Structure from Motion and Planar Homography

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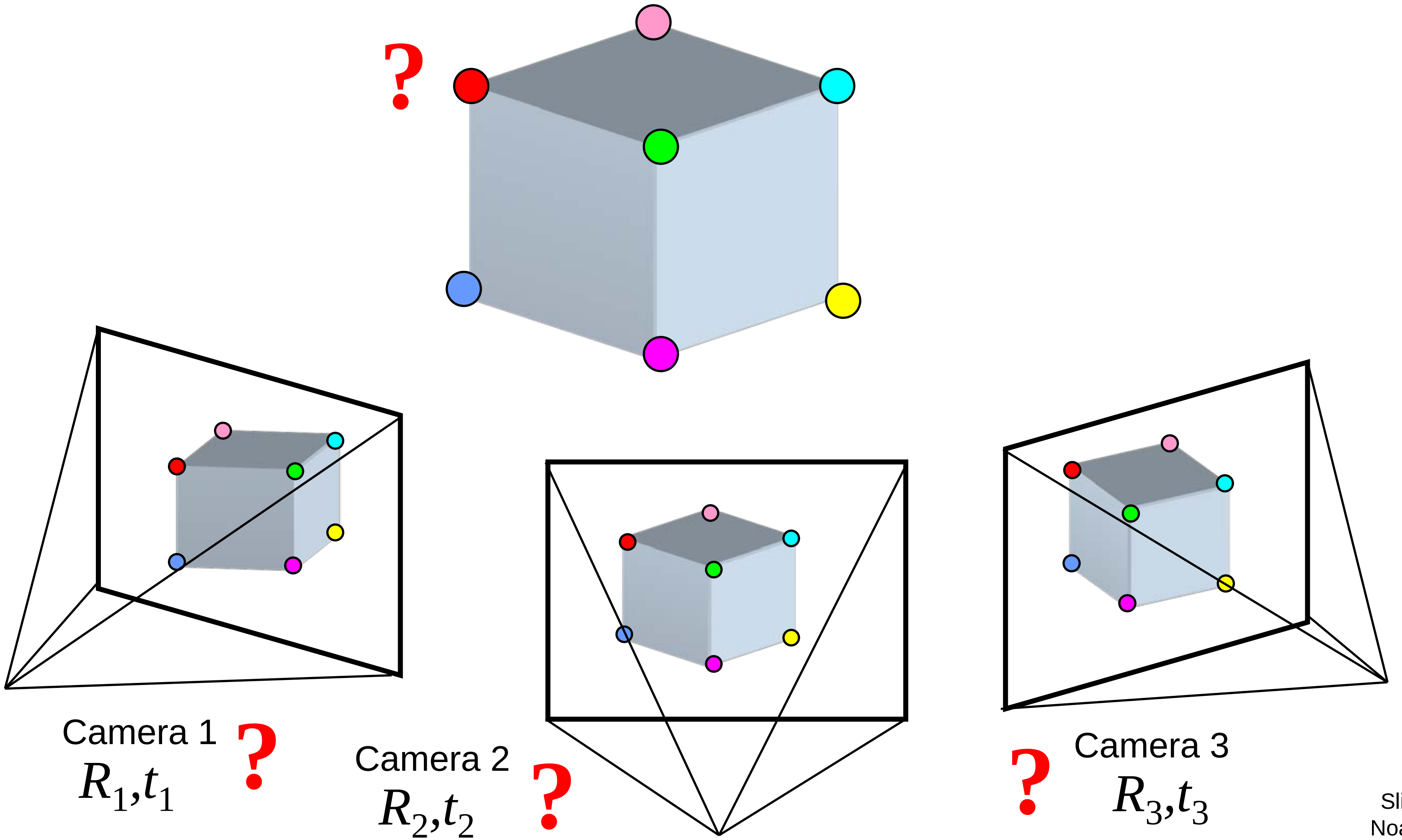
Slide credit to Svetlana Lazebnik and Robert T. Collins

Structure from motion

- Given a set of corresponding points in two or more images, compute the camera parameters and the 3D point coordinates



Slide credit: Noah Snavely

Overview: Structure from motion

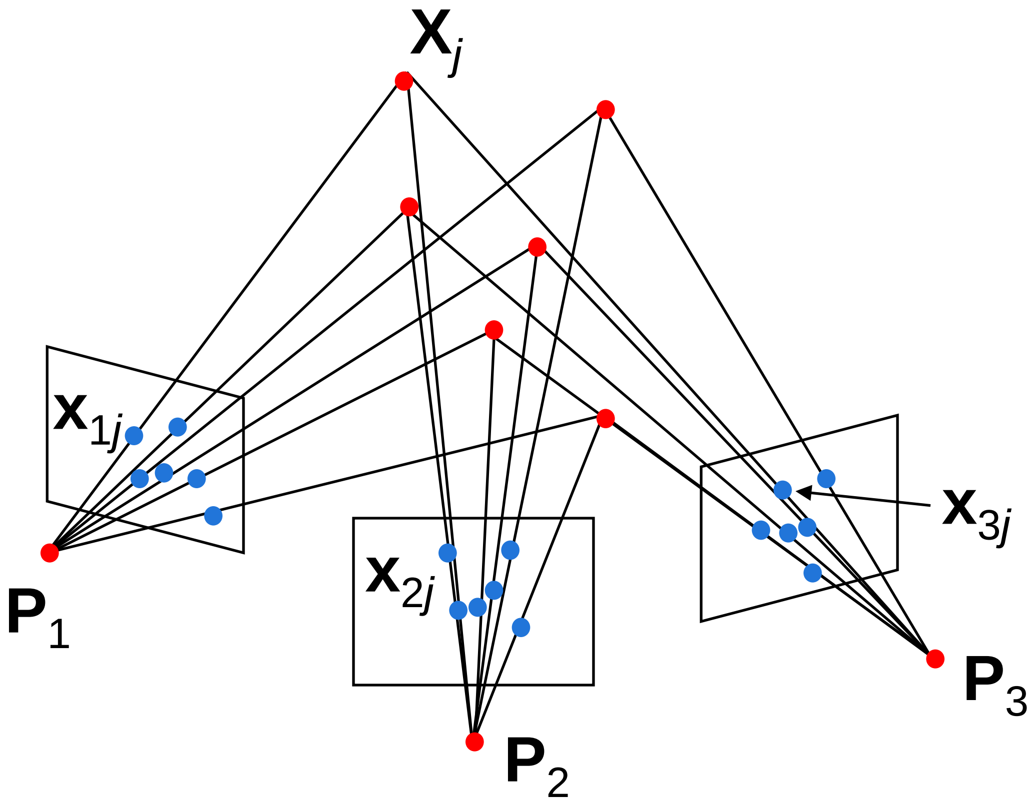
- Ambiguity
- Affine structure from motion
  - Factorization
- Dealing with missing data
  - Incremental structure from motion
- Projective structure from motion
  - Bundle adjustment
  - Self-calibration

Structure from motion

- Given: *m* images of *n* fixed 3D points

$$\mathbf{x}_{ij} = \mathbf{P}_i\,\mathbf{X}_j\,, \qquad i = 1, \ldots, m, \quad j = 1, \ldots, n$$

- Problem: estimate *m* projection matrices $\mathbf{P}_i$ and *n* 3D points $\mathbf{X}_j$ from the *mn* correspondences $\mathbf{x}_{ij}$



Structure from motion ambiguity

- If we scale the entire scene by some factor *k* and, at the same time, scale the camera matrices by the factor of 1/*k*, the projections of the scene points in the image remain exactly the same:

$$\mathbf{x} = \mathbf{P}\mathbf{X} = \left(\frac{1}{k}\mathbf{P}\right)(k\,\mathbf{X})$$

It is impossible to recover the absolute scale of the scene!

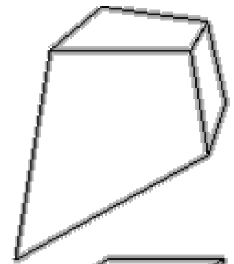
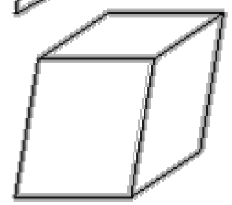
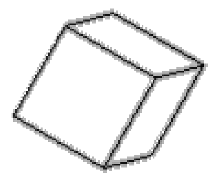
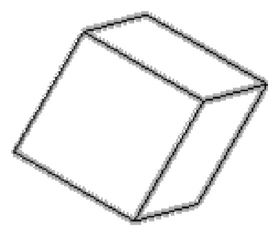
Structure from motion ambiguity

- If we scale the entire scene by some factor *k* and, at the same time, scale the camera matrices by the factor of 1/*k*, the projections of the scene points in the image remain exactly the same

- More generally, if we transform the scene using a transformation **Q** and apply the inverse transformation to the camera matrices, then the images do not change:

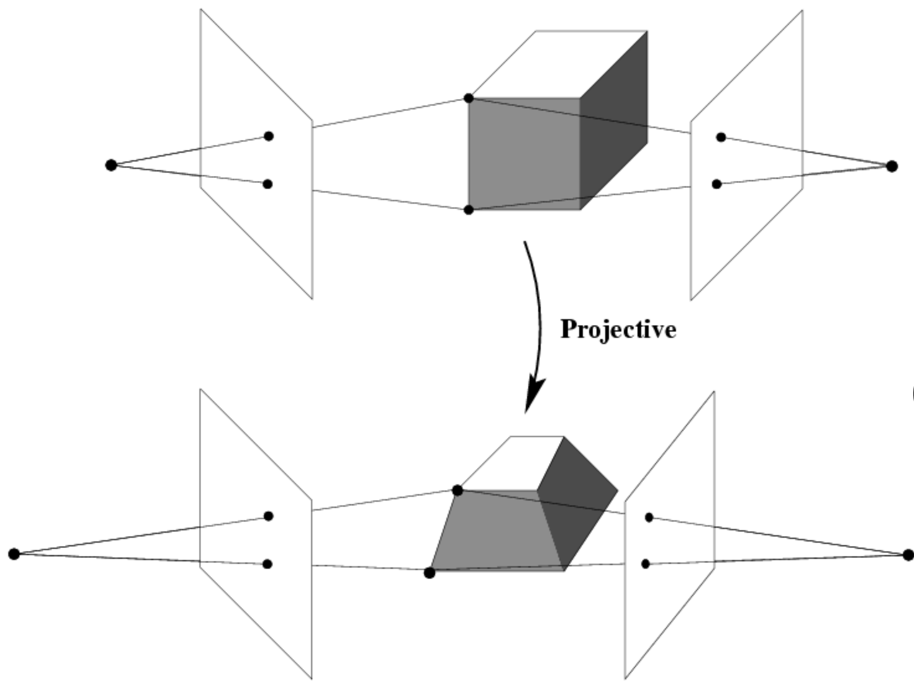
$$\mathbf{x} = \mathbf{P}\mathbf{X} = \left(\mathbf{P}\mathbf{Q}^{-1}\right)\left(\mathbf{Q}\mathbf{X}\right)$$

Types of ambiguity

Projective 15dof	$\begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$		Preserves intersection and tangency
Affine 12dof	$\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$		Preserves parallellism, volume ratios
Similarity 7dof	$\begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$		Preserves angles, ratios of length
Euclidean 6dof	$\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$		Preserves angles, lengths

- With no constraints on the camera calibration matrix or on the scene, we get a *projective* reconstruction
- Need additional information to *upgrade* the reconstruction to affine, similarity, or Euclidean

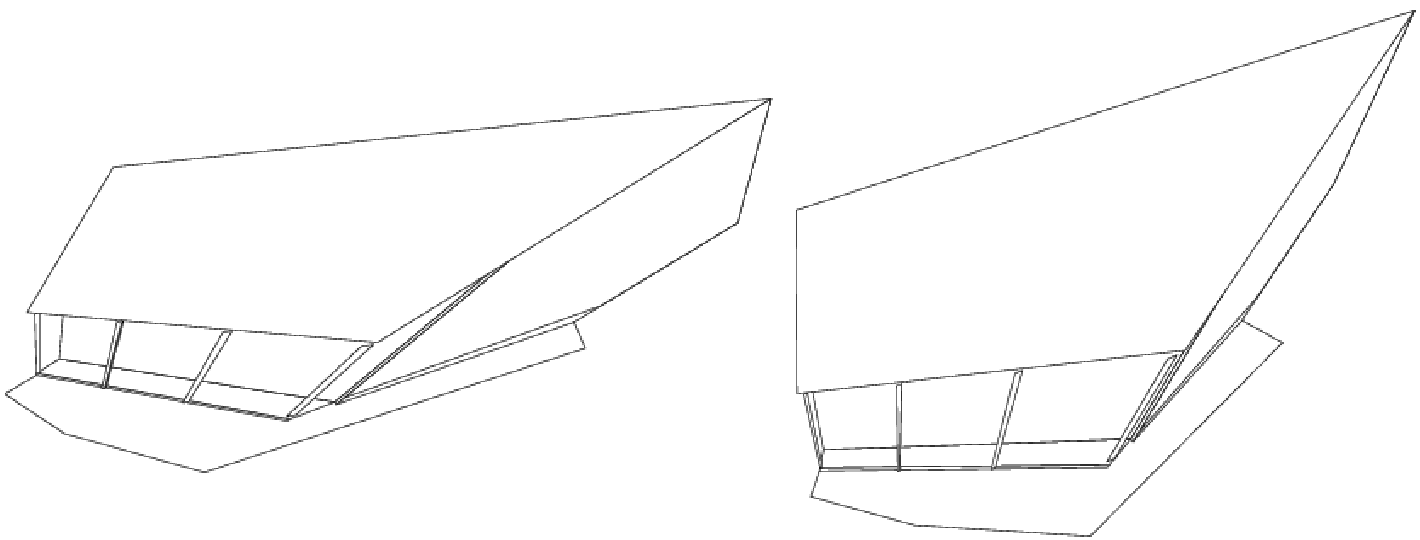
Projective ambiguity



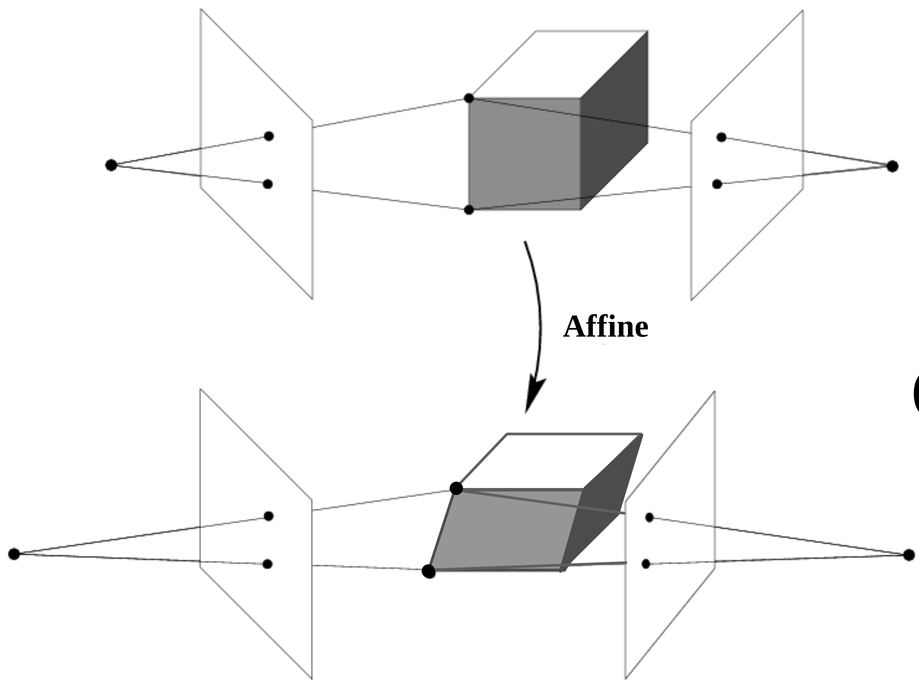
$$\mathbf{Q_p} = \begin{bmatrix} \mathrm{A} & \mathrm{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix}$$

$$\mathbf{x} = \mathbf{PX} = \left(\mathbf{PQ_P^{-1}}\right)\left(\mathbf{Q_P\,X}\right)$$

Projective ambiguity



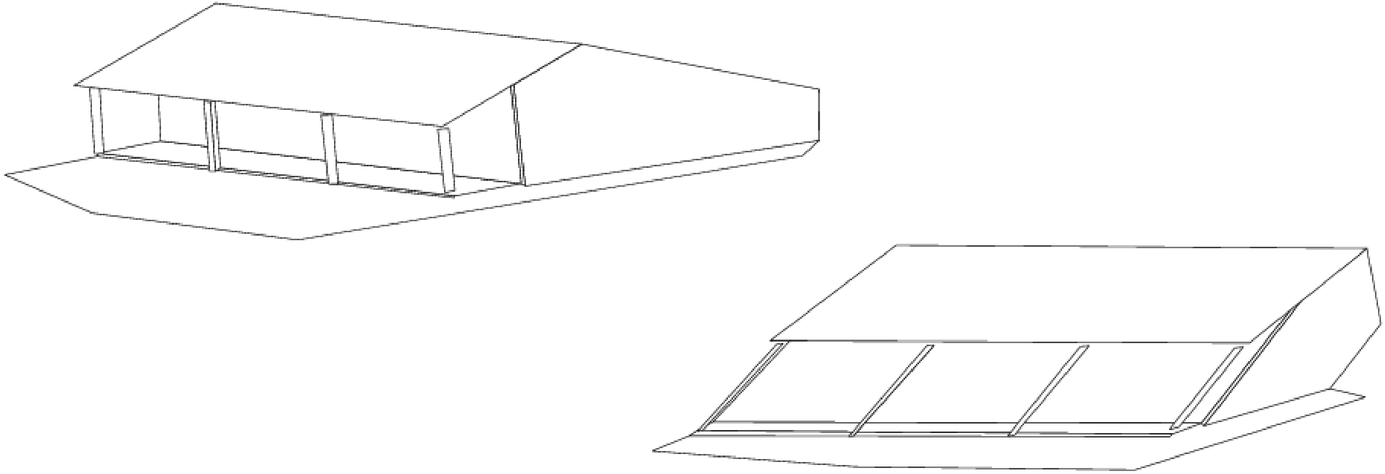
Affine ambiguity



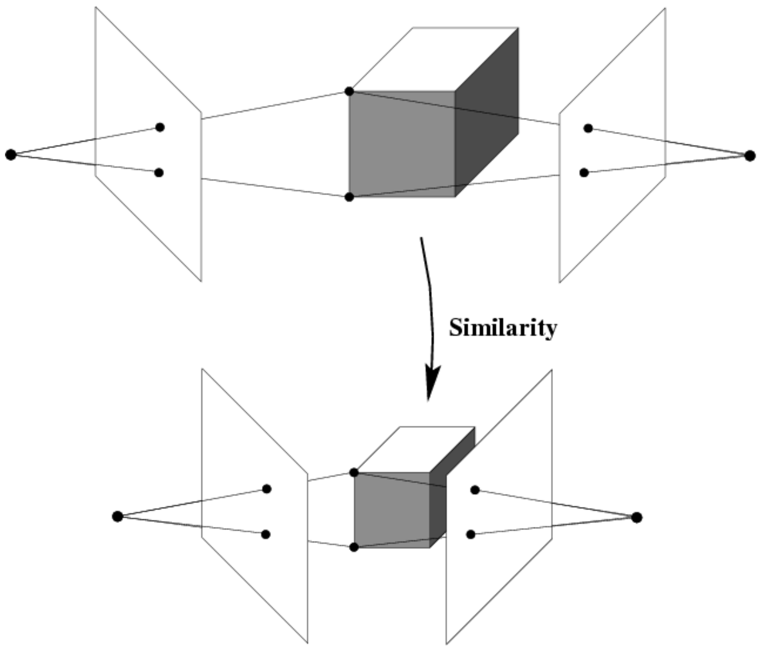
$$\mathbf{Q_A} = \begin{bmatrix} \mathrm{A} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{PX} = \left(\mathbf{PQ_A^{-1}}\right)\left(\mathbf{Q_A X}\right)$$

Affine ambiguity



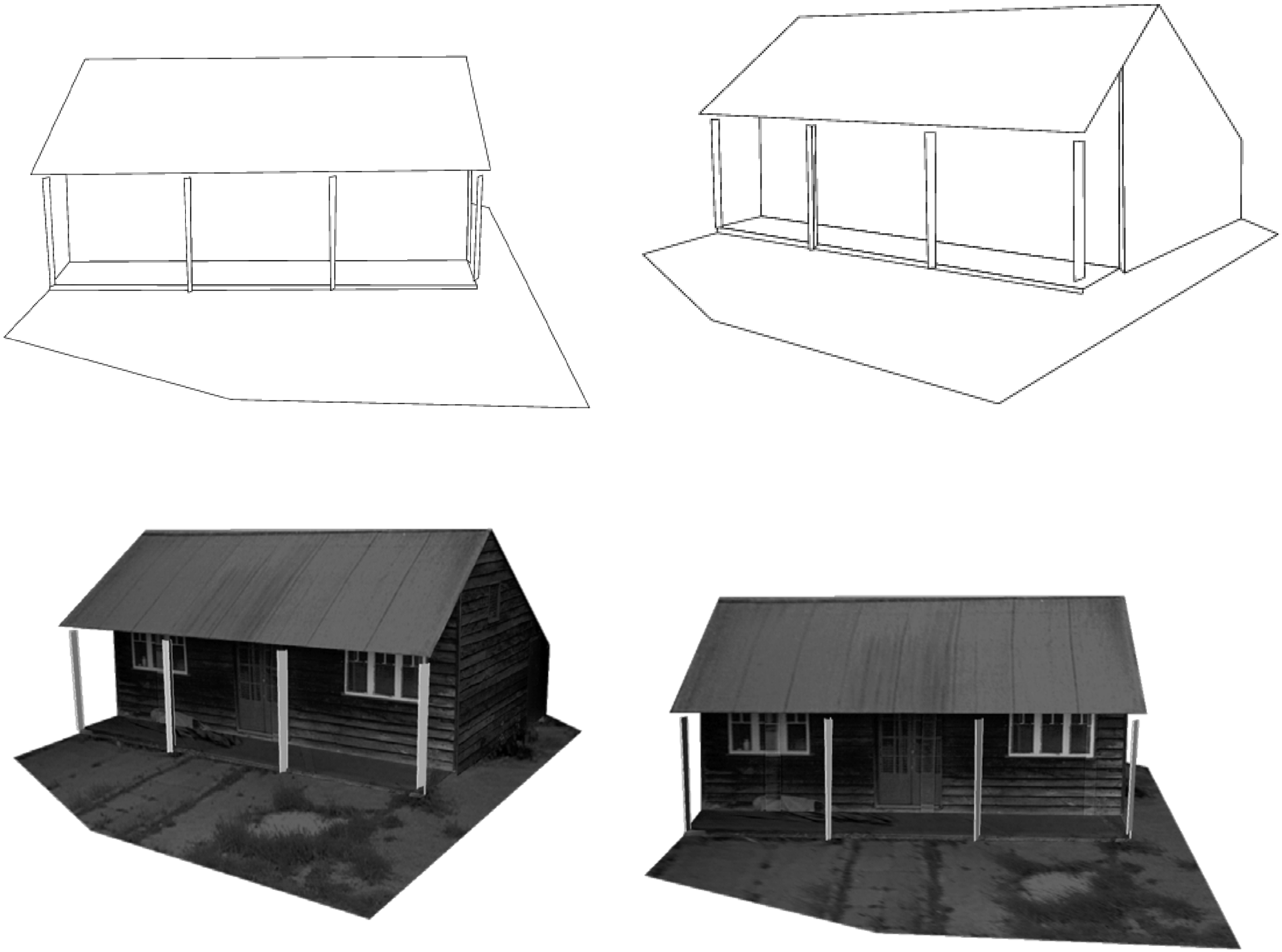
Similarity ambiguity



$$\mathbf{Q}_s = \begin{bmatrix} s\mathrm{R} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix}$$

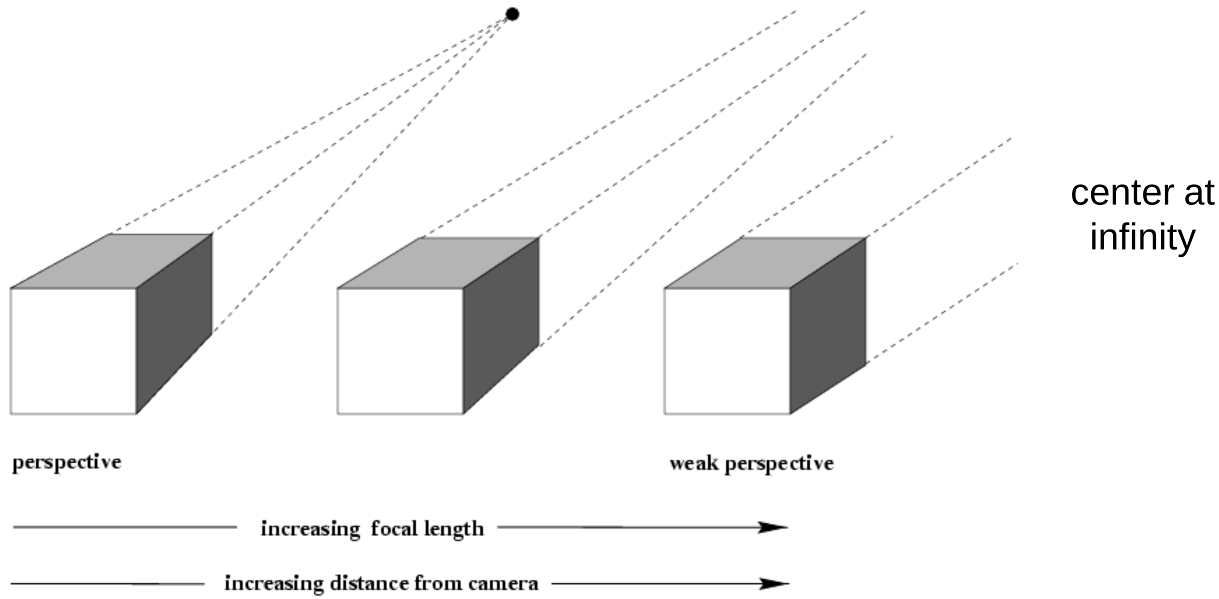
$$\mathbf{x} = \mathbf{P}\mathbf{X} = \left(\mathbf{P}\mathbf{Q}_{\mathrm{S}}^{-1}\right)\left(\mathbf{Q}_{\mathrm{S}}\mathbf{X}\right)$$

Similarity ambiguity



Structure from motion

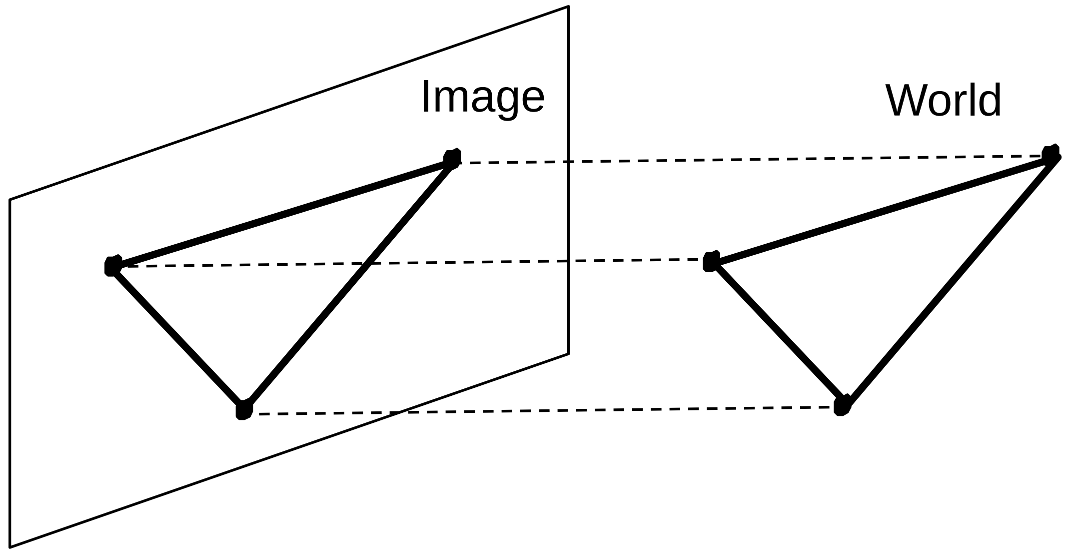
- Let's start with *affine cameras* (the math is easier)



Recall: Orthographic Projection

Special case of perspective projection

- Distance from center of projection to image plane is infinite



- Projection matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow (x, y)$$

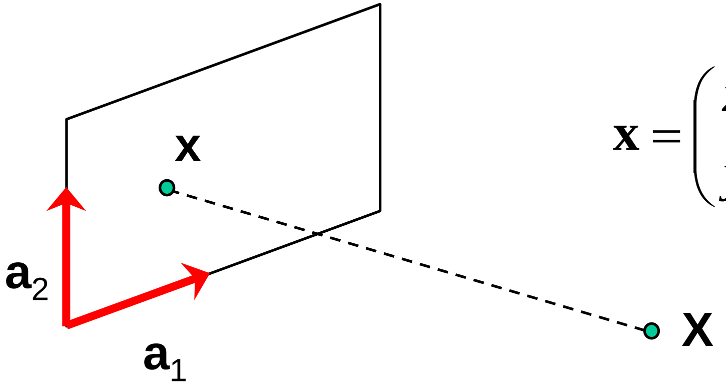
Slide by Steve Seitz

Affine cameras

- A general affine camera combines the effects of an affine transformation of the 3D space, orthographic projection, and an affine transformation of the image:

$$\mathbf{P} = [3\times 3\,\text{affine}]\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}[4\times 4\,\text{affine}] = \begin{bmatrix}a_{11} & a_{12} & a_{13} & b_1\\ a_{21} & a_{22} & a_{23} & b_2\\ 0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{b}\\ \mathbf{0} & 1\end{bmatrix}$$

- Affine projection is a linear mapping + translation in inhomogeneous coordinates



$$\mathbf{x} = \begin{pmatrix}x\\ y\end{pmatrix} = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\end{bmatrix}\begin{pmatrix}X\\ Y\\ Z\end{pmatrix} + \begin{pmatrix}b_1\\ b_2\end{pmatrix} = \mathbf{AX} + \mathbf{b}$$

Projection of world origin

Affine structure from motion

- Given: *m* images of *n* fixed 3D points:

$$\mathbf{x}_{ij} = \mathbf{A}_i \mathbf{X}_j + \mathbf{b}_i, \quad i = 1, \ldots, m, \; j = 1, \ldots, n$$

- Problem: use the *mn* correspondences $\mathbf{x}_{ij}$ to estimate *m* projection matrices $\mathbf{A}_i$ and translation vectors $\mathbf{b}_i$, and *n* points $\mathbf{X}_j$
- The reconstruction is defined up to an arbitrary *affine* transformation **Q** (12 degrees of freedom):

$$\begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{Q}^{-1}, \qquad \begin{pmatrix} \mathbf{X} \\ 1 \end{pmatrix} \rightarrow \mathbf{Q} \begin{pmatrix} \mathbf{X} \\ 1 \end{pmatrix}$$

- We have 2*mn* knowns and 8*m* + 3*n* unknowns (minus 12 dof for affine ambiguity)
- Thus, we must have 2*mn* >= 8*m* + 3*n* – 12
- For two views, we need four point correspondences

Affine structure from motion

- Centering: subtract the centroid of the image points

$$\hat{\mathbf{x}}_{ij} = \mathbf{x}_{ij} - \frac{1}{n}\sum_{k=1}^{n}\mathbf{x}_{ik} = \mathbf{A}_i\mathbf{X}_j + \mathbf{b}_i - \frac{1}{n}\sum_{k=1}^{n}\left(\mathbf{A}_i\mathbf{X}_k + \mathbf{b}_i\right)$$

$$= \mathbf{A}_i\left(\mathbf{X}_j - \frac{1}{n}\sum_{k=1}^{n}\mathbf{X}_k\right) = \mathbf{A}_i\hat{\mathbf{X}}_j$$

- For simplicity, assume that the origin of the world coordinate system is at the centroid of the 3D points
- After centering, each normalized point $\mathbf{x}_{ij}$ is related to the 3D point $\mathbf{X}_i$ by

$$\hat{\mathbf{x}}_{ij} = \mathbf{A}_i\mathbf{X}_j$$

Affine structure from motion

- Let’s create a $2m \times n$ data (measurement) matrix:

$$\mathbf{D} = \begin{bmatrix} \hat{\mathbf{x}}_{11} & \hat{\mathbf{x}}_{12} & \cdots & \hat{\mathbf{x}}_{1n} \\ \hat{\mathbf{x}}_{21} & \hat{\mathbf{x}}_{22} & \cdots & \hat{\mathbf{x}}_{2n} \\ & & \ddots & \\ \hat{\mathbf{x}}_{m1} & \hat{\mathbf{x}}_{m2} & \cdots & \hat{\mathbf{x}}_{mn} \end{bmatrix}$$

cameras
($2m$)

points (n)

C. Tomasi and T. Kanade. [Shape and motion from image streams under orthography: A factorization method.] *IJCV*, 9(2):137-154, November 1992.

Affine structure from motion

- Let's create a $2m \times n$ data (measurement) matrix:

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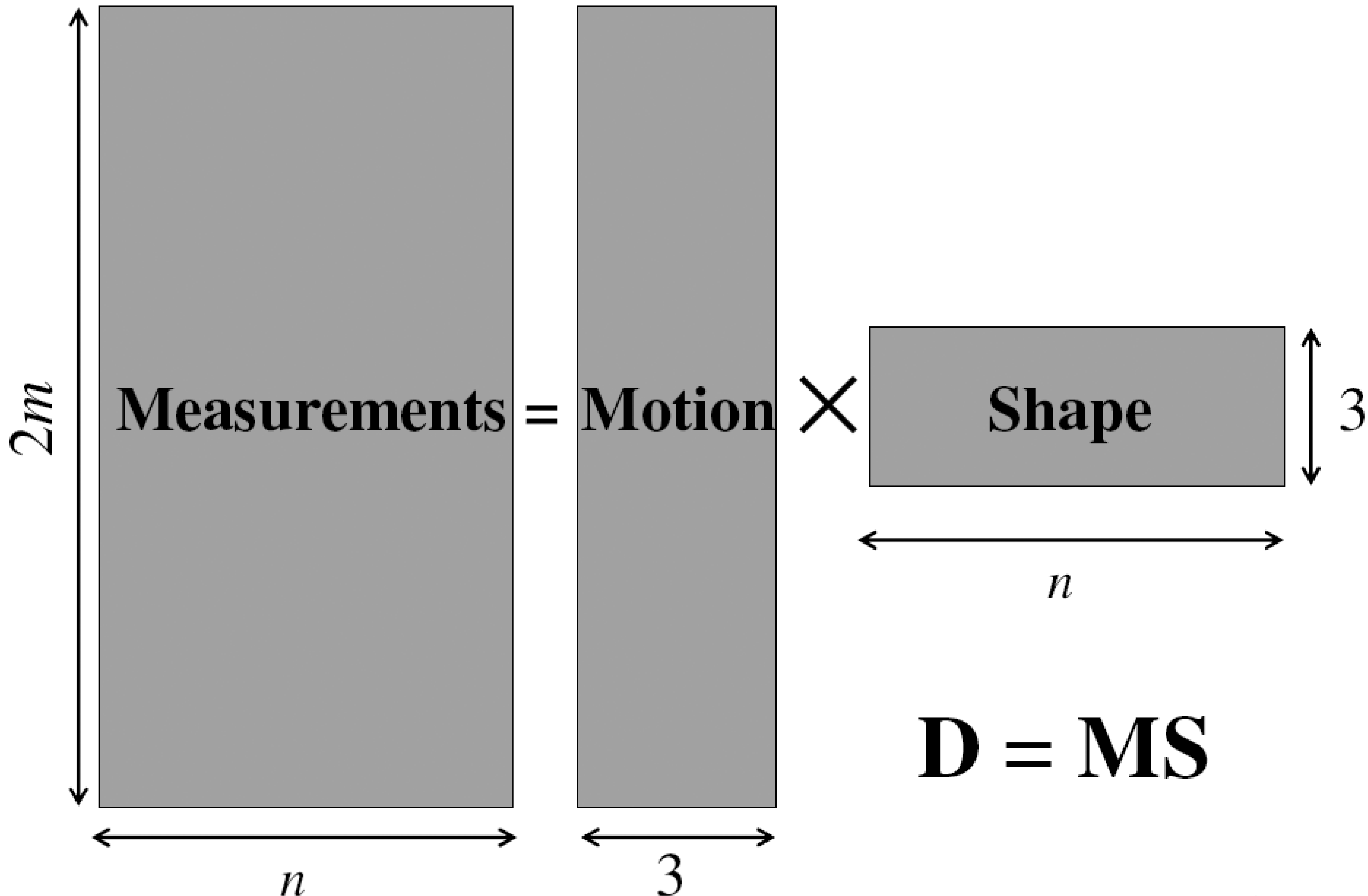
points ($3 \times n$)

cameras
($2m \times 3$)

The measurement matrix $\mathbf{D} = \mathbf{MS}$ must have rank 3!

C. Tomasi and T. Kanade. [Shape and motion from image streams under orthography: A factorization method.]() *IJCV*, 9(2):137-154, November 1992.

Factorizing the measurement matrix

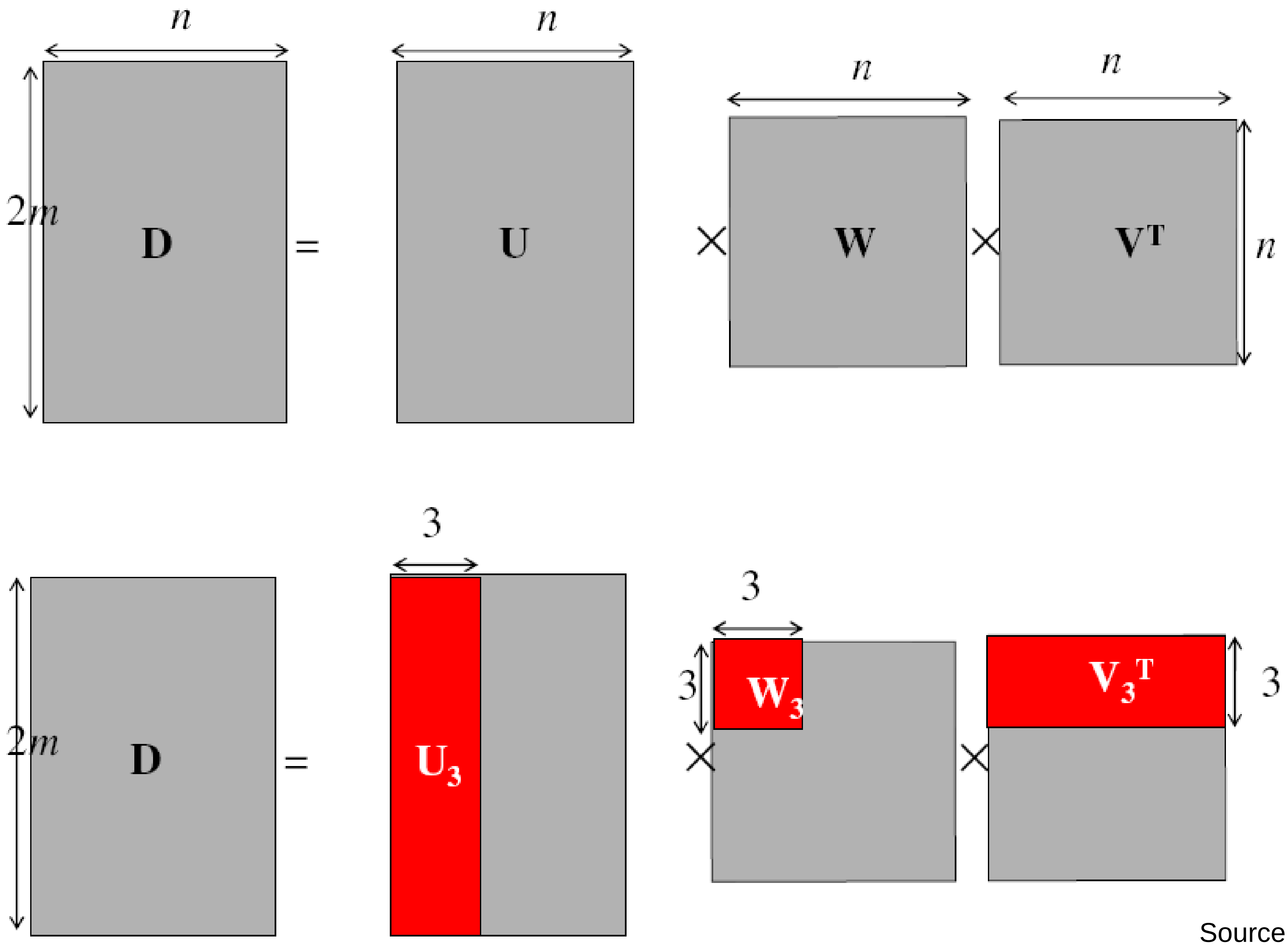


$$\mathbf{D} = \mathbf{MS}$$

Source: M. Hebert

Factorizing the measurement matrix

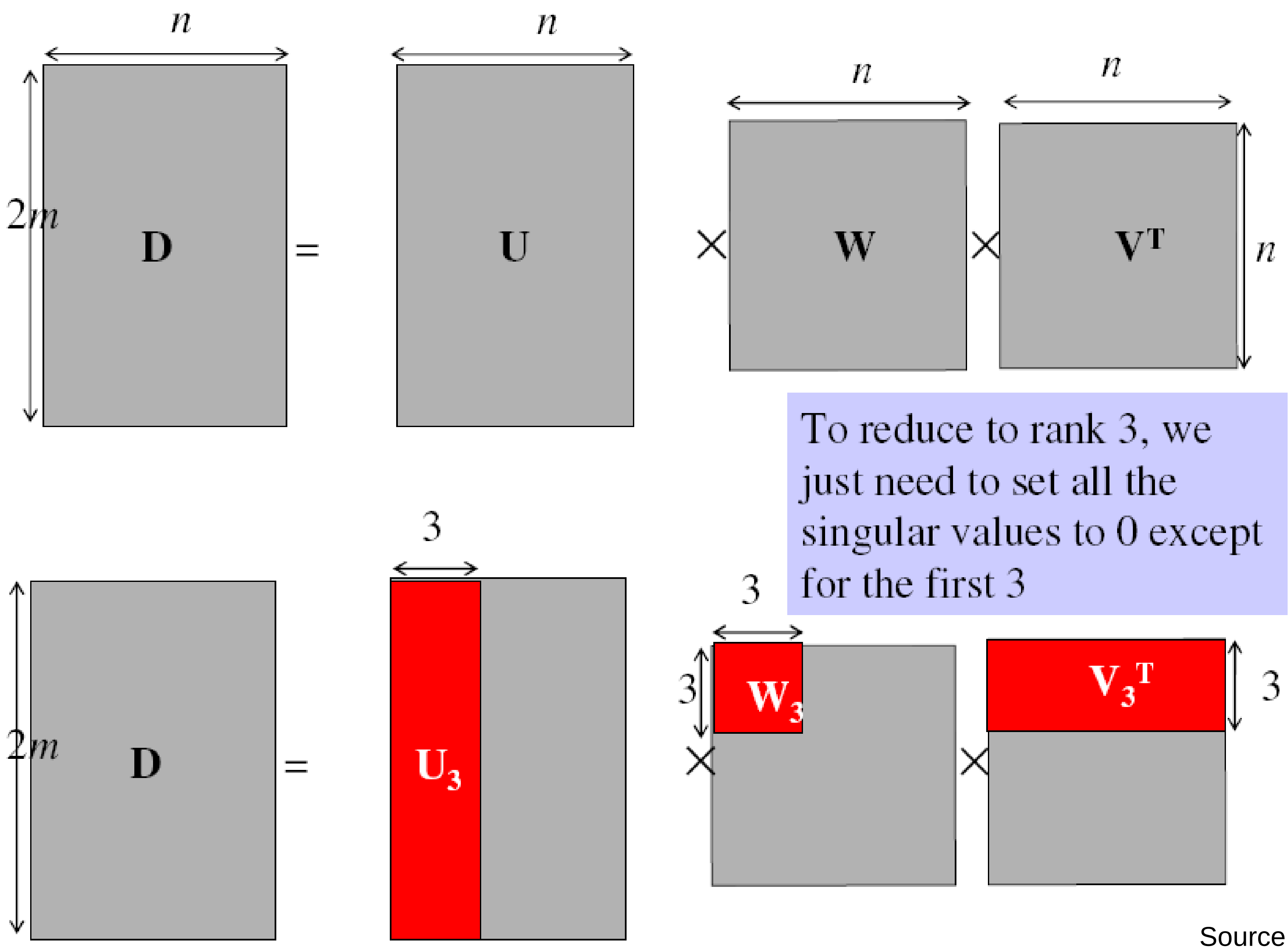
- Singular value decomposition of D:



Source: M. Hebert

Factorizing the measurement matrix

- Singular value decomposition of D:

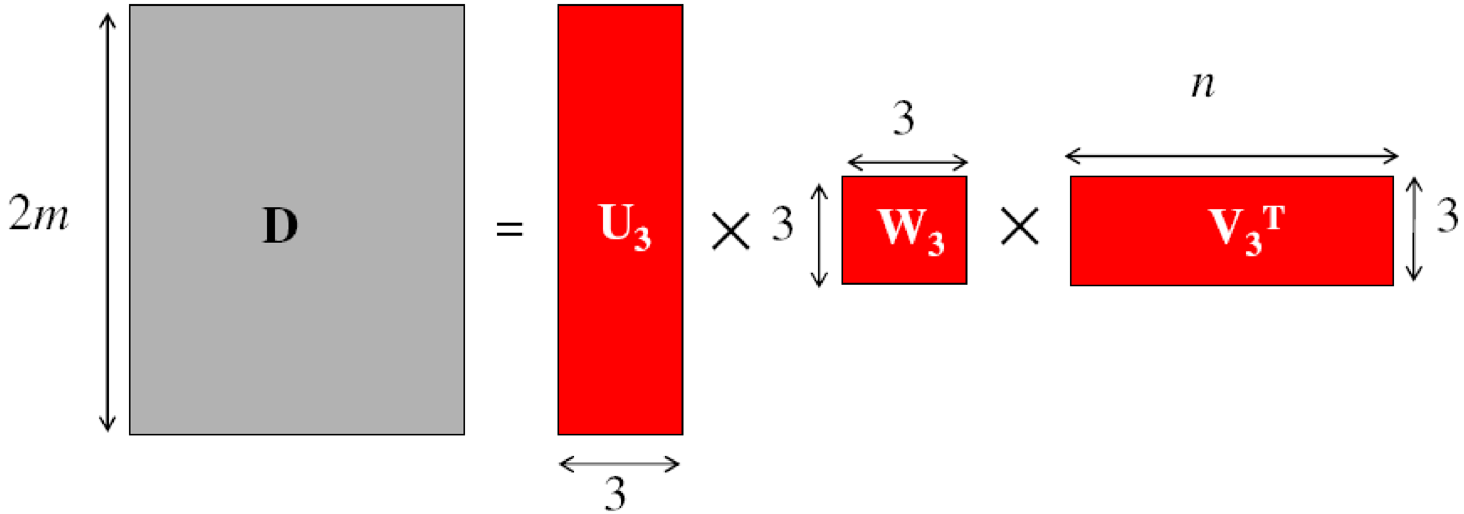


To reduce to rank 3, we just need to set all the singular values to 0 except for the first 3

Source: M. Hebert

Factorizing the measurement matrix

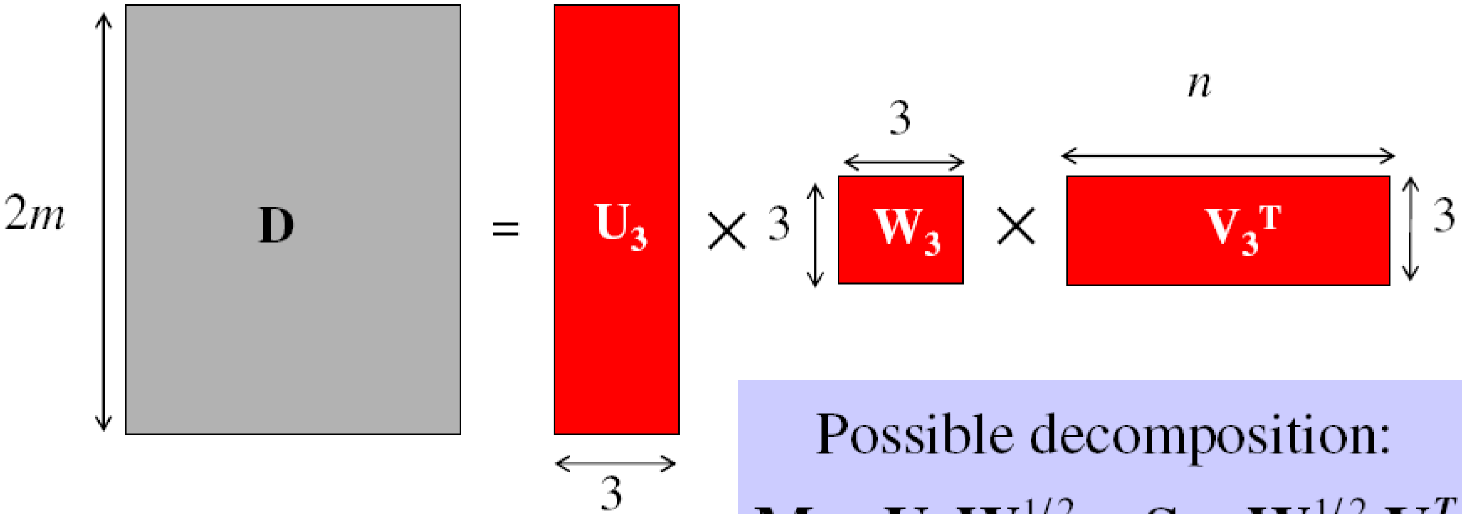
- Obtaining a factorization from SVD:



Source: M. Hebert

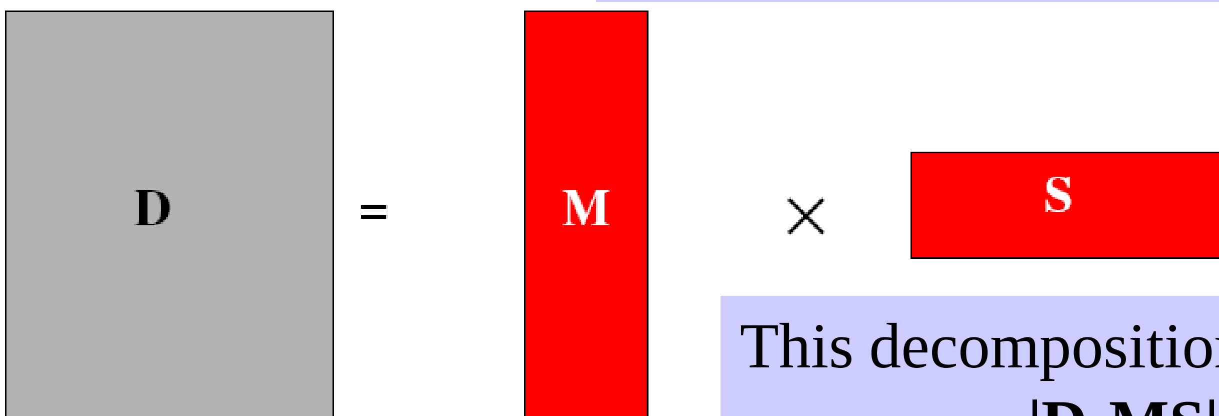
Factorizing the measurement matrix

- Obtaining a factorization from SVD:



Possible decomposition:

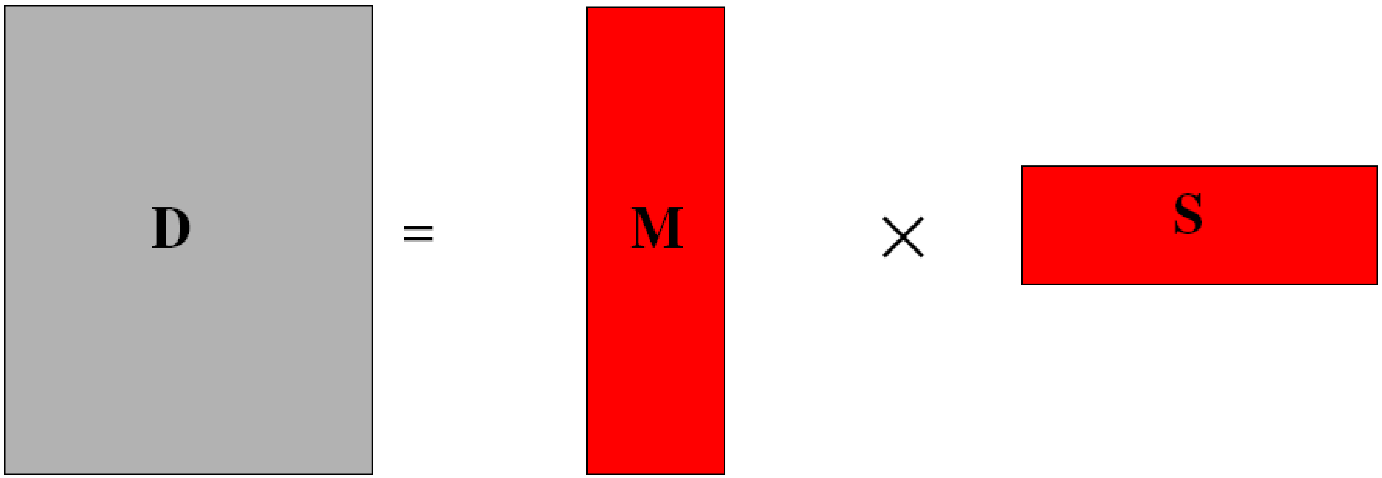
$$\mathbf{M} = \mathbf{U}_3 \mathbf{W}_3^{1/2} \qquad \mathbf{S} = \mathbf{W}_3^{1/2}\, \mathbf{V}_3^T$$



This decomposition minimizes $|\mathbf{D}\text{-}\mathbf{MS}|^2$

Source: M. Hebert

Affine ambiguity

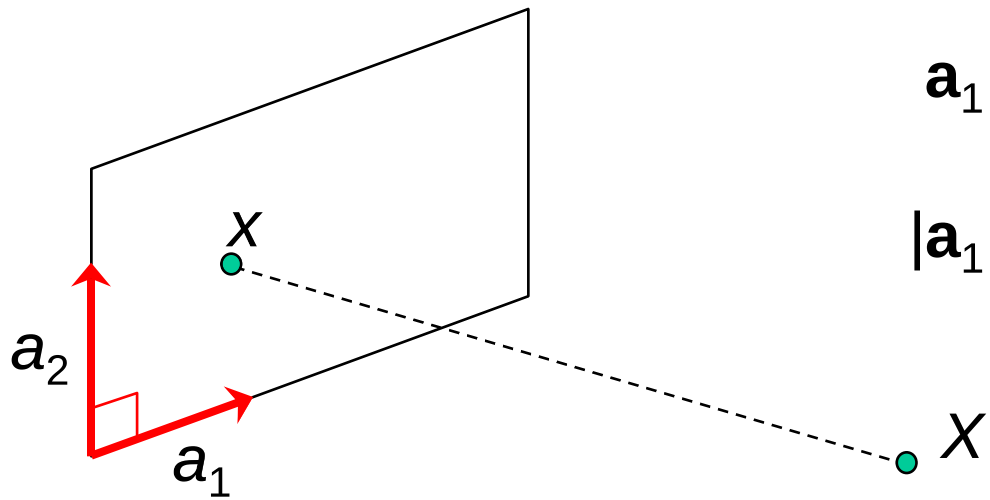


- The decomposition is not unique. We get the same **D** by using any 3×3 matrix **C** and applying the transformations $\mathbf{M} \rightarrow \mathbf{MC}$, $\mathbf{S} \rightarrow \mathbf{C}^{-1}\mathbf{S}$
- That is because we have only an affine transformation and we have not enforced any Euclidean constraints (like forcing the image axes to be perpendicular, for example)

Source: M. Hebert

Eliminating the affine ambiguity

- Orthographic: image axes are perpendicular and scale is 1



$$\mathbf{a}_1 \cdot \mathbf{a}_2 = 0$$

$$|\mathbf{a}_1|^2 = |\mathbf{a}_2|^2 = 1$$

- This translates into $3m$ equations in $\mathbf{L} = \mathbf{C}\mathbf{C}^T$:

$$\mathbf{A_i}\,\mathbf{L}\,\mathbf{A_i}^T = \mathbf{Id}, \qquad i = 1, \ldots, m$$

  - Solve for **L**
  - Recover **C** from **L** by Cholesky decomposition: $\mathbf{L} = \mathbf{C}\mathbf{C}^T$
  - Update **M** and **S**: $\mathbf{M} = \mathbf{MC}$, $\mathbf{S} = \mathbf{C}^{-1}\mathbf{S}$

Source: M. Hebert

Algorithm summary

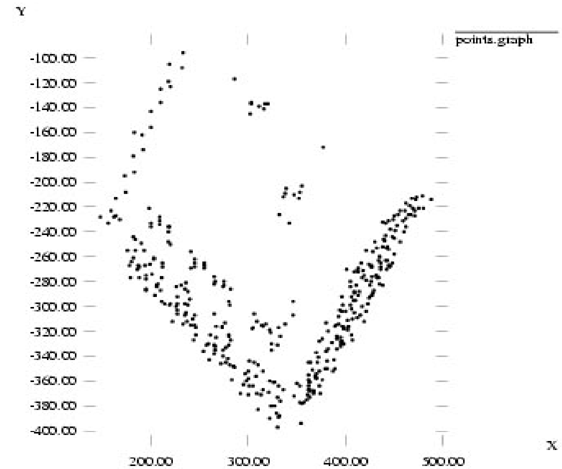
- Given: *m* images and *n* features $\mathbf{x}_{ij}$
- For each image *i, c*enter the feature coordinates
- Construct a $2m \times n$ measurement matrix **D**:
  - Column *j* contains the projection of point *j* in all views
  - Row *i* contains one coordinate of the projections of all the *n* points in image *i*
- Factorize **D**:
  - Compute SVD: $\mathbf{D} = \mathbf{U}\ \mathbf{W}\ \mathbf{V}^{\mathbf{T}}$
  - Create $\mathbf{U}_3$ by taking the first 3 columns of **U**
  - Create $\mathbf{V}_3$ by taking the first 3 columns of **V**
  - Create $\mathbf{W}_3$ by taking the upper left 3×3 block of **W**
- Create the motion and shape matrices:
  - $\mathbf{M} = \mathbf{U}_3\mathbf{W}_3^{½}$ and $\mathbf{S} = \mathbf{W}_3^{½}\ \mathbf{V}_3^{T}$ (**or** $\mathbf{M} = \mathbf{U}_3$ and $\mathbf{S} = \mathbf{W}_3\mathbf{V}_3^{T}$)
- Eliminate affine ambiguity

Source: M. Hebert

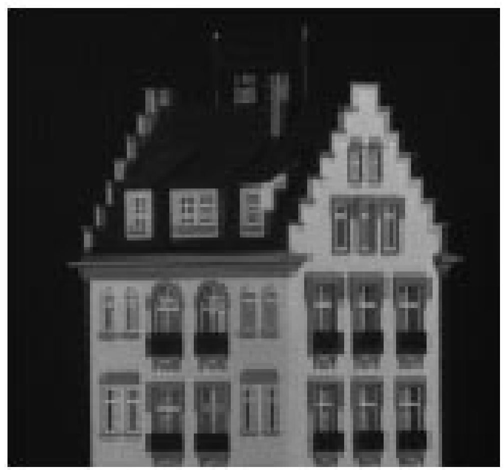
Reconstruction results

1

60



120



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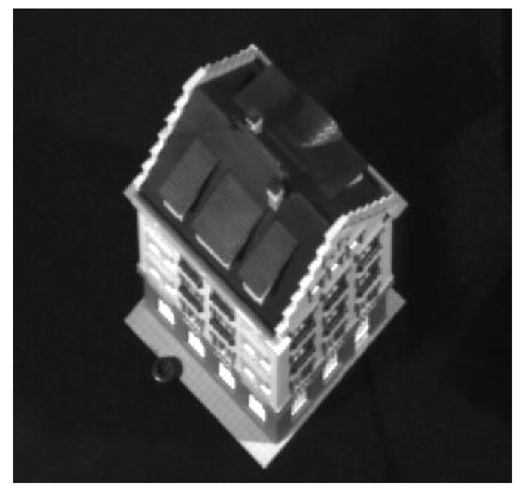
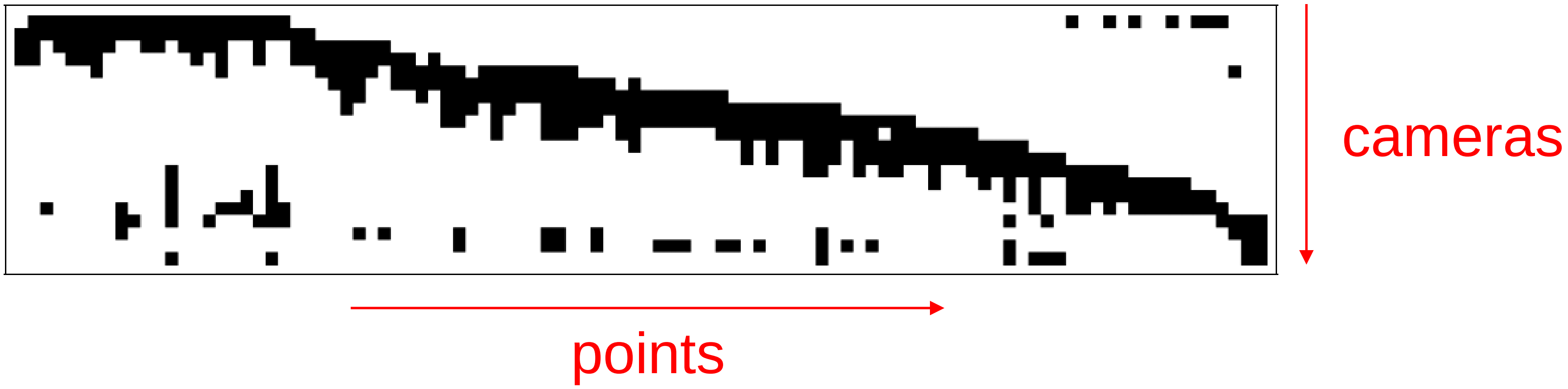


C. Tomasi and T. Kanade. [Shape and motion from image streams under orthography: A factorization method.](#) *IJCV*, 9(2):137-154, November 1992.

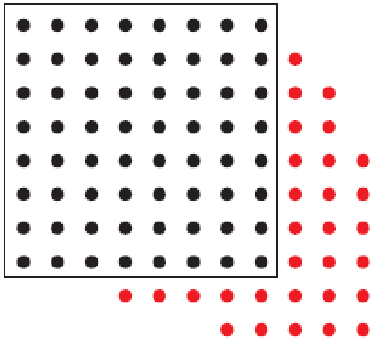
Dealing with missing data

- So far, we have assumed that all points are visible in all views
- In reality, the measurement matrix typically looks something like this:

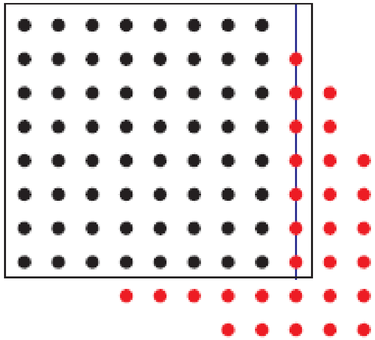


Dealing with missing data

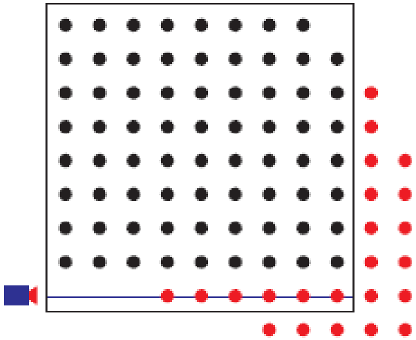
- Possible solution: decompose matrix into dense sub-blocks, factorize each sub-block, and fuse the results
  - Finding dense maximal sub-blocks of the matrix is NP-complete (equivalent to finding maximal cliques in a graph)
- Incremental bilinear refinement



(1) Perform factorization on a dense sub-block



(2) Solve for a new 3D point visible by at least two known cameras (linear least squares)



(3) Solve for a new camera that sees at least three known 3D points (linear least squares)

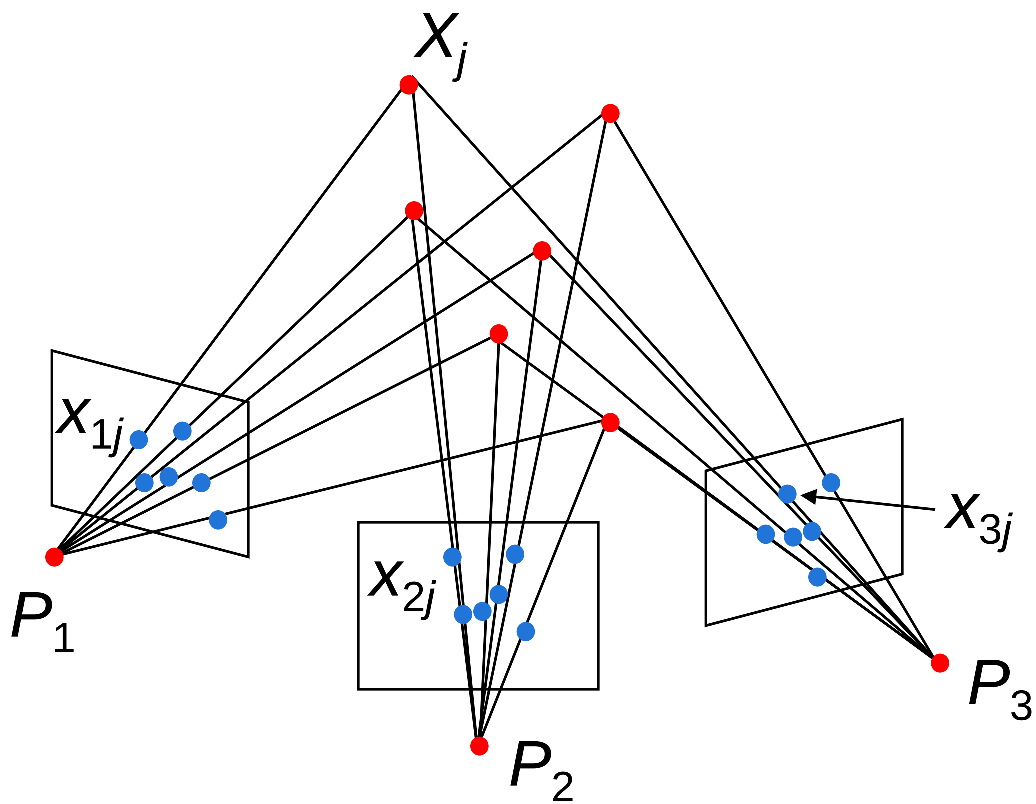
F. Rothganger, S. Lazebnik, C. Schmid, and J. Ponce. [Segmenting, Modeling, and Matching Video Clips Containing Multiple Moving Objects.]() PAMI 2007.

Projective structure from motion

- Given: *m* images of *n* fixed 3D points

$$\lambda_{ij}\mathbf{x}_{ij} = \mathbf{P}_i\mathbf{X}_j\,,\quad i = 1,\dots,m,\quad j = 1,\dots,n$$

- Problem: estimate *m* projection matrices $\mathbf{P}_i$ and *n* 3D points $\mathbf{X}_j$ from the *mn* correspondences $\mathbf{x}_{ij}$



Projective structure from motion

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$$\lambda_{ij}\mathbf{x}_{ij} = \mathbf{P}_i\,\mathbf{X}_j\,,\quad i = 1,\dots,m,\quad j = 1,\dots,n$$

- Problem: estimate *m* projection matrices $\mathbf{P}_i$ and *n* 3D points $\mathbf{X}_j$ from the *mn* correspondences $\mathbf{x}_{ij}$
- With no calibration info, cameras and points can only be recovered up to a 4x4 projective transformation **Q**:

$$\mathbf{X} \rightarrow \mathbf{QX},\ \mathbf{P} \rightarrow \mathbf{PQ}^{-1}$$

- We can solve for structure and motion when

$$2mn >= 11m + 3n - 15$$

- For two cameras, at least 7 points are needed

Projective SFM: Two-camera case

- Compute fundamental matrix $\mathbf{F}$ between the two views
- First camera matrix: $[\mathbf{I}|\mathbf{0}]$
- Second camera matrix: $[\mathbf{A}|\mathbf{b}]$
- Then $\mathbf{b}$ is the epipole ($\mathbf{F}^{T}\mathbf{b} = 0$), $\mathbf{A} = -[\mathbf{b}_{\times}]\mathbf{F}$

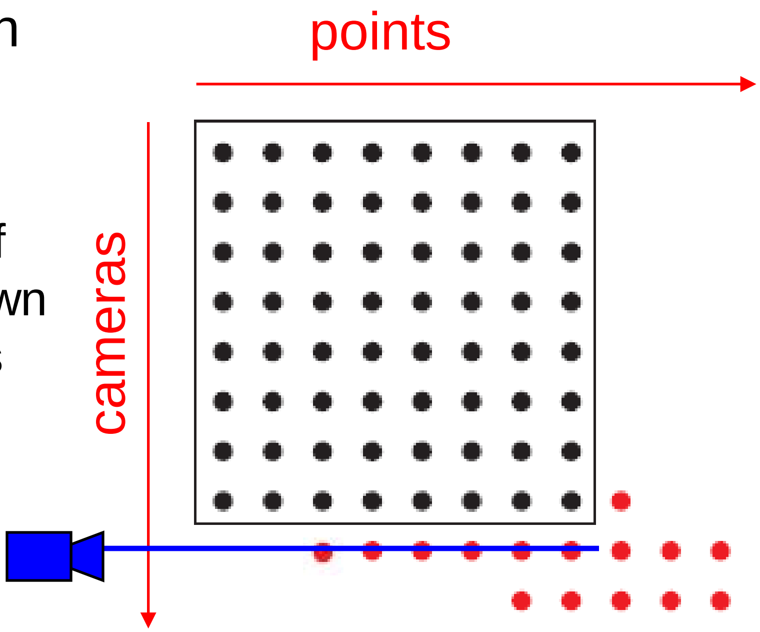
F&P sec. 8.3.2

Sequential structure from motion

- Initialize motion from two images using fundamental matrix

- Initialize structure by triangulation

- For each additional view:
  - Determine projection matrix of new camera using all the known 3D points that are visible in its image – *calibration*



Sequential structure from motion

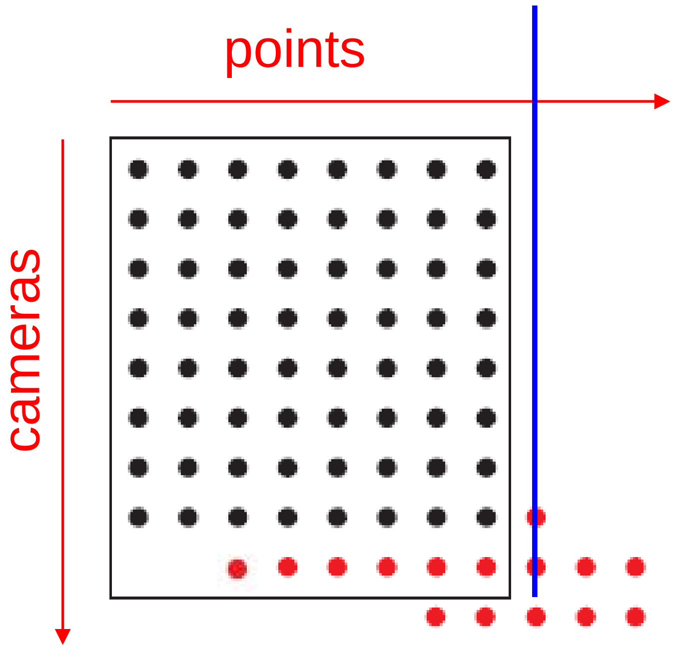
•Initialize motion from two images using fundamental matrix

•Initialize structure by triangulation

•For each additional view:

- Determine projection matrix of new camera using all the known 3D points that are visible in its image – *calibration*

- Refine and extend structure: compute new 3D points, re-optimize existing points that are also seen by this camera – *triangulation*



Sequential structure from motion

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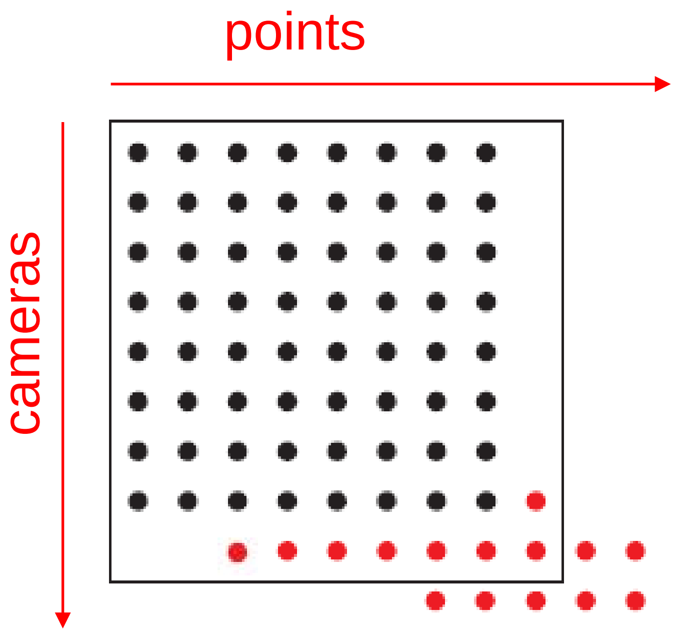
- Initialize structure by triangulation

- For each additional view:
  - Determine projection matrix of new camera using all the known 3D points that are visible in its image – *calibration*
  - Refine and extend structure: compute new 3D points, re-optimize existing points that are also seen by this camera – *triangulation*

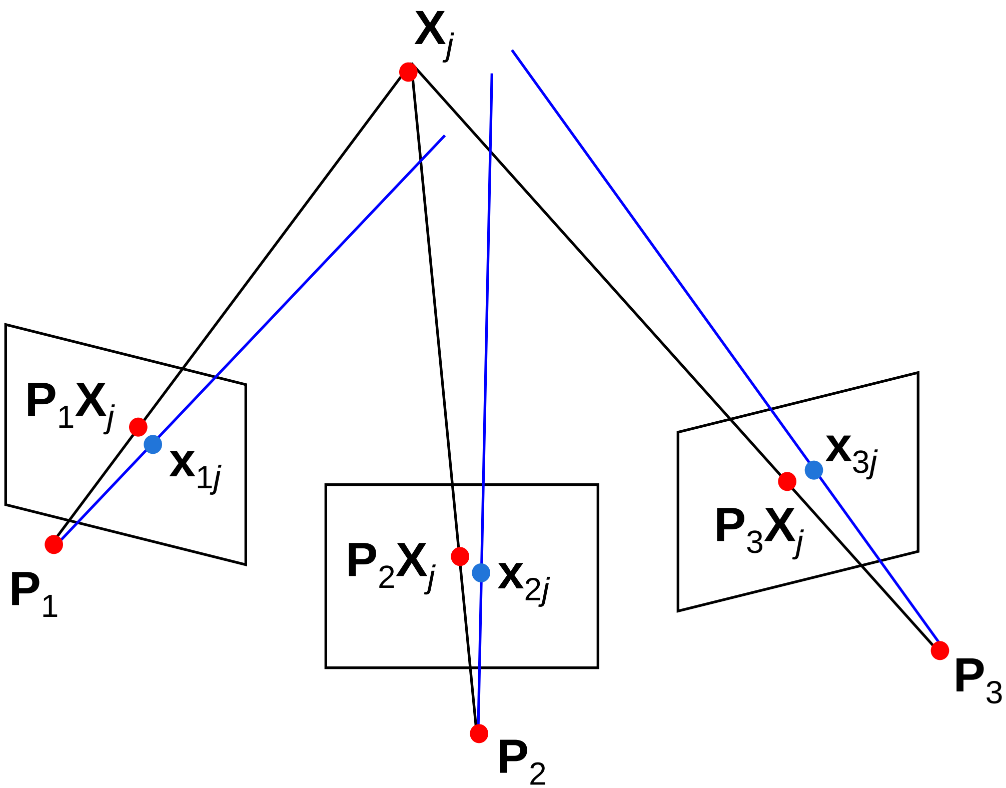
- Refine structure and motion: bundle adjustment



Bundle adjustment

- Non-linear method for refining structure and motion
- Minimizing reprojection error

$$E(\mathbf{P},\mathbf{X}) = \sum_{i=1}^{m}\sum_{j=1}^{n} D\left(\mathbf{x}_{ij}, \mathbf{P}_i\mathbf{X}_j\right)^2$$



Self-calibration

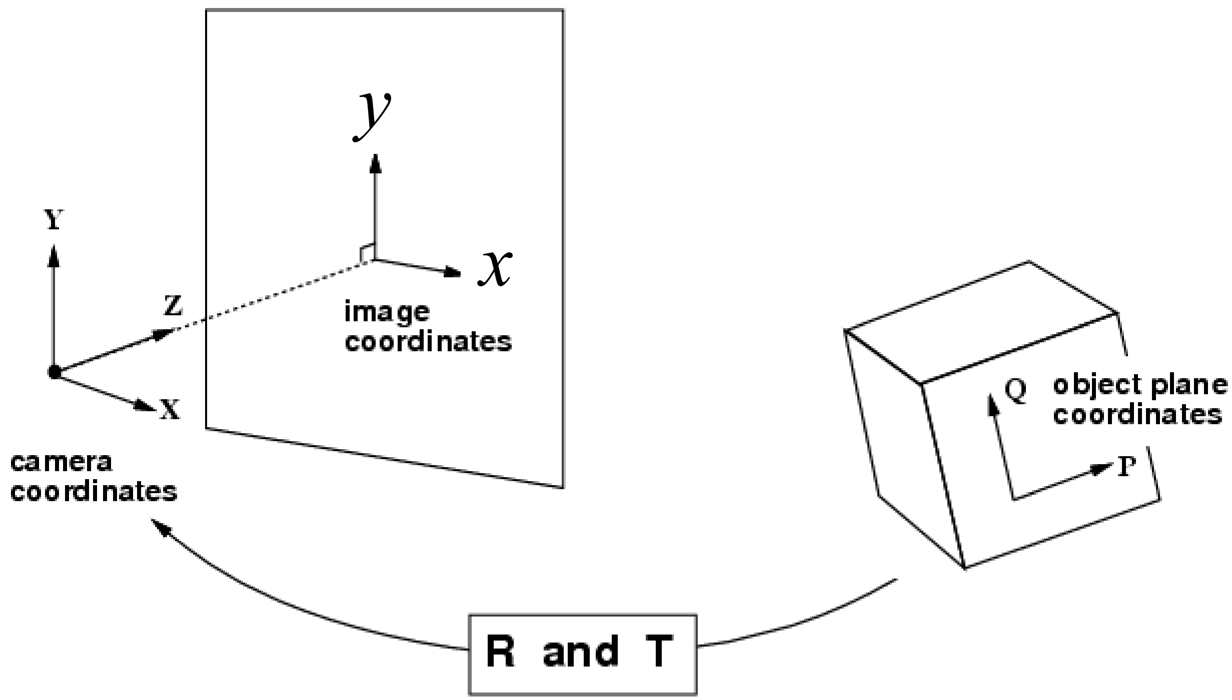
- Self-calibration (auto-calibration) is the process of determining intrinsic camera parameters directly from uncalibrated images
- For example, when the images are acquired by a single moving camera, we can use the constraint that the intrinsic parameter matrix remains fixed for all the images
  - Compute initial projective reconstruction and find 3D projective transformation matrix $\mathbf{Q}$ such that all camera matrices are in the form $\mathbf{P}_i = \mathbf{K}\,[\mathbf{R}_i \mid \mathbf{t}_i]$
- Can use constraints on the form of the calibration matrix: zero skew
- Can use vanishing points

Review: Structure from motion

- Ambiguity
- Affine structure from motion
  - Factorization
- Dealing with missing data
  - Incremental structure from motion
- Projective structure from motion
  - Bundle adjustment
  - Self-calibration

Planar Homography

Motivation:
Points on Planar Surface



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Review : Forward Projection

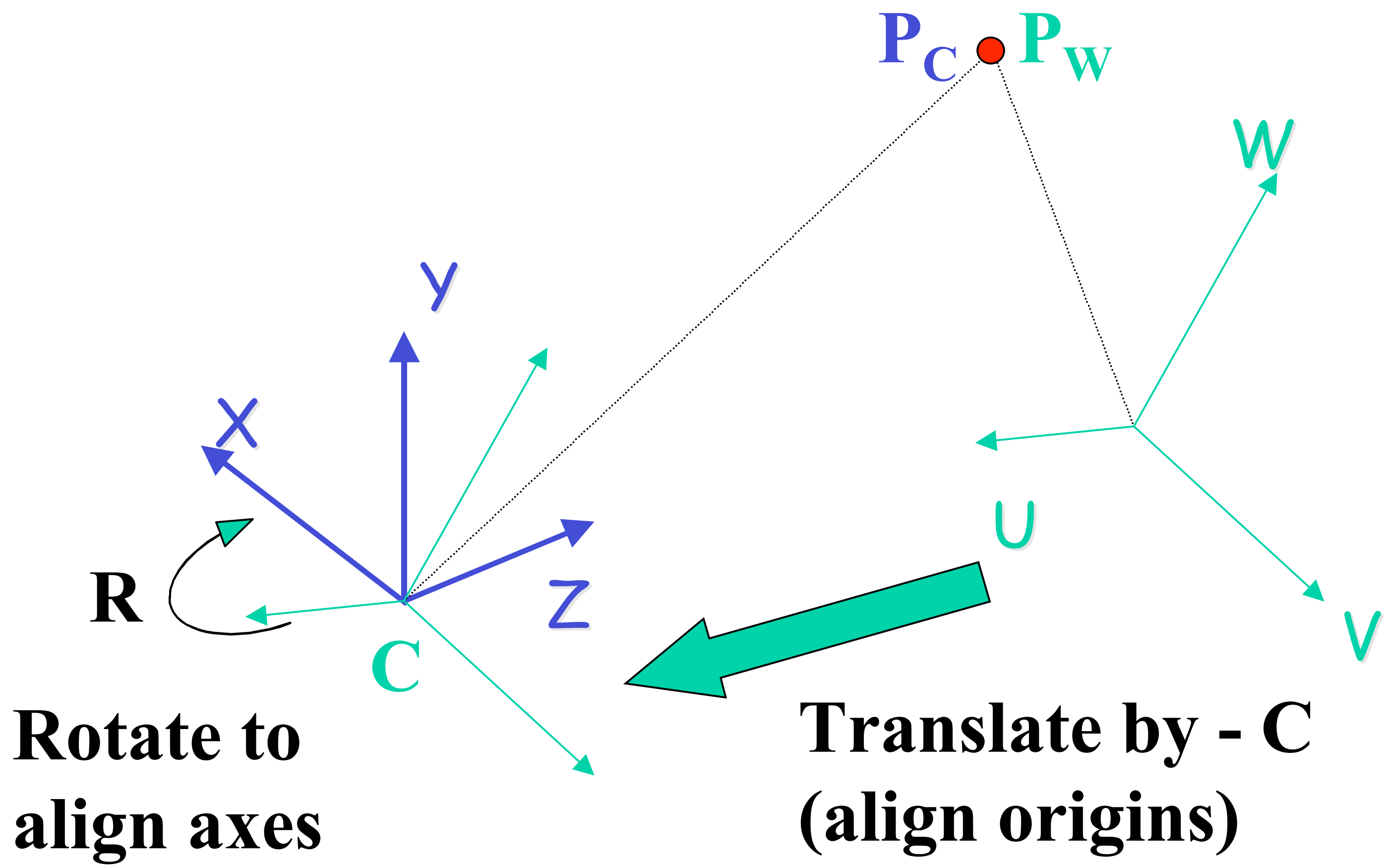
World Coords		Camera Coords		Film Coords		Pixel Coords

$$\begin{matrix} U \\ V \\ W \end{matrix} \xrightarrow{\;M_{ext}\;} \begin{matrix} X \\ Y \\ Z \end{matrix} \xrightarrow{\;M_{proj}\;} \begin{matrix} x \\ y \end{matrix} \xrightarrow{\;M_{aff}\;} \begin{matrix} u \\ v \end{matrix}$$

$$\begin{matrix} U \\ V \\ W \end{matrix} \qquad \begin{matrix} X \\ Y \\ Z \end{matrix} \xrightarrow{\;\;\;\;M_{int}\;\;\;\;} \begin{matrix} u \\ v \end{matrix}$$

$$\begin{matrix} U \\ V \\ W \end{matrix} \xrightarrow[\begin{matrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{31} & m_{33} & m_{34} \end{matrix}]{\;\;\;\;\;\;\;\;M\;\;\;\;\;\;\;\;} \begin{matrix} u \\ v \end{matrix}$$

World to Camera Transformation



$$P_C = R\,(P_W - C)$$
$$= R\,P_W + T$$

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Perspective Matrix Equation

(Camera Coordinates)

$$x = f\,\frac{X}{Z}$$

$$y = f\,\frac{Y}{Z}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$p = M_{\text{int}} \cdot P_{\text{C}}$$

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Film to Pixel Coords

2D affine transformation from film coords (x,y) to pixel coordinates (u,v):

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{M_{aff}}} \underbrace{\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{M_{proj}}} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

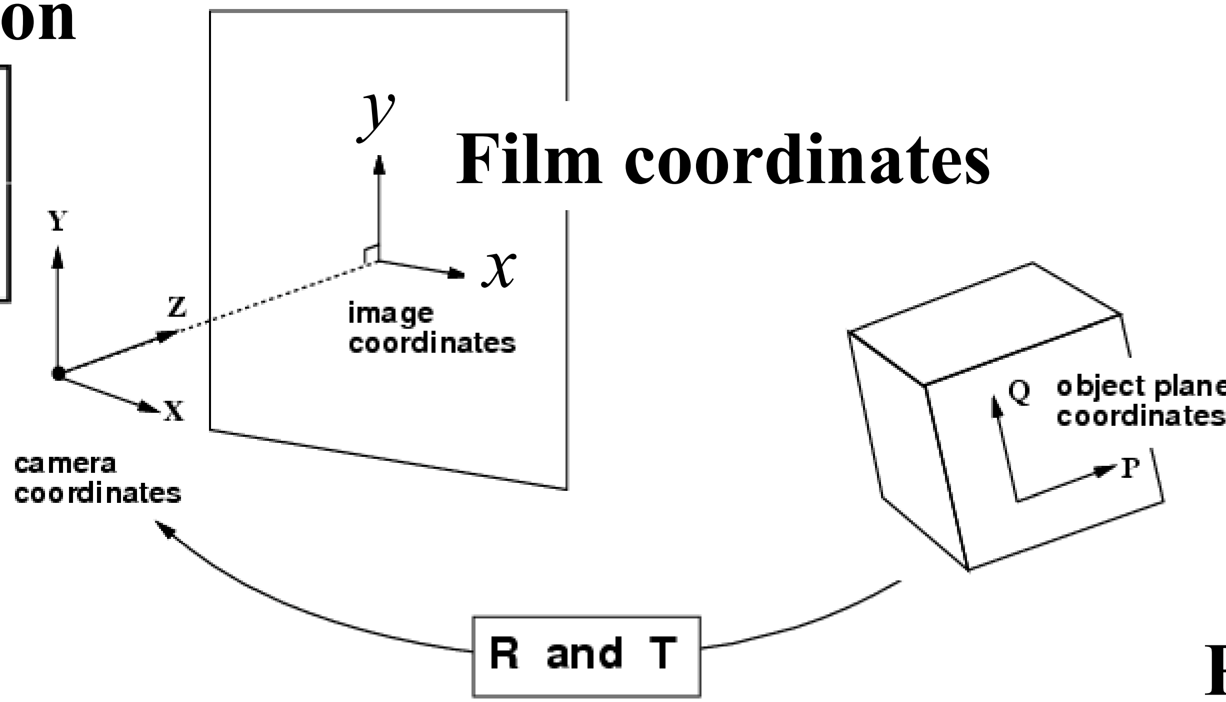
$$\mathbf{u} = \mathbf{M_{int}}\, \mathbf{P_C} = \mathbf{M_{aff}}\, \mathbf{M_{proj}}\, \mathbf{P_C}$$

Projection of Points on Planar Surface

Perspective projection

$$\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Film coordinates



$$\begin{bmatrix} p \\ q \\ 0 \\ 1 \end{bmatrix}$$

Point on plane

Rotation + Translation

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Projection of Planar Points

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \boxed{r_{13}} & t_x \\ r_{21} & r_{22} & \boxed{r_{23}} & t_y \\ r_{31} & r_{32} & \boxed{r_{33}} & t_z \\ 0 & 0 & \boxed{0} & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ \boxed{0} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & \boxed{0} \\ 0 & f & 0 & \boxed{0} \\ 0 & 0 & 1 & \boxed{0} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ r_{31} & r_{32} & t_z \\ \boxed{0} & \boxed{0} & \boxed{1} \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ r_{31} & r_{32} & t_z \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

Projection of Planar Points (cont)

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ r_{31} & r_{32} & t_z \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} fr_{11} & fr_{12} & ft_x \\ fr_{21} & fr_{22} & ft_y \\ r_{31} & r_{32} & t_z \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

Homography H (planar projective transformation)

Projection of Planar Points (cont)

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

Homography H (planar projective transformation)

Punchline: For planar surfaces, 3D to 2D perspective projection reduces to a 2D to 2D transformation.

Punchline2: This transformation is INVERTIBLE!

Special Case : Frontal Plane

What if the planar surface is perpendicular to the optic axis (Z axis of camera coord system)?

Then world rotation matrix simplies:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{matrix}} & \begin{matrix} t_x \\ t_y \\ t_z \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix} \begin{bmatrix} p \\ q \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Frontal Plane

So the homography for a frontal plane simplies:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} fr_{11} & fr_{12} & ft_x \\ fr_{21} & fr_{22} & ft_y \\ r_{31} & r_{32} & t_z \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \sim \begin{bmatrix} f\cos\theta & -f\sin\theta & ft_x \\ f\sin\theta & f\cos\theta & ft_y \\ 0 & 0 & t_z \end{bmatrix} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}$$

Similarity Transformation!

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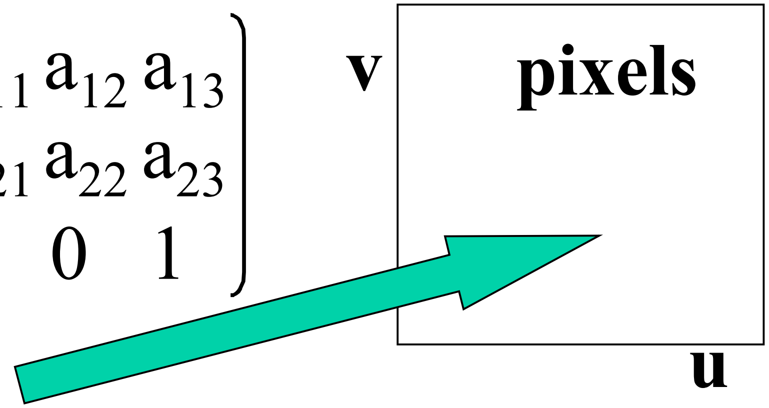
Convert to Pixel Coords

Internal camera parameters

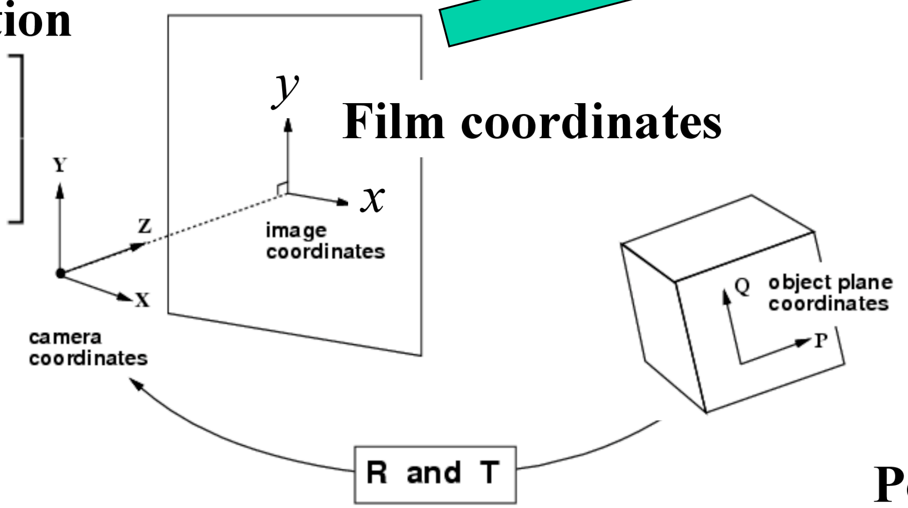
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{pmatrix}$$



Perspective projection

$$\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Film coordinates



$$\begin{bmatrix} p \\ q \\ 0 \\ 1 \end{bmatrix}$$

Point on plane

Rotation + Translation

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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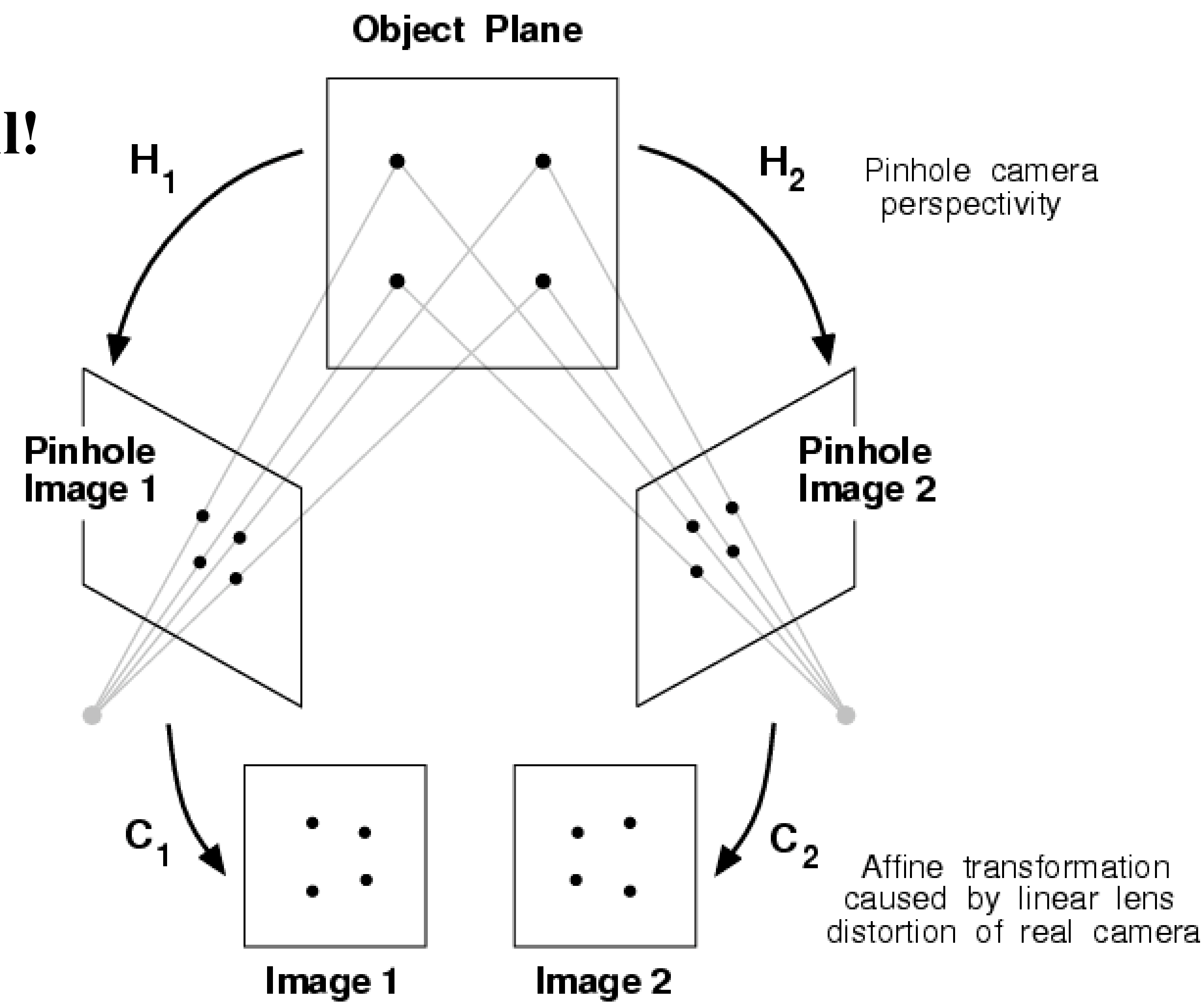
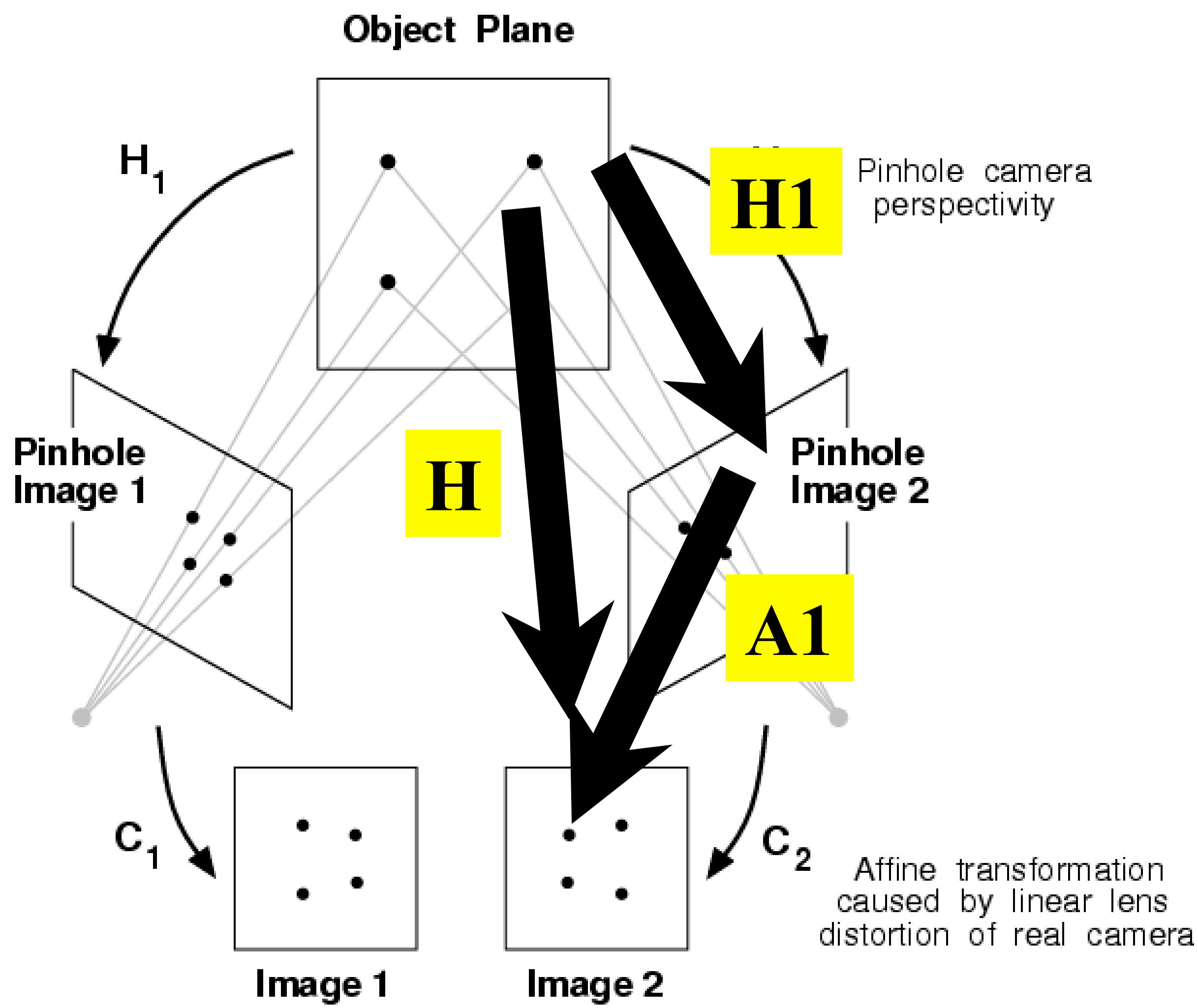
Planar Projection Diagram

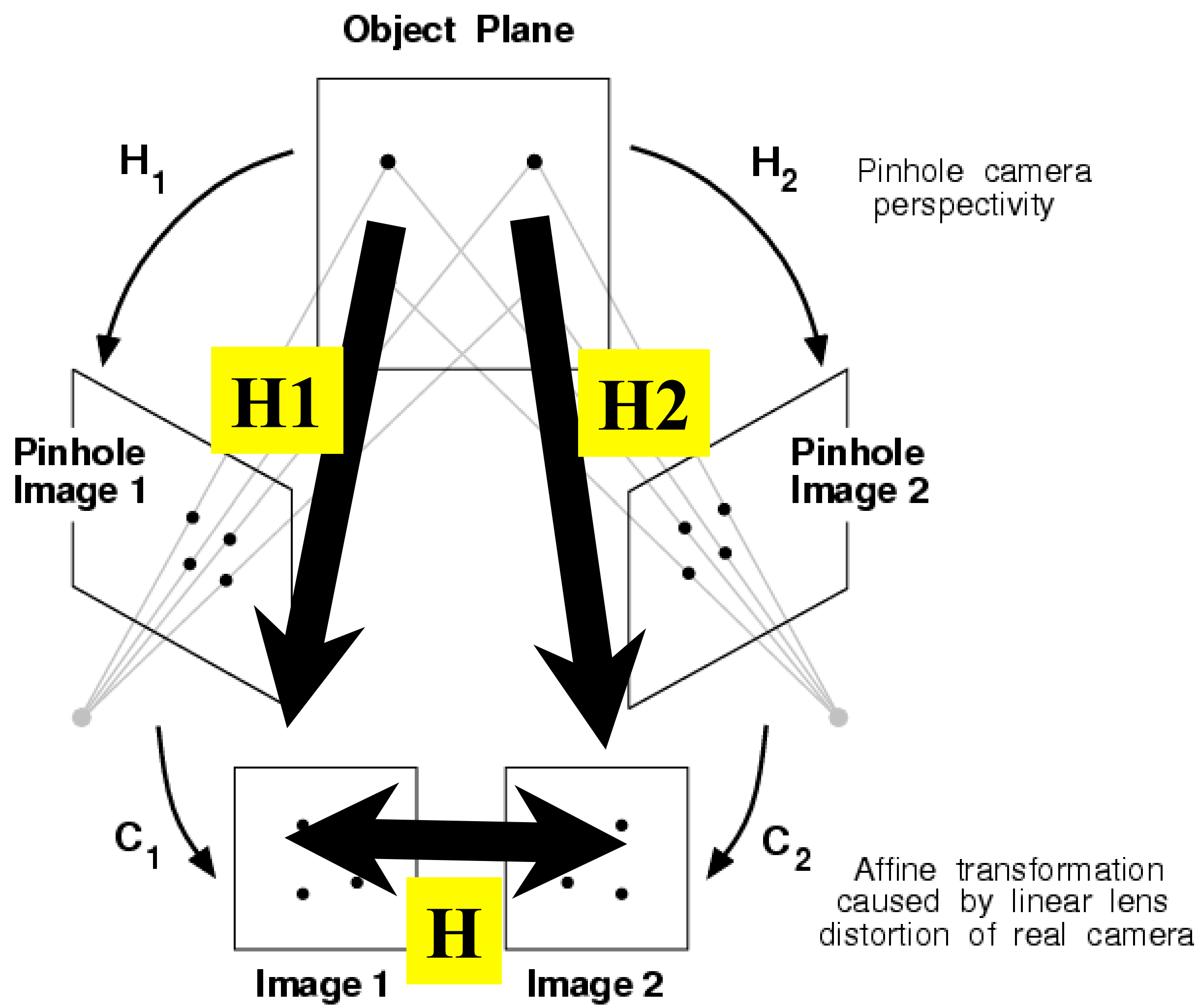
Here's where transformation groups get useful!



General Planar Projection



General Planar Projection



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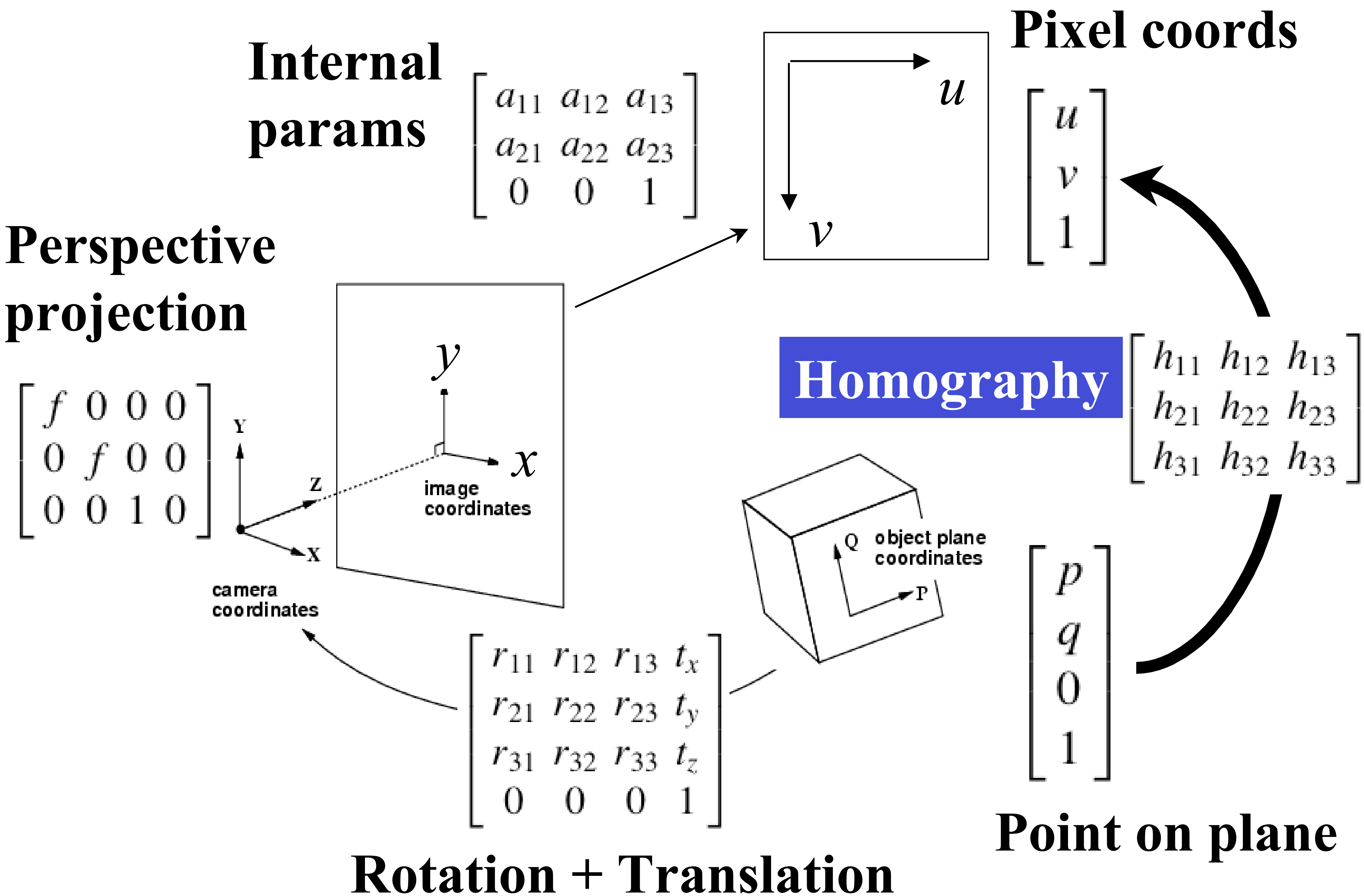
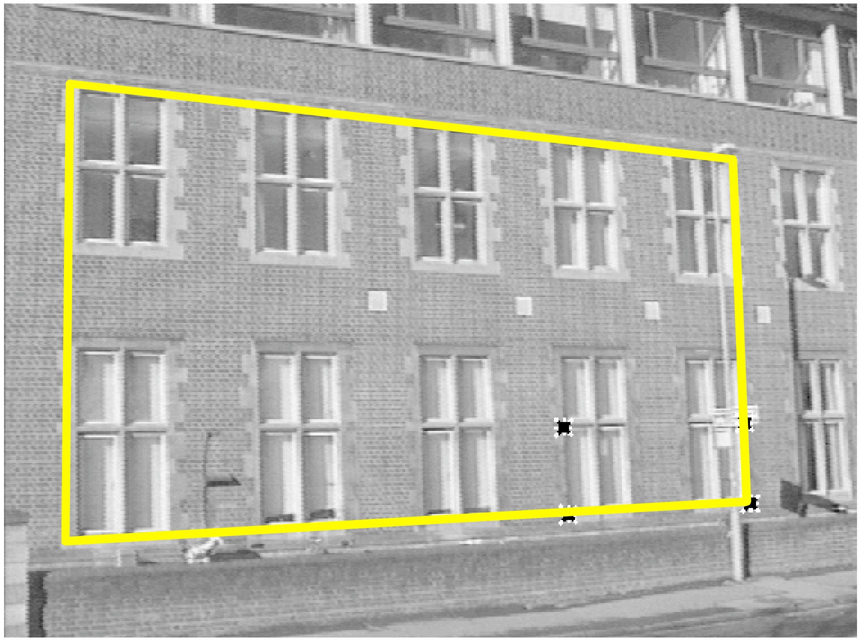
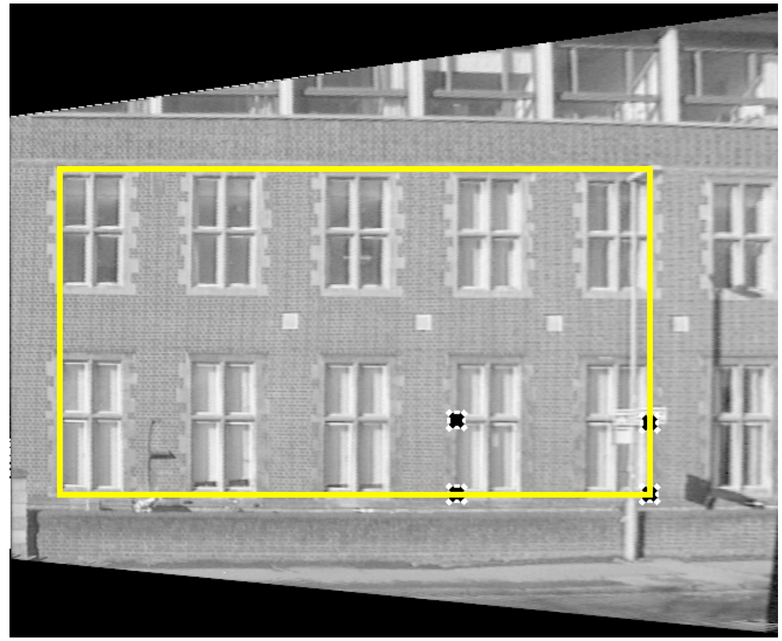
Summary: Planar Projection



Applying Homographies to Remove Perspective Distortion

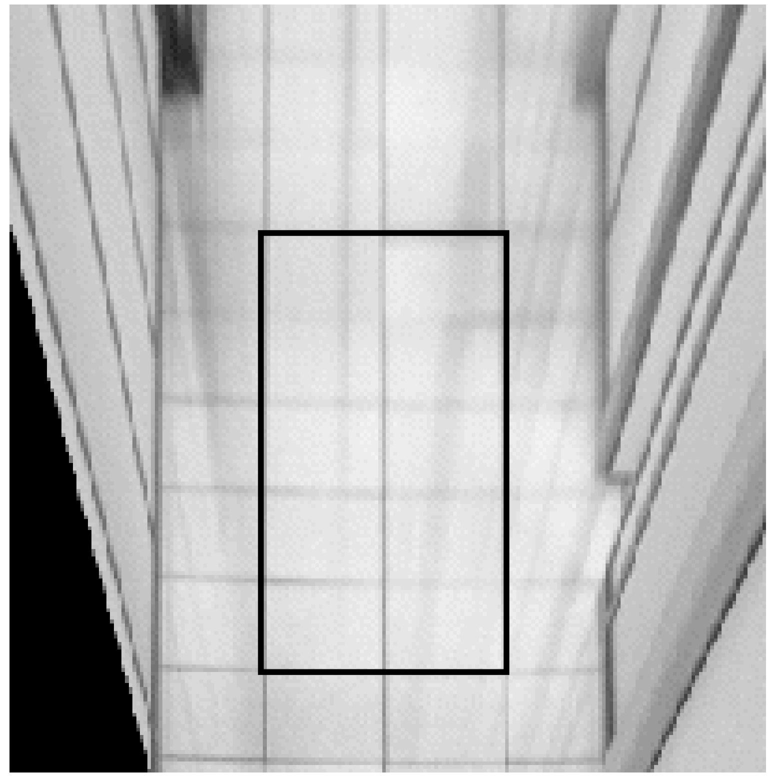




from Hartley & Zisserman

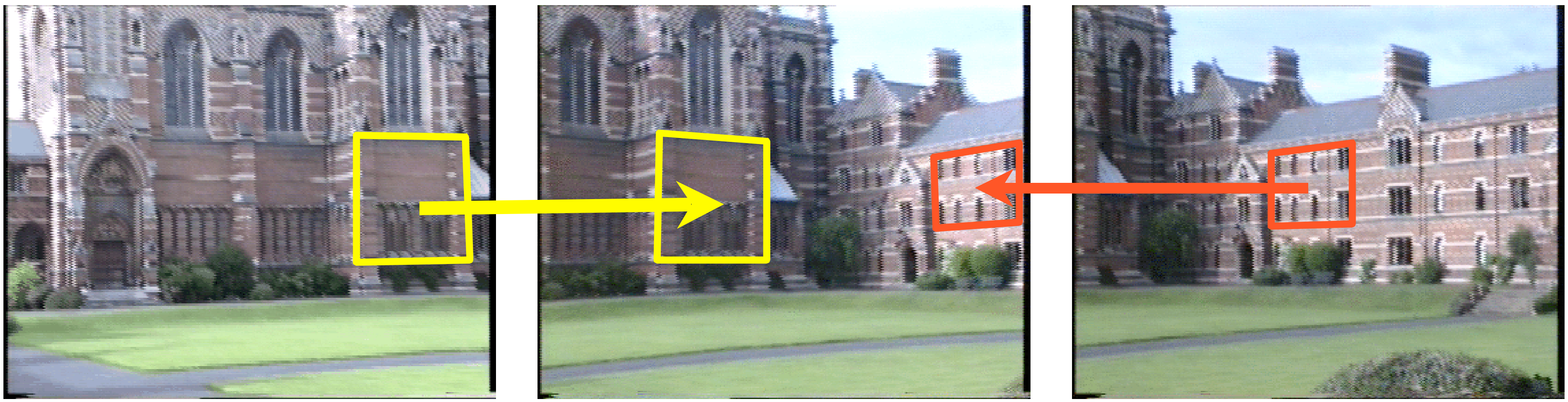
4 point correspondences suffice for the planar building facade

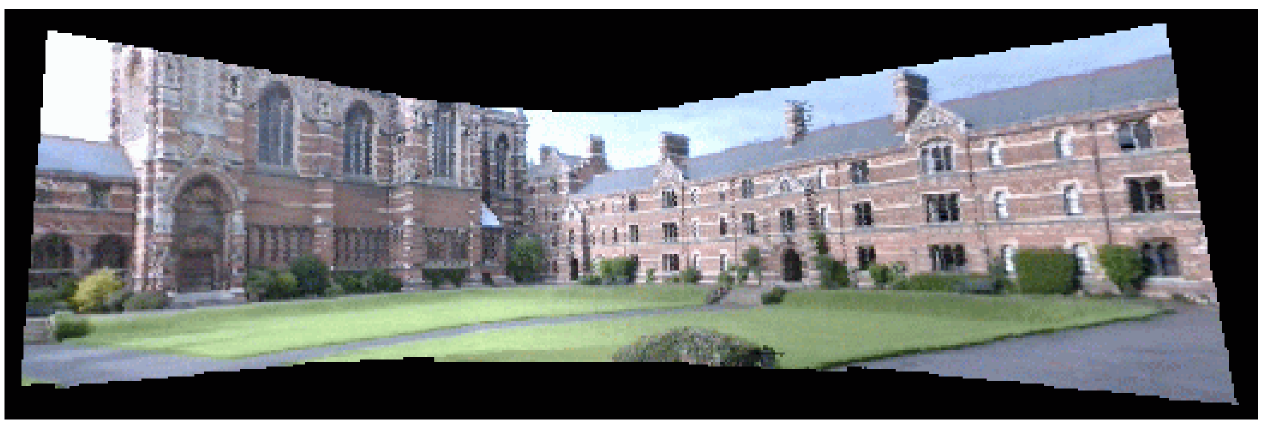
Homographies for Bird's-eye Views



from Hartley & Zisserman

Homographies for Mosaicing





from Hartley & Zisserman

Two Practical Issues

How to estimate the homography given four or more point correspondences (will derive L.S. solution now)

How to (un)warp image pixel values to produce a new picture (last class)

Estimating a Homography

Matrix Form:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \sim \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Equations:

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

Degrees of Freedom?

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \sim \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

There are 9 numbers $h_{11},\ldots,h_{33}$, so are there 9 DOF?

No. Note that we can multiply all h_{ij} by nonzero k without changing the equations:

$$x' = \frac{kh_{11}x + kh_{12}y + kh_{13}}{kh_{31}x + kh_{32}y + kh_{33}} \qquad x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$

$$y' = \frac{kh_{21}x + kh_{22}y + kh_{23}}{kh_{31}x + kh_{32}y + kh_{33}} \qquad y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

Enforcing 8 DOF

One approach: Set $h_{33} = 1$.

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + 1}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + 1}$$

Second approach: Impose unit vector constraint

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

Subject to the constraint: $h_{11}^2 + h_{12}^2 + h_{13}^2 + h_{21}^2 + h_{22}^2 + h_{23}^2 + h_{31}^2 + h_{32}^2 + h_{33}^2 = 1$

L.S. using Algebraic Distance

Setting $h_{33} = 1$

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + 1}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + 1}$$

Multiplying through by denominator

$$(h_{31}x + h_{32}y + 1)x' = h_{11}x + h_{12}y + h_{13}$$

$$(h_{31}x + h_{32}y + 1)y' = h_{21}x + h_{22}y + h_{23}$$

Rearrange

$$h_{11}x + h_{12}y + h_{13} - h_{31}xx' - h_{32}yx' = x'$$

$$h_{21}x + h_{22}y + h_{23} - h_{31}xy' - h_{32}yy' = y'$$

Algebraic Distance, h_{33}=1 (cont)

$$
\begin{array}{c c c c}
 & \textbf{2N x 8} & \textbf{8 x 1} & & \textbf{2N x 1} \\
\begin{array}{l} \textbf{Point 1} \\ \\ \textbf{Point 2} \\ \\ \textbf{Point 3} \\ \\ \textbf{Point 4} \\ \end{array} &
\begin{bmatrix}
x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1x'_1 & -y_1x'_1 \\
0 & 0 & 0 & x_1 & y_1 & 1 & -x_1y'_1 & -y_1y'_1 \\
x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2x'_2 & -y_2x'_2 \\
0 & 0 & 0 & x_2 & y_2 & 1 & -x_2y'_2 & -y_2y'_2 \\
x_3 & y_3 & 1 & 0 & 0 & 0 & -x_3x'_3 & -y_3x'_3 \\
0 & 0 & 0 & x_3 & y_3 & 1 & -x_3y'_3 & -y_3y'_3 \\
x_4 & y_4 & 1 & 0 & 0 & 0 & -x_4x'_4 & -y_4x'_4 \\
0 & 0 & 0 & x_4 & y_4 & 1 & -x_4y'_4 & -y_4y'_4
\end{bmatrix} &
\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} & = &
\begin{bmatrix} x'_1 \\ y'_1 \\ x'_2 \\ y'_2 \\ x'_3 \\ y'_3 \\ x'_4 \\ y'_4 \end{bmatrix} \\
\textbf{additional points} & \vdots & & & \vdots
\end{array}
$$

Algebraic Distance, h_{33}=1 (cont)

Linear equations

$$\underset{2N\times 8}{A}\ \underset{8\times 1}{h} = \underset{2N\times 1}{b}$$

Solve:

$$\overset{8\times 2N}{A^T}\ \overset{2N\times 8}{A}\ \overset{8\times 1}{h} = \overset{8\times 2N}{A^T}\ \overset{2N\times 1}{b}$$

$$\overbrace{(A^T\ A)}^{8\times 8}\ \overset{8\times 1}{h} = \overbrace{(A^T\ b)}^{8\times 1}$$

$$h = (A^T\ A)^{-1}\ (A^T\ b)$$

Matlab: h = A \ b

Caution: Numeric Conditioning

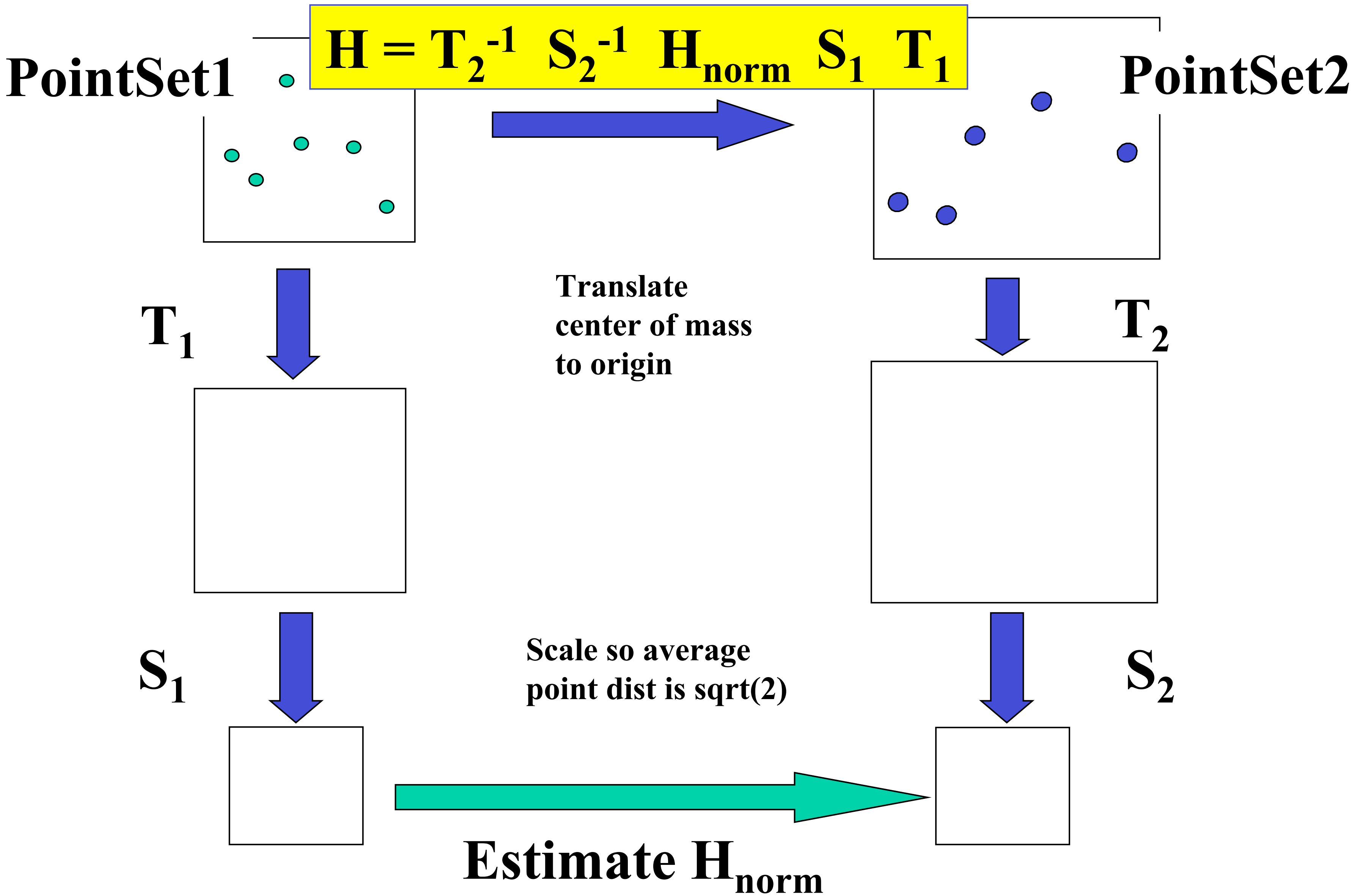
R.Hartley: “In Defense of the Eight Point Algorithm”

Observation: Linear estimation of projective transformation parameters from point correspondences often suffer from poor “conditioning” of the matrices involves. This means the solution is sensitive to noise in the points (<u>even if there are no outliers</u>).

To get better answers, precondition the matrices by performing a normalization of each point set by:

- translating center of mass to the origin
- scaling so that average distance of points from origin is sqrt(2).
- do this normalization to each point set independently

Hartley's PreConditioning



A More General Approach

What might be wrong with setting $h_{33} = 1$?

If h_{33} actually $= 0$, we can't get the right answer.

Algebraic Distance, ||h||=1

||h|| = 1

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

Multiplying through by denominator

$$(h_{31}x + h_{32}y + h_{33})x' = h_{11}x + h_{12}y + h_{13}$$

$$(h_{31}x + h_{32}y + h_{33})y' = h_{21}x + h_{22}y + h_{23}$$

Rearrange

$$h_{11}x + h_{12}y + h_{13} - h_{31}xx' - h_{32}yx' - h_{33}x' = 0$$

$$h_{21}x + h_{22}y + h_{23} - h_{31}xy' - h_{32}yy' - h_{33}y' = 0$$

Algebraic Distance, ||h||=1 (cont)

2N x 9 **9 x 1** **2N x 1**

4 POINTS

$$\underbrace{\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1x'_1 & -y_1x'_1 & -x'_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1y'_1 & -y_1y'_1 & -y'_1 \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix}}_{2N \times 9} \underbrace{\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix}}_{9 \times 1} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \\ \\ \\ \\ \end{bmatrix}}_{2N \times 1}$$

additional points

$$\vdots \qquad\qquad\qquad \vdots$$

Algebraic Distance, ||h||=1 (cont)

Homogeneous equations

$$\overset{\text{2Nx9}}{A}\quad \overset{\text{9x1}}{h} \quad = \quad \overset{\text{2Nx1}}{0}$$

Solve:

$$\overset{\text{9x2N}}{A^T}\quad \overset{\text{2Nx9}}{A}\quad \overset{\text{9x1}}{h} \quad = \quad \overset{\text{9x2N}}{A^T}\quad \overset{\text{2Nx1}}{0}$$

$$\overbrace{(A^T \quad A)}^{\text{9x9}}\quad \overset{\text{9x1}}{h} \quad = \quad \overset{\text{9x1}}{0}$$

SVD of $A^T A = \quad U \quad D \quad U^T$

Let h be the column of U (unit eigenvector) associated with the smallest eigenvalue in D. (if only 4 points, that eigenvalue will be 0)