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EQUILIBRIUM IN A PRODUCTION ECONOMY WITH AN INCOME TAX

BY WILBUR JOHN COLEMAN II[1]

A state-dependent income tax is incorporated into a stochastic, discrete-time, infinite-horizon production economy. Methods are developed for establishing the existence and uniqueness of an equilibrium, and for computing this equilibrium.

KEYWORDS: Tax policy, capital, existence, uniqueness, computation of the equilibrium.

1. INTRODUCTION

THIS PAPER DEVELOPS A MONOTONE-MAP METHOD to study the equilibrium in a production economy with an income tax. Aside from providing the means to examine economies distorted by an income tax, this method's principal strength is its ability to handle tax policies that depend on an economy's entire state vector. Specifying a tax policy to achieve a particular outcome often requires that tax rates depend on more than just an economy's exogenous variables, as is demonstrated by a tax policy designed to smooth tax revenues when aggregate income (an endogenous variable) is stochastic. In general, allowing a tax policy to depend on endogenous variables allows one to focus on an interaction between tax rates and the economy's equilibrium. Income, investment, and consumption clearly depend on the level of tax rates, but tax rates may also be chosen to depend on these variables. Computing an equilibrium then involves a simultaneous determination of consumption, for example, and tax rates. The objective of this paper is to develop a methodology for examining models that can address these types of issues.

The monotone map underlying this method refers to an operator that maps a set of consumption functions into itself. An equilibrium consumption function for the infinite-horizon economy is a fixed point of this operator, and the equilibrium consumption function n steps away from the horizon for a corresponding finite-horizon economy is computed by the nth iteration of this operator. This sequence of consumption functions is monotone, and by proving that this operator is continuous and that it maps a compact set of consumption functions into itself, one can prove that this monotone sequence converges uniformly to a fixed point. An additional restriction on the production function is sufficient to guarantee that this fixed point is indeed an equilibrium to the infinite-horizon economy. Moreover, with an additional restriction on utility the monotone map satisfies a definition of concavity. Along with a strengthened

[1]This paper is based on my dissertation, for which Robert E. Lucas, Jr., Lars P. Hansen, and Robert M. Townsend provided many helpful comments. I would also like to thank two anonymous referees, an editor, Edward J. Green, Ross E. Levine, and David B. Gordon for very useful criticisms. This paper should not be interpreted as reflecting the views of the Board of Governors of the Federal Reserve System or members of its staff.

definition of monotonicity, this concavity property ensures the equilibrium is unique.

Lucas and Stokey (1987) develop a monotone-map method to study an endowment economy with cash and credit goods. Stokey, Lucas, and Prescott (1989) apply methods similar to those developed here to study a deterministic production economy with a constant tax rate. As they note, however, their proof that an equilibrium exists does not extend to a stochastic setting. Also in a deterministic setting, but with a constant tax rate on wealth, Becker (1985) showed that the competitive equilibrium coincided with the unique solution to a concave optimization problem. Danthine and Donaldson (1986) showed that Becker's reformulation also works when production is stochastic.[2] In the literature just cited it thus has not been proven that there exists an equilibrium to a stochastic production economy with an income tax, which is the problem addressed in this paper.

2. THE MODEL

The model[3] consists of a large number of identical households and firms making decisions in a discrete-time infinite-horizon setting. Households begin each period with a fixed endowment of labor in addition to capital carried over from the previous period. During a period firms employ these factors in a constant-returns-to-scale production process to reproduce the single consumption-capital good. At the end of a period firms return the undepreciated portion of the capital to the households, and pay households for renting their capital and labor. The government taxes this rental income at a flat rate that depends on the state of the aggregate economy, and lump-sum redistributes the proceeds to the households. Households then decide how much to consume and how much to carry over as capital to the next period.

The state variables for the aggregate economy at time t consist of the aggregate capital-labor ratio X_t and the exogenous shock z_t.

ASSUMPTION 1: *$z_t \in Z$, where Z is finite. The shocks evolve according to the Markovian probabilities $\pi(z'|z) = Pr\{z_{t+1} = z'|z_t = z\}$.*

Let E_z denote the conditional expectation with respect to $\pi(\cdot|z)$, so that for any function ϕ: $Z \to \mathbb{R}_+$, $E_z[\phi(z')] = \sum_{z' \in Z} \phi(z')\pi(z'|z)$.

Each household assumes that the aggregate capital-labor ratio evolves recursively according to $X_{t+1} = g(X_t, z_t)$. Solving for the function g that is consistent with the households' behavior will be part of the equilibrium problem.

Denote the depreciation rate on capital by δ. With constant returns to scale in both factor inputs, the production process can be defined relative to a

[2] Current work by Kehoe, Levine, and Romer (1989) attempts to extend this approach, but they are unable to reformulate the type of problem considered here without using side conditions.

[3] I wish to thank an anonymous referee for suggesting that I motivate the model as a competitive equilibrium in the way done here.

per-labor production function f with the capital-labor ratio and an exogenous shock as arguments.

ASSUMPTION 2: $\delta \in (0,1]$. *The production function* $f:\mathbb{R}_+ \times Z \to \mathbb{R}_+$ *is, in its first argument, continuously differentiable, strictly increasing, strictly concave, and* $f(0,z)=0$, *all* z. *There exists some capital-labor ratio* $\bar{x} \in (0,\infty)$ *such that* $f(\bar{x},z)+(1-\delta)\bar{x} \leqslant \bar{x}$, *all* z, *and* $f(\bar{x},z)+(1-\delta)\bar{x}=\bar{x}$, *some* z.[4]

To shorten the notation, define $F(x,z)=f(x,z)+(1-\delta)x$, which equals gross production plus undepreciated capital. Define $K=[0,\bar{x}]$ and $K_{++}=(0,\bar{x}]$. Clearly $F:K\times Z \to K$, which will mean that $\bar{x}$ is the (unique) maximum maintainable capital-labor ratio. With this definition of $\bar{x}$, the law of motion for the aggregate capital-labor ratio can be restricted to functions $g:K_{++}\times Z \to K_{++}$.

Firms engage in perfect competition and thus they pay each factor its marginal product. Households inelastically supply their endowment of labor, so with a household's capital-labor ratio denoted by x_t its rental income per unit of labor is $f(X_t,z_t)+(x_t-X_t)f_1(X_t,z_t)$. Due to constant returns to scale a firm's total factor payment exhausts its output, so it does not matter who owns the firms.

The government provides each household with the lump-sum transfer per unit of labor equal to $d_t=d(X_t,z_t)$, $d:K_{++}\times Z \to K_{++}$, and taxes all income at the rate $\tau_t=\tau(X_t,z_t)$.

ASSUMPTION 3: *The tax function* $\tau:\mathbb{R}_+\times Z \to [0,1)$ *is continuous in its first argument, and* $(1-\tau)f_1$ *is strictly decreasing in its first argument.*

The tax policy, given by the function τ, is exogenously set by the government. The lump-sum transfer policy, given by the function d, is endogenously determined by the requirement that the government's budget be balanced each period. The government is thus not a net purchaser of goods, as it simply taxes and redistributes all the proceeds. Note that while the tax rate depends on the aggregate state variables, which in equilibrium equal the representative household's state variables, the households do not perceive the tax rate as dependent on its actions. This setup thus does not correspond to one with a progressive tax rate.[5]

[4]Note that no upper bound is imposed on the derivative at $x=0$, so this assumption does not exclude production functions where $f_1(0,z)<\infty$, all z, or $f_1(0,z)=\infty$, all z. A sufficient condition for $\bar{x}>0$ is $f_1(0,z)>1$, all z, and a sufficient condition for $\bar{x}<\infty$ is $\lim_{x\to\infty} f_1(x,z)=0$, all z.

[5]An extension that incorporates a progressive tax rate is straightforward. With an income y_t, let the tax rate be $\phi(y_t,X_t,z_t)$. Replace the first term on the right of equation (2.1) by $[1-\phi(f(X_t,z_t)+(x_t-X_t)f_1(X_t,z_t),X_t,z_t)][f(X_t,z_t)+(x_t-X_t)f_1(X_t,z_t)]$, and modify Assumption 3 to assume that ϕ is continuously differentiable in its first argument, that $(1-\phi(y,X,z))y$ is an increasing and concave function of y, and that $[1-\phi(f(x,z),x,z)-\phi_1(f(x,z),x,z)f(x,z)]f_1(x,z)$ is a strictly decreasing function of x. At the equilibrium, define $1-\tau(x,z)$ as the above term in brackets, and proceed as in Section 3.

Ending a period with after-tax income, undepreciated capital, and the government transfer, a household must decide on an amount c_t per unit of labor to consume, where the remainder x_{t+1} is the capital-labor ratio carried over to the next period:

$$x_{t+1} = (1-\tau(X_t, z_t))[f(X_t, z_t) + (x_t - X_t) f_1(X_t, z_t)] + (1-\delta)x_t + d(X_t, z_t) - c_t. \tag{2.1}$$

Each household's preferences are defined with respect to consumption per unit of labor. Expected discounted utility, defined over (feasible) consumption plans $C:\mathbb{R}_+ \times K_{++} \times Z \to \mathbb{R}_+$, is

$$E\left\{\sum_{t=0}^{\infty} \beta^t u(c_t)\right\}, \qquad c_t = C(x_t, X_t, z_t),$$

where $0 < \beta < 1$, (x_0, X_0, z_0) is known, the expectation is over sequences $\{z_t\}$, and the associated sequences $\{x_t\}$ and $\{X_t\}$ are given by (2.1) and g respectively.

Assumption 4: *The single-period utility function $u:\mathbb{R}_+ \to \mathbb{R}$ is bounded, continuously differentiable, strictly increasing, strictly concave, and $u'(0) = \infty$.*

Denote a household's state variables at the beginning of a period by $(x, X, z) \in \mathbb{R}_+ \times K_{++} \times Z$. The constraint set for the choice of consumption at the end of a period is

$$M(x, X, z) = [0, (1-\tau(X,z))(f(X,z) + (x-X) f_1(X,z)) + (1-\delta)x + d(X,z)],$$

and the value function V for a household's problem of choosing an optimal level of consumption satisfies the function equation

$$\begin{aligned} V(x, X, z) &= \sup_{c \in M(x,X,z)} \{u(c) + \beta E_z[V((1-\tau(X,z)) \\ &\qquad \times [f(X,z) + (x-X) f_1(X,z)] + (1-\delta)x \\ &\qquad + d(X,z) - c, g(X,z), z')]\}. \end{aligned} \tag{2.2}$$

Note that since $X > 0$, $M(x, X, z)$ is always a compact interval. Consider the Banach space of bounded, continuous, real-valued functions $v: \mathbb{R}_+ \times K_{++} \times Z \to \mathbb{R}$ equipped with the sup norm, and let $\mathscr{V}$ denote the subset of functions that are increasing (i.e., not strictly decreasing) and concave in their first argument.

Proposition 1: *Under Assumptions 1–4, given any continuous aggregate investment function $g:K_{++} \times Z \to K_{++}$ and any continuous transfer function d: $K_{++} \times Z \to K_{++}$, there exists a unique $V \in \mathscr{V}$ that satisfies (2.2). Moreover, this V is strictly increasing and strictly concave in its first argument. For each $(x, X, z) \in$*

$\mathbb{R}_+ \times K_{++} \times Z$, the supremum in (2.2) *is attained by a unique value $C(x, X, z)$, and the policy function C: $\mathbb{R}_+ \times K_{++} \times Z \to \mathbb{R}_+$ is continuous in its first argument.*

PROOF: This proof is standard in the literature (e.g., Stokey, Lucas, and Prescott (1989, Theorems 9.6–9.8)). *Q.E.D.*

Proposition 1 establishes that a household's problem has a solution for any continuous functions g and d. Consider, then, the following definition of what it means for g and d to be equilibrium functions.

DEFINITION: A *stationary equilibrium* consists of continuous functions (g, d) mapping $K_{++} \times Z$ into K_{++} such that (i) all tax revenues are lump-sum redistributed according to the transfer function $d = \tau f$ (note that this equality could have been imposed in (2.1)), and (ii) the aggregate investment function g is such that households choose to invest according to the same rule:

$$g(x, z) = F(x, z) - C(x, x, z),$$

where C is the policy function associated with the value function $V \in \mathscr{V}$ that satisfies (2.2).

To represent the equilibrium as the solution to first-order and envelope conditions, the policy function C must be an interior solution. For an arbitrary (g, d) this may not be so, as households could choose to consume all their income in a particular period and live off their labor income in the next period. This is not possible in equilibrium, however, as zero investment today leads to no output tomorrow.

PROPOSITION 2: *Under Assumptions* 1–4, *if (g, d) is an equilibrium with the associated policy function C and value function V, then $C(x, x, z)$, all $x > 0$, all z, lies in the nonempty interior $\mathring{M}(x, x, z)$ of $M(x, x, z)$, and V is continuously differentiable in its first argument at (x, x, z), all $x > 0$, all z.*

PROOF: Since $u'(0) = \infty$ and $f(0, z) = 0$, all z, $C(x, x, z) \in \mathring{M}(x, x, z)$, all $x > 0$, all z. By Benveniste and Scheinkman's theorem (1979, Theorem 1) the value function V is differentiable at (x, x, z), all $x > 0$, all z, with the derivative $V_1(x, x, z) = u'[C(x, x, z)][1 - \delta + (1 - \tau(x, z))f_1(x, z)]$. V_1 is continuous in its first argument since u' is continuous and C is continuous in its first argument. *Q.E.D.*

To shorten the notation, define $H(x, z) = 1 - \delta + (1 - \tau(x, z))f_1(x, z)$, which can be thought of as the after-tax marginal rate of substitution perceived by households. With the results of Propositions 1 and 2, at the equilibrium (g, d) and at $X = x$, the supremum in (2.2) satisfies the first-order condition

$$0 = u'(C(x, x, z)) - \beta E_z\big[V_1(F(x, z) - C(x, x, z), g(x, z), z')\big], \tag{2.3}$$

and the envelope condition is

$$V_1(x,x,z)=\beta E_z[V_1(F(x,z)-C(x,x,z),g(x,z),z')]H(x,z). \tag{2.4}$$

Define $c(x,z)=C(x,x,z)$. Substituting (2.3) into (2.4) yields $V_1(x,x,z)=u'(c(x,z))H(x,z)$, and substituting this into (2.3), using the equilibrium definition of g, yields

$$u'(c(x,z))=\beta E_z[u'(c(F(x,z)-c(x,z),z')) \times H(F(x,z)-c(x,z),z')]. \tag{2.5}$$

Any equilibrium can thus be represented as a consumption function that is a strictly positive solution ($c(x,z)>0$, all $x>0$, all z) to (2.5). Conversely, suppose that c is a strictly positive solution to (2.5), $g=F-c$, and $d=\tau f$. Is this (g,d) an equilibrium? For this g and d, define V as the unique solution to (2.2) with the associated policy function C. By construction $g(x,z)=F(x,z)-C(x,x,z)$ so (g,d) is an equilibrium. Note, however, that the zero consumption function $c=0$ is a solution to (2.5), but it is not an equilibrium, so it is important to explicitly deal with strictly positive solutions (it is easy to verify that $c=F$ is not a solution to (2.5)). The remainder of this paper proves the existence and uniqueness of a strictly positive solution to (2.5), and develops an algorithm to compute this solution.

3. EXISTENCE

In a deterministic setting, one can rule out solutions to (2.5) that are arbitrarily close to zero by showing that consumption is strictly positive at the steady state. In a stochastic setting, since there is no stationary point of consumption, it is difficult to rule out solutions to (2.5) that are arbitrarily close to zero. Since zero is a solution, any operator whose fixed points are solutions to (2.5) and that has zero in its domain had better have two solutions, one at zero and one that is strictly positive. Since the contraction-mapping theorem delivers a unique solution, it thus is not well suited for this problem. Most other existence theorems rely on the property of compactness for the set of candidate solutions, which, for infinite-dimensional spaces, is a considerably more difficult concept to deal with than the completeness requirement of the contraction-mapping fixed-point theorem. It turns out, however, that for the economy considered here a rather natural compact set of consumption functions can be constructed. This compactness property is also preserved under an operator that has a close association with the contraction mapping of a dynamic program. While this operator does not have the contraction property, it does have a monotonicity property which can be used to construct a sequence of consumption functions that converges to a strictly positive fixed point.

Define the set of consumption functions

$$C_F(K\times Z) = \left\{ c \colon \begin{array}{l} c\colon K\times Z\to K \text{ is continuous,} \\ 0\leqslant c(x,z)\leqslant F(x,z), \\ 0\leqslant c(y,z)-c(x,z)\leqslant F(y,z)-F(x,z) \text{ for } y\geqslant x. \end{array} \right.$$

Equip $C_F(K\times Z)$ with the sup norm. The third condition defining $C_F(K\times Z)$ is equivalent to requiring that consumption c and net investment $F-c$ be increasing functions of the capital-labor ratio x.

PROPOSITION 3: *Under Assumptions* 1–2, $C_F(K\times Z)$ *is convex and compact.*[6]

PROOF: Clearly $C_F(K\times Z)$ is convex. Since $|c(x,z)-c(y,z)|\leqslant|F(x,z)-F(y,z)|$, by the mean value theorem $C_F(K\times Z)$ is equicontinuous at every point $x>0$, all z. At $x=0$, for any $\varepsilon>0$ choose $\Delta>0$ such that $0<F(\Delta,z)\leqslant\varepsilon$, all z. With this Δ, whenever $0\leqslant y\leqslant\Delta$, $c\in C_F(K\times Z)$, it follows that $|c(0,z)-c(y,z)|\leqslant F(y,z)<\varepsilon$. $C_F(K\times Z)$ is thus equicontinuous, so by the Arzela-Ascoli theorem[7] $C_F(K\times Z)$ is compact. *Q.E.D.*

Define the operator A on the domain $C_F(K\times Z)$ by

$$u'((Ac)(x,z))=\beta E_z\big[u'(c(F(x,z)-(Ac)(x,z),z'))\times H(F(x,z)-(Ac)(x,z),z')\big]. \tag{3.1}$$

Clearly any strictly-positive fixed point of A is an equilibrium consumption function. The operator A has a well-defined meaning: for the finite-horizon version of this economy with the terminal consumption function fixed at c, $A^n(c)$ equals the equilibrium consumption function n steps away from the terminal date. One implication of what follows is thus the equivalence between the limit of the finite-horizon economies and the infinite-horizon economy. This result also establishes the close connection between A and the dynamic programming problem on which A is based.

The following proposition establishes that A is well defined, maps $C_F(K\times Z)$ into itself, and is continuous.

PROPOSITION 4: *Under Assumptions* 1–4, *for any* $c\in C_F(K\times Z)$, *a unique* $A(c)\in C_F(K\times Z)$ *exists. Moreover*, A *is continuous.*

[6] A metric space M is compact if every sequence in M contains a convergent subsequence. See Royden (1988, p. 194).

[7] The Arzela-Ascoli theorem states that a closed metric space of bounded real-valued functions defined on a compact set is compact if it is equicontinuous. A set of real-valued functions $C(K)$ defined on a metric space K is equicontinuous at $x\in K$ if for any $\varepsilon>0$ there exists a $\delta>0$ such that $|c(x)-c(y)|<\varepsilon$ whenever $d(x,y)<\delta$, $y\in K$, and $c\in C(K)$. $C(K)$ is equicontinuous if it is equicontinuous at every point, and uniformly equicontinuous if δ can be chosen independent of x. See Royden (1988, pp. 167–169) for the proof of the Arzela-Ascoli theorem and for the definition of equicontinuity.

PROOF: If $c(F(x,z),z')=0$ for some z', set $(Ac)(x,z)=0$, else define $(Ac)(x,z)$ as the y for which

$$\zeta(y,x,z)=\beta E_z\left[u'(c(F(x,z)-y,z'))H(F(x,z)-y,z')\right]-u'(y)$$

equals zero. ζ is negative for y close to 0, positive for y close to $F(x,z)$, and strictly increases as y increases. This proves the existence of a unique $A(c)$, which also proves that $A(c)$ is continuous. Since ζ increases with y and decreases with x, $A(c)$ is an increasing function of x, and by (3.1) $F-A(c)$ is an increasing function of x. Hence $A(C_F(K\times Z))\subset C_F(K\times Z)$.

Clearly $A(c_n)\rightarrow A(c)$ pointwise as $c_n\rightarrow c$. Since $C_F(K\times Z)$ is equicontinuous and $K\times Z$ is compact, this convergence is uniform,[8] which establishes that A is continuous. *Q.E.D.*

At this stage, proving the existence of a fixed point $c=A(c)$, $c\in C_F(K\times Z)$, is vacuous since we already know that $c=0$ is a fixed point. For this reason, Schauder's fixed-point theorem is inappropriate relative to the set $C_F(K\times Z)$. Consider, however, the following version of Tarski's fixed-point theorem: a continuous, monotone self-map A of a nonempty, partially-ordered, compact set M in which some element $m\in M$ is mapped downwards, $A(m)\leqslant m$, has a fixed point in M, and $A^n(m)$ converges to a maximal fixed point in the set $\{m^\dagger: m^\dagger\leqslant m, m^\dagger\in M\}$.[9] (A fixed point m^* in the set just defined is maximal in this set if any other fixed point m^{**} in this set must satisfy $m^{**}\leqslant m^*$.) This theorem contributes something since it asserts the existence of a maximal fixed point and provides a particular sequence that converges to it. For the economy considered here F will take the place of m, so this fixed point is maximal in the entire set $C_F(K\times Z)$, and thus I will simply refer to it as the maximal fixed point in $C_F(K\times Z)$.

To apply Tarski's fixed-point theorem, a partial ordering on $C_F(K\times Z)$ is required. Define the usual partial ordering by $\hat{c}\leqslant\tilde{c}$ if $\hat{c}(x,z)\leqslant\tilde{c}(x,z)$, all (x,z), with which the monotonicity of A can be established.

PROPOSITION 5: *Under Assumptions* 1–4, $(A,\leqslant)$ *is monotone*.

PROOF: Since $\hat{c}\leqslant\tilde{c}$ implies

$$u'((A\hat{c})(x,z))\geqslant\beta E_z\big[u'(\tilde{c}(F(x,z)-(A\hat{c})(x,z),z'))\times H(F(x,z)-(A\hat{c})(x,z),z')\big],$$

it follows that $A(\hat{c})\leqslant A(\tilde{c})$. *Q.E.D.*

[8] See Royden (1988, p. 169, Corollary 41).

[9] See Dugundji and Granas (1982, Theorem 1.4.2). The theorem only requires that every countable chain (i.e., every totally-ordered subset) has an infimum, which is satisfied if the set is compact. I wish to thank Darrell Duffie for bringing this theorem to my attention, and an anonymous referee for suggesting that I apply it in this case. In a previous version of this paper, I applied Schauder's theorem on a set that excluded $c=0$, where I used monotonicity to show that A mapped this set into itself.

The existence of the maximal fixed point in $C_F(K \times Z)$ as the limit of $A^n(F)$ can now be established.

PROPOSITION 6: *Under Assumptions* 1–4, *among the set of fixed points of* A *there exists one that is maximal in* $C_F(K \times Z)$, *and* $A^n(F)$ *converges uniformly to this maximal fixed point.*

PROOF: By Proposition 3 $C_F(K \times Z)$ is a compact set, and by Proposition 4 A is continuous and maps $C_F(K \times Z)$ into itself. By Proposition 5 $(A, \leqslant)$ is monotone. Clearly $A(F) \leqslant F$, so Tarski's theorem can be applied. Since $C_F(K \times Z)$ is a set of equicontinuous functions defined as a compact set, the convergence is uniform. *Q.E.D.*

Tarski's fixed-point theorem does not assert that the maximal fixed point in $C_F(K \times Z)$ is strictly positive, but it does assert that if $A^n(F)$ converges to 0, then no equilibrium in $C_F(K \times Z)$ exists. To prove that an equilibrium exists, one needs to show that $A^n(F)$ converges to a strictly positive function. This is easy to prove in the deterministic setting, as one can show that the steady-state value of consumption is strictly positive, and also that $A^n(F)$ evaluated at the stationary point of capital never falls below the steady-state value of consumption. This proof is completed by showing that if consumption is strictly positive for some capital-labor ratio, then it is strictly positive for every strictly positive capital-labor ratio. To generalize this proof to a stochastic setting, the following restriction on how the exogenous shocks enter the production and tax functions is sufficient.

ASSUMPTION 5: *There exists an* x_0 *such that, for all* z, $F(x_0, z) > x_0$ *and* $\beta E_z[H(x_0, z')] \leqslant 1$.

In the deterministic setting, Assumption 2 ensures that Assumption 5 holds since $F(x) > x$, all $x < \bar{x}$, and $H(\bar{x}) < 1$. In the stochastic setting, $H(\bar{x}, z) < 1$, all z, but the possibility $F(\bar{x}, z) < \bar{x}$, some z, requires that x_0 be somewhat smaller than $\bar{x}$. The condition $\beta E_z[H(x_0, z')] \leqslant 1$, however, requires that x_0 not be too small.

THEOREM 7: *Under Assumptions* 1–5, *a strictly positive fixed point exists.*

PROOF: I will first prove that the maximal fixed point is not zero, and then that it is strictly positive. Choose x_0 and α such that, for all z, $0 < \alpha < F(x_0, z) - x_0$ and $\beta E_z[H(x_0, z')] \leqslant 1$. Since F and τ satisfy Assumption 5, this can be done. To prove that the maximal fixed point is nonzero, it is sufficient to show that if $c(x_0, z) \geqslant \alpha$, all z, then $(Ac)(x_0, z) \geqslant \alpha$, all z. A sufficient condition for $(Ac)(x_0, z) \geqslant \alpha$ is

$$u'(\alpha) \geqslant \beta E_z\left[u'\left(c\left(F(x_0, z) - \alpha, z'\right)\right) H\left(F(x_0, z) - \alpha, z'\right)\right].$$

By construction $F(x_0, z) - \alpha > x_0$ and $c(x_0, z) \geqslant \alpha$, all z, so a sufficient condition for the above inequality is

$$u'(\alpha) \geqslant \beta E_z\left[u'(\alpha) H(x_0, z')\right].$$

This inequality is true by hypothesis, so a nonzero fixed point exists.

To prove that the maximal fixed point c^* is strictly positive, define x_1 as the largest x for which $c^*(x, z) = 0$ for some z, say $z^\dagger$. Suppose $x_1 > 0$. Since $F(x_1, z) > x_1$, all z (note that $x_1 < x_0$ since $c^*(x_0, z) \geqslant \alpha > 0$), the equilibrium condition

$$u'\left(c^*\left(x_1, z^\dagger\right)\right) = \beta E_{z^\dagger}\left[u'\left(c^*\left(F\left(x_1, z^\dagger\right) - c^*\left(x_1, z^\dagger\right), z'\right)\right) \times H\left(F\left(x_1, z^\dagger\right) - c^*\left(x_1, z^\dagger\right), z'\right)\right]$$

cannot hold, as the left side is unbounded while the right side is bounded. It must be that $x_1 = 0$. *Q.E.D.*

Since $A^n(F)$ converges to the maximal fixed point in $C_F(K \times Z)$, with Assumptions 1–5 it will converge to a strictly positive fixed point.

4. UNIQUENESS

To prove that an equilibrium is unique, consider extending the concavity-based argument for proving, for example, that a deterministic production function has a unique strictly positive fixed point. Suppose A satisfied the following property: for any strictly positive fixed point c of A and any $0 < t < 1$, $(Atc)(x, z) > t(Ac)(x, z)$, all $x > 0$, all z. Suppose, then, that two strictly positive fixed points c_1 and c_2 exist such that, for some $0 < t < 1$, $c_1 \geqslant tc_2$ and $c_1(x, z) = tc_2(x, z)$ for some $x > 0$ and some z. This would lead to the following contradiction: $c_1(x, z) = (Ac_1)(x, z) \geqslant (Atc_2)(x, z) > t(Ac_2)(x, z) = tc_2(x, z)$, all $x > 0$, all z. In general, however, no such $t > 0$ exists, as it cannot be ensured that a strictly positive lower bound for $c_1(x, z)/c_2(x, z)$ exists. The above concavity property of A is thus not sufficient to ensure that a strictly positive fixed point is unique. For any $x_0 > 0$, however, a lower bound for $c_1(x, z)/c_2(x, z)$ does exist over the region $x \geqslant x_0$. The idea pursued here is to somehow strengthen the property of monotonicity so as to obtain the above contradiction within this smaller domain.

Definition: The monotone function A: $C_F(K \times Z) \to C_F(K \times Z)$ is *pseudo-concave* if for any strictly positive $c \in C_F(K \times Z)$ and any $0 < t < 1$,

$$(Atc)(x, z) > t(Ac)(x, z), \qquad \text{all } x > 0, \text{all } z. \tag{4.1}$$

Definition: The function A: $C_F(K \times Z) \to C_F(K \times Z)$ is *x_0-monotone* if it is monotone and if for any strictly positive fixed point c_1 of A there exists some

$x_0 > 0$ such that the following is true: for any $0 \leqslant x_1 \leqslant x_0$ and any $c_2 \in C_F(K \times Z)$ such that $c_1(x, z) \geqslant c_2(x, z)$, all $x \geqslant x_1$, all z,

$$c_1(x, z) \geqslant (Ac_2)(x, z), \qquad \text{all } x \geqslant x_1, \text{ all } z. \tag{4.2}$$

If A is x_0-monotone then it is not necessary that $c_1(x, z) \geqslant c_2(x, z)$, all $x < x_1$, all z, for the monotonicity property of (4.2) to hold. With these definitions, the following uniqueness theorem can be proven.

THEOREM 8: *An x_0-monotone, pseudo-concave function A: $C_F(K \times Z) \to C_F(K \times Z)$ has at most one strictly positive fixed point $c \in C_F(K \times Z)$.*

PROOF: Suppose that two distinct strictly positive fixed points c_1 and c_2 exist. Without loss of generality, suppose $c_1(\hat{x}, \hat{z}) < c_2(\hat{x}, \hat{z})$ for some $\hat{x} > 0$ and some $\hat{z}$. Choose $0 < x_0 \leqslant \hat{x}$ and $0 < t < 1$ such that (i) $c_1(x, z) \geqslant tc_2(x, z)$, all $x \geqslant x_0$, all z, with equality for some (x, z), and (ii) $c_1(x, z) \geqslant (Atc_2)(x, z)$, all $x \geqslant x_0$, all z. Since A is x_0-monotone, this can be done. This leads to the contradiction

$$\begin{aligned} c_1(x, z) &\geqslant (Atc_2)(x, z) \\ &> t(Ac_2)(x, z) \\ &= tc_2(x, z), \qquad \text{all } x \geqslant x_0, \text{ all } z. \end{aligned}$$

Q.E.D.

Theorem 8 is similar to one proven by Krasnosel'skiĭ and Zabreĭko (1984, Theorem 46.1). They restrict the properties of A such that a $0 < t < 1$ can be chosen whereby the proof by contradiction of Theorem 8 is established even when $x_0 = 0$. The function A does not appear to satisfy their restrictions.

The following two lemmas establish that A is x_0-monotone. This requires the following assumption.

ASSUMPTION 6: $\beta H(0, z')\pi(z'|z) > 1$, *all* z, *all* z'.

As shown in the next lemma, this assumption ensures that net investment adds to the capital-labor ratio when the capital-labor ratio is low. One often assumes that $H(0, z) = \infty$, all z, which is obviously a sufficient condition for Assumption 6.

LEMMA 9: *With Assumptions* 1–4 *and* 6, *for any strictly positive fixed point c of A there exists some $x_0 > 0$ such that $F(x, z) - c(x, z) \geqslant x$, all $x \leqslant x_0$, all z.*

PROOF: Suppose c is a strictly positive fixed point of A and that for every $x_0 > 0$ there exists some $\hat{x} \leqslant x_0$ and some $\hat{z}$ such that $F(\hat{x}, \hat{z}) - c(\hat{x}, \hat{z}) < \hat{x}$. If this is true, then substituting $\hat{x}$ for $F(\hat{x}, \hat{z}) - c(\hat{x}, \hat{z})$ in (2.5) establishes that

$$u'(c(\hat{x}, \hat{z})) > \beta E_z[u'(c(\hat{x}, z'))H(\hat{x}, z')].$$

With Assumption 6, x_0 can be chosen sufficiently small so that this inequality implies

$$u'(c(\hat{x},\hat{z})) > \sum_{z'\in Z} u'(c(\hat{x},z')),$$

which is a contradiction. *Q.E.D.*

LEMMA 10: *Under Assumptions* 1–4 *and* 6, *A is x_0-monotone.*

PROOF: For any strictly positive fixed point c_1 of A, choose x_0 such that $F(x,z)-c_1(x,z)\geqslant x$, all $x\leqslant x_0$, all z. With the results of Lemma 9, this can be done. Choose any $0\leqslant x_1\leqslant x_0$ and $c_2\in C_F(K\times Z)$ such that $c_1(x,z)\geqslant c_2(x,z)$, all $x\geqslant x_1$, all z. Since $F(x,z)-c_1(x,z)\geqslant x_1$, all $x\geqslant x_1$, all z, it follows that $c_1[F(x,z)-c_1(x,z),z']\geqslant c_2[F(x,z)-c_1(x,z),z']$, all $x\geqslant x_1$, all z, all z', and thus

$$u'(c_1(x,z))\leqslant \beta E_z\big[u'(c_2(F(x,z)-c_1(x,z),z'))\times H(F(x,z)-c_1(x,z),z')\big],$$

all $x\geqslant x_1$, all z. This proves that $c_1(x,z)\geqslant (Ac_2)(x,z)$, all $x\geqslant x_1$, all z. *Q.E.D.*

To establish that A is pseudo-concave, the following assumption is sufficient.

ASSUMPTION 7: $u'(xy)=u'(x)u'(y)$, *all* $x,y\in K_{++}$.

Assumptions 4 and 7 are equivalent to constant relative risk aversion ($\log u'(x)= -\sigma\log x$) over the domain K_{++}, as proven in Eichhorn (1978, pp. 12–13).[10]

THEOREM 11: *Under Assumptions* 1–4 *and* 6–7, *A has at most one strictly positive fixed point.*

PROOF: Proposition 3 establishes that A: $C_F(K\times Z)\to C_F(K\times Z)$, Proposition 4 establishes that A is monotone, and Lemma 10 establishes that A is x_0-monotone. To show that A is pseudo-concave, in the functional equation determining $A(tc)$, substitute $tA(c)$ for $A(tc)$ to show that a sufficient condition for (4.1) is

$$u'(t(Ac)(x,z)) > \beta E_z\big[u'(tc(F(x,z)-t(Ac)(x,z),z'))\times H(F(x,z)-t(Ac)(x,z),z')\big],$$

all $x>0$, all z. Use Assumption 7 to cancel the first t in the argument of $u'(\cdot)$ on both sides of the inequality, and then use Assumption 3 to show that the

[10]Assumption 7 pertains only to the domain K_{++} so that the restriction in Assumption 4 that utility is bounded is not inconsistent with Assumption 7. This effectively restricts utility to be between linear and log utility over the domain K_{++}.

strict inequality holds. A is thus pseudo-concave, so by Theorem 8 there exists at most one strictly positive fixed point. *Q.E.D.*

5. COMPUTATION OF THE EQUILIBRIUM

Theorem 12 summarizes and slightly strengthens the results obtained so far.

THEOREM 12: *Under Assumptions* 1–7, *for any strictly positive* $c_0 \in C_F(K \times Z)$, *the sequence* $\{c_n\}$ *defined recursively by* $c_{n+1} = A(c_n)$, $n \geqslant 0$, *converges uniformly to the unique strictly positive fixed point* c^*.

PROOF: We already know that $A^n(F)$ converges uniformly to the unique strictly positive fixed point, so the proof is complete if for any strictly positive c_0 there exists a $\underline{c} \leqslant c_0$ that also converges uniformly to this fixed point. Choose an x_0 and α such that, for all z, $0 < \alpha < F(x_0, z) - x_0$, $\beta E_z[H(x_0, z')] \leqslant 1$, and $c_0(x_0, z) \geqslant \alpha$. Since F and τ satisfy Assumption 5 and c_0 is strictly positive, this can be done. Define $\underline{c}$ by

$$\underline{c} = \inf_c \{c\colon c(x_0, z) = \alpha, \text{ all } z, c \in C_F(K \times Z)\}.$$

Since $C_F(K \times Z)$ is compact, such a $\underline{c}$ exists. Clearly $\underline{c} \leqslant c_0$, and since A is monotone

$$A^n(\underline{c}) \leqslant A^n(c_0) \leqslant A^n(f).$$

$A(\underline{c}) \geqslant \underline{c}$ follows from the proof of Theorem 7, and a similar application of Tarski's fixed-point theorem guarantees that $A^n(\underline{c})$ converges uniformly to the fixed point c^*. *Q.E.D.*

As shown by Lucas and Stokey (1987, Theorem 3) in a different context, even without the additional assumptions guaranteeing uniqueness, if both $A^n(\underline{c})$ and $A^n(F)$ converge to the same solution c^*, then there does not exist another solution in the subset $\{c\colon \underline{c} \leqslant c \leqslant F\}$ of $C_F(K \times Z)$.

6. CONCLUDING REMARKS

This paper solved an infinite-horizon problem by proving that a sequence of finite-horizon problems converged to an equilibrium. Conceptually, this idea is not new. What is new here, however, is constructing this sequence via an operator A built up from first-order and envelope equations, and using the additional structure this reveals to prove certain properties of this sequence, or, equivalently, properties of the operator A. Since the recursive structure underlying the equilibrium equation (2.5) is common to a wide variety of models, for many of these models it may be possible to construct a similar operator A. The additional structure this reveals may differ from problem to problem, but the success of the application in this paper leads one to hope that this is a powerful method to study a rather large class of models.

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REFERENCES

BEALS, RICHARD, AND TJALLING C. KOOPMANS (1969): "Maximizing Stationary Utility in a Constant Technology," *SIAM Journal of Applied Mathematics*, 17, 1001–1015.

BECKER, ROBERT A. (1985): "Capital Income Taxation and Perfect Foresight," *Journal of Public Economics*, 26, 147–167.

BENVENISTE, L. M., AND J. A. SCHEINKMAN (1979): "On the Differentiability of the Value Function in Dynamic Models of Economics," *Econometrica*, 47, 727–732.

BLACKWELL, DAVID (1965): "Discounted Dynamic Programming," *Annals of Mathematical Statistics*, 36, 226–235.

BROCK, WILLIAM A., AND LEONARD J. MIRMAN (1972): "Optimal Economic Growth and Uncertainty: the Discounted Case," *Journal of Economic Theory*, 4, 497–513.

CASS, DAVID (1965): "Optimum Growth in an Aggregative Model of Capital Accumulation," *Review of Economic Studies*, 32, 233–240.

COLEMAN, WILBUR JOHN II (1987): "Money, Interest, and Capital," unpublished Ph.D. dissertation, University of Chicago, Dept. of Economics.

DANTHINE, JEAN-PIERRE, AND JOHN B. DONALDSON (1986): "A Note on the Effects of Capital Income Taxation on the Dynamics of a Competitive Economy," *Journal of Public Economics*, 28, 255–265.

DEBREU, G. (1954): "Valuation Equilibrium and Pareto Optimum," *Proceedings of the National Academy of Sciences of the U.S.A.*, 40, 588–592.

DUGUNDJI, JAMES, AND ANDRZEJ GRANAS (1982): *Fixed Point Theory*. Warsaw: Polish Scientific Publishers.

EICHHORN, WOLFGANG (1978): *Functional Equations in Economics*. Massachusetts: Addison-Wesley.

KEHOE, TIMOTHY J., DAVID K. LEVINE, AND PAUL M. ROMER (1989): "Characterizing Equilibria of Models with Externalities and Taxes as Solutions to Optimization Problems," unpublished manuscript, Federal Reserve Bank of Minneapolis.

KRASNOSEL'SKIĬ, M. A., AND P. P. ZABREĬKO (1984): *Geometrical Methods of Nonlinear Analysis*. Berlin: Springer-Verlag.

LEVHARI, D., AND T. SRINIVASAN (1969): "Optimal Savings Under Uncertainty," *Review of Economic Studies*, 36, 153–163.

LUCAS, ROBERT E., JR., AND NANCY L. STOKEY (1987): "Money and Interest in a Cash-in-Advance Economy," *Econometrica*, 55, 491–513.

ROYDEN, H. L. (1988): *Real Analysis*. 3rd ed. New York: Macmillan.

STOKEY, NANCY L., ROBERT E. LUCAS, JR., AND EDWARD C. PRESCOTT (1989): *Recursive Methods in Economic Dynamics*. Cambridge, Massachusetts: Harvard University Press.