Coding Exercise 3: Nash-in-Nash Bargaining

In this coding exercise students work with the Nash-in-Nash bargaining model. I have supplied `R` code comprising nn_starter.R, the main file with which the user interacts, and nn_functions.R, which defines a number of user-written functions. The first file calls the second.

Model:

In the model, each of three upstream "brands" sells to a single retailer. The input prices are determined by Nash-in-Nash bargaining. The retailer sets downstream prices taking as given the input prices and a logit demand system. Consumers choose among the brands and an outside good. Marginal costs are zero for each of the three brands. The retailer also does not incur marginal costs, aside from the input prices that it pays to the brands.

For notation, if all brands are sold by the retailer then the quantity sold of any one brand j is given by:

$$s_j = \frac{\exp\left(\xi_j + \alpha p_j\right)}{1 + \sum_{k \in \mathcal{J}} \exp(\xi_k + \alpha p_k)} \tag{1}$$

where $\alpha < 0$ and $\mathcal{J}$ is a set containing the three brands. With Nash-in-Nash bargaining, all three products are on the shelf in equilibrium, but the threat of removing one brand affects the equilibrium input prices. Thus, solving for equilibrium requires computing quantities with different sets of brands, and adjusting (1) accordingly. If brand j is not "on the shelf" then $s_j = 0$. The profit of brand j is:

$$\pi_j^B = w_j s_j$$

where w_j is the input price. The profit of the retailer is:

$$\pi^R = \sum_{k \in \mathcal{J}} (p_k - w_k) s_k$$

Finally, the bargaining parameter associated with brand j be θ_j.

With this in hand, the Nash product for brand k and the retailer equals:

$$\left(\pi_k^B(\mathcal{J}) - 0\right)^{\theta_k} \times \left(\pi^R(\mathcal{J}) - \pi^R(\mathcal{J} \backslash k)\right)^{(1-\theta_k)} \tag{2}$$

The left term has a disagreement value for brand k of zero. The right term has a disagreement value for the retailer equal to the profit that it would earn if it excludes the brand, taking into account that it could recapture some of brand k's sales with the other brands. In any Nash-in-Nash equilibrium, the input price w_k maximizes this Nash product conditional on the other input prices. That leads to the following first order condition:

$$\begin{aligned} 0 &= \theta_k \left(\pi_k^B(\mathcal{J}) - 0\right)^{(\theta_k - 1)} \frac{\partial \pi_k^B(\mathcal{J})}{\partial w_k} \times \left(\pi^R(\mathcal{J}) - \pi^R(\mathcal{J}\backslash k)\right)^{(1-\theta_k)} \\ &+ \left(\pi_k^B(\mathcal{J}) - 0\right)^{\theta_k} \times (1-\theta_k) \left(\pi^R(\mathcal{J}) - \pi^R(\mathcal{J}\backslash k)\right)^{(-\theta_k)} \frac{\partial \pi_k^R(\mathcal{J})}{\partial w_k} \end{aligned}$$

There are analogous first order conditions for each brand $j \in \mathcal{J}$. A vector of input prices constitutes a Nash-in-Nash equilibrium if it satisfies each of the first order conditions.

The Retailer's Pricing Problem (with Logit Tricks):

The retailer sets downstream prices that satisfy first order conditions for profit maximization:

$$p = w - \left(\frac{\partial s}{\partial p}\right)^{-1} s \tag{3}$$

where $p = (p_1, p_2, p_3)$, $w = (w_1, w_2, w_3)$, and $s = (s_1, s_2, s_3)$, keeping in mind that there are no costs other than input prices. With logit demand the demand derivatives are

$$\frac{\partial s_j}{\partial p_k} = \begin{cases} \alpha s_j (1 - s_j) & \text{if } j = k \\ -\alpha s_j s_k & \text{if } j \neq k \end{cases} \tag{4}$$

with $\alpha < 0$ as in our model. Therefore the first order conditions can be re-expressed as

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} - \frac{1}{\alpha} \begin{bmatrix} s_1(1-s_1) & -s_1 s_2 & -s_1 s_3 \\ -s_1 s_2 & s_2(1-s_2) & -s_2 s_3 \\ -s_1 s_3 & -s_2 s_3 & s_3(1-s_3) \end{bmatrix}^{-1} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

or, working through the matrix algebra,

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} - \frac{1}{\alpha} \begin{bmatrix} \frac{1}{1-sum(s)} \\ \frac{1}{1-sum(s)} \\ \frac{1}{1-sum(s)} \end{bmatrix}$$

And this is a remarkable result, one that is specific to logit demand. Notice two aspects of these FOCs:

1. The markup on each brand is the same. This is the "common markup" property of logit/Bertrand and it means that a profit-maximizing firm facing logit demand will set the same markup on all its products.

2. The markup is a function of the price parameter and the *combined* market share of the brands. This also is specific to logit/Bertrand models. In equilibrium, a firm with greater total market share has higher markups.

The code uses these tricks to compute the profit maximizing prices of the retailer for a given vector of input prices (functions "retail" and "retail_wrapper"). Using the common markup property, the search can a one-dimensional search over a markup rather than a $||\mathcal{J}||$-dimension search over prices. This save computation times—a trivial benefit here—but I suspect it also is somewhat more precise, which may help with the numeric derivatives. Something to have in your toolkit going forward.

Tasks for the Student:

Please do the following:

1. Use the initial parameterization in nn_starter.R, compute the Nash-in-Nash equilibrium, and summarize the results. This can be accomplished by adjusting the path in nn_starter.R and running the first 30 lines or so. There parameterization is: $\xi = (1, 2, 3)$, $\alpha = -1$, and $\theta = (0.5, 0.5, 0.5)$.

2. Explain how the nashinnash function calculates the outside option of the retailer. What steps are involved?

3. Explain how the nashinnash function calculates $\frac{\partial \pi_k^B(\mathcal{J})}{\partial w_k}$ and $\frac{\partial \pi^R(\mathcal{J})}{\partial w_k}$. Are results sensitive to the scaling factor? (1e-4 is used in the code.)

4. Change something about the parameterization and recompute equilibrium. Explain the economic intuition behind your results.

5. (Optional) Adjust the code so that it accommodates brands having non-zero marginal cost. Suppose that the marginal cost vector for brands is $c = (0, 0, 1)$, i.e., the third brand has a higher cost than it does in the initial parameterization. How does that affect the input price and profit of brand 2? Explain the economic intuition.