# 1. Optimal Growth

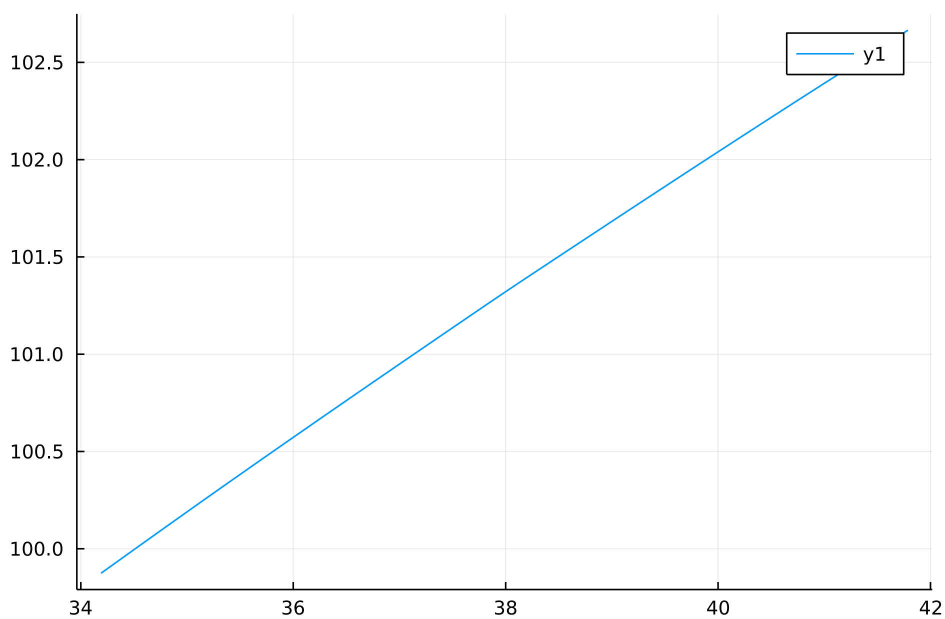
Problem:

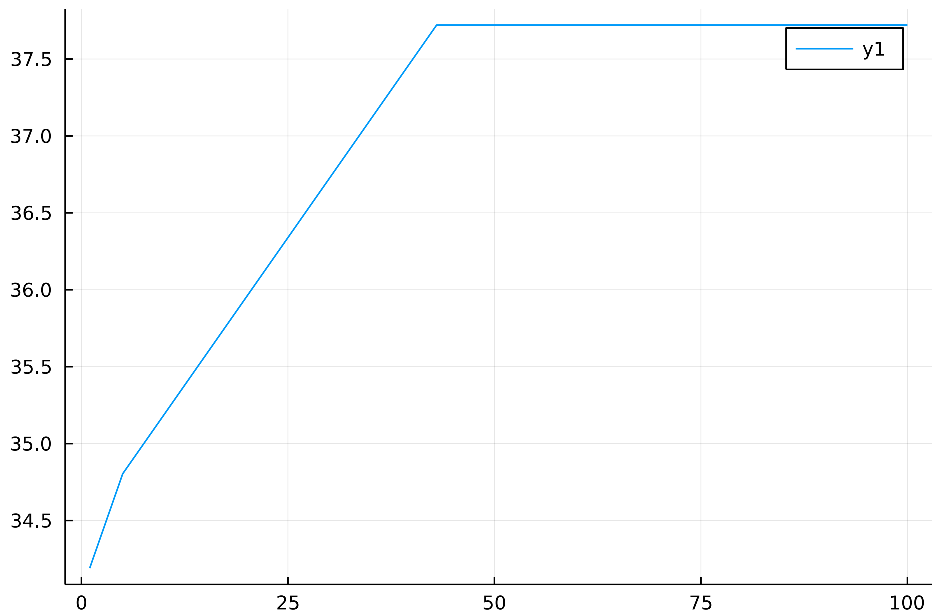
- $V(k) = \max_{c,k'} \left\{ log(c) + \beta V(k') \right\}$
- $c + k' = f(k) = k^\alpha + (1-\delta)k$
- $c, k' \in [0, f(k)]$

Solution:

- Euler: $c^{-1} = \beta * c'^{-1}(\alpha k'^{\alpha-1} + 1 - \delta)$
- Find $k_{ss}$ from $1 = \beta * (\alpha k_{ss}^{\alpha-1} + 1 - \delta)$
- Guess $V_0(k) = log(k)$ in $[k_{ss} * 0.9, k_{ss} * 1.1]$
- Iterate $V_{t+1}(k) = \max_{0 \le k' \le f(k)} \left\{ log(f(k) - k') + \beta V_t(k') \right\}$ until convergence
- Store policy rules $k'(k)$ and $c(k) = f(k) - k'(k)$
- From initial $k_0$ simulate economy with $k_{t+1} = k'(k_t)$ and $c_t = c(k_t)$

In [1]:
```
using LinearAlgebra, Plots

# Param
α = 0.36
δ = 0.025
β = 0.99

# Steady State
Kss = ((1/β+δ-1)/α)^(1/(α-1))
#print(Kss)

# Grids
K = Array(range(Kss*0.9,Kss*1.1,100))
V_ = log.(K)
V = log.(K)
KP = β*K;

# VFI
error = 1
while error>1e-10
    for (i,k) in enumerate(K)
        y = k^α+(1-δ)*k
        mask = ifelse.(K.<=y, 1.0, NaN)
        RHS = log.(y.-K.*mask)+β*V_.*mask
        RHS[isnan.(RHS)] .= -Inf
        v, kpi = findmax(RHS)
        KP[i] = K[kpi[1]]
        V[i] = v[1]
    end
    error = norm(V-V_)
    V_ = copy(V)
end
display(plot(K,V))
display(plot(K,KP))

# Simulate
k0 = Kss*0.9
T = 100
PATH = k0*Array(1:T)
for i in 1:T-1
    PATH[i+1] = KP[partialsortperm(abs.(K .- PATH[i]), 1)]
end
display(plot(PATH))
```




## 2. Stochastic Optimal Growth

Problem:

- $V(k, z) = \max_{c,k'} \{log(c) + \beta E[V(k', z')|z]\}$
- $c + k' = f(k, z) = zk^{\alpha} + (1 - \delta)k$
- $c, k' \in [0, f(k, z)]$
- $E[V(k', z')|z] = \sum_{z'} P_{zz'} V(k', z')$
- $P$ is known transition matrix, $Z$ is discrete shock space, $E[z] = 1$

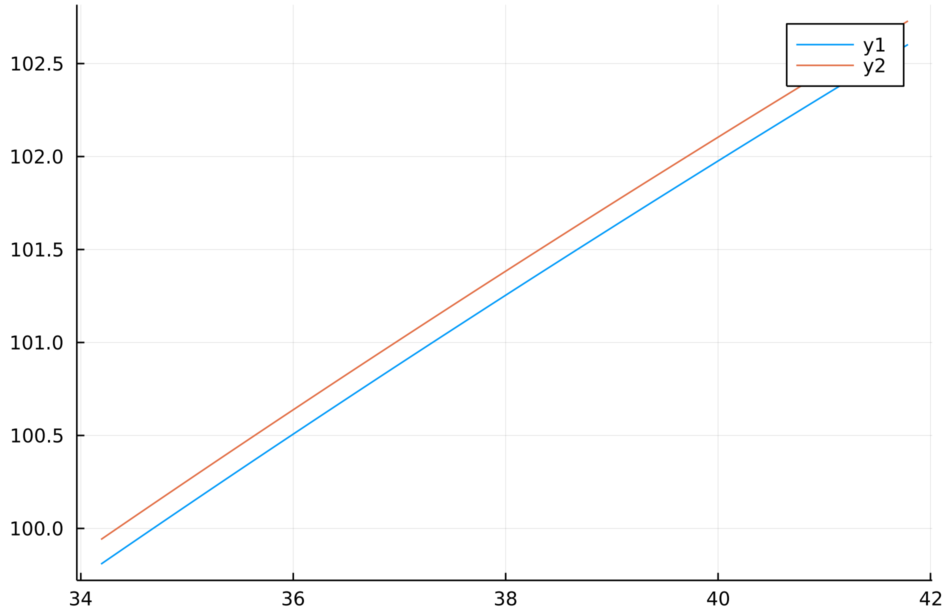
Solution:

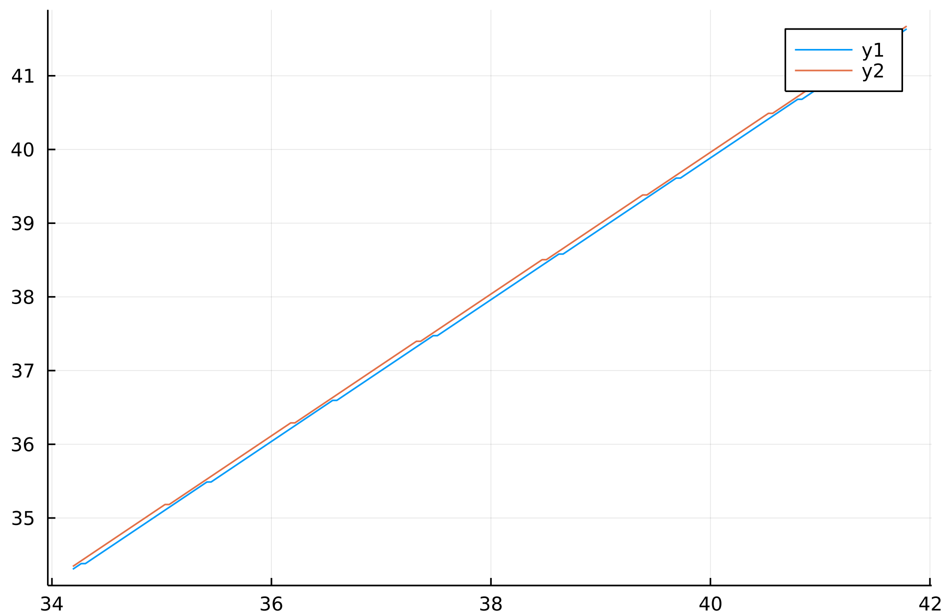
- Euler: $c^{-1} = \beta * E[c'^{-1}(z' \alpha k'^{\alpha-1} + 1 - \delta)|z]$
- Find $k_{ss}$ from $1 = \beta * (\alpha k_{ss}^{\alpha-1} + 1 - \delta)$
- Guess $V_0(k, z) = log(k)$ in $[k_{ss} * 0.9, k_{ss} * 1.1]$ x $Z$
- Iterate $V_{t+1}(k, z) = \max_{0 \leq k' \leq f(k)} \left\{log(f(k) - k') + E[\beta V_t(k', z')|z]\right\}$ until convergence
- Store policy rules $k'(k, z)$ and $c(k, z) = f(k, z) - k'(k, z)$
- From initial $k_0, z_0$ simulate economy with $k_{t+1} = k'(k_t, z_t)$ and $c_t = c(k_t, z_t)$

In [2]:
```
using PyPlot, LinearAlgebra

# Params
α = 0.36
δ = 0.025
β = 0.99
σ = 2.0
f(z,k) = z*(k^α)
u(c) = (c^(1-σ))/(1-σ)
Kss = ((1/β+δ-1)/α)^(1/(α-1))

# Grids
K= Array(range(Kss*0.9,Kss*1.1,200))
Z = [0.99 1.01]
P = Array([0.9 0.1; 0.1 0.9])
V_ = u.(K.*Z)
V = u.(K.*Z)
KP = β*(K.*Z);
CP = β*(K.*Z);

# VFI
error = 1
while error>1e-6
    for (ki,k) in enumerate(K)
        for (zi,z) in enumerate(Z)
            y = f(z,k)+(1-δ)*k
            mask = ifelse.(K.<=y, 1.0, NaN)
            RHS = log.(y.-K.*mask)+β*(V_[:, 1]*P[zi,1].+V_[:, 2].*P[zi,2]).*mask
            RHS[isnan.(RHS)] .= -Inf
            v, kpi = findmax(RHS)
            KP[ki,zi] = K[kpi[1]]
            V[ki, zi] = v[1]
            CP[ki, zi] = y-KP[ki,zi];
        end
    end
    error = norm(V-V_)
    V_ = copy(V)
end
```

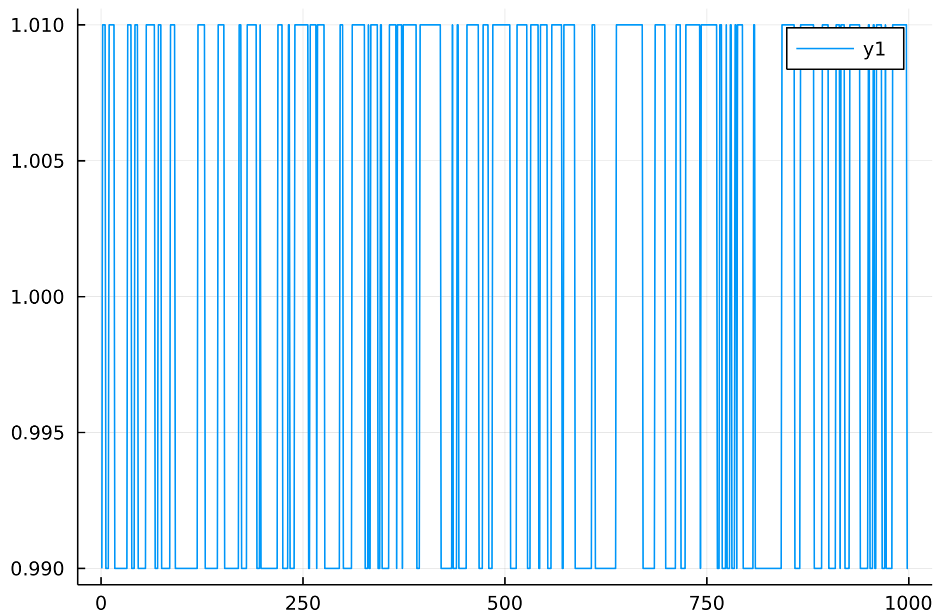
```
WARNING: using PyPlot.plot in module Main conflicts with an existing identifier.
```

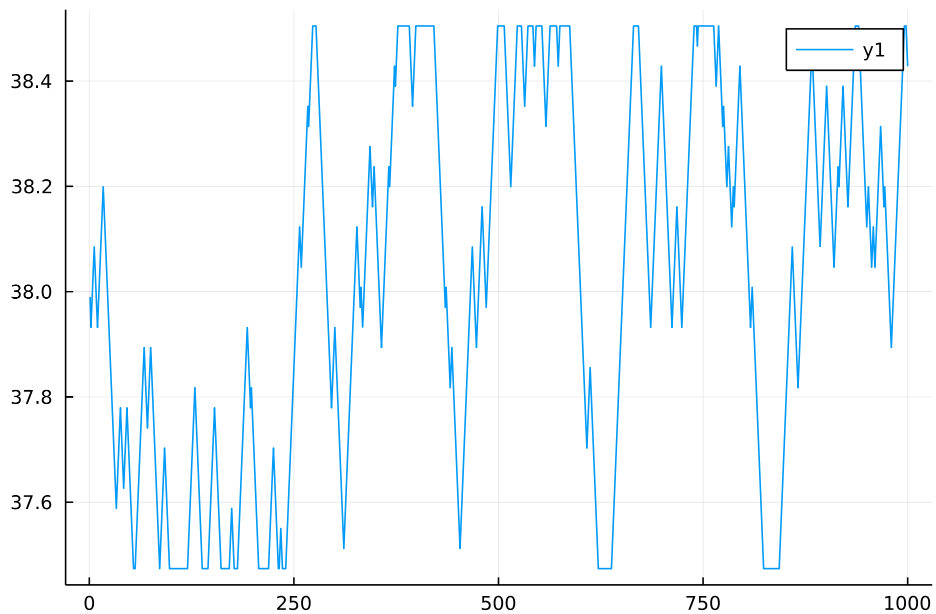
In [3]:
```
display(plot(K,V))
display(plot(K,KP))
```



In [4]:
```
# Simulate
using QuantEcon
k0 = Kss
z0 = Z[1]
T = 1000
ZIND = simulate(MarkovChain(P), T);
KIND = simulate(MarkovChain(P), T);
KPATH = k0*Array(1:T)
ZPATH = z0*Array(1:T-1);

for i in 1:T-1
    z = ZIND[i]
    KIND[i] = partialsortperm(abs.(K .- KPATH[i]), 1)
    KPATH[i+1] = KP[KIND[i], z]
    ZPATH[i] = Z[z]
    #println((z, KPATH[i], KPATH[i+1], ZPATH[i]))
end
display(plot(ZPATH))
display(plot(KPATH))
```



# 3. Optimal Growth (Coleman Operator)

Problem:

- $V(k) = \max_{c,k'} \left\{log(c) + \beta V(k')\right\}$
- $c + k' = f(k) = k^{\alpha}$ (full depreciation)
- $c, k' \in [0, y]$

Solution:

- Euler: $c^{-1} = \beta * c'^{-1} \alpha k'^{\alpha-1}$
- Find $k_{ss}$ from $1 = \beta * \alpha k_{ss}^{\alpha-1}$
- Guess $c_t(k) = \alpha k$ in $[k_{ss} * 0.9, k_{ss} * 1.1]$
- For each k, update $c_{t+1}(k)$ with $c$ that solves: $c^{-1} - \beta c_t(k^{\alpha} - c)^{-1} f'(k^{\alpha} - c) = 0$ until convergence
- Store policy rules $c(k)$ and $k'(k) = f(k) - c(k)$
- From initial $k_0$ simulate economy with $k_{t+1} = k'(k_t)$ and $c_t = c(k_t)$

In [5]:
```
using LinearAlgebra, Statistics
using BenchmarkTools, Interpolations, LaTeXStrings, Parameters, Plots, QuantEcon, Roots
using Optim, Random
using BenchmarkTools, Interpolations, Parameters, Plots, QuantEcon, Roots
```

In [6]:
```
function coleman!(Kg, g, K, β, up, f, fp)
    g_func = LinearInterpolation(K, g, extrapolation_bc = Line())
    for (i,k) in enumerate(K)
        #println(" ", (i,y))
        function h(c)
            kp = f(k) - c
            vals = up.(g_func.(kp)) .* fp(kp)
            return up(c) - β * mean(vals)
        end
        Kg[i] = find_zero(h, (1e-10, f(k)-1e-10))
    end
    return Kg
end

coleman(g, K, β, up, f, fp) = coleman!(similar(g), g, K, β, up, f, fp)
```

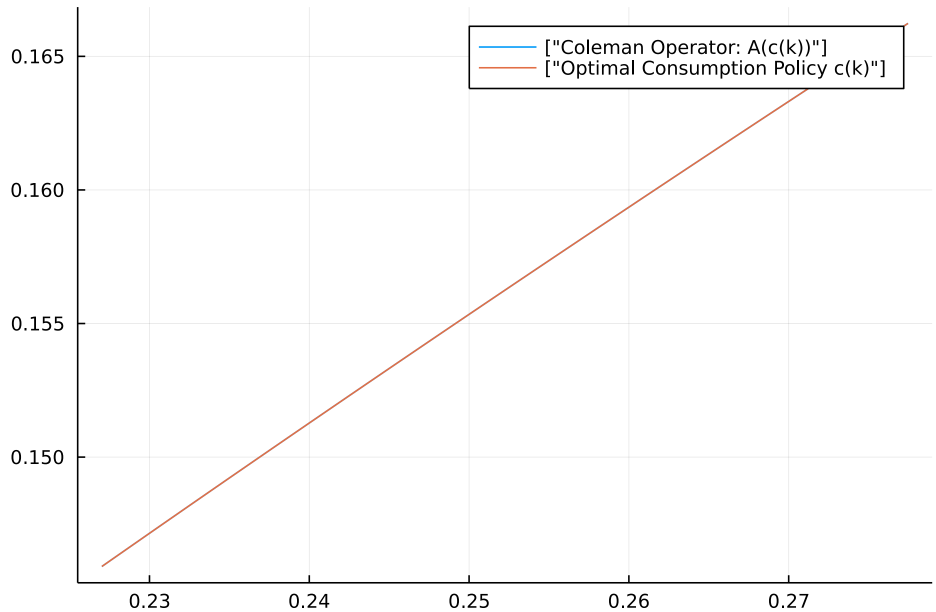
Out[6]:
```
coleman (generic function with 1 method)
```

In [7]:

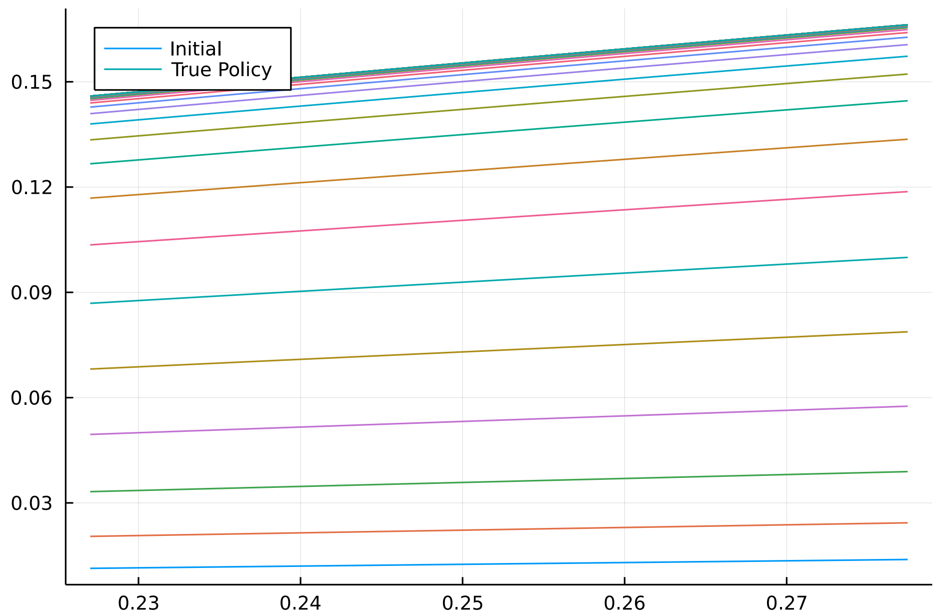
```
σ = 1.0
β = 0.95
α = 0.65
up(c) = c^(-σ)
f(k) = k^α
fp(k) = α*k^(α-1)
Kss = (1/(α*β))^(1/(α-1))
println(Kss)
K = Array(range(Kss*0.9, Kss*1.1, 200))

# Check Coleman Operator
cstar = (1-α*β).*f.(K) # True Policy
plot(K,cstar, label=["Coleman Operator: A(c(k))"])
plot!(K,coleman(cstar, K, β, up, f, fp), label=["Optimal Consumption Policy c(k)"])
```

```
0.2522434462207315
```

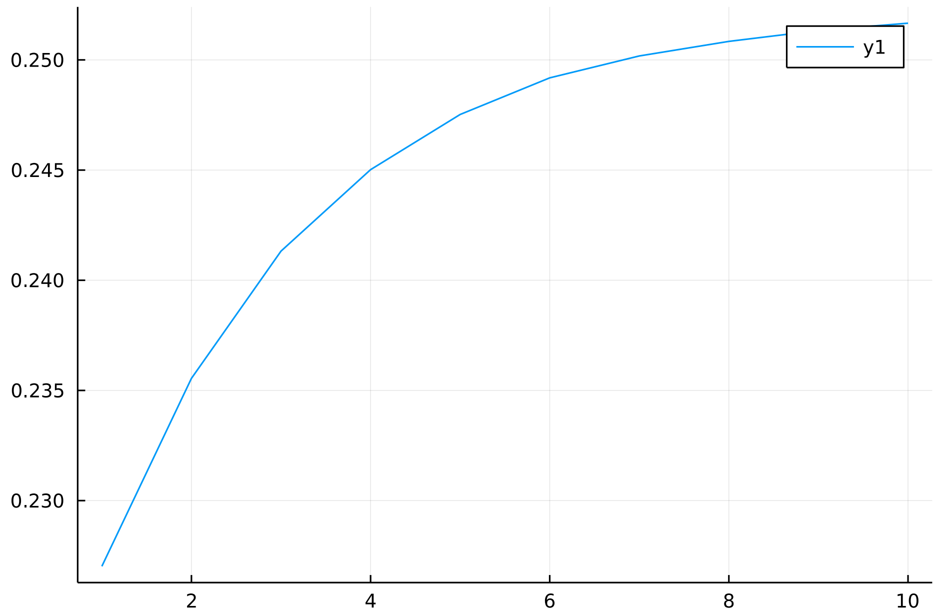
Out[7]:



In [8]:
```
function timeIteration(cstar, c_init)
    c = c_init
    plot(K,c,label = "Initial")
    for t in 1:100
        new_c = coleman(c, K, β, up, f, fp)
        c = new_c
        plot!(K, c, label=false)
    end
    plot!(K,cstar, label="True Policy")
    display(plot!(legend=:topleft))
    return c
end

c_init = (1-β)*K
cpolicy = timeIteration(cstar, c_init)
kpolicy = f.(K) - cpolicy;
```



In [9]:
```
# Simulate
k0 = Kss*0.9
T = 10
PATH = k0*Array(1:T)
for i in 1:T-1
    PATH[i+1] = kpolicy[partialsortperm(abs.(K .- PATH[i]), 1)]
end
display(plot(PATH))
```




## 4. Stochastic Optimal Growth (Coleman Operator)

Problem:

- $V(k,z) = \max_{c,k'}\{log(c) + \beta E[V(k',z')|z]\}$
- $c + k' = f(k,z) = zk^\alpha$
- $c, k' \in [0, f(k,z)]$
- $E[V(k',z')|z] = \sum_{z'} P_{zz'}V(k',z')$
- $P$ is known transition matrix, $Z$ is discrete shock space, $E[z] = 1$

Solution:

- Euler: $c^{-1} = \beta * E[c'^{-1}(z'\alpha k'^{\alpha-1})|z]$
- Find $k_{ss}$ from $1 = \beta * (\alpha k_{ss}^{\alpha-1})$
- Guess $c_t(k,z) = \alpha k$ in $[k_{ss} * 0.9, k_{ss} * 1.1]xZ$
- For each $k$, $z$, update $c_{t+1}(k,z)$ with $c$ that solves: $c^{-1} - \beta E[c_t(f(k,z) - c, z')^{-1} f_1(f(k,z) - c, z')|z] = 0$ until convergence
- Store policy rules $c(k,z)$ and $k'(k,z) = f(k,z) - c(k,z)$
- From initial $k_0$, $z_0$ simulate economy with $k_{t+1} = k'(k_t, z_t)$ and $c_t = c(k_t, z_t)$

In [10]:
```
using LinearAlgebra, Statistics
using BenchmarkTools, Interpolations, LaTeXStrings, Parameters, Plots, QuantEcon, Roots
using Optim, Random
using BenchmarkTools, Interpolations, Parameters, Plots, QuantEcon, Roots

σ = 1.0
β = 0.95
α = 0.65
up(c) = c^(-σ)
f(k,z) = z*k^α
fp(k,z) = z*α*k^(α-1)
Kss = (1/(α*β))^(1/(α-1))
println(Kss)
K = Array(range(Kss*0.9, Kss*1.1, 200))
Z = [0.99 1.01]
P = Array([0.9 0.1; 0.1 0.9])

function coleman!(Kg1, Kg2, g1, g2, K, β, up, f, fp)
    c1 = LinearInterpolation(K, g1, extrapolation_bc = Line()) # c(k,z1)
    c2 = LinearInterpolation(K, g2, extrapolation_bc = Line()) # c(k,z2)
    for (i,k) in enumerate(K)
        function h1(c) # Euler at k,z1
            kp = f(k,Z[1])-c
            val1 = up.(c1.(kp)) .* fp(kp,Z[1]) #z'=z[1]
            val2 = up.(c2.(kp)) .* fp(kp,Z[2]) #z'=z[2]
            return up(c) - β*(P[1,1]*val1+P[1,2]*val2)
        end
        function h2(c) # Euler at k,z2
            kp = f(k,Z[2])-c
            val1 = up.(c1.(kp)).* fp(kp,Z[1]) #z'=z[1]
            val2 = up.(c2.(kp)).* fp(kp,Z[2]) #z'=z[2]
            return up(c) - β*(P[2,1]*val1+P[2,2]*val2)
        end
        Kg1[i] = find_zero(h1, (1e-10, f(k,Z[1])-1e-10)) # Update c(k,z1)
        Kg2[i] = find_zero(h2, (1e-10, f(k,Z[2])-1e-10)) # Update c(k,z2)
    end
    return Kg1,Kg2
end
coleman(g1, g2, K, β, up, f, fp) = coleman!(similar(g1), similar(g2), g1, g2, K, β, up, f,
```

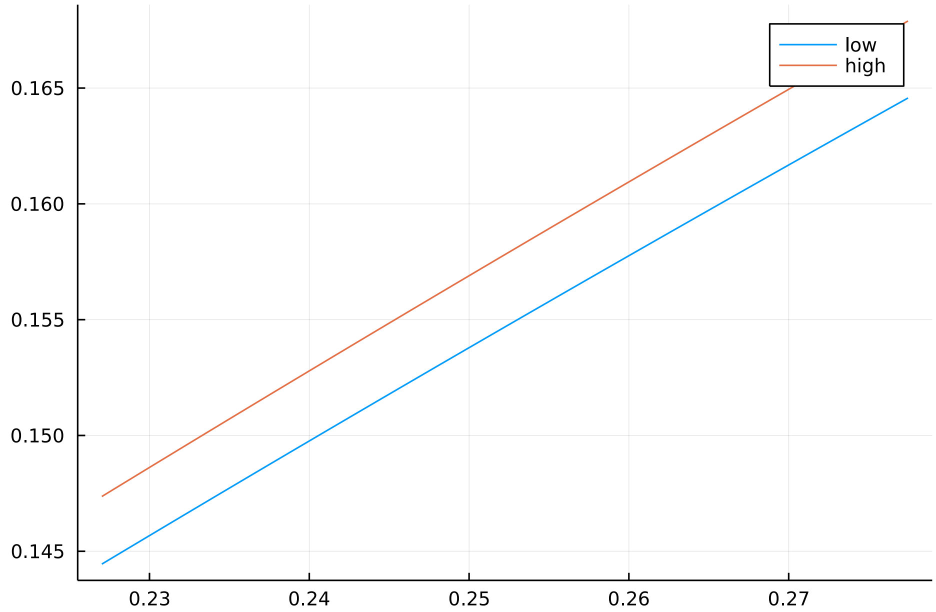
```
0.2522434462207315
```

Out[10]:
```
coleman (generic function with 2 methods)
```

In [11]:
```
function timeIteration(c1_init, c2_init)
    c1 = c1_init
    c2 = c2_init
    for t in 1:100
        new_c1, new_c2 = coleman(c1, c2, K, β, up, f, fp)
        c1 = new_c1
        c2 = new_c2
    end
    return c1,c2
end

c1,c2 = timeIteration((1-β).*Z[1].*K, (1-β).*Z[2].*K)
k1 = f.(K,Z[1]) - c1;
k2 = f.(K,Z[2]) - c2;
```

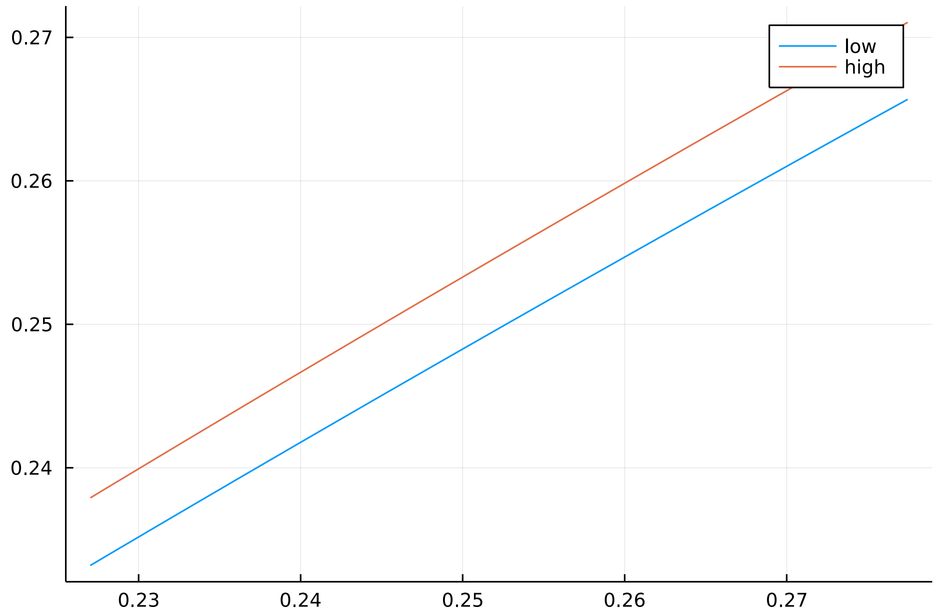
In [12]:
```
plot(K,c1, label="low")
plot!(K,c2,label="high")
```

Out[12]:




In [13]:
```
plot(K,k1,label="low")
plot!(K,k2,label="high")
```

Out[13]:




# 5. RBC Warmup Example in GDSGE

Problem:

- $V(k,z) = \max_{c,k'} \{u(c) + \beta E[V(k',z')|z]\}$
- u(c) = c^(1-\sigma)/(1-\sigma)
- $c + k' = f(k,z) = zk^{\alpha} + (1-\delta)k$
- $c, k' \in [0, f(k,z)]$
- $E[V(k',z')|z] = \sum_{z'} P_{zz'} V(k',z')$
- $P$ is known transition matrix, $Z$ is discrete shock space, $E[z] = 1$

Solution:

- Euler: $u'(c) = \beta * E[u'(c') * f'(k')|z]$
- Find $k_{ss}$ from $1 = \beta * (\alpha k_{ss}^{\alpha-1} + 1 - \delta)$
- Guess $c_t(k, z)$ in $[k_{ss} * 0.9, k_{ss} * 1.1]$x$Z$
- For each $k, z$, update $c_{t+1}(k, z)$ with $c$ that solves: $u'(c) - \beta E[u'(c_t(f(k, z) - c, z'))f_1(f(k, z) - c, z')|z] = 0$ until convergence
- Store policy rules $c(k, z)$ and $k'(k, z) = f(k, z) - c(k, z)$

In [14]:
```
using LinearAlgebra, Statistics
using BenchmarkTools, Interpolations, LaTeXStrings, Parameters, Plots, QuantEcon, Roots
using Optim, Random
using BenchmarkTools, Interpolations, Parameters, Plots, QuantEcon, Roots

σ = 2.0
β = 0.99
α = 0.36
δ = 0.025
up(c) = c^(-σ)
f(k,z) = z*k^α+(1-δ)*k
fp(k,z) = z*α*k^(α-1)+(1-δ)
Kss = ((1/β+δ-1)/α)^(1/(α-1))
println(Kss)
K = Array(range(Kss*0.9, Kss*1.1, 200))
Z = [0.99 1.01]
P = Array([0.9 0.1; 0.1 0.9])

function coleman!(Kg1, Kg2, g1, g2, K, β, up, f, fp)
    c1 = LinearInterpolation(K, g1, extrapolation_bc = Line()) # c(k,z1)
    c2 = LinearInterpolation(K, g2, extrapolation_bc = Line()) # c(k,z2)
    for (i,k) in enumerate(K)
        function h1(c) # Euler at k,z1
            kp = f(k,Z[1])-c
            val1 = up.(c1.(kp)) .* fp(kp,Z[1]) #z'=z[1]
            val2 = up.(c2.(kp)) .* fp(kp,Z[2]) #z'=z[2]
            return up(c) - β*(P[1,1]*val1+P[1,2]*val2)
        end
        function h2(c) # Euler at k,z2
            kp = f(k,Z[2])-c
            val1 = up.(c1.(kp)).* fp(kp,Z[1]) #z'=z[1]
            val2 = up.(c2.(kp)).* fp(kp,Z[2]) #z'=z[2]
            return up(c) - β*(P[2,1]*val1+P[2,2]*val2)
        end
        Kg1[i] = find_zero(h1, (1e-10, f(k,Z[1])-1e-10)) # Update c(k,z1)
        Kg2[i] = find_zero(h2, (1e-10, f(k,Z[2])-1e-10)) # Update c(k,z2)
    end
    return Kg1,Kg2
end
coleman(g1, g2, K, β, up, f, fp) = coleman!(similar(g1), similar(g2), g1, g2, K, β, up, f,
```

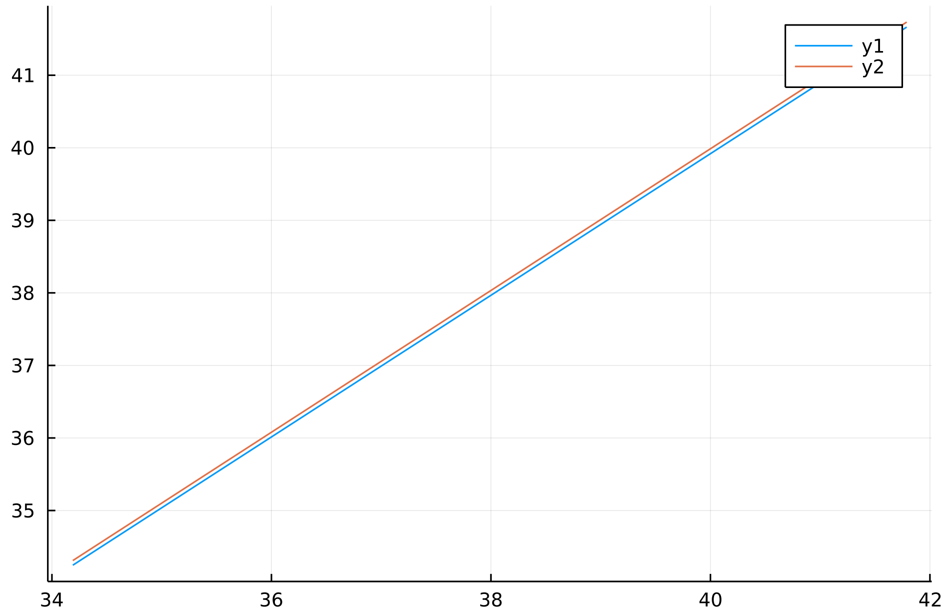
```
37.98925353815241
```

Out[14]:
```
coleman (generic function with 2 methods)
```

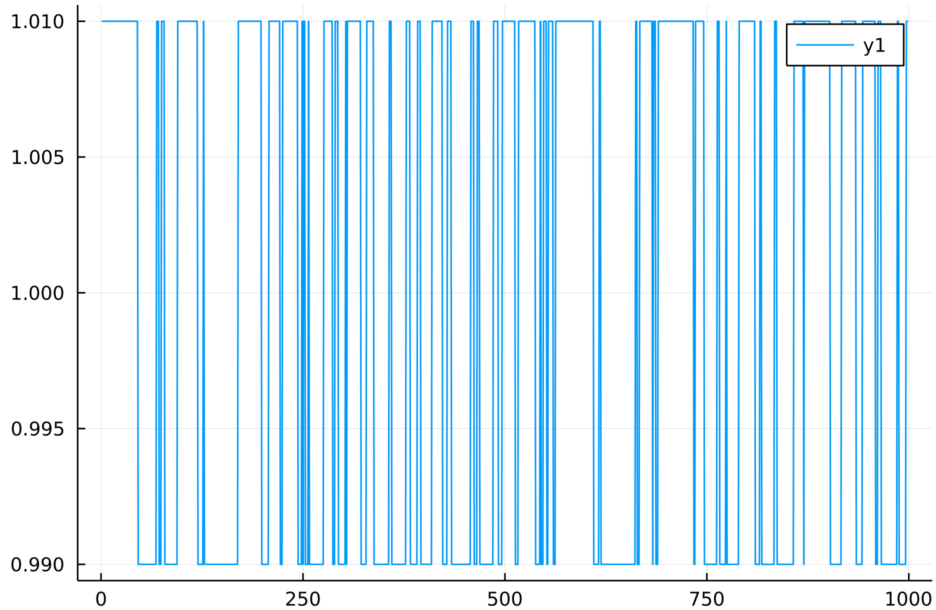
In [15]:
```
function timeIteration(c1_init, c2_init)
    c1 = c1_init
    c2 = c2_init
    error=1
    while error>1e-10
        new_c1, new_c2 = coleman(c1, c2, K, β, up, f, fp)
        error = norm(c1-new_c1)+norm(c2-new_c2)
        c1 = new_c1
        c2 = new_c2
    end
    return c1,c2
end
c1,c2 = timeIteration(1/13*K, 1/13*K)
k1 = f.(K,Z[1]) - c1;
k2 = f.(K,Z[2]) - c2;
```

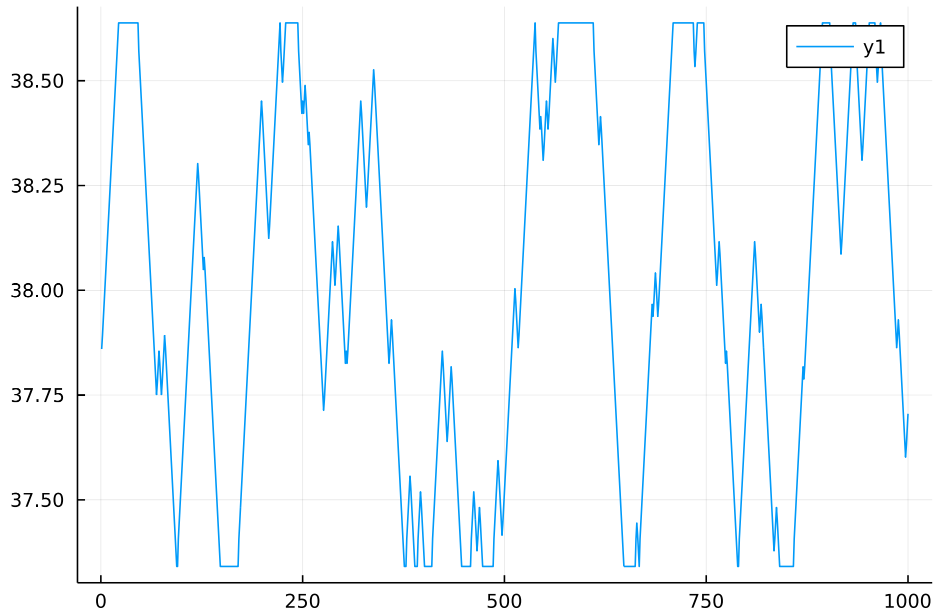
In [16]:
```
KP = hcat(k1,k2)
CP = hcat(c1,c2)
plot(K, KP)
```

Out[16]:



In [17]:
```
# Simulate
using QuantEcon
Z = [0.99 1.01]
P = Array([0.9 0.1; 0.1 0.9])
k0 = 37.85994455231658
z0 = Z[1]
T = 1000
ZIND = simulate(MarkovChain(P), T);
KIND = simulate(MarkovChain(P), T);
KPATH = k0*Array(1:T)
ZPATH = z0*Array(1:T-1);

for i in 1:T-1
    z = ZIND[i]
    KIND[i] = partialsortperm(abs.(K .- KPATH[i]), 1)
    KPATH[i+1] = KP[KIND[i], z]
    ZPATH[i] = Z[z]
    #println((z, KIND[i], KPATH[i], KPATH[i+1], ZPATH[i]))
end
display(plot(ZPATH))
display(plot(KPATH))
```



# 5. Coleman 1991: RBC + state-dependent income tax

Household:

- $V(k, K, z) = \max_{c,k'}\{u(c) + \beta E[V(k', K', z')|z]\}$
- $u(c) = c^{(1-\sigma)}/(1-\sigma)$
- $c + k' = (1 - \tau(K, z))(f(K, z) + (k - K)(f_1(K, z) + (1 - \delta)k + d(K, z) = y(k, K, z)$
- $c, k' \in [0, y(k, K, z)]$
- $f(k, z) = zk^{\alpha}$
- $f_1(k, z) = z\alpha k^{\alpha-1}$
- $F(k, z) = zk^{\alpha} + (1 - \delta)k$

Tax:

- Tax Policy: $\tau(K, z)$ s.t. $(1 - \tau(K, z))f_1(K, z)$ falling in $K$
- Transfer: $d(K, z) = \tau(K, z)$

Solution:

- Find $k_{ss}$ from $1 = \beta * (1 - \tau(k_{ss}, E[z])\alpha k_{ss}^{\alpha-1} + 1 - \delta)$
- Guess $c_t(k, z)$ in $[k_{ss} * 0.9, k_{ss} * 1.1]$x$Z$
- Define $H(k, z) = (1 - \delta) + (1 - \tau(k, z))f_1(k, z)$
- For each $k, z$, update $c_{t+1}(k, z)$ with $c$ that solves: $u'(c) - \beta E[u'(c_t(f(k, z) - c, z'))H(f(k, z) - c, z')|z] = 0$ until convergence
- Store policy rules $c(k, z)$ and $k'(k, z) = f(k, z) + (1 - \delta)k - c(k, z)$
- From initial $k_0, z_0$ simulate economy with $k_{t+1} = k'(k_t, z_t)$ and $c_t = c(k_t, z_t)$

In [18]:
```
using LinearAlgebra, Statistics
using BenchmarkTools, Interpolations, LaTeXStrings, Parameters, Plots, QuantEcon, Roots
using Optim, Random
using BenchmarkTools, Interpolations, Parameters, Plots, QuantEcon, Roots

σ = 2.0
β = 0.99
α = 0.36
δ = 0.025
up(c) = c^(-σ)
f(k,z) = z*k^α
F(k,z) = z*k^α+(1-δ)*k
fp(k,z) = z*α*k^(α-1)
τ(k,z) = 0.1 # simple case
H(k,z) = (1-δ)+(1-τ(k,z))*fp(k,z)
Kss = ((1/β+δ-1+1-τ(0,0))/α)^(1/(α-1))
println(Kss)
K = Array(range(Kss*0.9, Kss*1.1, 20))
Z = [0.99 1.01]
P = Array([0.9 0.1; 0.1 0.9])

function coleman!(Kg1, Kg2, g1, g2, K, β, up, f, fp)
    c1 = LinearInterpolation(K, g1, extrapolation_bc = Line()) # c(k,z1)
    c2 = LinearInterpolation(K, g2, extrapolation_bc = Line()) # c(k,z2)
    for (i,k) in enumerate(K)
        function h1(c) # Euler at k,z1
            kp = F(k,Z[1])-c
            val1 = up.(c1.(kp)) .* H(kp,Z[1]) #z'=z[1]
            val2 = up.(c2.(kp)) .* H(kp,Z[2]) #z'=z[2]
            return up(c) - β*(P[1,1]*val1+P[1,2]*val2)
        end
        function h2(c) # Euler at k,z2
            kp = F(k,Z[2])-c
            val1 = up.(c1.(kp)) .* H(kp,Z[1]) #z'=z[1]
            val2 = up.(c2.(kp)) .* H(kp,Z[2]) #z'=z[2]
            return up(c) - β*(P[2,1]*val1+P[2,2]*val2)
        end
        Kg1[i] = find_zero(h1, (1e-10, f(k,Z[1])-1e-10)) # Update c(k,z1)
        Kg2[i] = find_zero(h2, (1e-10, f(k,Z[2])-1e-10)) # Update c(k,z2)
    end
    return Kg1,Kg2
end
coleman(g1, g2, K, β, up, f, fp) = coleman!(similar(g1), similar(g2), g1, g2, K, β, up, f,
```

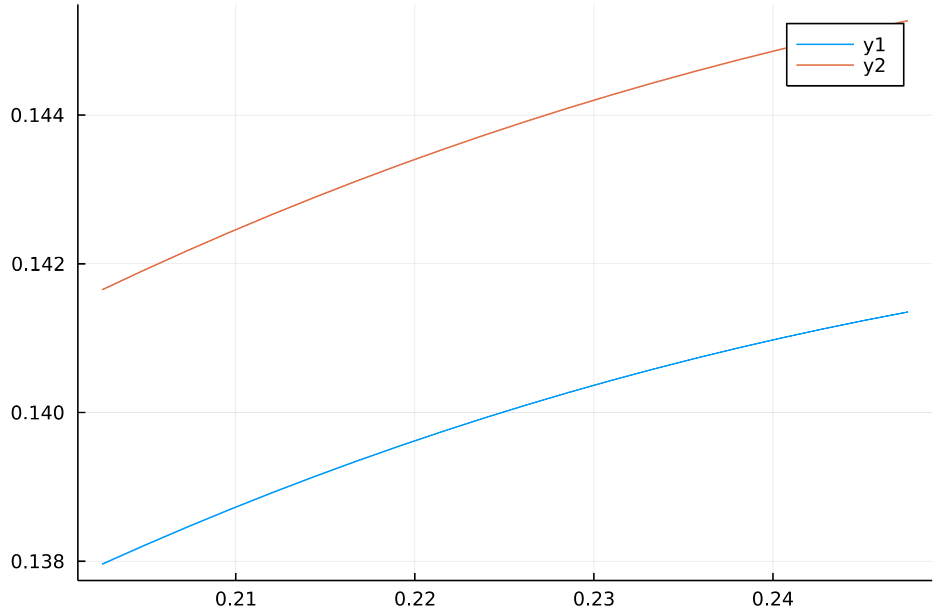
```
0.22503818185126323
```

Out[18]:
```
coleman (generic function with 2 methods)
```

In [19]:
```
function timeIteration(c1_init, c2_init)
    c1 = c1_init
    c2 = c2_init
    error=1
    while error>1e-10
        new_c1, new_c2 = coleman(c1, c2, K, β, up, f, fp)
        error = norm(c1-new_c1)+norm(c2-new_c2)
        c1 = new_c1
        c2 = new_c2
    end
    return c1,c2
end
c1,c2 = timeIteration(1/13*K, 1/13*K)
k1 = f.(K,Z[1]) - c1;
k2 = f.(K,Z[2]) - c2;
```

In [20]:
```
KP = hcat(k1,k2)
CP = hcat(c1,c2)
plot(K, KP)
```

Out[20]:




In [ ]: