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CS9.312 Introduction to Quantum Information and computation

Assignment-1

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Ans.2 :- Given → 1) $|\psi\rangle = |\phi_1\rangle + |\phi_2\rangle \Rightarrow \langle\psi| = \langle\phi_1| + \langle\phi_2|$
2) $|\chi\rangle = |\phi_1\rangle - |\phi_2\rangle \Rightarrow \langle\chi| = \langle\phi_1| - \langle\phi_2|$
3) $|\phi_1\rangle$ and $|\phi_2\rangle$ are not orthonormal

we are required to show that:

$$\langle\psi|\psi\rangle + \langle\chi|\chi\rangle = 2\langle\phi_1|\phi_1\rangle + 2\langle\phi_2|\phi_2\rangle$$

we know that $\langle\psi| = |\psi\rangle^\dagger$ — (1) (Transpose the ket vector This makes it a row vector and then complex conjugate the ket vector

Also, $|\psi\rangle \cdot |\psi\rangle = |\psi\rangle^\dagger |\psi\rangle = \langle\psi||\psi\rangle$ (from (1))

$= \langle\psi|\psi\rangle$

$\langle\psi|\psi\rangle = (\langle\phi_1| + \langle\phi_2|)(|\phi_1\rangle + |\phi_2\rangle)$ (from given (1))

$= \langle \phi_1 || \phi_1 \rangle + \langle \phi_1 || \phi_2 \rangle + \langle \phi_2 || \phi_1 \rangle + \langle \phi_2 || \phi_2 \rangle$

$= \langle \phi_1 | \phi_1 \rangle + \langle \phi_1 | \phi_2 \rangle + \langle \phi_2 | \phi_1 \rangle + \langle \phi_2 | \phi_2 \rangle \quad - (2)$

$$\langle \chi | \chi \rangle = (\langle \phi_1 | - \langle \phi_2 |)(|\phi_1\rangle - |\phi_2\rangle)$$

$$= (\langle \phi_1 || \phi_1 \rangle - \langle \phi_1 || \phi_2 \rangle - \langle \phi_2 || \phi_1 \rangle + \langle \phi_2 || \phi_2 \rangle)$$

$$= \langle \phi_1 | \phi_1 \rangle - \langle \phi_1 | \phi_2 \rangle - \langle \phi_2 | \phi_1 \rangle + \langle \phi_2 | \phi_2 \rangle \quad - (3)$$

Add the L.H.S and R.H.S of (2) and (3)

$$\langle \psi | \psi \rangle + \langle \chi | \chi \rangle = \langle \phi_1 | \phi_1 \rangle + \cancel{\langle \phi_1 | \phi_2 \rangle} + \cancel{\langle \phi_2 | \phi_1 \rangle} + \langle \phi_2 | \phi_2 \rangle + \langle \phi_1 | \phi_1 \rangle - \cancel{\langle \phi_1 | \phi_2 \rangle} - \cancel{\langle \phi_2 | \phi_1 \rangle} + \langle \phi_2 | \phi_2 \rangle$$

$$= \langle \phi_1 | \phi_1 \rangle + \langle \phi_1 | \phi_1 \rangle + \langle \phi_2 | \phi_2 \rangle + \langle \phi_2 | \phi_2 \rangle$$

$$= 2 \langle \phi_1 | \phi_1 \rangle + 2 \langle \phi_2 | \phi_2 \rangle$$

Hence, proved.

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Ans.4 <u>To Prove</u>: For a Hermitian operator ($\hat{A}^\dagger = \hat{A}$) all the eigenvalues are real

<u>Proof</u>: $\hat{H}$ is a Hermitian operator on an inner product space V over the field of complex numbers, C. That is, $\hat{H} = \hat{H}^\dagger$. Then, for an eigenvector $|x\rangle \in V$, $|x\rangle \neq |0\rangle$ and eigenvalue $\lambda \in C$.

$$\hat{H}|x\rangle = \lambda|x\rangle \quad - (1)$$

We know that $\forall\, |x\rangle, |y\rangle \in V$:

$$\langle x|\hat{A}|y\rangle = \langle y|\hat{A}^\dagger|x\rangle^* \quad - (3)$$

where * denotes the complex conjugate.

$\hat{H} = \hat{H}^\dagger$ gives: $\langle x|\hat{H}|y\rangle = \langle y|\hat{H}|x\rangle^*$

Now, $\langle x|\hat{H}|x\rangle = \langle x|(\hat{H}|x\rangle)$

$= \langle x|\lambda|x\rangle$ (from (1))

$= \lambda\langle x|x\rangle \quad - (2)$

$\langle x|\hat{H}|x\rangle = \langle x|\hat{H}^\dagger|x\rangle^*$ (from (3))

We know that $\hat{H} = \hat{H}^\dagger$

SO, $\langle x|\hat{H}|x\rangle = \langle x|\hat{H}|x\rangle^*$

Now, from (2)

$$\langle x|\hat{H}|x\rangle = \langle x|\hat{H}|x\rangle^* = (\lambda\langle x|x\rangle)^* \quad -(4)$$

Now, from (2) and (4)

$$\langle x|\hat{H}|x\rangle = \lambda\langle x|x\rangle = (\lambda\langle x|x\rangle)^*$$

$$\Rightarrow \lambda\langle x|x\rangle = (\lambda\langle x|x\rangle)^*$$

From the conjugate symmetry property of the inner product, we can see that:

$$\langle x|x\rangle = \langle x|x\rangle^*$$

which is true if and only if $\langle x|x\rangle \in R$

SO, $\lambda\langle x|x\rangle = \lambda^*\langle x|x\rangle$

$$\Rightarrow \lambda = \lambda^*$$

Therefore, $\lambda \in R$

To Prove: For a Hermitian operator ($\hat{A}^{\dagger} = \hat{A}$) the eigenvectors corresponding to different eigenvalues are orthogonal.

Proof: we know that $\hat{H}^{\dagger} = \hat{H}$ — (0)

Suppose,
$$H\vec{v}_1 = \mu_1 \vec{v}_1 \quad — (1)$$
$$H\vec{v}_2 = \mu_2 \vec{v}_2 \quad — (2)$$

with
$$H\vec{v} = \mu\vec{v},\ v \neq 0 \quad — (3)$$

we have,

$$\mu\langle v, v\rangle = \langle v, \mu v\rangle = \langle v, Hv\rangle = \langle H^{\dagger}v, v\rangle$$
$$= \langle Hv, v\rangle \text{ (from (0))} = \langle \mu v, v\rangle = \overline{\langle v, \mu v\rangle}$$
$$= \bar{\mu}\overline{\langle v, v\rangle} = \bar{\mu}\langle v, v\rangle \quad — (4)$$

(from the conjugate symmetry property of the inner product $\overline{\langle v, v\rangle} = \langle v, v\rangle$)

since $v \neq 0$, $\langle v, v\rangle \neq 0$ so we may divide it out of (4) and see that

$$\mu = \bar{\mu} \quad — (5)$$

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for any operator satisfying ⓪; this means that the eigenvalues of any operator satisfying ⓪ are real; therefore we may write

$$\mu_1 \langle v_1, v_2 \rangle = \langle \mu_1 v_1, v_2 \rangle = \langle H v_1, v_2 \rangle$$
$$= \underbrace{\langle v_1, H^{+} v_2 \rangle = \langle v_1, H v_2 \rangle}_{\text{from } ⓪} = \langle v_1, \mu_2 v_2 \rangle$$
$$= \mu_2 \langle v_1, v_2 \rangle \quad — ⑥$$

or

$$(\mu_1 - \mu_2) \langle v_1, v_2 \rangle = 0 \quad — ⑦$$

Assuming $\mu_1 \neq \mu_2$ (different eigenvalues as per question statement)

$$\langle v_1, v_2 \rangle = 0 \quad — ⑧$$

Thus, the vectors v_1, v_2 are orthogonal.

Ans. 1 Given: $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(a) eigenvalues of A:

First, we construct the matrix $A - \lambda I$

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$$A - \lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{pmatrix}$$

Then, we compute the determinant:

$$\det|A - \lambda I| = \det \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$= -\lambda(\lambda^2 - 1) - 1(-\lambda) = -\lambda^3 + \lambda + \lambda = 2\lambda - \lambda^3$$
$$= \lambda(2 - \lambda^2)$$

$$\Rightarrow \lambda = 0, \sqrt{2}, -\sqrt{2}$$

So, the eigenvalues of A are 0, $\sqrt{2}$ and $-\sqrt{2}$

Now, let us find eigenvectors corresponding to the eigenvalues 0, $\sqrt{2}$ and $-\sqrt{2}$

$\lambda = 0$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y \\ x+z \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow y = 0$

$\Rightarrow x+z = 0 \Rightarrow x = -z$

let $x = K \Rightarrow z = -K$

$y = 0$

$\underbrace{\text{Un-normalized}}_{\text{not-normalized}}$ eigenvector for $\lambda = 0$:

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} K \\ 0 \\ -K \end{bmatrix} = K \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Let us normalize the vector: To normalize the vector convert it to a "unit vector"

$$|X| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$$

$$\frac{X}{|X|} = K \begin{bmatrix} 1/\sqrt{2} \\ 0/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Divide the vector by scalar K.

$$= \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \quad \text{— } 1$$

$\lambda = \sqrt{2}$

$$\begin{bmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -\sqrt{2}x + y = 0 \quad \Rightarrow y = \sqrt{2}x$

$\Rightarrow x - \sqrt{2}y + z = 0$

$\Rightarrow y - \sqrt{2}z = 0 \quad \Rightarrow y = \sqrt{2}z$

Let $y = k \Rightarrow x = k/\sqrt{2}$

$\Rightarrow z = k/\sqrt{2}$

Not normalized eigenvector for $\lambda = \sqrt{2}$

$$Y = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} K/\sqrt{2} \\ K \\ K/\sqrt{2} \end{bmatrix} = K \begin{bmatrix} 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \end{bmatrix}$$

let us normalize the vector

$$|Y| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + (1)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + 1 + \frac{1}{2}}$$

$$= \sqrt{2}$$

$$\frac{Y}{|Y|} = K \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix}$$

Divide the vector by scalar K

$$\begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix} \quad - (2)$$

$\lambda = -\sqrt{2}$

$$\begin{bmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \sqrt{2}x + y = 0 \quad \Rightarrow y = -\sqrt{2}x$

$\Rightarrow x + \sqrt{2}y + z = 0$

$\Rightarrow y + \sqrt{2}z = 0 \quad \Rightarrow y = -\sqrt{2}z$

Let $y = K \Rightarrow x = -K/\sqrt{2}$

$\Rightarrow z = -K/\sqrt{2}$

Not normalized eigenvector for $\lambda = -\sqrt{2}$

$$Z = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -K/\sqrt{2} \\ K \\ -K/\sqrt{2} \end{bmatrix} = K \begin{bmatrix} -1/\sqrt{2} \\ 1 \\ -1/\sqrt{2} \end{bmatrix}$$

let us normalize the vector:

$$|Z| = \sqrt{\left(\frac{-1}{\sqrt{2}}\right)^2 + (1)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2} = \sqrt{2}$$

$$\frac{Z}{|Z|} = K \begin{bmatrix} -1/2 \\ 1/\sqrt{2} \\ -1/2 \end{bmatrix}$$

Divide the vector by scalar K:

$$\begin{bmatrix} -1/2 \\ 1/\sqrt{2} \\ -1/2 \end{bmatrix} \quad - ③$$

Normalized eigenvectors of A: (from ①, ②, ③)

$$\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/\sqrt{2} \\ -1/2 \end{bmatrix}.$$

$\{x_1, x_2, \ldots, x_n\}$ (in general) R^n

(b) Definition of orthonormal basis: (in our context)
A basis $B = \{x_1, x_2, x_3\}$ of R^3 will be an orthogonal basis if the elements of B are pairwise orthogonal that is $x_i \cdot x_j = 0 = 0$ whenever $i \neq j$. If in addition $x_i \cdot x_i = 1 \; \forall i$, then the basis is said to be an orthonormal basis

$|a_1\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \quad |a_2\rangle = \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix}, \quad |a_3\rangle = \begin{bmatrix} -1/2 \\ 1/\sqrt{2} \\ -1/2 \end{bmatrix}$

Let us first show that they form an orthonormal basis i.e $\langle a_j | a_k \rangle = \delta_{jk}$

$$\text{where } \delta_{jk} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

$$\langle a_1 | a_2 \rangle = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix}$$

$$= \left[\frac{1}{\sqrt{2}}\left(\frac{1}{2}\right) + 0\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{-1}{\sqrt{2}}\right) \right] = \left[\cancel{\frac{1}{2\sqrt{2}}} + 0 - \cancel{\frac{1}{2\sqrt{2}}} \right]$$

$$= [0] \quad — (1)$$

$$\langle a_2 | a_3 \rangle = \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/2 \end{bmatrix} \begin{bmatrix} -1/2 \\ 1/\sqrt{2} \\ -1/2 \end{bmatrix}$$

$$= \left[\frac{1}{2}\left(\frac{-1}{2}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{2}\left(\frac{1}{2}\right) \right]$$

$$= \left[\frac{-1}{4} + \frac{1}{2} - \frac{1}{4} \right] = \left[\frac{1}{2} - \frac{1}{2} \right]$$

$$= [0] \quad — (2)$$

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$$\langle a_1|a_3\rangle = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/2 \\ 1/\sqrt{2} \\ -1/2 \end{bmatrix}$$

$$= \left[\frac{1}{\sqrt{2}}\left(\frac{-1}{2}\right) + 0\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{2}\right)\right]$$

$$= \left[\cancel{\frac{-1}{2\sqrt{2}}} + 0 + \cancel{\frac{1}{2\sqrt{2}}}\right] = [0] \quad -③$$

$\therefore \langle a_1|a_2\rangle, \langle a_2|a_3\rangle, \langle a_1|a_3\rangle = 0$

Hence, the basis is an orthogonal basis. (from ①, ② and ③)

$$\langle a_1|a_1\rangle = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$= \left[\frac{1}{2} + 0 + \frac{1}{2}\right] = [1] \quad -④$$

$$\langle a_2|a_2\rangle = \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix}$$

$$= \left[\frac{1}{4} + \frac{1}{2} + \frac{1}{4}\right] = [1] \quad -⑤$$

$$\langle a_3|a_3\rangle = \begin{bmatrix} -1/2 & 1/\sqrt{2} & -1/2 \end{bmatrix} \begin{bmatrix} -1/2 \\ 1/\sqrt{2} \\ -1/2 \end{bmatrix}$$

$$= \left[\frac{1}{4} + \frac{1}{2} + \frac{1}{4}\right] = [1] \quad - ⑥$$

$\rightarrow$ from ④, ⑤, ⑥

$\therefore \langle a_1|a_1\rangle, \langle a_2|a_2\rangle, \langle a_3|a_3\rangle = 1$
Hence, the basis is an orthonormal basis

$\therefore \langle a_1|a_2\rangle, \langle a_2|a_3\rangle, \langle a_1|a_3\rangle = 0$
i.e $j \neq k$, so, $\delta_{jk} = 0$ and $\langle a_1|a_1\rangle$, $\langle a_2|a_2\rangle$, $\langle a_3|a_3\rangle = 1$ i.e $j = k$, so, $\delta_{jk} = 1$. Hence, proved that the eigenvectors denoted by $|a_1\rangle, |a_2\rangle, |a_3\rangle$ form an orthonormal basis.

Now, let's check if they form a complete basis, i.e $\sum_{j=1}^{3} |a_j\rangle\langle a_j| = I$

Now, we calculated $|a_1\rangle\langle a_1|$, $|a_2\rangle\langle a_2|$, $|a_3\rangle\langle a_3|$:

$$|a_1\rangle\langle a_1| = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix} \quad - ⑦$$

$$|a_2\rangle\langle a_2| = \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 & 1/2\sqrt{2} & 1/4 \\ 1/2\sqrt{2} & 1/2 & 1/2\sqrt{2} \\ 1/4 & 1/2\sqrt{2} & 1/4 \end{bmatrix} \quad -(8)$$

$$|a_3\rangle\langle a_3| = \begin{bmatrix} -1/2 \\ 1/\sqrt{2} \\ -1/2 \end{bmatrix} \begin{bmatrix} -1/2 & 1/\sqrt{2} & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 & -1/2\sqrt{2} & 1/4 \\ -1/2\sqrt{2} & 1/2 & -1/2\sqrt{2} \\ 1/4 & -1/2\sqrt{2} & 1/4 \end{bmatrix} \quad -(9)$$

Add equations (7), (8) and (9)

$$|a_1\rangle\langle a_1| + |a_2\rangle\langle a_2| + |a_3\rangle\langle a_3|$$

$$= \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix} + \begin{bmatrix} 1/4 & 1/2\sqrt{2} & 1/4 \\ 1/2\sqrt{2} & 1/2 & 1/2\sqrt{2} \\ 1/4 & 1/2\sqrt{2} & 1/4 \end{bmatrix}$$

$$+ \begin{bmatrix} 1/4 & -1/2\sqrt{2} & 1/4 \\ -1/2\sqrt{2} & 1/2 & -1/2\sqrt{2} \\ 1/4 & -1/2\sqrt{2} & 1/4 \end{bmatrix}$$

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$$= \begin{bmatrix} 1/2+1/4+1/4 & 0+1/2\sqrt{2}-1/2\sqrt{2} & -1/2+1/4+1/4 \\ 0+1/2\sqrt{2}-1/2\sqrt{2} & 0+1/2+1/2 & 0+1/2\sqrt{2}-1/2\sqrt{2} \\ -1/2+1/4+1/4 & 0+1/2\sqrt{2}-1/2\sqrt{2} & 1/2+1/4+1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, proved that they form a complete basis.

(c) Given: $|b_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|b_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $|b_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

We are required to find matrix U of the transformation from the basis $\{|a\rangle\}$ to $\{|b\rangle\}$

Suppose T is a linear transformation, $T: R^3 \to R^3$ and we are supposed to find matrix A of T such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$.

$$T\left(\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to A\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{bmatrix} 1/2+1/4+1/4 & 0+1/2\sqrt{2}-1/2\sqrt{2} & -1/2+1/4+1/4 \\ 0+1/2\sqrt{2}-1/2\sqrt{2} & 0+1/2+1/2 & 0+1/2\sqrt{2}-1/2\sqrt{2} \\ -1/2+1/4+1/4 & 0+1/2\sqrt{2}-1/2\sqrt{2} & 1/2+1/4+1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, proved that they form a complete basis.

cc) Given: $|b_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|b_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $|b_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

we are required to find matrix U of the transformation from the basis $\{|a\rangle\}$ to $\{|b\rangle\}$

suppose T is a linear transformation, $T: R^3 \to R^3$ and we are supposed to find matrix A of T such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$.

$$T\left(\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to A\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$T\left(\begin{bmatrix}1/2\\1/\sqrt{2}\\1/2\end{bmatrix}\right)=\begin{pmatrix}0\\1\\0\end{pmatrix}\rightarrow A\begin{bmatrix}1/2\\1/\sqrt{2}\\1/2\end{bmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$T\left(\begin{bmatrix}-1/2\\1/\sqrt{2}\\-1/2\end{bmatrix}\right)=\begin{pmatrix}0\\0\\1\end{pmatrix}\rightarrow A\begin{bmatrix}-1/2\\1/\sqrt{2}\\-1/2\end{bmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$A\overbrace{\begin{bmatrix}1/\sqrt{2} & 1/2 & -1/2\\0 & 1/\sqrt{2} & 1/\sqrt{2}\\-1/\sqrt{2} & 1/2 & -1/2\end{bmatrix}}^{B}=\overbrace{\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}}^{C}$$

$$AB=C \Rightarrow \underbrace{ABB^{-1}}_{\text{multiplied by } B^{-1} \text{ on both sides}}=CB^{-1} \Rightarrow A=CB^{-1}$$

$$\because C=I$$

$$\Rightarrow \boxed{A=B^{-1}}$$

$$\boxed{B^{-1}=\frac{adj(B)}{|B|}}$$

$$B = \begin{bmatrix} 1/\sqrt{2} & 1/2 & -1/2 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 & -1/2 \end{bmatrix}$$

$$|B| = \frac{1}{\sqrt{2}}\left(-\frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right) - \frac{1}{2}\left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right) = -\frac{1}{2} - \frac{1}{4} - \frac{1}{4} = -1$$

$$\operatorname{adj}(B) = \begin{bmatrix} +\left(\frac{-1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right) & -\frac{1}{2} & +\frac{1}{2} \\ -\left(\frac{-1}{4} + \frac{1}{4}\right) & +\left(\frac{-1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right) & -\left(\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\right) \\ +\left(\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\right) & -\frac{1}{2} & +\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} & -1/2 & 1/2 \\ 0 & -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/2 & 1/2 \end{bmatrix}$$

$$\text{adj}(B) = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/2 & -1/\sqrt{2} & -1/2 \\ 1/2 & -1/\sqrt{2} & 1/2 \end{bmatrix}$$

$$B^{-1} = \frac{\text{adj}(B)}{|B|} = -\text{adj}(B)$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} & 1/2 \\ -1/2 & 1/\sqrt{2} & -1/2 \end{bmatrix}$$

$A = CB^{-1}$ $(C = I)$, so, $A = B^{-1}$

Let $A = U$

Hence, the matrix U of the transformation from the basis $\{|a\rangle\}$ to $\{|b\rangle\}$ is:

$$U = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} & 1/2 \\ -1/2 & 1/\sqrt{2} & -1/2 \end{bmatrix}$$

Ans 3 Given: 1) The state $|\psi\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle + \frac{1}{\sqrt{10}}|\phi_3\rangle$

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2) $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ are orthonormal eigenstates of an operator $\hat{B}$.

3) $\hat{B}|\phi_n\rangle = n^2|\phi_n\rangle$

(a) we are required to check if $|\psi\rangle$ is normalized

If a state $|\psi\rangle$ is normalized then,

$$\langle\psi|\psi\rangle = \||\psi\rangle\|^2 = 1$$

$$\langle\psi|\psi\rangle = \left[\frac{1}{\sqrt{2}}\langle\phi_1| + \frac{1}{\sqrt{5}}\langle\phi_2| + \frac{1}{\sqrt{10}}\langle\phi_3|\right]^* \left[\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle + \frac{1}{\sqrt{10}}|\phi_3\rangle\right]$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2\langle\phi_1|\phi_1\rangle + \left(\frac{1}{\sqrt{5}}\right)^2\langle\phi_2|\phi_2\rangle + \left(\frac{1}{\sqrt{10}}\right)^2\langle\phi_3|\phi_3\rangle$$

$$= \frac{1}{2}(1) + \frac{1}{5}(1) + \frac{1}{10}(1) \quad \left(\because \langle\phi_1|\phi_1\rangle = 1, \langle\phi_2|\phi_2\rangle = 1, \langle\phi_3|\phi_3\rangle = 1\right)$$

$$= \frac{5+2+1}{10} = \frac{8}{10} = \frac{4}{5} \neq 1$$

where $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ are orthonormal eigenstates

Therefore, as $\langle\Psi|\Psi\rangle \neq 1$, $|\Psi\rangle$ is not normalized

(b) we are required to find the expectation value of $\hat{B}$ for the state $|\Psi\rangle$

Expectation value is defined as $\langle\hat{B}\rangle$

$$= \frac{\langle\Psi|\hat{B}|\Psi\rangle}{\langle\Psi|\Psi\rangle}$$

we calculated $\langle\Psi|\Psi\rangle$ in (a) which is $= \frac{4}{5}$

$$\hat{B}|\Psi\rangle = \hat{B}\left[\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle + \frac{1}{\sqrt{10}}|\phi_3\rangle\right]$$

$$= \frac{\hat{B}}{\sqrt{2}}|\phi_1\rangle + \frac{\hat{B}}{\sqrt{5}}|\phi_2\rangle + \frac{\hat{B}}{\sqrt{10}}|\phi_3\rangle$$

$$= \frac{(1)^2|\phi_1\rangle}{\sqrt{2}} + \frac{(2)^2|\phi_2\rangle}{\sqrt{5}} + \frac{(3)^2|\phi_3\rangle}{\sqrt{10}}$$

(from 3)

$$\hat{B}|\Psi\rangle = \frac{|\phi_1\rangle}{\sqrt{2}} + \frac{4}{\sqrt{5}}|\phi_2\rangle + \frac{9}{\sqrt{10}}|\phi_3\rangle$$

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$$\langle\psi|\hat{B}|\psi\rangle = \left(\frac{1}{\sqrt{2}}\langle\phi_1| + \frac{1}{\sqrt{5}}\langle\phi_2| + \frac{1}{\sqrt{10}}\langle\phi_3|\right)$$

$$\left(\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{4}{\sqrt{5}}|\phi_2\rangle + \frac{9}{\sqrt{10}}|\phi_3\rangle\right)$$

$$= \frac{1}{2}\langle\phi_1|\phi_1\rangle + \frac{4}{5}\langle\phi_2|\phi_2\rangle + \frac{9}{10}\langle\phi_3|\phi_3\rangle$$

$$= \frac{1}{2} + \frac{4}{5} + \frac{9}{10} = \frac{5+8+9}{10}$$

$$\frac{\langle\psi|\hat{B}|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{11\times5}{5\times4} = 2.75$$

(c) we are required to find the expectation value of $\hat{B}^2$ for the state $|\psi\rangle$

we calculate $\hat{B}|\psi\rangle$ for part (b), let's use it here

$$\hat{B}|\psi\rangle = \frac{|\phi_1\rangle}{\sqrt{2}} + \frac{4}{\sqrt{5}}|\phi_2\rangle + \frac{9}{\sqrt{10}}|\phi_3\rangle$$

multiplying L.H.S and R.H.S by $\hat{B}$

$$\hat{B}^2|\psi\rangle = \frac{\hat{B}|\phi_1\rangle}{\sqrt{2}} + \frac{4\hat{B}}{\sqrt{5}}|\phi_2\rangle + \frac{9\hat{B}}{\sqrt{10}}|\phi_3\rangle$$

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From given ③ $\hat{B}|\phi_n\rangle = n^2|\phi_n\rangle$, so, using that ①

$$\hat{B}^2|\Psi\rangle = \frac{(1)^2|\phi_1\rangle}{\sqrt{2}} + \frac{4\times4|\phi_2\rangle}{\sqrt{5}} + \frac{9\times(3)^2|\phi_3\rangle}{\sqrt{10}}$$

$$\hat{B}^2|\Psi\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{16}{\sqrt{5}}|\phi_2\rangle + \frac{81}{\sqrt{10}}|\phi_3\rangle$$

$$\langle\Psi|\hat{B}^2|\Psi\rangle = \left(\frac{1}{\sqrt{2}}\langle\phi_1| + \frac{1}{\sqrt{5}}\langle\phi_2| + \frac{1}{\sqrt{10}}\langle\phi_3|\right)$$

$$\left(\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{16}{\sqrt{5}}|\phi_2\rangle + \frac{81}{\sqrt{10}}|\phi_3\rangle\right)$$

$$= \frac{1}{2}\langle\phi_1|\phi_1\rangle + \frac{16}{5}\langle\phi_2|\phi_2\rangle + \frac{81}{10}\langle\phi_3|\phi_3\rangle$$

$$= \frac{1}{2} + \frac{16}{5} + \frac{81}{10} = \frac{5}{4}\left(\frac{5+32+81}{10}\right) = 11.8\left(\frac{5}{4}\right)$$

$$\frac{\langle\Psi|\hat{B}^2|\Psi\rangle}{\langle\Psi|\Psi\rangle} = 14.75$$