CS9.312 Introduction to Quantum Information and Computation

Assignment 2

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Ans.1 The Pauli spin matrices are as follows:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

→ (only listing the properties that will be used)

Properties of Pauli spin matrices are as follows:

1) Pauli matrices square to identity matrices:

$$\rightarrow \sigma_x^2 = \sigma_0^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\rightarrow \sigma_1^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\rightarrow \sigma_2^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\rightarrow \sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

we are required to check if the matrix $\begin{pmatrix} \sigma_0 & \sigma_1 \\ -i\sigma_2 & \sigma_3 \end{pmatrix}$ is unitary.

A matrix is unitary if its inverse is equal to its conjugate transpose, we can also write it as :

$$UU^{\dagger} = U^{\dagger}U = I$$

To check if a matrix is unitary, take the conjugate transpose of the matrix and multiply it by the matrix itself. If the result is the identity matrix, then the matrix is unitary.

$U^{\dagger} =$

STEP-1 : Take the complex conjugate of the given matrix

$$\begin{pmatrix} \sigma_0 & \sigma_1 \\ -i\sigma_2 & \sigma_3 \end{pmatrix}^{*} = \begin{pmatrix} \sigma_0^{*} & \sigma_1^{*} \\ (-i\sigma_2)^{*} & \sigma_3^{*} \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_0 & \sigma_1 \\ i\sigma_2 & \sigma_3 \end{pmatrix}$$

(As $\sigma_0, \sigma_1, \sigma_3$ are matrices with elements as real numbers, hence $\sigma_0^{*} = \sigma_0$, $\sigma_1^{*} = \sigma_1$, $\sigma_3^{*} = \sigma_3$)

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STEP-2: Take the transpose of the matrix obtained after taking the complex conjugate of the given matrix.

$$U^{\dagger} = \begin{pmatrix} \sigma_0 & i\sigma_2 \\ \sigma_1 & \sigma_3 \end{pmatrix}$$

Now, let's calculate $UU^{\dagger}$

$$UU^{\dagger} = \begin{pmatrix} \sigma_0 & \sigma_1 \\ -i\sigma_2 & \sigma_3 \end{pmatrix} \begin{pmatrix} \sigma_0 & i\sigma_2 \\ \sigma_1 & \sigma_3 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_0^2 + \sigma_1^2 & i\sigma_0\sigma_2 + \sigma_1\sigma_3 \\ -i\sigma_2\sigma_0 + \sigma_3\sigma_1 & \sigma_2^2 + \sigma_3^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2I & i\sigma_0\sigma_2 + \sigma_1\sigma_3 \\ -i\sigma_2\sigma_0 + \sigma_3\sigma_1 & 2I \end{pmatrix}$$

let's calculate $i\sigma_0\sigma_2 + \sigma_1\sigma_3$

$$= i\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

let's calculate $-i\sigma_2\sigma_0 + \sigma_3\sigma_1$

$$= -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

so, $UU^\dagger = \begin{pmatrix} 2I & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 2I \end{pmatrix}$

$$= \begin{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \neq I$$

So, the given matrix is not unitary.

Ans.2 We are required to show that opposite points on the Bloch sphere are orthogonal.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad \alpha, \beta \in C$$

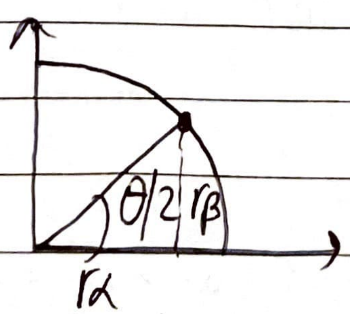
$$|\alpha|^2 + |\beta|^2 = 1$$

$$= r_\alpha e^{i\phi_\alpha}|0\rangle + r_\beta e^{i\phi_\beta}|1\rangle$$

Now, multiply the equation with a complex number that has an absolute value $= 1$, this number is $e^{-i\phi_\alpha}$, after doing this all physical properties of qubit stay unchanged, only the arguement of phase of both complex numbers changes.

$$= r_\alpha|0\rangle + r_\beta e^{i(\phi_\beta - \phi_\alpha)}|1\rangle$$

$$r_\alpha, r_\beta \in [0,1], \quad r_\alpha^2 + r_\beta^2 = 1$$




$$r_\alpha = \cos\frac{\theta}{2}$$

$$r_\beta = \sin\frac{\theta}{2}$$

$$\theta \in [0, \pi]$$

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$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i\phi}|1\rangle$ (I have ignored the global phase)

$\phi = \phi_\beta - \phi_\alpha$, $\phi \in [0, 2\pi]$

$\theta = 2\cos^{-1} r_\alpha$ $\theta \in [0, \pi]$
$= 2\sin^{-1} r_\beta$

Now, opposite points on the Bloch sphere
$\Rightarrow \theta_1 = 180°$ and $\theta_2 = 0°$ (by symmetry it works for any opposites)

$\Rightarrow |\psi_1\rangle = \cos\left(\frac{180°}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{180°}{2}\right)|1\rangle$ ($\rightarrow \frac{2\pi}{2} = \frac{\pi}{2}$, $\rightarrow \pi/2$, $\rightarrow \frac{\pi}{2}$, $\rightarrow \pi/2$)

$= \cos(90°)|0\rangle + e^{i\phi}\sin(90°)|1\rangle$
$= 0|0\rangle + e^{i\phi}1|1\rangle$
$= e^{i\phi}|1\rangle$

$\Rightarrow \langle\psi_1| = e^{-i\phi}\langle 1|$

$\Rightarrow |\psi_2\rangle = \cos\left(\frac{0°}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{0°}{2}\right)|1\rangle$ ($\rightarrow 0/2 = 0$, $\rightarrow 0$, $\rightarrow 0/2 = 0$, $\rightarrow 0$)

$= \cos(0°)|0\rangle + e^{i\phi}\sin(0°)|1\rangle$
$= |0\rangle$

The condition for two points $|\psi_1\rangle$ and $|\psi_2\rangle$ to be orthogonal is:

$\langle\psi_1|\psi_2\rangle = 0$

$= e^{-i\phi}\langle 1|0\rangle = 0$ ($\because \langle 1|0\rangle = 0$)

Hence, proved that opposite points on Bloch sphere are orthogonal.

Ans·3 Given: 1) An operator $\rho$ (4×4) matrix on the Hilbert space $C^4$ where

$$\rho = \frac{1}{4}(1-\varepsilon)I_4 + \varepsilon(|0\rangle \otimes |0\rangle)(\langle 0| \otimes \langle 0|)$$

2) $\varepsilon$ is a real parameter with $\varepsilon \in [0,1]$ and $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$

<u>solution</u>:

$$\rho = \frac{I_4}{4} - \frac{\varepsilon I_4}{4} + \varepsilon\left(\begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix}\right)\left([1\ 0][1\ 0]\right)$$

$$= \begin{bmatrix}1/4 & 0 & 0 & 0\\ 0 & 1/4 & 0 & 0\\ 0 & 0 & 1/4 & 0\\ 0 & 0 & 0 & 1/4\end{bmatrix} - \varepsilon\begin{bmatrix}1/4 & 0 & 0 & 0\\ 0 & 1/4 & 0 & 0\\ 0 & 0 & 1/4 & 0\\ 0 & 0 & 0 & 1/4\end{bmatrix}$$

$$+ \varepsilon\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)\left(\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}\right)$$

$$= \begin{bmatrix}1/4 & 0 & 0 & 0\\ 0 & 1/4 & 0 & 0\\ 0 & 0 & 1/4 & 0\\ 0 & 0 & 0 & 1/4\end{bmatrix} + \varepsilon\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$- \varepsilon\begin{bmatrix}1/4 & 0 & 0 & 0\\ 0 & 1/4 & 0 & 0\\ 0 & 0 & 1/4 & 0\\ 0 & 0 & 0 & 1/4\end{bmatrix}$$

$$= \begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix} + \varepsilon \begin{bmatrix} 3/4 & 0 & 0 & 0 \\ 0 & -1/4 & 0 & 0 \\ 0 & 0 & -1/4 & 0 \\ 0 & 0 & 0 & -1/4 \end{bmatrix}$$

$$\rho = \begin{bmatrix} (1+3\varepsilon)/4 & 0 & 0 & 0 \\ 0 & (1-\varepsilon)/4 & 0 & 0 \\ 0 & 0 & (1-\varepsilon)/4 & 0 \\ 0 & 0 & 0 & (1-\varepsilon)/4 \end{bmatrix}$$

For $\rho$ to be a density matrix:

1) $\rho \geq 0$ (positive semidefinite)
2) $\rho = \rho^T$ (Hermitian)
3) $\mathrm{Tr}[\rho] = 1$ (unit trace)

1) → $\rho \geq 0 \Rightarrow$ eigenvalues of $\rho$ should be $\geq 0$. In a diagonal matrix, diagonal elements are its eigenvalues.

So, for $\rho$, which is a diagonal matrix

$$\lambda_1 = \underbrace{(1+3\varepsilon)/4}_{\geq 1/4,\ \leq 1}, \quad \lambda_2 = \lambda_3 = \lambda_4 = \underbrace{(1-\varepsilon)/4}_{\geq 0,\ \leq 1/4}$$

$$\therefore \quad \varepsilon \in [0,1]$$

$\Rightarrow \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$

Hence, $\rho \geq 0$.

2) $\rho^{\dagger} = \begin{bmatrix} (1+3\varepsilon)/4 & 0 & 0 & 0 \\ 0 & (1-\varepsilon)/4 & 0 & 0 \\ 0 & 0 & (1-\varepsilon)/4 & 0 \\ 0 & 0 & 0 & (1-\varepsilon)/4 \end{bmatrix}$

$\Rightarrow \rho = \rho^{\dagger}$ (Hermitian)

3) $Tr[\rho] = \frac{(1+3\varepsilon)}{4} + \frac{(1-\varepsilon)}{4} + \frac{(1-\varepsilon)}{4} + \frac{(1-\varepsilon)}{4}$

$= \frac{1}{4}\left[4 + 3\varepsilon - 3\varepsilon\right] = 1$

$\Rightarrow Tr[\rho] = \underline{1}$ (unit trace)

Hence, $\rho$ defines a density matrix.

Ans. 4 Given: 1) states $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in the Hilbert space $C^2$

2) state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$ in the Hilbert space $C^4$.

3) $(\phi, \theta \in R)$ and,

$|\alpha\rangle = \cos\phi\,|0\rangle + \sin\phi\,|1\rangle$

$|\beta\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$

We are required to find the probability
$p(\phi, \theta) = |(\langle\alpha| \otimes \langle\beta|)|\psi\rangle|^2$

Solution:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix} - \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}\right)$$

$$|\alpha\rangle = \cos\phi\begin{bmatrix}1\\0\end{bmatrix} + \sin\phi\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\cos\phi\\\sin\phi\end{bmatrix}$$

$$|\beta\rangle = \cos\theta\begin{bmatrix}1\\0\end{bmatrix} + \sin\theta\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}$$

$$|\alpha\rangle \otimes |\beta\rangle = \begin{bmatrix}\cos\phi\\\sin\phi\end{bmatrix} \otimes \begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}$$

$$= \begin{bmatrix}\cos\phi\cos\theta\\\cos\phi\sin\theta\\\sin\phi\cos\theta\\\sin\phi\sin\theta\end{bmatrix}$$

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$$\langle\alpha| \otimes \langle\beta| = (|\alpha\rangle \otimes |\beta\rangle)^T$$

$$= [\cos\phi\cos\theta \quad \cos\phi\sin\theta \quad \sin\phi\cos\theta \quad \sin\phi\sin\theta]$$

$$(\langle\alpha| \otimes \langle\beta|)|\psi\rangle =$$

$$= \frac{1}{\sqrt{2}}[\cos\phi\cos\theta \quad \cos\phi\sin\theta \quad \sin\phi\cos\theta \quad \sin\phi\sin\theta]\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}[\cos\phi\sin\theta - \sin\phi\cos\theta]$$

$$= \frac{1}{\sqrt{2}}(\sin(\theta-\phi))$$

$$|(\langle\alpha| \otimes \langle\beta|)|\psi\rangle| = \left|\frac{1}{\sqrt{2}}\sin(\theta-\phi)\right|$$

$$|(\langle\alpha| \otimes \langle\beta|)|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\sin(\theta-\phi)\right|^2$$

$$= \boxed{\frac{1}{2}\sin^2(\theta-\phi) = p(\phi,\theta)}$$