Solving Two RBC Models with Matlab

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Abstract: This note covers two simple RBC models that help understand many stylized facts about economic fluctuation. The first benchmark model features a typical Cobb-Douglas production function and a frictionless two-sector economy with technology shocks only. The second simple model is otherwise standard but incorporates habit formation and government spending process. The note solves and linearizes optimal conditions, and establishes some facts with regard to business cycles. Supplementary materials with numerical method and Matlab codes will be provided at the workshop.

1 A Benchmark RBC Model

1.1 Introduction

This benchmark model is essentially the same as any textbook introductory RBC models[1]: two-sector frictionless economy, Cobb-Douglas technology and technology shock as sole source of fluctuation. We take this model as a benchmark to characterize some stylized facts about business cycles (and business cycle models) and to familiarize you with necessary steps to solve a model.

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[1]For example, models introduced in Long and Plosser(1983), Kydland and Prescott(1982), King and Rebelo(1999) among many other variations.

1.2 Firm

Representative firm maximizes its profit with production technology of a Cobb-Douglas form ($0 < \alpha < 1$):

$$y_t = a_t k_t^{\alpha} l_t^{1-\alpha}$$

There is perfect competition for labor and capital, in other word, the factors of production are paid their marginal products.
TFP, a_t, follows an AR(1) process given by ($0 < \rho < 1$):

$$\ln \tfrac{a_{t+1}}{a} = \rho \ln \tfrac{a_t}{a} + \epsilon_t, \quad \epsilon_t : N(0, \sigma)$$

1.3 Household

Representative household maximizes life time utility given by:

$$max E_t \textstyle\sum_{s=0}^{\infty} \beta^t [\tfrac{c_t^{1-\gamma}}{1-\gamma} - \tfrac{l_t^{1+\theta}}{1+\theta}]$$

where $0 < \beta < 1$, $\gamma > 0$, $\theta > 0$.
subject to budget constraint:

$$w_t l_t + r_t k_t = c_t + i_t$$

and capital law of motion($0 < \delta < 1$):

$$k_{t+1} = (1 - \delta) k_t + i_t$$

1.4 Optimal Conditions

$$\alpha \frac{y_t}{k_t} = r_t \tag{1}$$

$$(1 - \alpha) \frac{y_t}{l_t} = w_t \tag{2}$$

$$y_t = a_t k_t^{\alpha} l_t^{1-\alpha} \tag{3}$$

$$l_t^{\theta} = c_t^{-\gamma} w_t \tag{4}$$

$$c_t^{-\gamma} = \beta E_t c_{t+1}^{-\gamma}(r_{t+1} + 1 - \delta) \tag{5}$$

$$k_{t+1} = (1-\delta)k_t + i_t \tag{6}$$

$$y_t = c_t + i_t \tag{7}$$

$$\ln a_{t+1} = (1-\rho)\ln a + \rho \ln a_t + \epsilon_t \tag{8}$$

We can solve these 8 variables $[y; c; k; l; r; w; i; a]$ with 8 equations .
The next two steps are to solve the steady state and log-linearize above optimal conditions[2].

1.5 steady state

One can simply remove subscript of variables in all endogenous equations to obtain the steady state.(Steady state holds unconditionally for exogenous equations.)

$$\alpha \frac{y}{k} = r \tag{9}$$

$$(1-\alpha)\frac{y}{l} = w \tag{10}$$

$$y = ak^{\alpha}l^{1-\alpha} \tag{11}$$

$$l^{\theta} = c^{-\gamma}w \tag{12}$$

$$1 = \beta(r + 1 - \delta) \tag{13}$$

$$\delta k = i \tag{14}$$

$$y = c + i \tag{15}$$

$$\ln a = \ln a \tag{16}$$

[2]See Appendix 1 for log-linearization techniques

1.6 Log-Linearization

$$\hat{y_t} - \hat{k_t} - \hat{r_t} = 0 \tag{17}$$

$$0 = \hat{y_t} - \hat{l_t} - \hat{w_t} \tag{18}$$

$$0 = \hat{y_t} - \alpha\hat{k_t} - (1-\alpha)\hat{l_t} - \hat{a_t} \tag{19}$$

$$0 = \gamma\hat{c_t} + \theta\hat{l_t} - \hat{w_t} \tag{20}$$

$$E_t(\gamma\hat{c_{t+1}} - \hat{r_{t+1}}) = \gamma\hat{c_t} \tag{21}$$

$$k\hat{k_{t+1}} = (1-\delta)k\hat{k_t} + i\hat{i_t} \tag{22}$$

$$0 = y\hat{y_t} - c\hat{c_t} - i\hat{i_t} \tag{23}$$

$$\hat{a_{t+1}} - \epsilon_{t+1} = \rho\hat{a_t} \tag{24}$$

Define $\eta_{t+1} = \gamma\hat{c_{t+1}} - \hat{r_{t+1}} - E_t(\gamma\hat{c_{t+1}} - \hat{r_{t+1}})$, we re-write equation (22) as:

$$\gamma\hat{c_{t+1}} - \hat{r_{t+1}} = \gamma\hat{c_t} + \eta_{t+1}$$

We then put the log-linearized system into a matrix form:

$$G_0 Y_{t+1} = G_1 Y_t + \Psi\epsilon_{t+1} + \Pi\eta_{t+1}$$

Or into an extended form to allow foresight[3]:

$$G_0 Y^*_{t+1} = G_1 Y^*_t + \Psi\epsilon_{t+1} + \Pi\eta_{t+1}$$

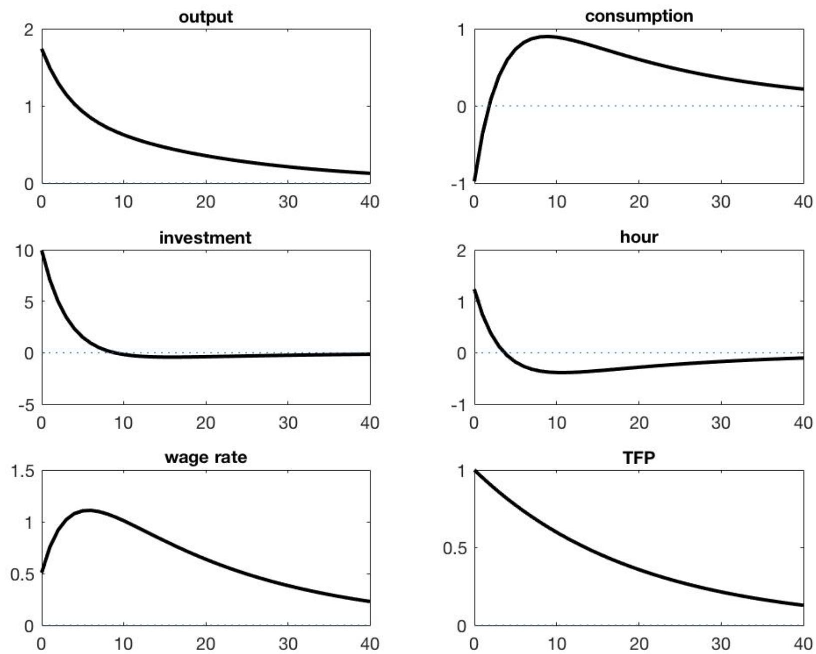
1.7 Calibration

Benchmark Economy (Economy A):

α	β	γ	θ	δ	ρ	a	σ
0.4	0.985	2	2	0.025	0.95	1	1

[3]This form also allows for second-order shocks. See Appendix 2 for a comparison of two matrix form.

1.8 Impulse Response



1.9 Discussion

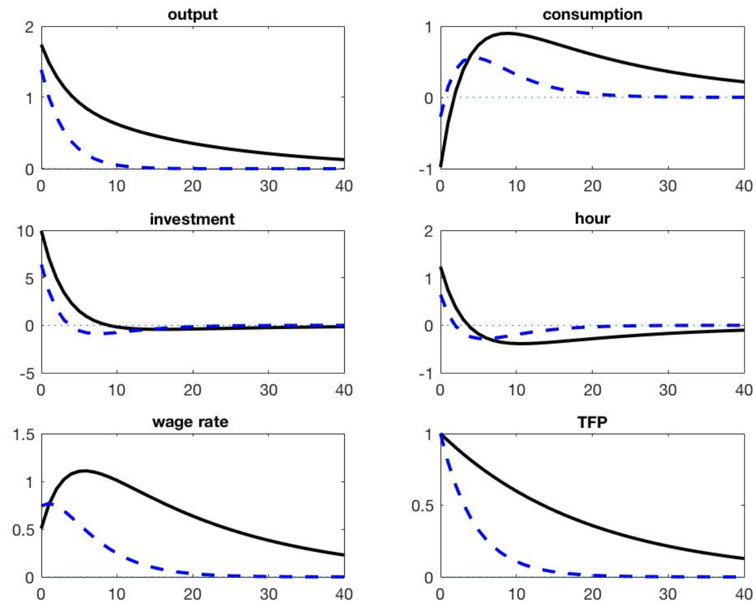
1) Impact of persistence of technology shock.

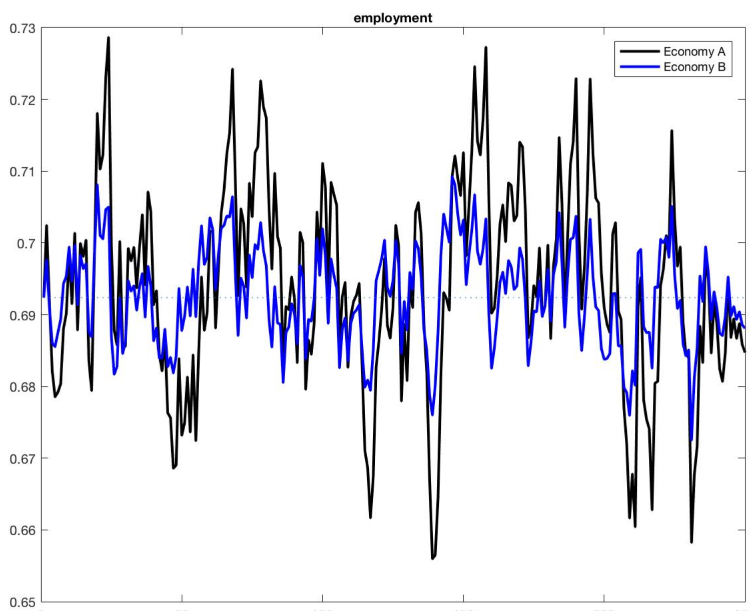
Economy A:

α	β	γ	θ	δ	ρ	a	σ
0.4	0.985	2	2	0.025	0.95	1	1

Economy B:

α	β	γ	θ	δ	ρ	a	σ
0.4	0.985	2	2	0.025	0.8	1	1





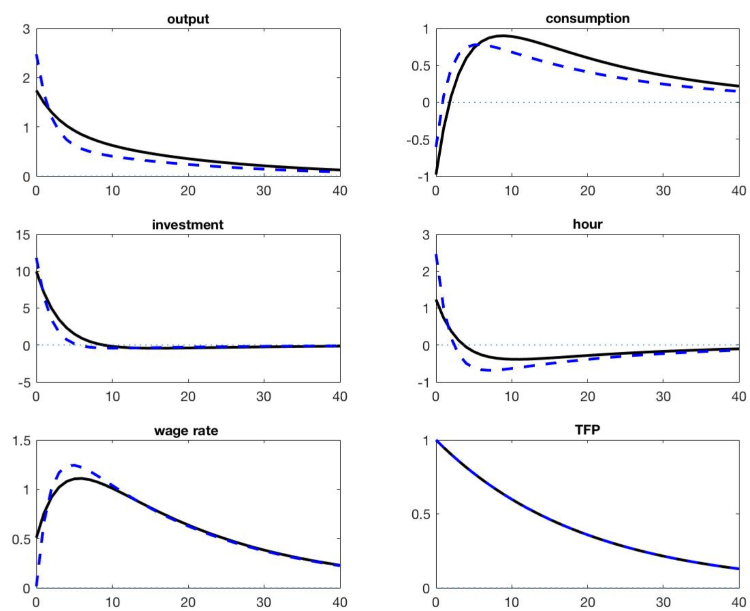
2) Impact of elasticity of labor supply.

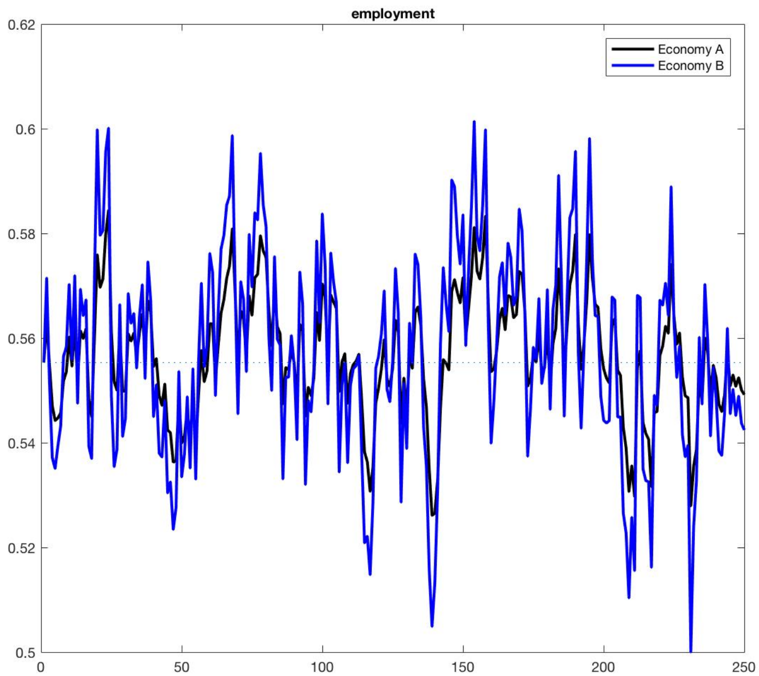
Economy A:

α	β	γ	θ	δ	ρ	a	σ
0.4	0.985	2	2	0.025	0.95	1	1

Economy C:

α	β	γ	θ	δ	ρ	a	σ
0.4	0.985	2	0.5	0.025	0.95	1	1





2 A Simple RBC Model with Fiscal Shock

2.1 Introduction

Upon building prototype RBC models, macroeconomists started using them to understand beyond business cycles phenomena. To see if prototype (simple) RBC models can be adequate, economists often compare empirical evidence with model implications of a model on a specific structural shock of interest. This section adds one common type of sluggish adjustments, habit formation, in a model with government spending process to illustrate how RBC models evolve over time and to account for the quantitative effects of government spending[4].

[4]Some of classic work include Baxter and King(1993), Burnside, Eichenbaum and Fisher(2004).

2.2 Household

Representative household maximizes life time utility given by:

$$maxE_t \sum_{s=0}^{\infty} \beta^t [\frac{(c_t - hC_{t-1})^{1-\gamma}}{1-\gamma} - \frac{l_t^{1+\theta}}{1+\theta}]$$

where $0 < \beta < 1$, $\gamma > 0$, $\theta > 0$.
subject to budget constraint:

$$(1 - \tau_t)(w_t l_t + r_t k_t) + r_t^b b_{t-1} + z_t = c_t + i_t + b_t$$

and capital law of motion $(0 < \delta < 1)$[5]:

$$k_{t+1} = (1 - \delta)k_t + i_t$$

2.3 Firm

Representative firm maximizes its profit with production technology of a Cobb-Douglas form $(0 < \alpha < 1)$:

$$y_t = a_t k_t^{\alpha} l_t^{1-\alpha}$$

There is perfect competition for labor and capital, in other word, the factors of production are paid their marginal products.
TFP, a_t, follows an AR(1) process given by $(0 < \rho < 1)$:

$$\ln \frac{a_{t+1}}{a} = \rho^a \ln \frac{a_t}{a} + \epsilon_t^a, \quad \epsilon_t^a : N(0, \sigma_a^2)$$

2.4 Government

Government collect tax and issue debt to pay for its purchase (g_t), transfers (z_t) and debt service. Its flow budget constraint therefore is:

$$tax_t + b_t = g_t + z_t + r_t^b b_{t-1}$$

[5] To add another sluggish adjustment we often incorporate a convex capital adjustment cost, for example, in a form of $k_{t+1} = (1 - \delta)k_t + [1 - S(\frac{i_t}{i_{t-1}})]i_t$.

where $tax_t = \tau_t(w_t l_t + r_t k_t) = \tau_t y_t$.
we focus on government spending effects; the lump-sum transfers are set to constant each period ($z_t = z$), and g_t follows a simple AR(1) process:

$$\ln \tfrac{g_{t+1}}{g} = \rho^g \ln \tfrac{g_t}{g} + \epsilon_t^g, \quad \epsilon_t^g : N(0, \sigma_g^2)$$

Tax rule: The income tax rate serves as the fiscal adjustment instrument and follows the rule:

$$\tau_t = \tau + \phi(b_{t-1} - b)$$

2.5 Optimal conditions

$$\alpha \frac{y_t}{k_t} = r_t \tag{25}$$

$$(1-\alpha)\frac{y_t}{l_t} = w_t \tag{26}$$

$$y_t = a_t k_t^{\alpha} l_t^{1-\alpha} \tag{27}$$

$$(c_t - hC_{t-1})^{-\gamma} = \lambda_t \tag{28}$$

$$l_t^{\theta} = \lambda_t (1-\tau_t) w_t \tag{29}$$

$$\lambda_t = \beta E_t[\lambda_{t+1}(1-\tau_{t+1}) r_{t+1} + \mu_{t+1}(1-\delta)] \tag{30}$$

$$\lambda_t = \beta E_t \lambda_{t+1} r_{t+1}^b \tag{31}$$

$$\lambda_t = \mu_t \tag{32}$$

$$k_{t+1} = (1-\delta)k_t + i_t \tag{33}$$

$$(1-\tau_t)y_t + r_t^b b_{t-1} + z = c_t + i_t + b_t \tag{34}$$

$$tax_t + b_t = g_t + z + r_t^b b_{t-1}$$

This equation together with equation (34) implies:

$$y_t = c_t + i_t + g_t \tag{35}$$

$$\tau_t = \tau + \phi(b_{t-1} - b) \tag{36}$$

$$\ln \frac{a_{t+1}}{a} = \rho^a \ln \frac{a_t}{a} + \epsilon_t^a \tag{37}$$

$$\ln \frac{g_{t+1}}{g} = \rho^g \ln \frac{g_t}{g} + \epsilon_t^g \tag{38}$$

There are 14 variables to be solved by above system[6]:

$$[y_t, c_t, i_t, k_t, l_t, b_t, r_t, w_t, r_t^b, \tau_t, \lambda_t, \mu_t, a_t, g_t]$$

We then solve the steady state and log-linearize the system as we did above.

2.6 Steady State

[7]

$$\alpha \frac{y}{k} = r \tag{39}$$

$$(1-\alpha)\frac{y}{l} = w \tag{40}$$

$$y = ak^{\alpha} l^{1-\alpha} \tag{41}$$

$$[(1-h)c]^{-\gamma} = \lambda \tag{42}$$

$$l^{\theta} = \lambda(1-\tau)w \tag{43}$$

$$1 = \beta[(1-\tau)r + (1-\delta)] \tag{44}$$

$$1 = \beta r^b \tag{45}$$

$$\lambda = \mu \tag{46}$$

$$\delta k = i \tag{47}$$

$$(1-\tau)y + r^b b + z = c + i + b \tag{48}$$

$$y = c + i + g \tag{49}$$

$$\tau = \tau \tag{50}$$

$$\ln \frac{a}{a} = \rho^a \ln \frac{a}{a} \tag{51}$$

$$\ln \frac{g}{g} = \rho^g \ln \frac{g}{g} \tag{52}$$

[6] Here we focus on a symmetric equilibrium where in equilibrium $c_t = C_t$.

[7] Condition (32) is utilized for simplification.

2.7 Log-linearization

[8]

$$0 = \hat{y}_t - \hat{k}_t - \hat{r}_t \tag{53}$$

$$0 = \hat{y}_t - \hat{l}_t - \hat{w}_t \tag{54}$$

$$0 = \hat{y}_t - \alpha\hat{k}_t - (1-\alpha)\hat{l}_t - \hat{a}_t \tag{55}$$

$$\frac{\gamma}{1-h}\widehat{c_{t+1}} + \widehat{\lambda_{t+1}} = \frac{\gamma h}{1-h}\hat{c}_t \tag{56}$$

$$0 = \theta\hat{l}_t - \hat{w}_t - \hat{\tau}_t - \hat{\lambda}_t \tag{57}$$

$$\widehat{\lambda_{t+1}} + \beta(1-\tau)r\widehat{\tau_{t+1}} + \beta(1-\tau)r\widehat{r_{t+1}} = \hat{\lambda}_t + \eta^1_{t+1}{}^{9} \tag{58}$$

$$\widehat{r^b_{t+1}} + \widehat{\lambda_{t+1}} = \hat{\lambda}_t + \eta^2_{t+1}{}^{10} \tag{59}$$

$$\hat{\lambda}_t = \hat{\mu}_t$$

$$k\widehat{k_{t+1}} = (1-\delta)k\hat{k}_t + i\hat{i}_t \tag{60}$$

$$(1-\tau)y\widehat{y_{t+1}} - c\widehat{c_{t+1}} - i\widehat{i_{t+1}} - b\widehat{b_{t+1}} - r^b b\widehat{b_{t+1}} + (1-\tau)y\widehat{\tau_{t+1}} = -r^b b\widehat{r^b_t} \tag{61}$$

$$0 = y\hat{y}_t - c\hat{c}_t - i\hat{i}_t - g\hat{g}_t \tag{62}$$

$$\tau\widehat{\tau_{t+1}} = \phi b\hat{b}_t \tag{63}$$

$$\widehat{a_{t+1}} - \epsilon^a_{t+1} = \rho^a\hat{a}_t \tag{64}$$

$$\widehat{g_{t+1}} - \epsilon^g_{t+1} = \rho^g\hat{g}_t \tag{65}$$

There are 13 variables to be solved by above system:

$$[y_t, c_t, i_t, k_t, l_t, b_t, r_t, w_t, r^b_t, \tau_t, \lambda_t, a_t, g_t]$$

[8] After utilizing condition (32), 13 variables remain.

[9] $\eta^1_{t+1} = \widehat{\lambda_{t+1}} + \beta(1-\tau)r\widehat{\tau_{t+1}} + \beta(1-\tau)r\widehat{r_{t+1}} - E_t[\widehat{\lambda_{t+1}} + \beta(1-\tau)r\widehat{\tau_{t+1}} + \beta(1-\tau)r\widehat{r_{t+1}}]$

[10] $\eta^2_{t+1} = \widehat{r^b_{t+1}} + \widehat{\lambda_{t+1}} - E_t[\widehat{r^b_{t+1}} + \widehat{\lambda_{t+1}}]$

2.8 Calibration

Benchmark Economy (Economy A):

α	β	γ	θ	δ	z	τ	ϕ	ρ^a	ρ^g	a	g	σ^a	σ^g
0.4	0.985	2	2	0.025		0.25	0.6	0.95	0.5	1		1	1

2.9 Impulse Response

Available at workshop.

2.10 Discussion

Available at workshop.

Reference

1.Baxter, Marianne, and Robert G. King. "Fiscal policy in general equilibrium." The American Economic Review, 1993: 315-334.
2.Burnside, Craig, Martin Eichenbaum, and Jonas DM Fisher."Fiscal shocks and their consequences" Journal of Economic theory 115.1, 2004: 89-117.
3.Long, John, Jr. and Charles Plosser,"Real Business Cycles" Journal of Political Economy, Vol. 91, 1983: 39-69.
4.King, Robert G., and Sergio T. Rebelo. "Resuscitating real business cycles." Handbook of macroeconomics, 1999: 927-1007.
5.Kydland Finn and Edward Prescott,"Time to Build and Aggregate Fluctuations" Econometrica, Vol. 50, November, 1982: 1345-1370.

Appendix 1: Techniques for Log-linearization

- single variable: $c_t = c(1 + \hat{c}_t)$
- two variables: $k_t l_t = kl(1 + \hat{k}_t + \hat{l}_t)$
- A quick way: $e^{\hat{c}_t} = 1 + \hat{c}_t$
- $c_t = ce^{\hat{c}_t} = c(1 + \hat{c}_t)$
- $k_t l_t = ke^{\hat{k}_t} le^{\hat{l}_t} = kle^{\hat{k}_t + \hat{l}_t} = kl(1 + \hat{k}_t + \hat{l}_t)$
- $k_t^a l_t^b = (ke^{\hat{k}_t})^a (le^{\hat{l}_t})^b = k^a l^b e^{a\hat{k}_t + b\hat{l}_t} = k^a l^b (1 + a\hat{k}_t + b\hat{l}_t)$

Appendix 2: Equivalence of Two Matrix Forms

$$\hat{a_{t+1}} = \rho \hat{a}_t + \epsilon_{t+1} \tag{66}$$

is equivalent to above two equations:

$$\hat{a_{t+1}} - u\hat{a_{t+1}} = \rho \hat{a}_t \tag{67}$$

$$u\hat{a_{t+1}} = \sigma \hat{\epsilon_{t+1}}, \sigma = s.e.(shock) \tag{68}$$