Assignment 4 Due Date: Monday February 15th

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I. Learning how to learn Can you think of a "machine learning algorithm" that could make the decisions given in the El-Gamal an Grether (1995) "Are People Bayesian?" experiment? Further, can you think of a way to teach this machine using "training data" that you provide to it? The machine couldi also be a real person, if you are trying to show examples to a person where you draw balls with replacement as in the El-Gamal and Grether experiment but instead of failing to reveal the actual bingo cage from which the sample of 6 balls with replacement were drawn (as El-Gamal and Grether did, to try to discourage any "learning about how to learn" by the subjects in the experiment), suppose you showed the subjects (or the machine learning algorithm) a number of random draws from the bingo cages for different prior values and as well as the *actual bingo cage* from which the balls were actually drawn? If you did this could you teach the human or machine learner how to behave rationally, i.e. according to Bayes Rule? Unfortunately, the data from the El Gamal and Grether experiment did not record which bingo cage the actual sample was drawn from. It only reports the total number of balls marked "N" that were drawn. *HINT:* The first step is to *program a training data generator* where for different prior probabilities of drawing from cage A (with 4 balls marked N) and cage B (with 3 balls marked N, the other 3 marked G) you also draw 6 balls from the chosen urn with replacement. If you then treat randomly chosen cage A or B from which the balls were drawn as the "dependent variable" and the result from randomly sampling 6 balls with replacement and the prior probability as "x variables" can you train a "machine learning algorithm" to "learn how to learn" and with enough "training instances" will this machine learning algorithm learn behave like a Bayesian learner? **HINT:** consider the model of subjective beliefs that subjects might have given in equation (19) of the answers to problem set 3. This is a parametric model of subjective beliefs that depend on three unknown parameters, (a,b,c) that includes Bayes Rule as a special case when (a,b,c) take the specific values given in equations (20), (21), and (22) of the answers to problem set 3. Now consider estimating these parameters by maximum likelihood given data from a random number generator that I provided you. So you generate "training data" using this computer program and then use this training data to estimate the (a,b,c) parameters by maximum likelihood. In machine learning terminology, you are train the machine to make classifications using an approach called *supervised learning* (i.e you are training the machine based on training instances where you show the machine the "right answer" to each training instance). I want you to generate samples of increasing sizes say, $N = 10, 100, 1000, 10000$ and show whether, with more training, the machine learning algorithm will converge to Bayesian learning, and if so, how fast is the *rate of convergence* to Bayesian decision making? Stated differently, approximately how many training instances N are necessary to train the machine learning algorithm to be "close" to Bayesian decision making (you might quantify this by asking how large N needs to be so that the maximum likelihood estimates $(\hat{a},\hat{b},\hat{c})$ are with in a ball of radius 0.01 around the true values corresponding to Bayes Rule (call them (a^*,b^*,c^*) where the starred values are given by equations (20), (21) and (22) of my answers to problem set 3) with probability at least 95%? **Further hint:** Here you might think of using asymptotics, and the fact that the maximum likeihood estimator is asymptotically normal. **Further hint:** write down the log-likeihood function for the binary logit model of "subjective probability" for choosing bingo cage A given in equation (19) of the answer to problem set 3, and the gradient and hessian for this log-likelihood. Then program this up as a Matlab function *learning_to_learn.blogit* inside the static class of methods in the file *learning_to_learn.m* and then run the Matlab program *train_machine.m* that I provided as a further hint on one way to do this problem. Do a 3-D scatterplot showing your parameter estimates and the true values for various training sample sizes suggested above.

II. The data from the experimental study by El-Gamala and Grether "Are People Bayesian?" (JASA 1995) are provided in 8 plain text files ending in "pay" on the web server. Below is the explanation of the structure of these files from David Grether:

"The data for the December 1995 JASA article 'Are People Bayesian' are contained in six files. The files are named for the school viz. csula,oxy,pcc and ucla and the treatment 'no' and 'pay'. The subjects in the 'no' files were paid a flat fee for participating and those in the 'pay' files received a bonus if a randomly chosen response was correct. Thus the file usclno.pay should have data for subjects from ucla who were paid a flat fee. Each file is organized as follows. The first line is a list of numbers 2, 3 and 4 giving the ordered list of the number of chances out of 6 for drawing cage A (the cage with 4 Ns and 2Gs). The second line consists of integers from 0 to 6 giving for each round of the experiment the number of Ns drawn out of 6 draws. The remaining lines contain a subject number and a list of 0s and 1s which give the subjects' responses with 1 meaning they subject picked A and 0 coded for B."

Unfortunately, the data files did not record which of the two cages the balls were actually drawn from, and thus we do not know which guesses by the subjects were correct and which were incorrect and thus how much the subjects would have earned. But despite this, these data are enough to try to replicate their results and try to estimate alternative structural models of subject choice.

A. First estimate the *homogeneous model* of *cutoff rules* and *error rates* in section of their paper and see if you can replicate their finding that for the UCLA students, Bayes rule is the maximum likelihood estimate of the cutoff rule (e.g. $(\hat{c}_1,\hat{c}_2,\hat{c}_3) = (4,3,2)$) and the "error rate" is $\hat{\varepsilon} = .308$. Are the asymptotic properties of this maximum likelihood estimator "standard"? (that is, what are the "standard errors" for c and ε and why weren't they reported in table 2 of their paper)?

B. Using Algorithm A of their paper see if yuou can replicate their result in table 2 that if we allow *heterogeneous cutoff rules* to be used by subjects, but we restrict the search over the set of possible cutoff rules to *allow only two cutoff rules* that the maximum likelihood estimates in table 2 for UCLA subjects are $\hat{c}^1 = (4,3,2)$ (Bayes rule) used by 71 subjects and $\hat{c}^2 = (3,3,3)$ (the "representativeness" rule) is used by 26 subjects. Note that the error rate for this specification falls to $\hat{\varepsilon} = 0.261$. Would it be possible to estimate *subject specific error rates?* If so, try extending their model to estimate subject-specific error rates and characterize the distribution of estimated error rates in the subject pool.

C. One shortcoming of their model is that it assumes if the a choice is "consistent" with a given cutoff rule c then they assume that the subject actually used that rule, whereas if the choice cannot be explained by the cutoff rule, then they assume it was the result of a random guess. However if a subject is randomly guessing, isn't it possible that some of their guesses could be consistent with a cutoff rule just by chance? If we do not observe when a subject is just guessing and when they are "really" using a cutoff rule, how could we adjust their model to account for this? What about the possibility of "trembles"? That is, some models of behavior posit that person intends to submit a response according to a given cutoff rule, but due to a random mistake in their response, they misrecord the choice they really intended to make with some probability. Is a model with "trembles" equivalent to the model that El-Gamal and Grether present in their paper?

D. Another shortcoming of their paper is that there is no explicit way in which the subject compensation enters the choice that subjects choose. They do show in table 2 that all subjects who were paid when they guessed the urn correctly had a lower error rate than subjects that were just paid a flat fee for participating in the experiment, regardless of the number of correct guesses they made. Can you outline a different model where subjects choices is governed by choosing the urn that gives them the highest *expected payoff*?

E. Specifically, consider a random utility model with risk-neutrality, where we assume that a subject chooses urn A if $R\Pi(A|n,\pi)+\sigma\varepsilon(A) \geq R\Pi(B|n,\pi)+\sigma\varepsilon(B)$, where $(\varepsilon(A),\varepsilon(B))$ are random utilities shocks associated with the subject's choice of either urn A or B, and R is the monetarypayoff the subject receives for guessing the urn from which the sample is drawn correctly, n is the number of balls marked G in the sample of 6 balls drawn from the urn and π is the prior probability that the balls would be drawn from urn A (where π takes only 3 values in this experiment, $(1/3,1/2,2/3)$). Can you think of a *flexible functional form* for the *subjective probability* $\Pi(A|n,\pi)$ and $\Pi(B|n,\pi) = 1-\Pi(A|n,\pi)$ that represents a subject's subjective probability that in an experiment with outcome (n,π) the sample is drawn from urn A or B? Ideally, this specification should include Bayes Rule as a special case, so you could test a range of theories of how different subjects form subjective probabilities that includes the "rational" theory Bayes rule as a special case.

F. Given your specification above, can you estimate the parameters of your random utility model and the scaling parameter σ? Start by estimating a homogeneous model, i.e. where you assume all subjects have the same subjective probability $\Pi(A|n,\pi)$ and the same scale parameter σ. Show that this RUM specification leads to the following *conditional choice probability* $P(d|x)$ where in this case d indicates the choice of either urn A or B by the subject and $x=(n,\pi)$

$$P(d|x) = \frac{\exp\{R\Pi(d|n,\pi)/\sigma\}}{\exp\{R\Pi(A|n,\pi)/\sigma\}+\exp\{R\Pi(B|n,\pi)/\sigma\}}, \quad d \in \{A,B\} \tag{1}$$

Note the important distinction in this model between $\Pi(A|n,\pi)$ (your model of the subject's subjective probability that a sample with n G balls when the prior is π was drawn from urn A) and $P(A|n,\pi)$ (the binomial logit choice probability that the subject will choose urn A after seeing n G balls when the prior is π). How well does this model fit the data compared to El-Gamal and Grether's "baseline" cutoff rule model you replicated in part A above? Can you estimate *subject specific* $P(A|n,\pi)$ and σ values or modify their Algorithm A to estimate a fixed number of "types" of subjects differing in their subjective probabilities and σ values?

G. Consider the *Heckman Singer* approach for incorporating *unobserved heterogeneity* into discrete choice models. Suppose you have parameters θ that characterize your flexible functional form for $P(A|n,\pi)$ so you can write it as $P(A|n,\pi,\theta)$. Suppose there are a finite number of "types" of subjects (indexed by their type, τ) so each type is characterized by the pair $(\theta_\tau,\sigma_\tau)$. Then for a given subject i, $i=1,\ldots,N$ who makes a sequence of T choices $(d_1^i,\ldots,d_T^i)$ in response to seeing $\{(n_t,\pi_t)\}$, $t=1,\ldots,T$ where n_t is the number of G balls drawn in the t^{th} round of the experiment under prior π_t of drawing from urn A, and $d_t=A$ if the subject chose urn A or $d_t=B$ otherwise, the likelihood for a type τ person is

$$L(\{d_t^i\}|\theta_\tau,\sigma_t) = \prod_{t=1}^{T} P(d_t^i|n_t,\pi_t,\theta_\tau,\sigma_t) \tag{2}$$

where $P(d_t^i|n_t,\pi_t,\theta,\sigma)$ is the logit probability that subject i makes choice d_t^i in the t^{th} round of the experiment where this logit choice probability is given in equation (1) above when your parameterization of the subject's subjective probability that the data comes from urn A is $\Pi(A|n_t,\pi_t,\theta)$ and σ is the extreme value scaling parameter. Then let $f(\tau)$ be the probability that a given subject has type τ. Then the Heckman Singer approach for estimating a 2 type model involves estimating a parameter vector $(\theta_{\tau_1},\sigma_{\tau_1},\theta_{\tau_2},\sigma_{\tau_2},f(\tau_1))$ using the following likelihood function

$$L(\theta_{\tau_1},\sigma_{\tau_1},\theta_{\tau_2},\sigma_{\tau_2},f(\tau)) = \prod_{i=1}^{N}\left[L(\{d_t^i\}|\theta_{\tau_1},\sigma_{\tau_1})f(\tau_1)+L(\{d_t^i\}|\theta_{\tau_2},\sigma_{\tau_2})(1-f(\tau_1))\right] \qquad (3)$$

The Heckman-Singer approach thus also allows for two types of subjects (who have possibly different subjective probability functions $\Pi(d|n,\pi,\theta_{\tau_1})$ and $\Pi(d|n,\pi,\theta_{\tau_2})$, respectively, and also two different extreme value scale parameters σ_{τ_1} and σ_{τ_2}, resepctively) but it differs from the way El-Gamal and Grether allowed for subject heterogeneity. Discuss these two approaches and for those of you who are really ambitious substantial extra credit is given if you manage to actually estimate the Heckman Singer specification above and compare the resulting model in terms of how well it fits the data with El-Gamal and Grether's cutoff rule specification. Is there any way you can see of testing which of these two different approaches to modeling subject heterogeneity and subject decision making is the "correct" one?

H. So far, we have not considered a "cost of effort" to making a decision. But what if there is a mental cost to calculation? A subject can always minimize effort and make a random guess and get paid the reward R with some probability if their guess was correct. The models above assume subjects don't really have a cost of effort in responding in the experiment. Can you speculate on whether it might be possible to extend one of the above models discussed above to allow for a "cost of mental effort" and whether it would be possible to identify parameters relating to the cost of effort and thereby help to predict which subjects choose to just make random guesses, and which engage in costly effort to make the best possible decision?

I. Also the models considered so far do not consider the effect of "learning by doing" — i.e. that subjects may get better at this task with repetition, so their initial guesses are poorer than the ones they make toward the end of the experiment. Can you speculate how you might include experience effects and learning by doing into one of the models above and are there ways you could test whether subjects' decisions are "stationary" i.e. their decision rule does not change during the experiment, or whether there is evidence of "non-stationarity" so their decision making process seems to be changing in response to repeating the choices up to 20 times?

J. Finally consider the "reduced form" approach. This does not try to delve into specific theories about what subjects are doing or why but rather simply tries to estimate flexible models of the subjects' choice probabilities, $P(d|x)$. If you do not have a good idea of what subjects are doing in this experiment or why, what sort of reduced form model might you try to estimate to try to at least summarize subject behavior without trying to test or understand their mental decision making process? Without the constraints of having to derive a choice probability from some theory of individual decision making (including Bayes Rule), do you find it easier to estimate a reduced form model and will this model be more flexible and fit the data better than any of the structural models considered above?