Real Business Cycles with Irreversible Investment

Global and Deep Learning Solution Methods

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Outline

Model

Global Solution

Deep Learning Solution

Planner's Problem

$$V(k,z) = \max_{c,i}\{\frac{c^{1-\sigma}}{1-\sigma} + \beta E[V(k',z')|z]\}$$

$$c + i = zk^{\alpha}$$

$$k' = (1-\delta)k + i$$

$$i \geq \phi I_{ss} > 0$$

$$c, k' \in [0, zk^{\alpha} + (1-\delta)k]$$

$$z' \in \{z_h, z_l\} \sim P(.|z)$$

Optimality Conditions

FOCs for (c, c', k, k', i, μ) to be optimal solutions:

- Euler: $c^{-\sigma} - \mu = \beta E[(\alpha z' k^{\alpha-1} + (1-\delta))c'^{-\sigma} - (1-\delta)\mu' | z]$
- Budget: $c + i = zk^{\alpha}$
- Multiplier: $\mu \geq 0$ (positive when binding)
- Occasionally Binding Constraint: $i \geq \phi I_{ss}$
- Slackness: $\mu(i - \phi I_{ss}) = 0$

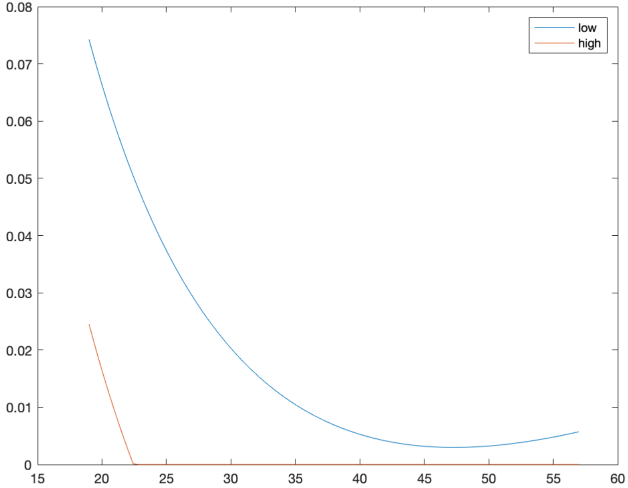
Parameters

- $\alpha = 0.36$
- $\beta = 0.99$
- $\sigma = 2.0$
- $\delta = 0.025$
- $\phi = 0.95$
- $Z = [0.99, 1.01]$
- $P = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$
- $K_{ss} = 37.9893$
- $I_{ss} = 0.9497$

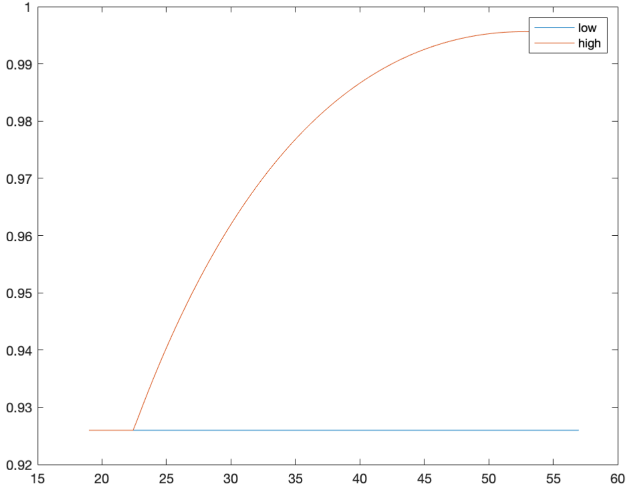
Global Solution

- Find the solution for last period T.
  - $i = \phi I_{ss}$
  - $\mu > 0$
  - $k' = (1-\delta)k + \phi I_{ss}$
  - $c = zk^{\alpha} - \phi I_{ss}$
- Use as guess for policy $c'(k,z), \mu'(k,z)$ in period $T-1$.
- Solve for $c(k,z), k'(k,z), \mu(k,z)$ from:
  - $c^{-\sigma} - \mu = \beta E[(\alpha z' k^{\alpha-1} + (1-\delta))c'^{-\sigma} - (1-\delta)\mu' | z]$
  - $c + k' = zk^{\alpha} + (1-\delta)k$
  - $\mu(k' - (1-\delta)k - \phi I_{ss}) = 0$
  - $\mu \geq 0$
- Update guess for $c'(k,z), \mu'(k,z)$ and repeat steps for $t = T-1, T-2, T-3...$ until convergence.

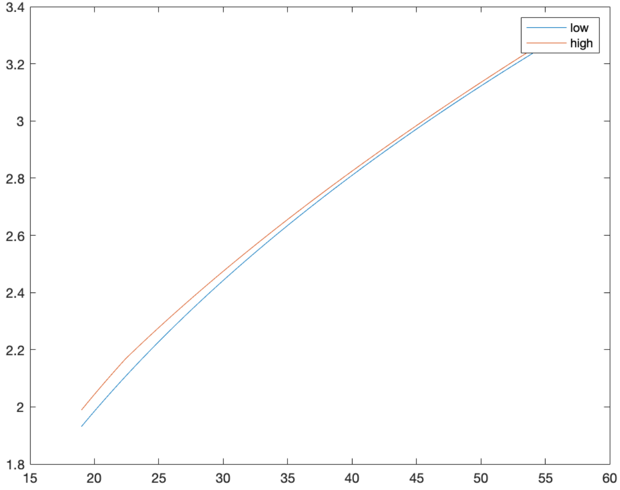
Multiplier: $\mu(k, z)$



Investment Function: $i(k, z)$



Consumption Function: $c(k, z)$



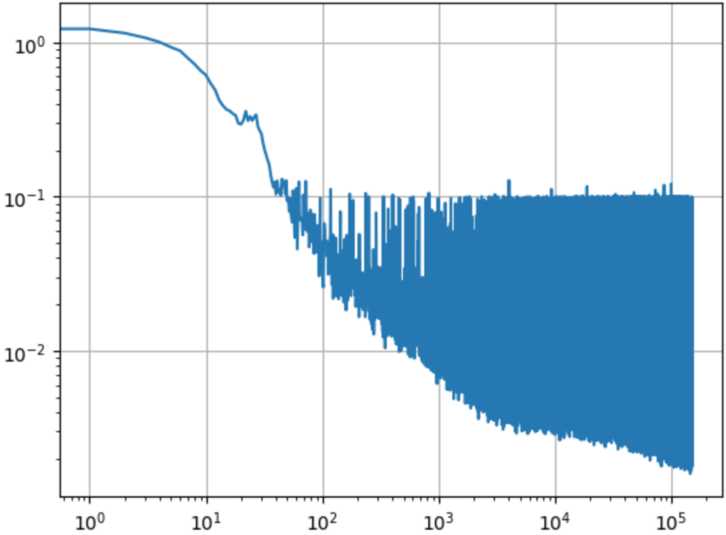
Deep Learning Solution Method

- Initialize functions - $c(k,z;\theta), h(k,z;\theta), i(k,z;\theta)$ as neural networks with parameters θ.
- All are nonnegative, with $c(k,z;\theta) > 0$.
- $h(k,z;\theta) = \frac{\mu(k,z;\theta)}{c^{-\sigma}}$ is the normalized multiplier.
- Fit the solution for last period T:
  - $i(k,z) = \phi I_{ss}$
  - $h(k,z) = 1$
  - $c = zk^{\alpha} - \phi I_{ss}$
- Use these functions to generate residuals:
  - $R1 = 1 - h - \beta E[c'^{-\sigma}(\alpha z' k^{\alpha-1} + (1-\delta) - (1-\delta)h')|z]$
  - $R2 = c + k' - zk^{\alpha} - (1-\delta)k$
  - $R3 = h(k' - (1-\delta)k - \phi I_{ss})$

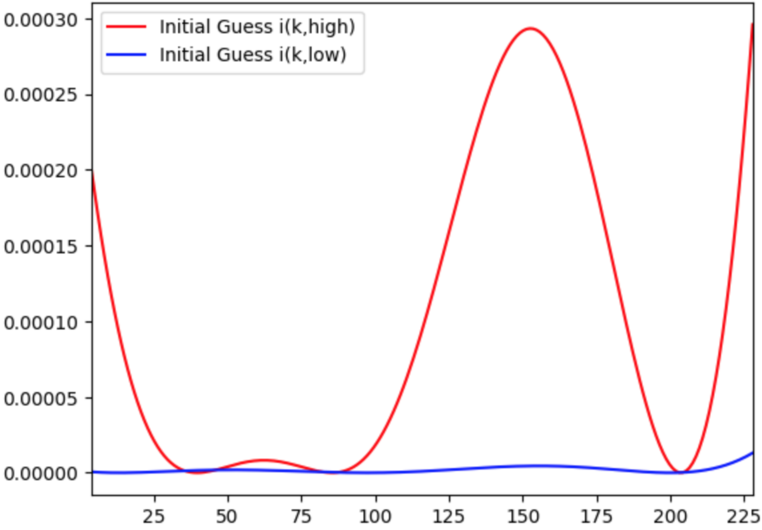
All-in-One Expectation Operator

- Construct residuals using a grid of k of size N.
- $J(\theta) = N^{-1} \sum_N (R1 * R1 + R2 * R2 + R3 * R3)$.
- Optimize the neural network i.e. update θ to minimize $J(\theta)$.
- Tricks that work:
  - Normalize the inputs to the neural network.
  - Bound the values of each policy function to be non-negative.
  - Normalize the multiplier.
  - Single loss function instead of optimizing different residuals separately.

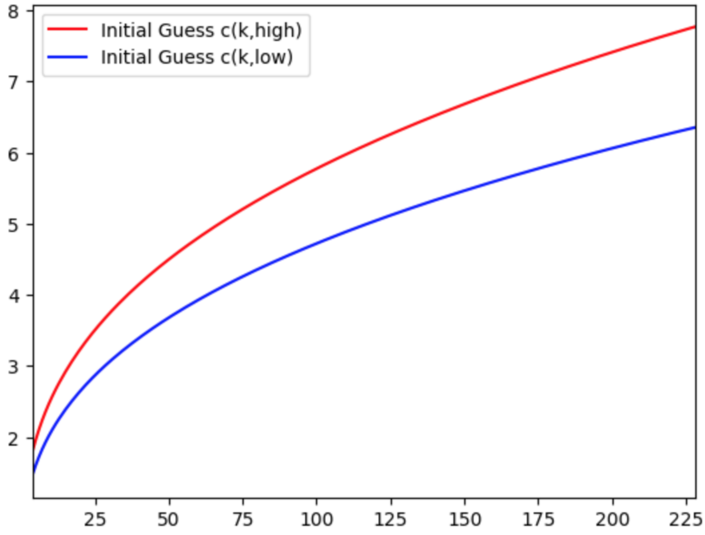
Training



Investment Function: $i(k, z) - I_{min}$



Consumption Function: $c(k, z)$



Conclusion

- Global methods are stable and fast because they use extra information in the Euler equation to approximate the solution.
- The range of problems to which we can apply deep learning solution methods extends to heterogenous agents and transitional dynamics.
- However deep learning solution methods are not yet evolved to the point where we can trust that the solution is correct. There are many ways in which training can stall.