ECON 619
Answers to problem set 4 on article "Are People Bayesian?" by El Gamal and Grether

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I. This question asked you to read this article and formalize mathematically the application of Bayes rule in this problem for deciding which of two bingo cages, A or B, a given sample of 6 balls *drawn with replacement* came from. Then I asked you to characterize Bayes rule under a different sampling rule, namely, where the subjects observe a sample of *4 balls drawn from the chosen urn without replacement.* In both decision problems subjects were compensated \$10 each time they correctly guessed the urn from which the balls were drawn from. If a subject is risk neutral and Bayesian, they will choose the urn for which the posterior probability of the urn given the sample of balls is the highest. The prior probability of drawning the balls from urn A is π_A and equals either $1/3$, $1/2$ or $2/3$ in this experiment.

Bayes rule for 6 balls drawn with replacement It is easy to see that with replacement the probability of drawing a ball labelled N is $2/3$ if the ball was drawn from urn (or bingo cage) A, and $1/2$ if the ball was drawn from urn B since urn A has 4 N balls and 2 G balls, and urn B has 3 N balls and 3 G balls. Thus we can write

$$\begin{aligned} p(N|A) &= \frac{2}{3} \\ p(N|B) &= \frac{1}{2} \end{aligned} \tag{1}$$

Now, in a sample of 6 balls drawn with replacment it is easy to see that the only relevant information is n, the number of balls labelled N. The random variable n takes on the possible values $\{0,1,\dots,6\}$ in each case. The conditional probability of n given the urn is binomial but with probability either $p_A = P(N|A)$ or $p_B = P(N|B)$. So we can write

$$\begin{aligned} p(n|A) &= \binom{6}{n} p_A^n (1-p_A)^{6-n} \\ p(n|B) &= \binom{6}{n} p_B^n (1-p_B)^{6-n} \end{aligned} \tag{2}$$

Using *Bayes Rule* it follows that the posterior probabilities for urns A and B are given by

$$\begin{aligned} p(A|n) &= \frac{p(n|A)\pi_A}{p(n|A)\pi_A + p(n|B)(1-\pi_A)} \\ p(B|n) &= \frac{p(n|B)(1-\pi_A)}{p(n|A)\pi_A + p(n|B)(1-\pi_A)} = 1 - p(A|n). \end{aligned} \tag{3}$$

If the subject is Bayesian and wants to choose the urn for which the expected payoff is the highest, they will simply choose the urn with the highest posterior probability, since the expected payoff from choosing the urns are

$$\begin{aligned} E\{\text{payoff}|\text{choose urn A}, n\} &= 10 \times p(A|n) + 0 \times (1 - p(A|n)) \\ E\{\text{payoff}|\text{choose urn B}, n\} &= 10 \times p(B|n) + 0 \times (1 - p(B|n)) \end{aligned} \tag{4}$$

so clearly to maximize the expected payoff, the subject should choose urn A if $p(A|n) \geq p(B|n)$ and choose urn B otherwise. This is equivalent to choosing urn A if the *posterior odds ratio for urn A relative to urn B is greater than 1.* This is given by

$$\frac{p(A|n)}{p(B|n)} = \left(\frac{p_A}{p_B}\right)^n \left(\frac{1-p_A}{1-p_B}\right)^{6-n} \frac{\pi_A}{1-\pi_A} \tag{5}$$

Equivalently, since $P(B|n) = 1 - P(A|n)$, the decision rule is to choose urn A when $P(A|n) \geq \frac{1}{2}$.

To be a bit more rigorous about this, we want to prove that the decision rule given above maximizes the expected payoff to the decision maker over all possible decision rules. Define the *information* provided to the decision maker as $x = (n, \pi)$ where n is the number of balls labelled N in the sample of 6 balls drawn (with replacement) from the chosen urn, and π is the probability that the sample was drawn from urn A (whose value was communicated to the subjects, though not the realization of the draw from the "prior cage" so the subjects did not observe which of urns A or B the sample of 6 balls was drawn from).

Let $\delta(x)$ be a possible decision rule which maps the possible realizations of x into the set $\{A, B\}$, where $\delta(x) = A$ denotes the choice of urn A and $\delta(x) = B$ denotes the choice of urn B. Let $\tilde{U}$ denote the urn that the sample was actually drawn from, so that $\tilde{U} = A$ denotes the event that the sample was drawn from urn A and $\tilde{U} = B$ when the sample was drawn from urn B. If the reward for choosing the urn correctly is R, the expected payoff from using decision rule δ is given by

$$RE\left\{I\{\delta(\tilde{x}) = A\}I\{\tilde{U} = A\} + I\{\delta(\tilde{x}) = B\}I\{\tilde{U} = B\}\right\}. \tag{6}$$

Using the Law of Iterated Expectations, we can write the expectation above as the expectation of the conditional expectation given $\tilde{x}$

$$E\left\{I\{\delta(\tilde{x}) = A\}I\{\tilde{U} = A\} + I\{\delta(\tilde{x}) = B\}I\{\tilde{U} = B\}|\tilde{x} = x\right\}. \tag{7}$$

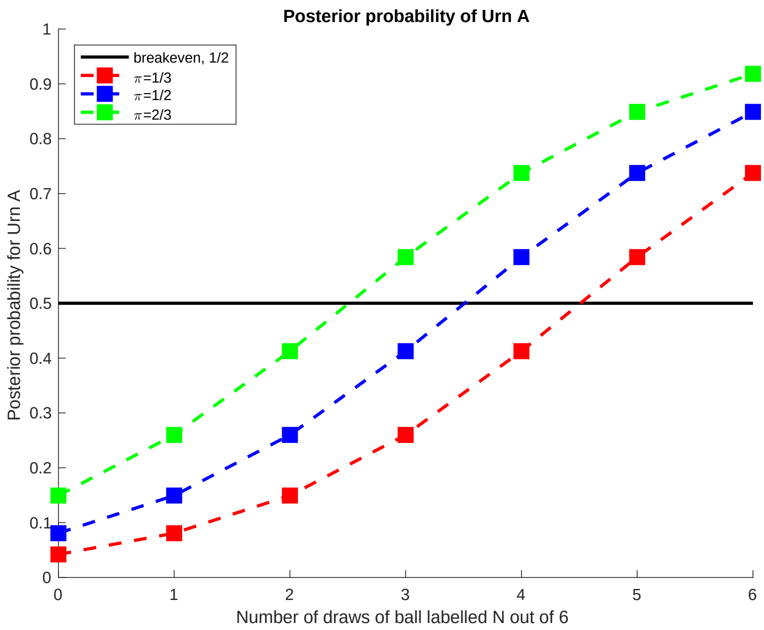
Let $P(A|x) = E\{I\{\tilde{U} = A\}|x\}$ be the conditional expectation of the indicator for the event that urn A was used to draw the sample of 6 balls. Similarly let $P(B|x)$ be the conditional expectation of the Bernoulli random variable $I\{\tilde{U} = B\}$. It is easy to see that these conditional probabilities are simply the probabilities we have already derived above using Bayes Rule, (3). So we have

$$
\begin{aligned}
& E\left\{I\{\delta(\tilde{x}) = A\}I\{\tilde{U} = A\} + I\{\delta(\tilde{x}) = B\}I\{\tilde{U} = B\}|\tilde{x} = x\right\} && (8) \\
= \; & I\{\delta(x) = A\}P(A|x) + I\{\delta(x) = B\}P(B|x) && (9) \\
= \; & I\{\delta(x) = A\}P(A|x) + I\{\delta(x) = B\}[1 - P(A|x)] && (10)
\end{aligned}
$$

It is easy to see that the optimal decision rule δ is the one that maximizes the conditional expectation above for each x, and thus $\delta(x) = A$ if $P(A|x) \geq P(B|x)$ and $\delta(x) = B$ otherwise. However this is equivalent to choosing urn A when the posterior probability is greater than $1/2$ since $P(B|x) = 1 - P(A|x)$, so we can write the optimal decision rule as

$$
\delta(x) = I\left\{P(A|x) \geq \frac{1}{2}\right\}. \tag{11}
$$

Note the important distinction between *objective probability* and *subjective probability* in this case. We ordinarily think of a Bayesian posterior to be a "subjective probability distribution" that reflects the subjective prior distribution that can differ from person to person. It is part of the Leonard Savage "subjective expected utility" view of choice under uncertainty, where people have subjective probability distributions that may not be the same as the "objective probability measure" (i.e. "true probability distribution") that is the "data generating mechanism" describing how observations/data are actually generated. However what the argument above shows if that the Bayesian posterior can in fact be an *objective probability distribution* when the prior that the decision maker has over the probability of drawing from urn A versus urn B is in fact the "true probability". This is what the El-Gamal and Grether experiment carefully tried to achieve using the device of the "prior cage" — i.e. to insure that subjects' subjective priors for the two urns coincides with the objective probability distribution for which urn was chosen to draw the sample of balls. In other branches of economics we also refer to this as *rational expectations,* i.e. the assumption that subjects are expected utility maximizer (or expected value maximizers, when they are risk-neutral) and their subjective beliefs coincide with the true probability distribution governing the uncertain outcomes (which technically is considered a choice under *risk* versus a choice under *uncertainty* since choices under risk are



those when the decision maker knows the true probability distribution governing the random payoff relevant outcomes, but choice under uncertainty is reserved for situation where there is no clear "objective probability measure" for events that everyone knows (i.e. is common knowledge), and thus different smart, otherwise rational people, may have different probability assessments for the same event (e.g. what is the probability Donald Trump will be impeached?). El-Gamal and Grether tried hard to take the subjective belief aspect out of the equation by using the prior cage drawing to create common knowledge of the prior probability of choosing each urn, thus their experiment is regarded as one of choice under risk rather than choice under uncertainty. What we showed above, with such rational expectations the Bayesian posterior belief is the optimal belief to have, i.e. it results in an optimal decision rule that maximizes the objective expected payoff from participating in the experiment. However if subjects are not fully rational, or do not fully understand probability theory, it may be that their *subjective belief* about which urn the sample was drawn from differs from the objectively rational belief, which is the Bayesian posterior that we have calculated above.

Using Matlab (source code provided in the `code` section of the Econ 612 website) I calculated the posterior odds ratios for each possible value of $n \in \{0, 1, \ldots, 6\}$ and plotted them below for the three different values of the prior $\pi_A \in \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$ in figure 1.

We see from figure 1 that when $\pi_A = \frac{1}{3}$ a Bayesian decision maker will select urn A if $n > 4$ since $P(A|n)$ is below $\frac{1}{2}$ for $n \in \{0,1,2,3,4\}$ (indicated by the horizontal black line at $p = 1/2$ in figure 1). When $\pi_A = \frac{1}{2}$, the Bayesian decision maker will choose urn A if $n > 3$, and this is because the decision maker factors in their prior belief that there is a 50/50 chance that the sample could have been drawn from urn A. Thus, even though a sample of with 3 balls marked N is more similar to urn B than to urn A (since urn B has 3 N balls and 3 G balls but urn A has 4 N balls and 2 G balls), the Bayesian decision maker still chooses urn A because of their prior belief that urn A would be chosen with probability $\frac{1}{2}$ as the urn from which the sample was drawn. When $\pi_A = \frac{2}{3}$, the decision maker has better than 50/50 confidence that the urn from which the sample was drawn was urn A and now the decision maker will choose urn A if $n > 2$. This indicates how the decisions of a Bayesian decision maker depends both on his/her prior beliefs and the data they observe. We will show below, if the subjects would have been shown more than 6 *IID* draws from each urn, the data would have greater weight than their prior beliefs and in the limit as the number of draws from an urn tends to ∞, the subjects should all of the weight on the data and none on the prior. For example, let M be the total number of balls drawn with replacement from the urn. Using a bit of algebra, you can show that the Bayesian decision maker bases his/her decision on the *sample mean* $\hat{p}_M = n/M$, and chooses urn A if

$$\frac{n}{M} \geq \frac{\log\left(\frac{1-p_B}{1-p_A}\right) + \log\left(\frac{(1-\pi_A)}{\pi_A}\right)/M}{\log\left(\frac{p_A}{1-p_A}\right) - \log\left(\frac{p_B}{1-p_B}\right)}. \tag{12}$$

Notice that the weight on the prior log-odds ratio, $\log(1-\pi_A/p_A))$ goes to 0 as $M \to \infty$ so the decision maker, asymptotically, simply compares the maximum likelihood probability estimate of the number of N balls in the sampled urn, n/M, to a threshold value that depends on p_A and p_B. If this estimated fraction is sufficiently high, then the decision maker chooses urn A (the urn with $p_A = 2/3$ N balls), otherwise the decision maker chooses urn B. This implies that as $M \to \infty$, the decision maker learns with certainty what the true probability of getting an N ball is, and so never makes a mistake. Evaluating the right hand side threshold in the equation above as $M \to \infty$ we see that the decision maker chooses urn A if $n/M \geq .5851$. Note that this differs from a naive cutoff that is halfway between the probability of drawing an N ball from urn A, $p_A = 2/3$, and the probability of drawing an N ball from urn B, $p_B = .5$: if you use naive Bayesian reasoning and your prior probability of drawing from urn A is $\pi_A = .5$, you might expect the cutoff to be equal to

$\pi_A\pi_A + (1-\pi_A)(1-p_A) = .5833$ and we see that the optimal cutoff based on the value of n such that $P(A|n) = 1/2$ is actually somewhat higher, and is equivalent to $n/M \simeq .5851$. In large samples the Law of Large Numbers implies that this cutoff converges to $p \geq .5851$ (which is the limit of the right hand side of inequality (12) above as $M \to \infty$), and the Bayesian posterior probability $P(A|n)$ converges with probability 1 to 1 if urn A was the urn that was used to draw a sample of size M or 0 if urn B was used to draw the sample of size M. That is, the Bayesian posterior is *consistent* and selects the actual urn used with probability 1 as $M \to \infty$. We can see that $n/M \to p_A$ with probability 1 if urn A was used, and thus with probability 1, for sufficiently large M we have $n/M \to p_A = 2/3 > .5851$ so the Bayesian posterior correctly selects urn A when it was used to draw the sample, and in the case where urn B was used to draw the sample $n/M \to p_B = 1/2 < .5851$ and thus with very high probability the decision maker will choose urn B in this case. That is, in large samples, a Bayesian decision maker will make very few mistakes in selecting the correct urn.

As an additional excerise, calculate the misclassification probability for the Bayesian decision maker and show that it converges to zero as $M \to \infty$. This is equivalent to proving that the Bayesian posterior probability $P(A|n)$ is *consistent* i.e. it converges to either 1 or 0 depending on whether urn A or urn B was used to draw the sample. (**Hint:** use the fact that $\tilde{n}$, the number of N balls, is binomial with parameters (M, p) where p is the true fraction of N balls in the urn.)

Bayes rule for 4 balls drawn without replacement This problem is more difficult because it is easy to see in this case that the draws from the chosen urn are no longer *IID* — independent and identically distributed. Instead the probability of drawning a ball marked N depends on the outcomes of previous draws and these probabilities are *sample dependent.* Unlike the case of sampling with replacement, the probability of drawing a ball marked N in each successive drawn from the urn changes based on the results of previous draws. In the case of sampling with replacement, the probabilities of these two sequences are the same, conditional on the urn from which they were drawn, because the sampling is *IID* in this case. However it turns out that for any given sample drawn from an urn without replacement, the probability of that sequence only depends on the number of N balls and G balls in the sequence, but not on the ordering of the N and G balls in the sample. The probability of getting n balls marked N and $4-n$ balls marked G when we do not care about the order in which the n N balls and $4-n$ G balls are drawn is given by a *hypergeometric distribution.* For the case of urn A the possible values of n are $n \in \{2,3,4\}$ and the probabilities of these three possible values of n are given by

$$p(n|A) = \frac{\binom{4}{n}\binom{2}{4-n}}{\binom{6}{4}} \tag{13}$$

and in the case of drawing from urn B, the possible values of n (i.e. the *support* of the distribution) is $n \in \{1,2,3\}$ and we have

$$p(n|B) = \frac{\binom{3}{n}\binom{3}{4-n}}{\binom{6}{4}}. \tag{14}$$

We could use the hypergeometric formulas above and substitute them into the formula for Bayes rule above to derive the posterior distribution $p(A|n)$, but I found it helpful to exhaustively write down the probabilities of the $2^4 = 16$ possible sequences of 4 sampled balls drawn from one or the other urns without replacement to verify that the probability of different sampled sequences only depends on the number n and not on the order of the N and G balls in any sequence that has n N balls and $4-n$ G balls.

$$\begin{aligned}
p_A(G,G,G,G) &= 0 \\
p_A(N,G,G,G) &= 0 \\
p_A(G,N,G,G) &= 0 \\
p_A(G,G,N,G) &= 0 \\
p_A(G,G,G,N) &= 0 \\
p_A(G,G,N,N) &= \frac{2}{6}\times\frac{1}{5}\times\frac{4}{4}\times\frac{3}{3} \\
p_A(G,N,G,N) &= \frac{2}{6}\times\frac{4}{5}\times\frac{1}{4}\times\frac{3}{3} \\
p_A(G,N,N,G) &= \frac{2}{6}\times\frac{4}{5}\times\frac{3}{4}\times\frac{1}{3} \\
p_A(N,G,N,N) &= \frac{4}{6}\times\frac{2}{5}\times\frac{3}{4}\times\frac{2}{3} \\
p_A(G,N,N,N) &= \frac{2}{6}\times\frac{4}{5}\times\frac{3}{4}\times\frac{2}{3} \\
p_A(N,N,N,G) &= \frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{2}{3} \\
p_A(N,N,N,N) &= \frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3} \\
p_A(N,N,G,G) &= \frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3} \\
p_A(N,G,N,G) &= \frac{4}{6}\times\frac{2}{5}\times\frac{3}{4}\times\frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
p_A(N,G,G,N) &= \frac{4}{6}\times\frac{2}{5}\times\frac{1}{4}\times\frac{3}{3}\\
p_A(N,N,G,N) &= \frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{2}{3}
\end{aligned} \tag{15}$$

The probabilities of the 16 possible sequences of 4 ball samples from urn B are given below

$$\begin{aligned}
p_B(G,G,G,G) &= 0\\
p_B(N,G,G,G) &= \frac{3}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}\\
p_B(G,N,G,G) &= \frac{3}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}\\
p_B(G,G,N,G) &= \frac{3}{6}\times\frac{2}{5}\times\frac{3}{4}\times\frac{1}{3}\\
p_B(G,G,G,N) &= \frac{3}{6}\times\frac{2}{5}\times\frac{1}{4}\times\frac{3}{3}\\
p_B(G,G,N,N) &= \frac{3}{6}\times\frac{2}{5}\times\frac{1}{4}\times\frac{2}{3}\\
p_B(G,N,G,N) &= \frac{3}{6}\times\frac{3}{5}\times\frac{3}{4}\times\frac{2}{3}\\
p_B(G,N,N,G) &= \frac{3}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{2}{3}\\
p_B(N,G,N,N) &= \frac{3}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}\\
p_B(G,N,N,N) &= \frac{3}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}\\
p_B(N,N,N,G) &= \frac{3}{6}\times\frac{2}{5}\times\frac{1}{4}\times\frac{3}{3}\\
p_B(N,N,N,N) &= 0\\
p_B(N,N,G,G) &= \frac{3}{6}\times\frac{2}{5}\times\frac{3}{4}\times\frac{2}{3}\\
p_B(N,G,N,G) &= \frac{3}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{2}{3}\\
p_B(N,G,G,N) &= \frac{3}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{2}{3}\\
p_B(N,N,G,N) &= \frac{3}{6}\times\frac{2}{5}\times\frac{3}{4}\times\frac{1}{3}
\end{aligned} \tag{16}$$

It should be easy for you to check these various probabilities and how I derived them. For example it is easy to see that $p_B(N,N,N,N) = 0$ because urn B only has 3 N balls so it is impossible to draw a sample of 4 N balls when drawing without replacement. For the same reason it is easy to see that $P_B(G,G,G,G) = 0$ too. Similarly for urn A, it is not possible to draw a sample with 3 or more G balls when sampling without replacement because this urn only has 2 G balls. The other positive probabilities can be derived by thinking of the probability of the samle as a product of conditional probabilities. For example to see how I derived $p_B(N,G,G,G) = \frac{3}{6}\times\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}$, note that for the first draw, this urn has 3 N balls and 3 G balls so there is a probability of $\frac{1}{2} = \frac{3}{6}$ that the first ball drawn without replacement is labelled N (in fact it is easy to see that there is always $\frac{1}{2}$ probability of drawning the first ball as an N or G ball). Then the urn has only 5 balls after taking out this first N ball, so on the 2nd draw the urn has 3 G balls and 2 N balls, and thus the probability that the 2nd ball drawn is G is $\frac{3}{5}$, and so on.

Table 1 below displays the posterior odds ratios in this case for the three prior probabilities $\pi_A \in \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$. We put NA in the cell for the sample (G,G,G,G) because it cannot happen regardless of which urn is chosen as the urn to draw the sample from. We can see that if the sample is (N,N,N,N) then the posterior odds ratio is ∞, i.e. we can conclude with certainty that the sample was drawn from urn A since there is zero probability of drawing a sample of 4 N balls without replacement from urn B which has 3 N and 3 G balls. Conversely, we are also certain that samples with 3 G balls could not have been drawn without replacement from urn A since it has only 2 G balls. In the remaining cells of the table, we see the posterior odds lead the decision maker to choose urn A under the prior probability $\pi_A = \frac{1}{3}$ when there are 3 or more N balls in the sample. When the prior probability of drawing from urn A increase to $\pi_A = \frac{1}{2}$, then the decision maker will choose urn A if the sample has 2 or more N balls in it, and when $\pi_A = \frac{2}{3}$, the decision maker has the same decision rule as for the prior $\pi_A = \frac{1}{2}$: the main difference is that the latter decision maker is more confident about their decision in the sense that their posterior odds ratio for urn B is twice as high in the samples where the decision maker chooses urn A.

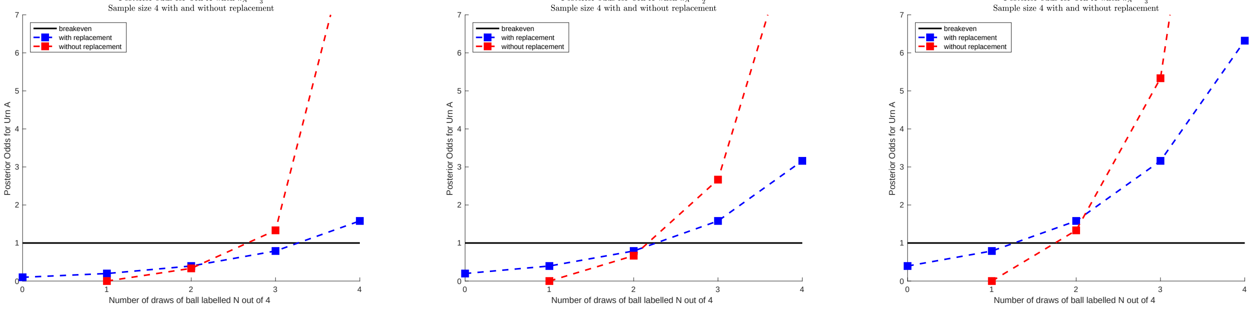
We can now conclude from this table how sampling without replacement and the difference in sample size affects the decision rule. In the case of the smallest prior probability that the sample would be drawn from urn A, $\pi_A = \frac{1}{3}$, we see that when the sample size is 4 and the balls are drawn without replacement the optimal decision is to choose urn A if the number of N balls is 3 or 4, whereas if the balls are sampled with replacement, then the decision maker will also choose urn A only when all 4 draws are N. If the prior probability is $\pi_A = \frac{1}{2}$ then the decision maker chooses urn A if the sample has 3 or 4 balls labelled N, regardless of whether the sampling is done with or without replacement. Finally, when the prior probability of choosing urn A is $\pi_A = \frac{2}{3}$, then the decision maker chooses urn A if there are 2 or more balls marked N, again regardless of whether the sampling is done with or without replacement. Thus, sampling without replacement only affects the cutoff threshold when the prior probability π_A is sufficiently low.

Of course there are two different things going on at the same time here: the sample size is changing

Table 1: Posterior Odds Ratios for urn A, samples of 4 draws without replacement

Sample	$\pi_A = \frac{1}{3}$	$\pi_A = \frac{1}{2}$	$\pi_A = \frac{2}{3}$
(G,G,G,G)	NA	NA	NA
(N,G,G,G)	0	0	0
(G,N,G,G)	0	0	0
(G,G,N,G)	0	0	0
(G,G,G,N)	0	0	0
(G,G,N,N)	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{4}{3}$
(G,N,G,N)	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{4}{3}$
(G,N,N,G)	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{4}{3}$
(N,N,G,G)	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{4}{3}$
(N,G,N,G)	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{4}{3}$
(N,G,G,N)	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{4}{3}$
(N,N,G,N)	$\frac{4}{3}$	$\frac{8}{3}$	$\frac{16}{3}$
(N,G,N,N)	$\frac{4}{3}$	$\frac{8}{3}$	$\frac{16}{3}$
(G,N,N,N)	$\frac{4}{3}$	$\frac{8}{3}$	$\frac{16}{3}$
(N,N,N,G)	$\frac{4}{3}$	$\frac{8}{3}$	$\frac{16}{3}$
(N,N,N,N)	∞	∞	∞

Figure 1: Effect of Sampling Method (with or without replacement) on decision rules



(4 observations versus 6 in the original experiment) and sampling with and without replacement. Figure 2 allows us to compare the effect of the sampling method for a sample size of 4, and comparing figures 1 and 2, we can see the effect of sample size when the sampling is done with replacement. With a sample size of 6 the thresholds for choosing urn A are 4, 3, 2, respectively as $\pi_A = 1/3$, $\pi_A = 1/2$ and $\pi_A = 2/3$, respectively. With a sample size of 4 figure 2 shows the threholds are 3, 2, 1, respectively. As we note above, when sampling is done without replacement, it is impossible to get a sample with $n = 0$ N balls from urn B, so the posterior odds ratio is undefined for this value of n and we do not plot it in the figure. Also when $n = 4$, the posterior odds ratio is infinity when sampling is done without replacement, since we are certain that such a sample can only have come from urn A. But for the other values $n \in \{1,2,3\}$ we can compare the two posterior odds ratios for different values of the prior probability π_A.

In the remainder of this answer, given the (long) answer to the first part of the question, we will now focus on answering the questions below.

A. In their paper, the find that subject pools who were paid on a flat fee basis had smaller proportions that were estimated to be following Bayesian decision rules compared to subject pools who were paid a bonus of $10 for choosing the correct cage. Can you show that of all possible decision rules a subject might adopt for choosing between the two cages, that one based on Bayes rule maximizes the subject's expected payments when they are payed a bonus for selecting the correct cage?

answer We already answered this above, see equation (11) above.

B. Can you formulate an alternative econometric model than the one El-Gamal and Grether formulated that reflects the payment scheme and might be able to predict their finding that paying subjects a bonus for selecting the right cage resulted in a higher fraction of subjects adopting the Bayesian decision rule?

answer We can think of subjects following a binary choice model in the McFadden extreme value/logit tradition that reflects the rewards to answering correctly and the subject's *subjective probability* that urn A is the "correct" urn (i.e. the one used to draw the sample). Let $x = (n, \pi)$ be the information that the subject has when making their choice and let $P_s(A|x)$ be the (as yet to be defined) subjective probability that urn A is the correct urn, so $1 - P_s(A|x)$ is the probability that urn B is the correct urn. If we add extreme value shocks that reflect "distractions" that subjects experience in choosing an urn, then we can write a binary logit model $P(A|x)$ to represent the probability that a subject selects urn A, as

$$P(A|x) = \frac{\exp\{RP_s(A|x)/\sigma\}}{\exp\{RP_s(A|x)/\sigma\} + \exp\{R(1 - P_s(A|x)/\sigma\}} \tag{17}$$

$$= \frac{1}{1 + \exp\{R[1 - 2P_s(A|x)]/\sigma\}} \tag{18}$$

If we assume that the subject is a rational Bayesian decision maker, then $P_s(A|x)$ equals the posterior probability derived from Bayes Rule in equation (3) above. Then as $\sigma \downarrow 0$, it is not hard to show that the choice (18) converges to the optimal Bayesian decision rule (11) from part A. But when there is "noise" represented by $\sigma > 0$ in the choice problem there will be random deviations but unlike the random 50/50 deviations that El-Gamal and Grether contemplated, the effect of such random shocks will be greatest when $P_s(A|x)$ is close to $1/2$ but will have little effect when $P_s(A|x)$ is close to 1 or 0 when the subject is pretty sure that the sample was drawn from either urn A or urn B, respectively. The El-Gamal and Grether model does not reflect the implication that the effect of the random errors should be much smaller when the subject is pretty sure about which urn the sample came from (as reflected by the case where $P_s(A|x)$ is close to 0 or 1, versus the "uncertain" case where $P_s(A|x)$ is close to $1/2$), and thus when the subject is pretty certain the choice probability will approach either $P(A|x) = 1/(1+\exp\{R/\sigma\})$ as $P_s(A|x)$ is close to 0 (so the subject chooses urn B with high probability), or $P(A|x) = 1/(1+\exp\{-R/\sigma\}$ when $P_s(A|x)$ is close to 1, so the subject is predicted to choose urn A with probability also close to 1.

The logit model can also explain why subjects were more likely to behave according to Bayes rule and have lower estimated "error rates" when the subjects were paid $R = 10$ dollars for correctly selecting the urn on each experiment trial, compared to the case that the subjects were paid on a flat rate per trial regardless of whether they selected the correct urn or not. In the logit choice probability (18) we see that the probabilities are dependent on the reward R and when $R = 0$ the subjects may well flip a coin since in that case $P(A|x) = 1/2$ for all x. This prediction may be too extreme in the other direction since El-Gamal and Grether did not find that subjects behaved completely randomly under the flat-rate payment treatment. Here the logit model can also help to explain this finding as well. If we reinterpret R to be the dollar equivalent value of the subject's "internal pride from correctly choosing the urn", then the subject may receive an internal psychological reward for choosing correctly even if there is no explicit financial reward for doing so.

What specification would be natural for the subjective probability $P_s(A|x)$? We would like a specification that nests Bayes Rule (3) as a special case. We can again use our friend the logit model to write the Bayesian posterior formula for $P(A|x) = P(A|n,\pi_A)$ in equation (3) as follows

$$P(A|n,\pi_A) = \frac{1}{1+\exp\{a+bn+c[\log(1-\pi_A)-\log(\pi_A)]\}} \tag{19}$$

where

$$\begin{aligned} a &= 6[\log(1-p_B)-\log(1-p_A)] && (20)\\ b &= \log\left(\frac{p_B}{1-p_B}\right)-\log\left(\frac{p_A}{1-p_A}\right) && (21)\\ c &= 1 && (22) \end{aligned}$$

By varying the coefficients (a,b,c) in the logit specification for the posterior probability $P(A|n,\pi_A)$ we obtain a candidate parametric family for the subjects' subjective probabilities $P_s(A|n,\pi_A)$ that includes Bayes Rule (3) as a special case. We can also consider estimating richer specifications that include quadratic terms in n or flexible specifications where we estimate coefficients on dummy variables for the 7 possible values of n. For example individuals who discount or ignore the prior information may have subjective probabilities that can be approximated with a small or zero value for c. Other individuals who over or under-react to the number n of N balls in the sample. For a Bayesian with $p_B = 1/2$ and $p_A = 2/3$ we have $b = -\log(2)$ and an individual who overreacts to the sample may have a value of b that is less than $-\log(2)$. Finally the constant term a should equal $6[\log(3)-\log(2)]$, and values that are higher than this imply an individual who is systematically underestimating the probability that the balls are drawn from urn A and vice versa for a value of a that is less than this.

This model can be further extended to include a "cost of effort" that a subject may expend to do more involved calculations to calculate the posterior probability, including "thinking harder" to produce a more refined subjective probability assessment $P_s(A|n,\pi_A)$. We could imagine a subject making an initial choice about whether to form an easy "snap judgement" about which urn the sample came from, versus investing more energy and effort to come up with a more refined subjective probability estimate. This "upper level" cost/benefit decision would trade off the expected gains in terms of payoff from expending more effort to produce a more accurate estimate of the probability a given sample comes from urn A, versus the effort involved in doing that. Suppose we let e denote the monetary equivalent of this cost of effort. We treat this as a fixed investment rather than a continuous choice by the subject. The expected benefit of incurring the effort cost e to produce a more informed subjective probability assessment $P_s(A|n,\pi_A)$, allowing for the extreme value shocks, can be written as

$$E\{U|e>0\} = \sigma \log\left(\exp\{R(P_s(A|n,\pi_A)/\sigma\} + \exp\{R(1-P_s(A|n,\pi_A))/\sigma\}\right) - e. \tag{23}$$

We can consider a much easier psychological calculation where the subject ignores the information about the prior probability π_A and makes a decision based only on n, such as a rule of the form: "choose urn A if $n/6 \geq a$ where a is a threshold value such as $a = 1/2$ or $a = .58$, etc. We can provide a following "smooth approximation" to this simple type of threshold rule using the probability $P_s(A|n)$ given by

$$P_s(A|n) = \frac{1}{1+\exp\left\{\left(\frac{n}{6}-a\right)/\sigma\right\}}. \tag{24}$$

Assume that the effort is zero for using a simple threshold rule as a basis for the subjective probability such as the formula above. Then the expected payoff from exerting no or little effort to make a decision is

$$E\{U|e=0\} = \sigma \log\left(\exp\{R(P_s(A|n)/\sigma\} + \exp\{R(1-P_s(A|n))/\sigma\}\right). \tag{25}$$

Consider a subject making a first stage decision about whether to exert effort and incorporate the prior information π_a to produce a better estimate of $P(A|n,\pi_A)$ that depends on the full information the subject receives, (n,π_A), versus a simpler easier decision that ignores π_A and is based on a simple threshold rule as described above. This can be represented by a binary logit model with probability of exerting effort $P(e>0)$ to make a more informed decision, given by

$$P(e>0) = \frac{\exp\{E\{U|e>0\}/\sigma_0\}}{\exp\{E\{U|e=0\}/\sigma_0\} + \exp\{E\{U|e>0\}/\sigma_0\}}. \tag{26}$$

where σ_0 is an upper level extreme-value scale parameter. This model can be viewed as a nested logit specification where the upper level represents the binary choice of whether or not to exert effort to to produce a better estimate of the probability that the sample is from urn A. Given this upper level choice, the lower level choice is to decide whether to say the sample is from urn A or urn B. Though we do not actually observe the upper level choice, it may be possible to estimate the parameters of this model including the effort cost e by using a marginal likelihood that "integrates out" the unobserved upper level choice of effort.

But note the huge paradox here! To calculate whether it is worth incurring the effort e to form a more informed probability estimate, the individual would actually have to do it! So the *ex ante* choice of whether to expend effort must itself come from some expectation of what the gains are without actually undertaking the effort to think about the problem. I will not pursue this avenue further, since it starts to reveal some of the paradoxes of rationality that we will ultimately have to confront in order to provide deeper theories of how people actually learn, think, make decisions, etc.

C. In their 1995 paper, El-Gamal and Grether assumed that the *error probability* ε was the same for all subjects. Why did they make this assumption? Is it a plausible assumption? Can you relax this assumption in their model and estimate *subject-specific error rates* ε_i? What sort of econometric problems might arise if you try to do this?

answer See their likelihood function equation on the top left hand column of page 1140:

$$f^{c,s}(x_1^s,\ldots,x_{t_s}^s) = \left(1-\frac{\varepsilon}{2}\right)^{X_c^s}\left(\frac{\varepsilon}{2}\right)^{t_s-X_c^s} \tag{27}$$

where recall that c is a particular cutoff rule, s is an index for a particular subject who participates in t_s trials of the choice experiment and $(x_1^s,\ldots,x_{t_s}^s)$ denote the signals that the subject observed in these t_s independent trials (as well as the choices d_t^s they made, which is a_t^s in their notation, see page 1139) and X_c^s is the total number of choices in the t_s trials for subject s that were predicted by the cutoff rule c. It is easy to see that using calculus that is possible to maximize this likelihood for $\varepsilon_s(c)$, i.e. a subject-specific error rate conditional on using threshold rule c, which is given by

$$\hat{\varepsilon}_s(c) = 2\left(1-\frac{X_c^s}{t_s}\right). \tag{28}$$

Note that if $X_c^s = t_s$, so that all of the decisions agree with the threshold rule c, then $\hat{\varepsilon}_s(c) = 0$. If $X_c^s = 0$ there is a problem in the estimator above since the formula implies an error rate greater than 1, so we have to modify the above as follows

$$\hat{\varepsilon}_s(c) = \min\left[1, 2\left(1-\frac{X_c^s}{t_s}\right)\right]. \tag{29}$$

This subject-specific error rate can be plugged back into the likelihood to get a *concentrated likelihood* and then we can iterate a discrete search over the finite number of possible threshold rules c for each subject to find the one that maximizes the likelihood for each subject, resulting in a subject-specific estimated threshold rule, $\hat{c}_s$ and corresponding error rate $\hat{\varepsilon}_s(\hat{c}_s)$. I am not sure why El-Gamal and Grether did not follow this approach in their paper.

D. Is the asymptotic theory of their "EC" (estimation-classification) maximum likelihood algorithm "standard" in the sense of resulting in $\sqrt{N}$-consistent asymptotically normal parameter estimates, or is some other type of asymptotics appropriate here? Why were no standard errors reported for their estimated decision rules?

answer the reason is that some of the parameters are *discrete* rather than continuous. The threshold rule parameters c are triplets of integers and so there are only a finite number of possible parameters, unlike a standard maximum likelihood problem where there are a continuum of possible values when the parameters are any possible element of a Euclidean space. The usual asymptotic theory of maximum likelihood is based on a Taylor series expansion about the true value, but this cannot be done for discrete valued parameters such as c. For a discrete (finite) parameter space, the appropriate distribution allowing for samplying error will be a multinomial distribution of the possible parameter values, with the greatest probability on the "true" parameter value. A theory similar to that for hypothesis testing using the log-likelihood ratio can be used so show that the maximum likelihood estimator will put a probability close to 1 on the true parameter value and near zero on the other non-true values, and the probabilities that the MLE estimator equals one of the non-true values of the parameters will converge to zero exponentially fast. It is a similar argument to the one we used to show that the posterior probability that the sample came from urn A converges either to 1 or to zero exponentially fast. So it is a "non-standard" asymptotics that applies in this case and this is why reporting standard errors for the estimated c values is not appropriate in this case.

E. The authors seemed to have intentionally *not* provided sequential feedback to the subjects about whether they chose a correct cage or not. Why do you think they did this? If there was feedback, presumably it would help the subjects to dynamically improve their ability to make correct predictions — i.e. the feedback could help the subjects *learn how to learn.* But even without feedback, do you think the subjects were learning and improving in their ability to play this "classification game"? If so, can you think of any tests you might perform to provide evidence for or against the hypothesis that there was "subject learning" going on? In particular, do you think the error rates in the first several classification decisions a give subject makes might be higher than in the last decisions that they make, resulting from some sort of *learning by doing?* If so, what sort of model might be appropriate to reflect such learning by doing, and how might additional feedback, such as sequential feedback after a subject makes each decision about the actual cage the sample was drawn from?

answer I think they did this because they did not want to open the can of worms about how to model how humans learn and improve over time. By providing minimal feedback, I believe the authors were hoping to observe subjects in their "natural state" before they would have much chance to learn and adapt and improve their peformance in this decision task. But it is a fundamental property of the "human neural net" for performance to improve with repetition on any task that the human thinks is important to learn. Even without any feedback along the way, the subjects' brains were probably expending subconscious effort to try to model the decision problem and consider alternative ways to think about it. In the spirit of Daniel Kahneman's book *Thinking: Fast and Slow* where there is a more implusive Type 1 subsystem in the brain that is good at making snap judgements but also a slower, more deliberative Type 2 subsystem that takes more energy and time but is better in considering the validity and potentially overriding the snap judgements made by the Type 1 subsystem. I have already tried to formalize simple versions of this Type 1/Type 2 subsystem dichotomy in the answer to part B above and note the paradoxes that one can run into without expending a lot more of my time/energy to think of a better, more plausible model. But some types of learning seem to be automatic and not require a huge amount of effort. It is what we call "learning by doing." Whether it requires some effort or little effort, it is plausible that decision making will change and become better/more sophisticated over the course of the experimental trials. If there are enough trials per subject, one way to test for learning is to divide the sample and estimate the parameters for the first third of the decisions made by subjects and then separately estimate the model and parameters based on the last third of the decisions. If there is learning by doing, we would expect that estimated error rates would be lower in the latter sample, and the estimated subjective probability $P_s(A|n,\pi_A)$ would approximate the objective Bayes rule formula given in equation (3) (or the logit equivalent, (19)) better using data from the last 1/3 of subject decisions. However beyond allowing the coefficients (a,b,c) to vary with cumulative experience in the experiment (say indexed by the number of trials done by the subject so far, t) it is hard to know how to model this learning by doing. This is another area where economics and psychology start to intersect, and where we can learn a great deal from Nobel prize winning psychologists such as Daniel Kahneman.