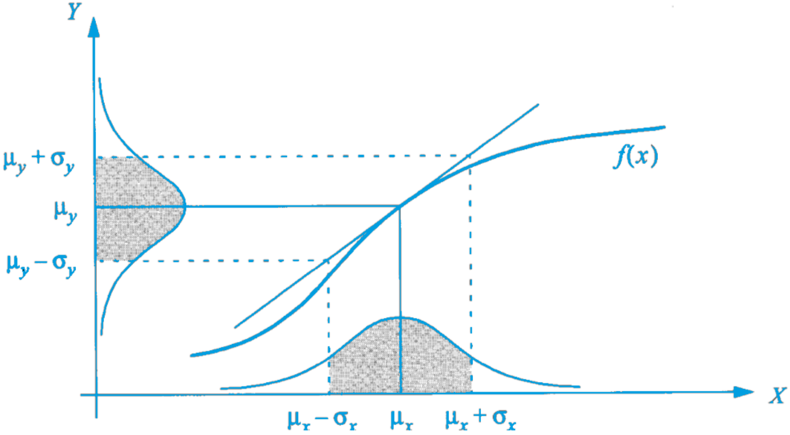
$$\begin{aligned} E[y_j] &= \int\int\int \dots \int (\sum_i a_{ij} x_i) p(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n + \\ &\int\int\int \dots \int b_j p(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \\ &= \sum_i a_{ij} \int\int\int \dots \int x_i p(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n + b_j \\ &= \sum_i a_{ij} E[x_i] + b \\ &\implies E[\mathbf{y}] = AE[\mathbf{x}] + \mathbf{b} \end{aligned}$$

# Covariance in the Linear Case

$$\begin{aligned}
\mathrm{cov}(\mathbf{y}) &= E[((A\mathbf{x} + b) - (AE[\mathbf{x}] + \mathbf{b}))\,((A\mathbf{x} + b) - (AE[\mathbf{x}] + \mathbf{b}))^T] \\
&= E[(A\mathbf{x} - AE[\mathbf{x}])(A\mathbf{x} - AE[\mathbf{x}])^T] \\
&= E[A(\mathbf{x} - E[\mathbf{x}])(\mathbf{x} - E[\mathbf{x}])^T A^T] \\
&= A\,\mathrm{cov}(\mathbf{x})A^T
\end{aligned}$$

# Nonlinear Case: Approximate

Use a Taylor expansion $f(x+h) = f(x_0) + hf'(x_0) + \epsilon$



# General Nonlinear Case: $\mu$ and $\operatorname{cov}$ Nonlinear

$$\mathbf{y} = F(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{pmatrix}$$

Via Taylor, we get

$$F(\mathbf{x}+\mathbf{h}) = F(\mathbf{x}) + \mathrm{Jac}(F)\mathbf{h} + O(\|\mathbf{h}\|)$$
$$E[\mathbf{y}] = F(E[\mathbf{x}])$$
$$\operatorname{cov}(\mathbf{y}) = \mathrm{Jac}(F)|_{\mathbf{x}} \operatorname{cov}(\mathbf{x})\, \mathrm{Jac}(F)^T|_{\mathbf{x}}$$

where $\mathrm{Jac}(F)$ is the **Jacobian** of $F$

# Covariance Example

Assume that a laser scanner measures polar coordinates $(d, \alpha)$, such that the measurements of $d$ and $\alpha$ are normally distributed, i.e. $d \sim \mathcal{N}(\mu_d, \sigma_d^2)$, $\alpha \sim \mathcal{N}(\mu_\alpha, \sigma_\alpha^2)$

The measurements have to be mapped to Cartesian $(x, y)$ coordinates via
$F\left((d,\alpha)^T\right) = (d\cos(\alpha), d\sin(\alpha))^T$

How does the original covariance matrix change after the conversion?

$$F\begin{pmatrix} d \\ \alpha \end{pmatrix} = \begin{pmatrix} d\cos(\alpha) \\ d\sin(\alpha) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix} = F\begin{pmatrix} \mu_d \\ \mu_\alpha \end{pmatrix} = \begin{pmatrix} \mu_d\cos(\mu_\alpha) \\ \mu_d\sin(\mu_\alpha) \end{pmatrix} = \begin{pmatrix} d\cos(\alpha) \\ d\sin(\alpha) \end{pmatrix}$$

$$\begin{aligned}
\nabla F = &\begin{pmatrix} \cos(\alpha) & -d\sin(\alpha) \\ \sin(\alpha) & d\cos(\alpha) \end{pmatrix} \\
\implies \operatorname{cov}\begin{pmatrix} x \\ y \end{pmatrix} &= \nabla F \operatorname{cov}\begin{pmatrix} d \\ \alpha \end{pmatrix} \nabla F^T \\
&= \begin{pmatrix} \cos(\alpha) & -d\sin(\alpha) \\ \sin(\alpha) & d\cos(\alpha) \end{pmatrix} \begin{pmatrix} \sigma_d^2 & 0 \\ 0 & \sigma_\alpha^2 \end{pmatrix} \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -d\sin(\alpha) & d\cos(\alpha) \end{pmatrix} \\
&= \begin{pmatrix} \sigma_d^2\cos^2(\alpha) + d^2\sigma_\alpha^2\sin^2(\alpha) & (\sigma_d^2 - d^2\sigma_\alpha^2)\cos(\alpha)\sin(\alpha) \\ (\sigma_d^2 - d^2\sigma_\alpha^2)\cos(\alpha)\sin(\alpha) & \sigma_d^2\sin^2(\alpha) + d^2\sigma_\alpha^2\cos^2(\alpha) \end{pmatrix}
\end{aligned}$$

# Covariance in Error Propagation

Let $F(x, y)$ and a set of $N$ data pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)$ be given. We can then compute the empirical means $\bar{x}$, $\bar{y}$ and the empirical variances $s_x$, $s_y$ as usual

Assume $x_1, \ldots, x_N$ are close to $\bar{x}$ (same for $y$); then

$$G_i = G(x_i, y_i) \approx G(\bar{x}, \bar{y}) + \left.\frac{\partial G}{\partial x}\right|_{\mu_x} (x_i - \bar{x}) + \left.\frac{\partial G}{\partial y}\right|_{\mu_y} (y_i - \bar{y})$$

$$\begin{aligned}
\bar{G} &= \frac{1}{N} \sum G_i = \frac{1}{N} \sum \left( G(\bar{x}, \bar{y}) + \left.\frac{\partial G}{\partial x}\right|_{\mu_x} (x_i - \bar{x}) + \left.\frac{\partial G}{\partial y}\right|_{\mu_y} (y_i - \bar{y}) \right) \\
&= G(\bar{x}, \bar{y}) + \frac{1}{N} \left.\frac{\partial G}{\partial x}\right|_{\mu_x} \underbrace{\sum (x_i - \bar{x})}_{=0} + \frac{1}{N} \left.\frac{\partial G}{\partial y}\right|_{\mu_y} \underbrace{\sum (y_i - \bar{y})}_{=0} \\
&= G(\bar{x}, \bar{y})
\end{aligned}$$

# Variance of $G$

$$
\begin{aligned}
s_G^2 &= \frac{1}{N}\sum (G_i - \bar{G})^2 \\
&\approx \frac{1}{N}\sum \left( \bar{G} + \left.\frac{\partial G}{\partial x}\right|_{\bar{x},\bar{y}} (x_i - \bar{x}) + \left.\frac{\partial G}{\partial y}\right|_{\bar{x},\bar{y}} (y_i - \bar{y}) - \bar{G} \right)^2 \\
&= \left( \left.\frac{\partial G}{\partial x}\right|_{\bar{x},\bar{y}} \right)^2 \frac{1}{N}\sum (x_i - \bar{x})^2 + \left( \left.\frac{\partial G}{\partial y}\right|_{\bar{x},\bar{y}} \right)^2 \frac{1}{N}\sum (y_i - \bar{y})^2 \\
&\quad + 2\left.\frac{\partial G}{\partial x}\right|_{\bar{x},\bar{y}} \left.\frac{\partial G}{\partial y}\right|_{\bar{x},\bar{y}} \underbrace{\frac{1}{N}\sum (x_i - \bar{x})(y_i - \bar{y})}_{\text{empirical covariance}} \\
&= \left( \left.\frac{\partial G}{\partial x}\right|_{\bar{x},\bar{y}} \right)^2 s_x^2 + \left( \left.\frac{\partial G}{\partial y}\right|_{\bar{x},\bar{y}} \right)^2 s_y^2 + 2\left.\frac{\partial G}{\partial x}\right|_{\bar{x},\bar{y}} \left.\frac{\partial G}{\partial y}\right|_{\bar{x},\bar{y}} s_{xy}
\end{aligned}
$$

# Variance of $G$ When $x$ and $y$ are Independent

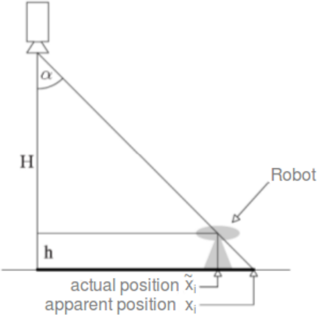
If $x$ and $y$ are independent, then $s_{xy} \approx 0$

In this case, we have

$$s_G^2 = \sum_i s_{z_i}^2 \left( \frac{\partial G}{\partial z_i} \right)^2$$

# Parallax During Observation

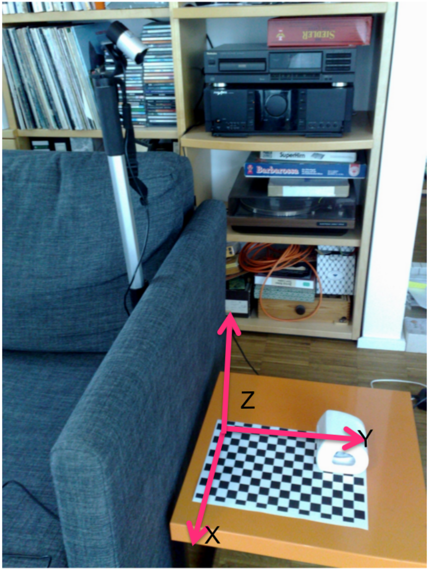
Color circles as markers




Correction of the position discrepancy caused by the perspective

$$\tilde{x}_i = x_i \left(1 - \frac{h}{H}\right)$$

# Camera Setup During Calibration




The frame used during calibration (and extrinsic parameter finding) should be aligned as shown in the figure

If ${}^{c}R_1$ and ${}^{c}\mathbf{t}_1$ - the rotation and translation of the camera - are provided, the position of a robot in the camera frame can be calculated as

$$ {}^{c}\mathbf{X} = {}^{c}R_1 \mathbf{X} + {}^{c}\mathbf{t}_1 $$

# “Point” Observations

Just like the circles in the parallax example, an LED is mapped to many points

Where is the robot actually?

How about observed points that are outside the depth of field (DoF) and are thus depicted as “circles of confusion” instead of points?