Recreational mathematics

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Introduction

Recreational mathematics

Recreational mathematics is an umbrella term, referring to mathematical puzzles and mathematical games.

Not all problems in this field require a knowledge of advanced mathematics, and thus, recreational mathematics often attracts the curiosity of non-mathematicians, and inspires their further study of mathematics.

Topics

This genre of mathematics includes logic puzzles and other puzzles that require deductive reasoning, the aesthetics of mathematics, and peculiar or amusing stories and coincidences about mathematics and mathematicians. Some of the more well-known topics in recreational mathematics are magic squares and fractals.

Mathematical games

Mathematical games are multiplayer games whose rules, strategies, and outcomes can be studied and explained by mathematics. The players of the game may not need to use mathematics in order to play mathematical games. For example, Mancala is a mathematical game, because mathematicians can study it using combinatorial game theory, even though no mathematics is necessary in order to play it.

Mathematical puzzles

Mathematical puzzles require mathematics in order to solve them. They have specific rules, as do multiplayer games, but mathematical puzzles don't usually involve competition between two or more players. Instead, in order to solve such a puzzle, the solver must find a solution that satisfies the given conditions.

Logic puzzles are a common type of mathematical puzzle. Conway's Game of Life and fractals are also considered mathematical puzzles, even though the solver only interacts with them by providing a set of initial conditions.

Sometimes, mathematical puzzles are referred to as mathematical games as well.

Others

Other curiosities and pastimes of non-trivial mathematical interest:

- Juggling (juggling patterns)
- Origami (many mathematical results, some deep)
- Cat's cradle and other string figures

Publications

- The *Journal of Recreational Mathematics* is the largest publication on this topic.
- *Mathematical Games* was the title of a long-running column on the subject by Martin Gardner, in Scientific American. He inspired several new generations of mathematicians and scientists, through his interest in mathematical recreations. *Mathematical Games* was succeeded by *Metamagical Themas*, a similarly distinguished, but shorter-running, column by Douglas Hofstadter, then by *Mathematical Recreations*, a column by Ian Stewart, and most recently *Puzzling Adventures* by Dennis Shasha.

In popular culture

- In the *Doctor Who* episode "42", the Doctor completes a sequence of happy primes, then complains that schools no longer teach recreational mathematics.
- *The Curious Incident of the Dog in the Nighttime*, a book about a young boy with Asperger syndrome, discusses many mathematical games and puzzles.

People

The foremost advocates of recreational mathematics have included:

- Lewis Carroll, author, mathematician and puzzlist
- John Horton Conway, mathematician and inventor of Conway's Game of life
- Henry Dudeney, regarded as England's greatest puzzlist
- Martin Gardner, author of *Mathematical Games*, a long running column in Scientific American
- Sam Loyd, regarded as America's greatest puzzlist
- Clifford A. Pickover, author of numerous books on recreational mathematics
- Marilyn vos Savant, author of "Ask Marilyn", a long running column in *PARADE*
- Malba Tahan, pseudonym of Júlio César de Mello e Souza, author of several books figuring recreational mathematics, including *The Man Who Counted*

Further reading

- W. W. Rouse Ball and H.S.M. Coxeter (1987). *Mathematical Recreations and Essays*, Thirteenth Edition, Dover. ISBN 0-486-25357-0.
- Henry E. Dudeney (1967). *536 Puzzles and Curious Problems. Charles Scribner's sons*. ISBN 0-684-71755-7.
- Sam Loyd (1959. 2 Vols.). in Martin Gardner: The Mathematical Puzzles of Sam Loyd. Dover. OCLC 5720955.
- Raymond M. Smullyan (1991). *The Lady or the Tiger? And Other Logic Puzzles*. Oxford University Press. ISBN 0-19-286136-0.

External links

- Mathematical treasure hunt on the Internet [1] by CIJM (for highschool students and general public)
- QEDcat - fun mathematical resources [2] by Burkard Polster and Marty Ross.
- mathpuzzle.com [3] by Ed Pegg, Jr.
- Puzzles of the Month [4] by Gianni A. Sarcone
- The Unreasonable Utility of Recreational Mathematics [5] by David Singmaster
- Nick's Mathematical Puzzles [6]
- Knot a Braid of Links [7]
- Project Eureka - collection of mathematical problems and puzzles [8]
- A+Click: Math problems with difficulty level adapted to students' age [9]

References

[1] http://treasurehunt.cijm.org
[2] http://www.qedcat.com
[3] http://www.mathpuzzle.com/
[4] http://www.archimedes-lab.org/page10b.html
[5] http://anduin.eldar.org/%7Eproblemi/singmast/ecmutil.html
[6] http://www.qbyte.org/puzzles/
[7] http://camel.math.ca/Kabol/
[8] http://projecteureka.org/
[9] http://www.aplusclick.com/map.htm

Mathematical puzzles

Mathematical puzzle

Mathematical puzzles make up an integral part of recreational mathematics. They have specific rules as do multiplayer games, but they do not usually involve competition between two or more players. Instead, to solve such a puzzle, the solver must find a solution that satisfies the given conditions. Mathematical puzzles require mathematics to solve them. Logic puzzles are a common type of mathematical puzzle.

Conway's Game of Life and fractals, as two examples, may also be considered mathematical puzzles even though the solver interacts with them only at the beginning by providing a set of initial conditions. After these conditions are set, the rules of the puzzle determine all subsequent changes and moves.

List of mathematical puzzles

The following categories are not disjoint; some puzzles fall into more than one category.

Numbers, arithmetic, and algebra

- Cross-figures or Cross number Puzzle
- Dyson numbers
- Four fours
- Feynman Long Division Puzzles
- Pirate loot problem
- Verbal arithmetics

Combinatorial

- Cryptograms
- Fifteen Puzzle
- Rubik's Cube and other sequential movement puzzles
- Sudoku
- Think-a-Dot
- Tower of Hanoi

Analytical or differential

- Ant on a rubber rope

  See also: Zeno's paradoxes

Probability

- Monty Hall problem

Tiling, packing, and dissection

- Mutilated chessboard problem
- Packing problem
- Pentominoes tiling
- Tangram
- Slothouber–Graatsma puzzle
- Conway puzzle
- Bedlam cube
- Soma cube
- T puzzle

Involves a board

- Conway's Game of Life
- Sudoku
- Mutilated chessboard problem
- Peg solitaire

Chessboard tasks

- Eight queens puzzle
- Knight's Tour
- No-three-in-line problem

Topology, knots, graph theory

The fields of knot theory and topology, especially their non-intuitive conclusions, are often seen as a part of recreational mathematics.

- Disentanglement puzzles
- Seven Bridges of Königsberg
- Water, gas, and electricity

Mechanical

- Rubik's Cube
- Think-a-Dot

0-player puzzles

- Flexagon
- Conway's Game of Life
- Polyominoes

External links

- Historical Math Problems/Puzzles [1] at Convergence [2]

References

[1] http://mathdl.maa.org/convergence/1/?pa=content&sa=browseNode&categoryId=9
[2] http://mathdl.maa.org/convergence/1/

Four fours

Four fours is a mathematical puzzle. The goal of four fours is to find the simplest mathematical expression for every whole number from 0 to some maximum, using only common mathematical symbols and the digit four (no other digit is allowed). Most versions of four fours require that each expression have exactly four fours, but some variations require that each expression have the minimum number of fours.

Rules

There are many variations of four fours; their primary difference is which mathematical symbols are allowed. Essentially all variations at least allow addition ("+"), subtraction ("−"), multiplication ("×"), division ("÷"), and parentheses, as well as concatenation (e.g., "44" is allowed). Most also allow the factorial ("!"), exponentiation (e.g. "44^4"), the decimal digit (".") and the square root operation, although sometimes square root is specifically excluded on the grounds that there is an implied "2" for the second root. Other operations allowed by some variations include subfactorial, ("!" before the number: !4 equals 9), overline (an infinitely repeated digit), an arbitrary root power, the gamma function (Γ(), where Γ(*x*) = (*x* − 1)!), and percent ("%"). Thus 4/4% = 100 and Γ(4)=6. A common use of the overline in this problem is for this value:

$$.\overline{4} = .4444... = 4/9$$

Typically the "log" operators are not allowed, since there is a way to trivially create any number using them. Paul Bourke credits Ben Rudiak-Gould with this description of how natural logarithms (ln()) can be used to represent any positive integer *n* as:

$$n = -\sqrt{4}\frac{\ln\left[\left(\ln\underbrace{\sqrt{\sqrt{\cdots\sqrt{4}}}}_{n}\right)/\ln 4\right]}{\ln 4}$$

Additional variants (usually no longer called "four fours") replace the set of digits ("4, 4, 4, 4") with some other set of digits, say of the birthyear of someone. For example, a variant using "1975" would require each expression to use one 1, one 9, one 7, and one 5.

Solutions

Here is a set of four fours solutions for the numbers 0 through 20, using typical rules. Some alternate solutions are listed here, although there are actually many more correct solutions

- 0 = 4+4-4-4
- 1 = 4/4+4-4
- 2 = 4 ÷ 4 + 4 ÷ 4
- 3 = (4 + 4 + 4) ÷ 4
- 4 = 4 ×(4 − 4) + 4
- 5 = (4 × 4 + 4) ÷ 4
- 6 = (4 + 4) ÷ 4 + 4 = 4.4 + 4 × .4
- 7 = 44 ÷ 4 − 4 = 4 + 4 − (4 ÷ 4)
- 8 = 4 + 4.4 − .4 = 4 + 4 + 4 − 4
- 9 = 4 + 4 + 4 ÷ 4
- 10 = (44 − 4) / 4 = 44 ÷ 4.4 = 4 + √4 + √4 + √4
- 11 = 4 ÷ .4 + 4 ÷ 4
- 12 = (44 + 4) ÷ 4
- 13 = 4! − 44 ÷ 4= (4 − .4) ÷ .4 + 4
- 14 = 4 × (4 − .4) − .4 = 4 ÷ .4 + √4 + √4
- 15 = 44 ÷ 4 + 4 = 4 × 4 − 4 ÷ 4
- 16 = .4 × (44 − 4) = 4 × 4 × 4 ÷ 4 = 4 + 4 + 4 + 4 = √4 × √4 × √4 × √4
- 17 = 4 × 4 + 4 ÷ 4
- 18 = 4 × 4 + 4 ÷ √4
- 19 = 4! − 4 − (4 ÷ 4)
- 20 = 4 × (4 ÷ 4 + 4)

There are also many other ways to find the answer for all of these.

Note that numbers with values less than one are not usually written with a leading zero. For example, "0.4" is usually written as ".4". This is because "0" is a digit, and in this puzzle only the digit "4" can be used.

A given number will generally have many possible solutions; any solution that meets the rules is acceptable. Some variations prefer the "fewest" number of operations, or prefer some operations to others. Others simply prefer "interesting" solutions, i.e., a surprising way to reach the goal.

Certain numbers, such as 113 and 123, are particularly difficult to solve under typical rules. For 113, Wheeler suggests Γ(Γ(4)) −(4! + 4)/4. For 123, Wheeler suggests the expression:

$$\sqrt{\sqrt{\sqrt{\left(\sqrt{4}/.4\right)^{4!}}}} - \sqrt{4}.$$

The first printed occurrence of this activity is in "Mathematical Recreations and Essays" by W. W. Rouse Ball published in 1892. In this book it is described as a "traditional recreation".

Algorithmics of the problem

This problem and its generalizations (like the five fives and the six sixes problem, both shown below) may be solved by a simple algorithm. The basic ingredients are hash tables that map rationals to strings. In these tables, the keys are the numbers being represented by some admissible combination of operators and the chosen digit *d*, e.g. four, and the values are strings that contain the actual formula. There is one table for each number *n* of occurrences of *d*. For example, when *d=4*, the hash table for two occurrences of *d* would contain the key-value pair *8* and **4+4**, and the one for three occurrences, the key-value pair *2* and **(4+4)/4** (strings shown in bold).

The task is then reduced to recursively computing these hash tables for increasing *n*, starting from *n=1* and continuing up to e.g. *n=4*. The tables for *n=1* and *n=2* are special, because they contain primitive entries that are not the combination of other, smaller formulas, and hence they must be initialized properly, like so (for *n=1*)

```
        T[4]     := "4";
        T[4/10]  := ".4";
        T[4/9]   := ".4...";
```

and

```
         T[44] := "44";.
```

(for *n=2*). Now there are two ways in which new entries may arise, either as a combination of existing ones through a binary operator, or by applying the factorial or square root operators (which does not use additional instances of *d*). The first case is treated by iterating over all pairs of subexpressions that use a total of *n* instances of *d*. For example, when *n=4*, we would check pairs *(a,b)* with *a* containing one instance of *d* and *b* three, and with *a* containing two instances of *d* and *b* two as well. We would then enter *a+b, a-b, b-a, a*b, a/b, b/a)* into the hash hable, including parenthesis, for *n=4*. Here the sets *A* and *B* that contain *a* and *b* are calculated recursively, with *n=1* and *n=2* being the base case. Memoization is used to ensure that every hash table is only computed once.

The second case (factorials and roots) is treated with the help of an auxiliary function, which is invoked every time a value *v* is recorded. This function computes nested factorials and roots of *v* up to some maximum depth, restricted to rationals.

The last phase of the algorithm consists in iterating over the keys of the table for the desired value of *n* and extracting and sorting those keys that are integers. This algorithm was used to calculate the five fives and six sixes examples shown below. The more compact formula (in the sense of number of characters in the corresponding value) was chosen every time a key occurred more than once.

Excerpt from the solution to the five fives problem

```
139 = ((((5+(5/5)))!/5)-5)
140 = (.5*(5+(5*55)))
141 = ((5)!+((5+(5+.5))/.5))
142 = ((5)!+((55/.5)/5))
143 = ((((5+(5/5)))!-5)/5)
144 = ((((55/5)-5))!/5)
145 = ((5*(5+(5*5)))-5)
146 = ((5)!+((5/5)+(5*5)))
147 = ((5)!+((.5*55)-.5))
148 = ((5)!+(.5+(.5*55)))
149 = (5+(((5+(5/5)))!/5))
```

Excerpt from the solution to the six sixes problem

In the table below, the notation .6... represents the value 6/9 or 2/3 (recurring decimal 6).

```
241 = ((.6+((6+6)*(6+6)))/.6)
242 = ((6*(6+(6*6)))-(6/.6))
243 = (6+((6*(.6*66))-.6))
244 = (.6...*(6+(6*(66-6))))
245 = ((((6)!+((6)!+66))/6)-6)
```

```
246 = (66+(6*((6*6)-6)))
247 = (66+((6+((6)!/.6...))/6))
248 = (6*(6+(6*(6-(.6.../6)))))
249 = (.6+(6*(6+((6*6)-.6))))
250 = (((6*(6*6))-66)/.6)
251 = ((6*(6+(6*6)))-(6/6))
252 = (66+(66+((6)!/6)))
253 = ((6/6)+(6*(6+(6*6))))
254 = ((.6...*((6*66)-6))-6)
255 = ((((6*6)+66)/.6)/.6...)
256 = (6*(6*(6-(6/(.6-6)))))
257 = (6+(((6)!+((6)!+66))/6))
258 = ((6)!-(66+(6*66)))
259 = ((((6*6)+((6)!/6))-.6)/.6)
260 = ((66+(((6)!/.6)/6))-6)
```

See also

- Krypto (game)

External links

- Bourke, Paul. *Four Fours Problem*. [1]
- Carver, Ruth. *Four Fours Puzzle* at MathForum.org [2]
- 4444 (Four Fours) Eyegate Gallery [3]

References

[1] http://astronomy.swin.edu.au/~pbourke/fun/4444
[2] http://mathforum.org/ruth/four4s.puzzle.html
[3] http://www.eyegate.com/showgal.php?id=7

Verbal arithmetic

Verbal arithmetic, also known as **alphametics**, **cryptarithmetic**, **crypt-arithmetic**, **cryptarithm** or **word addition**, is a type of mathematical game consisting of a mathematical equation among unknown numbers, whose digits are represented by letters. The goal is to identify the value of each letter. The name can be extended to puzzles that use non-alphabetic symbols instead of letters.

The equation is typically a basic operation of arithmetic, such as addition, multiplication, or division. The classic example, published in the July 1924 issue of Strand Magazine by Henry Dudeney,[1] is:

$$\begin{array}{cccccc} & & S & E & N & D \\ + & & M & O & R & E \\ \hline = & M & O & N & E & Y \end{array}$$

The solution to this puzzle is O = 0, M = 1, Y = 2, E = 5, N = 6, D = 7, R = 8, and S = 9.

Traditionally, each letter should represent a different digit, and (as in ordinary arithmetic notation) the leading digit of a multi-digit number must not be zero. A good puzzle should have a unique solution, and the letters should make up a cute phrase (as in the example above).

Verbal arithmetic can be useful as a motivation and source of exercises in the teaching of algebra.

History

Verbal arithmetic puzzles are quite old and their inventor is not known. An example in *The American Agriculturist*[2] of 1864 disproves the popular notion that it was invented by Sam Loyd. The name **crypt-arithmetic** was coined by puzzlist Minos (pseudonym of Maurice Vatriquant) in the May 1931 issue of Sphinx, a Belgian magazine of recreational mathematics. In the 1955, J. A. H. Hunter introduced the word "alphametic" to designate cryptarithms, such as Dudeney's, whose letters form meaningful words or phrases.[3] ,,,

Solving cryptarithms

Solving a cryptarithm by hand usually involves a mix of deductions and exhaustive tests of possibilities. For instance, the following sequence of deductions solves Dudeney's SEND + MORE = MONEY puzzle above (columns are numbered from right to left):

$$\begin{array}{cccccc} & & S & E & N & D \\ + & & M & O & R & E \\ \hline = & M & O & N & E & Y \end{array}$$

1. From column 5, **M = 1** since it is the only carry-over possible from the sum of two single digit numbers in column 4.
2. To produce a carry from column 4 to column 5, S + M is at least 9, so S is 8 or 9, so S + M is 9 or 10, and so O is 0 or 1. But M = 1, so **O = 0**.
3. If there were a carry from column 3 to column 4 then E = 9 and so N = 0. But O = 0, so there is no carry, and **S = 9**.
4. If there were no carry from column 2 to column 3 then E = N, which is impossible. Therefore there is a carry and N = E + 1.
5. If there were no carry from column 1 to column 2, then N + R = E mod 10, and N = E + 1, so E + 1 + R = E mod 10, so R = 9. But S = 9, so there must be a carry from column 1 to column 2 and **R = 8**.
6. To produce a carry from column 1 to column 2, we must have D + E = 10 + Y. As Y cannot be 0 or 1, D + E is at least 12. As D is at most 7, then E is at least 5. Also, N is at most 7, and N = E + 1. So E is 5 or 6.
7. If E were 6 then to make D + E at least 12, D would have to be 7. But N = E + 1, so N would also be 7, which is impossible. Therefore **E = 5** and **N = 6**.

8. To make D + E at least 12 we must have **D = 7**, and so **Y = 2**.

The use of modular arithmetic often helps. For example, use of mod-10 arithmetic allows the columns of an addition problem to be treated as simultaneous equations, while the use of mod-2 arithmetic allows inferences based on the parity of the variables.

In computer science, cryptarithms provide good examples to illustrate the brute force method, and algorithms that generate all permutations of *m* choices from *n* possibilities. For example, the Dudeney puzzle above can be solved by testing all assignments of eight values among the digits 0 to 9 to the eight letters S,E,N,D,M,O,R,Y, giving 1,814,400 possibilities. They provide also good examples for backtracking paradigm of algorithm design.

Other information

When generalized to arbitrary bases, the problem of determining if a cryptarithm has a solution is NP-complete.[4] (The generalization is necessary for the hardness result because in base 10, there are only 10! possible assignments of digits to letters, and these can be checked against the puzzle in linear time.)

Alphametics can be combined with other number puzzles such as Sudoku and Kakuro to create cryptic Sudoku and Kakuro.

See also

- Diophantine equation
- Mathematical puzzles
- Permutation
- Puzzles

References

[1] H. E. Dudeney, in *Strand Magazine* vol. 68 (July 1924), pp. 97 and 214.
[2] *American Agriculturist* **23** (12): pp. 349. December 1864
[3] J. A. H. Hunter, in the Toronto *Globe and Mail* (27 October 1955), p. 27.
[4] David Eppstein (1987). "On the NP-completeness of cryptarithms" (http://www.ics.uci.edu/~eppstein/pubs/Epp-SN-87.pdf). *SIGACT News* **18** (3): 38–40. doi:10.1145/24658.24662. .

- Martin Gardner, *Mathematics, Magic, and Mystery*. Dover (1956)
- Journal of Recreational Mathematics, has a regular alphametics column.
- Jack van der Elsen, *Alphametics*. Maastricht (1998)
- Kahan S., Have some sums to solve: The complete alphametics book, Baywood Publishing, (1978)
- Yang X. S., Cryptic Kakuro and Cross Sums Sudoku, Exposure Publishing, (2006)
- Brooke M. One Hundred & Fifty Puzzles in Crypt-Arithmetic. New York: Dover, (1963)

External links

- Alphametic Solver written in Python (http://code.activestate.com/recipes/576615/)
- Cryptarithms (http://www.cut-the-knot.org/cryptarithms/st_crypto.shtml) at cut-the-knot
- Weisstein, Eric W., " Alphametic (http://mathworld.wolfram.com/Alphametic.html)" from MathWorld.
- Weisstein, Eric W., " Cryptarithmetic (http://mathworld.wolfram.com/Cryptarithmetic.html)" from MathWorld.
- Alphametics and Cryptarithms (http://www.mathematik.uni-bielefeld.de/~sillke/PUZZLES/ALPHAMETIC/)
- An on-line tool to create Alphametics and Cryptarithms (http://www.iread.it/cryptarithms.php)

Feynman Long Division Puzzles

Physicist Richard Feynman sent the following puzzle for his father attached to a letter to his mother in 1939 [1] [2] .

Each digit of a long division has been replaced by a dot or the letter *A* (which stands for a unique digit). None of the dots are the same as the *A* digit. The goal is to reconstruct the original figures. The algorithm commonly used in the US calls for the quotient to be in the first line, the divisor to be in front of the) and the dividend behind it. The quotient is thus (....A..):(.A.) = (..A.)

Here is the division:

```
           . . A .
.A.) . . . . A . .
     . . A A
       . . . A
       . . A
       . . . .
       . A . .
         . . . .
         . . . .
               0
```

However, Feynman was not the author of this particular puzzle since the same skeleton division [3] had been previously proposed as problem E217 in the May 1936 issue of the American Mathematical Monthly by W. F. Cheney, Jr. and its solution by M. J. Turner was later published in the February 1937 issue of the same journal[4] , long before Richard Feynman's letter.

References

[1] "Don't you have time to think" - UK edition (http://www.amazon.com/dp/0141021136/), ISBN 0141021136

[2] "Perfectly reasonable deviations from the beaten path" - US edition (http://www.amazon.com/dp/B000NIJ4E2/), ISBN 0738206369

[3] Weisstein, Eric W., " Skeleton Division (http://mathworld.wolfram.com/SkeletonDivision.html)" from MathWorld.

[4] W. F. Cheney, Jr. (1936). "Problems for Solution: E211-E217" (http://links.jstor.org/sici?sici=0002-9890(193605)43:5<304:PFSE>2.0.CO;2-Q). *American Mathematical Monthly* (Mathematical Association of America) **43** (5): 304–305. doi:10.2307/2301207. . Solutions (http://links.jstor.org/sici?sici=0002-9890(193702)44:2<105:E>2.0.CO;2-H) by W. F. Cheney, Jr. and M. J. Turner (among other solvers but there is no mention of R. P. Feynman), Vol. 44, No. 2 (Feb., 1937), pp. 105-106.

External links

- AMM E217 by W. F. Cheney, Jr. (http://problemcorner.org:591/problemcorner.org/FMPro?-db=problems.fp5&-format=revealcomments.html&-lay=all&ProblemName=AMM&ProblemName=E217&-recid=36005&-find=)
- Daily Feynman Long Division Puzzles (http://dailyfeynmanlongdivisionpuzzles.blogspot.com/)
- FeynmanPuzzle - Kwiki (http://wiki.keithl.com/?FeynmanPuzzle)
- Screw Sudoku ... Do Feynman Long Division (http://thevariableman.blogspot.com/2007/03/screw-suduko-do-feynman-long-division.html)

Cross-figure

A **cross-figure** (also variously called **cross number puzzle** or **figure logic**) is a puzzle similar to a crossword in structure, but with entries which consist of numbers rather than words, with individual digits being entered in the blank cells. The numbers can be clued in various ways:

- The clue can make it possible to find the number required directly, by using general knowledge (e.g. "Date of the Battle of Hastings") or arithmetic (e.g. "27 times 79") or other mathematical facts (e.g. "Seventh prime number")
- The clue may require arithmetic to be applied to another answer or answers (e.g. "25 across times 3" or "9 down minus 3 across")
- The clue may indicate possible answers but make it impossible to give the correct one without using crosslights (e.g. "A prime number")
- One answer may be related to another in a non-determinate way (e.g. "A multiple of 24 down" or "5 across with its digits rearranged")
- Some entries may either not be clued at all, or refer to another clue (e.g. 7 down may be clued as "See 13 down" if 13 down reads "7 down plus 5")
- Entries may be grouped together for clueing purposes, e.g. "1 across, 12 across and 17 across together contain all the digits except 0"
- Some cross-figures use an algebraic type of clue, with various letters taking unknown values (e.g. "A - 2B, where neither A nor B is known in advance)
- Another special type of puzzle uses a real-world situation such as a family outing and base most clues on this (e.g. "Time taken to travel from Ayville to Beetown")

Cross-figures which use mostly the first type of clue may be used for educational purposes, but most enthusiasts would agree that this clue type should be used rarely, if at all. Without this type a cross-figure may superficially seem to be impossible to solve, since no answer can apparently be filled in until another has first been found, which without the first type of clue appears impossible. However, if a different approach is adopted where, instead of trying to find complete answers (as would be done for a crossword) one gradually narrows down the possibilities for individual cells (or, in some cases, whole answers) then the problem becomes tractable. For example, if 12 across and 7 down both have three digits and the clue for 12 across is "7 down times 2", one can work out that (i) the last digit of 12 across must be even, (ii) the first digit of 7 down must be 1, 2, 3 or 4, and (iii) the first digit of 12 across must be between 2 and 9 inclusive. (It is an implicit rule of cross-figures that numbers cannot start with 0; however, some puzzles explicitly allow this) By continuing to apply this sort of argument, a solution can eventually be found. Another implicit rule of cross-figures is that no two answers should be the same (in cross-figures allowing numbers to start with 0, 0123 and 123 may be considered different.)

A curious feature of cross-figures is that it makes perfect sense for the setter of a puzzle to try to solve it him or herself. Indeed, the setter should ideally do this (without direct reference to the answer) as it is essentially the only way to find out if the puzzle has a single unique solution. Alternatively, there are computer programs available that can be used for this purpose; however, they may not make it clear how difficult the puzzle is.

Given that some basic mathematical knowledge is needed to solve cross-figures, they are much less popular than crosswords. As a result, very few books of them have ever been published. Dell Magazines publishes a magazine called *Math Puzzles and Logic Problems* six times a year which generally contains as many as a dozen of these puzzles, which they name "Figure Logics". A magazine called *Figure it Out*, which was dedicated to number puzzles, included some, but it was very short-lived. This also explains why cross-figures have fewer established conventions than crosswords (especially cryptic crosswords). One exception is the use of the semicolon (;) to attach two strings of numbers together, for example 1234;5678 becomes 12345678. Some cross-figures voluntarily ignore this option and other "non-mathematical" approaches (e.g. palindromic numbers and repunits) where same result can be achieved through algebraic means.

External links

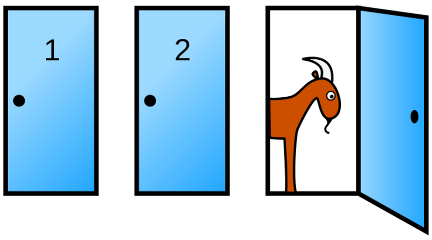
- "On Crossnumber Puzzles and The Lucas-Bonaccio Farm 1998" [1]
- The Little Pigley Farm crossnumber puzzle and its history by Joel Pomerantz [2]

References

[1] http://scisun.sci.ccny.cuny.edu/~wyscc/CrossNumber.pdf
[2] http://jig.joelpomerantz.com/fun/dogsmead.html

Monty Hall problem

The **Monty Hall problem** is a probability puzzle loosely based on the American television game show *Let's Make a Deal*. The name comes from the show's original host, Monty Hall. The problem is also called the **Monty Hall paradox**, as it is a *veridical paradox* in that the result appears absurd but is demonstrably true.



In search of a new car, the player picks a door, say 1. The game host then opens one of the other doors, say 3, to reveal a goat and offers to let the player pick door 2 instead of door 1.

The problem was originally posed in a letter by Steve Selvin to the *American Statistician* in 1975. A well-known statement of the problem was published in Marilyn vos Savant's "Ask Marilyn" column in *Parade* magazine in 1990:

> Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?
>
> —Whitaker/vos Savant 1990

Although not explicitly stated in this version, solutions are almost always based on the additional assumptions that the car is initially equally likely to be behind each door and that the host must open a door showing a goat, must randomly choose which door to open if both hide goats, and must make the offer to switch.

As the player cannot be certain which of the two remaining unopened doors is the winning door, and initially all doors were equally likely, most people assume that each of two remaining closed doors has an equal probability and conclude that switching does not matter; hence the usual answer is "stay with your original door". However, under standard assumptions, the player should switch—doing so doubles the overall probability of winning the car from 1/3 to 2/3.

The Monty Hall problem, in its usual interpretation, is mathematically equivalent to the earlier Three Prisoners problem, and both bear some similarity to the much older Bertrand's box paradox. These and other problems involving unequal distributions of probability are notoriously difficult for people to solve correctly; when the Monty Hall problem appeared in *Parade*, approximately 10,000 readers, including nearly 1,000 with PhDs, wrote to the magazine claiming the published solution ("switch!") was wrong. Numerous psychological studies examine how these kinds of problems are perceived. Even when given a completely unambiguous statement of the Monty Hall problem, explanations, simulations, and formal mathematical proofs, many people still meet the correct answer with disbelief.

Problem

Steve Selvin described a problem loosely based on the game show *Let's Make a Deal* in a letter to the *American Statistician* in 1975 (Selvin 1975a). In a subsequent letter he dubbed it the "Monty Hall problem" (Selvin 1975b). In 1990 the problem was published in its most well-known form in a letter to Marilyn vos Savant's "Ask Marilyn" column in *Parade*:

> Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which he knows has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?
>
> —Whitaker 1990

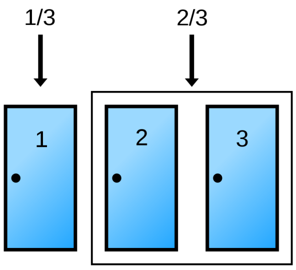
Certain aspects of the host's behavior are not specified in this wording of the problem. For example it is not clear if the host considers the position of the prize in deciding whether to open a particular door or is required to open a door under all circumstances (Mueser and Granberg 1999). Almost all sources make the additional assumptions that the car is initially equally likely to be behind each door, that the host must open a door showing a goat, and that he must make the offer to switch. Many sources add to this the assumption that the host chooses at random which door to open if both hide goats, often but not always meaning by that, at random with equal probabilities. The resulting set of assumptions gives what is called "the standard problem" by many sources (Barbeau 2000:87). According to Krauss and Wang (2003:10), even if these assumptions are not explicitly stated, people generally assume them to be the case. A fully unambiguous, mathematically explicit version of the standard problem is:

> Suppose you're on a game show and you're given the choice of three doors [and will win what is behind the chosen door]. Behind one door is a car; behind the others, goats [unwanted booby prizes]. The car and the goats were placed randomly behind the doors before the show. The rules of the game show are as follows: After you have chosen a door, the door remains closed for the time being. The game show host, Monty Hall, who knows what is behind the doors, now has to open one of the two remaining doors, and the door he opens must have a goat behind it. If both remaining doors have goats behind them, he chooses one [uniformly] at random. After Monty Hall opens a door with a goat, he will ask you to decide whether you want to stay with your first choice or to switch to the last remaining door. Imagine that you chose Door 1 and the host opens Door 3, which has a goat. He then asks you "Do you want to switch to Door Number 2?" Is it to your advantage to change your choice?
>
> —Krauss and Wang 2003:10

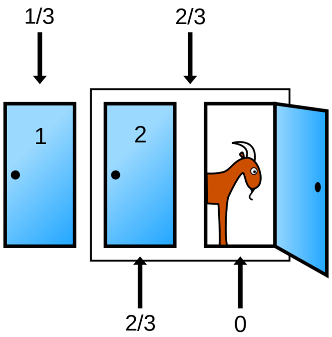
Solutions

There are several approaches to solving the Monty Hall problem all giving the same result—that a player who swaps has a 2/3 chance of winning the car. Most popular sources present solutions based on simple probabilistic reasoning. A different approach to solving the problem, commonly used in mathematical sources, is to treat it as a conditional probability problem.

Simple solutions



Player's pick has a 1/3 chance while the other two doors have 1/3 chance each, for a combined 2/3 chance.



Player's pick remains a 1/3 chance, while the other two doors a combined 2/3 chance, 2/3 for the still unopened one and 0 for the one the host opened.

The solution presented by vos Savant in *Parade* (vos Savant 1990b) shows the three possible arrangements of one car and two goats behind three doors and the result of switching or staying after initially picking Door 1 in each case:

Door 1	Door 2	Door 3	result if switching	result if staying
Car	Goat	Goat	**Goat**	**Car**
Goat	Car	Goat	**Car**	**Goat**
Goat	Goat	Car	**Car**	**Goat**

A player who stays with the initial choice wins in only one out of three of these equally likely possibilities, while a player who switches wins in two out of three. The probability of winning by staying with the initial choice is therefore 1/3, while the probability of winning by switching is 2/3.

An even simpler solution is to reason that switching loses if and only if the player initially picks the car, which happens with probability 1/3, so switching must win with probability 2/3. (Carlton 2005).

Another way to understand the solution is to consider the two original unchosen doors together. Instead of one door being opened and shown to be a losing door, an equivalent action is to combine the two unchosen doors into one since the player cannot choose the opened door (Adams 1990; Devlin 2003; Williams 2004; Stibel et al., 2008).

As Cecil Adams puts it (Adams 1990), "Monty is saying in effect: you can keep your one door or you can have the other two doors." The player therefore has the choice of either sticking with the original choice of door, or choosing the sum of the contents of the two other doors, as the 2/3 chance of hiding the car has not been changed by the opening of one of these doors.

As Keith Devlin says (Devlin 2003), "By opening his door, Monty is saying to the contestant 'There are two doors you did not choose, and the probability that the prize is behind one of them is 2/3. I'll help you by using my knowledge of where the prize is to open one of those two doors to show you that it does not hide the prize. You can now take advantage of this additional information. Your choice of door A has a chance of 1 in 3 of being the winner. I have not changed that. But by eliminating door C, I have shown you that the probability that door B hides the prize is 2 in 3.'"

Aids to understanding

Why the probability is not 1/2

The critical fact is that the host does not always have a choice (whether random or not) between the two remaining doors. He always chooses a door that he knows hides a goat after the contestant has made their choice. He always can do this, since he knows the location of the car in advance. If the host chooses completely at random when he has a choice, and if the car is initially likely to be behind any of the three doors, it turns out that the host's choice does not affect the probability that the car is behind the contestant's door. But even if the host has a bias to one door or another when he has a choice, the probability that the car is behind the contestant's door, so it turns out, can never exceed 1/2, again as long as initially all doors are equally likely. It is never unfavourable to switch, while on average it definitely does pay off to switch. The contestant will be asked if he wants to switch, and there was a 1 in 3 chance that his original choice hides a car and a 2 in 3 chance that his original choice hides a goat. By opening another door and revealing a goat the host has removed one of the two other doors, and the 1 in 3 chance of the contestant's initial door hiding a car means there is a 2 in 3 chance that the other closed door will turn out to hide a car.

This is different from a scenario where the host simply always chooses between the two other doors completely at random and hence there is a possibility (with a 1 in 3 chance) that he will reveal the car. In this instance the revelation of a goat would mean that the chance of the contestant's original choice being the car would go up to 1 in 2. This difference can be demonstrated by contrasting the original problem with a variation that appeared in vos Savant's column in November 2006. In this version, the host forgets which door hides the car. He opens one of the doors at random and is relieved when a goat is revealed. Asked whether the contestant should switch, vos Savant correctly replied, "If the host is clueless, it makes no difference whether you stay or switch. If he knows, switch" (vos Savant, 2006).

Increasing the number of doors

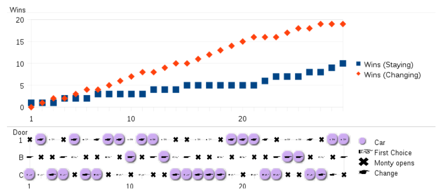
It may be easier to appreciate the solution by considering the same problem with 1,000,000 doors instead of just three (vos Savant 1990). In this case there are 999,999 doors with goats behind them and one door with a prize. The player picks a door. The game host then opens 999,998 of the other doors revealing 999,998 goats—imagine the host starting with the first door and going down a line of 1,000,000 doors, opening each one, skipping over only the player's door and one other door. The host then offers the player the chance to switch to the only other unopened door. On average, in 999,999 out of 1,000,000 times the other door will contain the prize, as 999,999 out of 1,000,000 times the player first picked a door with a goat. A rational player should switch. Intuitively speaking, the player should ask how likely is it, that given a million doors, he or she managed to pick the right one. The example can be used to show how the likelihood of success by switching is equal to (1 minus the likelihood of picking correctly the first time) for any given number of doors. It is important to remember, however, that this is based on the assumption that the host knows where the prize is and must not open a door that contains that prize, randomly selecting which other door to leave closed if the contestant manages to select the prize door initially.

This example can also be used to illustrate the opposite situation in which the host does *not* know where the prize is and opens doors randomly. There is a 999,999/1,000,000 probability that the contestant selects wrong initially, and the prize is behind one of the other doors. If the host goes about randomly opening doors not knowing where the prize is, the probability is likely that the host will reveal the prize before two doors are left (the contestant's choice and one other) to switch between. This is analogous to the game play on another game show, *Deal or No Deal*; In that game, the contestant chooses a numbered briefcase and then randomly opens the other cases one at a time.

Stibel et al. (2008) propose working memory demand is taxed during the Monty Hall problem and that this forces people to "collapse" their choices into two equally probable options. They report that when increasing the number of options to over 7 choices (7 doors) people tend to switch more often; however most still incorrectly judge the probability of success at 50/50.

Simulation

A simple way to demonstrate that a switching strategy really does win two out of three times on the average is to simulate the game with playing cards (Gardner 1959b; vos Savant 1996:8). Three cards from an ordinary deck are used to represent the three doors; one 'special' card such as the Ace of Spades should represent the door with the car, and ordinary cards, such as the two red twos, represent the goat doors.



Simulation of 30 outcomes of the Monty Hall problem

The simulation, using the following procedure, can be repeated several times to simulate multiple rounds of the game. One card is dealt face-down at random to the 'player', to represent the door the player picks initially. Then, looking at the remaining two cards, at least one of which must be a red two, the 'host' discards a red two. If the card remaining in the host's hand is the Ace of Spades, this is recorded as a round where the player would have won by switching; if the host is holding a red two, the round is recorded as one where staying would have won.

By the law of large numbers, this experiment is likely to approximate the probability of winning, and running the experiment over enough rounds should not only verify that the player *does* win by switching two times out of three, but show why. After one card has been dealt to the player, it is *already determined* whether switching will win the round for the player; and two times out of three the Ace of Spades is in the host's hand.

If this is not convincing, the simulation can be done with the entire deck, dealing one card to the player and keeping the other 51 (Gardner 1959b; Adams 1990). In this variant the Ace of Spades goes to the host 51 times out of 52, and stays with the host no matter how many *non*-Ace cards are discarded.

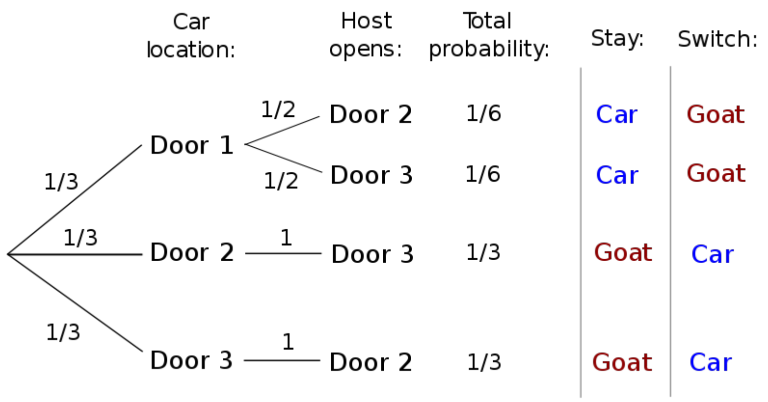
Another simulation, suggested by vos Savant, employs the "host" hiding a penny, representing the car, under one of three cups, representing the doors; or hiding a pea under one of three shells.

Conditional probability solution

The simple solutions show in various ways that a contestant who is going to switch will win the car with probability 2/3, and hence that switching is a winning strategy. Some sources, however, state that although the simple solutions give a correct numerical answer, they are incomplete or solve the wrong problem. These sources consider the question: given that the contestant has chosen Door 1 and given that the host has opened Door 3, revealing a goat, what is now the probability that the car is behind Door 2?

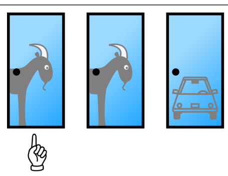
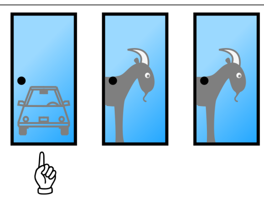
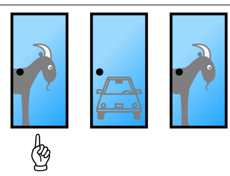
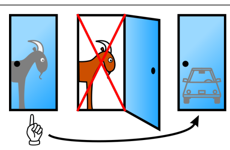
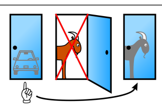
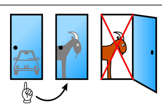
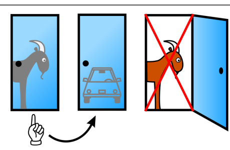
In particular, Morgan et al. (1991) state that many popular solutions are incomplete because they do not explicitly address their interpretation of vos Savant's rewording of Whitaker's original question (Seymann). The popular solutions correctly show that the probability of winning for a player who always switches is 2/3, but without additional reasoning this does not necessarily mean the probability of winning by switching is 2/3 *given which door the player has chosen and which door the host opens*. That probability is a conditional probability (Selvin 1975b; Morgan et al. 1991; Gillman 1992; Grinstead and Snell 2006:137). The difference is whether the analysis is of the average probability over all possible combinations of initial player choice and door the host opens, or of only one specific case—to be specific, the case where the player picks Door 1 and the host opens Door 3. Another way to express the difference is whether the player must decide to switch *before* the host opens a door, or is allowed to decide *after* seeing which door the host opens (Gillman 1992); either way, the player is interested in the probability of winning at the time they make their decision. Although the conditional and unconditional probabilities are both 2/3 for the problem statement with all details completely specified - in particular a completely random choice by the host of which door to open when he has a choice - the conditional probability may differ from the overall probability and the latter is not determined without a complete specification of the problem (Gill 2010). However as long as the initial choice has probability 1/3 of being correct, it is never to the contestants' disadvantage to switch, as the conditional probability of winning by switching is always at least 1/2.

The conditional probability of winning by switching given which door the host opens can be determined referring to the expanded figure below, or to an equivalent decision tree as shown to the right (Chun 1991; Grinstead and Snell 2006:137-138), or formally derived as in the mathematical formulation section below. For example, the player wins if the host opens Door 3 and the player switches and the car is behind Door 2, and this has probability 1/3. The player loses if the host opens Door 3 and the player switches and the car is behind Door 1, and this has probability 1/6. These are the only possibilities given host opens Door 3 and player switches. The overall probability that the host opens Door 3 is their sum, and we convert the two probabilities just found to conditional probabilities by dividing them by their sum. Therefore, the conditional probability of winning by switching given the player picks Door 1 and the host opens Door 3 is (1/3)/(1/3 + 1/6), which is 2/3.



Tree showing the probability of every possible outcome if the player initially picks Door 1

This analysis depends on the constraint in the explicit problem statement that the host chooses uniformly at random which door to open after the player has initially selected the car (1/6 = 1/2 * 1/3). If the host's choice to open Door 3 was made with probability q instead of probability 1/2, then the conditional probability of winning by switching becomes (1/3)/(1/3 + q * 1/3)). The extreme cases q=0, q=1 give conditional probabilities of 1 and 1/2 respectively; q=1/2 gives 2/3. If q is unknown then the conditional probability is unknown too, but still it is always at least 1/2 and on average, over the possible conditions, equal to the unconditional probability 2/3.

Car hidden behind Door 3	**Car hidden behind Door 1**		**Car hidden behind Door 2**
Player initially picks Door 1			
			
Host must open Door 2	Host randomly opens either goat door		Host must open Door 3
			
Probability 1/3	Probability 1/6	Probability 1/6	Probability 1/3

Switching wins	Switching loses	Switching loses	Switching wins
If the host has opened Door 2, switching wins twice as often as staying		If the host has opened Door 3, switching wins twice as often as staying	

Mathematical formulation

The above solution may be formally proven using Bayes' theorem, similar to Gill, 2002, Henze, 1997, and many others. Different authors use different formal notations, but the one below may be regarded as typical. Consider the discrete random variables:

$C \in \{1, 2, 3\}$: the number of the door hiding the **C**ar,

$S \in \{1, 2, 3\}$: the number of the door **S**elected by the player, and

$H \in \{1, 2, 3\}$: the number of the door opened by the **H**ost.

As the host's placement of the car is random, all values of *C* are equally likely. The initial (unconditional) probability of *C* is then

$P(C) = \frac{1}{3}$, for every value of *C*.

Further, as the initial choice of the player is independent of the placement of the car, variables *C* and *S* are independent. Hence the conditional probability of *C* given *S* is

$P(C|S) = P(C)$, for every value of *C* and *S*.

The host's behavior is reflected by the values of the conditional probability of *H* given *C* and *S*:

$$P(H|C,S) = \begin{cases} 0 & \text{if } H = S, \text{(the host cannot open the door picked by the player)} \\ 0 & \text{if } H = C, \text{(the host cannot open a door with a car behind it)} \\ 1/2 & \text{if } S = C, \text{(the two doors with no car are equally likely to be opened)} \\ 1 & \text{if } H \neq C \text{ and } S \neq C, \text{(there is only one door available to open)} \end{cases}$$

The player can then use Bayes' rule to compute the probability of finding the car behind any door, after the initial selection and the host's opening of one. This is the conditional probability of *C* given *H* and *S*:

$$P(C|H,S) = \frac{P(H|C,S)P(C|S)}{P(H|S)},$$

where the denominator is computed as the marginal probability

$$P(H|S) = \sum_{C=1}^{3} P(H,C|S) = \sum_{C=1}^{3} P(H|C,S)P(C|S).$$

Thus, if the player initially selects Door 1, and the host opens Door 3, the probability of winning by switching is

$$P(C=2|H=3,S=1) = \frac{1 \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + 1 \times \frac{1}{3} + 0 \times \frac{1}{3}} = \frac{2}{3}.$$

Sources of confusion

When first presented with the Monty Hall problem an overwhelming majority of people assume that each door has an equal probability and conclude that switching does not matter (Mueser and Granberg, 1999). Out of 228 subjects in one study, only 13% chose to switch (Granberg and Brown, 1995:713). In her book *The Power of Logical Thinking*, vos Savant (1996:15) quotes cognitive psychologist Massimo Piattelli-Palmarini as saying "... no other statistical puzzle comes so close to fooling all the people all the time" and "that even Nobel physicists systematically give the wrong answer, and that they *insist* on it, and they are ready to berate in print those who propose the right answer." Interestingly, pigeons make mistakes and learn from mistakes, and experiments, Herbranson and Schroeder, 2010, show that they rapidly learn to always switch, unlike humans.

Most statements of the problem, notably the one in *Parade Magazine*, do not match the rules of the actual game show (Krauss and Wang, 2003:9), and do not fully specify the host's behavior or that the car's location is randomly selected (Granberg and Brown, 1995:712). Krauss and Wang (2003:10) conjecture that people make the standard assumptions even if they are not explicitly stated. Although these issues are mathematically significant, even when controlling for these factors nearly all people still think each of the two unopened doors has an equal probability and conclude switching does not matter (Mueser and Granberg, 1999). This "equal probability" assumption is a deeply rooted intuition (Falk 1992:202). People strongly tend to think probability is evenly distributed across as many unknowns as are present, whether it is or not (Fox and Levav, 2004:637).

In addition to the "equal probability" intuition, a competing and deeply rooted intuition is that *revealing information that is already known does not affect probabilities*. Although this is a true statement, it is not true that just knowing the host can open one of the two unchosen doors to show a goat necessarily means that opening a specific door cannot affect the probability that the car is behind the initially-chosen door. If the car is initially placed behind the doors with equal probability and the host chooses uniformly at random between doors hiding a goat (as is the case in the standard interpretation) this probability indeed remains unchanged, but if the host can choose non-randomly between such doors then the specific door that the host opens reveals additional information. The host can always open a door revealing a goat *and* (in the standard interpretation of the problem) the probability that the car is behind the initially-chosen door does not change, but it is *not because* of the former that the latter is true. Solutions based on the assertion that the host's actions cannot affect the probability that the car is behind the initially-chosen door are very persuasive, but lead to the correct answer only if the problem is completely symmetrical with respect to both the initial car placement and how the host chooses between two goats (Falk 1992:207,213).

According to Morgan et al. (1991) "The distinction between the conditional and unconditional situations here seems to confound many." That is, they, and some others, interpret the usual wording of the problem statement as asking about the conditional probability of winning given which door is opened by the host, as opposed to the overall or unconditional probability. These are mathematically different questions and can have different answers depending on how the host chooses which door to open when the player's initial choice is the car (Morgan et al., 1991; Gillman 1992). For example, if the host opens Door 3 whenever possible then the probability of winning by switching for players initially choosing Door 1 is 2/3 overall, but only 1/2 if the host opens Door 3. In its usual form the problem statement does not specify this detail of the host's behavior, nor make clear whether a conditional or an unconditional answer is required, making the answer that switching wins the car with probability 2/3 equally vague. Many commonly presented solutions address the unconditional probability, ignoring which door was chosen by the player and which door opened by the host; Morgan et al. call these "false solutions" (1991). Others, such as Behrends (2008), conclude that "One must consider the matter with care to see that both analyses are correct."

Variants – slightly modified problems

Other host behaviors

The version of the Monty Hall problem published in *Parade* in 1990 did not specifically state that the host would always open another door, or always offer a choice to switch, or even never open the door revealing the car. However, vos Savant made it clear in her second followup column that the intended host's behavior could only be what led to the 2/3 probability she gave as her original answer. "Anything else is a different question" (vos Savant, 1991). "Virtually all of my critics understood the intended scenario. I personally read nearly three thousand letters (out of the many additional thousands that arrived) and found nearly every one insisting simply that because two options remained (or an equivalent error), the chances were even. Very few raised questions about ambiguity, and the letters actually published in the column were not among those few." (vos Savant, 1996) The answer follows if the car is placed randomly behind any door, the host must open a door revealing a goat regardless of the player's initial choice and, if two doors are available, chooses which one to open randomly (Mueser and Granberg, 1999). The table below shows a variety of OTHER possible host behaviors and the impact on the success of switching.

Determining the player's best strategy within a given set of other rules the host must follow is the type of problem studied in game theory. For example, if the host is not required to make the offer to switch the player may suspect the host is malicious and makes the offers more often if the player has initially selected the car. In general, the answer to this sort of question depends on the specific assumptions made about the host's behaviour, and might range from "ignore the host completely" to 'toss a coin and switch if it comes up heads', see the last row of the table below.

Morgan et al. (1991) and Gillman (1992) both show a more general solution where the car is (uniformly) randomly placed but the host is not constrained to pick uniformly randomly if the player has initially selected the car, which is how they both interpret the well known statement of the problem in *Parade* despite the author's disclaimers. Both changed the wording of the *Parade* version to emphasize that point when they restated the problem. They consider a scenario where the host chooses between revealing two goats with a preference expressed as a probability q, having a value between 0 and 1. If the host picks randomly q would be 1/2 and switching wins with probability 2/3 regardless of which door the host opens. If the player picks Door 1 and the host's preference for Door 3 is q, then in the case where the host opens Door 3 switching wins with probability 1/3 if the car is behind Door 2 and loses with probability $(1/3)q$ if the car is behind Door 1. The conditional probability of winning by switching *given the host opens Door 3* is therefore $(1/3)/(1/3 + (1/3)q)$ which simplifies to $1/(1+q)$. Since q can vary between 0 and 1 this conditional probability can vary between 1/2 and 1. This means even without constraining the host to pick randomly if the player initially selects the car, the player is never worse off switching. However, it is important to note that neither source suggests the player knows what the value of q is, so the player cannot attribute a probability other than the 2/3 that vos Savant assumed was implicit.

Possible host behaviors in unspecified problem

Host behavior	Result
"Monty from Hell": The host offers the option to switch only when the player's initial choice is the winning door (Tierney 1991).	Switching always yields a goat.
"Angelic Monty": The host offers the option to switch only when the player has chosen incorrectly (Granberg 1996:185).	Switching always wins the car.
"Monty Fall" or "Ignorant Monty": The host does not know what lies behind the doors, and opens one at random that happens not to reveal the car (Granberg and Brown, 1995:712) (Rosenthal, 2008).	Switching wins the car half of the time.
The host knows what lies behind the doors, and (before the player's choice) chooses at random which goat to reveal. He offers the option to switch only when the player's choice happens to differ from his.	Switching wins the car half of the time.
The host always reveals a goat and always offers a switch. If he has a choice, he chooses the leftmost goat with probability p (which may depend on the player's initial choice) and the rightmost door with probability $q=1-p$. (Morgan et al. 1991) (Rosenthal, 2008).	If the host opens the rightmost door, switching wins with probability $1/(1+q)$.
The host acts as noted in the specific version of the problem.	Switching wins the car two-thirds of the time. (Special case of the above with $p=q=½$)
The host opens a door and makes the offer to switch 100% of the time if the contestant initially picked the car, and 50% the time if she didn't. (Mueser and Granberg 1999)	Switching wins 1/2 the time at the Nash equilibrium.

Four-stage two-player game-theoretic (Gill, 2010, Gill, 2011). The player is playing against the show organisers (TV station) which includes the host. First stage: organizers choose a door (choice kept secret from player). Second stage: player makes a preliminary choice of door. Third stage: host opens a door. Fourth stage: player makes a final choice. The player wants to win the car, the TV station wants to keep it. This is a zero-sum two-person game. By von Neumann's theorem from game theory, if we allow both parties fully randomized strategies there exists a minimax solution or Nash equilibrium (Mueser and Granberg 1999).	Minimax solution (Nash equilibrium): car is first hidden uniformly at random and host later chooses uniform random door to open without revealing the car and different from player's door; player first chooses uniform random door and later always switches to other closed door. With his strategy, the player has a win-chance of at least 2/3, however the TV station plays; with the TV station's strategy, the TV station will lose with probability at most 2/3, however the player plays. The fact that these two strategies match (at least 2/3, at most 2/3) proves that they form the minimax solution.
As previous, but now host has option not to open a door at all.	Minimax solution (Nash equilibrium): car is first hidden uniformly at random and host later never opens a door; player first chooses a door uniformly at random and later never switches. Player's strategy guarantees a win-chance of at least 1/3. TV station's strategy guarantees a lose-chance of at most 1/3.

N doors

D. L. Ferguson (1975 in a letter to Selvin cited in Selvin 1975b) suggests an *N* door generalization of the original problem in which the host opens *p* losing doors and then offers the player the opportunity to switch; in this variant switching wins with probability $(N-1)/[N(N-p-1)]$. If the host opens even a single door the player is better off switching, but the advantage approaches zero as *N* grows large (Granberg 1996:188). At the other extreme, if the host opens all but one losing door the probability of winning by switching approaches 1.

Bapeswara Rao and Rao (1992) suggest a different *N* door version where the host opens a losing door different from the player's current pick and gives the player an opportunity to switch after each door is opened until only two doors remain. With four doors the optimal strategy is to pick once and switch only when two doors remain. With *N* doors this strategy wins with probability $(N-1)/N$ and is asserted to be optimal.

Quantum version

A quantum version of the paradox illustrates some points about the relation between classical or non-quantum information and quantum information, as encoded in the states of quantum mechanical systems. The formulation is loosely based on Quantum game theory. The three doors are replaced by a quantum system allowing three alternatives; opening a door and looking behind it is translated as making a particular measurement. The rules can be stated in this language, and once again the choice for the player is to stick with the initial choice, or change to another "orthogonal" option. The latter strategy turns out to double the chances, just as in the classical case. However, if the show host has not randomized the position of the prize in a fully quantum mechanical way, the player can do even better, and can sometimes even win the prize with certainty (Flitney and Abbott 2002, D'Ariano et al. 2002).

History of the problem

The earliest of several probability puzzles related to the Monty Hall problem is Bertrand's box paradox, posed by Joseph Bertrand in 1889 in his *Calcul des probabilités* (Barbeau 1993). In this puzzle there are three boxes: a box containing two gold coins, a box with two silver coins, and a box with one of each. After choosing a box at random and withdrawing one coin at random that happens to be a gold coin, the question is what is the probability that the other coin is gold. As in the Monty Hall problem the intuitive answer is 1/2, but the probability is actually 2/3.

The Three Prisoners problem, published in Martin Gardner's *Mathematical Games* column in *Scientific American* in 1959 (1959a, 1959b), is equivalent to the Monty Hall problem. This problem involves three condemned prisoners, a random one of whom has been secretly chosen to be pardoned. One of the prisoners begs the warden to tell him the name of one of the others who will be executed, arguing that this reveals no information about his own fate but increases his chances of being pardoned from 1/3 to 1/2. The warden obliges, (secretly) flipping a coin to decide which name to provide if the prisoner who is asking is the one being pardoned. The question is whether knowing the warden's answer changes the prisoner's chances of being pardoned. This problem is equivalent to the Monty Hall problem; the prisoner asking the question still has a 1/3 chance of being pardoned but his unnamed cohort has a 2/3 chance.

Steve Selvin posed the Monty Hall problem in a pair of letters to the *American Statistician* in 1975 (1975a, 1975b). The first letter presented the problem in a version close to its presentation in *Parade* 15 years later. The second appears to be the first use of the term "Monty Hall problem". The problem is actually an extrapolation from the game show. Monty Hall *did* open a wrong door to build excitement, but offered a known lesser prize—such as $100 cash—rather than a choice to switch doors. As Monty Hall wrote to Selvin:

> And if you ever get on my show, the rules hold fast for you—no trading boxes after the selection.
>
> —Hall 1975

A version of the problem very similar to the one that appeared three years later in *Parade* was published in 1987 in the Puzzles section of *The Journal of Economic Perspectives* (Nalebuff 1987). Nalebuff, as later writers in mathematical economics, sees the problem as a simple and amusing exercise in game theory.

Phillip Martin's article in a 1989 issue of *Bridge Today* magazine titled "The Monty Hall Trap" (Martin 1989) presented Selvin's problem as an example of what Martin calls the probability trap of treating non-random information as if it were random, and relates this to concepts in the game of bridge.

A restated version of Selvin's problem appeared in Marilyn vos Savant's *Ask Marilyn* question-and-answer column of *Parade* in September 1990 (vos Savant 1990). Though vos Savant gave the correct answer that switching would win two-thirds of the time, she estimates the magazine received 10,000 letters including close to 1,000 signed by PhDs, many on letterheads of mathematics and science departments, declaring that her solution was wrong (Tierney 1991). Due to the overwhelming response, *Parade* published an unprecedented four columns on the problem (vos Savant 1996:xv). As a result of the publicity the problem earned the alternative name Marilyn and the Goats.

In November 1990, an equally contentious discussion of vos Savant's article took place in Cecil Adams's column *The Straight Dope* (Adams 1990). Adams initially answered, incorrectly, that the chances for the two remaining doors must each be one in two. After a reader wrote in to correct the mathematics of Adams' analysis, Adams agreed that mathematically, he had been wrong, but said that the *Parade* version left critical constraints unstated, and without those constraints, the chances of winning by switching were not necessarily 2/3. Numerous readers, however, wrote in to claim that Adams had been "right the first time" and that the correct chances were one in two.

The *Parade* column and its response received considerable attention in the press, including a front page story in the *New York Times* (Tierney 1991) in which Monty Hall himself was interviewed. He appeared to understand the problem, giving the reporter a demonstration with car keys and explaining how actual game play on *Let's Make a Deal* differed from the rules of the puzzle.

Over 40 papers have been published about this problem in academic journals and the popular press (Mueser and Granberg 1999). Barbeau 2000 contains a survey of the academic literature pertaining to the Monty Hall problem and other closely related problems.

The problem continues to resurface outside of academia. The syndicated NPR program *Car Talk* featured it as one of their weekly "Puzzlers," and the answer they featured was quite clearly explained as the correct one (Magliozzi and Magliozzi, 1998). An account of the Hungarian mathematician Paul Erdős's first encounter of the problem can be found in *The Man Who Loved Only Numbers*—like many others, he initially got it wrong. The problem is discussed, from the perspective of a boy with Asperger syndrome, in *The Curious Incident of the Dog in the Night-time*, a 2003 novel by Mark Haddon. The problem is also addressed in a lecture by the character Charlie Eppes in an episode of the CBS drama *NUMB3RS* (Episode 1.13) and in Derren Brown's 2006 book *Tricks Of The Mind*. Penn Jillette explained the Monty Hall Problem on the "Luck" episode of Bob Dylan's Theme Time Radio Hour radio series. The

Monty Hall problem appears in the film *21* (Bloch 2008). Economist M. Keith Chen identified a potential flaw in hundreds of experiments related to cognitive dissonance that use an analysis with issues similar to those involved in the Monty Hall problem (Tierney 2008).

See also

- Bayes' theorem: The Monty Hall problem
- Principle of restricted choice (bridge)

Similar problems

- Bertrand's box paradox (also known as the three-cards problem)
- Boy or Girl paradox
- Three Prisoners problem
- Two envelopes problem

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External links

- The Game Show Problem (http://www.marilynvossavant.com/articles/gameshow.html)–the original question and responses on Marilyn vos Savant's web site
- Monty Hall (http://www.dmoz.org/Science/Math/Recreations/Famous_Problems/Monty_Hall//) at the Open Directory Project
- " Monty Hall Paradox (http://demonstrations.wolfram.com/MontyHallParadox/)" by Matthew R. McDougal, The Wolfram Demonstrations Project (simulation)
- The Monty Hall Problem (http://www.nytimes.com/2008/04/08/science/08monty.html) at The New York Times (simulation)
- The Door Keeper Game (http://dl.dropbox.com/u/548740/www/DoorKeeperGame/index.html) (simulation)

Logic puzzles

Logic puzzle

A **logic puzzle** is a puzzle deriving from the mathematics field of deduction.

History

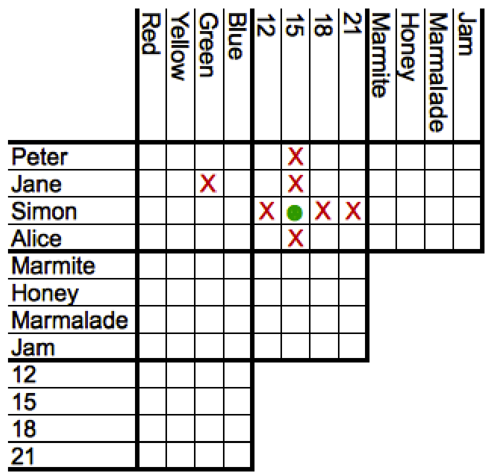
The logic puzzle was first produced by Charles Lutwidge Dodgson, who is better known under his pen name Lewis Carroll, the author of *Alice's Adventures in Wonderland*. In his book *The Game of Logic* he introduced a game to solve problems such as confirming the conclusion "Some greyhounds are not fat" from the statements "No fat creatures run well" and "Some greyhounds run well". Puzzles like this, where we are given a list of premises and asked what can be deduced from them, are known as syllogisms. Dodgson goes on to construct much more complex puzzles consisting of up to 8 premises.

In the second half of the 20th century mathematician Raymond M. Smullyan has continued and expanded the branch of logic puzzles with books such as *The Lady or the Tiger?*, *To Mock a Mockingbird* and *Alice in Puzzle-Land*. He popularized the "knights and knaves" puzzles, which involve knights, who always tell the truth, and knaves, who always lie.

There are also logic puzzles that are completely non-verbal in nature. Some popular forms include Sudoku, which involves using deduction to correctly place numbers in a grid; the nonogram, also called "Paint by Numbers", which involves using deduction to correctly fill in a grid with black-and-white squares to produce a picture; and logic mazes, which involve using deduction to figure out the rules of a maze.

Logic grid puzzles

Another form of logic puzzle, popular among puzzle enthusiasts and available in large magazines dedicated to the subject, is a format in which the set-up to a scenario is given, as well as the object (for example, determine who bid to as "logic grid" puzzles. The most famous example may be the so-called Zebra Puzzle, which asks the question *Who Owned the Zebra?*.



Example logic puzzle grid, with the information that Simon is 15 and Jane does not like green filled in.

Common in logic puzzle magazines are derivatives of the logic grid puzzle called "table puzzles" that are deduced in the same manner as grid puzzles, but lack the grid either because a grid would be too large, or because some other visual aid is provided. For example, a map of a town might be present in lieu of a grid in a puzzle about the location of different shops.

This type of puzzle is often included on the Law School Admissions Test (LSAT).

See also

- Category:Logic puzzles, a list of different logic puzzles
- List of puzzle video games

External links

- Puzzles [1] at the Open Directory Project
- Hankies, Snarks, and Triangles [2] - Ivars Peterson's MathTrek

References

[1] http://www.dmoz.org/Games/Puzzles/
[2] http://www.maa.org/mathland/mathland_1_13.html

River crossing puzzle

A **river crossing puzzle** is a type of transport puzzle in which the object is to carry items from one river bank to another. The difficulty of the puzzle may arise from restrictions on which or how many items can be transported at the same time, or from which or how many items may be safely left together.[1] The setting may vary cosmetically, for example, by replacing the river by a bridge.[1] The earliest known river-crossing problems occur in the manuscript *Propositiones ad Acuendos Juvenes* (English: *Problems to sharpen the young*), traditionally said to be written by Alcuin. The earliest copies of this manuscript date from the 9th century; it contains three river-crossing problems, including the fox, goose and bag of beans puzzle and the jealous husbands problem.[2]

Well-known river-crossing puzzles include:

- The fox, goose and bag of beans puzzle, in which a farmer must transport a fox, goose and bag of beans from one side of a river to another using a boat which can only hold one item in addition to the farmer, subject to the constraints that the fox cannot be left alone with the goose, and the goose cannot be left alone with the beans.
- The jealous husbands problem, in which three married couples must cross a river using a boat which can hold at most two people, subject to the constraint that no woman can be in the presence of another man unless her husband is also present. This is equivalent to the missionaries and cannibals problem, in which three missionaries and three cannibals must cross the river, with the constraint that at any time when both missionaries and cannibals are standing on either bank, the cannibals on that bank may not outnumber the missionaries.
- The bridge and torch problem.
- *Propositio de viro et muliere ponderantibus plaustrum*. In this problem, also occurring in *Propositiones ad Acuendos Juvenes*, a man and a woman of equal weight, together with two children, each of half their weight, wish to cross a river using a boat which can only carry the weight of one adult.[3]

These problems may be analyzed using graph-theoretic methods,[4] [5] by dynamic programming,[6] or by integer programming.[3]

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Fox, goose and bag of beans puzzle

The **fox, goose and bag of beans puzzle** is a river-crossing puzzle. It dates back to at least the 9th century[1] , and has entered the folklore of a number of ethnic groups[2] .

The story

Once upon a time a farmer went to market and purchased a fox, a goose, and a bag of beans. On his way home, the farmer came to the bank of a river and hired a boat. But in crossing the river by boat, the farmer could carry only himself and a single one of his purchases - the fox, the goose, or the bag of the beans.

If left alone, the fox would eat the goose, and the goose would eat the beans.

The farmer's challenge was to carry himself and his purchases to the far bank of the river, leaving each purchase intact. How did he do it?

Solution

The first step must be to bring the goose across the river, as any other will result in the goose or the beans being eaten. When the farmer returns to the original side, he has the choice of bringing either the fox or the beans across. If he brings the fox across, he must then return to bring the beans over, resulting in the fox eating the goose. If he brings the beans across, he will need to return to get the fox, resulting in the beans being eaten. Here he has a dilemma, solved by bringing the fox (or the beans) over *and bringing the goose back*. Now he can bring the beans (or the fox) over, leaving the goose, and finally return to fetch the goose.

His actions in the solution are summarised in the following steps:

1. Bring goose over
2. Return
3. Bring fox or beans over
4. Bring goose back
5. Bring beans or fox over
6. Return
7. Bring goose over

Thus there are seven crossings, four forward and three back.

Occurrence and variations

The puzzle is one of a number of river crossing puzzles, where the object is to move a set of items across a river subject to various restrictions.

In the earliest known occurrence of this problem, in the medieval manuscript *Propositiones ad Acuendos Juvenes*, the three objects are a wolf, a goat, and a cabbage. Other cosmetic variations of the puzzle also exist, such as wolf, sheep, and cabbage;[3] ;[2] , p. 26 fox, chicken, and grain;[4] ,fox, goose and corn[5] and panther, pig, and porridge.[6] The logic of the puzzle, in which there are three objects, *A*, *B*, and *C*, such that neither *A* and *B* nor *B* and *C* can be left together, remains the same.

The puzzle has been found in the folklore of African-Americans, Cameroon, the Cape Verde Islands, Denmark, Ethiopia, Ghana, Italy, Russia, Romania, Scotland, the Sudan, Uganda, Zambia, and Zimbabwe.[2] , pp. 26–27;[7] It has been given the index number H506.3 in Stith Thompson's motif index of folk literature, and is ATU 1579 in the Aarne-Thompson-Uther classification system.[8]

The puzzle was a favorite of Lewis Carroll,[9] and has been reprinted in various collections of recreational mathematics.[2] , p. 26.

A variation of the puzzle also appears in the Nintendo DS puzzle game Professor Layton and the Curious Village.

In some parts of Africa, variations on the puzzle have been found in which the boat can carry two objects instead of only one. When the puzzle is weakened in this way it is possible to introduce the extra constraint that no two items, including *A* and *C*, can be left together.[2] , p. 27.

See also

- Missionaries and cannibals problem
- Transport puzzle

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External links

- Goat, Cabbage and Wolf (http://www.cut-the-knot.org/ctk/GoatCabbageWolf.shtml) A Java simulation

Missionaries and cannibals problem

The **missionaries and cannibals problem**, and the closely related **jealous husbands problem**, are classic river-crossing problems.[1] The missionaries and cannibals problem is a well-known toy problem in artificial intelligence, where it was used by Saul Amarel as an example of problem representation.[2] [3]

The problem

In the missionaries and cannibals problem, three missionaries and three cannibals must cross a river using a boat which can carry at most two people, under the constraint that, for both banks, if there are missionaries present on the bank, they cannot be outnumbered by cannibals (if they were, the cannibals would eat the missionaries.) The boat cannot cross the river by itself with no people on board.[1]

In the jealous husbands problem, the missionaries and cannibals become three married couples, with the constraint that no woman can be in the presence of another man unless her husband is also present. Under this constraint, there cannot be both women and men present on a bank with women outnumbering men, since if there were, some woman would be husbandless. Therefore, upon changing women to cannibals and men to missionaries, any solution to the jealous husbands problem will also become a solution to the missionaries and cannibals problem.[1]

Solution

The earliest solution known to the jealous husbands problem, using 11 one-way trips, is as follows. The married couples are represented as α (male) and *a* (female), β and *b*, and γ and *c*.[4] , p. 291.

Trip number	Starting bank	Travel	Ending bank
(start)	α a β b γ c		
1	β b γ c	α a →	
2	β b γ c	← α	a
3	α β γ	bc →	a
4	α β γ	← a	b c
5	α a	β γ →	b c
6	α a	← β b	γ c
7	a b	α β →	γ c
8	a b	← c	α β γ
9	b	a c →	α β γ
10	b	← β	α a γ c
11		β b →	α a γ c
(finish)			α a β b γ c

This is a shortest solution to the problem, but is not the only shortest solution.[4] , p. 291.

If however, only one man can get out of the boat at a time and husbands must be on the shore to count as with his wife as opposed to just being in the boat at the shore: move 5 to 6 is impossible, for as soon as γ has stepped out *b* on the shore won't be with her husband, despite him being just in the boat.

As mentioned previously, this solution to the jealous husbands problem will become a solution to the missionaries and cannibals problem upon replacing men by missionaries and women by cannibals. In this case we may neglect the individual identities of the missionaries and cannibals. The solution just given is still shortest, and is one of four shortest solutions.[5]

If a woman in the boat at the shore (but not *on* the shore) counts as being by herself (i.e. not in the presence of any men on the shore), then this puzzle can be solved in 9 one-way trips:

Trip number	Starting bank	Travel	Ending bank
(start)	α a β b γ c		
1	β b γ c	α a $\rightarrow$	
2	β b γ c	$\leftarrow$ a	α
3	β γ c	ab $\rightarrow$	α
4	β γ c	$\leftarrow$ b	α a
5	γ c	β b $\rightarrow$	α a
6	γ c	$\leftarrow$ b	α a β
7	γ	bc $\rightarrow$	α a β
8	γ	$\leftarrow$ c	α a β b
9		γ c $\rightarrow$	α a β b
(finish)			α a β b γ c

Variations

An obvious generalization is to vary the number of jealous couples (or missionaries and cannibals), the capacity of the boat, or both. If the boat holds 2 people, then 2 couples require 5 trips; with 4 or more couples, the problem has no solution.[6] If the boat can hold 3 people, then up to 5 couples can cross; if the boat can hold 4 people, any number of couples can cross.[4] , p. 300.

If an island is added in the middle of the river, then any number of couples can cross using a two-person boat. If crossings from bank to bank are not allowed, then $8n-6$ one-way trips are required to ferry n couples across the river;[1] , p. 76 if they are allowed, then $4n+1$ trips are required if n exceeds 4, although a minimal solution requires only 16 trips if n equals 4.[1] , p. 79. If the jealous couples are replaced by missionaries and cannibals, the number of trips required does not change if crossings from bank to bank are not allowed; if they are however the number of trips decreases to $4n-1$, assuming that n is at least 3.[1] , p. 81.

History

The first known appearance of the jealous husbands problem is in the medieval text *Propositiones ad Acuendos Juvenes*, usually attributed to Alcuin (died 804.) In Alcuin's formulation the couples are brothers and sisters, but the constraint is still the same—no woman can be in the company of another man unless her brother is present.[1] , p. 74. From the 13th to the 15th century, the problem became known throughout Northern Europe, with the couples now being husbands and wives.[4] , pp. 291–293. The problem was later put in the form of masters and valets; the formulation with missionaries and cannibals did not appear until the end of the 19th century.[1] , p. 81 Varying the number of couples and the size of the boat was considered at the beginning of the 16th century.[4] , p. 296. Cadet de Fontenay considered placing an island in the middle of the river in 1879; this variant of the problem, with a two-person boat, was completely solved by Ian Pressman and David Singmaster in 1989.[1]

See also

- Fox, goose and bag of beans puzzle
- Transport puzzle
- Circumscription (logic)

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Bridge and torch problem

The **bridge and torch problem** (also known as *The Midnight Train*[1] and *Dangerous crossing*[2]) is a logic puzzle that deals with 4 people, a bridge and a torch. It is one of the category of river crossing puzzles, where a number of objects must move across a river, with some constraints.[3]

Story

Four people come to a river in the night. There is a narrow bridge, but it can only hold two people at a time. Because it's night, the torch has to be used when crossing the bridge. Person A can cross the bridge in 1 minute, B in 2 minutes, C in 5 minutes, and D in 8 minutes. When two people cross the bridge together, they must move at the slower person's pace. The question is, can they all get across the bridge in 15 minutes or less?[2]

Solution

An obvious first idea is that the cost of returning the torch to the people waiting to cross is an unavoidable expense which should be minimized. This strategy makes A the torch bearer, shuttling each person across the bridge:[4]

Elapsed Time	Starting Side	Action	Ending Side
0 minutes	A B C D		
2 minutes	C D	A and B cross forward, taking 2 minutes	A B
3 minutes	A C D	A returns, taking 1 minute	B
8 minutes	D	A and C cross forward, taking 5 minutes	A B C
9 minutes	A D	A returns, taking 1 minute	B C
17 minutes		A and D cross forward, taking 8 minutes	A B C D

This strategy does not permit a crossing in 15 minutes. To find the correct solution, one must realize that forcing the two slowest people to cross individually wastes time which can be saved if they both cross together:[4]

Elapsed Time	Starting Side	Action	Ending Side
0 minutes	A B C D		
2 minutes	C D	A and B cross forward, taking 2 minutes	A B
3 minutes	A C D	A returns, taking 1 minute	B
11 minutes	A	C and D cross forward, taking 8 minutes	B C D
13 minutes	A B	B returns, taking 2 minutes	C D
15 minutes		A and B cross forward, taking 2 minutes	A B C D

Variations and history

Several variations exist, with cosmetic variations such as differently named people, or variation in the crossing times or time limit.[5] The torch itself may expire in a short time and so serve as the time limit. In a variation called *The Midnight Train*, for example, person D needs 10 minutes instead of 8 to cross the bridge, and persons A, B, C and D, now called the four Gabrianni brothers, have 17 minutes to catch the midnight train.[1]

The puzzle is known to have appeared as early as 1981, in the book *Super Strategies For Puzzles and Games*. In this version of the puzzle, A, B, C and D take 5, 10, 20, and 25 minutes, respectively, to cross, and the time limit is 60 minutes.[6] [7] In all these variations, the structure and solution of the puzzle remain the same.

In the case where there are an arbitrary number of people with arbitrary crossing times, and the capacity of the bridge remains equal to two people, the problem has been completely analyzed by graph-theoretic methods.[4]

This problem has been used as a method to compare the usability of programming languages.[8]

External links

- Slides of the Capacity C Torch Problem [9]
- Paper discussing the Capacity C Torch Problem [10]

See also

- River crossing puzzle

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Bulls and cows

Bulls and Cows -- also known as **Cows and Bulls** or **Pigs and Bulls** or **Bulls and Cleots** -- is an old code-breaking paper and pencil game for two players, predating the similar commercially-marketed board game Master Mind.

4digits, an Open Source version of Bulls and Cows.

It is a game with numbers that may date back a century or more. It is played by two opponents. The game can also be played with 3 digits instead of 4.

On a sheet of paper, the players each write a 4-digit secret number. The digits must be all different. Then, in turn, the players try to guess their opponent's number who gives the number of matches. If the matching digits are on their right positions, they are "bulls", if on different positions, they are "cows". Example:

- Secret number: 4271
- Opponent's try: 1234
- Answer: 1 bull and 2 cows. (The bull is "2", the cows are "4" and "1".)

The first one to reveal the other's secret number wins the game. As the "first one to try" has a logical advantage, on every game the "first" player changes. In some places, the winner of the previous game will play "second". Sometimes, if the "first" player finds the number, the "second" has one more move to make and if he also succeeds, the result is even.

The secret numbers for bulls and cows are usually 4-digit-numbers, but the game can be played with 3 to 6 digit numbers (in every case it is more difficult than with 4).

The game may also be played by two teams of 2-3 players. The players of every team discuss before making their move, much like in chess.

A computer program **moo**, written in 1970 by J. M. Grochow at MIT in the PL/I computer language for the Multics operating system, was amongst the first Bulls and Cows computer implementations, inspired by a similar program written by Frank King in 1968 and running on the Cambridge University mainframe. Because the game has simple rules, while it is difficult and entertaining, there are many computer variants; it is often included in telephones and PDAs.

See also

- Mastermind — a similar game with colored pegs instead of digits
- Jotto — a similar game with words

External links

- Page with the PL/1 code for Moo by J.M. Grochow [1]
- Description of Bulls and Cows with numbers, and a Web playable version [2]
- Description of Bulls and Cows with words, and a Web playable version [3]
- Knuth, D. E. "The Computer as a Master Mind." J. Recr. Math. 9, 1-6, 1976-77 [4]
- Computer version of the game "Bulls and cows" [5]

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[4] http://www.dcc.fc.up.pt/~sssousa/RM09101.pdf
[5] http://mastermind.idhost.kz/

Induction puzzles

Induction Puzzles are logic puzzles which are solved via the application of the principle of induction. In most cases, the puzzle's scenario will involve several participants with reasoning capability (typically people) and the solution to the puzzle will be based on identifying what would happen in an obvious case, and then repeating the reasoning that: "as soon as one of the participants realises that the obvious case has *not* happened, they can eliminate it from their reasoning, so creating a new obvious case".

Typical tell-tale features of these puzzles include any puzzle in which each participant has a given piece of information about all other participants but not themselves. Also, usually some kind of hint is given to suggest that the participants can trust each others intelligence.

Examples

The King's Wise Men: The King called the three wisest men in the country to his court to decide who would become his new advisor. He placed a hat on each of their heads, such that each wise man could see all of the other hats, but none of them could see their own. Each hat was either white or blue. The king gave his word to the wise men that at least one of them was wearing a blue hat - in other words, there could be one, two, or three blue hats, but not zero. The king also announced that the contest would be fair to all three men. The wise men were also forbidden to speak to each other. The king declared that whichever man stood up first and announced the color of his own hat would become his new advisor. The wise men sat for a very long time before one stood up and correctly announced the answer. What did he say, and how did he work it out?

Josephine's Problem: In Josephine's Kingdom every woman has to take a logic exam before being allowed to marry. Every marrying woman knows about the fidelity of every man in the Kingdom *except* for her own husband, and etiquette demands that no woman should tell another about the fidelity of her husband. Also, a gunshot fired in any house in the Kingdom will be heard in any other house. Queen Josephine announced that unfaithful men had been discovered in the Kingdom, and that any woman knowing her husband to be unfaithful was required to shoot him at midnight following the day after she discovered his infidelity. How did the wives manage this?

Alice at the Convention of Logicians: At the Secret Convention of Logicians, the Master Logician placed a band on each attendee's head, such that everyone else could see it but the person themselves could not. There were many, many different colors of band. The Logicians all sat in a circle, and the Master instructed them that a bell was to be rung in the forest at regular intervals: at the moment when a Logician knew the color on his own forehead, he was to leave at the next bell. Anyone who left at the wrong bell was clearly not a true Logician but an evil infiltrator and would be thrown out of the Convention post haste; but the Master reassures the group by stating that the puzzle would not be impossible for anybody present. How did they do it?

Solutions

The King's Wise Men: This is one of the simplest induction puzzles and one of the clearest indicators to the method used.

- Suppose that you are one of the wise men. Looking at the other wise men, you see they are both wearing white hats. Since there are only three hats in total and the king specified that there was at least one blue hat, you would immediately know that your own hat must be blue.
- Now suppose that you see the other wise men, and one is wearing a white hat and the other is wearing a blue hat. If your own hat was white, then the man you can see wearing the blue hat would be himself seeing two white hats and would - by the logic above - have immediately declared his hat colour. If he doesn't do this, it can only be because *your* hat isn't white, therefore it must be blue.
- Now suppose that you see the other wise men and both are wearing blue hats. You can't work anything out from this. However, if your own hat was white, then one of the two other wise men would be seeing a blue and a white hat, and would have declared his hat colour by the rule above. Thus, if he hasn't done so, he must also be seeing two blue hats and thus your hat must be blue.

Please note, that this problem has a subtle but major flaw: time. Exactly how long should one of the King's Wise Men wait, before inferring anything from the (lack of) action he sees in the other two wise men? This ugly flaw is rightly eliminated in the statement of Josephine's Problem (aka the 'marital infidelity' problem, with its midnights) and the Alice at the Convention of Logicians problem (with its 'regular intervals'). The flaw can also be eliminated by the men realising only someone wearing a blue hat could win, and thus all hats must be blue for it to be a fair test.

Josephine's Problem: This is another good example of a general case.

- If there is only 1 unfaithful husband, then every woman in the Kingdom knows that except for his wife, who believes that everyone is faithful. Thus, as soon as she hears from the Queen that unfaithful men exist, she knows her husband must be unfaithful, and shoots him.
- If there are 2 unfaithful husbands, then both their wives believe there is only 1 unfaithful husband (the other one). Thus, they will expect that the case above will apply, and that the other husband's wife will shoot him at midnight on the next day. When no gunshot is heard, they will realise that the case above did *not* apply, thus there must be more than 1 unfaithful husband and (since they know that everyone else is faithful) the extra one must be their own husband.
- If there are 3 unfaithful husbands, each of their wives believes there to be only 2, so they will expect that the case above will apply and both husbands will be shot on the second day. When they hear no gunshot, they will realize that the case above did *not* apply, thus there must be more than 2 unfaithful husbands and as before their own husband is the only candidate to be the extra one.
- In general, if there are n unfaithful husbands, each of their wives will believe there to be $n-1$ and will expect to hear a gunshot at midnight on the $n-1$th day. When they don't, they know their own husband was the nth.

This problem is also known as the Cheating Husbands Problem, the Unfaithful Wives Problem or the Muddy Children Problem.

Please note that strictly, the process needs to be explicitly terminated, otherwise all husbands who survive the first night of shooting will be shot at midnight the next day. For example, Queen Josephine could announce, on the day after all the unfaithful husbands have been eliminated, that there are no longer any unfaithful husbands left. Otherwise, the day after the shooting all wives know of zero unfaithful husbands and if the assumption that unfaithful husbands exist is not removed, the argument relating to exactly one unfaithful husband, as given above, will apply - each wife with a surviving husband will deduce that her husband is guilty and shoot him.

Alice at the convention of Logicians: This is general induction plus a leap of logic.

- *Leap of logic:* Every colour must appear at least twice around the circle. This is because the Master stated that it would not be impossible for any Logician to solve the puzzle. If any colour existed only once around the circle, the Logician who bore it would have no way of knowing that the colour even existed in the problem, and it would be impossible for them to answer.
- Each of the Logicians can look around the circle and count the number of times they see each colour. Suppose that you are one of the Logicians and you see another colour only once. Since you know each colour must exist at least twice around the circle, the only explanation for a singleton colour is that it is the colour of your own band. For the same reason, there can only be one such singleton colour, and so you would leave on the first bell.
- Likewise any Logicians who see another colour only once should be able to determine their own colour, and will either leave with dignity or be thrown out as an infiltrator. Equivalently, any colour for which there are only two bands of that colour will be eliminated after the first bell has rung. Thereafter there must be at least three bands of any remaining colour.
- Suppose you do not see any colour once, but you do see a colour twice. If these were the only bands of this colour, then these two Logicians ought to have left at the first bell. Since they didn't, that can only be because your own band is the same colour, so you can leave at the second bell.
- The induction proceeds following the same pattern as used in Josephine's Problem.

See also

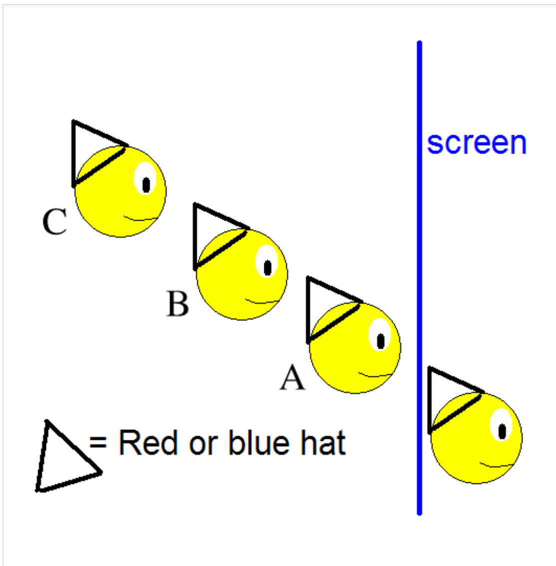
- Prisoners and hats puzzle — in-depth look at this induction puzzle
- Epistemic logic

Prisoners and hats puzzle

The **prisoners and hats puzzle** is an induction puzzle (a kind of logic puzzle) that involves reasoning about the actions of other people, drawing in aspects of Game theory. There are many variations, but the central theme remains the same. Not to be confused with the similar Hat Puzzle.

The puzzle

According to the story, four prisoners are arrested for a crime, but the jail is full and the jailer has nowhere to put them. He eventually comes up with the solution of giving them a puzzle so if they succeed they can go free but if they fail they are executed.



The jailer puts three of the men sitting in a line. The fourth man is put behind a screen (or in a separate room). He gives all four men party hats (as in diagram). The jailer explains that there are two red and two blue hats. The prisoners can see the hats in front of them but not on themselves or behind. The fourth man behind the screen can't see or be seen by any other prisoner. No communication between the men is allowed.

If any prisoner can figure out and say (out loud) to the jailer what colour hat he has on his head *all four prisoners go free*. The puzzle is to find how the prisoners can escape.

The solution

For the sake of explanation let's label the prisoners in line order A B and C. Thus B can see A (and his hat colour) and C can see A and B.

The prisoners know that there are only two hats of each colour. So if C observes that A and B have hats of the same colour, C would deduce that his own hat is the opposite colour. However, if A and B have hats of different colours, then C can say nothing. The key is that prisoner B, after allowing an appropriate interval, and knowing what C would do, can deduce that if C says nothing the hats on A and B must be different. Being able to see A's hat he can deduce his own hat colour. (The fourth prisoner is irrelevant to the puzzle: his only purpose is to wear the fourth hat).

In common with many puzzles of this type, the solution relies on the assumption that all participants are totally rational and are intelligent enough to make the appropriate deductions.

After solving this puzzle, some insight into the nature of communication can be gained by pondering whether the meaningful silence of prisoner C violates the "No communication" rule (given that communication is usually defined as the "transfer of information").

Variants

Four-Hat Variant

In a variant of this puzzle there are 3 hats of one colour and only 1 hat of another, and the 3 prisoners can see each other i.e. A sees B & C, B sees A & C and C sees A & B. (D again not to be seen and only there to wear the last hat)

The solution

There are two cases: in the trivial case, one of the three prisoners wears the single off-colour hat, and thus the other two can easily deduce the colour of theirs. In the non-trivial case, the three prisoners wear hats of the same colour, while D wears the off-colour hat. After a while, all four prisoners should be able to deduce that, since none of the others was able to state the colour of his own hat, D must wear the off-colour hat.

Five-Hat Variant

In another variant, only three prisoners and five hats (supposedly two black and three white) are involved. The three prisoners are ordered to stand in a straight line facing the front, with A in front and C at the back. They are told that there will be two black hats and three white hats. One hat is then put on each prisoner's head; each prisoner can only see the hats of the people in front of him and not on his own. The first prisoner that is able to announce the colour of his hat correctly will be released. No communication between the prisoners is allowed. After some time, only A is able to announce (correctly) that his hat is white. Why is that so?

The solution

Assuming that A wears a black hat:

- If B wears a black hat as well, C can immediately tell that he is wearing a white hat after looking at the two black hats in front of him.
- If B does not wear a black hat, C will be unable to tell the colour of his hat (since there is a black and a white). Hence, B can deduce from A's black hat and C's response that he (B) is not wearing a black hat (otherwise the above situation will happen) and is therefore wearing a white hat.

This therefore proves that A must not be wearing a black hat.

Three-Hat Variant

In this variant there are 3 prisoners and 3 hats. Each prisoner is assigned a random hat, either red or blue. Each person can see the hats of two others, but not their own. On a cue, they each have to guess their own hat color or pass. They win release if at least one person guessed correctly and none guessed incorrectly (passing is neither correct nor incorrect).

This puzzle doesn't have a 100% winning strategy, so the question is: What is the best strategy? Which strategy has the highest probability of winning?

If you think of colors of hats as bits, this problem has some important applications in coding theory.

The solution and the discussion of this puzzle can be found here [1] (also a solution to the analogous 7-hat puzzle) and other 3 variants are available on this Logic Puzzles [2] page (they are called Masters of Logic I-IV).

Ten-Hat Variant

In this variant there are 10 prisoners and 10 hats. Each prisoner is assigned a random hat, either red or blue, but the number of each color hat is not known to the prisoners. The prisoners will be lined up single file where each can see the hats in front of him but not behind. Starting with the prisoner in the back of the line and moving forward, they must each, in turn, say only one word which must be "red" or "blue". If the word matches their hat color they are released, if not, they are killed on the spot. A friendly guard warns them of this test one hour beforehand and tells them that they can formulate a plan where by following the stated rules, 9 of the 10 prisoners will definitely survive, and 1 has a 50/50 chance of survival. What is the plan to achieve the goal?

Ten-Hat Solution

The prisoners can use a binary code where each blue hat = 0 and each red hat = 1. The prisoner in the back of the line adds up all the values and if the sum is even he says "blue" (blue being =0 and therefore even) and if the sum is odd he says "red". This prisoner has a 50/50 chance of having the hat color that he said, but each subsequent prisoner can calculate his own color by adding up the hats in front (and behind after hearing the answers [excluding the prisoner in the back]) and comparing it to the initial answer given by the prisoner in the back of the line. The total number of red hats has to be an even or odd number matching the initial even or odd answer given by the prisoner in back.

Countably Infinite-Hat Variant

In this variant, a countably infinite number of prisoners, each with an unknown and randomly assigned red or blue hat line up single file line. Each prisoner faces away from the beginning of the line, and each prisoner can see all the hats in front of him, and none of the hats behind. Starting from the beginning of the line, each prisoner must correctly identify the color of his hat or he is killed on the spot. As before, the prisoners have a chance to meet beforehand, but unlike before, once in line, no prisoner can hear what the other prisoners say. The question is, is there a way to ensure that only finitely many prisoners are killed?

Countably Infinite-Hat Solution

If one accepts the axiom of choice, the answer is yes. In fact, even if we allow an uncountable number of different colors for the hats and an uncountable number of prisoners, the axiom of choice provides a solution that guarantees that only finitely many prisoners must die. The solution for the two color case is as follows, and the solution for the uncountably infinite color case is essentially the same:

The prisoners standing in line form a sequence of 0's and 1's, where 0 is taken to represent blue, and 1 is taken to represent red. Before they are put into the line, the prisoners define the following equivalence relation over all possible sequences that they might be put into: Two sequences are equivalent if they are identical after a finite number of entries. From this equivalence relation, the prisoners get a collection of equivalence classes. Using the axiom of choice, they select and memorize a representative sequence from each equivalence class.

When they are put into their line, each prisoner can see what equivalence class the *actual* sequence of hats belongs to. They then proceed guessing their hat color as if they were in the *representative* sequence from the appropriate equivalence class. Since the actual sequence and the representative sequence are in the same equivalence class, their entries are the same after finitely many prisoners. All prisoners after this point are saved.

Since the prisoners have no information about the color of their own hat and would make the same guess whichever color it has, each prisoner has a 50% chance of being killed. It may seem paradoxical that an infinite number of prisoners each have an even chance of being killed and yet it is certain that only a finite number are killed. However, there is no contradiction here, since this finite number can be arbitrarily large and no probability can be assigned to any particular number being killed.

This is easiest to see for the case of zero prisoners being killed. This happens if and only if the actual sequence is one of the selected representative sequences. If the sequences of 0's and 1's are viewed as binary representations of a real number between 0 and 1, the representative sequences form a non-measurable set. (This set is similar to a Vitali set, the only difference being that equivalence classes are formed with respect to numbers with finite binary representations rather than all rational numbers.) Hence no probability can be assigned to the event of zero prisoners being killed. The argument is similar for other finite numbers of prisoners being killed, corresponding to a finite number of variations of each representative.

Better Countably Infinite-Hat Solution

The previous solution guaranteed that only a finite number of prisoners could be killed. If there are a finite number of hat colors, then there is a better solution, one in which only the life of the first person is at risk.

Instead of guessing their hat color as if they were in the *representative* sequence from the appropriate equivalence class, the first person will use his answer to communicate the differences between the representative sequence and the actual sequence. When there are only two hat colors, he computes the parity of number of differences. When there are some finite number of hats n, he computes the sum of the differences (between the representative sequence and the actual sequence) modulo n. Since there are only a finite number of differences and a finite number of hats, this is well defined. Given this information, each prisoner, after listening to all of the previous answers, will be able to correctly determine his own hat color.

See also

- Hat Puzzle

References

[1] http://www.relisoft.com/science/hats.html
[2] http://brainden.com/logic-puzzles.htm

Hat puzzle

The **hat puzzle** is a classic logic problem, attributed to Todd Ebert, in his 1998 Ph.D. thesis at the University of California, Santa Barbara. It is a strategy question about a cooperative game, which has been shown to have connections to algebraic coding theory.[1]

Question

A team of N players, where N is at least three, are randomly assigned hats that are equally likely to be white or black, under conditions such that:

1. There is at least one black and one white hat assigned;
2. Each team player can see the hats of the other team members, but cannot see their own hat;
3. Team members cannot communicate in any way.

Hypothetically, there are N prisoners, but there was not enough space for all of them. The jailer decides to give them a test, and if all of them succeed in answering it, he will release them, whereas if any one of them answers incorrectly, then he will kill all of them. He describes the test as follows:

> I will put a hat, either white or black, on the head of each of you. You can see others' hats, but you can't see your own hat. You are given 20 minutes. I will place at least one white hat and at least one black hat. At least one of you should tell me the colour of the hat on your head. You can't signal to others or give a hint or anything like that. You should say only WHITE or BLACK. You can go and discuss for a while now.

All of them go and discuss for some time. And after they come back, he starts the test. Interestingly, each of them answers correctly and hence all are released.

The question is, is there a pre-agreed team strategy for guessing, which will improve on an expected value of 0.5, for the score of the team, when it is awarded as 1 if there is at least one correct hat colour guess and no incorrect guess, and 0 otherwise?

Answer

Each prisoner counts the number of white hats and black hats they see, and waits for $10N$ seconds, where N is the lower of the two numbers. After that time, (providing the other prisoners are all doing the same thing), he must be wearing the hat of the colour which he is seeing fewer of.

- Everyone wearing a white hat will see $W-1$ white hats and B black hats.
- Everyone wearing a black hat will see $B-1$ black hats and W white hats.
- If $W<B$ then all white hats will say 'White' at the same time ($10(W-1)$ seconds), and everyone else knows they are wearing black.
- If $B<W$ then all black hats will say 'Black' at the same time ($10(B-1)$ seconds), and everyone else knows they are wearing white.
- If $W=B$ then all prisoners will know their colour at the same time. ($10(B-1)$ seconds, or $10(W-1)$ seconds, they are equivalent).

Explanation

Four Prisoners

Suppose that there are 4 prisoners, among which two wear black hats and other two wear white.

```
          W       W                    W       W

                          or

          W       B                    B       B
```

Now, the first one sees two black hats, and a white hat. Had he seen three black hats, he could have easily told that his hat is a white hat, since there should at least be one white hat. Same is true for the second person who is also wearing a white hat. So, all of them are in dilemma. After 10 seconds, each of them thinks that, since the others are not able to deduce their own hat colour, he must be wearing a white hat. So they say WHITE. At about the same time, by similarity, the other two persons say BLACK.

Six Prisoners

Consider that there are 6 prisoners, among which three wear black hats and other three wear white.

```
        W                         W                         W

  B            B            B            W            W            W

                     or                        or

  B            B            B            B            B            B

        B                         B                         B
```

Now, the first one sees three black hats, and two white hats. Had he seen five black hats, he could have easily told that his hat is a white hat. Also, if he had seen 4 black hats, and a white hat, after 10 seconds, he could have guessed that he must be wearing a white hat since the other person wearing white hat is unable to deduce his hat's colour. (See Four Prisoners). But now he is still in dilemma and is unable to judge the colour even after 20 seconds. So, after 20 seconds, all the persons wearing white hats say WHITE and all the prisoners wearing black hats say BLACK.

Hamming Codes

There is a variation of the puzzle in which the players can answer 'I don't know' as a third choice, but all players must guess simultaneously, ruling out the previous time-dependent solution. The players win if at least one answers correctly, and none answer incorrectly. One strategy for solving this version of hat problem employs Hamming codes, which are commonly used to detect and correct errors in data transmission. Here, one can't solve the problem in the sense that the players will win in any case, but the probability for winning will be much higher than 50%, depending on the number of players in the puzzle configuration (e.g. winning probability of 87.5% for 7 players when using Hamming codes.)

This strategy can be applied to team sizes of $N = 2^k - 1$ (e.g. 3, 7, 15) and achieves an expected value of $\frac{2^k - 1}{2^k}$. Notably, the Hamming code strategy yields greater expected values for larger values of N.

In this version of the problem, any individual guess has a 50% chance of being right. However, the Hamming code approach works by concentrating wrong guesses together onto certain distributions of hats. For some cases, all the players will guess incorrectly; whereas for the other cases, only one player will guess, but correctly. While half of all guesses are still incorrect, this results in the players winning more than 50% of the time.

A simple example of this type of solution with three players is instructive. With three players, there are eight possibilities, two of which are involve all players having the same color hat, and the other six where two players have one color and the other player has the other color.

The players can guarantee that they win in the latter cases (75% of the time) with the following strategy:

1. Any player who observes two hats of two different colors remains silent (or says "I don't know.")
2. Any player who observes two hats of the same color guesses the opposite color.

In the two cases when all three players have the same hat color, they will all guess incorrectly. But in the other six cases, only one player will guess, correctly, that his hat is the opposite of his fellow players'.

Further reading

- Andrew Liu, "Two Applications of a Hamming Code" *The College Mathematics Journal* 40 (2009) 2-5.
- Christopher Hardin and Alan D. Taylor, An introduction to infinite hat problems [2]. Mathematical Intelligencer, 30(4), 2008.

[1] Biography of Todd Ebert at California State University, Long Beach (http://www.csulb.edu/colleges/coe/cecs/views/personnel/fulltime/ebert.shtml)

[2] http://www.math.union.edu/~hardinc/pub/introinf.pdf

See also

- Induction Puzzles
- Prisoners and hats puzzle

Knights and Knaves

Knights and Knaves is a type of logic puzzle devised by Raymond Smullyan.

On a fictional island, all inhabitants are either knights, who always tell the truth, or knaves, who always lie. The puzzles involve a visitor to the island who meets small groups of inhabitants. Usually the aim is for the visitor to deduce the inhabitants' type from their statements, but some puzzles of this type ask for other facts to be deduced. The puzzle may also be to determine a yes/no question which the visitor can ask in order to discover what he needs to know.

An early example of this type of puzzle involves three inhabitants referred to as A, B and C. The visitor asks A what type he is, but does not hear A's answer. B then says "A said that he is a knave" and C says "Don't believe B: he is lying!" To solve the puzzle, note that no inhabitant can say that he is a knave. Therefore B's statement must be untrue, so he is a knave, making C's statement true, so he is a knight. Since A's answer invariably would be "I'm a knight", it is not possible to determine whether A is a knight or knave from the information provided.

In some variations, inhabitants may also be alternators, who alternate between lying and telling the truth, or normals, who can say whatever they want (as in the case of Knight/Knave/Spy puzzles). A further complication is that the inhabitants may answer yes/no questions in their own language, and the visitor knows that "bal" and "da" mean "yes" and "no" but does not know which is which. These types of puzzles were a major inspiration for what has become known as "the hardest logic puzzle ever".

Some Examples of "Knights and Knaves" puzzles

A large class of elementary logical puzzles can be solved using the laws of Boolean algebra and logic truth tables. Familiarity with boolean algebra and its simplification process will help with understanding the following examples.

John and Bill are residents of the island of knights and knaves.

Question 1

John says: We are both knaves.

Who is what?

Question 2

John: If (and only if) Bill is a knave, then I am a knave.

Bill: We are of different kinds.

Who is who?

Question 3

Here is a rendition of perhaps the most famous of this type of puzzle:

John and Bill are standing at a fork in the road. You know that one of them is a knight and the other a knave, but you don't know which. You also know that one road leads to Death, and the other leads to Freedom. By asking one yes/no question, can you determine the road to Freedom?

This version of the puzzle was further popularised by a scene in the 1986 fantasy film, Labyrinth, in which Sarah (Jennifer Connelly) finds herself faced with two doors each guarded by a two-headed knight. One door leads to the castle at the centre of the labyrinth, and one to certain doom. It had also appeared some ten years previously, in a very similar form, in the *Doctor Who* story *Pyramids of Mars*.

Solution to Question 1

John is a knave and Bill is a knight.

John's statement is: "Both John and Bill are knaves."

If John were a knight, he would not be able to say that he was a knave since he would be lying. John being a knave makes the statement "Both John and Bill are knaves." false. Hence either John is a knight or Bill is a knight. Since John can't be a knight, Bill must be a knight for the statement to hold.

Solution for Question 2

John is a knave and Bill is a knight.

In this scenario, John is saying the equivalent of "we are not of different kinds" (that is, either they are both knights, or they are both knaves). Bill is contradicting him, saying "we are of different kinds". Since they are making contradictory statements, one must be a knight and one must be a knave. Since that is exactly what Bill said, Bill must be the knight, and John is the knave.

Solution to Question 3

There are several ways to find out which way leads to freedom. One alternative is asking the following question: "Will the other man tell me that your path leads to freedom?"

If the man says "No", then the path does lead to freedom, if he says "Yes", then it does not. The following logic is used to solve the problem.

If the question is asked of the knight and the knight's path leads to freedom, he will say "No", truthfully answering that the knave would lie and say "No". If the knight's path does not lead to freedom he will say "Yes", since the knave would say that the path leads to freedom.

If the question is asked of the knave and the knave's path leads to freedom he will say "no" since the knight would say "yes" it does lead to freedom. If the Knave's path does not lead to freedom he would say Yes since the Knight would tell you "No" it doesn't lead to freedom.

The reasoning behind this is that, whichever guardian the questioner asks, one would not know whether the guardian was telling the truth or not. Therefore one must create a situation where they receive both the truth *and* a lie applied one to the other. Therefore if they ask the Knight, they will receive the truth about a lie; if they ask the Knave then they will receive a lie about the truth.

Note that the above solution requires that each of them know that the other is a knight/knave. An alternate solution is to ask of either man, "What would your answer be if I asked you if your path leads to freedom?"

If the man says "Yes", then the path leads to freedom, if he says "No", then it does not. The reason is fairly easy to understand, and is as follows:

If you ask the knight if their path leads to freedom, they will answer truthfully, with "yes" if it does, and "no" if it does not. They will also answer this question truthfully, again stating correctly if the path led to freedom or not.

If you ask the knave if their path leads to freedom, they will answer falsely about their answer, with "no" if it does, and "yes" if it does not. However, when asked this question, they will lie about what their false answer would be, in a sense, lying about their lie. They would answer correctly, with their first lie canceling out the second.

This question forces the knight to say a truth about a truth, and the knave to say a lie about a lie, resulting, in either case, with the truth.

Another alternative is to ask: "Is either one of the following statements correct? You are a Knight and at the same time this is the path to freedom; or you are a knave and this is not the path to freedom". More alternatives for a question to ask can be found using Boolean algebra.

References

- George Boolos, John P. Burgess, Richard C. Jeffrey, *Logic, logic, and logic* (Harvard University Press, 1999).

External links

- An automated Knights and Knaves puzzle generator [1]
- A note on some philosophical implications of the Knights and Knaves puzzle for the concept of knowability [2]
- A complete list and analysis of Knight, Knave, and Spy puzzles, where spies are able to lie or tell the truth. [3]

References

[1] http://www.hku.hk/cgi-bin/philodep/knight/puzzle
[2] http://www.nottingham.ac.uk/journals/analysis/preprints/COOK.pdf
[3] http://newheiser.googlepages.com/knightsandknaves

The Hardest Logic Puzzle Ever

The Hardest Logic Puzzle Ever is a title coined by American philosopher and logician George Boolos in an article published in *The Harvard Review of Philosophy* (an Italian translation was published earlier in the newspaper *La Repubblica*, under the title ***L'indovinello più difficile del mondo***) for the following Raymond Smullyan inspired logic puzzle:

> Three gods A, B, and C are called, in some order, True, False, and Random. True always speaks truly, False always speaks falsely, but whether Random speaks truly or falsely is a completely random matter. Your task is to determine the identities of A, B, and C by asking three yes-no questions; each question must be put to exactly one god. The gods understand English, but will answer all questions in their own language, in which the words for *yes* and *no* are 'da' and 'ja', in some order. You do not know which word means which.

Boolos provides the following clarifications:[1]

> - It could be that some god gets asked more than one question (and hence that some god is not asked any question at all).
> - What the second question is, and to which god it is put, may depend on the answer to the first question. (And of course similarly for the third question.)
> - Whether Random speaks truly or not should be thought of as depending on the flip of a coin hidden in his brain: if the coin comes down heads, he speaks truly; if tails, falsely.
> - Random will answer 'da' or 'ja' when asked any yes-no question.[1]

History

Boolos credits the logician Raymond Smullyan as the originator of the puzzle and John McCarthy with adding the difficulty of not knowing what 'da' and 'ja' mean. Related puzzles can be found throughout Smullyan's writings, e.g. in *What is the Name of This Book?*, pp. 149–156, he describes a Haitian island where half the inhabitants are zombies (who always lie) and half are humans (who always tell the truth) and explains that "the situation is enormously complicated by the fact that although all the natives understand English perfectly, an ancient taboo of the island forbids them ever to use non-native words in their speech. Hence whenever you ask them a yes-no question, they reply 'Bal' or 'Da'—one of which means *yes* and the other *no*. The trouble is that we do not know which of 'Bal' or 'Da' means *yes* and which means *no*". There are other related puzzles in *The Riddle of Sheherazade*.[2]

More generally this puzzle is based on Smullyan's famous Knights and Knaves puzzles (e.g. on a fictional island, all inhabitants are either knights, who always tell the truth, or knaves, who always lie. The puzzles involve a visitor to the island who must ask a number of yes/no questions in order to discover what he needs to know). A version of these puzzles was popularized by a scene in the 1986 fantasy film, *Labyrinth*. There are two doors with two guards. One guard lies and one guard does not. One door leads to the castle and the other leads to "certain death." The puzzle is to find out which door leads to the castle by asking one of the guards one question. In the movie the protagonist, named Sarah, does this by asking, "Would he [the other guard] tell me that this door leads to the castle?"

The solution

Boolos provided his solution in the same article in which he introduced the puzzle. Boolos states that the "first move is to find a god that you can be certain is not Random, and hence is either True or False".[1] There are many different questions that will achieve this result. One strategy is to use complicated logical connectives in your questions (either biconditionals or some equivalent construction).

Boolos' question was:

- Does 'da' mean *yes* if and only if you are True if and only if B is Random?[1]

Equivalently:

- Are an odd number of the following statements true: you are False, 'da' means *yes*, B is Random?

It was observed by Roberts (2001) -- and independently by Rabern and Rabern (2008) -- that the puzzle's solution can be simplified by using certain counterfactuals.[2] [3] The key to this solution is that, for any yes/no question Q, asking either True or False the question

- If I asked you Q, would you say 'ja'?

results in the answer 'ja' if the truthful answer to Q is *yes*, and the answer 'da' if the truthful answer to Q is *no* (Rabern and Rabern (2008) call this result the embedded question lemma). The reason it works can be seen by looking at the eight possible cases.

- Assume that 'ja' means *yes* and 'da' means *no*.

(i) True is asked and responds with 'ja'. Since he is telling the truth the truthful answer to Q is 'ja', which means *yes*.

(ii) True is asked and responds with 'da'. Since he is telling the truth the truthful answer to Q is 'da', which means *no*.

(iii) False is asked and responds with 'ja'. Since he is lying it follows that if you asked him Q he would instead answer 'da'. He would be lying, so the truthful answer to Q is 'ja', which means *yes*.

(iv) False is asked and responds with 'da'. Since he is lying it follows that if you asked him Q he would in fact answer 'ja'. He would be lying, so the truthful answer to Q is 'da', which means *no*.

- Assume 'ja' means *no* and 'da' means *yes*.

(v) True is asked and responds with 'ja'. Since he is telling the truth the truthful answer to Q is 'da', which means *yes*.

(vi) True is asked and responds with 'da'. Since he is telling the truth the truthful answer to Q is 'ja', which means *no*.

(vii) False is asked and responds with 'ja'. Since he is lying it follows that if you asked him Q he would in fact answer 'ja'. He would be lying, so the truthful answer to Q is 'da', which means *yes*.

(viii) False is asked and responds with 'da'. Since he is lying it follows that if you asked him Q he would instead answer 'da'. He would be lying, so the truthful answer to Q is 'ja', which means *no*.

Using this fact, one may proceed as follows.[2]

- Ask god B, "If I asked you 'Is A Random?', would you say 'ja'?". If B answers 'ja', then either B is Random (and is answering randomly), or B is not Random and the answer indicates that A is indeed Random. Either way, C is not Random. If B answers 'da', then either B is Random (and is answering randomly), or B is not Random and the answer indicates that A is not Random. Either way, A is not Random.
- Go to the god who was identified as *not* being Random by the previous question (either A or C), and ask him: "If I asked you 'Are you True?', would you say 'ja'?". Since he is not Random, an answer of 'ja' indicates that he is True and an answer of 'da' indicates that he is False.

- Ask the same god the question: "If I asked you 'Is B Random?', would you say 'ja'?". If the answer is 'ja' then B is Random; if the answer is 'da' then the god you have not yet spoken to is Random. The remaining god can be identified by elimination.

Random's behaviour

Most readers of the puzzle assume that Random will provide completely random answers to any question asked of him; however, Rabern and Rabern (2008) have pointed out that the puzzle does not actually state this.[2] And in fact, Boolos' third clarifying remark explicitly refutes this assumption.

- Whether Random speaks truly or not should be thought of as depending on the flip of a coin hidden in his brain: if the coin comes down heads, he speaks truly; if tails, falsely.

This says that Random randomly acts as a false-teller or a truth-teller, not that Random answers randomly.

A small change to the question above yields a question which will always elicit a meaningful answer from Random. The change is as follows:

  - If I asked you Q *in your current mental state*, would you say 'ja'?[2]

This effectively extracts the truth-teller and liar personalities from Random and forces him to be only one of them. By doing so the puzzle becomes completely trivial, that is, truthful answers can be easily obtained.

- 1. Ask god A, "If I asked you 'Are you Random?' in your current mental state, would you say 'ja'?"
- 2a. If A answers 'ja', then A is Random: Ask god B, "If I asked you 'Are you True?', would you say 'ja'?"
  - If B answers 'ja', then B is True and C is False.
  - If B answers 'da', then B is False and C is True. In both cases, the puzzle is solved.
- 2b. If A answers 'da', then A is not Random: Ask god A, "If I asked you 'Are you True?', would you say 'ja'?"
  - If A answers 'ja', then A is True.
  - If A answers 'da', then A is False.
- 3. Ask god A, "If I asked you 'Is B Random?', would you say 'ja'?"
  - If A answers 'ja', then B is Random, and C is the opposite of A.
  - If A answers 'da', then C is Random, and B is the opposite of A.

Rabern and Rabern (2008) suggest making an amendment to Boolos' original puzzle so that Random is actually random. The modification is to replace Boolos' third clarifying remark to with the following:[2]

- Whether Random says 'ja' or 'da' should be thought of as depending on the flip of a coin hidden in his brain: if the coin comes down heads, he says 'ja'; if tails, he says 'da'.

With this modification, the puzzle's solution demands the more careful god-interrogation given at the end of the *The Solution* section.

Unanswerable questions and exploding god-heads

In *A simple solution to the hardest logic puzzle ever*,[2] B. Rabern and L. Rabern develop the puzzle further by pointing out that it is not the case that 'ja' and 'da' are the only possible answers a god can give. It is also possible for a god to be unable to answer at all. For example, if the question "Are you going to answer this question with the word that means *no* in your language?" is put to True, he cannot answer truthfully. (The paper represents this as his *head exploding*, "...they are infallible gods! They have but one recourse – their heads explode.") Allowing the "exploding head" case gives yet another solution of the puzzle and introduces the possibility of solving the puzzle (modified and original) in just two questions rather than three. In support of a two-question solution to the puzzle, the authors solve a similar simpler puzzle using just two questions.

  - Three gods A, B, and C are called, in some order, Zephyr, Eurus, and Aeolus. The gods always speak truly. Your task is to determine the identities of A, B, and C by asking yes-no questions; each question must be put to exactly one god. The gods understand English and will answer in English.

Note that this puzzle is trivially solved with three questions. Furthermore, to solve the puzzle in two questions, the following lemma is proved.

> **Tempered Liar Lemma.** If we ask A "Is it the case that {[(you are going to answer 'no' to this question) AND (B is Zephyr)] OR (B is Eurus) }?", a response of 'yes' indicates that B is Eurus, a response of 'no' indicates that B is Aeolus, and an exploding head indicates that B is Zephyr. Hence we can determine the identity of B in one question.

Using this lemma it is simple to solve the puzzle in two questions. Rabern and Rabern (2008) use a similar trick (tempering the liar's paradox) to solve the original puzzle in just two questions. In ``How to solve the hardest logic puzzle ever in two questions" G. Uzquiano uses these techniques to provide a two question solution to the amended puzzle.[4] [5] Two question solutions to both the original and amended puzzle take advantage of the fact that some gods have an inability to answer certain questions. Neither True nor False can provide an answer to the following question.

- Would you answer the same as Random would to the question `Is Dushanbe in Kirghizia?'?

Since the amended Random answers in a truly random manner, neither True nor False can predict whether Random would answer `ja' or `da' to the question of whether Dushanbe is in Kirghizia. Given this ignorance they will be unable to tell the truth or lie -- they will therefore remain silent. Random, however, who spouts random nonsense, will have no problem spouting off either `ja' or `da'. Uzquiano (2010) exploits this asymmetry to provide a two question solution to the modified puzzle. Yet, one might assume that the gods have an ``oracular ability to predict Random's answers even before the coin flip in Random's brain?"[4] In this case, a two question solution is still available by using self-referential questions of the style employed in Rabern and Rabern (2008).

- Would you answer `ja' to the question of whether you would answer `da' to this question?

Here again neither True nor False are able to answer this question given their commitments of truth-telling and lying, respectively. They are forced to answer`ja' just in case the answer they are committed to give is `da' and this they cannot do. Just as before they will suffer a head explosion. In contrast, Random will mindlessly spout his nonsense and randomly answer `ja' or `da'. Uzquiano (2010) also uses this asymmetry to provide a two question solution to the modified puzzle.[4] [5]

See also

- George Boolos
- Raymond Smullyan
- Knights and Knaves
- Logic Puzzle

Notes

[1] George Boolos, *The Hardest Logic Puzzle Ever* (Harvard Review of Philosophy, 6:62-65, 1996).

[2] Brian Rabern and Landon Rabern, *A simple solution to the hardest logic puzzle ever*, (Analysis 68 (298), 105–112, April 2008).

[3] T.S. Roberts, *Some thoughts about the hardest logic puzzle ever* (Journal of Philosophical Logic 30:609–612(4), December 2001).

[4] Gabriel Uzquiano. *How to solve the hardest logic puzzle ever in two questions*, (Analysis 70(1), 39-44, January 2010).

[5] Brian Rabern and Landon Rabern. "In defense of the two question solution to the hardest logic puzzle ever", (unpublished).

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- T.S. Roberts, *Some thoughts about the hardest logic puzzle ever* (Journal of Philosophical Logic 30:609–612(4), December 2001).
- Brian Rabern and Landon Rabern, *A simple solution to the hardest logic puzzle ever* (Analysis 68(2), 105–112, April 2008).
- Gabriel Uzquiano, *How to solve the hardest logic puzzle ever in two questions*, (Analysis 70(1), 39-44, January 2010).
- Raymond Smullyan, *What is the Name of This Book?* (Prentice Hall, Englewood Cliffs, NJ, 1978).
- Raymond Smullyan, *The Riddle of Sheherazade* (A. A. Knopf, Inc., New York, 1997).

External links

- George Boolos. The hardest logic puzzle ever. The Harvard Review of Philosophy, 6:62–65, 1996. (http://www.hcs.harvard.edu/~hrp/issues/1996/Boolos.pdf)
- T.S. Roberts. Some thoughts about the hardest logic puzzle ever. Journal of Philosophical Logic, 30:609–612(4), December 2001. (http://www.springerlink.com/content/v43v5431h2324888/)
- Brian Rabern and Landon Rabern. A simple solution to the hardest logic puzzle ever. Analysis 68 (298), 105–112, April 2008. (http://www.blackwell-synergy.com/doi/abs/10.1111/j.1467-8284.2007.00723.x)
- Tom Ellis. Even harder than the hardest logic puzzle ever. (http://www.srcf.ucam.org/~te233/maths/puzzles/evenharder.html)
- Brian Rabern and Landon Rabern. In defense of the two question solution to the hardest logic puzzle ever. (http://philrsss.anu.edu.au/~brian/defense_HLPE.pdf)
- Gabriel Uzquiano. How to solve the hardest logic puzzle ever in two questions. (http://users.ox.ac.uk/~sfop0198/Three Gods.pdf)
- Walter Carnielli. Contrafactuais, contradição e o enigma lógico mais difícil do mundo. Revista Omnia Lumina. (http://www.revistaomnialumina.com/?p=57)

Three way duel

A **truel** is a neologism for a duel between three opponents, in which players can fire at one another in an attempt to eliminate them while surviving themselves.[1]

Game theory overview

A variety of forms of truels have been studied. Features that determine the nature of a truel include[1]

- the probability of each player hitting their chosen targets (often not assumed to be the same for each player)
- whether the players shoot simultaneously or sequentially, and, if sequentially, whether the shooting order is predetermined, or determined at random from among the survivors;
- the number of bullets each player has (in particular, whether this is finite or infinite);
- whether or not intentionally missing is allowed.

There is usually a general assumption that each player in the truel wants to survive, and will behave logically in a manner that maximizes individual probabilities of survival.[1]

In the widely studied form, the three have different probabilities of hitting their target.[1]

If a single bullet is used, the probabilities of hitting the target are equal and deliberate missing is allowed, the best strategy for the first shooter is to deliberately miss. Since he is now disarmed, the next shooter will have no reason to shoot the first one and so will shoot at the third shooter. While the second shooter might miss deliberately, there would then be the risk that the third one would shoot him. If the first shooter does not deliberately miss, he will presumably be shot by whichever shooter remained.

If an unlimited number of bullets are used, then deliberate missing may be the best strategy for a duelist with lower accuracy than both opponents. If both have better than 50% success rate, he should continue to miss until one of his opponents kills the other. Then he will get the first shot at the remaining opponent. But if the "middle" opponent is weak, it can better to team up with him until the strongest is eliminated. The details depend on the firing order. For example, if the order is P, Q, R, with respective probabilities $p > q > r$, and it is R's turn, R should waste his shot if:

$$p < q.(1+q)/(1 - q + q^2)$$

but not do so if:

$$(1-q)(1 - q + q^2).p^2 - q.(1-q).(1 + 2q).p - q^3 > 0$$

In between, R should waste his shot if:

$$r > p(p - q - p.q - q^2 + p.q^2)/(p^2(1-q) + q^2(1-p)^2)$$

History

The earliest known mention of three-person "duels" appears to have been in A. P. Herbert's play "Fat King Melon" (1927). An extensive bibliography has been compiled by D. Marc Kilgour [2] ; see also [3] The word "truel" was introduced in Martin Shubik's 1964 book "Game Theory and Related Approaches to Social Behavior", page 43, and independently in Richard Epstein's 1967 book "Theory of Gambling and Statistical Logic", page 343.

Truels in popular culture

In one of the most famous spaghetti westerns, *The Good, the Bad and the Ugly*, the final showdown is played out to be a climactic truel between the three main characters: Blondie ("The Good"), Angel Eyes ("The Bad"), and Tuco ("The Ugly"). The standoff remains a signature piece for director Sergio Leone and one of the best-known scenes in film history.

The climactic ending to the 1992 film *Reservoir Dogs* has a truel between the characters Mr. White, Nice Guy Eddie, and Joe Cabot in which only Mr. White survives.

The truel is also parodied at the climax of the film *Dead Again*.

A truel with swords is fought between Jack Sparrow, Will Turner, and James Norrington in the 2006 film *Pirates of the Caribbean: Dead Man's Chest*; all three characters survive.

The short film Truel [4] explores the idea of a three-way duel.

References

[1] Kilgour, D. Marc; Brams, Stephen J. (December 1997), "The Truel" (http://jstor.org/stable/2691167), *Mathematics Magazine* **70** (5): 315–326, doi:10.2307/2691167,

[2] "The truel list", http://info.wlu.ca/~wwwmath/faculty/kilgour/truel.htm Retrieved February 12, 2010

[3] "Last Person Standing" Puzzle Hint, http://numb3rs.wolfram.com/puzzles/hint.html Retrieved December 23, 2008.

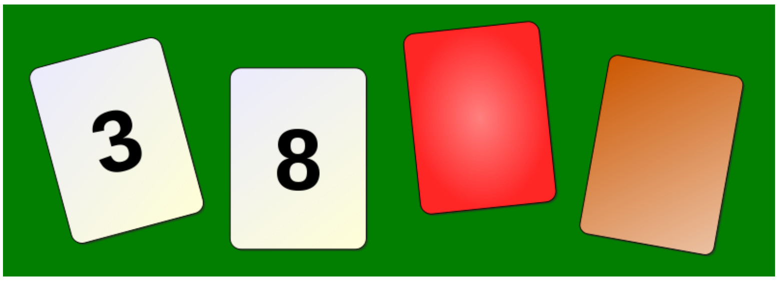
[4] http://www.imdb.com/title/tt0211074/

See also

- Mexican standoff

Wason selection task

Devised in 1966 by Peter Cathcart Wason,[1] [2] the **Wason selection task**, one of the most famous tasks in the psychology of reasoning,[3] is a logic puzzle which is formally equivalent to the following question:



Which card(s) must be turned over to test the idea that if a card shows an even number on one face, then its opposite face is red?

> *You are shown a set of four cards placed on a table, each of which has a number on one side and a colored patch on the other side. The visible faces of the cards show 3, 8, red and brown. Which card(s) should you turn over in order to test the truth of the proposition that if a card shows an even number on one face, then its opposite face is red?*

A response which identifies a card which need not be inverted, or a response which fails to identify a card which needs to be inverted, are both incorrect. Note that the original task dealt with numbers (even, odd) and letters (vowels, consonants).

Solution

The correct response was to turn the cards showing 8 and brown, but no other card. Remember how the proposition was stated: "*If* the card shows an even number on one face, *then* its opposite face is red." Only a card which has an even number on one face *and* which is *not* red on the other face can invalidate this rule. If we turn over the card labelled "3" and find that it is red, this does not invalidate the rule. Likewise, if we turn over the red card and find that it has the label "3", this also does not break the rule. On the other hand, if the brown card has the label "4", this invalidates the rule: it has an even number, but is not red. The interpretation of "if" here is that of the material conditional in classical logic.

Explanations of performance on the task

In Wason's study, not even 10% of subjects found the correct solution.[4] This result was replicated in 1993.[5]

Some authors have argued that participants do not read "if... then..." as the material conditional, since the natural language conditional is not the material conditional.[6] [7] (See also the paradoxes of the material conditional for more information.) However one interesting feature of the task is how participants react when the classical logic solution is explained:

> A psychologist, not very well disposed toward logic, once confessed to me that despite all problems in short-term inferences like the Wason Card Task, there was also the undeniable fact that he had never met an experimental subject who did not understand the logical solution when it was explained to him, and then agreed that it was correct.[8]

The selection task tends to produce the "correct" response when presented in a context of social relations. For example, if the rule used is "If you are drinking alcohol then you must be over 18", and the cards have an age on one side and beverage on the other, e.g., "17", "beer", "22", "coke", most people have no difficulty in selecting the correct cards ("17" and "beer").

This suggests a principle to distinguish Wason tasks which people find easy from those that they find difficult: namely, that a Wason task proves to be easier if the rule to be tested is one of social exchange (*in order to receive benefit X you need to fulfill condition Y*) and the subject is asked to police the rule, but is more difficult otherwise. Such a distinction, if empirically borne out, would support the contention of evolutionary psychologists that certain features of human psychology may be mechanisms that have evolved, through natural selection, to solve specific problems of social interaction, rather than expressions of general intelligence.[9] Alternatively, it could just mean that there are some linguistic contexts in which people tend to interpret "if" as a material conditional, and other linguistic contexts in which its most common vernacular meaning is different.

See also

- Psychology of reasoning
- Confirmation bias
- Logic
- Necessary and sufficient conditions
- Reasoning
- Evolutionary psychology

References

[1] Wason, P. C. (1966). "Reasoning". In Foss, B. M.. *New horizons in psychology*. Harmondsworth: Penguin.
[2] Wason, P. C.; Shapiro, Diana (1971). "Natural and contrived experience in a reasoning problem". *Quarterly Journal of Experimental Psychology* **23**: 63–71. doi:10.1080/00335557143000068.
[3] "The Wason selection task has often been claimed to be the single most investigated experimental paradigm in the psychology of reasoning". Ken Manktelow (1999). Reasoning and Thinking, Hove: Psychology Press, p. 8
[4] P.C. Wason (1977). "Self-contradictions." In: P.N. Johnson-Laird & P.C. Wason (eds.) "Thinking: Readings in cognitive science." Cambridge: Cambridge University Press.
[5] J.St.B.T. Evans et al. (1993). "Human reasoning: The psychology of deduction." Hove: Lawrence Erlbaum Associates.
[6] Oaksford, M., & Chater, N. (1994). A rational analysis of the selection task as optimal data selection. *Psychological Review*, 101, 608-631.
[7] Stenning, K. and van Lambalgen, M. (2004). A little logic goes a long way: basing experiment on semantic theory in the cognitive science of conditional reasoning. *Cognitive Science*, 28(4):481–530.
[8] Johan van Benthem (2008). Logic and reasoning: do the facts matter? *Studia Logica*, 88(1), 67-84
[9] Cosmides, L.; Tooby, J. (1992). Barkow *et al.*. ed. *Cognitive Adaptions for Social Exchange*. New York: Oxford University Press. (http://www.psych.ucsb.edu/research/cep/papers/Cogadapt.pdf)

Further reading

- Barkow, Jerome H.; Leda Cosmides, John Tooby (1995). *The adapted mind: evolutionary psychology and the generation of culture*. Oxford University Press US. pp. 181–184. ISBN 9780195101072.

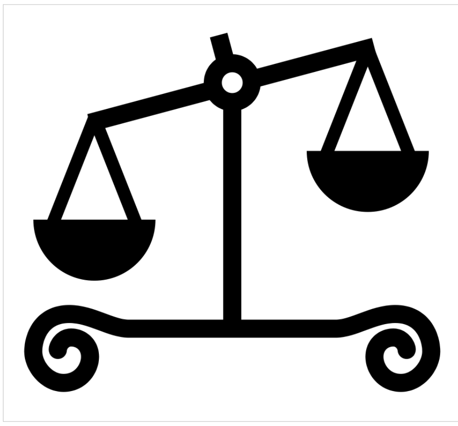
External links

- Here is the general structure of a Wason selection task (http://www.psych.ucsb.edu/research/cep/socex/wason.htm) — from the Center for Evolutionary Psychology at the University of California, Santa Barbara (http://www.psych.ucsb.edu/research/cep/)
- CogLab: Wason Selection (http://coglab.wadsworth.com/experiments/WasonSelection.shtml) — from Wadsworth CogLab 2.0 Cognitive Psychology Online Laboratory (http://coglab.wadsworth.com/)
- Elementary My Dear Wason (http://www.philosophyexperiments.com/wason/Default.aspx) - interactive version of Wason Selection Task at PhilosophyExperiments.Com

Balance puzzle

A number of logic puzzles exist that are based on the balancing of similar-looking items, often coins, to determine which one is of a different value within a limited number of uses of the balance scales. These differ from puzzles where items are assigned weights, in that only the relative mass of these items is relevant.

Premise



A well-known example has nine (or fewer) items, say coins (or balls), that are identical in weight save for one, which in this example we will say is lighter than the others—a counterfeit (an oddball). The difference is only perceptible by using a pair of scales balance, but only the coins themselves can be weighed, and it can only be used twice in total.

Is it possible to isolate the counterfeit coin with only two weighings?

Solution

To find a solution to the problem we first consider the maximum number of marbles from which one can find the lighter one in just one weighing. The maximum number possible is Three Marbles. To find the lighter one we can compare any two marbles, leaving the third marble unweighed. If the two marbles tested weigh the same then the lighter marble must be the one not on the balance - otherwise it is the one indicated as lighter by the balance.

Now, assume we have three marbles wrapped in a bigger marble shaped box. In one move, we can find which of the three marble shaped boxes is lighter (this box would contain the lighter marble) and, in the second weighing, as was shown above, we can find which of the three marbles within the box is lighter. So in two weighings we can find a single light marble from a set of $3 * 3 = 9$.

Note that we could reason along the same line, further, to see that in three weighings one can find the odd-lighter marble amongst 27 marbles and in 4 weighings, from 81 marbles.

The twelve-coin problem

A more complex version exists where there are twelve coins, eleven of which are identical and one of which is different, but it is not known whether it is heavier or lighter than the others. This time the balance may be used three times to isolate the unique coin and determine its weight relative to the others.

Solution

The procedure is less straightforward for this problem, and the second and third weighings depend on what has happened previously, although that need not be the case (see below).

- Four coins are put on each side. There are two possibilities:

  1. One side is heavier than the other. If this is the case, remove three coins from the heavier side, move three coins from the lighter side to the heavier side, and place three coins that were not weighed the first time on the lighter side. (Remember which coins are which.) There are three possibilities:

     1.a) The same side that was heavier the first time is still heavier. This means that either the coin that stayed there is heavier or that the coin that stayed on the lighter side is lighter. Balancing one of these against one of the other ten coins will reveal which of these is true, thus solving the puzzle.

     1.b) The side that was heavier the first time is lighter the second time. This means that one of the three coins that went from the lighter side to the heavier side is the light coin. For the third attempt, weigh two of these coins against each other: if one is lighter, it is the unique coin; if they balance, the third coin is the light one.

     1.c) Both sides are even. This means the one of the three coins that was removed from the heavier side is the heavy coin. For the third attempt, weigh two of these coins against each other: if one is heavier, it is the unique coin; if they balance, the third coin is the heavy one.

  2. Both sides are even. If this is the case, all eight coins are identical and can be set aside. Take the four remaining coins and place three on one side of the balance. Place 3 of the 8 identical coins on the other side. There are three possibilities:

     2.a) The three remaining coins are lighter. In this case you now know that one of those three coins is the odd one out and that it is lighter. Take two of those three coins and weigh them against each other. If the balance tips then the lighter coin is the odd one out. If the two coins balance then the third coin not on the balance is the odd one out and it is lighter.

     2.b) The three remaining coins are heavier. In this case you now know that one of those three coins is the odd one out and that it is heavier. Take two of those three coins and weigh them against each other. If the balance tips then the heavier coin is the odd one out. If the two coins balance then the third coin not on the balance is the odd one out and it is heavier.

     2.c) The three remaining coins balance. In this case you know that the unweighed coin is the odd one out. Weigh the remaining coin against one of the other 11 coins and this will tell you whether it is heavier or lighter.

With some outside the box thinking, such as assuming that there are authentic (genuine) coins at hand, a solution may be found quicker. In fact if there is one authentic coin for reference then the suspect coins can be thirteen. Number the coins from 1 to 13 and the authentic coin number 0 and perform these weightings in any order:

- 0,1,4,5,6 against 7,10,11,12,13
- 0,2,4,10,11 against 5,8,9,12,13
- 0,3,8,10,12 against 6,7,9,11,13

If only one weighting is off balance then it must be one of the coins 1,2,3 which only appear in one weighting. If all weightings are off balance then it is one of the coins 10-13 that appear in all weightings. Picking out the one counterfeit coin corresponding to each of the 27 outcomes is always possible (13 coins one either too heavy or too light is 26 possibilities) except when all weightings are abllanced, in which case there is no counterfeit coin. If coins 0 and 13 are deleted from the weightings they give one generic solution to the 12 coin question.

In literature

Niobe, the protagonist of Piers Anthony's novel *With a Tangled Skein*, must solve the twelve-coin variation of this puzzle to find her son in Hell: Satan has disguised the son to look identical to eleven other demons, and he is heavier or lighter depending on whether he is cursed to lie or able to speak truthfully. The solution in the book follows the given example 1.c.

External links

- A playable example of the first puzzle [1]
- A playable example of the second puzzle [2]
- *Two-pan balance and generalized counterfeit coin problem* [3]

References

[1] http://www.puzzles.com/PuzzlePlayground/HeavyWeight/HeavyWeight.htm
[2] http://nrich.maths.org/public/viewer.php?obj_id=5796
[3] http://math.uni.lodz.pl/~andkom/Marcel/Kule-en.pdf

Wine/water mixing problem

In the **wine/water mixing problem**, one starts with two barrels, one holding wine and the other an equal volume of water. A cup of wine is taken from the wine barrel and added to the water. A cup of the wine/water mixture is then returned to the wine barrel, so that the volumes in the barrels are again equal. The question is then posed—which of the two mixtures is purer?[1]

The problem can be solved without resorting to computation. It is not necessary to state the volumes of wine and water, as long as they are equal. The volume of the cup is irrelevant, as is any stirring of the mixtures. Also, any number of transfers can be made, as long as the volume of liquid in each barrel is the same at the end.[2]

Solution

The mixtures should be visualised as separated into their water and wine components. Conservation of substance then implies that the volume of wine in the barrel holding mostly water has to be equal to the volume of water in the barrel holding mostly wine.[2] To help in grasping this, the wine and water may be represented by, say, 100 red and 100 white marbles, respectively. If 20, say, red marbles are mixed in with the white marbles, and 20 marbles of any color are returned to the red container, then there will again be 100 marbles in each container. If there are now x white marbles in the red container, then there must be x red marbles in the white container. The mixtures will therefore be of equal purity. An example is shown below.

Time Step	Red Marble Container	Action	White Marble Container
0	100 (all red)		100 (all white)
1		20 (all red) →	
2	80 (all red)		120 (100 white, 20 red)
3		← 20 (16 white, 4 red)	
4	100 (84 red, 16 white)		100 (84 white, 16 red)

History

This puzzle was mentioned by W. W. Rouse Ball in the third, 1896, edition of his book *Mathematical Recreations And Problems Of Past And Present Times*, and is said to have been a favorite problem of Lewis Carroll.[3] [4]

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[3] p. 662, Mathematical Recreations And Problems Of Past And Present Times, David Singmaster, pp. 653–663 in *Landmark Writings in Western Mathematics 1640-1940*, edited by Ivor Grattan-Guinness, Roger Cooke, et al., Elsevier: 2005, ISBN 0444508716.
[4] *Mathematical recreations and problems of past and present times*, W. W. Rouse Ball, London and New York: Macmillan, 1896, 3rd ed.

Zebra Puzzle

The **zebra puzzle** is a well-known logic puzzle. It is often called *Einstein's Puzzle* or *Einstein's Riddle* because it is said to have been invented by Albert Einstein as a boy, [1] with the claim that Einstein said "only 2 percent of the world's population can solve it."[2] The puzzle is also sometimes attributed to Lewis Carroll.[3] However, there is no known evidence for Einstein's or Carroll's authorship; and the original puzzle cited below mentions brands of cigarette, such as Kools, that did not exist during Carroll's lifetime or Einstein's boyhood.

There are several versions of this puzzle. The version below is quoted from the first known publication in Life International magazine on December 17, 1962. The March 25, 1963 issue contained the solution given below and the names of several hundred solvers from around the world.

Text of the original puzzle

1. There are five houses.
2. The Englishman lives in the red house.
3. The Spaniard owns the dog.
4. Coffee is drunk in the green house.
5. The Ukrainian drinks tea.
6. The green house is immediately to the right of the ivory house.
7. The Old Gold smoker owns snails.
8. Kools are smoked in the yellow house.
9. Milk is drunk in the middle house.
10. The Norwegian lives in the first house.
11. The man who smokes Chesterfields lives in the house next to the man with the fox.
12. Kools are smoked in the house next to the house where the horse is kept. (should be "... in a house ...", see Discussion section)

13. The Lucky Strike smoker drinks orange juice.
14. The Japanese smokes Parliaments.
15. The Norwegian lives next to the blue house.

Now, who drinks water? Who owns the zebra?

In the interest of clarity, it must be added that each of the five houses is painted a different color, and their inhabitants are of different national extractions, own different pets, drink different beverages and smoke different brands of American cigarets [sic]. One other thing: in statement 6, *right* means *your* right.

– Life International, *December 17, 1962*

The premises leave out some details, notably that the houses are in a row.

Since neither water nor a zebra is mentioned in the clues, there exists a reductive solution to the puzzle, namely that no one owns a zebra or drinks water. If, however, the questions are read as "Given that one resident drinks water, which is it?" and "Given that one resident owns a zebra, which is it?" then the puzzle becomes a non-trivial challenge to inferential logic. (A frequent variant of the puzzle asks "Who owns the fish?")

It is possible not only to deduce the answers to the two questions but to figure out who lives where, in what color house, keeping what pet, drinking what drink, and smoking what brand of cigarettes.

Rule 12 leads to a contradiction. It should have read "Kools are smoked in *a* house next to the house where the horse is kept." (Note *a* instead of *the*.) The text above has been kept as it is, as it is meant to be a presentation of the text of the puzzle as originally published.

Solution

House	**1**	**2**	**3**	**4**	**5**
Color	Yellow	Blue	Red	Ivory	Green
Nationality	Norwegian	Ukrainian	Englishman	Spaniard	Japanese
Drink	**Water**	Tea	Milk	Orange juice	Coffee
Smoke	Kools	Chesterfield	Old Gold	Lucky Strike	Parliament
Pet	Fox	Horse	Snails	Dog	**Zebra**

Here are some deductive steps that can be followed to derive the solution. A useful method is to try to fit known relationships into a table and eliminate possibilities. Key deductions are in italics.

Step 1

We are told the Norwegian lives in the 1st house (10). It does not matter whether this is counted from the left or from the right. We just need to know the order, not the direction.

From (10) and (15), the 2nd house is blue. What color is the 1st house? Not green or ivory, because they have to be next to each other (6 and the 2nd house is blue). Not red, because the Englishman lives there (2). Therefore *the 1st house is yellow*.

It follows that Kools are smoked in the 1st house (8) and the Horse is kept in the 2nd house (12).

So what is drunk by the Norwegian in the 1st, yellow, Kools-filled house? Not tea since the Ukrainian drinks that (5). Not coffee since that is drunk in the green house (4). Not milk since that is drunk in the 3rd house (9). Not orange juice since the drinker of orange juice smokes Lucky Strikes (13). Therefore it is water (the missing beverage) that is drunk by the Norwegian.

Step 2

So what is smoked in the 2nd, blue house where we know a Horse is also kept?

Not Kools which are smoked in the 1st house (8). Not Old Gold since that house must have snails (7).

Let's suppose Lucky Strikes are smoked here, which means orange juice is drunk here (13). Then consider: Who lives here? Not the Norwegian since he lives in the 1st House (10). Not the Englishman since he lives in a red house (2). Not the Spaniard since he owns a dog (3). Not the Ukrainian since he drinks tea (4). Not the Japanese who smokes Parliaments (14). Since this is an impossible situation, Lucky Strikes are not smoked in the 2nd house.

Let's suppose Parliaments are smoked here, which means the Japanese man lives here (14). Then consider: What is drunk here? Not tea since the Ukrainian drinks that (5). Not coffee since that is drunk in the green house (4). Not milk since that is drunk in the 3rd house (9). Not orange juice since the drinker of that smokes Lucky Strike (13). Again, since this is an impossible situation, Parliaments are not smoked in the 2nd house.

Therefore, *Chesterfields are smoked in the 2nd house*.

So who smokes Chesterfields and keeps a Horse in the 2nd, blue house? Not the Norwegian who lives in the 1st House (10). Not the Englishman who lives in a red house (2). Not the Spaniard who owns a dog (3). Not the Japanese who smokes Parliaments (14). Therefore, the Ukrainian lives in the 2nd House, where he drinks tea (5)!

Step 3

Since Chesterfields are smoked in the 2nd house, we know from (11) that the fox is kept in either the 1st house or the 3rd house.

Let us first assume that the fox is kept in the 3rd house. Then consider: what is drunk by the man who smokes Old Golds and keeps snails (7)? We have already ruled out water and tea from the above steps. It cannot be orange juice since the drinker of that smokes Lucky Strikes (13). It cannot be milk because that is drunk in the 3rd house (9), where we have assumed a fox is kept. This leaves coffee, which we know is drunk in the green house (4).

So if the fox is kept in the 3rd house, then someone smokes Old Golds, keeps snails and drinks coffee in a green house. Who can this person be? Not the Norwegian who lives in the 1st house (10). Not the Ukrainian who drinks tea (5). Not the Englishman who lives in a red house (2). Not the Japanese who smokes Parliaments (14). Not the Spaniard who owns a dog (3).

This is impossible. So it follows that the fox is not kept in the 3rd house, but *in the 1st house*.

Step 4

From what we have found so far, we know that coffee and orange juice are drunk in the 4th and 5th houses. It doesn't matter which is drunk in which; we will just call them the coffee house and the orange juice house.

So where does the man who smokes Old Gold and keeps snails live? Not the orange juice house since Lucky Strike is smoked there (13).

Suppose this man lives in the coffee house. Then we have someone who smokes Old Gold, keeps snails and drinks coffee in a green (4) house. Again, by the same reasoning in step 3, this is impossible.

Therefore, the Old Gold-smoking, Snail-keeping man lives in the 3rd house.

It follows that Parliaments are smoked in the green, coffee-drinking house, by the Japanese man (14). This means the Spaniard must be the one who drinks orange juice, smokes Lucky Strikes and keeps a dog. By extension, the Englishman must live in the 3rd house, which is red. By process of elimination, the Spaniard's house is the ivory one.

By now we have filled in every variable except one, and it is clear that the Japanese is the one who keeps the zebra.

Right-to-left solution

The above solution assumed that the *first* house is the *leftmost* house. If we assume that the first house is the rightmost house, we find the following solution. Again the Japanese keeps the zebra, and the Norwegian drinks water.

House	**5**	**4**	**3**	**2**	**1**
Color	Ivory	Green	Red	Blue	Yellow
Nationality	Spaniard	Japanese	Englishman	Ukrainian	Norwegian
Drink	Orange juice	Coffee	Milk	Tea	**Water**
Smoke	Lucky Strike	Parliament	Old Gold	Chesterfield	Kools
Pet	Dog	**Zebra**	Snails	Horse	Fox

Other versions

Other versions of the puzzle have one or more of the following differences to the original puzzle:

1. Some colors, nationalities, cigarette brands and pets are substituted for other ones and the clues are given in different order. These do not change the logic of the puzzle.
2. One rule says that the green house is on the left of the white house, instead of on the right of it. This change has the same effect as numbering the houses from right to left instead of left to right (see section right-to-left solution above). It results only in swapping of the two corresponding houses with all their properties, but makes the puzzle a bit easier. It is also important to note that the omission of the word *immediately*, as in *immediately to the left/right of the white house*, leads to multiple solutions to the puzzle.
3. The clue "The man who smokes Blend has a neighbor who drinks water" is redundant and therefore makes the puzzle even easier. Originally, this clue was only in form of the question "Who drinks water?"

When given to children, the cigarette brands are often replaced by snacks eaten in each house.

1. The British person lives in the red house.
2. The Swede keeps dogs as pets.
3. The Dane drinks tea.
4. The green house is on the left of the white house.
5. The green homeowner drinks coffee.
6. The man who smokes Pall Mall keeps birds.
7. The owner of the yellow house smokes Dunhill.
8. The man living in the center house drinks milk.
9. The Norwegian lives in the first house.
10. The man who smokes Blend lives next to the one who keeps cats.
11. The man who keeps the horse lives next to the man who smokes Dunhill.
12. The man who smokes Bluemaster drinks beer.
13. The German smokes Prince.
14. The Norwegian lives next to the blue house.
15. The man who smokes Blend has a neighbor who drinks water.

Question: Who owns the fish?

1. The Englishman lives in the red house.
2. The Spaniard owns the dog.
3. Coffee is drunk in the green house.
4. The Ukrainian drinks tea.

5. The green house is immediately to the right of the ivory house.
6. The Ford driver owns the snail.
7. A Toyota driver lives in the yellow house.
8. Milk is drunk in the middle house.
9. The Norwegian lives in the first house to the left.
10. The man who drives the Chevy lives in the house next to the man with the fox.
11. A Toyota is parked next to the house where the horse is kept.
12. The Dodge owner drinks orange juice.
13. The Japanese owns a Porsche.
14. The Norwegian lives next to the blue house.

References

[1] Stangroom, Jeremy (2009). *Einstein's Riddle: Riddles, Paradoxes, and Conundrums to Stretch Your Mind*. Bloomsbury USA. pp. 10–11. ISBN 978-1596916654.

[2] Einstein's Riddle! (http://www.naute.com/puzzles/puzzle13.phtml), naute.com

[3] James Little, Cormac Gebruers, Derek Bridge, & Eugene Freuder. "Capturing Constraint Programming Experience: A Case-Based Approach" (http://www.cs.ucc.ie/~dgb/papers/Little-Et-Al-2002.pdf) (PDF). Cork Constraint Computation Centre, University College, Cork, Ireland. . Retrieved 2009-09-05.

External links

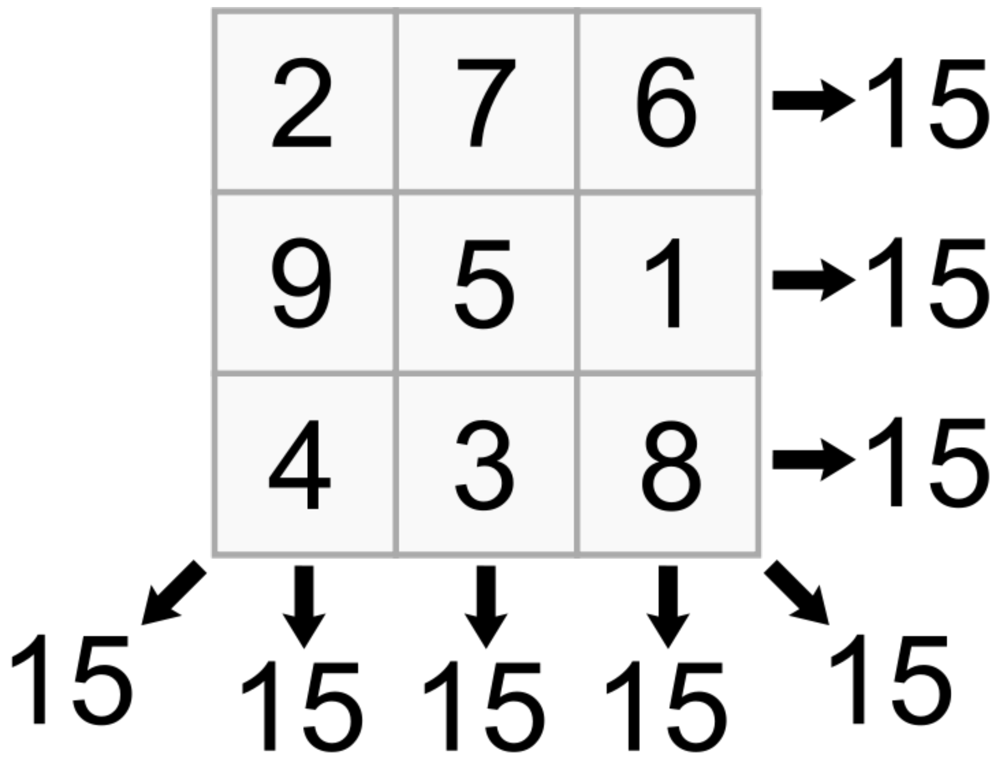
- Generator of logic puzzles similar to the Zebra Puzzle (http://www.mensus.net/brain/logic.shtml)
- Formulation and solution in ALCOIF Description Logic (http://www.iis.nsk.su/persons/ponom/ontologies/)
- SWI Prolog Solution (http://sandbox.rulemaker.net/ngps/119)
- Oz/Mozart Solution (http://www.mozart-oz.org/documentation/fdt/node23.html)
- Common Lisp and C/C++ solutions (http://www.weitz.de/einstein.html)
- English solution (PDF) (http://www.atkielski.com/ESLPublic/Game - Who Owns the Fish Group Version.pdf) - structured as a group activity for teaching English as a foreign language
- http://www.sfr-fresh.com/unix/misc/glpk-4.42.tar.gz:a/glpk-4.42/examples/zebra.mod A GNU MathProg model for solving the problem, included with the GNU Linear Programming Kit
- Solution to Einstein's puzzle using the method of elimination (http://csl.sublevel3.org/einsteins_problem/)

Magic squares

Magic square

In recreational mathematics, a **magic square** of order n is an arrangement of n^2 numbers, usually distinct integers, in a square, such that the n numbers in all rows, all columns, and both diagonals sum to the same constant.[1] A **normal** magic square contains the integers from 1 to n^2. The term "magic square" is also sometimes used to refer to any of various types of word square.

Normal magic squares exist for all orders $n \geq 1$ except $n = 2$, although the case $n = 1$ is trivial, consisting of a single cell containing the number 1. The smallest nontrivial case, shown below, is of order 3.

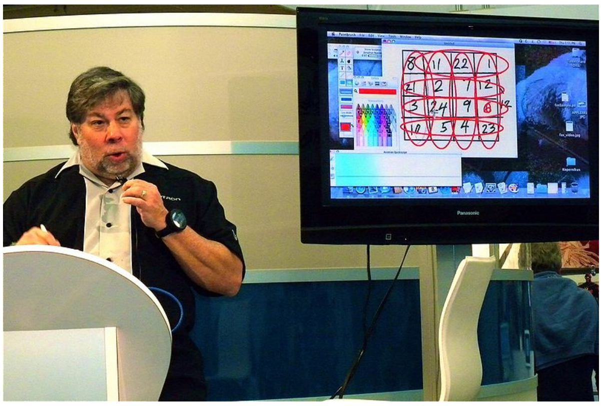


The constant sum in every row, column and diagonal is called the magic constant or magic sum, M. The magic constant of a normal magic square depends only on n and has the value

$$M = \frac{n(n^2+1)}{2}.$$

For normal magic squares of order $n = 3, 4, 5, \ldots$, the magic constants are:

15, 34, 65, 111, 175, 260, ... (sequence A006003 in OEIS).



At Macworld Expo 2009, Apple Inc. cofounder Steve Wozniak created a magic square after asking his audience to suggest a random number for the magic constant.

History

Magic squares were known to Chinese mathematicians, as early as 650 BCE[2] and Arab mathematicians, possibly as early as the 7th century, when the Arabs conquered northwestern parts of the Indian subcontinent and learned Indian mathematics and astronomy, including other aspects of combinatorial mathematics. The first magic squares of order 5 and 6 appear in an encyclopedia from Baghdad *circa* 983 CE, the *Encyclopedia of the Brethren of Purity* (*Rasa'il Ihkwan al-Safa*); simpler magic squares were known to several earlier Arab mathematicians.[2] Some of these squares were later used in conjunction with magic letters as in (Shams Al-ma'arif) to assist Arab illusionists and magicians.[3]

Original script from Shams Al-ma'arif.

Lo Shu square (3×3 magic square)

Chinese literature dating from as early as 650 BC tells the legend of Lo Shu or "scroll of the river Lo".[2] In ancient China there was a huge flood. The great king Yu (禹) tried to channel the water out to sea where then emerged from the water a turtle with a curious figure/pattern on its shell; circular dots of numbers which were arranged in a three by three grid pattern such that the sum of the numbers in each row, column and diagonal was the same: 15, which is also the number of days in each of the 24 cycles of the Chinese solar year. This pattern, in a certain way, was used by the people in controlling the river.

Printed version of the previous manuscript. Eastern Arabic numerals were used.

4	9	2
3	5	7
8	1	6

The Lo Shu Square, as the magic square on the turtle shell is called, is the unique normal magic square of order three in which 1 is at the bottom and 2 is in the upper right corner. Every normal magic square of order three is obtained from the Lo Shu by rotation or reflection.

The Square of Lo Shu is also referred to as the Magic Square of Saturn or Chronos.

Cultural significance

Magic squares have fascinated humanity throughout the ages, and have been around for over 4,120 years. They are found in a number of cultures, including Egypt and India, engraved on stone or metal and worn as talismans, the belief being that magic squares had astrological and divinatory qualities, their usage ensuring longevity and prevention of diseases.

The Kubera-Kolam is a floor painting used in India which is in the form of a magic square of order three. It is essentially the same as the Lo Shu Square, but with 19 added to each number, giving a magic constant of 72.

23	28	21
22	24	26
27	20	25

Persia

Although a definitive judgement of early history of magic squares is not available, it has been suggested that magic squares are probably of pre-Islamic Persian origin.[4] The study of magic squares in medieval Islam in Persia is however common, and supposedly, came after the introduction of Chess in Persia.[5] For instance in the tenth century, the Persian mathematician Buzjani has left a manuscript on page 33 of which there is a series of magic squares, which are filled by numbers in arithmetic progression in such a way that the sums on each line, column and diagonal are equal.[6]

Arabia

Magic squares were known to Islamic mathematicians, possibly as early as the 7th century, when the Arabs came into contact with Indian culture, and learned Indian mathematics and astronomy, including other aspects of combinatorial mathematics. It has also been suggested that the idea came via China. The first magic squares of order 5 and 6 appear in an encyclopedia from Baghdad *circa* 983 AD, the Rasa'il Ikhwan al-Safa (the Encyclopedia of the Brethren of Purity); simpler magic squares were known to several earlier Arab mathematicians.[2]

The Arab mathematician Ahmad al-Buni, who worked on magic squares around 1250 A.D., attributed mystical properties to them, although no details of these supposed properties are known. There are also references to the use of magic squares in astrological calculations, a practice that seems to have originated with the Arabs.[2]

India

The 3x3 magic square was used as part of rituals in India from vedic times, and continues to be used to date.The Ganesh yantra is a 3x3 magic square. A well known early 4x4 magic square in India can be seen in Khajuraho in the Parshvanath Jain temple. It dates from the 10th century.[7]

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

This is referred to as the Chautisa Yantra, since each row, column, diagonal, 2x2 sub-square, the corners of each 3x3 and 4x4 square, the two sets of four symmetrical numbers (1+11+16+6 and 2+12+15+5), and the sum of the middle two entries of the two outer columns and rows (12+1+6+15 and 2+16+11+5), sums to 34.

Europe

In 1300, building on the work of the Arab Al-Buni, Greek Byzantine scholar Manuel Moschopoulos wrote a mathematical treatise on the subject of magic squares, leaving out the mysticism of his predecessors.[8] Moschopoulos is thought to be the first Westerner to have written on the subject. In the 1450s the Italian Luca Pacioli studied magic squares and collected a large number of examples.[2]

In about 1510 Heinrich Cornelius Agrippa wrote *De Occulta Philosophia*, drawing on the Hermetic and magical works of Marsilio Ficino and Pico della Mirandola, and in it he expounded on the magical virtues of seven magical squares of orders 3 to 9, each associated with one of the astrological planets. This book was very influential throughout Europe until the counter-reformation, and Agrippa's magic squares, sometimes called Kameas, continue to be used within modern ceremonial magic in much the same way as he first prescribed.[2] [9]

Saturn=15		
4	9	2
3	5	7
8	1	6

Jupiter=34			
4	14	15	1
9	7	6	12
5	11	10	8
16	2	3	13

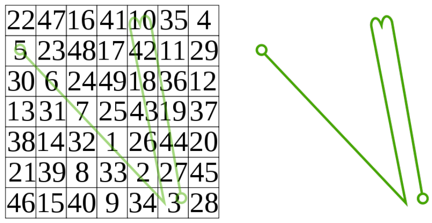
Mars=65				
11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

Sol=111					
6	32	3	34	35	1
7	11	27	28	8	30
19	14	16	15	23	24
18	20	22	21	17	13
25	29	10	9	26	12
36	5	33	4	2	31

Venus=175						
22	47	16	41	10	35	4
5	23	48	17	42	11	29
30	6	24	49	18	36	12
13	31	7	25	43	19	37
38	14	32	1	26	44	20
21	39	8	33	2	27	45
46	15	40	9	34	3	28

Mercury=260							
8	58	59	5	4	62	63	1
49	15	14	52	53	11	10	56
41	23	22	44	45	19	18	48
32	34	35	29	28	38	39	25
40	26	27	37	36	30	31	33
17	47	46	20	21	43	42	24
9	55	54	12	13	51	50	16
64	2	3	61	60	6	7	57

Luna=369								
37	78	29	70	21	62	13	54	5
6	38	79	30	71	22	63	14	46
47	7	39	80	31	72	23	55	15
16	48	8	40	81	32	64	24	56
57	17	49	9	41	73	33	65	25
26	58	18	50	1	42	74	34	66
67	27	59	10	51	2	43	75	35
36	68	19	60	11	52	3	44	76
77	28	69	20	61	12	53	4	45



Hagiel = הגיאל = 5; 3; 10; 1(10); 30(3)

The derivation of the sigil of Hagiel, the planetary intelligence of Venus, drawn on the magic square of Venus. Each Hebrew letter provides a numerical value, giving the vertices of the sigil.

The most common use for these Kameas is to provide a pattern upon which to construct the sigils of spirits, angels or demons; the letters of the entity's name are converted into numbers, and lines are traced through the pattern that these successive numbers make on the kamea. In a magical context, the term *magic square* is also applied to a variety of word squares or number squares found in magical grimoires, including some that do not follow any obvious pattern, and even those with differing numbers of rows and columns. They are generally intended for use as talismans. For instance the following squares are: The Sator square, one of the most famous magic squares found in a number of grimoires including the *Key of Solomon*; a square "to overcome envy", from *The Book of Power*;[10] and two squares from *The Book of the Sacred Magic of Abramelin the Mage*, the first to cause the illusion of a superb palace to appear, and the second to be worn on the head of a child during an angelic invocation:

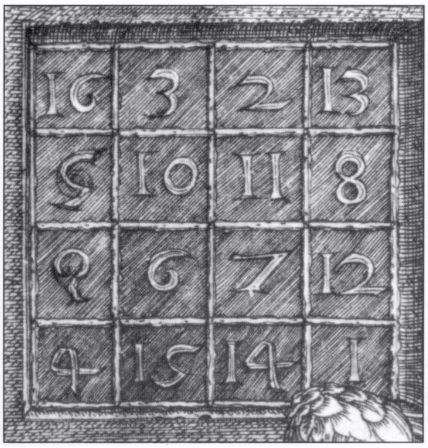
S	A	T	O	R
A	R	E	P	O
T	E	N	E	T
O	P	E	R	A
R	O	T	A	S

6	66	848	938
8	11	544	839
1	11	383	839
2	73	774	447

H	E	S	E	B
E	Q	A	L	
S				
E		G		
B				

A	D	A	M
D	A	R	A
A	R	A	D
M	A	D	A

Albrecht Dürer's magic square



Detail of *Melencolia I*

The order-4 magic square in Albrecht Dürer's engraving *Melencolia I* is believed to be the first seen in European art. It is very similar to Yang Hui's square, which was created in China about 250 years before Dürer's time. The sum 34 can be found in the rows, columns, diagonals, each of the quadrants, the center four squares, and the corner squares(of the 4x4 as well as the four contained 3x3 grids). This sum can also be found in the four outer numbers clockwise from the corners (3+8+14+9) and likewise the four counter-clockwise (the locations of four queens in the two solutions of the 4 queens puzzle [11]), the two sets of four symmetrical numbers (2+8+9+15 and 3+5+12+14), the sum of the middle two entries of the two outer columns and rows (5+9+8+12 and 3+2+15+14), and in four kite or cross shaped quartets(3+5+11+15, 2+10+8+14, 3+9+7+15, and 2+6+12+14). The two numbers in the middle of the bottom row give the date of the engraving: 1514. The numbers 1 and 4 at either side of the date correspond to, in English, the letters 'A' and 'D' which are the initials of the artist.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Dürer's magic square can also be extended to a magic cube.[12]

Dürer's magic square and his Melencolia I both also played large roles in Dan Brown's 2009 novel, The Lost Symbol.

Sagrada Família magic square

The Passion façade of the Sagrada Família church in Barcelona, designed by sculptor Josep Subirachs, features a 4×4 magic square:

The magic constant of the square is 33, the age of Jesus at the time of the Passion. Structurally, it is very similar to the Melancholia magic square, but it has had the numbers in four of the cells reduced by 1.

A magic square on the Sagrada Família church façade

1	14	14	4
11	7	6	9
8	10	10	5
13	2	3	15

While having the same pattern of summation, this is not a *normal* magic square as above, as two numbers (10 and 14) are duplicated and two (12 and 16) are absent, failing the 1→n^2 rule.

Similarly to Dürer's magic square, the Sagrada Familia's magic square can also be extended to a magic cube.[13]

Types and construction

There are many ways to construct magic squares, but the standard (and most simple) way is to follow certain configurations/formulas which generate regular patterns. Magic squares exist for all values of *n*, with only one exception: it is impossible to construct a magic square of order 2. Magic squares can be classified into three types: odd, doubly even (*n* divisible by four) and singly even (*n* even, but not divisible by four). Odd and doubly even magic squares are easy to generate; the construction of singly even magic squares is more difficult but several methods exist, including the LUX method for magic squares (due to John Horton Conway) and the Strachey method for magic squares.

Group theory was also used for constructing new magic squares of a given order from one of them, please see.[14]

The number of different *n*×*n* magic squares for *n* from 1 to 5, not counting rotations and reflections:

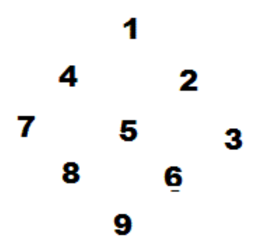
1, 0, 1, 880, 275305224 (sequence A006052 [15] in OEIS).

The number for $n = 6$ has been estimated to 1.7745×10^{19}.

Method for constructing a magic square of odd order

A method for constructing magic squares of odd order was published by the French diplomat de la Loubère in his book *A new historical relation of the kingdom of Siam* (Du Royaume de Siam, 1693), under the chapter entitled *The problem of the magical square according to the Indians*.[16] The method operates as follows:

Starting from the central column of the first row with the number 1, the fundamental movement for filling the squares is diagonally up and right, one step at a time. If a filled square is encountered, one moves vertically down one square instead, then continuing as before. When a move would leave the square, it is wrapped around to the last row or first column, respectively.



Yang Hui's construction method

step 1		
	1	
	.	
	.	

step 2		
	1	
	.	
		2

step 3		
	1	
3		
		2

step 4		
	1	
3		
4		2

step 5		
	1	
3	5	
4		2

step 6		
	1	6
3	5	
4		2

step 7		
	1	6
3	5	7
4		2

step 8		
8	1	6
3	5	7
4		2

step 9		
8	1	6
3	5	7
4	9	2

Starting from other squares rather than the central column of the first row is possible, but then only the row and column sums will be identical and result in a magic sum, whereas the diagonal sums will differ. The result will thus be a semimagic square and not a true magic square. Moving in directions other than north east can also result in magic squares.

Order 3		
8	1	6
3	5	7
4	9	2

Order 5				
17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Order 9								
47	58	69	80	1	12	23	34	45
57	68	79	9	11	22	33	44	46
67	78	8	10	21	32	43	54	56
77	7	18	20	31	42	53	55	66
6	17	19	30	41	52	63	65	76
16	27	29	40	51	62	64	75	5
26	28	39	50	61	72	74	4	15
36	38	49	60	71	73	3	14	25
37	48	59	70	81	2	13	24	35

The following formulae help construct magic squares of odd order

Order n				
Squares (n)	Last No.	Middle No.	Sum (M)	I_{th} row and J_{th} column No.
n	n^2	$\frac{n^2+1}{2}$	$\left(\frac{n^2+1}{2}\right)n$	$n((I+J-1+\left\lfloor\frac{n}{2}\right\rfloor) \bmod n) + ((I+2J-2) \bmod n) + 1$

Example:

Order 5			
Squares (n)	Last No.	Middle No.	Sum (M)
5	25	13	65

The "*Middle Number*" is always in the diagonal bottom left to top right.
The "*Last Number*" is always opposite the number **1** in an outside column or row.

A method of constructing a magic square of doubly even order

Doubly even means that n is an even multiple of an even integer; or 4p (e.g. 4, 8, 12), where p is an integer.

Generic pattern All the numbers are written in order from left to right across each row in turn, starting from the top left hand corner. The resulting square is also known as a mystic square. Numbers are then either retained in the same place or interchanged with their diametrically opposite numbers in a certain regular pattern. In the magic square of order four, the numbers in the four central squares and one square at each corner are retained in the same place and the others are interchanged with their diametrically opposite numbers.

A construction of a magic square of order 4 Go left to right through the square filling counting and filling in on the diagonals only. Then continue by going left to right from the top left of the table and fill in counting down from 16 to 1. As shown below.

***M* = Order 4**			
1			4
	6	7	
	10	11	
13			16

***M* = Order 4**			
1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

An extension of the above example for Orders 8 and 12 First generate a "truth" table, where a '1' indicates selecting from the square where the numbers are written in order 1 to n^2 (left-to-right, top-to-bottom), and a '0' indicates selecting from the square where the numbers are written in reverse order n^2 to 1. For M = 4, the "truth" table is as shown below, (third matrix from left.)

***M* = Order 4**			
1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

***M* = Order 4**			
16	15	14	13
12	11	10	9
8	7	6	5
4	3	2	1

***M* = Order 4**			
1	0	0	1
0	1	1	0
0	1	1	0
1	0	0	1

***M* = Order 4**			
1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

Note that a) there are equal number of '1's and '0's; b) each row and each column are "palindromic"; c) the left- and right-halves are mirror images; and d) the top- and bottom-halves are mirror images (c & d imply b.) The truth table can be denoted as (9, 6, 6, 9) for simplicity (1-nibble per row, 4 rows.) Similarly, for M=8, two choices for the truth table are (A5, 5A, A5, 5A, 5A, A5, 5A, A5) or (99, 66, 66, 99, 99, 66, 66, 99) (2-nibbles per row, 8 rows.) For M=12, the truth table (E07, E07, E07, 1F8, 1F8, 1F8, 1F8, 1F8, 1F8, E07, E07, E07) yields a magic square (3-nibbles per row, 12 rows.) It is possible to count the number of choices one has based on the truth table, taking rotational symmetries into account.

Medjig-method of constructing magic squares of even number of rows

This method is based on a 2006 published mathematical game called medjig (author: Willem Barink, editor: Philos-Spiele). The pieces of the medjig puzzle are squares divided in four quadrants on which the numbers 0, 1, 2 and 3 are dotted in all sequences. There are 18 squares, with each sequence occurring 3 times. The aim of the puzzle is to take 9 squares out of the collection and arrange them in a 3 x 3 "medjig-square" in such a way that each row and column formed by the quadrants sums to 9, along with the two long diagonals.

The medjig method of constructing a magic square of order 6 is as follows:

- Construct any 3 x 3 medjig-square (ignoring the original game's limit on the number of times that a given sequence is used).
- Take the 3 x 3 magic square and divide each of its squares into four quadrants.
- Fill these quadrants with the four numbers from 1 to 36 that equal the original number modulo 9, i.e. x+9y where x is the original number and y is a number from 0 to 3, following the pattern of the medjig-square.

Example:

Order 3		
8	1	6
3	5	7
4	9	2

Medjig 3 x 3					
2	3	0	2	0	2
1	0	3	1	3	1
3	1	1	2	2	0
0	2	0	3	3	1
3	2	2	0	0	2
0	1	3	1	1	3

Order 6					
26	35	1	19	6	24
17	8	28	10	33	15
30	12	14	23	25	7
3	21	5	32	34	16
31	22	27	9	2	20
4	13	36	18	11	29

Similarly, for any larger integer N, a magic square of order 2N can be constructed from any N x N medjig-square with each row, column, and long diagonal summing to (N-1)*N/2, and any N x N magic square (using the four numbers from 1 to 4N^2 that equal the original number modulo N^2).

Construction of panmagic squares

Any number p in the order-n square can be uniquely written in the form p = an + r, with r chosen from {1,...,n}. Note that due to this restriction, a and r are *not* the usual quotient and remainder of dividing p by n. Consequently the problem of constructing can be split in two problems easier to solve. So, construct two matching square grids of order n satisfying panmagic properties, one for the a-numbers (0,..., n−1), and one for the r-numbers (1,...,n). This requires a lot of puzzling, but can be done. When successful, combine them into one panmagic square. Van den Essen and many others supposed this was also the way Benjamin Franklin (1706–1790) constructed his famous Franklin squares. Three panmagic squares are shown below. The first two squares have been constructed April 2007 by Barink, the third one is some years older, and comes from Donald Morris, who used, as he supposes, the Franklin way of construction.

Order 8, sum 260							
62	4	13	51	46	20	29	35
5	59	54	12	21	43	38	28
52	14	3	61	36	30	19	45
11	53	60	6	27	37	44	22
64	2	15	49	48	18	31	33
7	57	56	10	23	41	40	26
50	16	1	63	34	32	17	47
9	55	58	8	25	39	42	24

Order 12, sum 870											
138	8	17	127	114	32	41	103	90	56	65	79
19	125	140	6	43	101	116	30	67	77	92	54
128	18	7	137	104	42	31	113	80	66	55	89
5	139	126	20	29	115	102	44	53	91	78	68
136	10	15	129	112	34	39	105	88	58	63	81
21	123	142	4	45	99	118	28	69	75	94	52
130	16	9	135	106	40	33	111	82	64	57	87
3	141	124	22	27	117	100	46	51	93	76	70
134	12	13	131	110	36	37	107	86	60	61	83
23	121	144	2	47	97	120	26	71	73	96	50
132	14	11	133	108	38	35	109	84	62	59	85
1	143	122	24	25	119	98	48	49	95	74	72

Order 12, sum 870											
1	120	121	48	85	72	73	60	97	24	25	144
142	27	22	99	58	75	70	87	46	123	118	3
11	110	131	38	95	62	83	50	107	14	35	134
136	33	16	105	52	81	64	93	40	129	112	9
8	113	128	41	92	65	80	53	104	17	32	137
138	31	18	103	54	79	66	91	42	127	114	7
5	116	125	44	89	68	77	56	101	20	29	140
139	30	19	102	55	78	67	90	43	126	115	6
12	109	132	37	96	61	84	49	108	13	36	133
135	34	15	106	51	82	63	94	39	130	111	10
2	119	122	47	86	71	74	59	98	23	26	143
141	28	21	100	57	76	69	88	45	124	117	4

The order 8 square satisfies all panmagic properties, including the Franklin ones. It consists of 4 perfectly panmagic 4x4 units. Note that both order 12 squares show the property that any row or column can be divided in three parts having a sum of 290 (= 1/3 of the total sum of a row or column). This property compensates the absence of the more standard panmagic Franklin property that any 1/2 row or column shows the sum of 1/2 of the total. For the rest the order 12 squares differ a lot.The Barink 12x12 square is composed of 9 perfectly panmagic 4x4 units, moreover any 4 consecutive numbers starting on any odd place in a row or column show a sum of 290. The Morris 12x12 square lacks these properties, but on the contrary shows constant Franklin diagonals. For a better understanding of the constructing decompose the squares as described above, and see how it was done. And note the difference between the Barink constructions on the one hand, and the Morris/Franklin construction on the other hand.

In the book *Mathematics* in the Time-Life Science Library Series, magic squares by Euler and Franklin are shown. Franklin designed this one so that any four-square subset (any four contiguous squares that form a larger square, or any four squares equidistant from the center) total 130. In Euler's square, the rows and columns each total 260, and halfway they total 130—and a chess knight, making its L-shaped moves on the square, can touch all 64 boxes in consecutive numerical order.

Construction similar to the Kronecker Product

There is a method reminiscent of the Kronecker product of two matrices, that builds an nm x nm magic square from an n x n magic square and an m x m magic square.[17]

The construction of a magic square using genetic algorithms

A magic square can be constructed using genetic algorithms.[18] This is an elegant trial and error process in which an initial population of magic squares with random values are generated. The *fitnesses* of these individual magic square are calculated based on the "flatness" of the magic square, that is, the degree of deviation in the sums of the rows, columns, and diagonals. The population of magic squares will interbreed (exchange values) in a manner coherent to genetics, based on the *fitness* score of the magic squares. Thus, magic squares with a higher *fitness* score will have a higher likelihood of reproducing. In the interbreeding process where the magic squares exchange their values, a mutation factor is introduced, imitating genetic mutation in nature. This mutation will be included or naturally excluded from the solution depending on their contribution to the fitness of the magic square. The next generation of the magic square population is again calculated for their fitness, and this process continues until a solution has been found.

Generalizations

Extra constraints

Certain extra restrictions can be imposed on magic squares. If not only the main diagonals but also the broken diagonals sum to the magic constant, the result is a panmagic square. If raising each number to certain powers yields another magic square, the result is a bimagic, a trimagic, or, in general, a multimagic square.

Different constraints

Sometimes the rules for magic squares are relaxed, so that only the rows and columns but not necessarily the diagonals sum to the magic constant (this is usually called a **semimagic square**).

In heterosquares and antimagic squares, the $2n + 2$ sums must all be *different*.

Multiplicative magic squares

Instead of *adding* the numbers in each row, column and diagonal, one can apply some other operation. For example, a multiplicative magic square has a constant *product* of numbers. A multiplicative magic square can be derived from an additive magic square by raising 2 (or any other integer) to the power of each element. For example, the original Lo-Shu magic square becomes:

***M* = 32768**		
16	512	4
8	32	128
256	2	64

Other examples of multiplicative magic squares include:

***M* = 216**		
2	9	12
36	6	1
3	4	18

***M* = 6720**			
1	6	20	56
40	28	2	3
14	5	24	4
12	8	7	10

Ali Skalli's non iterative method of construction is also applicable to multiplicative magic squares. On the 7x7 example below, the products of each line, each column and each diagonal is 6,227,020,800.

Skalli multiplicative 7 x 7						
27	50	66	84	13	2	32
24	52	3	40	54	70	11
56	9	20	44	36	65	6
55	72	91	1	16	36	30
4	24	45	60	77	12	26
10	22	48	39	5	48	63
78	7	8	18	40	33	60

Multiplicative magic squares of complex numbers

Still using Ali Skalli's non iterative method, it is possible to produce an infinity of multiplicative magic squares of complex numbers[19] belonging to $\mathbb{C}$ set. On the example below, the real and imaginary parts are integer numbers, but they can also belong to the entire set of real numbers $\mathbb{R}$. The product is: **−352,507,340,640 − 400,599,719,520 *i***.

Skalli multiplicative 7 x 7 of complex numbers						
21+14i	−70+30i	−93−9i	−105−217i	16+50i	4−14i	14−8i
63−35i	28+114i	−14i	2+6i	3−11i	211+357i	−123−87i
31−15i	13−13i	−103+69i	−261−213i	49−49i	−46+2i	−6+2i
102−84i	−28−14i	43+247i	−10−2i	5+9i	31−27i	−77+91i
−22−6i	7+7i	8+14i	50+20i	−525−492i	−28−42i	−73+17i
54+68i	138−165i	−56−98i	−63+35i	4−8i	2−4i	70−53i
24+22i	−46−16i	6−4i	17+20i	110+160i	84−189i	42−14i

Other magic shapes

Other shapes than squares can be considered, resulting, for example, in magic stars and magic hexagons. Going up in dimension results in magic cubes, magic tesseracts and other magic hypercubes.

Edward Shineman has developed yet another design in the shape of magic diamonds.

Combined extensions

One can combine two or more of the above extensions, resulting in such objects as *multiplicative multimagic hypercubes*. Little seems to be known about this subject.

Related problems

Over the years, many mathematicians, including Euler and Cayley have worked on magic squares, and discovered fascinating relations.

Magic square of primes

Rudolf Ondrejka (1928–2001) discovered the following 3x3 magic square of primes, in this case nine Chen primes:

17	89	71
113	59	5
47	29	101

The Green–Tao theorem implies that there are arbitrarily large magic squares consisting of primes.

Using Ali Skalli's non-iterative method of magic squares construction, it is easy to create magic squares of primes[20] of any dimension. In the example below, many symmetries appear (including all sorts of crosses), as well as the horizontal and vertical translations of all those. The magic constant is 13665.

Skalli Primes 5 x 5				
2087	2633	2803	2753	3389
2843	2729	3347	2099	2647
3359	2113	2687	2819	2687
2663	2777	2699	3373	2153
2713	3413	2129	2621	2789

It is believed that an infinite number of Skalli's magic squares of prime exist, but no demonstration exists to date. However, it is possible to easily produce a considerable number of them, not calculable in the absence of demonstration.

n-Queens problem

In 1992, Demirörs, Rafraf, and Tanik published a method for converting some magic squares into *n*-queens solutions, and vice versa.[21]

See also

- Arithmetic sequence
- Antimagic square
- Bimagic square
- Eight queens puzzle
- Freudenthal magic square
- Heterosquare
- Hexagonal Tortoise Problem
- Latin square
- Multimagic square (also known as a **Satanic square**)
- Magic series
- Magic cube
- Magic cube classes
- Magic star
- Magic tesseract
- Magic hypercube
- Most-perfect magic square
- Nasik magic hypercube
- Panmagic square (also known as a **Diabolic square**)
- Prime reciprocal magic square
- Room square
- Sator Arepo Tenet Opera Rotas
- Square matrices
- Sudoku
- Tetramagic square
- Trimagic square
- Unsolved problems in mathematics
- Vedic square
- Word square
- Yang Hui

- John R. Hendricks

Notes

[1] " Magic Square (http://demonstrations.wolfram.com/MagicSquare/)" by Onkar Singh, Wolfram Demonstrations Project.

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[3] The most famous Arabic book in magics, named "Shams Al-ma'arif (Arabic: فراعملا سمش باتك), for Ahmed bin Ali Al-boni who died about 1225 (622 AH).reprinted in Beirut in 1985

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[5] Helaine Selin, Ubiratan D'Ambrosio, *Mathematics Across Cultures: The History of Non-western Mathematics*, Published by Springer, 2001, ISBN 140200260, p. 160.

[6] Sesiano, J., *Abūal-Wafā\rasp's treatise on magic squares* (french), Z. Gesch. Arab.-Islam. Wiss. 12 (1998), 121--244.

[7] Magic Squares and Cubes By William Symes Andrews, 1908, Open court publish company

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[12] " Magic cube with Dürer's square (http://sites.google.com/site/aliskalligvaen/home-page/-magic-cube-with-duerer-s-square)" Ali Skalli's magic squares and magic cubes

[13] " Magic cube with Gaudi's square (http://sites.google.com/site/aliskalligvaen/home-page/-magic-cube-with-gaudi-s-square)" Ali Skalli's magic squares and magic cubes

[14] http://www.gaspalou.fr/magic-squares/index.htm

[15] http://en.wikipedia.org/wiki/Oeis%3Aa006052

[16] *Mathematical Circles Squared" By Phillip E. Johnson, Howard Whitley Eves, p.22*

[17] Hartley, M. "Making Big Magic Squares" (http://www.dr-mikes-math-games-for-kids.com/making-big-magic-squares.html).

[18] Evolving a Magic Square using Genetic Algorithms (http://www.dcs.napier.ac.uk/~benp/summerschool/jdemos/herdy/magic_problem2.html)

[19] " 8x8 multiplicative magic square of complex numbers (http://sites.google.com/site/aliskalligvaen/home-page/-multiplicative-of-complex-numbers-8x8)" Ali Skalli's magic squares and magic cubes

[20] " magic square of primes (http://sites.google.com/site/aliskalligvaen/home-page/-5x5-square-of-prime-numbers)" Ali Skalli's magic squares and magic cubes

[21] O. Demirörs, N. Rafraf, and M.M. Tanik. Obtaining *n*-queens solutions from magic squares and constructing magic squares from *n*-queens solutions. Journal of Recreational Mathematics, 24:272–280, 1992

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- A 'perfect' magic square (http://www.doermann.com/square/index.html) presented as a magic trick (Online Generator - Magic Square 4x4 using Javascript)
- Magic Squares of Order 4,5,6, and some theory (http://www.hbmeyer.de/backtrack/mag4en.htm)
- Evolving a Magic Square using Genetic Algorithms (http://www.dcs.napier.ac.uk/~benp/summerschool/jdemos/herdy/magic_problem2.html)

- Magic Square Museum (http://www.magic-square-museum.com/): the first Second Life museum about Magic Square. Vulcano (89,35,25)
- Magic squares and magic cubes (http://sites.google.com/site/aliskalligvaen/home-page): examples of magic squares and magic cubes built with Ali Skalli's non iterative method
- Puzzle-generator based on Magic Squares evolved by spectral variation of a given numerical sequence (http://thomas-rode.lima-city.de)

Further reading

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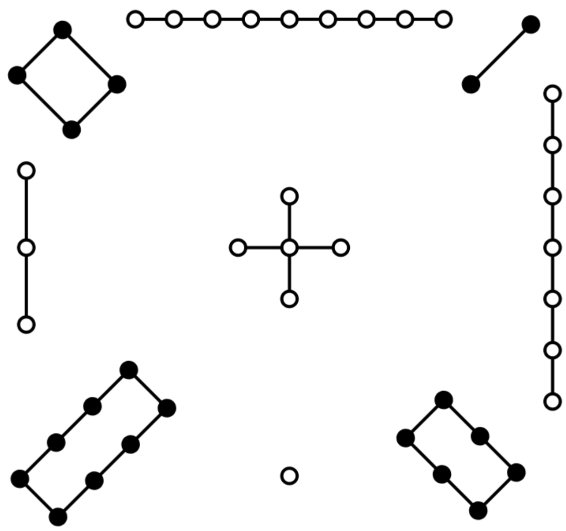
Lo Shu Square

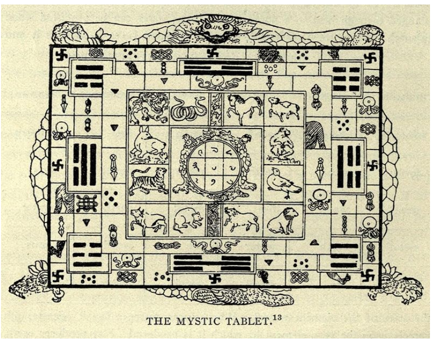

The 洛書 luòshū.

Lo Shu Square (simplified Chinese: 洛书; traditional Chinese: 洛書; pinyin: *luò shū*; also written 雒書; literally: Luo (River) Book/Scroll) or the **Nine Halls Diagram** (simplified Chinese: 九宫图; traditional Chinese: 九宮圖; pinyin: *jiǔ gōng tú*), is the unique normal magic square of order three. Lo Shu is part of the legacy of the most ancient Chinese mathematical and divinatory (Yi Jing 易經) traditions, and is an important emblem in Feng Shui (風水), the art of geomancy concerned with the placement of objects in relation to the flow of qi (氣) 'natural energy'.

Chinese legends concerning the pre-historic Emperor Yu (夏禹) tell of the Lo Shu, often in connection with the Ho Tu (河圖) figure and 8 trigrams. In ancient China there was a huge deluge: the people offered sacrifices to the god of one of the flooding rivers, the Lo river (洛水), to try to calm his anger. A magical turtle emerged from the water with the curious and decidedly unnatural (for a turtle shell) Lo Shu pattern on its shell: circular dots giving unitary (base 1) representations (figurate numbers) of the integers one through nine are arranged in a three-by-three grid.

4	9	2
3	5	7
8	1	6

Modern representation of the *Lo Shu* square as a magic square



The Lo Shu square on the back of a small turtle (in the center), surrounded by the signs of the Chinese Zodiac and the Eight trigrams, all carried by a large turtle (which, presumably, stands for the Dragon horse that had earlier revealed the trigrams to Fu Xi). A Tibetan design.

The odd and even numbers alternate in the periphery of the Lo Shu pattern; the 4 even numbers are at the four corners, and the 5 odd numbers (outnumbering the even numbers by one) form a cross in the center of the square. The sums in each of the 3 rows, in each of the 3 columns, and in both diagonals, are all 15 (fifteen is the number of days in each of the 24 cycles of the Chinese solar year). Since 5 is in the center cell, the sum of any two other cells that are directly through the 5 from each other is 10 (e.g., opposite corners add up to 10, the number of the Ho Tu (河圖)).

The Lo Shu is sometimes connected numerologically with the Ba Gua 八卦 "8 trigrams", which can be arranged in the 8 outer cells, reminiscent of circular trigram diagrams. Because north is placed at the bottom of maps in China, the 3x3 magic square having number 1 at the bottom and 9 at the top is used in preference to the other rotations/reflections. As seen in the "Later Heaven" arrangement, 1 and 9 correspond with ☵ Kǎn 水 "Water" and ☲ Lí 火 "Fire" respectively. In the "Early Heaven" arrangement, they would correspond with ☷ Kūn 地 "Earth" and ☰ Qián 天 "Heaven" respectively. Like the Ho Tu (河圖), the Lo Shu square, in conjunction with the 8 trigrams, is sometimes used as a mandalic representation important in Feng Shui (風水) geomancy.

External links

- Lo Shu Square: Definition, Nature and History [1]

References

[1] http://www.taliscope.com/Collection_en.html

Frénicle standard form

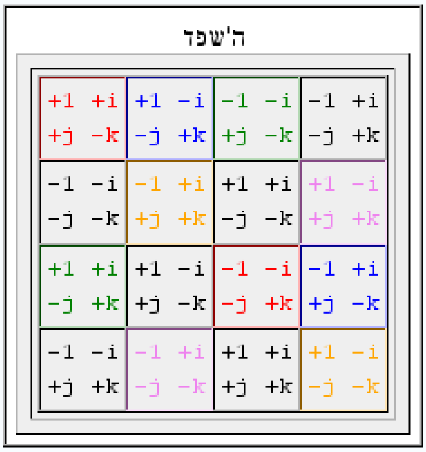
A magic square is in **Frénicle standard form**, named for Bernard Frénicle de Bessy, if the following two conditions apply:

1. the element at position [1,1] (top left corner) is the smallest of the four corner elements; and
2. the element at position [1,2] (top edge, second from left) is smaller than the element in [2,1].

This standard form was devised since a magic square remains "essentially similar" if it is rotated or transposed, or flipped so that the order of rows is reversed — there exists 8 different magic squares sharing one standard form. For example, the following magic squares are all essentially similar, with only the final square being in Frénicle standard form:

```
8 1 6   8 3 4    4 9 2   4 3 8    6 7 2   6 1 8    2 9 4   2 7 6
3 5 7   1 5 9    3 5 7   9 5 1    1 5 9   7 5 3    7 5 3   9 5 1
4 9 2   6 7 2    8 1 6   2 7 6    8 3 4   2 9 4    6 1 8   4 3 8
```

Generalising the concept of essentially different squares



384

For each group of magic squares one might identify the corresponding group of automorphisms, the group of transformations preserving the special properties of this group of magic squares. This way one can identify the number of different magic square classes.

example: From the perspective of Galois theory the most-perfect magic squares are not distinguishable. This means that the number of elemnts in the associated Galois group is 1. Please compare A051235 *Number of essentially different most-perfect pandiagonal magic squares of order 4n.* [1] with A000012 *The simplest sequence of positive numbers: the all 1's sequence.* [2]

References

[1] http://www.research.att.com/~njas/sequences/A051235
[2] http://www.research.att.com/~njas/sequences/A000012

Associative magic square

An **associative magic square** is a magic square for which every pair of numbers symmetrically opposite to the center sum up to the same value.

External links

- Associative magic square [1], MathWorld

References

[1] http://mathworld.wolfram.com/AssociativeMagicSquare.html

Most-perfect magic square

Most-perfect magic square from the Parshvanath Jain temple in Khajuraho

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

transcription of the indian numerals

A **most-perfect magic square** of order n is a magic square containing the numbers 1 to n^2 with two additional properties:

1. Each 2×2 subsquare sums to $2s$, where $s = n^2 + 1$.

2. All pairs of integers distant $n/2$ along a (major) diagonal sum to s.

Examples

Two 12×12 most-perfect magic squares can be obtained adding 1 to each element of:

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]	[,11]	[,12]
[1,]	64	92	81	94	48	77	67	63	50	61	83	78
[2,]	31	99	14	97	47	114	28	128	45	130	12	113
[3,]	24	132	41	134	8	117	27	103	10	101	43	118
[4,]	23	107	6	105	39	122	20	136	37	138	4	121
[5,]	16	140	33	142	0	125	19	111	2	109	35	126
[6,]	75	55	58	53	91	70	72	84	89	86	56	69
[7,]	76	80	93	82	60	65	79	51	62	49	95	66
[8,]	115	15	98	13	131	30	112	44	129	46	96	29
[9,]	116	40	133	42	100	25	119	11	102	9	135	26
[10,]	123	7	106	5	139	22	120	36	137	38	104	21
[11,]	124	32	141	34	108	17	127	3	110	1	143	18
[12,]	71	59	54	57	87	74	68	88	85	90	52	73

	[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]	[,11]	[,12]
[1,]	4	113	14	131	3	121	31	138	21	120	32	130
[2,]	136	33	126	15	137	25	109	8	119	26	108	16
[3,]	73	44	83	62	72	52	100	69	90	51	101	61
[4,]	64	105	54	87	65	97	37	80	47	98	36	88
[5,]	1	116	11	134	0	124	28	141	18	123	29	133
[6,]	103	66	93	48	104	58	76	41	86	59	75	49
[7,]	112	5	122	23	111	13	139	30	129	12	140	22
[8,]	34	135	24	117	35	127	7	110	17	128	6	118
[9,]	43	74	53	92	42	82	70	99	60	81	71	91
[10,]	106	63	96	45	107	55	79	38	89	56	78	46
[11,]	115	2	125	20	114	10	142	27	132	9	143	19
[12,]	67	102	57	84	68	94	40	77	50	95	39	85

Properties

All most-perfect magic squares are panmagic squares.

Apart from the trivial case of the first order square, most-perfect magic squares are all of order $4n$. In their book, Kathleen Ollerenshaw and David S. Brée give a method of construction and enumeration of all most-perfect magic squares. They also show that there is a one-to-one correspondence between reversible magic squares and most-perfect magic squares.

For $n = 36$, there are about 2.7×10^{44} essentially different most-perfect magic squares.

References

- Kathleen Ollerenshaw, David S. Brée: *Most-perfect Pandiagonal Magic Squares: Their Construction and Enumeration*, Southend-on-Sea : Institute of Mathematics and its Applications, 1998, 186 pages, ISBN 0-905091-06-X
- T.V.Padmakumar, *Number Theory and Magic Squares*, Sura books [1], India, 2008, 128 pages, ISBN 978-81-8449-321-4

External links

- A051235: Number of essentially different most-perfect pandiagonal magic squares of order 4n [1] from The On-Line Encyclopedia of Integer Sequences

References

[1] http://www.surabooks.com/ProductDisplay.asp?RefID=2800

Panmagic square

A **pandiagonal magic square** or **panmagic square** (also **diabolic square**, **diabolical square** or **diabolical magic square**) is a magic square with the additional property that the broken diagonals, i.e. the diagonals that wrap round at the edges of the square, also add up to the magic constant.

A pandiagonal magic square remains pandiagonally magic not only under rotation or reflection, but also if a row or column is moved from one side of the square to the opposite side. As such, an $n\times n$ pandiagonal magic square can be regarded as having$8n^2$ orientations.

4×4 panmagic squares

The smallest non-trivial pandiagonal magic squares are 4×4 squares.

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

In 4×4 panmagic squares, the magic constant of 34 can be seen in a number of patterns in addition to the rows, columns and diagonals:

- Any of the sixteen 2×2 squares, including those that wrap around the edges of the whole square, e.g. 14+11+4+5, 1+12+15+6
- The corners of any 3×3 square, e.g. 8+12+5+9
- Any pair of horizontally or vertically adjacent numbers, together with the corresponding pair displaced by a (2, 2) vector, e.g. 1+8+16+9

Thus of the 86 possible sums adding to 34, 52 of them form regular patterns, compared with 10 for an ordinary 4×4 magic square.

There are only three distinct 4×4 pandiagonal magic squares, namely the one above and the following:

1	12	7	14
8	13	2	11
10	3	16	5
15	6	9	4

1	8	11	14
12	13	2	7
6	3	16	9
15	10	5	4

In any 4×4 pandiagonal magic square, any two numbers at the opposite corners of a 3×3 square add up to 17. Consequently, no 4×4 panmagic squares are associative.

5×5 panmagic squares

There are many 5×5 pandiagonal magic squares. Unlike 4×4 panmagic squares, these can be associative. The following is a 5×5 associative panmagic square:

20	8	21	14	2
11	4	17	10	23
7	25	13	1	19
3	16	9	22	15
24	12	5	18	6

Ina addition to the rows, columns, and diagonals, a 5×5 pandiagonal magic square also shows its magic sum in four "quincunx" patterns, which in the above example are:

17+25+13+1+9 = 65 (center plus adjacent row and column squares)

21+7+13+19+5 = 65 (center plus the remaining row and column squares)

4+10+13+16+22 = 65 (center plus diagonally adjacent squares)

20+2+13+24+6 = 65 (center plus the remaining squares on its diagonals)

Each of these quincunxes can be translated to other positions in the square by cyclic permutation of the rows and columns (wrapping around), which in a pandiagonal magic square does not affect the equality of the magic sums. This leads to 100 quincunx sums, including broken quincunxes analogous to broken diagonals.

The quincunx sums can be proved by taking linear combinations of the row, column, and diagonal sums. Consider the panmagic square

A	B	C	D	E
F	G	H	I	J
K	L	M	N	O
P	Q	R	S	T
U	V	W	X	Y

with magic sum Z. To prove the quincunx sum A+E+M+U+Y = Z (corresponding to the 20+2+13+24+6 = 65 example given above), one adds together the following:

3 times each of the diagonal sums A+G+M+S+Y and E+I+M+Q+U The diagonal sums A+J+N+R+V, B+H+N+T+U, D+H+L+P+Y,and E+F+L+R+X The row sums A+B+C+D+E and U+V+W+X+Y

From this sum the following are subtracted:

The row sums F+G+H+I+J and P+Q+R+S+T The column sum C+H+M+R+W Twice each of the column sums B+G+L+Q+V and D+I+N+S+X.

The net result is 5A+5E+5M+5U+5Y = 5Z, which divided by 5 gives the quincunx sum. Similar linear combinations can be constructed for the other quincunx patterns H+L+M+N+R, C+K+M+O+W, and G+I+M+Q+S.

External links

- Panmagic Square at MathWorld [1]

References

- W. S. Andrews, *Magic Squares and Cubes*. New York: Dover, 1960. Originally printed in 1917. See especially Chapter X.

References

[1] http://mathworld.wolfram.com/PanmagicSquare.html

Bimagic square

In mathematics, a **bimagic square** is a magic square that also remains magic if all of the numbers it contains are squared. The first known bimagic square has order 8 and magic constant 260; it has been conjectured by Bensen and Jacoby that no nontrivial bimagic squares of order less than 8 exist. This was shown for magic squares containing the elements 1 to n^2 by Boyer and Trump.

However, J. R. Hendricks was able to show in 1998 that no bimagic square of order 3 exists, save for the trivial bimagic square containing the same number nine times. The proof is fairly simple: let the following be our bimagic square.

a	b	c
d	e	f
g	h	i

It is well known that a property of magic squares is that $a+i=2e$. Similarly, $a^2+i^2=2e^2$. Therefore $(a-i)^2=2(a^2+i^2)-(a+i)^2=4e^2-4e^2=0$. It follows that $a=e=i$. The same holds for all lines going through the center.

For 4×4 squares, Luke Pebody was able to show by similar methods that the only 4×4 bimagic squares (up to symmetry) are of the form

a	b	c	d
c	d	a	b
d	c	b	a
b	a	d	c

or

a	a	b	b
b	b	a	a
a	a	b	b
b	b	a	a

An 8×8 bimagic square.

16	41	36	5	27	62	55	18
26	63	54	19	13	44	33	8
1	40	45	12	22	51	58	31
23	50	59	30	4	37	48	9
38	3	10	47	49	24	29	60
52	21	32	57	39	2	11	46
43	14	7	34	64	25	20	53
61	28	17	56	42	15	6	35

Nontrivial bimagic squares are now (2010) known for any order from eight to 64. Thanks to Li Wen of China, created the first known bimagic squares of orders 34, 37, 38, 41, 43, 46, 47, 53, 58, 59, 61, 62 filling the gaps of the last unknown orders.

See also

- Magic square
- Trimagic square
- Multimagic square
- Magic cube
- Bimagic cube
- Trimagic cube
- Multimagic cube

External links

- Aale de Winkel's listing of all 80 bimagic squares of order 8 [1].

References

[1] http://www.magichypercubes.com/Encyclopedia/DataBase/BiMagicSquare_08.html

Trimagic square

In mathematics, a **trimagic square** is a magic square that also remains magic if all of the numbers it contains are squared or cubed. Trimagic squares of orders 12, 32, 64, 81 and 128 have been discovered so far; the only known trimagic square of order 12, given below, was found in June 2002 by German mathematician Walter Trump.

1	22	33	41	62	66	79	83	104	112	123	144
9	119	45	115	107	93	52	38	30	100	26	136
75	141	35	48	57	14	131	88	97	110	4	70
74	8	106	49	12	43	102	133	96	39	137	71
140	101	124	42	60	37	108	85	103	21	44	5
122	76	142	86	67	126	19	78	59	3	69	23
55	27	95	135	130	89	56	15	10	50	118	90
132	117	68	91	11	99	46	134	54	77	28	13
73	64	2	121	109	32	113	36	24	143	81	72
58	98	84	116	138	16	129	7	29	61	47	87
80	34	105	6	92	127	18	53	139	40	111	65
51	63	31	20	25	128	17	120	125	114	82	94

See also

- Magic square
- Bimagic square
- Multimagic square
- Magic cube
- Bimagic cube
- Trimagic cube
- Multimagic cube

Multimagic square

In mathematics, a ***P*-multimagic square** (also known as a **satanic square**) is a magic square that remains magic even if all its numbers are replaced by their *k*th power for $1 \leq k \leq P$. Thus, a magic square is bimagic if it is 2-multimagic, and trimagic if it is 3-multimagic.

The first 4-magic square, of order 512, was constructed in May 2001 by André Viricel and Christian Boyer; about one month later, in June 2001, Viricel and Boyer presented the first 5-magic square, of order 1024. They also presented a 4-magic square of order 256 in January 2003, and another 5-magic square, of order 729, was constructed in June 2003 by Chinese mathematician Li Wen.

The smallest known normal satanic square, shown below, has order 8.

5	31	35	60	57	34	8	30
19	9	53	46	47	56	18	12
16	22	42	39	52	61	27	1
63	37	25	24	3	14	44	50
26	4	64	49	38	43	13	23
41	51	15	2	21	28	62	40
54	48	20	11	10	17	55	45
36	58	6	29	32	7	33	59

This magic square has a magic constant of 260. Raising every number to the second power yields the following magic square with a sum of 11180.

25	961	1225	3600	3249	1156	64	900
361	81	2809	2116	2209	3136	324	144
256	484	1764	1521	2704	3721	729	1
3969	1369	625	576	9	196	1936	2500
676	16	4096	2401	1444	1849	169	529
1681	2601	225	4	441	784	3844	1600
2916	2304	400	121	100	289	3025	2025
1296	3364	36	841	1024	49	1089	3481

See also

- Magic square
- Diabolic square
- Magic cube
- Multimagic cube

External links

- multimagie.com [1]
- puzzled.nl [2]

References

[1] http://www.multimagie.com/indexengl.htm
[2] http://www.puzzled.nl/

Prime reciprocal magic square

A **prime reciprocal magic square** is a magic square using the digits of the reciprocal of a prime number.

Consider a number divided into one, like 1/3 or 1/7. In base ten, the remainder, and so the digits, of 1/3 repeats at once: 0·3333... However, the remainders of 1/7 repeat over six, or 7-1, digits: 1/7 = 0·142857142857142857... If you examine the multiples of 1/7, you can see that each is a cyclic permutation of these six digits:

```
1/7 = 0·1 4 2 8 5 7...
2/7 = 0·2 8 5 7 1 4...
3/7 = 0·4 2 8 5 7 1...
4/7 = 0·5 7 1 4 2 8...
5/7 = 0·7 1 4 2 8 5...
6/7 = 0·8 5 7 1 4 2...
```

If the digits are laid out as a square, it is obvious that each row will sum to 1+4+2+8+5+7, or 27, and only slightly less obvious that each column will also do so, and consequently we have a magic square:

```
1 4 2 8 5 7
2 8 5 7 1 4
4 2 8 5 7 1
5 7 1 4 2 8
7 1 4 2 8 5
8 5 7 1 4 2
```

However, neither diagonal sums to 27, but all other prime reciprocals in base ten with maximum period of p-1 produce squares in which all rows and columns sum to the same total.

Other properties of Prime Reciprocals: Midy's Theorem Midy's theorem

The repeating pattern of an even number of digits [7-1, 11-1, 13-1, 17-1, 19-1, 29-1, ...] in the quotients when broken in half are the nines-compliment of each half:

```
1/7 = 0.142,857,142,857 ...
     +0.857,142
      ---------
      0.999,999
```

```
1/11 = 0.09090,90909 ...
      +0.90909,09090
       -----
       0.99999,99999
```

```
1/13 = 0.076,923 076,923 ...
      +0.923,076
       ---------
       0.999,999
```

```
1/17 = 0.05882352,94117647
      +0.94117647,05882352
       ------------------
        0.99999999,99999999
```

```
1/19 = 0.052631578,947368421 ...
      +0.947368421,052631578
        ---------------------
        0.999999999,999999999
```

Ekidhikena Purvena From: Swami_Bharati_Krishna_Tirtha's_Vedic_mathematics#By_one_more_than_the_one_before

Concerning the number of decimal places shifted in the quotient per multiple of 1/19:

```
01/19 = 0.052631578,947368421
02/19 = 0.1052631578,94736842
04/19 = 0.21052631578,9473684
08/19 = 0.421052631578,947368
16/19 = 0.8421052631578,94736
```

A factor of 2 in the numerator produces a shift of one decimal place to the right in the quotient.

```
In the square from 1/19, with maximum period 18 and row-and-column
total of 81, both diagonals  also sum to 81, and this square is
therefore fully magic:
```

```
01/19 = 0·0 5 2 6 3 1 5 7 8 9 4 7 3 6 8 4 2 1...
02/19 = 0·1 0 5 2 6 3 1 5 7 8 9 4 7 3 6 8 4 2...
03/19 = 0·1 5 7 8 9 4 7 3 6 8 4 2 1 0 5 2 6 3...
04/19 = 0·2 1 0 5 2 6 3 1 5 7 8 9 4 7 3 6 8 4...
05/19 = 0·2 6 3 1 5 7 8 9 4 7 3 6 8 4 2 1 0 5...
06/19 = 0·3 1 5 7 8 9 4 7 3 6 8 4 2 1 0 5 2 6...
07/19 = 0·3 6 8 4 2 1 0 5 2 6 3 1 5 7 8 9 4 7...
08/19 = 0·4 2 1 0 5 2 6 3 1 5 7 8 9 4 7 3 6 8...
09/19 = 0·4 7 3 6 8 4 2 1 0 5 2 6 3 1 5 7 8 9...
10/19 = 0·5 2 6 3 1 5 7 8 9 4 7 3 6 8 4 2 1 0...
11/19 = 0·5 7 8 9 4 7 3 6 8 4 2 1 0 5 2 6 3 1...
12/19 = 0·6 3 1 5 7 8 9 4 7 3 6 8 4 2 1 0 5 2...
13/19 = 0·6 8 4 2 1 0 5 2 6 3 1 5 7 8 9 4 7 3...
14/19 = 0·7 3 6 8 4 2 1 0 5 2 6 3 1 5 7 8 9 4...
15/19 = 0·7 8 9 4 7 3 6 8 4 2 1 0 5 2 6 3 1 5...
16/19 = 0·8 4 2 1 0 5 2 6 3 1 5 7 8 9 4 7 3 6...
17/19 = 0·8 9 4 7 3 6 8 4 2 1 0 5 2 6 3 1 5 7...
18/19 = 0·9 4 7 3 6 8 4 2 1 0 5 2 6 3 1 5 7 8...
```

[1]

The same phenomenon occurs with other primes in other bases, and the following table lists some of them, giving the prime, base, and magic total (derived from the formula base-1 x prime-1 / 2):

Prime	Base	Total
19	10	81
53	12	286
53	34	858
59	2	29
67	2	33
83	2	41
89	19	792
167	68	5,561
199	41	3,960
199	150	14,751
211	2	105
223	3	222
293	147	21,316
307	5	612
383	10	1,719
389	360	69,646
397	5	792
421	338	70,770
487	6	1,215
503	420	105,169
587	368	107,531
593	3	592
631	87	27,090
677	407	137,228
757	759	286,524
787	13	4,716
811	3	810
977	1,222	595,848
1,033	11	5,160
1,187	135	79,462
1,307	5	2,612
1,499	11	7,490
1,877	19	16,884
1,933	146	140,070
2,011	26	25,125
2,027	2	1,013
2,141	63	66,340
2,539	2	1,269

3,187	97	152,928
3,373	11	16,860
3,659	126	228,625
3,947	35	67,082
4,261	2	2,130
4,813	2	2,406
5,647	75	208,902
6,113	3	6,112
6,277	2	3,138
7,283	2	3,641
8,387	2	4,193

See also

- cyclic number

References

Rademacher, H. and Toeplitz, O. The Enjoyment of Mathematics: Selections from Mathematics for the Amateur. Princeton, NJ: Princeton University Press, pp. 158–160, 1957.

Weisstein, Eric W. "Midy's Theorem." From MathWorld—A Wolfram Web Resource. http://mathworld.wolfram.com/MidysTheorem.html

References

[1] http://upload.wikimedia.org/wikipedia/commons/5/59/MgkSqr_1_over_19_Deva.tif

Heterosquare

A **heterosquare** of order n is an arrangement of the integers 1 to n^2 in a square, such that the rows, columns, and diagonals all sum to different values. There are no heterosquares of order 2, but heterosquares exist for any order $n \geq 3$.

1	2	3
8	9	4
7	6	5

Order 3

2	1	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Order 4

1	2	3	4	5
16	17	18	19	6
15	24	25	20	7
14	23	22	21	8
13	12	11	10	9

Order 5

Heterosquares are easily constructed, as shown in the above examples. If n is odd, filling the square in a spiral pattern will produce a heterosquare. And if n is even, a heterosquare results from writing the numbers 1 to n^2 in order, then exchanging 1 and 2.

It is strongly suspected that there are exactly 3120 essentially different heterosquares of order 3.

An antimagic square is a special kind of heterosquare where the $2n + 2$ row, column and diagonal sums are *consecutive* integers.

Antimagic square

An **antimagic square** of order n is an arrangement of the numbers 1 to n^2 in a square, such that the n rows, the n columns and the two diagonals form a sequence of $2n + 2$ consecutive integers. The smallest antimagic squares have order 4.

2	15	5	13
16	3	7	12
9	8	14	1
6	4	11	10

1	13	3	12
15	9	4	10
7	2	16	8
14	6	11	5

In each of these two antimagic squares of order 4, the rows, columns and diagonals sum to ten different numbers in the range 29–38.

Antimagic squares form a subset of heterosquares which simply have each row, column and diagonal sum different. They contrast with magic squares where each sum is the same.

A **sparse antimagic square** (SAM) is a square matrix of size n by n of nonnegative integers whose nonzero entries are the consecutive integers $1, \ldots, m$ for some $m \leq n^2$, and whose row-sums and column-sums constitute a set of consecutive integers[1] . If the diagonals are included in the set of consecutive integers, the array is known as a **sparse totally anti-magic square** (STAM). Note that a STAM is not necessarily a SAM, and vice-versa.

Some open problems

- How many antimagic squares of a given order exist?
- Do antimagic squares exist for all orders greater than 3?
- Is there a simple proof that no antimagic square of order 3 exists?

References

[1] Gray, I. D.; MacDougall, J.A. (2006). "Sparse anti-magic squares and vertex-magic labelings of bipartite graphs". *Discrete Mathematics* **306**: 2878–2892. doi:10.1016/j.disc.2006.04.032.

External links

- Mathworld (http://mathworld.wolfram.com/AntimagicSquare.html)

Mystic square

The square array of the integers 1 through n^2 that is generated when a method for constructing a 4 × 4 magic square is generalized was called a **mystic square** by Joel B. Wolowelsky and David Shakow in their article describing a method for constructing a magic square whose order is a multiple of 4.[1] A 4 × 4 magic square can be constructed by writing out the numbers from 1 to 16 consecutively in a 4 × 4 matrix and then interchanging those numbers on the diagonals that are equidistant from the center. (Figure 1). The sum of each row, column and diagonal is 34, the "magic number" for a 4 × 4 magic square. In general, the "magic number" for an $n \times n$ magic square is $n(n2 + 1)/2$.

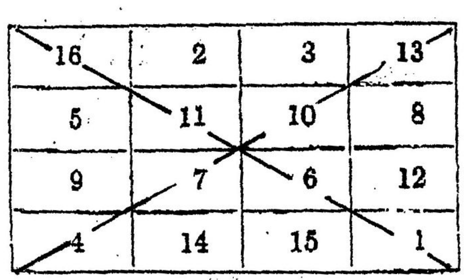


FIG. 1

Properties of a mystic square

As seen in the example for a 6 × 6 square (Figure 2), the properties of the mystic square are related to those of a 6 × 6 magic square. The sum of the diagonals is 111, the magic number for a 6 × 6 magic square. The sums of the rows increase arithmetically with a common difference of 12 and an average of 111. The columns also increase arithmetically with a common difference of 2 and an average of 111. The quotient of the two common differences is 6. This pattern proves true for all values of n. For the special case of $n = 4$ (where the mystic square is already a magic square), the quotient of the common differences is the indeterminate 0/0, which may be assigned the value 4 for consistency.

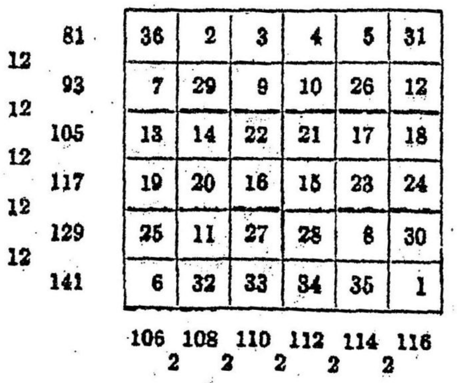


Fig. 2

Converting an $n \times n$ mystic square to a magic square when n is a multiple of 4

As illustrated in the case where $n = 8$, the method consists of changing the position of the numbers that lie of the sides of the square that is formed by joining the midpoints of the sides of the mystic square (Figure 3). Each of these lines is first “reflected” with the number on the opposite end of the same line (Figure 4). These numbers are in turn reflected “across the board” (Figure 5). This produces an 8x8 Magic Square.

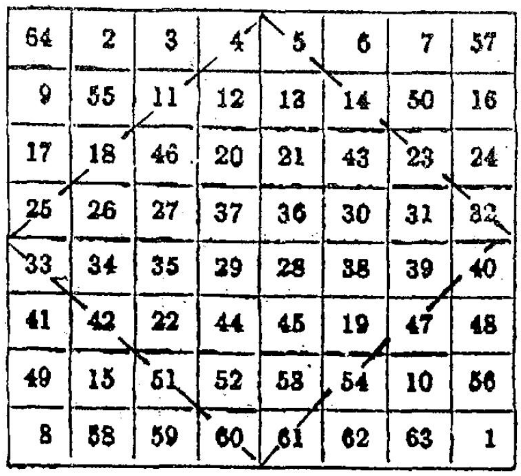


Fig. 3

In general, $(n/4) - 1$ reflection lines are required to convert an $n \times n$ mystic square into a magic square. When applying this method to a 12×12 mystic square, two reflection lines are necessary (Figure 6). Note that each reflection line must contain n terms. In the case of the 12×12 illustrated here, each second set (4, 15, 26, 37) contains only 4 terms, and so must be completed by adding two terms (54, 65). (In the case of a 4×4 mystic square, 0 reflection lines are required.)

64	2	3	25	32	6	7	57
9	55	18	12	13	23	50	16
17	11	46	20	21	43	14	24
4	26	27	37	36	30	31	5
60	34	35	29	28	38	39	61
41	51	22	44	45	19	54	48
49	15	42	52	53	47	10	56
8	58	59	33	40	62	63	1

FIG. 4

64	2	3	61	60	6	7	57
9	55	54	12	13	51	50	16
17	47	46	20	21	43	42	24
40	26	27	37	36	30	31	33
32	34	35	29	28	38	39	25
41	23	22	44	45	19	18	48
49	15	14	52	53	11	10	56
8	58	59	5	4	62	63	1

FIG. 5

144	2	3	4	5	6	7	8	9	10	11	138
13	131	15	16	17	18	19	20	21	22	122	24
25	26	118	28	29	30	31	32	33	111	35	36
37	38	39	105	41	42	43	44	100	46	47	48
49	50	51	52	92	54	55	89	57	58	59	60
61	62	63	64	65	70	78	68	69	70	71	72
73	74	75	76	77	67	66	60	81	82	83	84
85	86	87	88	51	90	91	53	93	94	95	96
97	98	99	45	101	102	103	104	40	106	107	108
109	110	34	112	113	114	115	116	117	27	119	120
121	23	123	124	125	126	127	128	128	130	14	131
12	134	135	136	137	138	139	140	141	142	143	1

FIG. 6

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Latin square

In the combinatorics and statistics, a **Latin square** is an $n \times n$ table filled with n different symbols in such a way that each symbol occurs exactly once in each row and exactly once in each column. Here is an example:

Displaying a 7×7 Latin square, this stained glass window honors Ronald Fisher, whose *Design of Experiments* discussed Latin squares. Fisher's student, A. W. F. Edwards, designed this window for Caius College, Cambridge.

1	2	3
2	3	1
3	1	2

In the design of experiments, Latin squares are a special case of ***row-column* designs** for two blocking factors:[1] Many row-column designs are constructed by concatenating Latin squares.[2] In algebra, Latin squares are generalizations of groups; in fact, Latin squares are characterized as being the multiplication tables (Cayley tables) of quasigroups. Other applications include error correcting codes.

The name Latin square originates from Leonhard Euler, who used Latin characters as symbols.

A Latin square is said to be *reduced* (also, *normalized* or *in standard form*) if its first row and first column are in natural order. For example, the Latin square above is reduced because both its first row and its first column are 1,2,3 (rather than 3,1,2 or any other order). We can make any Latin square reduced by permuting (reordering) the rows and columns.

Properties

Orthogonal array representation

If each entry of an $n \times n$ Latin square is written as a triple (r,c,s), where r is the row, c is the column, and s is the symbol, we obtain a set of n^2 triples called the orthogonal array representation of the square. For example, the orthogonal array representation of the first Latin square displayed above is:

{ (1,1,1),(1,2,2),(1,3,3),(2,1,2),(2,2,3),(2,3,1),(3,1,3),(3,2,1),(3,3,2) },

where for example the triple (2,3,1) means that in row 2 and column 3 there is the symbol 1. The definition of a Latin square can be written in terms of orthogonal arrays:

- A Latin square is the set of all triples (r,c,s), where $1 \le r, c, s \le n$, such that all ordered pairs (r,c) are distinct, all ordered pairs (r,s) are distinct, and all ordered pairs (c,s) are distinct.

For any Latin square, there are n^2 triples since choosing any two uniquely determines the third. (Otherwise, an ordered pair would appear more than once in the Latin square.)

The orthogonal array representation shows that rows, columns and symbols play rather similar roles, as will be made clear below.

Equivalence classes of Latin squares

Many operations on a Latin square produce another Latin square (for example, turning it upside down).

If we permute the rows, permute the columns, and permute the names of the symbols of a Latin square, we obtain a new Latin square said to be *isotopic* to the first. Isotopism is an equivalence relation, so the set of all Latin squares is divided into subsets, called *isotopy classes*, such that two squares in the same class are isotopic and two squares in different classes are not isotopic.

Another type of operation is easiest to explain using the orthogonal array representation of the Latin square. If we systematically and consistently reorder the three items in each triple, another orthogonal array (and, thus, another Latin square) is obtained. For example, we can replace each triple (r,c,s) by (c,r,s) which corresponds to transposing the square (reflecting about its main diagonal), or we could replace each triple (r,c,s) by (c,s,r), which is a more complicated operation. Altogether there are 6 possibilities including "do nothing", giving us 6 Latin squares called the conjugates (also parastrophes) of the original square.

Finally, we can combine these two equivalence operations: two Latin squares are said to be paratopic, also main class isotopic, if one of them is isotopic to a conjugate of the other. This is again an equivalence relation, with the equivalence classes called main classes, *species*, or paratopy classes. Each main class contains up to 6 isotopy classes.

Number

There is no known easily-computable formula for the number $L(n)$ of $n \times n$ Latin squares with symbols 1,2,...,n. The most accurate upper and lower bounds known for large n are far apart. One classic result is

$$\prod_{k=1}^{n} (k!)^{n/k} \geq L(n) \geq \frac{(n!)^{2n}}{n^{n^2}}$$

(this given by van Lint and Wilson).

Here we will give all the known exact values. It can be seen that the numbers grow exceedingly quickly. For each n, the number of Latin squares altogether (sequence A002860 [3] in OEIS) is n! (n-1)! times the number of reduced Latin squares (sequence A000315 [4] in OEIS).

The numbers of Latin squares of various sizes

n	*reduced Latin squares of size n*	*all Latin squares of size n*
1	1	1
2	1	2
3	1	12
4	4	576
5	56	161280
6	9408	812851200
7	16942080	61479419904000
8	535281401856	108776032459082956800
9	377597570964258816	5524751496156892842531225600
10	7580721483160132811489280	9982437658213039871725064756920320000
11	5363937773277371298119673540771840	776966836171770144107444346734230682311065600000

For each *n*, each isotopy class (sequence A040082 [5] in OEIS) contains up to $(n!)^3$ Latin squares (the exact number varies), while each main class (sequence A003090 [6] in OEIS) contains either 1, 2, 3 or 6 isotopy classes.

Equivalence classes of Latin squares

n	*main classes*	*isotopy classes*
1	1	1
2	1	1
3	1	1
4	2	2
5	2	2
6	12	22
7	147	564
8	283657	1676267
9	19270853541	115618721533
10	34817397894749939	208904371354363006
11	2036029552582883134196099	12216177315369229261482540

Examples

We give one example of a Latin square from each main class up to order 5.

$$\begin{bmatrix}1\end{bmatrix} \quad \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix} \quad \begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\3 & 1 & 2\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 3 & 4\\2 & 1 & 4 & 3\\3 & 4 & 1 & 2\\4 & 3 & 2 & 1\end{bmatrix} \quad \begin{bmatrix}1 & 2 & 3 & 4\\2 & 4 & 1 & 3\\3 & 1 & 4 & 2\\4 & 3 & 2 & 1\end{bmatrix}$$

$$\begin{bmatrix}1&2&3&4&5\\2&3&5&1&4\\3&5&4&2&1\\4&1&2&5&3\\5&4&1&3&2\end{bmatrix}\quad\begin{bmatrix}1&2&3&4&5\\2&4&1&5&3\\3&5&4&2&1\\4&1&5&3&2\\5&3&2&1&4\end{bmatrix}$$

They present, respectively, the multiplication tables of the following groups:

- $\{0\}$ – the trivial 1-element group
- $\mathbb{Z}_2$ – the binary group
- $\mathbb{Z}_3$ – cyclic group of order 3
- $\mathbb{Z}_2 \times \mathbb{Z}_2$ – the Klein four-group
- $\mathbb{Z}_4$ – cyclic group of order 4
- $\mathbb{Z}_5$ – cyclic group of order 5
- the last one is an example of a quasigroup, or rather a loop, which is not associative.

Applications

Error correcting codes

Sets of Latin squares that are orthogonal to each other have found an application as error correcting codes in situations where communication is disturbed by more types of noise than simple white noise, such as when attempting to transmit broadband Internet over powerlines.[7] [8] [9]

Firstly, the message is sent by using several frequencies, or channels, a common method that makes the signal less vulnerable to noise at any one specific frequency. A letter in the message to be sent is encoded by sending a series of signals at different frequencies at successive time intervals. In the example below, the letters A to L are encoded by sending signals at four different frequencies, in four time slots. The letter C for instance, is encoded by first sending at frequency 3, then 4, 1 and 2.

$$\begin{array}{c}A\\B\\C\\D\end{array}\begin{bmatrix}1&2&3&4\\2&1&4&3\\3&4&1&2\\4&3&2&1\end{bmatrix}\quad\begin{array}{c}E\\F\\G\\H\end{array}\begin{bmatrix}1&3&4&2\\2&4&3&1\\3&1&2&4\\4&2&1&3\end{bmatrix}\quad\begin{array}{c}I\\J\\K\\L\end{array}\begin{bmatrix}1&4&2&3\\2&3&1&4\\3&2&4&1\\4&1&3&2\end{bmatrix}$$

The encoding of the twelve letters are formed from three Latin squares that are orthogonal to each other. Now imagine that there's added noise in channels 1 and 2 during the whole transmission. The letter A would then be picked up as:

$$12\quad 12\quad 123\quad 124$$

In other words, in the first slot we receive signals from both frequency 1 and frequency 2; while the third slot has signals from frequencies 1, 2 and 3. Because of the noise, we can no longer tell if the first two slots were 1,1 or 1,2 or 2,1 or 2,2. But the 1,2 case is the only one that yields a sequence matching a letter in the above table, the letter A. Similarly, we may imagine a burst of static over all frequencies in the third slot:

$$1\quad 2\quad 1234\quad 4$$

Again, we are able to infer from the table of encodings that it must have been the letter A being transmitted. The number of errors this code can spot is one less than the number of time slots. It has also been proved that if the number of frequencies is a prime or a power of a prime, the orthogonal Latin squares produce error detecting codes that are as efficient as possible.

Mathematical puzzles

The problem of determining if a partially filled square can be completed to form a Latin square is NP-complete.[10]

The popular *Sudoku* puzzles are a special case of Latin squares; any solution to a *Sudoku* puzzle is a Latin square. *Sudoku* imposes the additional restriction that nine particular 3×3 adjacent subsquares must also contain the digits 1–9 (in the standard version). The more recent KenKen puzzles are also examples of latin squares.

Heraldry

The Latin square also figures in the blazon of the arms of the Statistical Society of Canada.[11] Also, it appears in the logo of the International Biometric Society.[12]

See also

- Latin hypercube sampling
- Graeco-Latin square
- Magic Square
- Problems in Latin squares
- Small Latin squares and quasigroups
- Mathematics of Sudoku
- Futoshiki
- Rook's graph, a graph that has Latin squares as its colorings.
- Eight queens puzzle
- Block design
- Word square

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External links

- Latin Squares (http://eom.springer.de/L/l057620.htm) in the Encyclopaedia of Mathematics
- Latin Squares in Java (http://www.cut-the-knot.org/Curriculum/Algebra/Latin.shtml) at cut-the-knot
- Infinite Latin Square (Java) (http://www.cut-the-knot.org/Curriculum/Combinatorics/InfiniteLatinSquare.shtml) at cut-the-knot
- Magic Square in Latin Square (http://www.muljadi.org/MagicSudoku.htm)

Graeco-Latin square

In mathematics, a **Graeco-Latin square** or **Euler square** or **orthogonal Latin squares** of order *n* over two sets *S* and *T*, each consisting of *n* symbols, is an $n \times n$ arrangement of cells, each cell containing an ordered pair (s,t), where $s \in S$ and a $t \in T$, such that every row and every column contains each $s \in \mathrm{S}$ exactly once, and that no two cells contain the same ordered pair of symbols.

Aα	Bγ	Cβ
Bβ	Cα	Aγ
Cγ	Aβ	Bα

Orthogonal Latin squares of order 3

The arrangement of the Latin characters alone and of the Greek characters alone each forms a Latin square. A Graeco-Latin square can therefore be decomposed into two "orthogonal" Latin squares. Orthogonality here means that every pair (s, t) from the Cartesian product $S \times T$ occurs exactly once.

Aα	Bδ	Cβ	Dε	Eγ
Bβ	Cε	Dγ	Eα	Aδ
Cγ	Dα	Eδ	Aβ	Bε
Dδ	Eβ	Aε	Bγ	Cα
Eε	Aγ	Bα	Cδ	Dβ

Orthogonal Latin squares of order 5

History

Orthogonal Latin squares have been known to predate Euler. As described by Donald Knuth in Volume 4 of TAOCP, the construction of 4x4 set was published by Jacques Ozanam in 1725 (in *Recreation mathematiques et physiques*) as a puzzle involving playing cards. The problem was to take all aces, kings, queens and jacks from a standard deck of cards, and arrange them in a 4x4 grid such that each row and each column contained all four suits as well as one of each face value. This problem has several solutions.

A common variant of this problem was to arrange the 16 cards so that, in addition to the row and column constraints, each diagonal contains all four face values and all four suits as well. As described by Martin Gardner in *Gardner's Workout*, the number of distinct solutions to this problem was incorrectly estimated by Rouse Ball to be 72, and persisted many years before it was shown to be 144 by Kathleen Ollerenshaw. Each of the 144 solutions has 8 reflections and rotations, giving 1152 solutions in total. The 144x8 solutions can be categorized into the following two classes:

Solution	Normal form
Solution #1	A♠ K♥ Q♦ J♣ Q♣ J♦ A♥ K♠ J♥ Q♠ K♣ A♦ K♦ A♣ J♠ Q♥
Solution #2	A♠ K♥ Q♦ J♣ J♦ Q♣ K♠ A♥ K♣ A♦ J♥ Q♠ Q♥ J♠ A♣ K♦

For each of the two solutions, 24x24 = 576 solutions can be derived by permuting the four suits and the four face values independently. No permutation will convert the two solutions into each other.

The solution set can be seen to be complete through this proof outline:

1. Without loss of generality, let us choose the card in the top left corner to be A♠.
2. Now, in the second row, the first two squares can be neither ace nor spades, due to being on the same column or diagonal respectively. Therefore, one of the remaining two squares must be an ace, and the other must be a spade, since the card A♠ itself cannot be repeated.
3. If we choose the cell in the second row, third column to be an ace, and propagate the constraints, we will get Solution #1 above, up to a permutation of the remaining suits and face values.
4. Conversely, if we choose the (2,3) cell to be a spade, and propagate the constraints, we will get Solution #2 above, up to a permutation of the remaining suits and face values.
5. Since no other possibilities exist for (2,3), the solution set is complete.

Euler's work and conjecture

Orthogonal Latin squares were studied in detail by Leonhard Euler, who took the two sets to be $S = \{A, B, C, \ldots\}$, the first n upper-case letters from the Latin alphabet, and $T = \{\alpha , \beta, \gamma, \ldots\}$, the first n lower-case letters from the Greek alphabet—hence the name Graeco-Latin square.

In the 1780s Euler demonstrated methods for constructing Graeco-Latin squares where n is odd or a multiple of 4. Observing that no order-2 square exists and unable to construct an order-6 square (see thirty-six officers problem), he conjectured that none exist for any oddly even number $n \equiv 2 \pmod 4$. Indeed, the non-existence of order-6 squares was definitely confirmed in 1901 by Gaston Tarry through exhaustive enumeration of all possible arrangements of symbols. However, Euler's conjecture resisted solution for a very long time.

Counterexamples to the conjecture of Euler

In 1959, R.C. Bose and S. S. Shrikhande constructed some counterexamples (dubbed the *Euler spoilers*) of order 22 using mathematical insights. Then E. T. Parker found a counterexample of order 10 through computer search on UNIVAC (this was one of the earliest combinatorics problems solved on a digital computer).

In 1960, Parker, Bose, and Shrikhande showed Euler's conjecture to be false for all $n \geq 10$. Thus, Graeco-Latin squares exist for all orders $n \geq 3$ except $n = 6$.

Applications

Graeco-Latin squares are used in the design of experiments, tournament scheduling and constructing magic squares. The French writer Georges Perec structured his 1978 novel Life: A User's Manual around a 10×10 orthogonal square.

Mutually orthogonal Latin squares

Mutually orthogonal Latin squares arise in various problems. A set of Latin squares is called mutually orthogonal if every pair of its element Latin squares is orthogonal to each other.

Any two of text, foreground color, background color and typeface form a pair of orthogonal Latin squares:

fjords	jawbox	phlegm	qiviut	zincky
zincky	fjords	jawbox	phlegm	qiviut
qiviut	zincky	fjords	jawbox	phlegm
phlegm	qiviut	zincky	fjords	jawbox
jawbox	phlegm	qiviut	zincky	fjords

The above table shows 4 mutually orthogonal Latin squares of order 5, representing respectively:

- the text: *fjords*, *jawbox*, *phlegm*, *qiviut*, and *zincky*
- the foreground color: white, red, lime, blue, and yellow
- the background color: black, maroon, teal, navy, and silver
- the typeface: serif (Georgia / Times Roman), sans-serif (Verdana / Helvetica), monospace (Courier New), cursive (Comic Sans), and fantasy (Impact).

Due to the Latin square property, each row and each column has all five texts, all five foregrounds, all five backgrounds, and all five typefaces.

Due to the mutually orthogonal property, there is exactly one instance somewhere in the table for any pair of elements, such as (white foreground, monospace), or (*fjords*, navy background) etc., and also all possible such pairs of values and dimensions are represented exactly once each.

The above table therefore allows for testing 5 values each of 4 different dimensions in only 25 observations instead of 625 observations. Note that the five 6-letter words (*fjords*, *jawbox*, *phlegm*, *qiviut*, and *zincky*) between them cover all 26 letters of the alphabet at least once each. The table therefore allows for examining each letter of the alphabet in five different typefaces, foreground colors, and background colors.

Due to a close relation between orthogonal Latin squares and combinatorial designs, every pair of distinct cells in the 5x5 table will have exactly one of the following properties in common:

- a common row, or
- a common column, or
- a common text, or
- a common typeface, or
- a common background color, or
- a common foreground color.

In each category, every cell has 4 neighbors (4 neighbors in the same row with nothing else in common, 4 in the same column, etc.), giving 6 * 4 = 24 neighbors, which makes it a complete graph with 6 different edge colors.

The number of mutually orthogonal latin squares

The number of mutually orthogonal Latin squares that may exist for a given order n is not known for general n, and is an area of research in combinatorics. It is known that the maximum number of MOLS for any n cannot exceed (n-1), and this upper bound is achieved when n is a power of a prime number. The minimum is known to be 2 for all n except for n = 1, 2 or 6, where it is 1. For general composite numbers, the number of MOLS is not known. The first few values starting with n = 2, 3, 4... are 1, 2, 3, 4, 1, 6, 7, 8, ... (sequence A001438 [1] in OEIS).

See also

- Block design
- Blocking (statistics)
- Combinatorial design
- Design of experiments
- Hyper-Graeco-Latin square design
- Randomized block design

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External links

- Euler's work on Latin Squares and Euler Squares [2] at Convergence [2]
- Java Tool which assists in constructing Graeco-Latin squares (it does not construct them by itself) [3] at cut-the-knot
- *Anything but square: from magic squares to Sudoku* [4]

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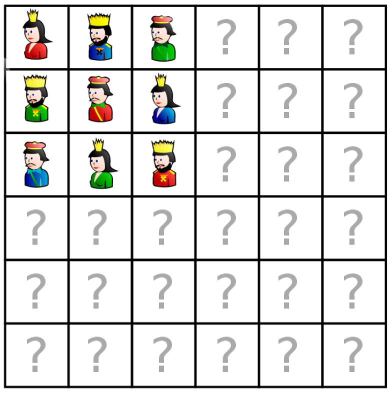
Thirty-six officers problem



The **thirty-six officers problem** is a mathematical puzzle proposed by Leonhard Euler in 1782.[1] [2]

The problem asks if it is possible to arrange 6 regiments consisting of 6 officers each of different ranks in a 6 × 6 square so that no rank or regiment will be repeated in any row or column. Such an arrangement would form a Graeco-Latin square. Euler correctly conjectured there was no solution to this problem, and Gaston Tarry proved this in 1901;[3] but the problem has led to important work in combinatorics.[4]

Besides the 6 by 6 case the only other case where the equivalent problem has no solution is the 2 by 2 case, i.e. when there are 4 officers.

References

[1] Euler, L., *Recherches sur une nouvelle espece de quarres magiques* (1782).

[2] P. A. MacMahon (1902). "Magic Squares and Other Problems on a Chess Board" (http://books.google.com/?id=cuudxJgEnyEC&pg=PA54&dq=euler+36-officers). *Proceedings of the Royal Institution of Great Britain* **XVII**: 50–63. .

[3] G. Terry (1900–1901). "Le Probléme de 36 Officiers" (http://books.google.com/?id=qzkDAAAAIAAJ&q=36+"Recherches+sur+une+nouvelle+espece+de+quarres+magiques"&dq=36+"Recherches+sur+une+nouvelle+espece+de+quarres+magiques"). *Comptes Rendu de l' Association Française pour l' Avancement de Science Naturel* (National Academy of Sciences) **1**: 122–123 & 2170–2203. .

[4] Dougherty, Steven. "36 Officer Problem." Steven Dougherty's Euler Page (http://academic.scranton.edu/faculty/doughertys1/euler.htm). 4 Aug 2006.

External links

- Euler's Officer Problem (http://mathdl.maa.org/convergence/1/?pa=content&sa=viewDocument&nodeId=1434&bodyId=1599) at Convergence
- (http://www.thinkfun.com/PRODUCT.ASPX?PageNo=PRODUCT&Catalog=By+Category&Category=7SERPZLR&ProductId=6830) Cube36. A puzzle that uses this problem.

Magic series

A **magic series** is a set of distinct positive numbers which add up to the magic sum of a magic square, thus potentially making up a line in a magic square.

So, in an $n \times n$ magic square using the numbers from 1 to n^2, a magic series is a set of n distinct numbers adding up to $n(n^2+1)/2$. For $n = 2$, there are just two magic series, 1+4 and 2+3, and there is no magic square. The eight magic series when $n = 3$ all appear in the rows, columns and diagonals of a 3×3 magic square.

Maurice Kraitchik gave the number of magic series up to $n = 7$ in *Mathematical Recreations* in 1942 (sequence A052456 [1] in OEIS). In 2002, Henry Bottomley extended this up to $n = 36$ and independently Walter Trump up to $n = 32$. In 2005, Trump extended this to $n = 54$ (over 2×10^{111}) while Bottomley gave an experimental approximation for the numbers of magic series:

$$\frac{1}{\pi} \cdot \sqrt{\frac{3}{e}} \cdot \frac{(en)^n}{n^3 - \frac{3}{5}n^2 + \frac{2}{7}n}$$

In July 2006, Robert Gerbicz extended this sequence up to $n = 150$.

External links

- Walter Trump's pages on magic series [2]
- Number of magic series up to order 150 [3]

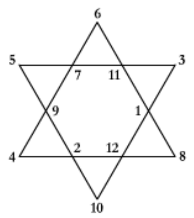
References

[1] http://en.wikipedia.org/wiki/Oeis%3Aa052456
[2] http://www.trump.de/magic-squares/magic-series/index.html
[3] http://robert.gerbicz.googlepages.com/magic.txt

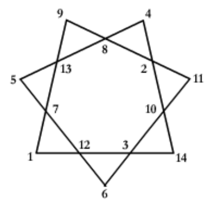
Magic star

An *n*-pointed **magic star** is a star polygon with Schläfli symbol {*n*/2} in which numbers are placed at each of the *n* vertices and *n* intersections, such that the four numbers on each line sum to the same magic constant. A **normal** magic star contains the consecutive integers 1 to 2*n*. The magic constant of an *n*-pointed normal magic star is $M = 4n + 2$.

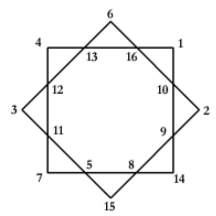
No star polygons with fewer than 5 points exist, and the construction of a normal 5-pointed magic star turns out to be impossible. The smallest examples of normal magic stars are therefore 6-pointed. Some examples are given below. Notice that for specific values of *n*, the *n*-pointed magic stars are also known as *magic hexagram* etc.



Magic hexagram
$M = 26$



Magic heptagram
$M = 30$



Magic octagram
$M = 34$

See also

- Magic square

External links

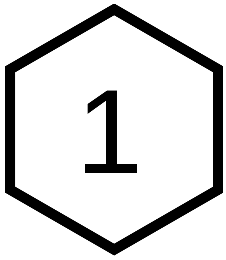
- Marian Trenkler's Magic Stars [1]
- Gianni Sarcone's Magic Star [2]

References

[1] http://math.ku.sk/~trenkler/MagicStars.doc
[2] http://www.archimedes-lab.org/pzm58b.html

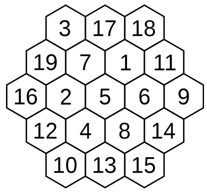
Magic hexagon

A **magic hexagon** of order *n* is an arrangement of numbers in a centered hexagonal pattern with *n* cells on each edge, in such a way that the numbers in each row, in all three directions, sum to the same magic constant. A **normal** magic hexagon contains the consecutive integers from 1 to $3n^2 - 3n + 1$. It turns out that magic hexagons exist only for $n = 1$ (which is trivial) and $n = 3$. Moreover, the solution of order 3 is essentially unique.[1] Meng also gave a less intricate constructive proof.[2]



Order 1

$M = 1$

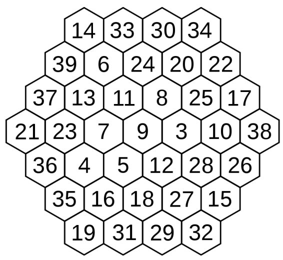


Order n=3

$M = 38$

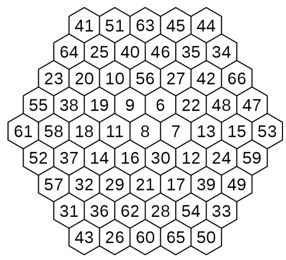
The order-3 magic hexagon has been published many times as a 'new' discovery. An early reference, and possibly the first discoverer, is Ernst von Haselberg (1887).

Although there are no normal magical hexagons with order greater than 3, certain abnormal ones do exist. In this case, abnormal means starting the sequence of numbers other than with 1. Arsen Zahray discovered these order 4 and 5 hexagons:



Order 4

$M = 111$

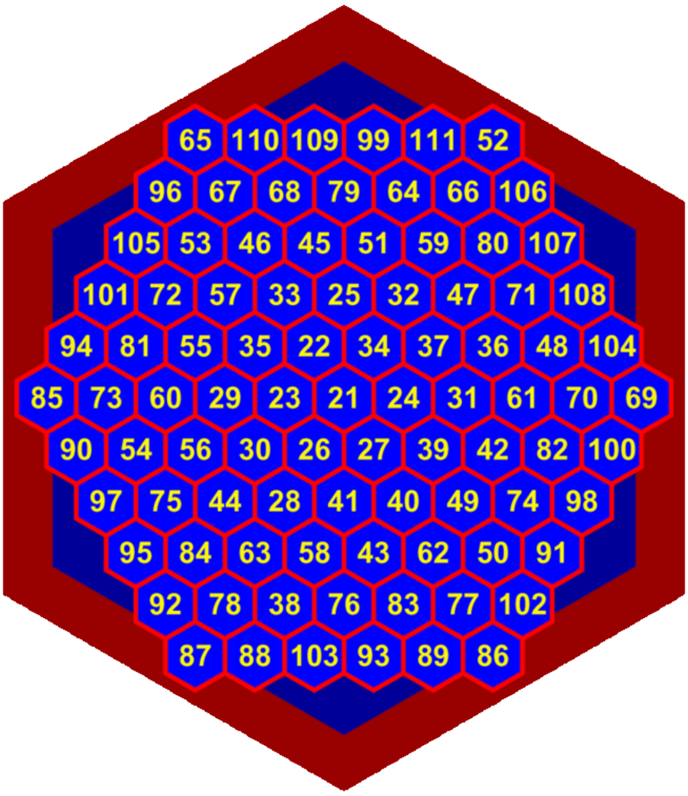


Order 5

$M = 244$

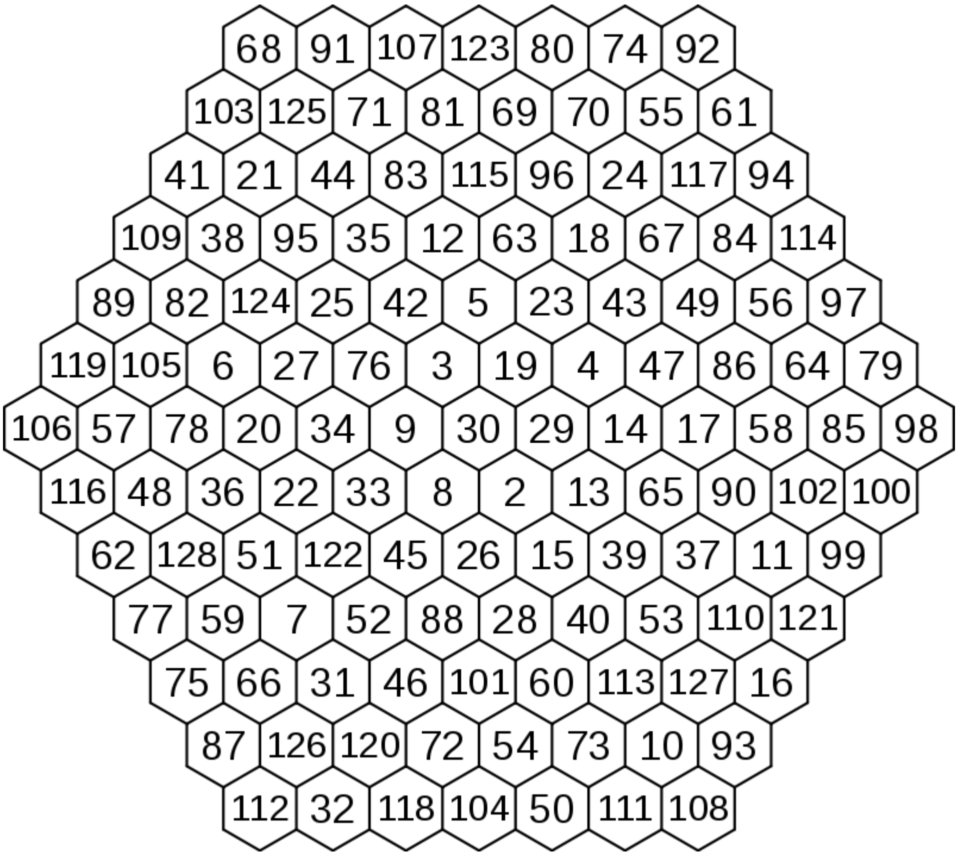
The order 4 hexagon starts with 3 and ends with 39, its rows summing to 111. The order 5 hexagon starts with 6 and ends with 66 and sums to 244.

An order 6 hexagon can be seen below. It was created by Louis Hoelbling, October 11, 2004:



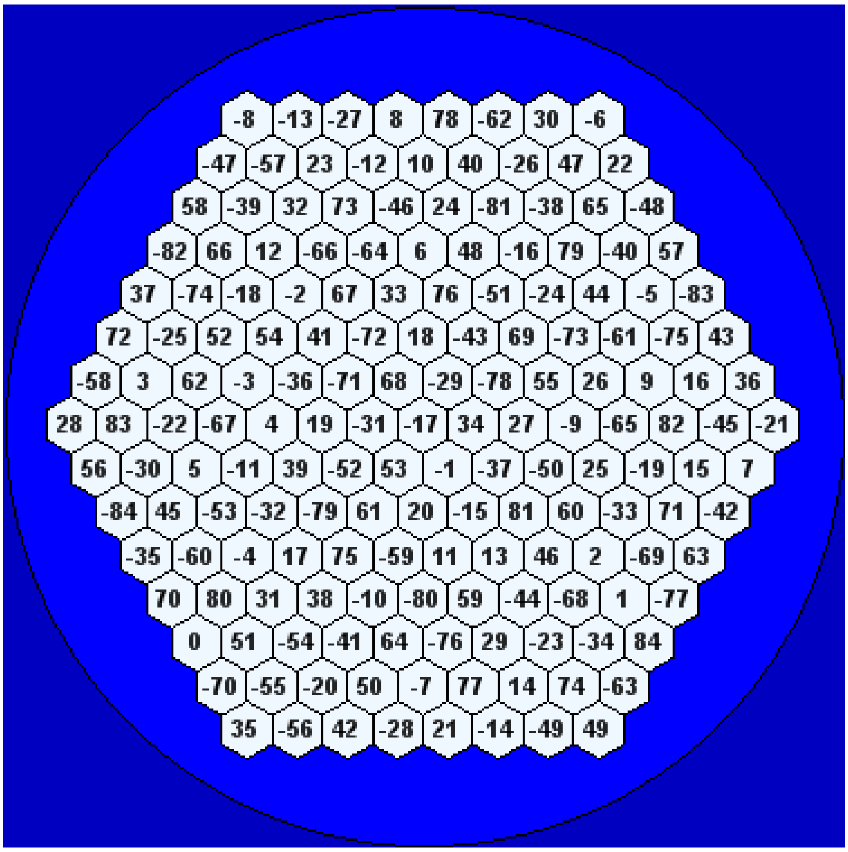
It starts with 21, ends with 111, and its sum is 546.

The largest magic hexagon so far was discovered using simulated annealing by Arsen Zahray on 22 March 2006:



It starts with 2, ends with 128 and its sum is 635.

However, a slightly larger, order 8 magic hexagon was generated by Louis K. Hoelbling on February 5, 2006:



It starts with -84 and ends with 84, and its sum is 0.

Proof

Here is a proof sketch that no normal magic hexagons exist except those of order 1 and 3.

The magic constant *M* of a normal magic hexagon can be determined as follows. The numbers in the hexagon are consecutive, and run from 1 to (3n^2-3n+1). Hence their sum is a triangular number, namely

There are $r = (2n - 1)$ rows running along any given direction (E-W, NE-SW, or NW-SE). Each of these rows sum up to the same number *M*. Therefore:

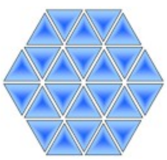
Rewriting this as

$$32M = 72n^3 - 108n^2 + 90n - 27 + \frac{5}{2n-1}$$

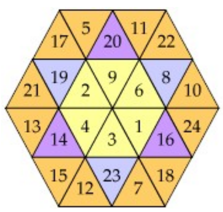
shows that $5/(2n - 1)$ must be an integer. The only $n \geq 1$ that meet this condition are $n = 1$ and $n = 3$.

Another type of magic hexagon

Hexagons can also be constructed with triangles, as the following diagrams show.



Order 2



Order 2 with numbers 1–24

This type of configuration can be called a T-hexagon and it has many more properties than the hexagon of hexagons.

As with the above, the rows of triangles run in three directions and there are 24 triangles in a T-hexagon of order 2. In general, a T-hexagon of order *n* has $6n^2$triangles. The sum of all these numbers is given by:

$$S = \frac{6n^2(6n^2+1)}{2}$$

If we try to construct a magic T-hexagon of side *n*, we have to choose *n* to be even, because there are *r* = 2*n* rows so the sum in each row must be

$$M = \frac{S}{R} = \frac{3n^2(6n^2+1)}{2n}$$

For this to be an integer, *n* has to be even. To date, magic T-hexagons of order 2, 4, 6 and 8 have been discovered. The first was a magic T-hexagon of order 2, discovered by John Baker on 13 September 2003. Since that time, John has been collaborating with David King, who discovered that there are 59,674,527 non-congruent magic T-hexagons of order 2.

Magic T-hexagons have a number of properties in common with magic squares, but they also have their own special features. The most surprising of these is that the sum of the numbers in the triangles that point upwards is the same as the sum of those in triangles that point downwards (no matter how large the T-hexagon). In the above example,

$$\begin{aligned}&17+20+22+21+2+6+10+14+3+16+12+7\\&=5+11+19+9+8+13+4+1+24+15+23+18\\&=150\end{aligned}$$

To find out more about magic T-hexagons, visit Hexagonia [3] or the Hall of Hexagons [4].

Notes

[1] <Trigg, C. W. "A Unique Magic Hexagon" (http://www.mathematik.uni-bielefeld.de/~sillke/PUZZLES/magic-hexagon-trigg), *Recreational Mathematics Magazine*, January–February 1964. Retrieved on 2009-12-16.

[2] <Meng, F. "Research into the Order 3 Magic Hexagon" (http://www.yau-awards.org/English/N/N92-Research into the Order 3 Magic Hexagon.pdf), *Shing-Tung Yau Awards*, October 2008. Retrieved on 2009-12-16.

[3] http://www.naturalmaths.com.au/hexagonia

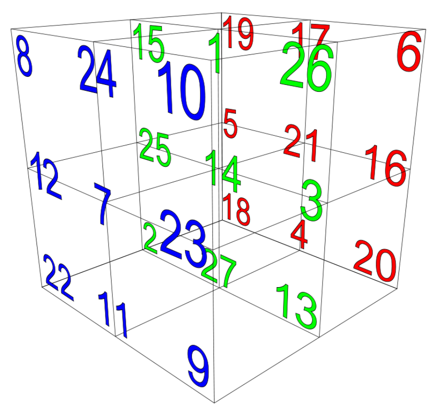
[4] http://www.drking.plus.com/hexagons/

References

- Baker. J. E. and King, D. R. (2004) "The use of visual schema to find properties of a hexagon" Visual Mathematics, Volume 5, Number 3
- Baker, J. E. and Baker, A. J. (2004) "The hexagon, nature's choice" Archimedes, Volume 4

Magic cube

In mathematics, a **magic cube** is the 3-dimensional equivalent of a magic square, that is, a number of integers arranged in a n x n x n pattern such that the sum of the numbers on each row, each column, each pillar and the four main space diagonals is equal to a single number, the so-called magic constant of the cube, denoted $M_3(n)$. It can be shown that if a magic cube consists of the numbers 1, 2, ..., n^3, then it has magic constant (sequence A027441 [1] in OEIS)



An example of a 3 × 3 × 3 magic cube. In this example, no slice is a magic square. In this case, the cube is classed as a simple magic cube.

$$M_3(n) = \frac{n(n^3+1)}{2}.$$

If, in addition, the numbers on every cross section diagonal also sum up to the cube's magic number, the cube is called a perfect magic cube; otherwise, it is called a semiperfect magic cube. The number n is called the order of the magic cube. If the sums of numbers on a magic cube's broken space diagonals also equal the cube's magic number, the cube is called a pandiagonal cube.

Alternate definition

In recent years, an alternate definition for the perfect magic cube has gradually come into use. It is based on the fact that a pandiagonal magic square has traditionally been called **perfect**, because all possible lines sum correctly. This is not the case with the above definition for the cube.

Bimagic cubes and beyond

As in the case of magic squares, a bimagic cube has the additional property of remaining a magic cube when all of the entries are squared, a trimagic cube remains a magic cube under both the operations of squaring the entries and of cubing the entries. (Only two of these are known, as of 2005.) A tetramagic cube remains a magic cube when the entries are squared, cubed, or raised to the fourth power.

Magic cubes based on Dürer's and Gaudi Magic squares

A magic cube can be built with the constraint of a given magic square appearing on one of its faces Magic cube with the magic square of Dürer [2], and Magic cube with the magic square of Gaudi [3]

See also

- Perfect magic cube
- Semiperfect magic cube
- Multimagic cube
- Magic tesseract
- Magic hypercube
- Magic hypercubes
- Magic cube class
- Asymptotic magic hyper-tesseract
- Nasik magic hypercube
- John R. Hendricks

External links

- MathWorld: Magic Cube [4]
- Harvey Heinz: All about Magic Cubes [5]
- Marian Trenkler: Magic p-dimensional cubes [6]
- Marian Trenkler: An algorithm for making magic cubes [7]
- Ali Skalli's magic squares and magic cubes [8]

References

[1] http://en.wikipedia.org/wiki/Oeis%3Aa027441
[2] http://sites.google.com/site/aliskalligvaen/home-page/-magic-cube-with-duerer-s-square
[3] http://sites.google.com/site/aliskalligvaen/home-page/-magic-cube-with-gaudi-s-square
[4] http://mathworld.wolfram.com/MagicCube.html
[5] http://members.shaw.ca/hdhcubes/index.htm
[6] http://math.ku.sk/~trenkler/aa-cub-01.pdf
[7] http://math.ku.sk/~trenkler/05-MagicCube.pdf
[8] http://sites.google.com/site/aliskalligvaen/home-page

Magic cube classes

Every magic cube may be assigned to one of six **magic cube classes**, based on the cube characteristics.

This new system is more precise in defining magic cubes. But possibly of more importance, it is consistent for all orders and all dimensions of magic hypercubes.

Minimum requirements for a cube to be magic are: All rows, columns, pillars, and 4 triagonals must sum to the same value.

The six classes

- **Simple:**

The minimum requirements for a magic cube are: All rows, columns, pillars, and 4 triagonals must sum to the same value. A Simple magic cube contains no magic squares or not enough to qualify for the next class.
The smallest normal simple magic cube is order 3. Minimum correct summations required = $3m^2 + 4$

- **Diagonal:**

Each of the $3m$ planar arrays must be a simple magic square. The 6 oblique squares are also simple magic. The smallest normal diagonal magic cube is order 5.
These squares were referred to as 'Perfect' by Gardner and others! At the same time he referred to Langman's 1962 pandiagonal cube also as 'Perfect'.
Christian Boyer and Walter Trump now consider this *and* the next two classes to be *Perfect*. (See *Alternate Perfect* below).
A. H. Frost referred to all but the simple class as **Nasik** cubes.
The smallest normal diagonal magic cube is order 5. See Diagonal magic cube. Minimum correct summations required = $3m^2 + 6m + 4$

- **Pantriagonal:**

All $4m^2$ pantriagonals must sum correctly (that is 4 one-segment, $12(m-1)$ two-segment, and $4(m-2)(m-1)$ three-segment). There may be some simple AND/OR pandiagonal magic squares, but not enough to satisfy any other classification.
The smallest normal pantriagonal magic cube is order 4. See Pantriagonal magic cube.
Minimum correct summations required = $7m^2$. All pan-r-agonals sum correctly for r = 1 and 3.

- **PantriagDiag:**

A cube of this class was first constructed in late 2004 by Mitsutoshi Nakamura. This cube is a combination Pantriagonal magic cube and Diagonal magic cube. Therefore, all main and broken triagonals sum correctly, and it contains $3m$ planar simple magic squares. In addition, all 6 oblique squares are pandiagonal magic squares. The only such cube constructed so far is order 8. It is not known what other orders are possible. See Pantriagdiag magic cube.
Minimum correct summations required = $7m^2 + 6m$

- **Pandiagonal:**

ALL $3m$ planar arrays must be pandiagonal magic squares. The 6 oblique squares are always magic (usually simple magic). Several of them MAY be pandiagonal magic. Gardner also called this (Langman's pandiagonal) a 'perfect' cube, presumably not realizing it was a higher class then Myer's cube. See previous note re Boyer and Trump.
The smallest normal pandiagonal magic cube is order 7. See Pandiagonal magic cube.
Minimum correct summations required = $9m^2 + 4$. All pan-r-agonals sum correctly for r = 1 and 2.

- **Perfect:**

ALL $3m$ planar arrays must be pandiagonal magic squares. In addition, ALL pantriagonals must sum correctly. These two conditions combine to provide a total of 9m pandiagonal magic squares.

The smallest normal perfect magic cube is order 8. See Perfect magic cube.

Nasik; A. H. Frost (1866) referred to all but the simple magic cube as Nasik!
C. Planck (1905) redefined *Nasik* to mean magic hypercubes of any order or dimension in which all possible lines summed correctly.
i.e. ***Nasik*** is a **preferred alternate**, and less ambiguous term for the *perfect* class.
Minimum correct summations required = $13m^2$. All pan-r-agonals sum correctly for r = 1, 2 and 3.

Alternate Perfect Note that the above is a relatively new definition of *perfect*. Until about 1995 there was much confusion about what constituted a *perfect* magic cube (see the discussion under **diagonal:**)
. Included below are references and links to discussions of the old definition
With the popularity of personal computers it became easier to examine the finer details of magic cubes. Also more and more work was being done with higher dimension magic Hypercubes. For example, John Hendricks constructed the world's first **Nasik** magic tesseract in 2000. Classed as a perfect magic tesseract by Hendricks definition.

Generalized for All Dimensions

A magic hypercube of dimension n is perfect if all pan-n-agonals sum correctly. Then all lower dimension hypercubes contained in it are also perfect.
For dimension 2, The Pandiagonal Magic Square has been called *perfect* for many years. This is consistent with the perfect (nasik) definitions given above for the cube. In this dimension, there is no ambiguity because there are only two classes of magic square, simple and perfect.
In the case of 4 dimensions, the magic tesseract, Mitsutoshi Nakamura has determined that there are 18 classes. He has determined their characteristics and constructed examples of each. And in this dimension also, the *Perfect (nasik)* magic tesseract has all possible lines summing correctly and all cubes and squares contained in it are also nasik magic.

Another definition and a table

Proper: A Proper magic cube is a magic cube belonging to one of the six classes of magic cube, but containing exactly the minimum requirements for that class of cube. i.e. a proper simple or pantriagonal magic cube would contain no magic squares, a proper diagonal magic cube would contain exactly $3m + 6$ simple magic squares, etc. This term was coined by Mitsutoshi Nakamura in April, 2004.

Class of magic cube	Smallest possible order	Lines summing correctly to S $S = \frac{m(m^3+1)}{2}$				Simple magic squares		Pandiagonal (Nasik) magic squares	
(r-agonal ->)		Ortho. 1	Diag. 2	Triag. 3	Total	Planar	Oblique	Planar	Oblique
Simple	3	$3m^2$	--	4	$3m^2 + 4$	--	--	--	--
Diagonal	5	$3m^2$	$6m$	4	$3m^2 + 6m + 4$	$3m$	6	--	--
Pantriagonal	4	$3m^2$	--	$4m^2$	$7m^2$	--	--	--	--
PantriagDiag	8?	$3m^2$	$6m$	$4m^2$	$7m^2 + 6m$	$3m$	0	--	6
Pandiagonal	7	$3m^2$	$6m^2$	4	$9m^2 + 4$	--	6	$3m$	--
Perfect (Nasik)	8	$3m^2$	$6m^2$	$4m^2$	$13m^2$	--	--	$3m$	$6m$

Minimium lines (and magic squares) required for each class of magic cube

Notes for table

1. For the diagonal or pandiagonal classes, one or possibly 2 of the 6 oblique magic squares may be pandiagonal magic. All but 6 of the oblique squares are ‘broken’. This is analogous to the broken diagonals in a pandiagonal magic square. i.e. Broken diagonals are 1-D in a 2_D square; broken oblique squares are 2-D in a 3-D cube.
2. The table shows the minimum lines or squares required for each class (i.e. Proper). Usually there are more, but not enough of one type to qualify for the next class.

See also

- Magic hypercube
- Nasik magic hypercube
- Panmagic square
- Space diagonal
- John R. Hendricks

Further reading

- Frost, Dr. A. H., On the General Properties of Nasik Cubes, QJM 15, 1878, pp 93–123
- Planck, C., The Theory of Paths Nasik, Printed for private circulation, A.J. Lawrence, Printer, Rugby,(England), 1905
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- Hendricks, John R., Magic Squares to Tesseracts by Computer, Self-published 1999. 0-9684700-0-9
- Hendricks, John R., Perfect n-Dimensional Magic Hypercubes of Order 2n, Self-published 1999. 0-9684700-4-1
- Clifford A. Pickover (2002). *The Zen of Magic Squares, Circles and Stars*. Princeton Univ. Press, 2002, 0-691-07041-5. pp 101–121

External links

Cube classes

- Christian Boyer: Perfect Magic Cubes [1]
- Harvey Heinz: Perfect Magic Hypercubes [2]
- Harvey Heinz: 6 Classes of Cubes [3]
- Walter Trump: Search for Smallest [4]

Perfect Cube

- Aale de Winkel: Magic Encyclopedia [5]
- A long quote from C. Plank (1917) on the subject of *nasik* as a substitute term for *perfect*. [6]

Tesseract Classes

- The Square, Cube, and Tesseract Classes [7]

References

[1] http://cboyer.club.fr/multimagie/index.htm
[2] http://members.shaw.ca/hdhcubes/cube_perfect.htm
[3] http://members.shaw.ca/hdhcubes/index.htm#
[4] http://www.trump.de/magic-squares/magic-cubes/cubes-1.html
[5] http://www.magichypercubes.com/Encyclopedia/
[6] http://members.shaw.ca/hdhcubes/cube_define.htm#Theory%20of%20Paths%20Nasik
[7] http://members.shaw.ca/tesseracts/t_classes.htm

Perfect magic cube

In mathematics, a **perfect magic cube** is a magic cube in which not only the columns, rows, pillars and main space diagonals, but also the cross section diagonals sum up to the cube's magic constant.

Perfect magic cubes of order one are trivial; cubes of orders two to four can be proven not to exist, and cubes of orders five and six were first discovered by Walter Trump and Christian Boyer on November 13 and September 1, 2003, respectively. A perfect magic cube of order seven was given by A. H. Frost in 1866, and on March 11, 1875, an article was published in the Cincinnati Commercial newspaper on the discovery of a perfect magic cube of order 8 by Gustavus Frankenstein. Perfect magic cubes of orders nine and eleven have also been constructed. The first perfect cube of order 10 has been constructed in 1988. (Li Wen, China)

An alternative definition

In recent years, an alternative definition for the perfect magic cube was proposed by John R. Hendricks. It is based on the fact that a pandiagonal magic square has traditionally been called 'perfect', because all possible lines sum correctly. This is not the case with the above definition for the cube. See Nasik magic hypercube for an unambiguous alternative term

This same reasoning may be applied to hypercubes of any dimension. Simply stated; if all possible lines of m cells (m = order) sum correctly, the hypercube is perfect. All lower dimension hypercubes contained in this hypercube will then also be perfect. This is not the case with the original definition, which does not require that the planar and diagonal squares be a pandiagonal magic cube.

The original definition is applicable only to magic cubes, not tesseracts, dimension 5 cubes, etc.

Example: A perfect magic cube of order 8 has 244 correct lines by the **old** definition, but 832 correct lines by this **new** definition.

Order 8 is the smallest possible perfect magic cube. None can exist for double odd orders.

Gabriel Arnoux constructed an order 17 perfect magic cube in 1887. F.A.P.Barnard published order 8 and order 11 perfect cubes in 1888.

By the modern (Hendricks) definition, there are actually six classes of magic cube; simple magic cube, pantriagonal magic cube, diagonal magic cube, pantriagdiag magic cube, pandiagonal magic cube, and perfect magic cube.

Nasik; A. H. Frost (1866) referred to all but the simple magic cube as Nasik! C. Planck (1905) redefined Nasik to mean magic hypercubes of any order or dimension in which all possible lines summed correctly.

i.e. Nasik is an alternative, and unambiguous term for the perfect class of any dimension of magic hypercube.

First known Perfect Magic Cube

Walter Trump and Christian Boyer, 2003-11-13

This cube consists of all numbers from 1 to 125. The sum of the 5 numbers in each of the 25 rows, 25 columns, 25 pillars, 30 diagonals and 4 triagonals (space diagonals) equals the magic constant 315.

1° level - ***2° level***

$$\begin{bmatrix} 25 & 16 & 80 & 104 & 90 \\ 115 & 98 & 4 & 1 & 97 \\ 42 & 111 & 85 & 2 & 75 \\ 66 & 72 & 27 & 102 & 48 \\ 67 & 18 & 119 & 106 & 5 \end{bmatrix} \quad \begin{bmatrix} 91 & 77 & 71 & 6 & 70 \\ 52 & 64 & 117 & 69 & 13 \\ 30 & 118 & 21 & 123 & 23 \\ 26 & 39 & 92 & 44 & 114 \\ 116 & 17 & 14 & 73 & 95 \end{bmatrix}$$

3° level - ***4° level***

$$\begin{bmatrix} 47 & 61 & 45 & 76 & 86 \\ 107 & 43 & 38 & 33 & 94 \\ 89 & 68 & (63) & 58 & 37 \\ 32 & 93 & 88 & 83 & 19 \\ 40 & 50 & 81 & 65 & 79 \end{bmatrix} \quad \begin{bmatrix} 31 & 53 & 112 & 109 & 10 \\ 12 & 82 & 34 & 87 & 100 \\ 103 & 3 & 105 & 8 & 96 \\ 113 & 57 & 9 & 62 & 74 \\ 56 & 120 & 55 & 49 & 35 \end{bmatrix}$$

5° level -

$$\begin{bmatrix} 121 & 108 & 7 & 20 & 59 \\ 29 & 28 & 122 & 125 & 11 \\ 51 & 15 & 41 & 124 & 84 \\ 78 & 54 & 99 & 24 & 60 \\ 36 & 110 & 46 & 22 & 101 \end{bmatrix}$$

See also

- Magic cube classes
- Nasik magic hypercube
- John R. Hendricks

References

Frost, Dr. A. H., On the General Properties of Nasik Cubes, QJM 15, 1878, pp 93–123

Planck, C., The Theory of Paths Nasik, Printed for private circulation, A.J. Lawrence, Printer, Rugby,(England), 1905

H.D, Heinz & J.R. Hendricks, *Magic Square Lexicon: Illustrated*, hdh, 2000, 0-9687985-0-0

External links

- Walter Trump: Perfect magic cube of order 6 found [4]
- Christian Boyer: Perfect magic cubes [1]
- MathWorld news: Perfect magic cube of order 5 discovered [2]
- MathWorld: Perfect magic cube [3]
- Harvey Heinz: Perfect Magic Hypercubes [2]
- Aale de Winkel: The Magic Encyclopedia [5]
- Impossibility Proof for doubly-odd order Pandiagonal and Perfect hypercubes [4]

References

[1] http://perso.club-internet.fr/cboyer/multimagie/English/Perfectcubes.htm
[2] http://mathworld.wolfram.com/news/2003-11-18/magiccube/
[3] http://mathworld.wolfram.com/PerfectMagicCube.html
[4] http://members.shaw.ca/hdhcubes/cube_update-1.htm#Pandiagonal%20impossibility%20proof

Bimagic cube

In mathematics, a **bimagic cube** is a magic cube that also remains magic if all of the numbers it contains are squared. The only known example of a bimagic cube was given by John Hendricks in 2000; it has order 25 and magic constant 195325.

See also

- Magic cube
- Trimagic cube
- Multimagic cube
- Magic square
- Bimagic square
- Trimagic square
- Multimagic square

References

- Weisstein, Eric W., "Bimagic Cube [1]" from MathWorld.

References

[1] http://mathworld.wolfram.com/BimagicCube.html

Multimagic cube

In mathematics, a ***P*-multimagic cube** is a magic cube that remains magic even if all its numbers are replaced by their k-th power for $1 \leq k \leq P$. Thus, a magic cube is bimagic if and only if it is 2-multimagic, and trimagic if and only if it is 3-multimagic.

See also

- Magic square
- Multimagic square

Magic tesseract

In mathematics, a **magic tesseract** is the 4-dimensional counterpart of a magic square and magic cube, that is, a number of integers arranged in an $n \times n \times n \times n$ pattern such that the sum of the numbers on each pillar (along any axis) as well as the main space diagonals is equal to a single number, the so-called magic constant of the tesseract, denoted $M_4(n)$. It can be shown that if a magic tesseract consists of the numbers 1, 2, ..., n^4, then it has magic constant (sequence A021003 [1] in OEIS)

$$M_4(n) = \frac{n(n^4+1)}{2}.$$

The number n is called the order of the magic tesseract.

Perfect magic tesseract

If, in addition, the numbers on every cross section diagonal also sum up to the tesseract's magic constant, the tesseract is called a **perfect magic tesseract**; otherwise, it is called a **semiperfect magic tesseract**.

Alternative Definition

The above assumes that one of the older definitions for perfect magic cubes is used. See Magic Cube Classes. The ***Universal Classification System for Hypercubes*** (John R. Hendricks) requires that for any dimension hypercube, *all* possible lines sum correctly for the hypercube to be considered *perfect* magic. Because of the confusion with the term *perfect*, ***nasik*** is now the preferred term for *any* magic hypercube where ***all*** possible lines sum to *S*. Nasik was defined in this manner by C. Planck in 1905. A nasik magic tesseract has 40 lines of *m* numbers passing through each of the m^4 cells.

The smallest possible *nasik* magic tesseract is of order 16; its magic constant is 524296. The first one was discovered by retired meteorologist John R. Hendricks from British Columbia in 1999 with the help of Cliff Pickover at the IBM Thomas J. Watson Research Center in Yorktown Heights, New York after about ten hours of computing time on an IBM IntelliStation computer system.

See also

- Magic hypercube
- Magic hypercubes
- Nasik magic hypercube
- John R. Hendricks

References

- Andrews, W.S., *Magic Squares and Cubes*, Dover, Publ., 1960, this is a facsimile of an Open Court 1917 edition. Two essays on 'octahedrons' (pages 351 - 375 written by Kingsley and Planck.
- Hendricks, John R., *Magic Squares to Tesseract by Computer*, Self-published, 1998, 0-9684700-0-9
- Hendricks, John R., *All Third-Order Magic Tesseracts*, Self-published, 1999, 0-9684700-2-5
- Hendricks, John R., *Perfect n-Dimensional Magic Hypercubes of Order* 2^n, Self-published, 1999, 0-9684700-4-1.
- Heinz, H.D., & Hendricks, J.R., *Magic Square Lexicon: Illustrated*, HDH, 2000, 0-9687985-0-0
- Unfortunately, all of Hendricks books (except the Lexicon) are now out-of-print. Some are available for download in PDF from his web site. All are available at the Strens Recreational Mathematics Collection (Univ. of C.)

External links

- John Hendricks Math [2]
- University of Calgary, Strens Rec. Math Collections [3]
- 11 pages about magic tesseracts [4]

References

[1] http://en.wikipedia.org/wiki/Oeis%3Aa021003
[2] http://members.shaw.ca/johnhendricksmath/
[3] http://www.ucalgary.ca/lib-old/sfgate/strens/index.html
[4] http://members.shaw.ca/tesseracts/

Magic hypercube

In mathematics, a **magic hypercube** is the *k*-dimensional generalization of magic squares, magic cubes and magic tesseracts; that is, a number of integers arranged in an $n \times n \times n \times ... \times n$ pattern such that the sum of the numbers on each pillar (along any axis) as well as the main space diagonals is equal to a single number, the so-called magic constant of the hypercube, denoted $M_k(n)$. It can be shown that if a magic hypercube consists of the numbers 1, 2, ..., n^k, then it has magic number

$$M_k(n) = \frac{n(n^k+1)}{2}$$

If, in addition, the numbers on every cross section diagonal also sum up to the hypercube's magic number, the hypercube is called a perfect magic hypercube; otherwise, it is called a semiperfect magic hypercube. The number *n* is called the order of the magic hypercube.

Five-, six-, seven- and eight-dimensional magic hypercubes of order three have been constructed by J. R. Hendricks.

Marian Trenkler proved the following theorem: A *p*-dimensional magic hypercube of order *n* exists if and only if $p > 1$ and *n* is different from 2 or $p = 1$. A construction of a magic hypercube follows from the proof.

The R programming language includes a module, `library(magic)`, that will create magic hypercubes of any dimension (with *n* a multiple of 4).

Change to more modern conventions here-after (basically k ==> n and n ==> m)

Conventions

It is customary to denote the dimension with the letter 'n' and the order of a hypercube with the letter 'm'.

- **(*n*) Dimension** : the number of directions within a hypercube.
- **(*m*) Order** : the number of numbers along a direction.

Further: In this article the analytical number range $[0..m^n\text{-}1]$ is being used. For the regular number range $[1..m^n]$ you can add 1 to each number. This has absolutely no effect on the properties of the hypercube.

Notations

in order to keep things in hand a special notation was developed:

- **$[\ _k i;\ k=[0..n-1];\ i=[0..m-1]\ ]$**: positions within the hypercube
- **$<\ _k i;\ k=[0..n-1];\ i=[0..m-1]\ >$**: vector through the hypercube

Note: The notation for position can also be used for the value on that position. There where it is appropriate dimension and order can be added to it thus forming: ${}^n[_k i]_m$

As is indicated 'k' runs through the dimensions, while the coordinate 'i' runs through all possible values, when values 'i' are outside the range it is simply moved back into the range by adding or subtracting appropriate multiples of m, as the magic hypercube resides in n-dimensional modular space.

There can be multiple 'k' between bracket, these can't have the same value, though in undetermined order, which explains the equality of:

$$[\ _1 i,\ _k j\ ] = [\ _k j,\ _1 i\ ]$$

Of course given 'k' also one value 'i' is referred to.
When a specific coordinate value is mentioned the other values can be taken as 0, which is especially the case when the amount of 'k's are limited using pe. #k=1 as in:

$$[_k 1\ ;\ \#k=1] = [_k 1\ _j 0\ ;\ \#k=1;\ \#j=n-1]\ \text{("axial"-neighbor of } [_k 0])$$

(#j=n-1 can be left unspecified) j now runs through all the values in [0..k-1,k+1..n-1].

Further: without restrictions specified 'k' as well as 'i' run through all possible values, in combinations same letters assume same values. Thus makes it possible to specify a particular line within the hypercube (see r-agonal in pathfinder section)

Note: as far as I now this notation is not in general use yet(?), Hypercubes are not generally analyzed in this particular manner.

Further: "**perm(0..n-1)**" specifies a permutation of the n numbers 0..n-1.

Construction

Besides more specific constructions two more general construction method are noticeable:

KnightJump construction

This construction generalizes the movement of the chessboard horses (vectors <1,2>, <1,-2>, <-1,2>, <-1,-2>) to more general movements (vectors $<_k i>$). The method starts at the position P_0 and further numbers are sequentially placed at positions V_0 further until (after m steps) a position is reached that is already occupied, a further vector is needed to find the next free position. Thus the method is specified by the n by n+1 matrix:

$$[P_0,\ V_0\ ..\ V_{n-1}]$$

This positions the number 'k' at position:

$$P_k = P_0 + {}_{l=0}\sum^{n-1} ((k\backslash m^l)\%m)\ V_l;\ k = 0\ ..\ m^n-1.$$

C. Planck gives in his 1905 article "**The theory of Path Nasiks**" [1] conditions to create with this method "Path Nasik" (or modern {perfect}) hypercubes.

Latin prescription construction

(modular equations). This method is also specified by an n by n+1 matrix. However this time it multiplies the n+1 vector [x_0,..,x_{n-1},1], After this multiplication the result is taken modulus m to achieve the n (Latin) hypercubes:

$$LP_k = (\ {}_{l=0}\sum^{n-1} LP_{k,l}\ x_l + LP_{k,n}\ )\ \%\ m$$

of radix m numbers (also called "**digits**"). On these LP_k's "**digit changing**" (?i.e. Basic manipulation) are generally applied before these LP_k's are combined into the hypercube:

$${}^nH_m = {}_{k=0}\sum^{n-1} LP_k\ m^k$$

J.R.Hendricks often uses modular equation, conditions to make hypercubes of various quality can be found on http://www.magichypercubes.com/Encyclopedia [2] at several places (especially p-section)

Both methods fill the hypercube with numbers, the knight-jump guarantees (given appropriate vectors) that every number is present. The Latin prescription only if the components are orthogonal (no two digits occupying the same position)

Multiplication

Amongst the various ways of compounding, the multiplication[3] can be considered as the most basic of these methods. The **basic multiplication** is given by:

$$^{n}H_{m}1 * {}^{n}H_{m}2 : {}^{n}[_{k}i]_{m}1m_{2} = {}^{n}[[[_{k}i \setminus m_{2}]_{m}1m_{1}{}^{n}]_{m}2 + [_{k}i \% m_{2}]_{m}2]_{m}1_{m}2$$

Most compounding methods can be viewed as variations of the above, As most qualfiers are invariant under multiplication one can for example place any aspectial variant of $^{n}H_{m}2$ in the above equation, besides that on the result one can apply a manipulation to improve quality. Thus one can specify pe the J. R. Hendricks / M. Trenklar doubling. These things go beyond the scope of this article.

Aspects

A huper cube knows **n! 2ⁿ** Aspectial variants, which are obtained by coordinate reflection ($[_{k}i]$ --> $[_{k}(-i)]$) and coordinate permutations ($[_{k}i]$ --> $[_{perm[k]}i]$) effectively giving the Aspectial variant:

$$^{n}H_{m}{}^{\sim R\ perm(0..n-1)}; \ R = \sum_{k=0}^{n-1} ((reflect(k)) \ ? \ 2^{k} : 0) \ ; \ perm(0..n-1) \text{ a permutation of } 0..n-1$$

Where reflect(k) true iff coordinate k is being reflected, only then 2^{k} is added to R. As is easy to see only n coordinates can be reflected explaining 2^{n}, the n! permutation of n coordinates explains the other factor to the total amount of "Aspectial variants"!

Aspectial variants are generally seen as being equal. Thus any hypercube can be represented shown in **"normal position"** by:

$$[_{k}0] = \min([_{k}\theta \ ; \ \theta \ \varepsilon \ \{-1,0\}]) \ \text{(by reflection)}$$
$$[_{k}1 \ ; \ \#k=1] < [_{k+1}1 \ ; \ \#k=1] \ ; \ k = 0..n-2 \ \text{(by coordinate permutation)}$$

(explicitely stated here: $[_{k}0]$ the minimum of all corner points. The axial neighbour sequentially based on axial number)

Basic manipulations

Besides more specific manipulations, the following are of more general nature

- **#[perm(0..n-1)]** : component permutation
- **^[perm(0..n-1)]** : coordinate permutation (n == 2: transpose)
- **$\_2^{axis}$[perm(0..m-1)]** : monagonal permutation (axis ε [0..n-1])
- **=[perm(0..m-1)]** : digit change

Note: '#', '^', '_' and '=' are essential part of the notation and used as manipulation selectors.

Component permutation

Defined as the exchange of components, thus varying the factor m^{k} in $m^{perm(k)}$, because there are n component hypercubes the permutation is over these n components

Coordinate permutation

The exchange of coordinate $[_{k}i]$ into $[_{perm(k)}i]$, because of n coordinates a permutation over these n directions is required.
The term **transpose** (usually denoted by ^{t}) is used with two dimensional matrices, in general though perhaps "coordinate permutation" might be preferable.

Monagonal permutation

Defined as the change of [$_k$**i**] into [$_k$**perm(i)**] alongside the given "axial"-direction. Equal permutation along various axes can be combined by adding the factors 2^{axis}. Thus defining all kinds of r-agonal permutations for any r. Easy to see that all possibilities are given by the corresponding permutation of m numbers.

Noted be that **reflection** is the special case:

```
~R = _R[n-1,..,0]
```

Further when all the axes undergo the same ;permutation (R = 2^n-1) an **n-agonal permutation** is achieved, In this special case the 'R' is usually omitted so:

$$\_[perm(0..n-1)] = \_(2^n-1)[perm(0..n-1)]$$

Digitchanging

Usually being applied at component level and can be seen as given by **[$_k$i]** in **perm([$_k$i]**) since a component is filled with radix m digits, a permutation over m numbers is an appropriate manner to denote these.

Pathfinders

J. R. Hendricks called the directions within a hypercubes "**pathfinders**", these directions are simplest denoted in a ternary number system as:

$$Pf_p \text{ where: } p = \sum_{k=0}^{n-1} ({}_k i + 1)\ 3^k \iff \langle {}_k i \rangle \ ;\ i\ \varepsilon\ \{-1,0,1\}$$

This gives 3^n directions. since every direction is traversed both ways one can limit to the upper half [(3^n-1)/2,..,3^n-1)] of the full range.

With these pathfinders any line to be summed over (or r-agonal) can be specified:

$$[\ {}_j0\ {}_kp\ {}_lq\ ;\ \#j=1\ \#k=r-1\ ;\ k > j\ ]\ \langle\ {}_j1\ {}_k\theta\ {}_l0\ ;\ \theta\ \varepsilon\ \{-1,1\}\ \rangle\ \ ;\ p,q\ \varepsilon\ [0,..,m-1]$$

which specifies all (broken) r-agonals, p and q ranges could be omitted from this description. The main (unbroken) r-agonals are thus given by the slight modification of the above:

$$[\ {}_j0\ {}_k0\ {}_l{-1}\ {}_sp\ ;\ \#j=1\ \#k+\#l=r-1\ ;\ k,l > j\ ]\ \langle\ {}_j1\ {}_k1\ {}_l{-1}\ {}_s0\ \rangle$$

Qualifications

A hypercube ^nH_m with numbers in the analytical numberrange [0..m^n-1] has the magic sum:

$$^nS_m = m\ (m^n - 1)\ /\ 2.$$

Besides more specific qualifications the following are the most important, "summing" of course stands for "summing correctly to the magic sum"

- {**r-agonal**} : all main (unbroken) r-agonals are summing.
- {**pan r-agonal**} : all (unbroken and broken) r-agonals are summing.
- {**magic**} : {1-agonal n-agonal}
- {**perfect**} : {pan r-agonal; r = 1..n}

Note: This series doesn't start with 0 since a nill-agonal doesn't exist, the numbers correspond with the usual name-calling: 1-agonal = monagonal, 2-agonal = diagonal, 3-agonal = triagonal etc.. Aside from this the number correspond to the amount of "-1" and "1" in the corresponding pathfinder.

In case the hypercube also sum when all the numbers are raised to the power p one gets p-multimagic hypercubes. The above qualifiers are simply prepended onto the p-multimagic qualifier. This defines qualifications as {r-agonal

2-magic}. Here also "2-" is usually replaced by "bi", "3-" by "tri" etc. ("1-magic" would be "monomagic" but "mono" is usually omitted). The sum for p-Multimagic hypercubes can be found by using Faulhaber's formula and divide it by m^{n-1}.

Also "magic" (i.e. {1-agonal n-agonal}) is usually assumed, the Trump/Boyer {diagonal} cube is technically seen {1-agonal 2-agonal 3-agonal}.

Nasik magic hypercube gives arguments for using {**nasik**} as synonymous to {**perfect**}. The strange generalization of square 'perfect' to using it synonymous to {diagonal} in cubes is however also resolve by putting curly brackets around qualifiers, so {**perfect**} means {pan r-agonal; r = 1..n} (as mentioned above).

some minor qualifications are:

- {^{n}**compact**} : {all order 2 subhyper cubes sum to $2^n\ {}^{n}S_m$ / m}
- {^{n}**complete**} : {all pairs halve an n-agonal apart sum equal (to $(m^n - 1)$)}

{^{n}**compact**} might be put in notation as : $\mathbf{{}_{(k)}\Sigma\ [_j i + {}_k 1] = 2^n\ {}^{n}S_m / m}$.
{^{n}**complete**} can simply written as: $\mathbf{[_j i] + [_j i + {}_k(m/2)\ ;\ \#k{=}n\ ] = m^n - 1}$.
Where:
$_{(k)}\Sigma$ is symbolic for summing all possible k's, there are 2^n possibilities for $_k1$.
$[_j i + {}_k1]$ expresses $[_j i]$ and all its r-agonal neighbors.
for {complete} the complement of $[_j i]$ is at position $[_j i + {}_k(m/2)\ ;\ \#k{=}n\ ]$.
for squares: {^{2}**compact** ^{2}**complete**} is the "modern/alternative qualification" of what Dame Kathleen Ollerenshaw called most-perfect magic square, {^{n}compact ^{n}complete} is the qualifier for the feature in more than 2 dimensions
Caution: some people seems to equate {compact} with {^{2}compact} instead of {^{n}compact}. Since this introductory article is not the place to discuss these kind of issues I put in the dimensional pre-superscript ^{n} to both these qualifiers (which are defined as shown)
consequences of {^{n}compact} is that several figures also sum since they can be formed by adding/subtracting order 2 sub-hyper cubes. Issues like these go beyond this articles scope.

Special hypercubes

The following hypercubes serve special purposes;

The "normal hypercube"

$$^{n}N_m\ :\ [_k i]\ =\ {}_{k=0}\Sigma^{n-1}\ {}_k i\ m^k$$

This hypercube can be seen as the source of all numbers. A procedure called "Dynamic numbering" [4] makes use of the isomorphism of every hypercube with this normal, changing the source, changes the hypercube. Usually these sources are limited to direct products of normal hypercubes or normal hyperbeams (defined as having possibly other orders along the various directions).

The "constant 1"

```
n1m : [ki] = 1
```

The hypercube that is usually added to change the here used "analytic" number range into the "regular" number range. Other constant hypercubes are of course multiples of this one.

See also

- Magic hyperbeam
- Nasik magic hypercube
- Space diagonal
- John R. Hendricks

File format

Based on XML, the file format Xml-Hypercubes [5] is developed to describe various hypercubes to ensure human readability as well as programmatical usability. Besides full listings the format offers the ability to invoke mentioned constructions (amongst others)

References

[1] http://www.magichypercubes.com/Encyclopedia/k/PathNasiks.zip
[2] http://www.magichypercubes.com/Encyclopedia
[3] this is a n-dimensional version of (pe.): Alan Adler magic square multiplication (http://mathforum.org/alejandre/magic.square/adler/product.html)
[4] http://www.magichypercubes.com/Encyclopedia/d/DynamicNumbering.html
[5] http://www.magichypercubes.com/Encyclopedia/x/XmlHypercubes.html

External links

- The Magic Encyclopedia (http://www.magichypercubes.com/Encyclopedia/index.html) Articles by Aale de Winkel
- Magic Cubes - Introduction (http://members.shaw.ca/hdhcubes/) by Harvey D. Heinz
- Magic Cubes and Hypercubes - References (http://math.ku.sk/~trenkler/Cube-Ref.html) Collected by Marian Trenkler
  - An algorithm for making magic cubes (http://math.ku.sk/~trenkler/05-MagicCube.pdf) by Marian Trenkler
- multimagie.com (http://www.multimagie.com/) Articles by Christian Boyer

Further reading

- J.R.Hendricks: Magic Squares to Tesseract by Computer, Self-published, 1998, 0-9684700-0-9
- Planck, C., M.A.,M.R.C.S., The Theory of Paths Nasik, 1905, printed for private circulation. Introductory letter to the paper

Nasik magic hypercube

A ***Nasik magic hypercube*** is a magic hypercube with the added restriction that all possible lines through each cell sum correctly to $S = \frac{m(m^n+1)}{2}$ where S = the magic constant, m = the order and n = the dimension, of the hypercube.

Or, to put it more concisely, all pan-r-agonals sum correctly for r = 1...n.

The above definition is the same as the Hendricks definition of ***perfect***, but different than the Boyer/Trump definition. See Perfect magic cube Because of the confusion over the term perfect when used with reference to magic squares, magic cubes, and in general magic hypercubes, I am proposing the above as an **unambiguous term**. Following is an attempt to use the magic cube as a specific example.

A ***Nasik magic cube*** is a magic cube with the added restriction that all 13m possible lines sum correctly to the magic constant. This class of magic cube is commonly called perfect (John Hendricks definition.). See Magic cube classes. However, the term *perfect* is ambiguous because it is also used for other types of magic cubes. Perfect magic cube demonstrates just one example of this.
The term *nasik* would apply to all dimensions of magic hypercubes in which the number of correctly summing paths (lines) through any cell of the hypercube is $P = (3^n - 1)/2$

A *pandiagonal* magic square then would be a *nasik* square because 4 magic line pass through each of the m^2cells. This was A.H. Frost's original definition of nasik.
A *nasik* magic cube would have 13 magic lines passing through each of it's m^3 cells. (This cube also contains 9m pandiagonal magic squares of order m.)
A *nasik* magic tesseract would have 40 lines passing through each of it's m^4 cells.
And so on.

Background support

In 1866 and 1878, Rev. A. H. Frost coined the term *Nasik* for the type of magic square we commonly call *pandiagonal* and often call *perfect*. He then demonstrated the concept with an order-7 cube we now class as *pandiagonal*, and an order-8 cube we class as *pantriagonal*.[1] [2]
In another 1878 paper he showed another *pandiagonal* magic cube and a cube where all 13m lines sum correctly[3] i.e. Hendricks *perfect*.[4] He referred to all of these cubes as ***nasik***!
In 1905 Dr. Planck expanded on the nasik idea in his Theory of Paths Nasik. In the introductory to his paper, he wrote;

> Analogy suggest that in the higher dimensions we ought to employ the term nasik as implying the existence of magic summations parallel to any diagonal, and not restrict it to diagonals in sections parallel to the plane faces. The term is used in this wider sense throughout the present paper.
>
> – C. Planck, M.A.,M.R.C.S., The Theory of Paths Nasik, 1905[5]

In 1917, Dr. Planck wrote again on this subject.

> It is not difficult to perceive that if we push the Nasik analogy to higher dimensions the number of magic directions through any cell of a k-fold must be ½(3^k-1).
>
> – W. S. Andrews, Magic Squares and Cubes, Dover Publ., 1917, page 366 [6]

In 1939, B. Rosser and R. J. Walker published a series of papers on diabolic (perfect) magic squares and cubes. They specifically mentioned that these cubes contained 13m^2 correctly summing lines. They also had 3m pandiagonal magic squares parallel to the faces of the cube, and 6m pandiagonal magic squares parallel to the triagonal planes.[7]

Conclusion

If the term ***nasik*** is adopted as the definition for a magic hypercube where all possible lines sum correctly, there will no longer be confusion over what exactly is a Perfect magic cube. And, as in Hendricks definition of perfect, all pan-*r*-agonals sum correctly, and all lower dimension hypercubes contained in it are ***nasik*** (Hendricks perfect).

See also

Magic hypercube

Magic hypercubes

Magic cube

Magic cube classes

Perfect magic cube

Magic tesseract

John R. Hendricks

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[1] Frost, A. H., Invention of Magic Cubes, *Quarterly Journal of Mathematics*, 7,1866, pp92-102
[2] Frost, A. H., *On the General Properties of Nasik Squares*, QJM, 15, 1878, pp 34-49
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[4] Heinz, H.D., and Hendricks, J.R., *Magic Square Lexicon: Illustrated*, 2000, 0-9687985-0-0 pp 119-122
[5] Planck, C., M.A.,M.R.C.S., *The Theory of Paths Nasik*, 1905, printed for private circulation. Introductory letter to the paper.
[6] Andrews, W. S., Magic Squares and Cubes, Dover Publ. 1917. Essay pages 363-375 written by C. Planck
[7] Rosser, B. and Walker, R. J., *Magic Squares: Published papers and Supplement*, 1939. A bound volume at Cornell University, catalogued as QA 165 R82+pt.1-4

External links

- History, definitions, and examples of perfect magic cubes and other dimensions. (http://members.shaw.ca/hdhcubes/cube_perfect.htm)
- An alternative definition of Perfect, with history of recent discoveries (http://multimagie.com/indexengl.htm)
- More on this alternative definition. (http://www.trump.de/magic-squares/magic-cubes/cubes-1.html)
- A Magic Hypercube encyclopedia with a broad range of material (http://www.magichypercubes.com/Encyclopedia/index.html)
- A Unified classification system for hypercubes (http://members.shaw.ca/hdhcubes/index.htm#6 Classes of Cubes)
- An ambitious ongoing work on classifications of magic cubes and tesseracts (http://homepage2.nifty.com/googol/magcube/en/)
- A variety of John R. Hendricks material, written under his direction (http://members.shaw.ca/johnhendricksmath/)

Categories

Magic hyperbeam

A **magic hyperbeam** (**n-dimensional magic rectangle**) is a variation on a magic hypercube where the orders along each direction may be different. As such a **magic hyperbeam** generalises the two dimensional **magic rectangle** and the three dimensional **magic beam**, a series that mimics the series magic square, magic cube and magic hypercube. This article will mimic the magic hypercubes article in close detail, and just as that article serves merely as an introduction to the topic.

Conventions

It is customary to denote the dimension with the letter 'n' and the orders of a hyperbeam with the letter 'm' (appended with the subscripted number of the direction it applies to).

- (***n***) **Dimension** : the amount of directions within a hyperbeam.
- ($\boldsymbol{m_k}$) **Order** : the amount of numbers along kth monagonal $k = 0, ..., n - 1$.

Further: In this article the analytical number range $[0..{}_{k=0}\prod^{n-1} m_k - 1]$ is being used.

Notations

in order to keep things in hand a special notation was developed:

- **[$_k i$; k=[0..n-1]; i=[0..m_k-1]]**: positions within the hyperbeam
- **< $_k i$; k=[0..n-1]; i=[0..m_k-1] >**: vectors through the hyperbeam

Note: The notation for position can also be used for the value on that position. There where it is appropriate dimension and orders can be added to it thus forming: $^n[_k i]_m 0,..,m_{n-1}$

Construction

Basic

Description of more general methods might be put here, I don't often create hyperbeams, so I don't know whether Knightjump or Latin Prescription work here. Other more adhoc methods suffice on occasion I need a hyperbeam.

Multiplication

Amongst the various ways of compounding, the multiplication[1] can be considered as the most basic of these methods. The **basic multiplication** is given by:

```
nB(m..)1 * nB(m..)2 : n[ki](m..)1(m..)2 = n[ <sub>k</sub>i \ m<sub>k2</sub>(m..)1 k=0∏n-1 mk1](m..)2 + [ki % mk2](m..)2](m..)1(m..)2
```

(m..) abbreviates: $m_0,..,m_{n-1}$.
$(m..)_1(m..)_2$ abbreviates: $m_0 1 m_0 2,..,m_{n-1} 1 m_{n-1} 2$.

Curiosities

all orders are either even or odd

A fact that can be easily seen since the magic sums are:

$$S_k = m_k \left({}_{j=0}\prod^{n-1} m_j - 1\right) / 2$$

When any of the orders m_k is even, the product is even and thus the only way S_k turns out integer is when all m_k are even.

Thus suffices: all m_k are either even or odd.

This is with the exception of m_k=1 of course, which allows for general identities like:

- $N_m^{\ t} = N_{m,1} * N_{1,m}$
- $N_m = N_{1,m} * N_{m,1}$

Which goes beyond the scope of this introductory article

Only one direction with order = 2

since any number has but one complement only one of the directions can have $m_k = 2$.

Aspects

A hyperbeam knows $\mathbf{2^n}$ Aspectial variants, which are obtained by coördinate reflection ($[_k i]$ --> $[_k(-i)]$) effectively giving the Aspectial variant:

$${}^{n}B_{(m_0..m_{n-1})}{}^{\sim R} \; ; \; R = {}_{k=0}\sum^{n-1} \; ((\mathrm{reflect}(k)) \; ? \; 2^k : 0) \; ;$$

Where reflect(k) true iff coördinate k is being reflected, only then 2^k is added to R.

In case one views different orientations of the beam as equal one could view the number of aspects $\mathbf{n!\ 2^n}$ just as with the magic hypercubes, directions with equal orders contribute factors depending on the hyperbeam's orders. This goes beyond the scope of this article.

Basic manipulations

Besides more specific manipulations, the following are of more general nature

- **^[perm(0..n-1)]** : coördinate permutation (n == 2: transpose)
- **_2$^{\mathbf{axis}}$[perm(0..m-1)]** : monagonal permutation (axis ε [0..n-1])

Note: '^' and '_' are essential part of the notation and used as manipulation selectors.

Coördinate permutation

The exchange of coördinaat $[_{\mathbf{k}} i]$ into $[_{\mathbf{perm(k)}} i]$, because of n coördinates a permutation over these n directions is required.

The term **transpose** (usually denoted by ^{t}) is used with two dimensional matrices, in general though perhaps "coördinaatpermutation" might be preferable.

Monagonal permutation

Defined as the change of $[_k\mathbf{i}]$ into $[_k\mathbf{perm(i)}]$ alongside the given "axial"-direction. Equal permutation along various axes with equal orders can be combined by adding the factors 2^{axis}. Thus defining all kinds of r-agonal permutations for any r. Easy to see that all possibilities are given by the corresponding permutation of m numbers.

normal position

In case no restrictions are considered on the n-agonals a magic hyperbeam can be represented shown in **"normal position"** by:

$$[_k i] < [_k (i+1)] \; ; \; i = 0..m_k-2 \text{ (by monagonal permutation)}$$

Qualification

Qualifying the hyperbeam is less developed then it is on the magic hypercubes in fact only the k'th monagonal direction need to sum to:

$$S_k = m_k \; ({}_{j=0}\prod^{n-1} m_j - 1) / 2$$

for all k = 0..n-1 for the hyperbeam to be qualified {**magic**}

When the orders are not relatively prime the n-agonal sum can be restricted to:

$$S = \mathrm{lcm}(m_i \; ; \; i = 0..n-1) \; ({}_{j=0}\prod^{n-1} m_j - 1) / 2$$

with all orders relatively prime this reaches its maximum:

$$S_{max} = {}_{j=0}\prod^{n-1} m_j \; ({}_{j=0}\prod^{n-1} m_j - 1) / 2$$

Special hyperbeams

The following hyperbeams serve special purposes:

The "normal hyperbeam"

$${}^{n}N_{m}0,..,m_{n-1} \; : \; [_k i] = {}_{k=0}\sum^{n-1} {}_k i \; m_k{}^k$$

This hyperbeam can be seen as the source of all numbers. A procedure called "Dynamic numbering" [4] makes use of the isomorphism of every hyperbeam with this normal, changing the source, changes the hyperbeam. Basic multiplications of normal hyperbeams play a special role with the "Dynamic numbering" [4] of magic hypercubes of order ${}_{k=0}\prod^{n-1} m^k$.

The "constant 1"

$$^n1_{m}0,..,m_{n-1} : [_ki] = 1$$

The hyperbeam that is usually added to change the here used "analytic" numberrange into the "regular" numberrange. Other constant hyperbeams are of course multiples of this one.

See also

- magic hypercubes

References

[1] this is a hyperbeam version of (pe.): Alan Adler magic square multiplication (http://mathforum.org/alejandre/magic.square/adler/product.html)

External links

- http://www.magichypercubes.com/Encyclopedia (http://www.magichypercubes.com/Encyclopedia)
- J.R.Hendricks (http://members.shaw.ca/johnhendricksmath/)
- Marián Trenklar Cube-Ref.html (http://math.ku.sk/~trenkler/Cube-Ref.html)
- Mitsutoshi Nakamura: Rectangles (http://homepage2.nifty.com/googol/magcube/en/rectangles.htm)

Further reading

- Thomas R. Hagedorn, On the existence of magic n-dimensional rectangles, Discrete Mathematics 207 (1999), 53-63.
- Thomas R. Hagedorn, Magic retangles revisited, *Discrete Mathematics* 207 (1999), 65-72.
- Marián Trenkler, Magic rectangles, The Mathematical Gazette 83(1999), 102-105.
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Number placement puzzles

Sudoku

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

A Sudoku puzzle...

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

...and its solution numbers marked in red

Sudoku (Japanese: 数独 *listen*) (English pronunciation: /suːˈdoʊkuː/ ***soo**-DOH-**koo***) is a logic-based,[1] [2] combinatorial[3] number-placement puzzle. The objective is to fill a 9×9 grid with digits so that each column, each row, and each of the nine 3×3 sub-grids that compose the grid (also called "boxes", "blocks", "regions", or "sub-squares") contains all of the digits from 1 to 9. The puzzle setter provides a partially completed grid, which typically has a unique solution.

Completed puzzles are always a type of Latin square with an additional constraint on the contents of individual regions. For example, the same single integer may not appear twice in the same 9x9 playing board row or column or in any of the nine 3x3 subregions of the 9x9 playing board.[4]

The puzzle was popularized in 1986 by the Japanese puzzle company Nikoli, under the name Sudoku, meaning *single number*.[5] It became an international hit in 2005.[6]

History


DIVERTISSEMENTS QUOTIDIENS

N° 3879 — CARRÉ MAGIQUE DIABOLIQUE

Par M. B. Meyniel

Compléter le carré ci-dessous en employant les neuf premiers nombres chacun neuf fois de manière que les horizontales, les verticales et les deux grandes diagonales donnent toujours à l'addition le même total.

Ce carré devra être diabolique, c'est-à-dire que le carré restera magique si l'on place une ligne horizontale ou une colonne verticale à la suite de toutes les autres.

N° 3865 — MATHÉMATIQUES

Par M. Adolphe R.

Solution

Solutions justes

Les solutions et les envois de problèmes inédits doivent être adressés, dans la huitaine, au rédacteur soussigné.

FÉLIX ANDRÉ.

PROGRAMME

From *La France* newspaper, July 6, 1895.

Number puzzles appeared in newspapers in the late 19th century, when French puzzle setters began experimenting with removing numbers from magic squares. *Le Siècle*, a Paris-based daily, published a partially completed 9×9 magic square with 3×3 sub-squares on November 19, 1892.[7] It was not a Sudoku because it contained double-digit numbers and required arithmetic rather than logic to solve, but it shared key characteristics: each row, column and sub-square added up to the same number.

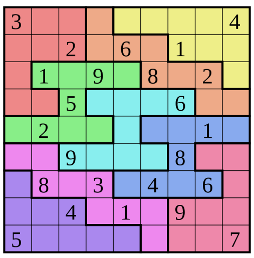
On July 6, 1895, *Le Siècle's* rival, *La France*, refined the puzzle so that it was almost a modern Sudoku. It simplified the 9×9 magic square puzzle so that each row, column and broken diagonals contained only the numbers 1–9, but did not mark the sub-squares. Although they are unmarked, each 3×3 sub-square does indeed comprise the numbers 1–9 and the additional constraint on the broken diagonals leads to only one solution.[8]

These weekly puzzles were a feature of French newspapers such as *L'Echo de Paris* for about a decade but disappeared about the time of the First World War.[9]

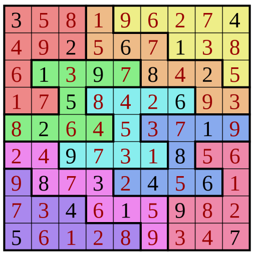
According to Will Shortz, the modern Sudoku was most likely designed anonymously by Howard Garns, a 74-year-old retired architect and freelance puzzle constructor from Indiana, and first published in 1979 by Dell Magazines as *Number Place* (the earliest known examples of modern Sudoku). Garns' name was always present on the list of contributors in issues of *Dell Pencil Puzzles and Word Games* that included *Number Place*, and was always absent from issues that did not.[10] He died in 1989 before getting a chance to see his creation as a worldwide phenomenon.[10] It is unclear if Garns was familiar with any of the French newspapers listed above.

The puzzle was introduced in Japan by Nikoli in the paper *Monthly Nikolist* in April 1984[10] as *Sūji wa dokushin ni kagiru* (数字は独身に限る), which can be translated as "the digits must be single" or "the digits are limited to one occurrence." (In Japanese,"dokushin" means an "unmarried person".) At a later date, the name was abbreviated to *Sudoku* by Maki Kaji (鍜治　真起 *Kaji Maki*), taking only the first kanji of compound words to form a shorter version.[10] In 1986, Nikoli introduced two innovations: the number of givens was restricted to no more than 32, and puzzles became "symmetrical" (meaning the givens were distributed in rotationally symmetric cells). It is now published in mainstream Japanese periodicals, such as the *Asahi Shimbun*.

Variants

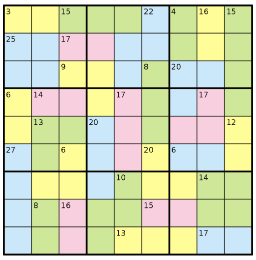


A nonomino or Jigsaw Sudoku puzzle, as seen in the *Sunday Telegraph*



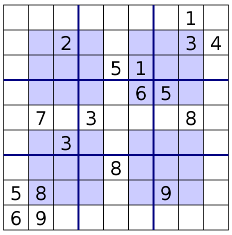
Solution numbers in red for above puzzle

Although the 9×9 grid with 3×3 regions is by far the most common, variations abound. Sample puzzles can be 4×4 grids with 2×2 regions; 5×5 grids with pentomino regions have been published under the name *Logi-5*; the World Puzzle Championship has featured a 6×6 grid with 2×3 regions and a 7×7 grid with six heptomino regions and a disjoint region. Larger grids are also possible. The *Times* offers a 12×12-grid *Dodeka sudoku* with 12 regions of 4×3 squares. Dell regularly publishes 16×16 *Number Place Challenger* puzzles (the 16×16 variant often uses 1 through G rather than the 0 through F used in hexadecimal). Nikoli offers 25×25 *Sudoku the Giant* behemoths.

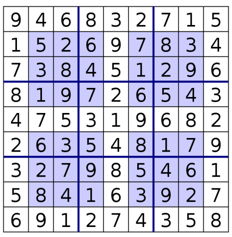


A Killer Sudoku puzzle

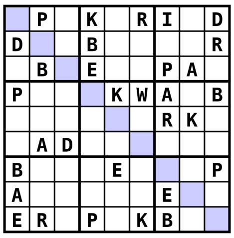
Solution for puzzle to the left



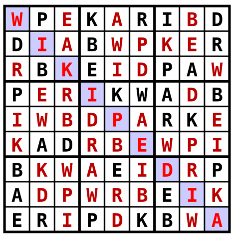
Hypersudoku puzzle



Solution numbers for puzzle to the left



A Wordoku puzzle



Solution in red for puzzle to the left

Another common variant is to add limits on the placement of numbers beyond the usual row, column, and box requirements. Often the limit takes the form of an extra "dimension"; the most common is to require the numbers in the main diagonals of the grid also to be unique. The aforementioned *Number Place Challenger* puzzles are all of this variant, as are the *Sudoku X* puzzles in the *Daily Mail*, which use 6×6 grids.

A variant named "Mini Sudoku" appears in the American newspaper *USA Today*, which is played on a 6×6 grid with 3×2 regions. The object is the same as standard Sudoku, but the puzzle only uses the numbers 1 through 6.

Another variant is the combination of Sudoku with Kakuro on a 9 × 9 grid, called Cross Sums Sudoku, in which clues are given in terms of cross sums. The clues can also be given by cryptic alphametics in which each letter represents a single digit from 0 to 9. An excellent example is NUMBER+NUMBER=KAKURO which has a unique solution 186925+186925=373850. Another example is SUDOKU=IS*FUNNY whose solution is 426972=34*12558.

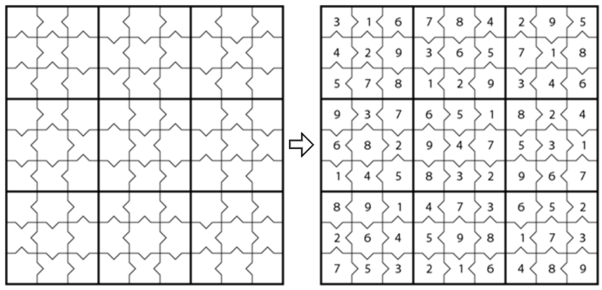
Killer Sudoku combines elements of Sudoku with Kakuro—usually, no initial numbers are given, but the 9×9 grid is divided into regions, each with a number that the sum of all numbers in the region must add up to and with no repeated numerals. These must be filled in while obeying the standard rules of Sudoku.

Hypersudoku is one of the most popular variants. It is published by newspapers and magazines around the world and is also known as "NRC Sudoku", "Windoku", "Hyper-Sudoku" and "4 Square Sudoku". The layout is identical to a normal Sudoku, but with additional interior areas defined in which the numbers 1 to 9 must appear. The solving algorithm is slightly different from the normal Sudoku puzzles because of the leverage on the overlapping squares. This overlap gives the player more information to logically reduce the possibilities in the remaining squares. The approach to playing is similar to Sudoku but with possibly more emphasis on scanning the squares and overlap rather than columns and rows.

Puzzles constructed from multiple Sudoku grids are common. Five 9×9 grids which overlap at the corner regions in the shape of a quincunx is known in Japan as Gattai 5 (five merged) Sudoku. In *The Times*, *The Age* and *The Sydney Morning Herald* this form of puzzle is known as *Samurai SuDoku*. The Baltimore Sun and the Toronto Star publish a puzzle of this variant (titled *High Five*) in their Sunday edition. Often, no givens are to be found in overlapping regions. Sequential grids, as opposed to overlapping, are also published, with values in specific locations in grids needing to be transferred to others.

Alphabetical variations have emerged, sometimes called **Wordoku**; there is no functional difference in the puzzle unless the letters spell something. Some variants, such as in the *TV Guide*, include a word reading along a main diagonal, row, or column once solved; determining the word in advance can be viewed as a solving aid. A Wordoku might contain other words, other than the main word. Like in the example to the left, the words "Kari", "Park" and "Per" could also be found in the solution. This might be avoided by e.g. substituting the character "R" with e.g. a "Q".

A tabletop version of Sudoku can be played with a standard 81-card Set deck (see Set game). A three-dimensional *Sudoku* puzzle was invented by Dion Church and published in the *Daily Telegraph* in May 2005. There is a Sudoku version of the Rubik's Cube named Sudoku Cube.



An example of Greater Than Sudoku.

There are many other variants. Some are different shapes in the arrangement of overlapping 9x9 grids, such as butterfly, windmill, or flower.[11] Others vary the logic for solving the grid. One of these is Greater Than Sudoku. In this a 3×3 grid of the Sudoku is given with 12 symbols of Greater Than (>) or Less Than (<) on the common line of the two adjacent numbers.[10] Another variant on the logic of solution is Clueless Sudoku, in which nine 9x9 Sudoku grids are themselves placed in a three-by-three array. The center cell in each 3x3 grid of all nine puzzles is left blank and form a tenth Sudoku puzzle without any cell completed; hence, "clueless".[11]

Mathematics of Sudoku

A completed *Sudoku* grid is a special type of Latin square with the additional property of no repeated values in any of the 9 blocks of contiguous 3×3 cells. The relationship between the two theories is now completely known, after Denis Berthier proved in his book *The Hidden Logic of Sudoku* (May 2007) that a first-order formula that does not mention blocks (also called boxes or regions) is valid for Sudoku if and only if it is valid for Latin Squares (this property is trivially true for the axioms and it can be extended to any formula). (Citation taken from p. 76 of the first edition: "*any block-free resolution rule is already valid in the theory of Latin Squares extended to candidates*" – which is restated more explicitly in the second edition, p. 86, as: "*a block-free formula is valid for Sudoku if and only if it is valid for Latin Squares*").

The first known calculation of the number of classic 9×9 *Sudoku* solution grids was posted on the USENET newsgroup *rec.puzzles* in September 2003[12] and is 6,670,903,752,021,072,936,960 (sequence A107739 [13] in OEIS), or approximately 6.67 x 10^{21}. This is roughly 1.2×10^{-6} times the number of 9×9 Latin squares. A detailed calculation of this figure was provided by Bertram Felgenhauer and Frazer Jarvis in 2005.[14] Various other grid sizes have also been enumerated—see the main article for details. The number of *essentially different* solutions, when symmetries such as rotation, reflection, permutation and relabelling are taken into account, was shown by Ed Russell and Frazer Jarvis to be just 5,472,730,538[15] (sequence A109741 [16] in OEIS).

The maximum number of givens provided while still not rendering a unique solution is four short of a full grid; if two instances of two numbers each are missing and the cells they are to occupy form the corners of an orthogonal rectangle, and exactly two of these cells are within one region, there are two ways the numbers can be assigned. Since this applies to Latin squares in general, most variants of *Sudoku* have the same maximum. The inverse problem—the fewest givens that render a solution unique—is unsolved, although the lowest number yet found for the standard variation without a symmetry constraint is 17, a number of which have been found by Japanese puzzle enthusiasts,[17] [18] and 18 with the givens in rotationally symmetric cells. Over 48,000 examples of Sudokus with 17 givens resulting in a unique solution are known.

In 2010 mathematicians Paul Newton and Stephen DeSalvo of the University of Southern California showed that the way that numbers are arranged in Sudoku puzzles is even more random than the number arrangements in randomly generated 9x9 matrices. This is because the rules of Sudoku exclude some random arrangements that have an innate symmetry.[19]

Recent popularity

In 1997, New Zealander and retired Hong Kong judge Wayne Gould, then in his early 50s, saw a partly completed puzzle in a Japanese bookshop. Over six years he developed a computer program to produce puzzles quickly. Knowing that British newspapers have a long history of publishing crosswords and other puzzles, he promoted *Sudoku* to *The Times* in Britain, which launched it on 12 November 2004 (calling it *Su Doku*). The first letter to *The Times* regarding Su Doku was published the following day on 13 November from Ian Payn of Brentford, complaining that the puzzle had caused him to miss his stop on the tube.[20]

The rapid rise of *Sudoku* in Britain from relative obscurity to a front-page feature in national newspapers attracted commentary in the media and parody (such as when *The Guardian*'s *G2* section advertised itself as the first newspaper supplement with a *Sudoku* grid on every page).[21] Recognizing the different psychological appeals of easy and difficult puzzles, *The Times* introduced both side by side on 20 June 2005. From July 2005, Channel 4 included a daily *Sudoku* game in their Teletext service. On 2 August, the BBC's programme guide *Radio Times* featured a weekly Super Sudoku which features a 16×16 grid.

In the United States, the first newspaper to publish a Sudoku puzzle by Wayne Gould was *The Conway Daily Sun* (New Hampshire), in 2004.[22]

The world's first live TV *Sudoku* show, 1 July 2005, Sky One.

The world's first live TV *Sudoku* show, *Sudoku Live*, was a puzzle contest first broadcast on 1 July 2005 on Sky One. It was presented by Carol Vorderman. Nine teams of nine players (with one celebrity in each team) representing geographical regions competed to solve a puzzle. Each player had a hand-held device for entering numbers corresponding to answers for four cells. Phil Kollin of Winchelsea, England was the series grand prize winner taking home over £23,000 over a series of games. The audience at home was in a separate interactive competition, which was won by Hannah Withey of Cheshire.

Later in 2005, the BBC launched *SUDO-Q*, a game show that combines Sudoku with general knowledge. However, it uses only 4×4 and 6×6 puzzles. Four seasons were produced, before the show ended in 2007.

In 2006, a Sudoku website published songwriter Peter Levy's Sudoku tribute song,[23] but quickly had to take down the mp3 due to heavy traffic. British and Australian radio picked up the song, which is to feature in a British-made Sudoku documentary. The Japanese Embassy also nominated the song for an award, with Levy doing talks with Sony in Japan to release the song as a single.[24]

Sudoku software is very popular on PCs, websites, and mobile phones. It comes with many distributions of Linux. Software has also been released on video game consoles, such as the Nintendo DS, PlayStation Portable, the Game Boy Advance, Xbox Live Arcade, the Nook e-book reader, several iPod models, and the iPhone. In fact, just two weeks after Apple, Inc. debuted the online App Store within its iTunes store on July 11, 2008, there were already nearly 30 different Sudoku games, created by various software developers, specifically for the iPhone and iPod Touch. One of the most popular video games featuring Sudoku is *Brain Age: Train Your Brain in Minutes a Day!*. Critically and commercially well received, it generated particular praise for its Sudoku implementation[25] [26] [27] and sold more than 8 million copies worldwide.[28] Due to its popularity, Nintendo made a second *Brain Age* game titled *Brain Age*^{2}, which has over 100 new Sudoku puzzles and other activities.

In June 2008 an Australian drugs-related jury trial costing over AU$1,000,000 was aborted when it was discovered that five of the twelve jurors had been playing Sudoku instead of listening to evidence.[29]

Competitions

- The first World Sudoku Championship was held in Lucca, Italy, from March 10–12, 2006. The winner was Jana Tylová of the Czech Republic.[30] The competition included numerous variants.[31]
- The second World Sudoku Championship was held in Prague from March 28 to April 1, 2007.[32] The individual champion was Thomas Snyder of the USA. The team champion was Japan.[33]
- The third World Sudoku Championship was held in Goa, India, from April 14–16, 2008. Thomas Snyder repeated as the individual overall champion, and also won the first ever Classic Trophy (a subset of the competition counting only classic Sudoku). The Czech Republic won the team competition.[34]
- The fourth World Sudoku Championship was held in Žilina, Slovakia, from April 24–27, 2009. After past champion Thomas Snyder of USA won the general qualification, Jan Mrozowski of Poland emerged from a 36-competitor playoff to become the new World Sudoku Champion. Host nation Slovakia emerged as the top team in a separate competition of three-membered squads.[35]
- The fifth World Sudoku Championship was held in Philadelphia, USA from April 29 – May 2, 2010. Jan Mrozowski of Poland successfully defended his world title in the individual competition while Germany won a separate team event. The puzzles were written by Thomas Snyder and Wei-Hwa Huang, both past US Sudoku champions.[36]
- In the United States, The Philadelphia Inquirer Sudoku National Championship has been held three times, each time offering a $10,000 prize to the advanced division winner and a spot on the U.S. National Sudoku Team traveling to the world championships. Puzzlemaster Will Shortz has served as tournament host. The winners of the event have been Thomas Snyder (2007),[37] Wei-Hwa Huang (2008), and Tammy McLeod (2009).[38] In the most recent event, the third place finalist in the advanced division, Eugene Varshavsky, performed quite poorly onstage after setting a very fast qualifying time on paper, which caught the attention of organizers and competitors including past champion Thomas Snyder who requested organizers reconsider his results due to a suspicion of cheating.[38] Following an investigation and a retest of Varshavsky, the organizers disqualified him and awarded Chris Narrikkattu third place.[38]

See also

- 36 cube
- Algorithmics of Sudoku
- Futoshiki
- Hidato
- Kakuro
- KenKen
- List of Nikoli puzzle types
- List of Sudoku terms and jargon
- Logic puzzle
- Mathematics of Sudoku
- Nonogram (aka Paint by numbers, O'ekaki)

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External links

- *Sudoku* (http://www.dmoz.org/Games/Puzzles/Brain_Teasers/Sudoku/) at the Open Directory Project – An active listing of *Sudoku* links.
- Father of Sudoku puzzles next move (http://news.bbc.co.uk/2/hi/asia-pacific/6745433.stm) BBC

Mathematics of Sudoku

The class of **Sudoku** puzzles consists of a partially completed row-column grid of cells partitioned into N regions each of size N cells, to be filled in using a prescribed set of N distinct symbols (typically the numbers {1, ..., N}), so that each row, column and region contains exactly one of each element of the set. The puzzle can be investigated using mathematics.

Overview

The mathematical analysis of Sudoku falls into two main areas: analyzing the properties of a) completed grids and b) puzzles. Grid analysis has largely focused on counting (enumerating) possible solutions for different *variants*. Puzzle analysis centers on the initial *given* values. The techniques used in either are largely the same: combinatorics and permutation group theory, augmented by the dexterous application of programming tools.

There are many Sudoku variants, (partially) characterized by the size (N) and shape of their regions. For classic Sudoku, N=9 and the regions are 3x3 squares (blocks). A rectangular Sudoku uses rectangular regions of row-column dimension R×C. For R×1 (and 1×C), i.e. where the region is a row or column, Sudoku becomes a Latin square.

Other Sudoku variants also exist, such as those with irregularly-shaped regions or with additional constraints (hypercube) or different (Samunamupure) constraint types. See Sudoku - Variants for a discussion of variants and Sudoku terms and jargon for an expanded listing.

The mathematics of Sudoku is a relatively new area of exploration, mirroring the popularity of Sudoku itself. NP-completeness was documented late 2002 [1] , enumeration results began appearing in May 2005 [2] .

In contrast with the two main mathematical approaches of Sudoku mentioned above, an approach resting on mathematical logic and dealing with the resolution of the puzzles from the viewpoint of a player has recently been proposed in Denis Berthier's book "The Hidden Logic of Sudoku". This formalizes certain mathematical symmetries of the game and elicits resolution rules based on them, such as "hidden xy-chains".

Mathematical context

The general problem of solving *Sudoku* puzzles on $n^2 \times n^2$ boards of $n \times n$ blocks is known to be NP-complete [1] . For n=3 (classical *Sudoku*), however, this result is of little relevance: algorithms such as Dancing Links can solve puzzles in fractions of a second.

Solving *Sudoku* puzzles can be expressed as a graph coloring problem. Consider the $9 \times 9 = 3^2 \times 3^2$ case. The aim of the puzzle in its standard form is to construct a proper 9-coloring of a particular graph, given a partial 9-coloring. The graph in question has 81 vertices, one vertex for each cell of the grid. The vertices can be labeled with the ordered pairs (x, y), where x and y are integers between 1 and 9. In this case, two distinct vertices labeled by (x, y) and (x', y') are joined by an edge if and only if:

- $x = x'$ (same column) or,
- $y = y'$ (same row) or,
- $\lceil x/3 \rceil = \lceil x'/3 \rceil$ and $\lceil y/3 \rceil = \lceil y'/3 \rceil$ (same 3 × 3 cell)

The puzzle is then completed by assigning an integer between 1 and 9 to each vertex, in such a way that vertices that are joined by an edge do not have the same integer assigned to them.

A valid *Sudoku* solution grid is also a Latin square. There are significantly fewer valid *Sudoku* solution grids than Latin squares because *Sudoku* imposes the additional regional constraint. Nonetheless, the number of valid *Sudoku* solution grids for the standard 9×9 grid was calculated by Bertram Felgenhauer and Frazer Jarvis in 2005 to be 6,670,903,752,021,072,936,960 [3] (sequence A107739 [13] in OEIS). This number is equal to $9! \times 72^2 \times 2^7 \times$

27,704,267,971, the last factor of which is prime. The result was derived through logic and brute force computation. Russell and Jarvis also showed that when symmetries were taken into account, there were 5,472,730,538 solutions [4] (sequence A109741 [16] in OEIS). The number of valid *Sudoku* solution grids for the 16×16 derivation is not known.

The maximum number of givens that can be provided while still not rendering the solution unique, regardless of variation, is four short of a full grid; if two instances of two numbers each are missing and the cells they are to occupy are the corners of an orthogonal rectangle, and exactly two of these cells are within one region, there are two ways the numbers can be added. The inverse of this—the fewest givens that render a solution unique—is an unsolved problem, although the lowest number yet found for the standard variation without a symmetry constraint is 17, a number of which have been found by Japanese puzzle enthusiasts [5] [6] , and 18 with the givens in rotationally symmetric cells.

Sudokus from group tables

As in the case of Latin squares the (addition- or) multiplication tables (Cayley tables) of finite groups can be used to construct Sudokus and related tables of numbers. Namely, one has to take subgroups and quotient groups into account:

Take for example $\mathbb{Z}_n \oplus \mathbb{Z}_n$ the group of pairs, adding each component separately modulo some n. By omitting one of the components, we suddenly find ourselves in $\mathbb{Z}_n$ (and this mapping is obviously compatible with the respective additions, i.e. it is a group homomorphism). One also says that the latter is a quotient group of the former, because some once different elements become equal in the new group. However, it is also a subgroup, because we can simply fill the missing component with 0 to get back to $\mathbb{Z}_n \oplus \mathbb{Z}_n$.
Under this view, we write down the example $n = 3$:

The addition table in

(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)	(2,0)	(2,1)	(2,2)
(0,1)	(0,2)	(0,0)	(1,1)	(1,2)	(1,0)	(2,1)	(2,2)	(2,0)
(0,2)	(0,0)	(0,1)	(1,2)	(1,0)	(1,1)	(2,2)	(2,0)	(2,1)
(1,0)	(1,1)	(1,2)	(2,0)	(2,1)	(2,2)	(0,0)	(0,1)	(0,2)
(1,1)	(1,2)	(1,0)	(2,1)	(2,2)	(2,0)	(0,1)	(0,2)	(0,0)
(1,2)	(1,0)	(1,1)	(2,2)	(2,0)	(2,1)	(0,2)	(0,0)	(0,1)
(2,0)	(2,1)	(2,2)	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)
(2,1)	(2,2)	(2,0)	(0,1)	(0,2)	(0,0)	(1,1)	(1,2)	(1,0)
(2,2)	(2,0)	(2,1)	(0,2)	(0,0)	(0,1)	(1,2)	(1,0)	(1,1)

Each Sudoku region looks the same on the second component (namely like the subgroup $\mathbb{Z}_3$), because these are added regardless of the first one. On the other hand, the first components are equal in each block, and if we imagine each block as one cell, these first components show the same pattern (namely the quotient group $\mathbb{Z}_3$). As already outlined in the article of Latin squares, this really is a Latin square of order 9.
Now, to yield a Sudoku, let us permute the rows (or equivalently the columns) in such a way, that each block is redistributed exactly once into each block - for example order them $1, 4, 7, 2, 5, 8, 3, 6, 9$. This of course preserves the Latin square property. Furthermore, in each block the lines have distinct first component by construction and each line in a block has distinct entries via the second component, because the blocks' second

components originally formed a Latin square of order 3(from the subgroup $\mathbb{Z}_3$).

Thus we really get a Sudoku (Rename the pairs to numbers 1...9 if you wish)! With the example and the row permutation above, we yield:

Generating a Sudoku

(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)	(2,0)	(2,1)	(2,2)
(1,0)	(1,1)	(1,2)	(2,0)	(2,1)	(2,2)	(0,0)	(0,1)	(0,2)
(2,0)	(2,1)	(2,2)	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)
(0,1)	(0,2)	(0,0)	(1,1)	(1,2)	(1,0)	(2,1)	(2,2)	(2,0)
(1,1)	(1,2)	(1,0)	(2,1)	(2,2)	(2,0)	(0,1)	(0,2)	(0,0)
(2,1)	(2,2)	(2,0)	(0,1)	(0,2)	(0,0)	(1,1)	(1,2)	(1,0)
(0,2)	(0,0)	(0,1)	(1,2)	(1,0)	(1,1)	(2,2)	(2,0)	(2,1)
(1,2)	(1,0)	(1,1)	(2,2)	(2,0)	(2,1)	(0,2)	(0,0)	(0,1)
(2,2)	(2,0)	(2,1)	(0,2)	(0,0)	(0,1)	(1,2)	(1,0)	(1,1)

For this method to work, one generally does not need a product of two equally-sized groups. A so-called short exact sequence of finite groups of appropriate size already does the job! Try for example the group $\mathbb{Z}_4$with quotient- and subgroup $\mathbb{Z}_2$. It seems clear (already from enumeration arguments), that not all Sudokus can be generated this way!

Terminology

A *puzzle* is a partially completed *grid*. The initial puzzle values are known as *givens* or *clues*. The *regions* are also called *boxes* or *blocks*. The use of the term *square* is generally avoided because of ambiguity. Horizontally adjacent blocks constitute a *band* (the vertical equivalent is called a *stack*).

A *proper* puzzle has a unique solution. The constraint *'each digit appears in each row, column and region'* is called the *One Rule*.

See basic terms [7] or the List of Sudoku terms and jargon for an expanded list of terminology.

Variants

Sudoku regions are polyominoes. Although the classic "3x3" Sudoku is made of square nonominoes, it is possible to apply the rules of Sudoku to puzzles of other sizes – the 2x2 and 4x4 square puzzles, for example. Only N^2xN^2 Sudoku puzzles can be tiled with square polyominoes. Another popular variant is made of rectangular regions – for example, 2x3 hexominoes tiled in a 6x6 grid. The following notation is used for discussing this variant. *R*x*C* denotes a rectangular region with *R* rows and *C* columns. The implied grid configuration has *R* blocks per band (*C* blocks per stack), *C*×*R* bands×stacks and grid dimensions *N*x*N*, with $N = R \cdot C$. Puzzles of size NxN, where N is prime can only be tiled with irregular N-ominoes. Each NxN grid can be tiled multiple ways with N-ominoes. Before enumerating the number of solutions to a Sudoku grid of size NxN, it is necessary to determine how many N-omino tilings exist for a given size (including the standard square tilings, as well as the rectangular tilings).

The size ordering of Sudoku puzzle can be used to define an integer series, e.g. for square Sudoku, the integer series of possible solutions ((sequence A107739 [13] in OEIS)).

Sudoku with square *N*x*N* regions are more symmetrical than immediately obvious from the *One Rule*. Each row and column intersects N regions and shares N cells with each. The number of bands and stacks also equals N. Rectangular Sudoku do not have these properties. The "3x3" Sudoku is additionally unique: N is also the number of row-column-region constraints from the *One Rule* (i.e. there are N=3 types of *units*).

See the List of Sudoku terms and jargon for an expanded list and classification of variants.

Definition of terms and labels

Term Labels

1	2	3	4	5	6	7	8	9
2	B1			B2			B3	
3								
4					c	B6		
5	B4			B5	5			
6			r	5	6			
7			B8			B9		
8	B7							
9								

Let

- s be a solution to a Sudoku grid with specific dimensions, satisfying the *One Rule* constraints
- S = {s}, be the set of all solutions
- |S|, the *cardinality* of S, is the number of elements in S, i.e. the number of solutions, also known as the *size* of S.

The number of solutions depends on the grid dimensions, rules applied and the definition of distinct solutions. For the 3×3 region grid, the conventions for labeling the rows, columns, blocks (boxes) are as shown. Bands are numbered top to bottom, stacks left to right. By extension the labeling scheme applies to any rectangular Sudoku grid.

Term labels for box-row and box-column *triplets* are also shown.

- triplet - an unordered combination of 3 values used in a box-row (or box-column), e.g. a triplet = {3, 5, 7} means the values 3, 5, 7 occur in a box-row (column) without specifying their location order. A triplet has 6 (3!) ordered permutations. By convention, triplet values are represented by their ordered digits. Triplet objects are labeled as:
- r*BR*, identifies a row triplet for box *B* and (grid) row *R*, e.g. r56 is the triplet for box 5, row 6, using the grid row label.
- c*BC*, identifies similarly a column triplet for box *B* and (grid) column row *C*.

The {a, b, c} notation also reflects that fact a triplet is a subset of the allowed digits. For regions of arbitrary dimension, the related object is known as a *minicol(umn)* or *minirow*.

Enumerating Sudoku solutions

The answer to the question 'How many Sudokus are there?' depends on the definition of when similar solutions are considered different.

Enumerating all possible Sudoku solutions

For the enumeration of *all* possible solutions, two solutions are considered distinct if any of their corresponding (81) cell values differ. Symmetry relations between similar solutions are ignored., e.g. the rotations of a solution are considered distinct. Symmetries play a significant role in the enumeration strategy, but not in the count of *all* possible solutions.

Enumerating the Sudoku 9×9 grid solutions directly

The first approach taken historically to enumerate Sudoku solutions ('Enumerating possible Sudoku grids' [2] by Felgenhauer and Jarvis) was to analyze the permutations of the top band used in valid solutions. Once the Band1 symmetries and equivalence classes for the partial grid solutions were identified, the completions of the lower two bands were constructed and counted for each equivalence class. Summing completions over the equivalence classes, weighted by class size, gives the total number of solutions as 6,670,903,752,021,072,936,960 (6.67×10^{21}). The value was subsequently confirmed numerous times independently. The Algorithm details section (below) describes the method.

Enumeration using band generation

A second enumeration technique based on *band generation* was later developed that is significantly less computationally intensive.

Enumerating essentially different Sudoku solutions

Two valid grids are *essentially* the same if one can be derived from the other.

Sudoku preserving symmetries

The following operations always translate a valid Sudoku grid into another valid grid: (values represent permutations for classic Sudoku)

- Relabeling symbols (9!)
- Band permutations (3!)
- Row permutations within a band ($3!^3$)
- Stack permutations (3!)
- Column permutations within a stack ($3!^3$)
- Reflection, transposition and rotation (2). (Given any transposition or quarter-turn rotation in conjunction with the above permutations, any combination of reflections, transpositions and rotations can be produced, so these operations only contribute a factor of 2.)

These operations define a symmetry relation between equivalent grids. Excluding relabeling, and with respect to the 81 grid cell values, the operations form a subgroup of the symmetric group S_{81}, of order $3!^8\times2 = 3359232$.

Identifying distinct solutions with Burnside's Lemma

For a solution, the set of equivalent solutions which can be reached using these operations (excluding relabeling), form an orbit of the symmetric group. The number of essentially different solutions is then the number of orbits, which can be computed using Burnside's lemma. The Burnside *fixed points* are solutions that differ only by relabeling. Using this technique, Jarvis/Russell [4] computed the number of essentially different (symmetrically distinct) solutions as 5,472,730,538.

Enumeration results

The number of ways of filling in a blank Sudoku grid was shown in May 2005 to be 6,670,903,752,021,072,936,960 (~6.67×10^{21}) (original announcement [8]). The paper 'Enumerating possible Sudoku grids' [2] , by Felgenhauer and Jarvis, describes the calculation.

Since then, enumeration results for many Sudoku variants have been calculated: these are summarised below.

Sudoku with rectangular regions

In the table, "Dimensions" are those of the regions (e.g. 3x3 in normal Sudoku). The "Rel Err" column indicates how a simple approximation, using the generalised method of Kevin Kilfoil [9] , compares to the true grid count: it is an underestimate in all cases evaluated so far.

Dimensions	Nr Grids	Attribution	Verified?	Rel Err
1x?	*see Latin squares*			n/a
2×2	288	various [10]	Yes	-11.1%
2×3	28200960 = c. 2.8×10^{7}	Pettersen [11]	Yes	-5.88%
2×4	29136487207403520 = c. 2.9×10^{16}	Russell [12]	Yes	-1.91%
2×5	1903816047972624930994913280000 = c. 1.9×10^{30}	Pettersen [13]	Yes	-0.375%
2×6	38296278920738107863746324732012492486187417600000 = c. 3.8×10^{49}	Pettersen [14]	No	-0.238%
3×3	6670903752021072936960 = c. 6.7×10^{21}	Felgenhauer/Jarvis [3]	Yes	-0.207%
3×4	81171437193104932746936103027318645818654720000 = c. 8.1×10^{46}	Pettersen / Silver [15]	No	-0.132%
3×5	**unknown**, estimated c. 3.5086×10^{84}	Silver [16]	n/a	
4×4	**unknown**, estimated c. 5.9584×10^{98}	Silver [17]	n/a	
4×5	**unknown**, estimated c. 3.1764×10^{175}	Silver [18]	n/a	
5×5	**unknown**, estimated c. 4.3648×10^{308}	Silver / Pettersen [19]	n/a	

The standard 3×3 calculation can be carried out in less than a second on a PC. The 3×4 (= 4×3) problem is much harder and took 2568 hours to solve, split over several computers.

Sudoku bands

For large (R,C), the method of Kevin Kilfoil [20] (generalised method:[9]) is used to estimate the number of grid completions. The method asserts that the Sudoku row and column constraints are, to first approximation, conditionally independent given the box constraint. Omitting a little algebra, this gives the Kilfoil-Silver-Pettersen formula:

$$\text{Number of Grids} \simeq \frac{b_{R,C}^{C} \times b_{C,R}^{R}}{(RC)!^{RC}}$$

where $b_{R,C}$ is the number of ways of completing a Sudoku band [7] of R horizontally adjacent $R \times C$ boxes. Petersen's algorithm [21] , as implemented by Silver [22] , is currently the fastest known technique for exact evaluation of these $b_{R,C}$.

The band counts **for problems whose full Sudoku grid-count is unknown** are listed below. As in the previous section, "Dimensions" are those of the regions.

Dimensions	Nr Bands	Attribution	Verified?
$2 \times C$	$(2C)!\,(C!)^2$	(obvious result)	Yes
$3 \times C$	$(3C)!(C!)^6 \sum_{k=0...C} \binom{C}{k}^3$	Pettersen [11]	Yes
$4 \times C$	(long expression: see below)	Pettersen [23]	Yes [24]
4×4	$16! \times 4!^{12} \times 1273431960$ = c. 9.7304×10^{38}	Silver [17]	Yes
4×5	$20! \times 5!^{12} \times 879491145024$ = c. 1.9078×10^{55}	Russell [25]	Yes
4×6	$24! \times 6!^{12} \times 677542845061056$ = c. 8.1589×10^{72}	Russell [26]	Yes
4×7	$28! \times 7!^{12} \times 563690747238465024$ = c. 4.6169×10^{91}	Russell [27]	Yes
(calculations up to 4×100 have been performed by Silver [28] , but are not listed here)			
5×3	$15! \times 3!^{20} \times 324408987992064$ = c. 1.5510×10^{42}	Silver [18]	Yes#
5×4	$20! \times 4!^{20} \times 518910423730214314176$ = c. 5.0751×10^{66}	Silver [18]	Yes#
5×5	$25! \times 5!^{20} \times 1165037550432885119709241344$ = c. 6.9280×10^{93}	Pettersen / Silver [29]	No
5×6	$30! \times 6!^{20} \times 3261734691836217181002772823310336$ = c. 1.2127×10^{123}	Pettersen / Silver [30]	No
5×7	$35! \times 7!^{20} \times 10664509989209199533282539525535793414144$ = c. 1.2325×10^{154}	Pettersen / Silver [31]	No
5×8	$40! \times 8!^{20} \times 39119312409010825966116046645368393936122855616$ = c. 4.1157×10^{186}	Pettersen / Silver [32]	No
5×9	$45! \times 9!^{20} \times 156805448016006165940259131378329076911634037242834944$ = c. 2.9406×10^{220}	Pettersen / Silver [33]	No
5×10	$50! \times 10!^{20} \times 67443174870122749266442113849022431593112673476558194874 7776$ = c. 3.2157×10^{255}	Pettersen / Silver [34]	No

: same author, different method

The expression for the 4×C case is:

where:

the outer summand is taken over all a,b,c such that $0<=a,b,c$ and $a+b+c=2C$

the inner summand is taken over all $k_{12},k_{13},k_{14},k_{23},k_{24},k_{34} \geq 0$ such that

$k_{12},k_{34} \leq a$ and

$k_{13},k_{24} \leq b$ and

$k_{14},k_{23} \leq c$ and

$k_{12}+k_{13}+k_{14} = a-k_{12}+k_{23}+k_{24} = b-k_{13}+c-k_{23}+k_{34} = c-k_{14}+b-k_{24}+a-k_{34} = C$

Sudoku with additional constraints

The following are all restrictions of 3x3 Sudokus. The type names have not been standardised: click on the attribution links to see the definitions.

Type	Nr Grids	Attribution	Verified?
3doku	104015259648	Stertenbrink [35]	Yes
Disjoint Groups	201105135151764480	Russell [36]	Yes
Hypercube	37739520	Stertenbrink [37]	Yes
Magic Sudoku	5971968	Stertenbrink [38]	Yes
Sudoku X	55613393399531520	Russell [39]	Yes
NRC Sudoku	6337174388428800	Brouwer [40]	Yes

All Sudokus remain valid (no repeated numbers in any row, column or region) under the action of the Sudoku preserving symmetries (see also Jarvis [4]). Some Sudokus are special in that some operations merely have the effect of relabelling the digits; several of these are enumerated below.

Transformation	Nr Grids	Verified?
Transposition	10980179804160	Indirectly
Quarter Turn	4737761280	Indirectly
Half Turn	56425064693760	Indirectly
Band cycling	5384326348800	Indirectly
Within-band row cycling	39007939461120	Indirectly

Further calculations of this ilk combine to show that the number of essentially different Sudoku grids is 5,472,730,538, for example as demonstrated by Jarvis / Russell [4] and verified by Pettersen [41] . Similar methods have been applied to the 2x3 case, where Jarvis / Russell [42] showed that there are 49 essentially different grids (see also the article by Bailey, Cameron and Connelly [43]), to the 2x4 case, where Russell [44] showed that there are 1,673,187 essentially different grids (verified by Pettersen [45]), and to the 2x5 case where Pettersen [46] showed that there are 4,743,933,602,050,718 essentially different grids (not verified).

Minimum number of givens

Proper puzzles have a unique solution. An **irreducible puzzle** (or **minimal puzzle**) is a proper puzzle from which no givens can be removed leaving it a proper puzzle (with a single solution). It is possible to construct minimal puzzles with different numbers of givens. This section discusses the minimum number of givens for proper puzzles.

Ordinary Sudoku

The lowest known is 17 givens in general Sudoku, or 18 when the positions of the givens are constrained to be half-turn rotationally symmetric. It is conjectured that these are the best possible, evidence for which stems from extensive randomised searching:

- Gordon Royle has compiled a list of (as of September 16, 2009) 48826 17-clue puzzles [6] , no two of which are isomorphic. None of these was isomorphic to a symmetric puzzle, nor contained a 16-clue puzzle.
- The most fruitful set of clue positions, in terms of number of distinct 17-clue puzzles they admit, from Royle's list have been exhaustively searched for 17-clue puzzles. All 36 puzzles found by this process were already in Royle's

list. All 34 puzzles on the next most fruitful set of clue positions were also in Royle's list.
- The most fruitful solution grids (in the same sense) have been exhaustively searched for 16-clue puzzles using CHECKER [47] with no success. This includes one, "strangely familiar" [48] , grid that yields exactly 29 different 17-clue puzzles, all of which had already been discovered by Royle's random search technique (suggesting once again that Royle's list of 17s is close to complete).

Sudoku with additional constraints

Additional constraints (here, on 3×3 Sudokus) lead to a smaller minimum number of clues.

- 3doku: *no results for this variant*
- Disjoint Groups: some 12-clue puzzles [49] have been demonstrated by Glenn Fowler. Later also 11-clue puzzles are found. It is not known if this is the best possible.
- Hypercube: various 8-clue puzzles [50] (the best possible) have been demonstrated by Guenter Stertenbrink.
- Magic Sudoku: a 7-clue example [51] has been provided by Guenter Stertenbrink. It is not known if this is the best possible.
- Sudoku X: a list of 1167 12-clue puzzles [52] has been collected by Ruud van der Werf. It is not known if this is the best possible.

1		2						
		3						
								4
	4			5				
	6			7				
							2	
	8							
						8		

Disjoint Groups: 11 clues.

- NRC Sudoku: an 11-clue example [40] has been provided by Andries Brouwer. It is not known if this is the best possible.
- 2-Quasi-Magic Sudoku: a 4-clue example [53] has been provided by Tony Forbes. It is suspected that this is the best possible.

Sudoku with irregular regions

"Du-sum-oh" [54] (a.k.a. "geometry number place") puzzles replace the 3×3 (or *R*×*C*) regions of Sudoku with irregular shapes of a fixed size. Bob Harris has proved [55] that it is always possible to create *N*-1 clue du-sum-ohs on an *N*×*N* grid, and has constructed several examples. Johan de Ruiter has proved [56] that for any *N*>3 there exist polyomino tilings that can not be turned into a Sudoku puzzle with *N* irregular shapes of size *N*.

Sum number place ("Killer Sudoku")

In sum number place (Samunamupure), the regions are of irregular shape and various sizes. The usual constraints of no repeated value in any row, column or region apply. The clues are given as sums of values within regions (e.g. a 4-cell region with sum 10 must consist of values 1,2,3,4 in some order). The minimum number of clues for Samunamupure is not known, nor even conjectured.

A variant on Miyuki Misawa's web site [57] replaces sums with relations: the clues are symbols **=**, **<** and **>** showing the relative values of (some but not all) adjacent region sums. She demonstrates an example with only eight relations. It is not known whether this is the best possible.

Method and algorithm details for the 9×9 grid direct enumeration

The approach described here was the historically first strategy employed to enumerate the Sudoku 9×9 grid solutions, as published by Felgenhauer and Jarvis in 'Enumerating possible Sudoku grids' [2] . The methods and algorithms used are very straight forward and provide a practical introduction to several mathematical concepts. The development is presented here for those wishing to explore these topics.

The strategy begins by analyzing the permutations of the top band used in valid solutions. Once the Band1 symmetries and equivalence class for the partial solutions are identified, the completions of the lower two bands are constructed and counted for each equivalence class. Summing the completions over the equivalence classes gives the total number of solutions as 6,670,903,752,021,072,936,960 (c. 6.67×10^{21}).

Counting the top band permutations

The Band1 algorithm proceeds as follows:

- Choose a canonical labeling of the digits by assigning values for B1, e.g.

```
1 2 3
4 5 6
7 8 9
```

Compute the rest of the Band1 permutations relative to the B1 canonical choice.

- Compute the permutations of B2 by partitioning the B1 cell values over the B2 row triplets. From the triplet combinations compute the B2 permutations. There are k=0..3 ways to choose the:

B1 r11 values for B2 r22, the rest must go to r23,

B1 r12 values for B2 r23, the rest must go to r21,

B1 r13 values for B2 r21, the rest must go to r22, i.e.

$$\text{N combinations for B2} = \sum_{k=0..3} \binom{3}{k}^3$$

(This expression may be generalized to any *R*×3 box band variant. (Pettersen [11]). Thus B2 contributes 56×6^3 permutations.

- The choices for B3 triplets are row-wise determined by the B1 B2 row triplets. B3 always contributes 6^3 permutations.

The permutations for Band1 are $9! \times 56 \times 6^6 = 9! \times 2612736 \sim 9.48 \times 10^{11}$.

Band1 permutation details

Triplet rBR(box/row) Labels

r 1 1	r 2 1	r 3 1
r 1 2	r 2 2	r 3 2
r 1 3	r 2 3	r 3 3

The permutations of B1 are the number of ways to relabel the 9 digits, 9! = 362880. Counting the permutations for B2 is more complicated, because the choices for B2 depend on the values in B1. (This is a visual representation of the expression given above.) The conditional calculation needs a branch (sub-calculation) for each alternative. Fortunately, there are just 4 cases for the top B2 triplet (r21): it contains either 0, 1, 2, or 3 of the digits from the B1 middle row triplet(r12). Once this B2 top row choice is made, the rest of the B2 combinations are fixed. The Band1 row triplet labels are shown on the right.

(Note: Conditional combinations becomes an increasingly difficult as the computation progresses through the grid. At this point the impact is minimal.)

Case 0 Matching Cells Triplets

1	2	3	7	8	9	4	5	6
4	5	6	1	2	3	7	8	9
7	8	9	4	5	6	1	2	3

Case 0: No Overlap. The choices for the triplets can be determined by elimination.

r21 can't be r11 or r12 so it must be = r13; r31 must be = r12 etc.

The Case 0 diagram shows this configuration, where the pink cells are triplet values that can be arranged in any order within the triplet. Each triplet has 3! = 6 permutations. The 6 triplets contribute 6^6 permutations.

Case 3: 3 Digits Match: triplet r21 = r12. The same logic as case 0 applies, but with a different triplet usage. Triplet r22 must be = r13, etc. The number of permutations is again 6^6. (Felgenhauer/Jarvis [2] call the cases 0 and 3 the *pure match* case.

Case 1 Match - Triplet Cell Options

1	2	3	3	3	2	3	2	1
4	5	6	1	3	2	3	2	1
7	8	9	1	2	1	3	2	1

Case 1: 1 Match for r21 from r12

In the Case 1 diagram, B1 cells show canonical values, which are color coded to show their row-wise distribution in B2 triplets. Colors reflect distribution but not location or values. For this case: the B2 top row triplet (r21) has 1 value from B1 middle triplet, the other colorings can now be deduced. E.g. the B2 bottom row triplet (r23) coloring is forced by r21: the other 2 B1 middle values must go to bottom, etc. Fill in the number of B2 options for each color, 3..1, beginning top left. The B3 color coding is omitted since the B3 choices are row-wise determined by B1, B2. B3 always contributes 3! permutations per row triplet, or 6^3 for the block.

For B2, the triplet values can appear in any position, so a 3! permutation factor still applies, for each triplet. However, since some of the values were paired relative to their origin, using the raw option counts would overcount the number of permutations, due to interchangeability within the pairing. The option counts need to be divided by the permuted size of their grouping (2), here 2!.= 2 (See n Choose k) The pair in each row cancels the 2s for the B2 option counts, leaving a B2 contribution of $3^3 \times 6^3$. The B2×B3 combined contribution is $3^3 \times 6^6$.

Case 2 Match - Triplet Cell Options

1	2	3	3	2	3	3	2	1
4	5	6	2	1	3	3	2	1
7	8	9	2	1	1	3	2	1

Case 2: 2 Matches for r21 from r12. The same logic as case 1 applies, but with the B2 option count column groupings reversed. Case 3 also contributes $3^3 \times 6^6$ permutations.

Totaling the 4 cases for Band1 B1..B3 gives $9! \times 2 \times (3^3 + 1) \times 6^6 = 9!\ 56 \times 6^6$ permutations.

Band1 symmetries and equivalence classes

Symmetries are used to reduce the computational effort to enumerate the Band1 permutations.

A symmetry in mathematics is an operation that preserves a quality of an object. For a Sudoku solution grid, a symmetry is a transformation whose result is also a solution grid. The following symmetries apply independently for the top band:

- Block B1 values may be relabeled, giving 9! permutations
- Blocks B1..3 may be interchanged, with 3!=6 permutations
- Rows 1..3 may be interchanged, with 3!=6 permutations
- Within each block, the 3 columns may be interchanged, giving 6^3 permutations.

Combined, the symmetries give $9! \times 6^5 = 362880 \times 7776$ equivalent permutations for each Band1 solution.

A symmetry defines an equivalence relation, here, between the solutions, and partitions the solutions into a set of equivalence classes. The Band1 row, column and block symmetries divide the 56*6^6 permutations into (not less than) 336 (56×6) equivalence classes with (up to) 6^5 permutations in each, and 9! relabeling permutations for each class. (*Min/Max* caveats apply since some permutations may not yield distinct elements due to relabeling.)

Since the solution for any member of an equivalence class can be generated from the solution of any other member, we only need to enumerate the solutions for a single member in order to enumerate all solutions over all classes. Let

- sb : be a valid permutation of the top band
- Sb = [sb] : be an equivalence class, relative to sb and some equivalence relation
- Sb.z = |Sb| : the size of Sb, be the number of sb elements (permutations) in [sb]
- Sb.n : be the number of Band2,3 completions for (any) sb in Sb
- {Sb} : be the set of all Sb equivalence classes relative to the equivalence relation
- {Sb}.z = |{Sb}| : be the number of equivalence classes

The total number of solutions *N* is then:

$$N = \sum_{\{Sb\}} \mathrm{Sb.z} \times \mathrm{Sb.n}$$

Solution and counting permutation symmetry

The Band1 symmetries (above) are *solution permutation symmetries* defined so that a permuted solution is also a solution. For the purpose of enumerating solutions, a *counting symmetry* for grid completion can be used to define band equivalence classes that yield a minimal number of classes.

Counting symmetry partitions valid Band1 permutations into classes that place the same completion constraints on lower bands; all members of a band *counting symmetry* equivalence class must have the same number of grid completions since the completion constraints are equivalent. Counting symmetry constraints are identified by the Band1 column triplets (a column value set, no implied element order). Using band counting symmetry, a minimal generating set of 44 equivalence classes [58] was established.

The following sequence demonstrates mapping a band configuration to a counting symmetry equivalence class. Begin with a valid band configuration.

1	2	3	5	8	6	7	4	9
4	5	6	9	1	7	8	2	3
7	8	9	4	3	2	5	1	6

Build column triplets by ordering the column values within each column. This is not a valid Sudoku band, but does place the same constraints on the lower bands as the example.

1	2	3	4	1	2	5	1	3
4	5	6	5	3	6	7	2	6
7	8	9	9	8	7	8	4	9

Construct an equivalence class ID from the B2, B3 column triplet values. Use column and box swaps to achieve the lowest lexicographical ID. The last figure shows the column and box ordering for the ID: 124 369 578 138 267 459. All Band1 permutations with this counting symmetry ID will have the same number of grid completions as the original example. An extension of this process can be used to build the largest possible *band counting symmetry* equivalence classes.

1	2	3	1	3	5	1	2	4
4	5	6	2	6	7	3	6	5
7	8	9	4	9	8	8	7	9

Note, while column triplets are used to construct and identify the equivalence classes, the class members themselves are the valid Band1 permutations: class size (Sb.z) reflects column triplet permutations compatible with the *One Rule* solution requirements. *Counting symmetry* is a completion property and applies only to a partial grid (band or stack). *Solution symmetry* for preserving solutions can be applied to either partial grids (bands, stacks) or full grid solutions. Lastly note, *counting symmetry* is more restrictive than simple numeric completion count equality: two (distinct) bands belong to the same *counting symmetry* equivalence class only if they impose equivalent completion constraints.

Band 1 reduction details

Symmetries group similar object into equivalence classes. Two numbers need to be distinguished for equivalence classes, and band symmetries as used here, a third:

- the number of equivalence classes ({Sb}.n).
- the cardinality, size or number of elements in an equivalence class, which may vary by class (Sb.z)
- the number of Band2,3 completions compatible with a member of a Band1 equivalence class (Sb.n)

The Band1 (6^5) symmetries divide the (56×6^6) Band1 valid permutations into (not less than) 336 (56×6) equivalence classes with (up to) 6^5 permutations each. The *not less than* and *up to* caveats are necessary, since some combinations of the transformations may not produce distinct results, when relabeling is required (see below). Consequently some equivalence classes may contain less than 6^5 distinct permutations and the theoretical minimum number of classes may not be achieved.

Each of the valid Band1 permutations can be expanded (completed) into a specific number of solutions with the Band2,3 permutations. By virtue of their similarity, each member of an equivalence class will have the same number of completions. Consequently we only need to construct the solutions for one member of each equivalence class and then multiply the number of solutions by the size of the equivalence class. We are still left with the task of identifying and calculating the size of each equivalence class. Further progress requires the dexterous application of computational techniques to catalogue (classify and count) the permutations into equivalence classes.

Felgenhauer/Jarvis [2] catalogued the Band1 permutations using lexicographical ordered IDs based on the ordered digits from blocks B2,3. Block 1 uses a canonical digit assignment and is not needed for a unique ID. Equivalence class identification and linkage uses the lowest ID within the class.

Application of the (2×6^2) B2,3 symmetry permutations produces 36288 (28×6^4) equivalence classes, each of size 72. Since the size is fixed, the computation only needs to find the 36288 equivalence class IDs. (Note: in this case, for any Band1 permutation, applying these permutations to achieve the lowest ID provides an index to the associated equivalence class.)

Application of the rest of the block, column and row symmetries provided further reduction, i.e. allocation of the 36288 IDs into fewer, larger equivalence classes. When the B1 canonical labeling is lost through a transformation, the result is relabeled to the canonical B1 usage and then catalogued under this ID. This approach generated 416 equivalence classes, somewhat less effective than the theoretical 336 minimum limit for a full reduction.

Application of *counting symmetry* patterns for *duplicate paired* digits achieved reduction to 174 and then to 71 equivalence classes. The introduction of equivalence classes based on *band counting symmetry* (subsequent to Felgenhauer/Jarvis by Russell [58]) reduced the equivalence classes to a minimum generating set of 44.

The diversity of the ~2.6×10^6, 56×6^6 Band1 permutations can be reduced to a set of 44 Band1 equivalence classes. Each of the 44 equivalence classes can be expanded to millions of distinct full solutions, but the entire solution space has a common origin in these 44. The 44 equivalence classes play a central role in other enumeration approaches as well, and speculation will return to the characteristics of the 44 classes when puzzle properties are explored later.

Band 2-3 completion and results

Enumerating the Sudoku solutions breaks into an initial setup stage and then into two nested loops. Initially all the valid Band1 permutations are grouped into equivalence classes, who each impose a common constraint on the Band2,3 completions. For each of the Band1 equivalence classes, all possible Band2,3 solutions need to be enumerated. An outer Band1 loop iterates over the 44 equivalence classes. In the inner loop, all lower band completions for each of the Band1 equivalence class are found and counted.

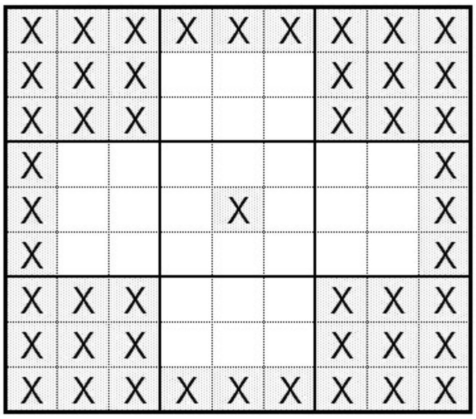
The computation required for the lower band solution search can be minimised by the same type of symmetry application used for Band1. There are 6! (720) permutations for the 6 values in column 1 of Band2,3. Applying the lower band (2) and row within band (6×6) permutations creates 10 equivalence classes of size 72. At this point, completing 10 sets of solutions for the remaining 48 cells with a recursive descent, backtracking algorithm is feasible with 2 GHz class PC so further simplification is not required to carry out the enumeration.

Using this approach, the number of ways of filling in a blank Sudoku grid was shown in May 2005 (original announcement [59]) to be 6,670,903,752,021,072,936,960 (6.67×10^{21}). The paper 'Enumerating possible Sudoku grids' [2] by Felgenhauer and Jarvis, describes the calculation.

The result, as confirmed by Russell [58] , also contains the distribution of solution counts for the 44 equivalence classes. The listed values are before application of the 9! factor for labeling and the two 72 factors ($72^2 = 5184$) for each of Stack 2,3 and Band2,3 permutations. The number of completions for each class is consistently on the order of 100,000,000, while the number of Band1 permutations covered by each class however varies from 4 − 3240. Within this wide size range, there are clearly two clusters. Ranked by size, the lower 33 classes average ~400 permutations/class, while the upper 11 average ~2100. The disparity in consistency between the distributions for size and number of completions or the separation into two clusters by size is yet to be examined.

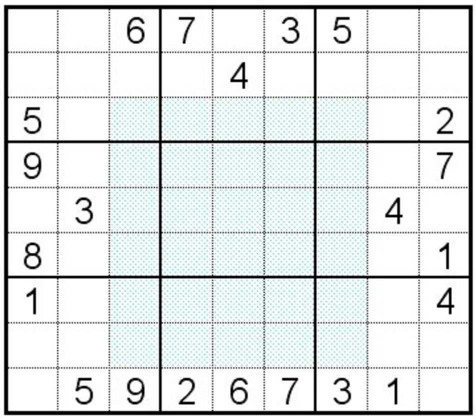
Constraints of Clue Geometry

It has been postulated that no proper sudoku can have clues limited to the range of positions in the pattern below.[60]



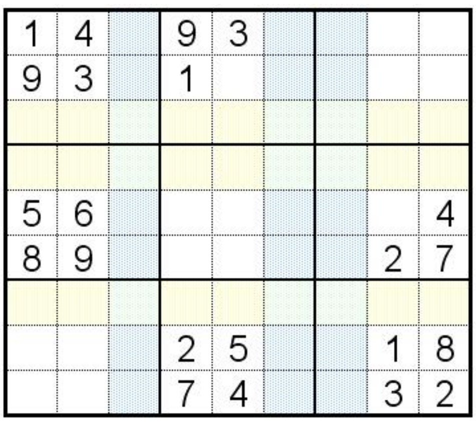
A range of clue positions insufficient for a proper sudoku

The largest rectangular orthogonal "hole" (region with no clues) in a proper sudoku is believed to be a rectangle of 30 cells (a 5 x 6 rectangular area).[61] [62] One such example is the following with 22 clues:



Sudoku with 30 cell (5 x 6) empty rectangle

The largest total number of empty groups (rows, columns, and squares) in a sudoku is believed to be nine. One example is the following; a sudoku which has 3 empty squares, 3 empty rows, and 3 empty columns and gives 22 clues.[63] [64]



Sudoku with nine empty groups

Automorphic Sudokus

Automorphic sudokus are sudoku puzzles which solve to an automorphic grid. A grid is automorphic if it can be transformed in a way that leads back to the original grid, when that same transformation would not otherwise lead back to the original grid. An example of a grid which is automorphic would be a grid which can be rotated 180 degrees resulting in a new grid where the new cell values are a permutation of the original grid. An example of an automorphic sudoku which solves to such a grid is below[65] .

			5	6			3	4
			7	8			5	6
1	2							
3	4						6	7
							8	9
4	5			2	3			
6	7			4	5			

Example of an Automorphic Sudoku

Notice that if this sudoku is rotated 180 degrees, and the clues relabeled with the permutation (123456789) -> (987654321), it returns to the same sudoku. Expressed another way, this sudoku has the property that every 180 degree rotational pair of clues (a, b) follows the rule (a) + (b) = 10.

Since this sudoku is automorphic, so too its solution grid must be automorphic. Furthermore, every cell which is solved has a symmetrical partner which is solved with the same technique (and the pair would take the form a + b = 10).

In this example the automorphism is easy to identify, but in general automorphism is not always obvious. There are several types of transformations of a sudoku, and therefore automorphism can take several different forms too. Among the population of all sudoku grids, those that are automorphic are rare. They are considered interesting because of their intrinsic mathematical symmetry.

See also

- Sudoku - main Sudoku article
- List of Sudoku terms and jargon
- Dancing Links - Donald Knuth
- Algorithmics of sudoku

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[4] http://www.afjarvis.staff.shef.ac.uk/sudoku/sudgroup.html
[5] http://www2.ic-net.or.jp/~takaken/auto/guest/bbs46.html
[6] Minimum Sudoku (http://www.csse.uwa.edu.au/~gordon/sudokumin.php) by Gordon Royle
[7] Basic terms (http://forum.enjoysudoku.com/viewtopic.php?t=4120) by "Pappocom" (Wayne Gould)
[8] http://forum.enjoysudoku.com/viewtopic.php?p=1721#p1721
[9] http://forum.enjoysudoku.com/post15920.html#p15920
[10] http://forum.enjoysudoku.com/post2992.html#p2992

[11] http://forum.enjoysudoku.com/post11444.html#p11444
[12] http://forum.enjoysudoku.com/post11874.html#p11874
[13] http://forum.enjoysudoku.com/post12191.html#p12191
[14] http://forum.enjoysudoku.com/viewtopic.php?p=37761#p37761
[15] http://forum.enjoysudoku.com/post25773.html#p25773
[16] http://forum.enjoysudoku.com/post17768.html#p17768
[17] http://forum.enjoysudoku.com/post15909.html#p15909
[18] http://forum.enjoysudoku.com/post17750.html#p17750
[19] http://forum.enjoysudoku.com/post19318.html#p19318
[20] http://forum.enjoysudoku.com/post788.html#p788
[21] http://forum.enjoysudoku.com/post18161.html#p18161
[22] http://forum.enjoysudoku.com/post20867.html#p20867
[23] http://forum.enjoysudoku.com/post16169.html#p16169
[24] http://forum.enjoysudoku.com/post18396.html#p18396
[25] http://forum.enjoysudoku.com/post16071.html#p16071
[26] http://forum.enjoysudoku.com/post16071.html#p16071
[27] http://forum.enjoysudoku.com/post16071.html#p16071
[28] http://forum.enjoysudoku.com/post21245.html#p21245
[29] http://forum.enjoysudoku.com/post19318.html#p19318
[30] http://forum.enjoysudoku.com/post19318.html#p19318
[31] http://forum.enjoysudoku.com/post19416.html#p19416
[32] http://forum.enjoysudoku.com/post20867.html#p20867
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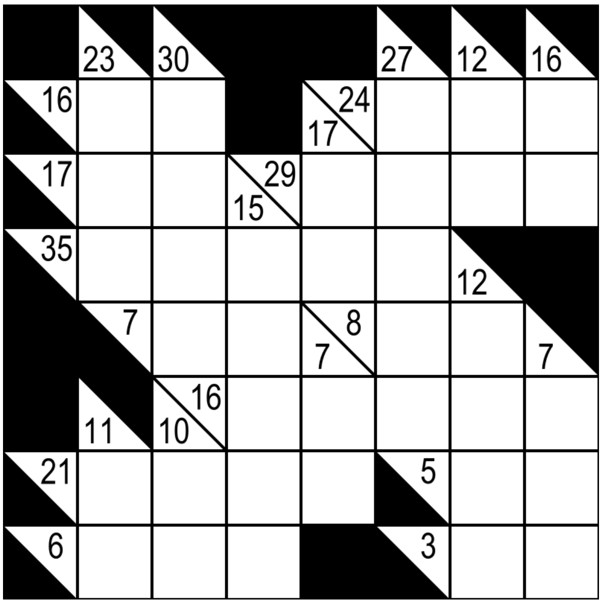
[34] http://forum.enjoysudoku.com/post20867.html#p20867
[35] http://forum.enjoysudoku.com/post11088.html#p11088
[36] http://forum.enjoysudoku.com/post2536.html#p2536
[37] http://forum.enjoysudoku.com/post11216.html#p11216
[38] http://www.setbb.com/phpbb/viewtopic.php?t=366&mforum=sudoku
[39] http://forum.enjoysudoku.com/post3366.html#p3366
[40] http://homepages.cwi.nl/~aeb/games/sudoku/nrc.html
[41] http://forum.enjoysudoku.com/post13388.html#p13388
[42] http://www.afjarvis.staff.shef.ac.uk/sudoku/sud23gp.html
[43] http://www.maths.qmul.ac.uk/~pjc/preprints/sudoku.pdf
[44] http://www.afjarvis.staff.shef.ac.uk/sudoku/sud24gp.html
[45] http://forum.enjoysudoku.com/post29983.html#p29983
[46] http://forum.enjoysudoku.com/post32483.html#p32483
[47] http://www.math.ie/checker.html
[48] http://www.csse.uwa.edu.au/~gordon/sudokupat.php?cn=3
[49] http://forum.enjoysudoku.com/post22062.html#p22062
[50] http://magictour.free.fr/sudoku6
[51] http://forum.enjoysudoku.com/post13320.html#p13320
[52] http://www.sudocue.net/minx.php
[53] http://anthony.d.forbes.googlepages.com/M215qm.pdf
[54] http://www.bumblebeagle.org/dusumoh/9x9/index.html
[55] http://www.bumblebeagle.org/dusumoh/proof/index.html
[56] http://www.liacs.nl/assets/Bachelorscripties/10-04-JohandeRuiter.pdf
[57] http://www7a.biglobe.ne.jp/~sumnumberplace/79790008/
[58] http://www.afjarvis.staff.shef.ac.uk/sudoku/ed44.html
[59] http://forum.enjoysudoku.com/post1721.html#p1721
[60] http://forum.enjoysudoku.com/viewtopic.php?t=5384&postdays=0&postorder=asc&start=0 *(ask for some patterns that they don't have puzzles)*
[61] http://forum.enjoysudoku.com/viewtopic.php?t=4209 *largest hole in a sudoko; largest empty space*
[62] http://www.flickr.com/photos/npcomplete/2471768905/ *large empty space*
[63] http://forum.enjoysudoku.com/post7486.html#p7486 *largest number of empty groups?*
[64] http://www.flickr.com/photos/npcomplete/2361922691/ *clues bunched in clusters*
[65] http://www.flickr.com/photos/npcomplete/2629305621/"(Six Dots with 5 x 5 Empty Hole)

External links

- Enumerating possible Sudoku grids (http://www.afjarvis.staff.shef.ac.uk/sudoku/sudoku.pdf) by Felgenhauer and Jarvis, describes the all-solutions calculation
- V. Elser's difference-map algorithm also solves Sudoku (http://plus.maths.org/latestnews/jan-apr06/sudoku/index.html)
- Sudoku Puzzle - an Exercise in Constraint Programming and Visual Prolog 7 (http://www.visual-prolog.com/vip/example/pdcExample/sudoku.htm) by Carsten Kehler Holst (in Visual Prolog)
- Sudoku Squares and Chromatic Polynomials (http://www.ams.org/notices/200706/tx070600708p.pdf) by Herzberg and Murty, treats Sudoku puzzles as vertex coloring problems in graph theory.

Kakuro

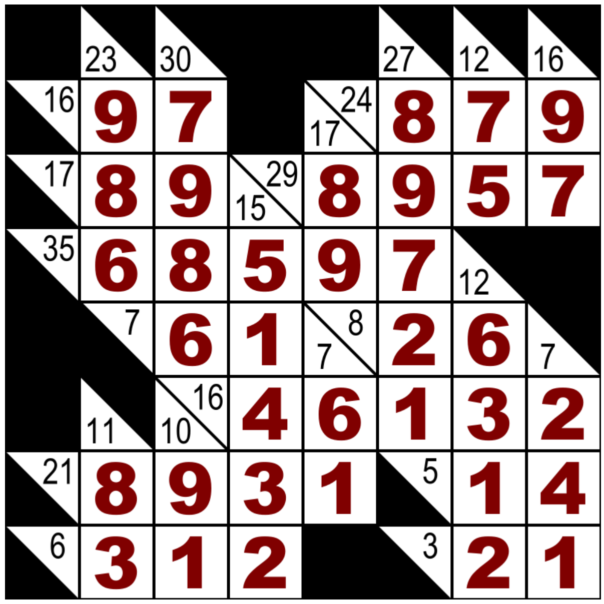
Kakuro or **Kakkuro** (Japanese: カックロ) is a kind of logic puzzle that is often referred to as a mathematical transliteration of the crossword. Kakuro puzzles are regular features in many math-and-logic puzzle publications in the United States. Dell Magazines came up with the original English name *Cross Sums* and other names such as *Cross Addition* have also been used, but the Japanese name *Kakuro,* abbreviation of Japanese *kasan kurosu,* (加算クロス, addition cross) seems to have gained general acceptance and the puzzles appear to be titled this way now in most publications. The popularity of Kakuro in Japan is immense, second only to Sudoku among Nikoli's famed logic-puzzle offerings.[1]



An easy Kakuro puzzle

The canonical Kakuro puzzle is played in a grid of filled and barred cells, "black" and "white" respectively. Puzzles are usually 16×16 in size, although these dimensions can vary widely. Apart from the top row and leftmost column which are entirely black, the grid is divided into "entries" — lines of white cells — by the black cells. The black cells contain a diagonal slash from upper-left to lower-right and a number in one or both halves, such that each horizontal entry has a number in the black half-cell to its immediate left and each vertical entry has a number in the black half-cell immediately above it. These numbers, borrowing crossword terminology, are commonly called "clues".

The object of the puzzle is to insert a digit from 1 to 9 inclusive into each white cell such that the sum of the numbers in each entry matches the clue associated with it and that no digit is duplicated in any entry. It is that lack of duplication that makes creating Kakuro puzzles with unique solutions possible, and which means solving a Kakuro puzzle involves investigating combinations more, compared to Sudoku in which the focus is on permutations. There is an unwritten rule for making Kakuro puzzles that each clue must have at least two numbers that add up to it. This is because including one number is mathematically trivial when solving Kakuro puzzles; one can simply disregard the number entirely and subtract it from the clue it indicates.



Solution for the above puzzle

At least one publisher[2] includes the constraint that a given combination of numbers can only be used once in each grid, but still markets the puzzles as plain Kakuro.

Some publishers prefer to print their Kakuro grids exactly like crossword grids, with no labeling in the black cells and instead numbering the entries, providing a separate list of the clues akin to a list of crossword clues. (This eliminates the row and column that are entirely black.) This is purely an issue of image and does not affect solving.

In discussing Kakuro puzzles and tactics, the typical shorthand for referring to an entry is "(clue, in numerals)-in-(number of cells in entry, spelled out)", such as "16-in-two" and "25-in-five". The exception is what would otherwise be called the "45-in-nine" — simply "45" is used, since the "-in-nine" is mathematically implied (nine cells is the longest possible entry, and since it cannot duplicate a digit it must consist of all the digits from 1 to 9 once). Curiously, "3-in-two", "4-in-two", "5-in-two", "43-in-eight", and "44-in-eight" are still frequently called as such, despite the "-in-two" and "-in-eight" being equally implied.

Solving techniques

Although brute-force guessing is of course possible, a better weapon is the understanding of the various combinatorial forms that entries can take for various pairings of clues and entry lengths. Those entries with sufficiently large or small clues for their length will have fewer possible combinations to consider, and by comparing them with entries that cross them, the proper permutation — or part of it — can be derived. The simplest example is where a 3-in-two crosses a 4-in-two: the 3-in-two must consist of '1' and '2' in some order; the 4-in-two (since '2' cannot be duplicated) must consist of '1' and '3' in some order. Therefore, their intersection must be '1', the only digit they have in common.

When solving longer sums there are additional ways to find clues to locating the correct digits. One such method would be to note where a few squares together share possible values thereby eliminating the possibility that other squares in that sum could have those values. For instance, if two 4-in-two clues cross with a longer sum, then the 1 and 3 in the solution must be in those two squares and those digits cannot be used elsewhere in that sum.

When solving sums which have a limited number of solution sets then that can lead to useful clues. For instance, a 30-in-seven sum only has two solution sets: {1,2,3,4,5,6,9} and {1,2,3,4,5,7,8}. If one of the squares in that sum can only take on the values of {8,9} (if the crossing clue is a 17-in-two sum, for example) then that not only becomes an indicator of which solution set fits this sum, it eliminates the possibility of any other digit in the sum being either of those two values, even before determining which of the two values fits in that square.

Another useful approach in more complex puzzles is to identify which square a digit goes in by eliminating other locations within the sum. If all of the crossing clues of a sum have many possible values, but it can be determined that there is only one square which could have a particular value which the sum in question must have, then whatever other possible values the crossing sum would allow, that intersection must be the isolated value. For example, a 36-in-eight sum must contain all digits except 9. If only one of the squares could take on the value of 2 then that must be the answer for that square.

A "box technique" can also be applied on occasion, when the geometry of the unfilled white cells at any given stage of solving lends itself to it: by summing the clues for a series of horizontal entries (subtracting out the values of any digits already added to those entries) and subtracting the clues for a mostly-overlapping series of vertical entries, the difference can reveal the value of a partial entry, often a single cell. This is possible due to the fact that addition is both associative and commutative.

It is common practice to mark potential values for cells in the cell corners until all but one have been proven impossible; for particularly challenging puzzles, sometimes entire ranges of values for cells are noted by solvers in the hope of eventually finding sufficient constraints to those ranges from crossing entries to be able to narrow the ranges to single values. Because of space constraints, instead of digits some solvers use a positional notation, where a potential numerical value is represented by a mark in a particular part of the cell, which makes it easy to place several potential values into a single cell. This also makes it easier to distinguish potential values from solution values.

Some solvers also use graph paper to try various digit combinations before writing them into the puzzle grids.

Mathematics of Kakuro

Kakuro puzzles are NP-complete[3]

There are two kinds of mathematical symmetry readily identifiable in kakuro puzzles: minimum and maximum constraints are duals, as are missing and required values.

All sum combinations can be represented using a bitmapped representation. This representation is useful for determining missing and required values using bitwise logic operations.

Variants

A relatively common variant of Kakuro is its logical successor, *Cross Products* (or *Cross Multiplication*), where the clues are the product of the digits in the entries rather than the sum. Dell Magazines has produced such puzzles, but also allowed repeating of digits aside from 1 due to space limitations in the number of digits in each product in a puzzle. On the other hand, puzzles by Games Magazines are more like crossword puzzles, allowing the implementation of the no-repeating digits rule.

Another variant is having a different range of values that are inserted in the cells, such as 1 to 12, instead of the standard 1 to 9.

A genuine combination of Sudoku and Kakuro is the so called "Cross Sums Sudoku" in which clues are given as cross sums on a standard 9 x 9 Sudoku grid. A relevant variant is the so-called "Cryptic Kakuro" where the clues are given in terms of alphametics and each number represents a digit from 1 to 9.

The final puzzle of the 2004 United States qualifier for the World Puzzle Championship is titled *Cross Number Sums Place*: it is a *Cross Sums* where every row and column of the grid (except the top row and leftmost column as usual) contains exactly nine white cells, none of which — even across multiple entries — are allowed to use the same digit twice, like a *Number Place* (*Sudoku*); in addition, small circles are printed on the borders between some white cells; numerically adjacent digits must be placed astride those circles, and may not appear orthogonally adjacent when not astride a circle.

See also

- Killer Sudoku, a variant of Sudoku which is solved using similar techniques.

References

[1] http://www.conceptispuzzles.com/index.aspx?uri=puzzle/kakuro/history
[2] Keesing Group B.V, publishing in Belgium, Denmark, France, and the Netherlands. (http://www.tazuku.nl/default.aspx?id=8bee251a-c830-4fc3-8e46-a9e3fbbdc44b) shows the restriction.
[3] Takahiro, S. (2001). The complexities of puzzles, cross sum and their another solution problems (ASP). Thesis for BSc, Department of Information Science, University of Tokyo.

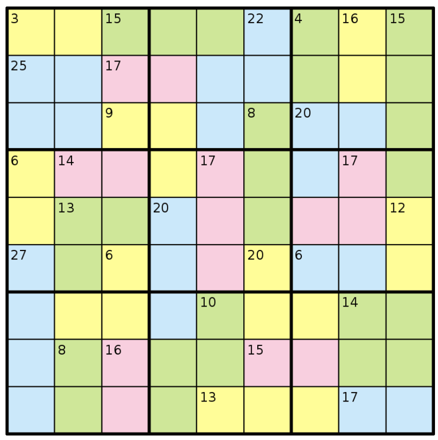
External links

- Tutorial at Nikoli (http://www.nikoli.co.jp/en/puzzles/kakuro/) (Macromedia Flash required)
- The New Grid on the Block (http://www.guardian.co.uk/g2/story/0,,1569223,00.html): *The Guardian* newspaper's introduction to Kakuro
- Table of combinations for use in Kakuro (http://www.kevinpluck.net/kakuro/KakuroCombinations.html)

Killer sudoku

Killer sudoku (also **killer su doku**, **sumdoku**, **sum doku**, **addoku**, or **samunamupure**) is a puzzle that combines elements of sudoku and kakuro. Despite the name, the simpler killer sudokus can be easier to solve than regular sudokus, depending on the solver's skill at mental arithmetic; the hardest ones, however, can take hours to crack.

A typical problem is shown on the right, using colors to define the groups of cells. More often, puzzles are printed in black and white, with thin dotted lines used to outline the "cages" (see below for terminology).



Example of a Killer Sudoku problem.

History

Killer sudokus were already an established variant of sudoku in Japan by the mid 1990s, where they were known as "samunamupure." The name stemmed from a Japanized form of the English words "sum number place." Killer sudokus were introduced to most of the English-speaking world by *The Times* in 2005.

Traditionally, as with regular sudoku puzzles, the grid layout is symmetrical around a diagonal, horizontal or vertical axis, or a quarter or half turn about the centre. This is a matter of aesthetics, though, rather than obligatory: many Japanese puzzle-makers will make small deviations from perfect symmetry for the sake of improving the puzzle. Other puzzle-makers may produce entirely asymmetrical puzzles.

Terminology

Cell

A single square that contains one number in the grid

Row

A horizontal line of 9 cells

Column

A vertical line of 9 cells

Nonet

A 3×3 grid of cells, as outlined by the bolder lines in the diagram above

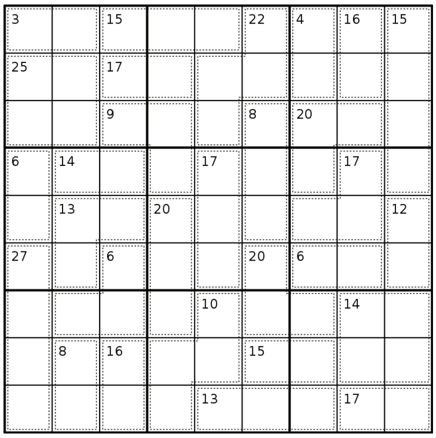
Cage

The grouping of cells denoted by a dotted line or by individual colours.

House

Any nonrepeating set of 9 cells: can be used as a general term for "row, column, or nonet" (or, in Killer X variants, "long diagonal")

Solution to the example above.



The same example problem, as it would be printed in black and white.

Rules

The objective is to fill the grid with numbers from 1 to 9 in a way that the following conditions are met:

- Each row, column, and nonet contains each number exactly once.
- The sum of all numbers in a cage must match the small number printed in its corner.
- No number appears more than once in a cage. (This is the standard rule for killer sudokus, and implies that no cage can include more than 9 cells.)

In 'Killer X', an additional rule is that each of the long diagonals contains each number once.

Duplicate cell ambiguity

By convention in Japan, killer sudoku cages do not include duplicate numbers. However, when *The Times* first introduced the killer sudoku on 31 August 2005, the newspaper did not make this rule explicit. Even though the vast majority of killer sudoku puzzles followed the rule anyway, English-speaking solvers scratched their heads over appropriate solving strategies given the ambiguity. On September 16, 2005 The Times added a new ruling that “Within each dotted-line shape, a digit CAN be repeated if the normal row, column and 3x3 box rules are not broken”. But on September 19 the rule changed to “Within each dotted-line shape, a digit CANNOT be repeated if the normal row, column and 3x3 box rules are not broken" - causing even more scratching of heads. This revised rule stuck and the world standard is no duplicates within cages.

Solving strategies

Fewest possible combinations

Generally the problem is best tackled starting from the extreme sums — cages with the largest or the smallest sums. This is because these have the fewest possible combinations. For example, 3 cells within the same house totalling 23 can only be 6, 8, and 9.

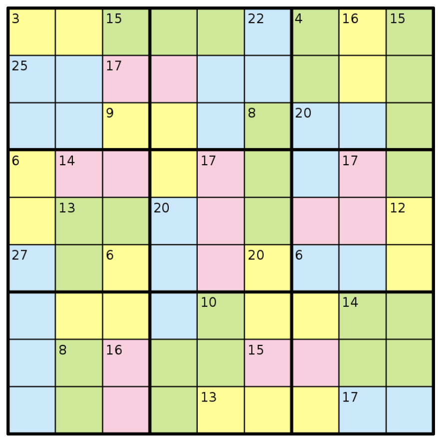
In the early stages of the game, the most common way to begin filling in numbers is to look at such low-sum or high-sum cages that form a 'straight line'. As the solver can infer from these that certain numbers are in a certain row or column, he can begin 'cross-hatching' across from them.

The 45 rule

A further technique can be derived from the knowledge that the numbers in all houses (rows, columns and nonets) add up to 45. By adding up the cages and single numbers in a particular house, the user can deduce the result of a single cell. If the cell calculated is within the house itself, it is referred to as an 'innie'; conversely if the cell is outside it, it is called an 'outie'. Even if this is not possible, advanced players may find it useful to derive the sum of two or three cells, then use other elimination techniques (see below for an example of this). The '45' technique can also be extended to calculate the innies or outies of N adjacent houses, as the difference between the cage-sums and N*45.

Initial analysis of the sample problem

Fewest possible combinations



The sample problem.

The two cells in the top left must be 1+2. The 3 cells to the right totaling 15 cannot therefore have either a 1 or a 2, so they must be either 3+4+8, 3+5+7, or 4+5+6.

The two vertical cells in the top left of the top right nonet cannot be 2+2 as that would mean duplicates, so they must be 1+3. The 1 cannot be in the top line as that conflicts with our first 2 cells therefore the top cell of this pair is 3 and the lower cell 1. This also means the 3 cell cage 15 to the left cannot contain a 3 and so is 4+5+6.

Similarly the neighbouring 16 must be 9+7.

The four cells in the top right cage (totaling 15) can only include one of 1, 3, 7, or 9 (if at all) because of the presence of 1, 3, 7, and 9 in the top right hand nonet. If any one of 1, 3, 7, or 9 is present then this must be the lone square in the nonet below. Therefore these 4 cells is one of 1+2+4+8 or 2+3+4+6.

The 2 cells in the middle of the left edge must be either 1+5 or 2+4. And so on.

45

Looking at the nonet on the left hand side in the middle, we can see that there are three cages which do not cross over into another nonet; these add up to 33, meaning that the sum of the remaining two cells must be 12. This does not seem particularly useful, but consider that the cell in the bottom right of the nonet is part of a 3-cage of 6; it can therefore only contain 1, 2 or 3. If it contained 1 or 2, the other cell would have to contain 11 or 10 respectively; this is impossible. It must, therefore, contain 3, and the other cell 9.

Cage total tables

The following tables list the possible combinations for various sums.

2 cells

```
03: 12
04: 13
05: 14 23
06: 15 24
07: 16 25 34
08: 17 26 35
09: 18 27 36 45
10: 19 28 37 46
11: 29 38 47 56
12: 39 48 57
13: 49 58 67
14: 59 68
15: 69 78
16: 79
17: 89
```

3 cells

```
06: 123
07: 124
08: 125 134
09: 126 135 234
10: 127 136 145 235
11: 128 137 146 236 245
12: 129 138 147 156 237 246 345
13: 139 148 157 238 247 256 346
14: 149 158 167 239 248 257 347 356
15: 159 168 249 258 267 348 357 456
16: 169 178 259 268 349 358 367 457
17: 179 269 278 359 368 458 467
18: 189 279 369 378 459 468 567
19: 289 379 469 478 568
20: 389 479 569 578
21: 489 579 678
22: 589 679
23: 689
```

```
24: 789
```

4 cells

```
10: 1234
11: 1235
12: 1236 1245
13: 1237 1246 1345
14: 1238 1247 1256 1346 2345
15: 1239 1248 1257 1347 1356 2346
16: 1249 1258 1267 1348 1357 1456 2347 2356
17: 1259 1268 1349 1358 1367 1457 2348 2357 2456
18: 1269 1278 1359 1368 1458 1467 2349 2358 2367 2457 3456
19: 1279 1369 1378 1459 1468 1567 2359 2368 2458 2467 3457
20: 1289 1379 1469 1478 1568 2369 2378 2459 2468 2567 3458 3467
21: 1389 1479 1569 1578 2379 2469 2478 2568 3459 3468 3567
22: 1489 1579 1678 2389 2479 2569 2578 3469 3478 3568 4567
23: 1589 1679 2489 2579 2678 3479 3569 3578 4568
24: 1689 2589 2679 3489 3579 3678 4569 4578
25: 1789 2689 3589 3679 4579 4678
26: 2789 3689 4589 4679 5678
27: 3789 4689 5679
28: 4789 5689
29: 5789
30: 6789
```

5 cells

```
15: 12345
16: 12346
17: 12347 12356
18: 12348 12357 12456
19: 12349 12358 12367 12457 13456
20: 12359 12368 12458 12467 13457 23456
21: 12369 12378 12459 12468 12567 13458 13467 23457
22: 12379 12469 12478 12568 13459 13468 13567 23458 23467
23: 12389 12479 12569 12578 13469 13478 13568 14567 23459 23468 23567
24: 12489 12579 12678 13479 13569 13578 14568 23469 23478 23568 24567
25: 12589 12679 13489 13579 13678 14569 14578 23479 23569 23578 24568 34567
26: 12689 13589 13679 14579 14678 23489 23579 23678 24569 24578 34568
27: 12789 13689 14589 14679 15678 23589 23679 24579 24678 34569 34578
28: 13789 14689 15679 23689 24589 24679 25678 34579 34678
29: 14789 15689 23789 24689 25679 34589 34679 35678
30: 15789 24789 25689 34689 35679 45678
31: 16789 25789 34789 35689 45679
32: 26789 35789 45689
33: 36789 45789
34: 46789
35: 56789
```

6 cells

```
21: 123456
22: 123457
23: 123458 123467
24: 123459 123468 123567
25: 123469 123478 123568 124567
26: 123479 123569 123578 124568 134567
27: 123489 123579 123678 124569 124578 134568 234567
28: 123589 123679 124579 124678 134569 134578 234568
29: 123689 124589 124679 125678 134579 134678 234569 234578
30: 123789 124689 125679 134589 134679 135678 234579 234678
31: 124789 125689 134689 135679 145678 234589 234679 235678
32: 125789 134789 135689 145679 234689 235679 245678
33: 126789 135789 145689 234789 235689 245679 345678
34: 136789 145789 235789 245689 345679
35: 146789 236789 245789 345689
36: 156789 246789 345789
37: 256789 346789
38: 356789
39: 456789
```

7 cells

```
28: 1234567
29: 1234568
30: 1234569 1234578
31: 1234579 1234678
32: 1234589 1234679 1235678
33: 1234689 1235679 1245678
34: 1234789 1235689 1245679 1345678
35: 1235789 1245689 1345679 2345678
36: 1236789 1245789 1345689 2345679
37: 1246789 1345789 2345689
38: 1256789 1346789 2345789
39: 1356789 2346789
40: 1456789 2356789
41: 2456789
42: 3456789
```

8 cells

```
36: 12345678
37: 12345679
38: 12345689
39: 12345789
40: 12346789
41: 12356789
42: 12456789
```

```
43: 13456789
44: 23456789
```

9 cells

```
45: 123456789
```

6, 7, and 8 cells

It is easiest to determine the combinations within large cages by means of complements. The table for *6 cell* cages is the complement of the *3 cell* table adding up to 45 minus the listed value; similarly, the *7 cell* table complements the *2 cell* table. An 8-cell cage is of course missing only one digit (45 minus the sum of the cage).

See also

- Kakuro

External links

- Too good for Fiendish? Then try Killer Su Doku [1] - article in *The Times*
- [2] seems to be the best free killer sudoku resource on the WWW. Some of the more difficult problems can take hours or even defeat the toughest solvers. Also features the variants 'Greater than sudoku' where the solver is not given the value in the box only that it is greater or less than it's neighbour and 'Greater than killer' which combines 'Greater than sudoku' with killer.

References

[1] http://www.timesonline.co.uk/article/0,,7-1757275,00.html
[2] http://www.killersudokuonline.com/index.html

KenKen

KenKen or **KenDoku** is a style of arithmetic and logic puzzle invented in 2004 by the Japanese math teacher Tetsuya Miyamoto, an innovator who says he practices "the art of teaching without teaching".[1] He intends the puzzles as an instruction-free method of training the brain.[2] The names **Calcudoku** and **Mathdoku** are sometimes used by those who don't have the rights to use the KenKen or KenDoku trademarks.[3]

The name derives from the Japanese for cleverness (賢 *ken, kashiko(i)*).[1]

As in sudoku, the goal of each puzzle is to fill a grid with digits — 1 through 4 for a 4×4 grid, 1 through 5 for a 5×5, etc. — so that no digit appears more than once in any row or column. Grids range in size from 3×3 to 9×9. Additionally, KenKen grids are divided into heavily outlined groups of cells — often called “cages” — and the numbers in the cells of each cage must produce a certain “target” number when combined using a specified mathematical operation (either addition, subtraction, multiplication or division). For example, a three-cell cage specifying addition and a target number of 6 in a 4×4 puzzle might be satisfied with the digits 1, 2, and 3. Digits may be repeated within a cage, as long as they are not in the same row or column. No operation is relevant for a single-cell cage: placing the "target" in the cell is the only possibility. The target number and operation appear in the upper left-hand corner of the cage.

In the English-language KenKen books of Will Shortz, the issue of the non-associativity of division and subtraction is addressed by restricting clues based on either of those operations to cages of only two cells. Some puzzle authors have not done this and have published puzzles that use more than two cells for these operations.

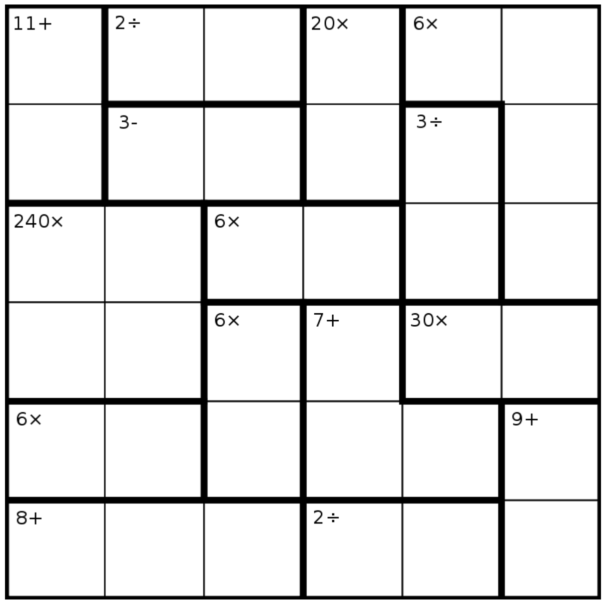
History

In 2005, toy inventor Robert Fuhrer encountered KenKen books published in Japan by the educational publisher Gakken Co., Ltd. and titled "Kashikoku naru Puzzle" (賢くなるパズル *kashikoku naru pazuru*, lit. "smartness puzzle").[2] Fuhrer's company Nextoy, LLC (now holder of a trademark on "KenKen" and "KenDoku" as a name for brain-training puzzles) and chess International Master Dr. David Levy helped bring the puzzles to the attention of Michael Harvey, features editor of *The Times* (London).[4] Harvey, impressed with what he calls its "depth and magnitude", arranged for publication of such puzzles, starting in March 2008, in *The Times*. Other papers, including the *New York Times*, followed suit. KenKen now appears in more than 40 newspapers nationwide, as well as numerous international publications.

Example

The objective is to fill the grid in with the digits 1 through 6 such that:

- Each row contains exactly one of each digit
- Each column contains exactly one of each digit
- Each bold-outlined group of cells is a cage containing digits which achieve the specified result using the specified mathematical operation: addition (+), subtraction (−), multiplication (×), and division (÷). (Unlike Killer Sudoku, digits may repeat within a group.)

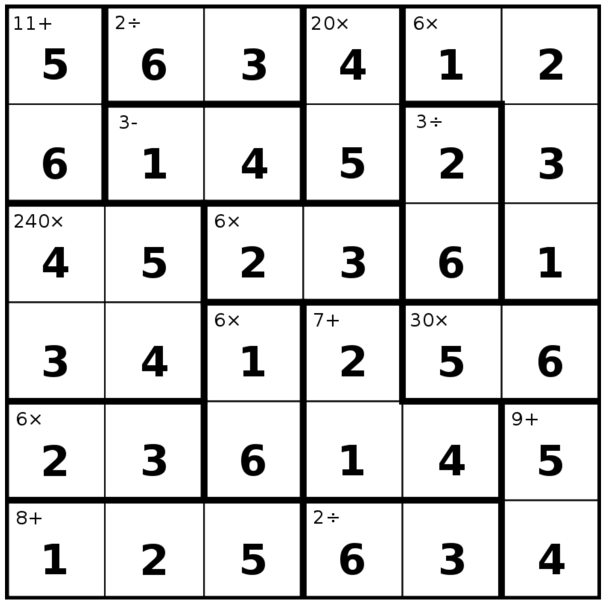


A typical KenKen problem.

Some of the techniques from Sudoku and Killer Sudoku can be used here, but much of the process involves the listing of all the possible options and eliminating the options one by one as other information requires.

In the example here:

- "11+" in the leftmost column can only be "5,6"
- "2÷" in the top row must be one of "1,2", "2,4" or "3,6"
- "20×" in the top row must be "4,5".
- "6×" in the top right must be "1,1,2,3". Therefore the two "1"s must be in separate columns, thus row 1 column 5 is a "1".
- "240×" on the left side is one of "6,5,4,2" or "3,5,4,4". Either way there is a five and it must be in the right pair of cells since we have "5,6" already in column 1.
- etc.



Solution to the above problem.

Extensions

More complex KenKen problems are formed using the principles described above but omitting the symbols +, −, × and ÷, thus leaving them as yet another unknown to be determined.

The restriction of puzzle size to the range two through nine is not absolute. A KenKen of size two is of little value even as an example, as it can immediately be solved by trying the two possibilities — ones on the "rising" diagonal and twos on the "falling" one, or vice versa. But extension beyond nine presents only difficulties of calculation with larger numbers, and the need, in recording possible values, to avoid confusing multi-digit numbers with items in a list of single digit ones.

References

[1] A New Puzzle Challenges Math Skills (http://www.nytimes.com/2009/02/09/arts/09ken.html?_r=1&em), *New York Times*, February 8, 2009

[2] Tetsuya Miyamoto creates KenKen. Train your brain (http://entertainment.timesonline.co.uk/tol/arts_and_entertainment/games_and_puzzles/article3599704.ece), *The Times*, 21 March 2008

[3] KenDoku renamed to CalcuDoku (http://www.conceptispuzzles.com/index.aspx?uri=info/news/285)

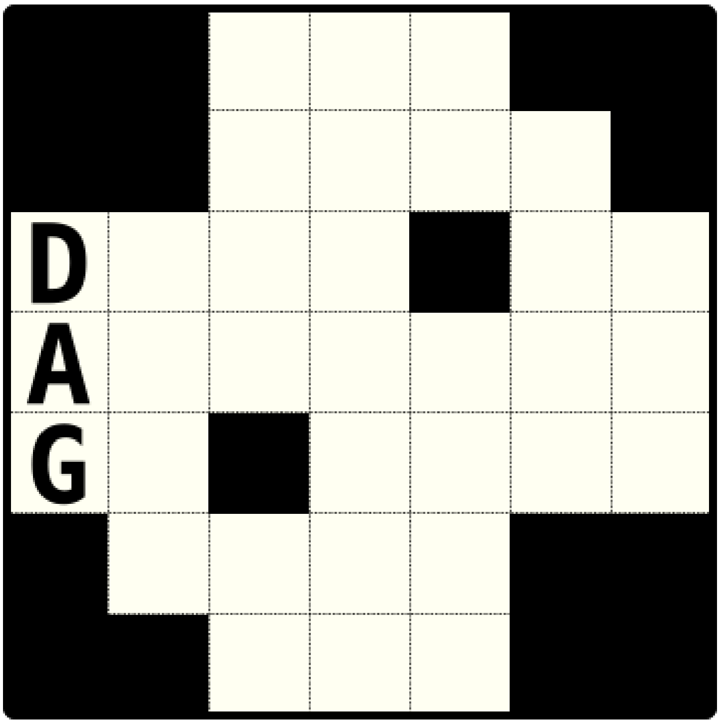
[4] Stephey, M. J. "The Next Sudoku?" Time Magazine 23 Mar. 2009: 72.

External links

- kenken.com (official site) (http://www.kenken.com)
- Puzzle Guru Will Shortz (http://www.time.com/time/arts/article/0,8599,1882455,00.html), *Time Magazine*, March 2, 2009
- PyKen (http://code.google.com/p/pyken) - an implementation that uses various extensions, such as exponentiation and complex arithmetic

Fill-In

Fill-Ins (also known as **Fill-It-Ins**) are a type of logic puzzle commonly printed in various puzzle magazines. While they superficially resemble crossword puzzles, they do not require any form of "outside knowledge"; unlike a crossword, one can solve a Fill-In in a non-native tongue, even one with an alternate character set.



Starting grid for a simple **Fill-In**. The word list is:
GI IO ON OR
DAG EVO OED REF
ARID CLEF CLOD DAIS DENS DOLE EDIT SILO
ARTICLE VESICLE

Rules

Fill-Ins are presented in two parts: the *grid*, which typically resembles an empty American crossword (with one word possibly already entered as a starting point) and the *word list.* The word list is traditionally sorted first by length and then alphabetically, so as to make finding a particular word as simple as possible.

The goal is to "fill in" the grid with the words in the word list. Words are entered horizontally and vertically, as in crosswords, but the word list gives no indication of location or direction; the puzzle aspect comes from determining the placement of all the words.

Solution methods

Solved grid for the above Fill-In.

Solving a Fill-In typically amounts to searching for words of a certain length with letters in specific places. If a starter word is given in the grid, it is often useful to use it as the beginning search point, but long words are more generally useful (especially if there's only a few of them in the word list). There are usually many three- and four-letter words in a puzzle; using longer words first often makes the placement of shorter words easier.

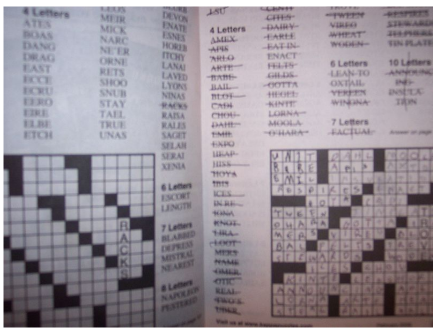
Raw "trial and error" is best used when there are only two or three words that can potentially fit at a given location; temporarily assume one of the words, and see if an impossible letter combination results. If so, that word is not the one that should go in the grid at that location. Unlike Sudoku, most moderately difficult Fill-Ins require a number of these "trial and error" explorations to solve.

Care must be given to marking out words that are not explicitly placed in the grid; this occurs when one fills in a vertical sequence of horizontal words, or vice versa. Forgetting to do this results in "extra words" and often makes the puzzle more difficult to solve.

Variations

A common variation on the standard Fill-In is using numbers, or a mix of numbers and letters, instead of specific words. As the words in the puzzle do not "mean anything," such puzzles are effectively identical to the standard type of puzzles, with more or less symbols.

Particularly large Fill-Ins may list 'Across' and 'Down' words separately, to make the search for matching words require less "trial and error."



A picture of the inside of a Fill-In.

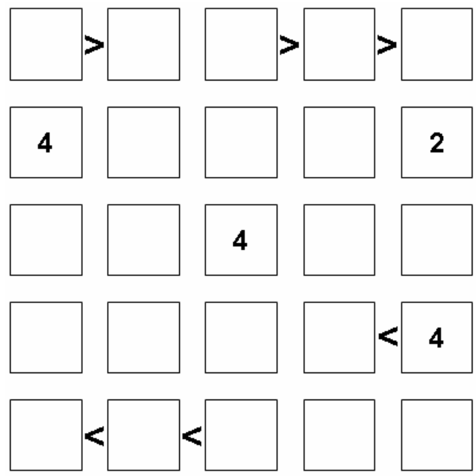
Also common is the replacement of a standard American crossword grid with a more open, "loose crossword" format (as is common with crosswords used in classrooms or generated by computer programs), where there is no symmetry and every symbol is not necessarily covered in both directions. Often the black squares are omitted in this form (unless needed to disambiguate a close cluster of squares); these are commonly marketed under the names *Frameworks* or *Kriss-Kross*.

A more complex variation on the Fill-In is the *Diagramless Fill-In*, where the grid is initially empty except for all instances of a single letter (typically a vowel). Like diagramless crosswords, these puzzles rely on the rotational symmetry of American crossword grids to help the solver determine the location of the various words.

Futoshiki

Futoshiki (不等式 *futōshiki*) or Unequal is a logic puzzle game from Japan. Its name means "inequality". It is also spelled **hutosiki** (using Kunrei-shiki romanization).

The puzzle is played on a square grid, such as 5 x 5. The objective is to place the numbers 1 to 5 (or whatever the dimensions are) such that each row, and column contains each of the digits 1 to 5. Some digits may be given at the start. In addition, inequality constraints are also initially specifed between some of the squares, such that one must be higher or lower than its neighbour. These constraints must be honoured as the grid is filled out.

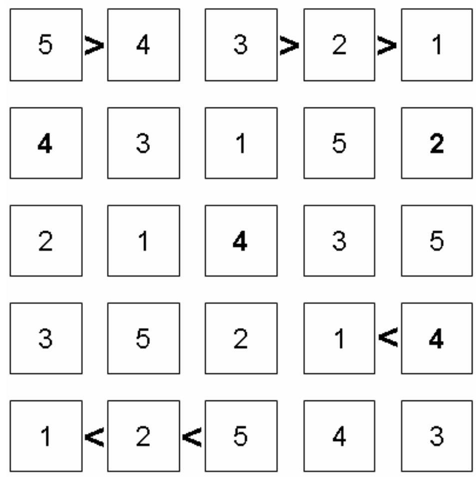


An example of a 5x5 Futoshiki puzzle ...

Solving the puzzle

Solving the puzzle requires a combination of logical techniques.[1] Numbers in each row and column restrict the number of possible values for each position, as do the inequalities.

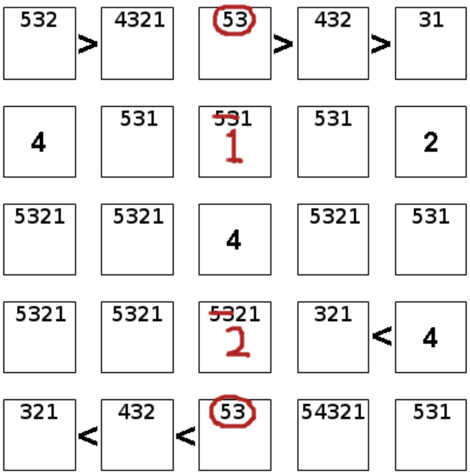
Once the table of possibilities has been determined, a crucial tactic to solve the puzzle involves "AB elimination", in which subsets are identified within a row whose range of values can be determined. For example, if the first two squares within a row must contain 1 or 2, then these numbers can be excluded from the remaining squares. Similarly, if the first three squares must contain 1 or 2; 1 or 3; and 1 or 2 or 3, then those remaining must contain other values (4 and 5 in a 5x5 puzzle).



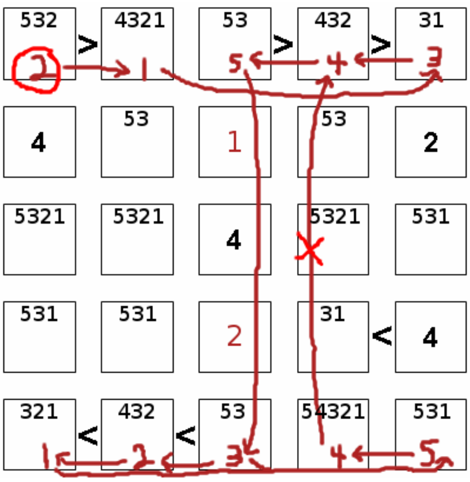
... and its solution

Another important technique is to work through the range of possibilities in open inequalities. A value on one side of an inequality determines others, which then can be worked through the puzzle until a contradiction is reached and the first value is excluded.

Additionally, many Futoshiki puzzles are promised to possess unique solutions. If this is strictly true, then regions of the form



The first step to solve the puzzle is to enumerate possible values based on inequalities and non-duplication within rows and columns. Then *AB elimination* may be usable to narrow down the range of possibilities. As shown here, the top and bottom positions in the center column must contain 5 and 3, so these can be excluded from the second and fourth positions.



Logical deduction within the inequalities can restrict the range of possibilities. As shown here, a 2 in the upper left corner requires a 1 in the second position due to the first inequality; but a 1 in the second position allows only a 3 in the fifth position ... and so on, until we conclude two 4s would need to be placed in the same column. Likewise a 3 in the upper left corner would require the top row to be 3 2 5 4 1 and the bottom again to be 1 2 3 4 5 - leading to the same contradiction. Only a 5 is permissible in the top left corner, from which we deduce 5 4 3 2 1 at top and 1 2 5 4 3 at bottom. The remainder of the solution is simple elimination.

```
A . B
. . .
```

```
B . A
```

cannot be present, unless an inequality or pre-filled number can specify which of the two numbers is B and which number is A.

A solved Futoshiki puzzle is a Latin square.

Futoshiki in the United Kingdom

The puzzle is published every Saturday in The Guardian and in The Daily Telegraph, and daily in The Times and also in the Dundee Courier.

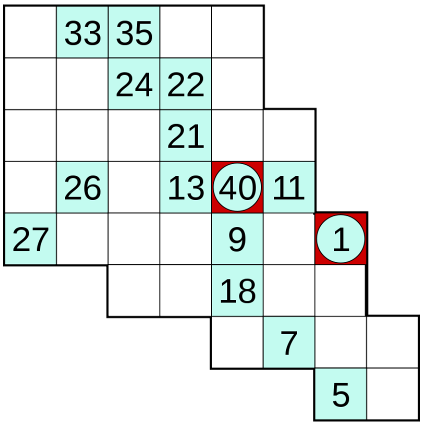
Notes

[1] "What strategy tips will help me solve Futoshiki puzzles?" (http://www.clarity-media.co.uk/puzzleanswer/faq-24). .

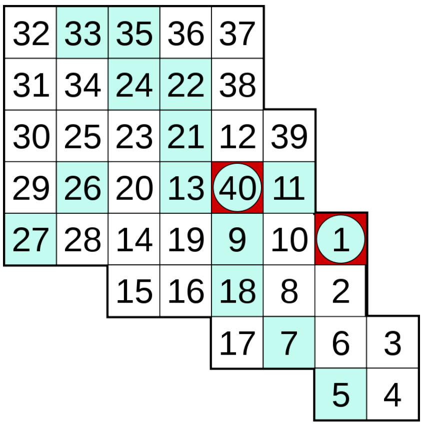
External links

- Guardian article on the launch of Futoshiki (http://www.guardian.co.uk/japan/story/0,,1884417,00.html)

Hidato



A Hidato puzzle...



...and its solution

Hidato is a logic puzzle game invented by Dr. Gyora Benedek, an Israeli mathematician. The goal of Hidato is to fill the grid with consecutive numbers that connect horizontally, vertically, or diagonally.

About the puzzle

In every Hidato puzzle the smallest and the highest number are listed on the grid. There are more numbers on the board to help to direct the player how to start the solution and to ensure that Hidato has only a single solution. It is usually played on a square grid like Sudoku or Kakuro but can also include irregular shaped grids like hearts, skulls, and so forth.

Hidato puzzles are published in newspapers such as the *Daily Mail* and *Detroit Free Press*.

Every Hidato puzzle has a unique solution that can be found by simple logic.

External links

- Hidato official web site [1]

References

[1] http://www.hidato.com/

Survo Puzzle

Survo puzzle is a kind of logic puzzle presented (in April 2006) and studied by Seppo Mustonen. [1] The name of the puzzle is associated to Mustonen's Survo system which is a general environment for statistical computing and related areas. [2]

In a Survo puzzle the task is to fill an $m \times n$ table by integers 1,2,...,$m{\cdot}n$ so that each of these numbers appears only once and their row and column sums are equal to integers given on the bottom and the right side of the table. Often some of the integers are given readily in the table in order to guarantee uniqueness of the solution and/or for making the task easier. [2]

To some extent, Survo puzzles resemble Sudoku and Kakuro puzzles. However, numbers used in the solution are not restricted to 1,2,...,9 and the size of puzzle grid is typically very small. Solving Survo puzzles is also related to making of magic squares. [3]

The degree of difficulty in solving Survo puzzles is strongly varying. Easy puzzles, meant for school children are pure exercises in addition and subtraction, while more demanding ones require also good logic reasoning. The hardest Survo puzzles cannot be solved without computers. [4]

Certain properties of Survo system like editorial computing and the COMB operation making e.g. restricted integer partitions, support solving of Survo puzzles.

Survo puzzles have been published regularly in Finland by Ilta-Sanomat and the Scientific magazine of the University of Helsinki from September 2006. Solving of Survo puzzles was one of the three main topics in the national entrance examination of the Finnish Universities in computer science (2009). [5]

Example

Here is a simple Survo puzzle with 3 rows and 4 columns:

	A	B	C	D	
1		**6**			**30**
2	**8**				**18**
3			**3**		**30**
	27	**16**	**10**	**25**	

Numbers 3, 6, and 8 are readily given. The task is to put remaining numbers of 1-12 (3×4=12) to their places so that the sums are correct.

The puzzle has a unique solution found stepwise as follows: The missing numbers are 1,2,4,5,7,9,10,11,12. Usually it is best to start from a row or a column with fewest missing numbers. In this case columns A, B, and C are such.

Column A is not favorable since the sum 19 of missing numbers can be presented according to the rules in several ways (e.g. 19 = 7 + 12 = 12 + 7 = 9 + 10 = 10 + 9). In the column B the sum of missing numbers is 10 having only one partition 10 = 1 + 9 since the other alternatives 10 = 2 + 8 = 3 + 7 = 4 + 6 are not accepted due to numbers already present in the table. Number 9 cannot be put in the row 2 since then the sum of this row would exceed the value 18. Therefore the only choice is to start the solution by

	A	B	C	D	
1		**6**			**30**
2	**8**	1			**18**
3		9	**3**		**30**
	27	**16**	**10**	**25**	

Now the column A has only one alternative 27 - 8 = 19 = 7 + 12 = 12 + 7. Number 7 cannot be in the row 1 because the sum of missing numbers in that row would be 30 - 7 - 6 = 17 and this allows no permitted partition. Thus we have

	A	B	C	D	
1	12	**6**			**30**
2	**8**	1			**18**
3	7	9	**3**		**30**
	27	**16**	**10**	**25**	

implying that the last number in the last row will be 30 - 7 - 9 -3 = 11:

	A	B	C	D	
1	12	**6**			**30**
2	**8**	1			**18**
3	7	9	**3**	11	**30**
	27	**16**	**10**	**25**	

In the first row the sum of the missing numbers is 30 - 12 - 6 = 12. Its only possible partition is 12 = 2 + 10 and so that number 2 will be in the column C; 10 in this position is too much for the column sum.

	A	B	C	D	
1	12	**6**	2	10	**30**
2	**8**	1			**18**
3	7	9	**3**	11	**30**
	27	**16**	**10**	**25**	

The solution is then easily completed as

	A	B	C	D	
1	12	**6**	2	10	**30**
2	**8**	1	5	4	**18**
3	7	9	**3**	11	**30**
	27	**16**	**10**	**25**	

Thus basic arithmetics and simple reasoning is enough for solving easy Survo puzzles like this one.

Properties of Survo puzzles

The rules of Survo puzzles are simpler than those of Sudoku. The grid is always rectangular or square and typically much smaller than in Sudoku and Kakuro. [6]

The solving strategies are varying depending on the difficulty of the puzzle. [6] In their simplest form, as in the following 2 × 3 case (degree of difficulty 0)

		3	**9**
	6		**12**
9	**7**	**5**	

Survo puzzles are suitable exercises in addition and subtraction. [6]

The open 3 × 4 Survo puzzle (degree of difficulty 150)

				24
				15
				39
21	**10**	**18**	**29**	

where none of the numbers are readily given, is much harder. Also it has only one solution.

The problem can be simplified by giving some of the numbers readily, for example, as

7		**5**		**24**
	1		**8**	**15**
		11		**39**
21	**10**	**18**	**29**	

which makes the task almost trivial (degree of difficulty 0). [6]

Assessing degree of difficulty

Measuring the degree of difficulty is based on the number of 'mutations' needed by the first solver program made by Mustonen in April 2006. This program works by using a partially randomized algorithm. [7]

The program starts by inserting the missing numbers randomly in the table and tries then to get the computed sums of rows and columns as close to the true ones as possible by exchanging elements in the table systematically. This trial leads either to a correct solution or (as in most cases) to dead end where the discrepancy between computed and true sums cannot be diminished systematically. In the latter case a 'mutation' is made by exchanging two or more numbers randomly. Thereafter the systematic procedure plus mutation is repeated until a true solution is found. In most cases, the mean number of mutations works as a crude measure for the level of difficulty of solving a Survo puzzle. This measure (MD) is computed as the mean number of mutations when the puzzle is solved 1000 times by starting from a randomized table. The distribution of the number of mutations comes close to a geometric distribution.

These numeric values are often converted to a 5-star scale as follows: [8]

MD

0 - 30	*
31 - 150	**
151 - 600	***
601 - 1500	****
1500 -	*****

The degree of difficulty given as an MD value is rather inaccurate and it may be even misleading when the solution is found by clever deductions or by creative guesswork. This measure works better when it is required that the solver also proves that the solution is unique.

Open Survo puzzles

A Survo puzzle is called open, if merely marginal sums are given. Two open $m \times n$ puzzles are considered essentially different if one of them cannot converted to another by interchanging rows and columns or by transposing when $m = n$. In these puzzles the row and column sums are distinct. The number of essentially different and uniquely solvable $m \times n$ Survo puzzles is denoted by $S(m,n)$. [7]

Reijo Sund was the first to pay attention to enumeration of open Survo puzzles. He calculated $S(3,3)=38$ by studying all 9! = 362880 possible 3 × 3 tables by the standard combinatorial and data handling program modules of Survo. Thereafter Mustonen found $S(3,4)=583$ by starting from all possible partitions of marginal sums and by using the first solver program. Petteri Kaski computed $S(4,4)=5327$ by converting the task into an exact cover problem.

Mustonen made in Summer 2007 a new solver program which confirms the previous results. The following $S(m,n)$ values have been determined by this new program: [9]

m/*n*	2	3	4	5	6	7	8	9	10
2	1	18	62	278	1146	5706	28707	154587	843476
3	18	38	583	5337	55815	617658			
4	62	583	5327	257773					
5	278	5337	257773						
6	1146	55815							
7	5706	617658							
8	28707								
9	154587								
10	843476								

Already computation of *S*(5,5) seems to be a very hard task on the basis of present knowledge.

Swapping method

The swapping method for solution of Survo puzzles has been created by combining the idea of the original solver program to the observation that the products of the marginal sums crudely indicate the positions of the correct numbers in the final solution. [10] The procedure is started by filling the original table by numbers 1,2,...,m·n according to sizes of these products and by computing row and column sums according to this initial setup. Depending on how these sums deviate from the true sums, it is tried to improve the solution by swapping two numbers at a time. When using the swapping method the nature of solving Survo puzzles becomes somewhat similar to that of Chess problems. By this method it is hardly possible to verify the uniqueness of the solution.

For example, a rather demanding 4 × 4 puzzle (MD=2050)

				51
				36
				32
				17
51	**42**	**26**	**17**	

is solved by 5 swaps. The initial setup is

					Sum	OK	error
	16	15	10	8	49	**51**	-2
	14	12	9	4	39	**36**	3
	13	11	6	3	33	**32**	1
	7	5	2	1	15	**17**	-2
Sum	50	43	27	16			
OK	**51**	**42**	**26**	**17**			
error	-1	1	1	-1			

and the solution is found by swaps (7,9) (10,12) (10,11) (15,16) (1,2). In the Survo system, a sucro /SP_SWAP takes care of bookkeeping needed in the swapping method.

Quick games

Solving of a hard Survo puzzle can take several hours. Solving Survo puzzles as quick games offers another kind of challenges. [4] The most demanding form of a quick game is available in the net as a Java applet. [11] In this quick game, open 5 × 5 puzzles are solved by selecting (or guessing) the numbers by mouse clicks. A wrong choice evokes a melodic musical interval. Its range and direction indicate the quality and the amount of the error. The target is to attain as high score as possible. The score grows by correct choices and it is decreased by wrong ones and by the time used for finding the final solution.

See also

- Sudoku
- Kakuro
- Magic square

References

[1] Aitola, Kerttu (2006): "Survo on täällä" ("Survo is here"). *Yliopisto* **54(12)**: 44–45.
[2] Mustonen, Seppo (2007): "Survo Crossings" (http://www.csc.fi/english/csc/publications/cscnews/back_issues/cscnews1_2007). *CSCnews* **1/2007**: 30–32.
[3] Vehkalahti, Kimmo (2007): "Some comments on magic squares and Survo puzzles" (http://www.helsinki.fi/~kvehkala/Kimmo_Vehkalahti_Windsor.pdf). The 16th International Workshop on Matrices and Statistics, University of Windsor, Canada, June 1–3, 2007.
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[5] "Tietojenkäsittelytieteen yhteisvalinta 22.5.2009, Tehtävä 3: Survo-ristikko" (http://www.tkt-yhteisvalinta.fi/valintakoe2009/tkt09_Tehtava3.pdf). ("National entrance examination in computer science, May 22nd 2009, Exercise 3: Survo Puzzle").
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[8] Mustonen, Seppo (2006-09-26): "Survo-ristikon vaikeuden arviointi" (http://www.survo.fi/ristikot/vaikeusarvio.html) ("Evaluating the degree of difficulty of a Survo Puzzle"). Retrieved on 2009-08-30.
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[10] Mustonen, Seppo (2007-07-09): "On the swapping method" (http://www.survo.fi/puzzles/swapm.html). Retrieved on 2009-08-30.
[11] "Survo Puzzle (5x5 quick game) as a Java applet" (http://www.survo.fi/java/quick5x5.html). Retrieved on 2009-08-30.

External links

- Survo Puzzles (http://www.survo.fi/puzzles/): Problems and solutions

Hitori

Hitori (Japanese for: Alone or one person) (ひとりにしてくれ *Hitori ni shite kure*; literally "leave me alone") is a type of logic puzzle published by Nikoli.

4	8	1	6	3	2	5	7
3	6	7	2	1	6	5	4
2	3	4	8	2	8	6	1
4	1	6	5	7	7	3	5
7	2	3	1	8	5	1	2
3	5	6	7	3	1	8	4
6	4	2	3	5	4	7	8
8	7	1	4	2	3	5	6

Example of an incomplete **Hitori** puzzle (see bottom of page for solution)

Rules

Hitori is played with a grid of squares or cells, and each cell contains a number. The objective is to eliminate numbers by filling in the squares such that remaining cells do not contain numbers that appear more than once in either a given row or column.

Filled-in cells cannot be horizontally or vertically adjacent, although they can be diagonally adjacent. The remaining un-filled cells must form a single component connected horizontally and vertically.

Solving techniques

- When it is confirmed that a cell must be black, one can see all orthogonally adjacent cells must not be black. Some players find it useful to circle any numbers which must be white as it makes the puzzle easier to read as you progress.
- If a number has been circled to show that it must be white, any cells containing the same number in that row and column must also be black.
- If a cell would separate a white area of the grid if it were painted black, the cell must be white.
- In a sequence of three identical, adjacent numbers; the centre number must be white. If one of the end numbers were white this would result in two adjacent filled in cells which is not allowed.
- In a sequence of two identical, adjacent numbers; if the same row or column contains another cell of the same number the number standing on its own must be black. If it were white this would result in two adjacent filled in cells which is not allowed.
- Any number that has two identical numbers on opposite sides of itself must be white, because one of the two identical numbers must be black, and it cannot be adjacent to another black cell.
- When four identical numbers are in a two by two square on the grid, two of them must be black along a diagonal. There are only two possible combinations, and it is sometimes possible to decide which is correct by determining if one variation will cut white squares off from the remainder of the grid.
- When four identical numbers form a square in the corner of a grid, the corner square and the one diagonally opposite must be black. The alternative would leave the corner square isolated from the other white numbers.

History

Hitori is an original puzzle of Nikoli; it first appeared in *Puzzle Communication Nikoli* in issue #29 (March 1990).

In media

- Episode 11 of xxxHolic: Kei is titled Hitori in reference to this.

See also

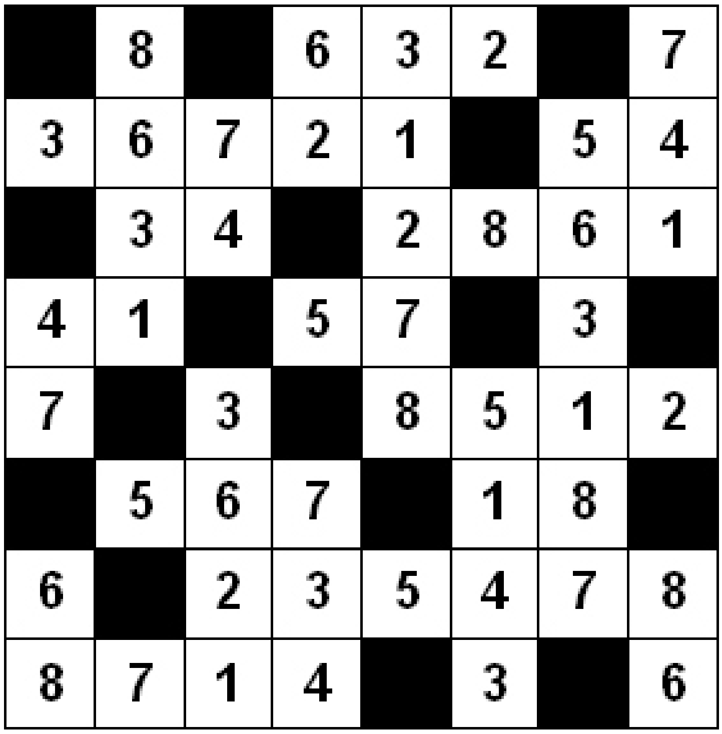
- List of Nikoli puzzle types
- Sudoku
- Kakuro

References

- Puzzle Cyclopedia, Nikoli, 2004. ISBN.

External links

- Sample Hitori puzzles [1] on the Nikoli web site
- Hitori tutorials [2] on the Nikoli website
- Hitori Number Puzzle Game [3] Play online at Funmin



Example of a completed **Hitori** puzzle (see top of page for incomplete puzzle)

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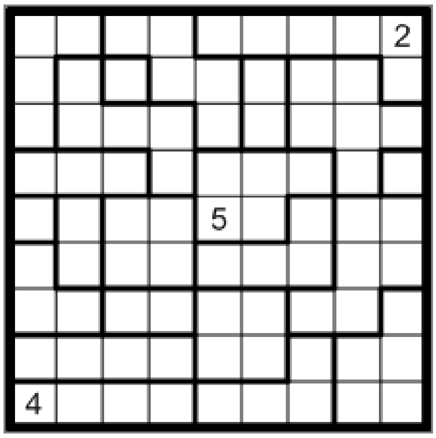
[1] http://www.nikoli.com/en/puzzles/hitori
[2] http://www.nikoli.com/en/puzzles/hitori/rule.html
[3] http://funmin.com/online-games/hitori/index.php

Ripple Effect

For the sociological term, see Ripple effect. For the episode of Stargate SG-1, see Ripple Effect (Stargate SG-1).

Ripple Effect (Japanese:波及効果 *Hakyuu Kouka*) is a logic puzzle published by Nikoli. As of 2007, two books consisting entirely of *Ripple Effect* puzzles have been published by Nikoli. The second was published on October 4, 2007.



Moderately difficult sample puzzle

Rules

Ripple Effect is played on a rectangular grid divided into polyominoes. The solver must place one positive integer into each cell of the grid - some of which may be given in advance - according to these rules:

- Every polyomino must contain the consecutive integers from 1 to the quantity of cells in that polyomino inclusive.
- If two identical numbers appear in the same row or column, at least that many cells with other numbers must separate them. For example, two cells both containing '1' may not be orthogonally adjacent, but must have at least one cell between them with a different number. Two cells marked '3' in the same row or column must have at least three cells with other numbers between them in that row or column, and so on.

History

Ripple Effect is an original puzzle of Nikoli; it first appeared in Puzzle Communication Nikoli #73 (May 1998).

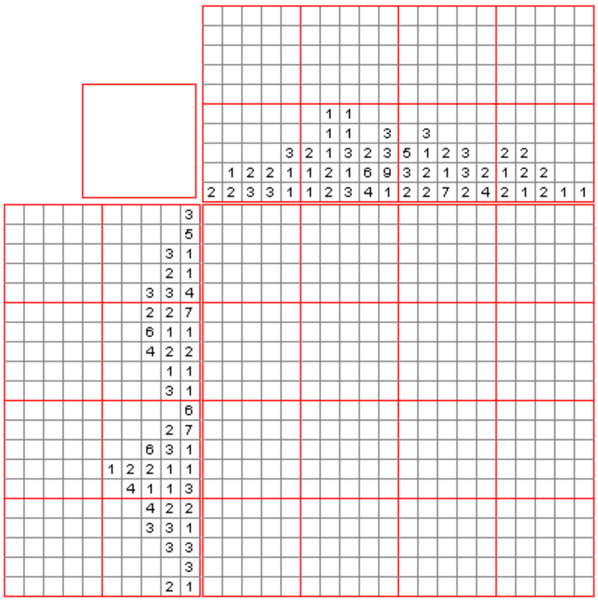
See also

- List of Nikoli puzzle types

Grid determination puzzles

Nonogram

Nonograms, also known as **Paint by Numbers** or **Griddlers** are picture logic puzzles in which cells in a grid have to be colored or left blank according to numbers given at the side of the grid to reveal a hidden picture. In this puzzle type, the numbers measure how many unbroken lines of filled-in squares there are in any given row or column. For example, a clue of "4 8 3" would mean there are sets of four, eight, and three filled squares, in that order, with at least one blank square between successive groups.



Example of a nonogram puzzle being solved. The steps of the process are grouped together a bit.

These puzzles are often black and white but can also have some colors. If they are colored, the number clues will also be colored in order to indicate the color of the squares. Two differently colored numbers may or may not have a space in between them. For example, a black four followed by a red two could mean four black spaces, some empty spaces, and two red spaces, or it could simply mean four black spaces followed immediately by two red ones.

There are no theoretical limits on the size of a nonogram, and they are also not restricted to square layouts.

Names

Nonograms are also known by many other names, including Paint by Numbers, Griddlers, Pic-a-Pix, Picross, PrismaPixels, Pixel Puzzles, Crucipixel, Edel, FigurePic, Grafilogika, Hanjie, Illust-Logic, Japanese Crosswords, Japanese Puzzles, Kare Karala!, Logic Art, Logic Square, Logicolor, Logik-Puzzles, Logimage, Obrazki logiczne, Zakódované obrázky, Maľované krížovky, Oekaki Logic, Oekaki-Mate, Paint Logic, Shchor Uftor, Gobelini, and Tsunamii. They have also been called Paint by Sudoku and Binary Coloring Books, although these names are entirely inaccurate.

History

Tetsuya Nishio (left) with Dave Green, president of Conceptis

In 1987, Non Ishida, a Japanese graphics editor, won a competition in Tokyo by designing grid pictures using skyscraper lights which are turned on or off. At the same time and with no connection, a professional Japanese puzzler named Tetsuya Nishio invented the same puzzles.

Paint by numbers puzzles started appearing in Japanese puzzle magazines. Nintendo picked up on this puzzle fad and in 1995 released two "Picross" (**Pic**ture **Cross**word) titles for the Game Boy and nine for the Super Famicom (eight of which were released in two-month intervals for the Nintendo Power Super Famicom Cartridge Writer as the "NP Picross" series) in Japan. Only one of these, *Mario's Picross* for the Game Boy, was released outside of Japan.

In 1988, Non Ishida published three picture grid puzzles in Japan under the name of "Window Art Puzzles".

In 1990, James Dalgety in the UK invented the name Nonograms after Non Ishida, and *The Sunday Telegraph* started publishing them on a weekly basis.

In 1993, First book of Nonograms was published by Non Ishida in Japan. The *Sunday Telegraph* published a dedicated puzzle book titled the "Book of Nonograms". Nonograms were also published in Sweden, United States (originally by *Games* magazine[1]), South Africa and other countries.

In 1995, paint by numbers started appearing on hand held electronic toys such as Game Boy and on other plastic puzzle toys. Increased popularity in Japan launched new publishers and by now there were several monthly magazines, some of which contained up to 100 puzzles.

In 1996, the Japanese arcade game Logic Pro was released by Deniam Corp, with a sequel released the following year.

In 1998, The *Sunday Telegraph* ran a competition to choose a new name for their puzzles. Griddlers was the winning name that readers chose.

In 1999, Paint by numbers were published by Sanoma Uitgevers in Holland, Puzzler Media (formerly British European Associated Publishers) in the UK and Nikui Rosh Puzzles in Israel.

In 2007, Nintendo released another version of Picross, this time for their Nintendo DS console.

Today, magazines with nonogram puzzles are published in the USA, UK, Germany, Netherlands, Italy, Hungary, Finland, Ukraine, and many other countries.

Solution techniques

In order to solve a puzzle, one needs to determine which cells are going to be boxes and which are going to be empty. Determining which cells are to be empty (called spaces) is as important as determining which are to be filled (called boxes). Later in the solving process, the spaces help to determine where a clue (continuing block of boxes and a number in the legend) may spread. Solvers usually use a dot or a cross to mark cells that are spaces for sure.

It is also important never to guess. Only cells that can be determined by logic should be filled. If guessing, a single error can spread over the entire field and completely ruin the solution. It usually comes to the surface only after a while, when it is very difficult to correct the puzzle. Usually, only advanced and experienced solvers are able to fix it completely and finish such ruined puzzles.

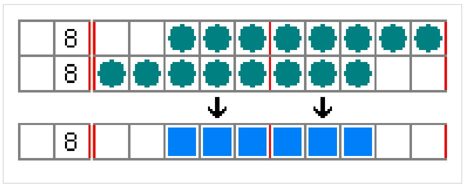
The hidden picture plays no part in the solving process. Even if it is obvious from the picture that a cell will be a box, it is usually treacherous to rely on it. The picture, however, may help find and eliminate an error.

Simpler puzzles can usually be solved by a reasoning on a single row only (or a single column) at each given time, to determine as many boxes and spaces on that row as possible. Then trying another row (or column), until there are rows that contain undetermined cells.

Some more difficult puzzles may also require several types of "what if?" reasoning that include more than one row (or column). This works on searching for contradictions: *When a cell cannot be a box, because some other cell would produce an error, it will definitely be a space. And vice versa.* Advanced solvers are sometimes able to search even deeper than into the first "what if?" reasoning. It takes, however, a lot of time to get some progress.

Simple boxes

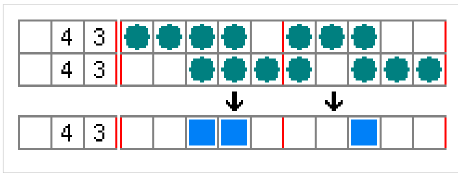
At the beginning of the solution a simple method can be used to determine as many boxes as possible. This method uses conjunctions of possible places for each block of boxes. For example, in a row of ten cells with only one clue of *8*, the bound block consisting of 8 boxes could spread from



- the right border, leaving two spaces to the left;
- the left border, leaving two spaces to the right;
- or somewhere in between.

In result, the block will spread for sure through the conjunction in the middle.

The same of course applies when there are more clues in the row. For example, in a row of ten cells with clues of *4* and *3*, the bound blocks of boxes could be

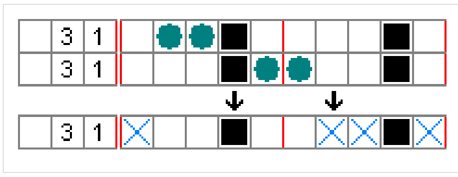


- crowded to the left, one next to the other, leaving two spaces to the right;
- crowded to the right, one just next to the other, leaving two spaces to the left;
- or somewhere between.

Consequently, the first block of four boxes definitely includes the third and fourth cells, while the second block of three boxes definitely includes the eighth cell. Boxes can therefore be placed in the third, fourth and eighth cells. Important note: When determining boxes in this way, boxes can be placed in cells only when the same block overlaps; in this example, although two blocks overlap in the sixth cell, they are different blocks, and so it cannot yet be said whether or not the sixth cell will contain a box.

Simple spaces

This method consists of determining spaces by searching for cells that are out of range of any possible blocks of boxes. For example, considering a row of ten cells with boxes in the fourth and ninth cell and with clues of *3* and *1*, the block bound to the clue *3* will spread through the fourth cell and clue *1* will be at the ninth cell.



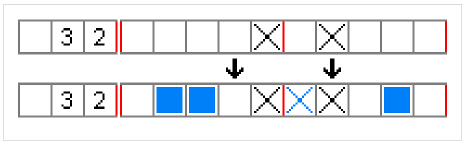
First, the clue *1* is complete and there will be a space at each side of the bound block.

Second, the clue *3* can only spread somewhere between the second cell and the sixth cell, because it always has to include the fourth cell; however, this may leave cells that may not be boxes in any case, i.e. the first and the seventh.

Note: In this example all blocks are accounted for; this is not always the case. The player must be careful for there may be clues or blocks that are not bound to each other yet.

Forcing

In this method, the significance of the spaces will be shown. A space placed somewhere in the middle of an uncompleted row may force a large block to one side or the other. Also, a gap that is too small for any possible block may be filled with spaces.

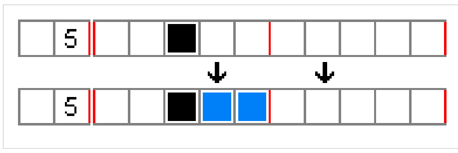


For example, considering a row of ten cells with spaces in the fifth and seventh cells and with clues of *3* and *2*:

- the clue of *3* would be forced to the left, because it could not fit anywhere else.
- the empty gap on the sixth cell is too small to accommodate clues like *2* or *3* and may be filled with spaces.
- finally, the clue of *2* will spread through the ninth cell according to method *Simple Boxes* above.

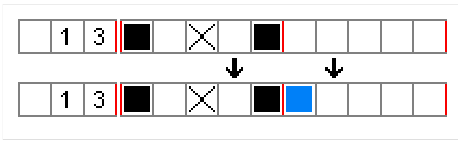
Glue

Sometimes, there is a box near the border that is not farther from the border than the length of the first clue. In this case, the first clue will spread through that box and will be forced outward from the border.



For example, considering a row of ten cells with a box in the third cell and with a clue of *5*, the clue of *5* will spread through the third cell and will continue to the fifth cell because of the border.

Note: This method may also work in the middle of a row, further away from the borders.



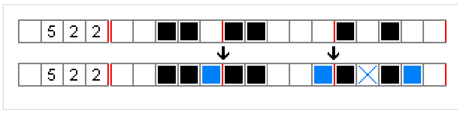
- A space may act as a border, if the first clue is forced to the right of that space.
- The *first* clue may also be preceded by some other clues, if all the clues are already bound to the left of the forcing space.

Joining and splitting

Boxes closer to each other may be sometimes joined together into one block or split by a space into several blocks. When there are two blocks with an empty cell between, this cell:

- will be a space, if joining the two blocks by a box would produce a too large block;
- and will be a box, if splitting the two blocks by a space would produce a too small block that does not have enough free cells around to spread through.

For example, considering a row of fifteen cells with boxes in the third, fourth, sixth, seventh, eleventh and thirteenth cell and with clues of *5*, *2* and *2*:



- the clue of *5* will join the first two blocks by a box into one large block, because a space would produce a block of only 4 boxes that is not enough there;
- and the clues of *2* will split the last two blocks by a space, because a box would produce a block of 3 continuous boxes, which is not allowed there.
- *Note: The illustration picture also shows how the clues of* 2 *will be further completed. This is, however, not part of the* Joining and splitting *technique, but the* Glue *technique described above.*

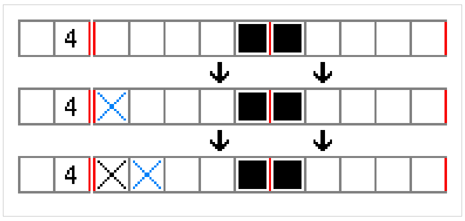
Punctuating

To solve the puzzle, it is usually also very important to enclose each bound and/or completed block of boxes immediately by separating spaces as described in *Simple spaces* method. Precise punctuating usually leads to more *Forcing* and may be vital for finishing the puzzle. *Note: The examples above did not do that only to remain simple.*

Mercury

Mercury is a special case of *Simple spaces* technique. Its name comes from the way mercury pulls back from the sides of a container.

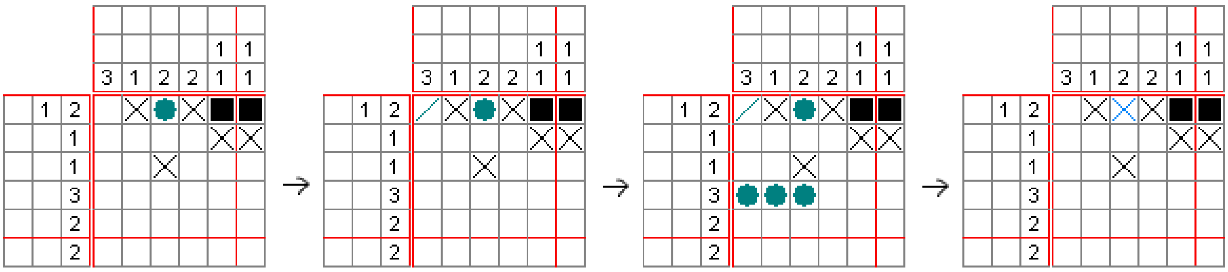
If there is a box in a row that is in the same distance from the border as the length of the first clue, the first cell will be a space. This is because the first clue would not fit to the left of the box. It will have to spread through that box, leaving the first cell behind. Furthermore, when the box is actually a block of more boxes to the right, there will be more spaces at the beginning of the row, determined by using this method several times.



Contradictions

Some more difficult puzzles may also require advanced reasoning. When all simple methods above are exhausted, searching for contradictions may help. It is wise to use a pencil (or other color) for that in order to be able to undo the last changes. The procedure includes:

1. Trying an empty cell to be a box (or then a space).
2. Using all available methods to solve as much as possible.
3. If an error is found, the tried cell will not be the box for sure. It will be a space (or a box, if space was tried).



In this example a box is tried in the first row, which leads to a space at the beginning of that row. The space then *forces* a box in the first column, which *glues* to a block of three boxes in the fourth row. However, that is wrong because the third column does not allow any boxes there, which leads to a conclusion that the tried cell must not be a box, so it must be a space.

The problem of this method is that there is no quick way to tell which empty cell to try first. Usually only a few cells lead to any progress, and the other cells lead to dead ends. Most worthy cells to start with may be:

- cells that have many non-empty neighbors;
- cells that are close to the borders or close to the blocks of spaces;
- cells that are within rows that consist of more non-empty cells.

Deeper recursion

Some puzzles may require to go deeper with searching for the contradictions. This is, however, not possible simply by a pen and pencil, because of the many possibilities that need to be searched. This method is practical for a computer to use, due to its practically limitless memory.

Multiple rows

In some cases, reasoning over a set of rows may also lead to the next step of the solution even without contradictions and deeper recursion. However, finding such sets is usually as difficult as finding contradictions.

Multiple solutions

There are puzzles that have several feasible solutions (one such is a picture of a simple chessboard). In these puzzles, all solutions are *correct* by the definition, but not all must give a reasonable picture.

Nonograms in computing

Solving nonogram puzzles is an NP-complete problem.[2] This means that there is no polynomial time algorithm that solves all nonogram puzzles unless P = NP.

However, certain classes of puzzles, such as those in which each row or column has only one block of cells and all cells are connected, may be solved in polynomial time by transforming the problem into an instance of 2-satisfiability.[3]

Other picture logic puzzles

Pentomino paint-by-numbers is a variant in which the twelve pentomino shapes must be placed in the grid, without touching each other (even diagonally).

Triddlers[4] are an offshoot that uses triangle shapes instead of squares.

Paint by pairs or **Link-a-Pix** consists of a grid, with numbers filling some squares; pairs of numbers must be located correctly and connected with a line filling a total of squares equal to that number. There is only one unique way to link all the squares in a properly-constructed puzzle. When completed, the squares that have lines are filled; the contrast with the blank squares reveals the picture. (As above, colored versions exist that involving matching numbers of the same color.)

Fill-a-Pix also uses a grid with numbers within. In this format, each number indicates how many of the squares immediately surrounding it, and itself, will be filled. A square marked "9," for example, will have all 8 surrounding squares and itself filled. If it is marked "0" those squares are all blank.

Maze-a-Pix uses a maze in a standard grid. When the single correct route from beginning to end is located, each 'square' of the solution is filled in (alternatively, all non-solution squares are filled in) to create the picture.

Tile Paint is another type of picture logic puzzle by Nikoli. It works like regular nonograms except that it only specifies the *total* number of squares in each row or column that will be filled in and irregular sections within the grid have borders around them that indicate that, if one of the squares within it is filled in, all of them must be filled in.

Video game versions

As noted above, the Game Boy saw its own version, titled *Mario's Picross*. The game was initially released in Japan on March 14, 1995 to decent success. However, the game failed to become a hit in the U.S. market, despite a heavy ad campaign by Nintendo. The game is of an escalating difficulty, with successive puzzle levels containing larger puzzles. Each puzzle has a limited amount of time to be cleared. Hints (line clears) may be requested at a time penalty, and mistakes made earn time penalties as well (the amount increasing for each mistake). *Mario's Picross 2* was released later for Game Boy and *Mario's Super Picross* for the Super Famicom, neither of which were translated for the U.S. market (*Mario's Super Picross* was, however, later released on the Wii Virtual Console's PAL service on September 14, 2007, as part of its Hanabi Festival). Both games introduced *Wario's Picross* as well, featuring Mario's nemesis in the role. These rounds vary by removing the hint function, and mistakes are not penalized — at the price that mistakes are not even revealed. These rounds can only be cleared when all correct boxes are marked, with no mistakes. The time limit was also removed. Nintendo also released eight *Picross* volumes on the Japanese Nintendo Power peripheral in Japan, each a new set of puzzles without the Mario characters.

More recently, Nintendo has released *Picross DS* for the Nintendo DS portable system. It contains several stages of varying difficulty, from 5x5 grids to 25x20 grids. Normal mode will tell you if you made an error (with a time penalty) and free mode will not tell you whether you made an error. A hint is available before starting the puzzle in all modes; the game reveals a complete row and column at random. Additional puzzles are available through Nintendo's Wi-Fi server; some of the original Mario Picross puzzles are available. Nintendo has been making new releases available bi-weekly. *Picross DS* was released in Europe and Australia on 11 May 2007 and in the United States on July 30, 2007 and has been received well by critics, labelling the game "Addictive". A 3D version of the game, titled *Picross 3D*, was also released for the DS in Japan in 2009 and internationally in 2010.

In Nancy Drew Shadow at the Water's Edge nonograms can be played, and a big grid nonogram is needed to solve to finish the game

iPhone/iPad/iOS apps

Several iPhone and iPad apps have surfaced since the app store opened.
CrossPix Magic [5] and CrossPix Magic Express [6] are iPad versions, from PurpleZoo Productions [7], which has a pleasant interface and various difficulty of puzzles.

Most notable iPhone version is Nonograms Pro [8], and iPad versions Nonograms [9] and NonogramsHDLite [10] by Hot Cocoa Games. These games capture the spirit of Nonograms in an easy to play interface.

Books

Several books of nonogram puzzles have been published in the US since 2006, to tie in with the sudoku craze. Titles include *Paint-doku*, *O'ekaki: Paint by Sudoku*, *The Essential Book of Hanjie* and *Crosspix*.

See also

- Battleship (puzzle)

References

[1] *Games Magazine Presents Paint by Numbers* (http://books.google.ca/books?id=K98BAAAACAAJ). Random House. 1994. ISBN 0812923847. .

[2] Ueda, Nobuhisa; Nagao, Tadaaki (1996), *NP-completeness results for NONOGRAM via Parsimonious Reductions* (http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.57.5277&rep=rep1&type=pdf), **TR96-0008**, Technical Report, Department of Computer Science, Tokyo Institute of Technology, , retrieved 2008-09-16

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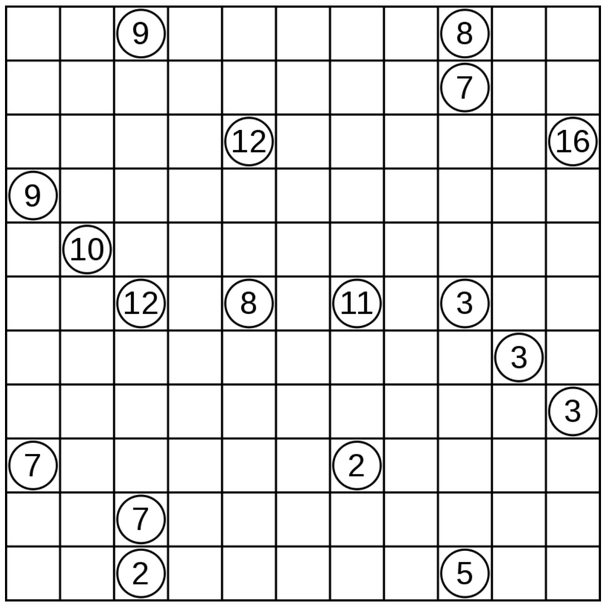
[4] "Triddlers rules and examples" (http://forum.griddlers.net/pages/t_rules). Griddlers.net. . Retrieved 1 January 2010.
[5] http://itunes.apple.com/us/app/crosspix-magic/id360039120?mt=8
[6] http://itunes.apple.com/us/app/crosspix-magic-express/id398916179?mt=8
[7] http://www.purplezoo.com
[8] http://itunes.apple.com/us/app/nonograms-pro/id384106476?mt=8
[9] http://itunes.apple.com/us/app/nonograms/id384120502?mt=8
[10] http://itunes.apple.com/us/app/nonogramshdlite/id395928335?mt=8

External links

- Nonograms (Brain teasers) (http://www.dmoz.org/Games/Puzzles/Brain_Teasers/Nonograms//) at the Open Directory Project
- Nonograms (Video games) (http://www.dmoz.org/Games/Video_Games/Puzzle/Nonograms//) at the Open Directory Project

Kuromasu

Kuromasu (Japanese:□どこ *kurodoko*) is a binary-determination logic puzzle published by Nikoli. As of 2005, one book consisting entirely of Kuromasu puzzles has been published by Nikoli.



Moderately difficult Kuromasu puzzle

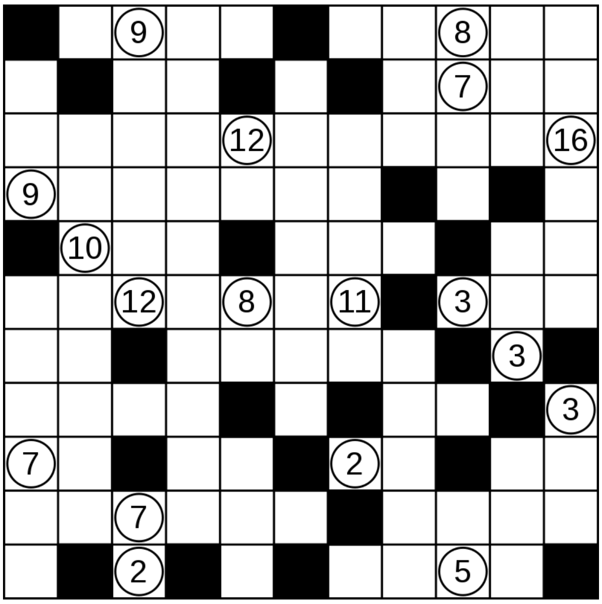
Rules

Kuromasu is played on a rectangular grid. Some of these cells have numbers in them. Each cell may be either black or white. The object is to determine what type each cell is.

The following rules determine which cells are which:

- Each number on the board represents the number of white cells that can be seen from that cell, including itself. A cell can be seen from another cell if they are in the same row or column, and there are no black cells between them in that row or column.
- Numbered cells may not be black.
- No two black cells may be horizontally or vertically adjacent.
- All the white cells must be connected horizontally or vertically.

Solution methods



Solution to puzzle

Any cell with a number in it must be white. This is very important. For example, suppose there is a 2 cell with another numbered cell next to it. Then clearly both the 2 cell and the other cell can be seen from the 2 cell. No other cells can be visible from the 2, or else we'd exceed the count. Therefore, all other neighboring cells to the 2 must be black. Also, the cell beyond the other numbered cell must be black. This is a good way to start some puzzles.

Suppose a 2 and another numbered cell or white cell are in the same row or column with just one space between them. Then the cell in the middle must be black, because if it were white, the 2 would be able to see at least 3 cells. This can also get you started on some puzzles quickly.

Contrariwise, if the number inside a cell is equal to the maximum number of cells it could possibly see, then all those cells must be white in order for that maximum to be possible. For example, in a 7×7 puzzle, the maximum number you can have in any cell is 13 (the cell itself, plus six others in the row, plus six other in the column). If a 13 appears in a cell of a 7×7 puzzle, all cells in the same row or column as the 13 must be white. This is often represented by placing dots in those cells.

There are other methods.

History

Kuromasu is an original puzzle of Nikoli; it first appeared in *Puzzle Communication Nikoli* #34 (June 1991). The English language Nikoli website uses Engrish to translate the name as "Where is Black Cells".

See also

- List of Nikoli puzzle types

References

- Nikoli's English page [1]

External links

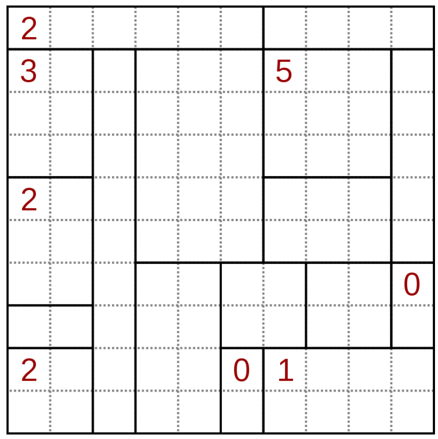
- A Java based Kuromasu Solver [2]
- Kuromasu Applet (German) [3]
- Comparing Methods for Solving Kuromasu Puzzles [4]
- A constraint programming solution in Prolog (French) [5]

References

[1] http://www.nikoli.co.jp/en/puzzles/where_is_black_cells/
[2] http://hoshan.org/kuromasu-solver-using-java
[3] http://www.janko.at/Applets/Kuromasu/index.htm
[4] http://www.liacs.nl/assets/Bachelorscripties/20-TimvanMeurs.pdf
[5] http://jfoutelet.developpez.com/articles/kuromasu/

Heyawake

Heyawake (Japanese: へやわけ, "divided rooms") is a binary-determination logic puzzle published by Nikoli. As of 2005, five books consisting entirely of *Heyawake* puzzles have been published by Nikoli. It first appeared in *Puzzle Communication Nikoli* #39 (September 1992).



Moderately easy sample puzzle (solution)

Rules

Heyawake is played on a rectangular grid of cells with no standard size; the grid is divided into variously sized rectangular "rooms" by bold lines following the edges of the cells. Some rooms may contain a single number, typically printed in their upper-left cell; as originally designed, every room was numbered, but this is rarely necessary for solving and is no longer followed.

Some of the cells in the puzzle are to be painted black; the object of the puzzle is to determine for each cell if it must be painted or must be left blank (remaining white). In practice, it is often easier to mark known "blank" cells in some way—for example, by placing a dot in the center of the cell.

The following rules determine which cells are which:

- Rule 1: Painted cells may never be orthogonally connected (they may not share a side, although they can touch diagonally).
- Rule 2: All white cells must be interconnected (form a single polyomino).
- Rule 3: A number indicates exactly how many painted cells there must be in that particular room.
- Rule 4: A room which has no number may contain any number of painted cells (including the possibility of zero cells).
- Rule 5: Where a straight (orthogonal) line of connected white cells is formed, it must not contain cells from more than two rooms—in other words, any such line of white cells which connects three or more rooms is forbidden.

Solution methods

Note that the first two rules also apply to (for example) *Hitori* puzzles, and thus these puzzles share some of their solving methods:

- If it is discovered that a cell is painted, it is immediately known that all of the four (orthogonally) adjacent cells must be white (from Rule 1).
- A section of (orthogonally) contiguous white cells cannot be cut off from the rest of the grid (from Rule 2). Black cells may not form a diagonal split across the grid nor a closed loop; any cell that would complete such a "short circuit" must be white instead.

More complex puzzles require combining Rule 1 and Rule 2 to make progress without guessing; the key is recognizing where the cells must assume one of two checkered patterns and one leads to a short circuit.

The remaining rules differentiate *Heyawake* from other "dynasty" puzzles:

- Rule 5 is the defining rule of the puzzle; black cells must be placed to prevent any (orthogonal) lines of white cells that cross two room borders ("spanners").
- Numbered rooms typically provide solvers a starting place, among other deductions. The following are the simplest examples of rooms defined at the onset:
  - A 2×2 room in the corner of the grid containing a '2' must have one painted cell in the grid corner and the second painted square diagonally outward from the corner. As painted squares may not share a side (Rule 1), the only alternative would disconnect the forced white cell in the corner, violating Rule 2.
  - A 2×3 room with the 3-cell side along a grid border containing a '3' must have a painted cell in the center of the 3-cell side along the border and the other two in the opposite corners of the room, for similar reasons to the above.
  - A 3×3 room containing a '5' must have a checkered pattern, with painted cells in all corners and the center.

Computational complexity

The computational complexity of Heyawake has been analyzed recently[1] : deciding for a given instance of Heyawake whether there exists a solution to the puzzle is NP-complete. An interpretation of this theoretical result in layman's terms is that this puzzle is as hard to solve as the Boolean satisfiability problem, which is a well studied difficult problem in computer science.

See also

List of Nikoli puzzle types

Notes

[1] M. Holzer, O. Ruepp (2007)

References

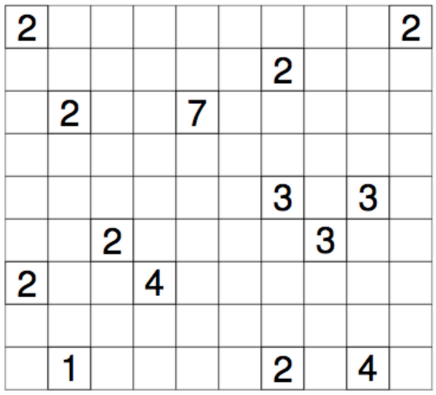
- Holzer, Markus; Ruepp, Oliver (2007). "The Troubles of Interior Design–A Complexity Analysis of the Game Heyawake". *Proceedings, 4th International Conference on Fun with Algorithms, LNCS 4475*. Springer, Berlin/Heidelberg. pp. 198–212. doi:10.1007/978-3-540-72914-3_18. ISBN 978-3-540-72913-6.

External links

- Nikoli's page on Heyawake (http://www.nikoli.co.jp/en/puzzles/heyawake/)

Nurikabe

Nurikabe (hiragana: ぬりかべ) is a binary determination puzzle named for an invisible wall in Japanese folklore that blocks roads and delays foot travel. Nurikabe was apparently invented and named by Nikoli; other names (and attempts at localization) for the puzzle include *Cell Structure* and *Islands in the Stream*.



Example of a moderately difficult 10x9 *Nurikabe* puzzle

Rules

The puzzle is played on a typically rectangular grid of cells, some of which contain numbers. The challenge is to construct a block maze (with no particular entrance or exit) subject to the following rules:

1. The "walls" are made of connected adjacent stone "blocks" in the grid of cells.
2. At the start of the puzzle, each numbered cell defines (and is one block in) a wall, and the number indicates how many blocks the wall must contain. The solver is not allowed to add any further walls beyond these.
3. Walls may not connect to each other, even if they have the same number.
4. Any cell which is not a block in a wall is part of "the maze."
5. The maze must be a single orthogonally contiguous whole: you must be able to reach any part of the maze from any other part by a series of adjacent moves through the maze.
6. The maze is not allowed to have any "rooms" -- meaning that the maze may not contain any 2x2 squares of non-block space. (On the other hand, the walls may contain 2x2 squares of blocks.)

Solvers will typically dot the non-numbered cells they've determined to be walls, and will shade in cells they've determined to be part of the maze. Because of this, the wall cells are often called "white" cells and the maze cells are often called "black" cells. In addition, in the "Islands in the Stream" localization, the "walls" are called "islands," the "maze" is called "the stream," and the ban on "rooms" is called a ban on "pools."

Like most other pure-logic puzzles, a unique solution is expected, and a grid containing random numbers is highly unlikely to provide a uniquely solvable *Nurikabe* puzzle.

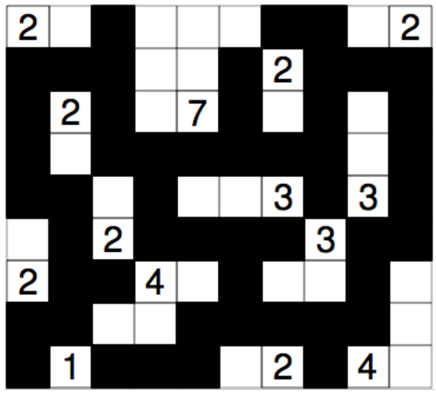
History

Nurikabe was first developed by "reenin (れーにん)," whose pen name is the Japanese pronunciation of "Lenin" and whose autonym can be read as such, in the 33rd issue of (Puzzle Communication) Nikoli at March 1991. It soon created a sensation, and has appeared in all issues of that publication from the 38th to the present.

As of 2005, seven books consisting entirely of *Nurikabe* puzzles have been published by Nikoli.

(This paragraph mainly depends on "Nikoli complete works of interesting-puzzles(ニコリ　オモロパズル大全集)." http://www.nikoli.co.jp/storage/addition/omopadaizen/)

Solution methods

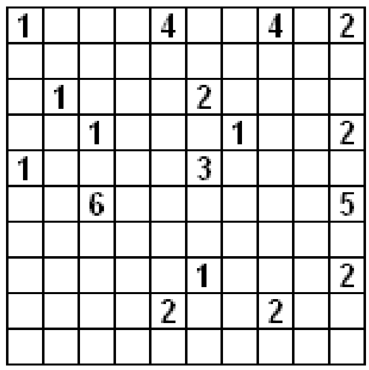


Solution to the example puzzle given above

No blind guessing should be required to solve a *Nurikabe* puzzle. Rather, a series of simple procedures and rules can be developed and followed, assuming the solver is sufficiently observant to find where to apply them.

The greatest mistake made by beginning solvers is to concentrate solely on determining black or white and not the other; most *Nurikabe* puzzles require going back and forth. Marking white cells may force other cells to be black lest a section of black be isolated, and vice versa. (Those familiar with Go can think of undetermined cells next to various regions as "liberties" and apply "atari" logic to determine how they must grow.) Oddly, the easiest rule to forget is the most basic one: all cells must be either black or white, so if it can be proved a cell isn't one, it must be the other.

Basic strategy



A Nurikabe puzzle being solved by a human. Dots represent the cells that are known to be white.

- Since two islands may only touch at corners, cells between two partial islands (numbers and adjacent white cells that don't total their numbers yet) must be black. This is often how to start a *Nurikabe* puzzle, by marking cells adjacent to two or more numbers as black.
- Once an island is "complete"—that is, it has all the white cells its number requires—all cells that share a side with it must be black. Obviously, any cells marked with '1' at the outset are complete islands unto themselves, and can be isolated with black at the beginning.
- Whenever three black cells form an "elbow"—an L-shape—the cell in the bend (diagonally in from the corner of the L) must be white. (The alternative is a "pool", for lack of a better term.)
- All black cells must eventually be connected. If there is a black region with only one possible way to connect to the rest of the board, the sole connecting pathway must be black.
- All white cells must eventually be part of exactly one island. If there is a white region that does not contain a number, and there is only one possible way for it to connect to a numbered white region, the sole connecting pathway must be white.
- Some puzzles will require the location of "unreachables"—cells that cannot be connected to any number, being either too far away from all of them or blocked by other numbers. Such cells must be black. Often, these cells will have only one route of connection to other black cells or will form an elbow whose required white cell (see previous bullet) can only reach one number, allowing further progress.

Advanced strategy

- If there is a square consisting of two black cells and two unknown cells, at least one of the two unknown cells must remain white according to the rules. Thus, if one of those two unknown cells (call it 'A') can only be connected to a numbered square by way of the other one (call it 'B'), then B must necessarily be white (and A may or may not be white).
- If an island of size N already has N-1 white cells identified, and there are only two remaining cells to choose from, and those two cells touch at their corners, then the cell between those two that is on the far side of the island must be black.

Related puzzles

The binary determination puzzles LITS and Mochikoro, also published by Nikoli, are similar to *Nurikabe* and employ similar solution methods. The binary determination puzzle Atsumari is similar to *Nurikabe* but based upon a hexagonal tiling rather than a square tiling.

See also

- Minesweeper
- Hashiwokakero
- List of Nikoli puzzle types

References

- Brandon McPhail, James D. Fix. Nurikabe is NP-Complete [1] NW Conference of the CSCC, 2004. Also presented at Reed Mathematics Colloquium, 2004.
- Markus Holzer, Andreas Klein and Martin Kutrib. On The NP-Completeness of The NURIKABE Pencil Puzzle and Variants Thereof [2]. Proceedings of the 3rd International Conference on Fun with Algorithms [3], 2004.

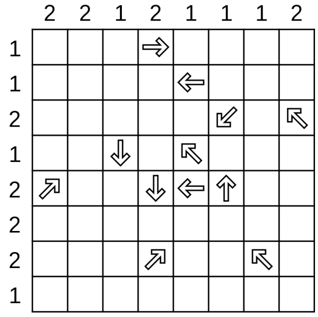
External links

- Nikoli's English page on *Nurikabe* [4]
- Some difficult *Nurikabe* puzzles [5]

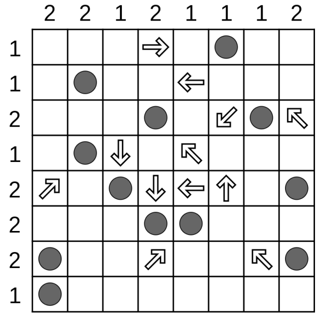
References

[1] http://www.cs.umass.edu/~mcphailb/papers/2004nurikabeposter.pdf
[2] http://cage.ugent.be/~klein/papers/nurikabe.pdf
[3] http://www.informatik.uni-trier.de/~ley/db/conf/fun/index.html
[4] http://www.nikoli.co.jp/en/puzzles/nurikabe/
[5] http://fabpedigree.com/nurikabe/

Shinro



A Shinro puzzle in its initial state...



...and with the hidden Holes displayed.

Shinro (しんろ) is a logic-based puzzle that has similarities to Sudoku and Minesweeper. The objective is to locate 12 hidden 'Holes' on an 8×8 grid. The board contains a variable number of arrows, each of which points to at least one Hole. A count of the number of Holes is given for each Row and Column.

Originally appearing in Japanese puzzle magazines, Shinro was popularized by its appearance in Southwest Airline's *Spirit* Magazine[1] . It has since spawned web-based and iPhone versions.

Name

New York-based puzzle-writing company Puzzability [2] has been credited[3] with coining the name Shinro in 2007. The name **Shinro** (しんろ) translates[4] to "compass bearing", referring to the arrows that point towards the Holes.

Availability

Websites:

- Southwest Airlines Spirit Magazine, Fun and Games section [5] Downloadable PDF with four puzzles
- Shinropuzzles website [6] Printable puzzles with solutions
- Sternenhimmel [7] (Babelfish translation [8]) German variation where each arrow points to *only one* Hole

iPhone:

- Shinro Mines [9]
- Jabeh [10] with video tutorial
- Sudoku Shinro [11]

See also

- Logic puzzle

Notes

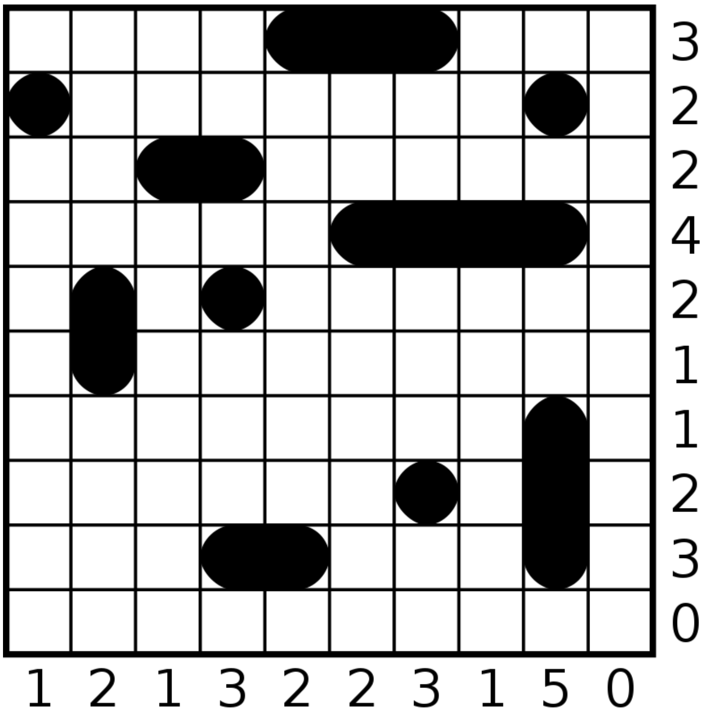
[1] Southwest Airlines Spirit Magazine, Fun and Games section (http://www.spiritmag.com/fun/index.php)
[2] http://www.puzzability.com/
[3] Shinropuzzles website (http://shinropuzzles.web.officelive.com) Credit given to Puzzability for coining the name 'Shinro'
[4] Google Translate translation (http://translate.google.com/translate_t?prev=hp&hl=en&js=y&text=□□□&file=&sl=ja&tl=en) of **Shinro** (しんろ)
[5] http://www.spiritmag.com/fun/index.php
[6] http://shinropuzzles.web.officelive.com/puzzles.aspx
[7] http://www.janko.at/Raetsel/Sternenhimmel/index.htm
[8] http://66.196.80.202/babelfish/translate_url_content?.intl=us&lp=de_en&trurl=http%3a%2f%2fbabelfish.altavista.com%2fbabelfish%2ftrurl_pagecontent%3flp%3dde_en%26url%3dhttp%3a%2f%2fwww.janko.at%2fRaetsel%2fSternenhimmel%2findex.htm
[9] http://www.benjiware.com/main/Shinro_Mines.html
[10] http://www.jabeh.org/
[11] http://web.me.com/romancini/Far_Apps/Shinro.html

External links

- Shinro tutorial (http://shinropuzzles.web.officelive.com/aboutus.aspx)
- Downloadable puzzles, Shinro tutorial, and a book (http://playshinro.com)

Battleship puzzle

The **Battleship** puzzle (sometimes called **Bimaru**, **Solitaire Battleships**. or **Battleship Solitaire**) is a logic puzzle based on the Battleship guessing game. It and its variants have appeared in several puzzle contests, including the World Puzzle Championship[1] , and puzzle magazines, such as *Games* magazine[2] .



Example of a solved Solitaire Battleships puzzle

Solitaire Battleship was invented in Argentina by Jaime Poniachik and was first featured in 1982 in the Spanish magazine Humor & Juegos. Battleship gained more widespread popularity after its international debut at the first World Puzzle Championship in New York City in 1992. Battleship appeared in Games magazine the following year and remains a regular feature of the magazine. Variants of Battleship have emerged since the puzzle's inclusion in the first World Puzzle Championship.

Battleship is played in a grid of squares that hides ships of different sizes. Numbers alongside the grid indicate how many squares in a row or column are occupied by part of a ship.

History

The solitaire version of Battleship was invented in Argentina in 1982 under the name Batalla Naval, with the first published puzzles appearing in 1982 in the Spanish magazine Humor & Juegos. Battleship was created by Jaime Poniachik, founder of Humor & Juegos, and Eduardo Abel Gimenez, Jorge Varlotta, and Daniel Samoilovich, who were editors of the magazine.

After 1982, no more Battleship puzzles were published until five years later in 1987, when Battleship puzzles were published in Juegos Para Gente De Mente, a renamed version of Humor & Juegos. The publishing company of Juegos Para Gente de Mente regularly publishes Battleship puzzles in its monthly magazine Enigmas Lógicos.

Battleship made its international debut at the first World Puzzle Championship in New York in 1992 and met with success. The next World Puzzle Championship in 1993 featured a variant of Battleship that omitted some of the row and column numbers. Other variants have emerged since then, including Hexagonal Battleship, 3D Battleship, and Diagonal Battleship.[3] [4]

Battleship was first published in Games magazine in 1993, the year after the first World Puzzle Championship.

Rules

In Battleship, an armada of battleships is hidden in a square grid of 10×10 small squares. The armada includes one battleship four squares long, two cruisers three squares long, three destroyers two squares long, and four submarines one square in size. Each ship occupies a number of contiguous squares on the grid, arranged horizontally or vertically. The boats are placed so that no boat touches any other boat, not even diagonally.

The goal of the puzzle is to discover where the ships are located. A grid may start with clues in the form of squares that have already been solved, showing a submarine, an end piece of a ship, a middle piece of a ship, or water. Each row and column also has a number beside it, indicating the number of squares occupied by ship parts in that row or column, respectively.[5]

Variants of the standard form of solitaire battleship have included using larger or smaller grids (with comparable changes in the size of the hidden armada), as well as using a hexagonal grid.

Strategy

The basic solving strategy for a Battleship puzzle is to add segments to incomplete ships where appropriate, draw water in squares that are known not to contain a ship segment, and to complete ships in a row or column whose number is the same as the number of unsolved squares in that row or column, respectively. More advanced strategies include looking for places where the largest ship that has not yet been located can fit into the grid, and looking for rows and columns that are almost complete and determining if there is only one way to complete them.[5]

Computers and Battleship

Battleship is an NP-complete problem.[6] In 1997, former contributing editor to the Battleship column in Games Magazine[7] Moshe Rubin released *Fathom It!*, a popular Windows implementation of Battleship.[8]

References

[1] http://wpc.puzzles.com/history/2000.htm
[2] http://www.gamesmagazine-online.com/
[3] http://www.mountainvistasoft.com/variations.htm
[4] http://www.conceptispuzzles.com/index.aspx?uri=puzzle/battleships/history
[5] Gordon, Peter; Mike Shenk. "Introduction". *Yubotu: Sink the Fleet in these Addictive Battleship Puzzles*. Conceptis Puzzles. New York City, NY: Sterling Publishing Company, Inc.. pp. 5–6. ISBN 1-4027-4189-8.

[6] Sevenster, M. 2004, ' Battleships as Decision Problem (http://www.mountainvistasoft.com/docs/BattleshipsAsDecidabilityProblem.pdf)', *ICGA Journal* [Electronic], Vol. 27, No. 3, pp.142-149. ISSN 1389-6911. Accessed: September 5, 2007
[7] http://www.mountainvistasoft.com/author.htm
[8] http://www.mountainvistasoft.com/reviews.htm

Further reading

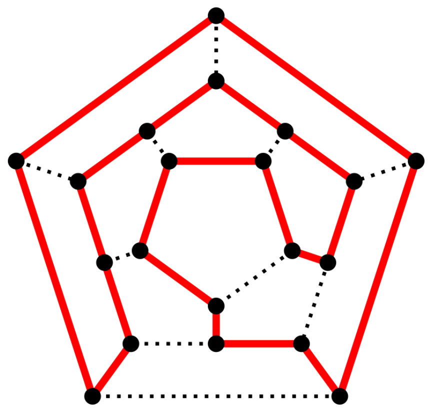
- Gordon, Peter; Mike Shenk. *Yubotu: Sink the Fleet in these Addictive Battleship Puzzles*. Conceptis Puzzles. New York City, NY: Sterling Publishing Company, Inc.. ISBN 1-4027-4189-8.
- Gordon, Peter; Mike Shenk. *Solitaire Battleships: 108 Challenging Logic Puzzles*.

External links

- The Battleship Omnibus (http://www.mountainvistasoft.com/omnibus.htm) - Extensive information on variants, competitions, and strategies.
- Battleships at Conceptis Puzzles (http://www.conceptispuzzles.com/index.aspx?uri=puzzle/battleships) - History, rules, tutorial, techniques.

Connection puzzles

Icosian game



A Hamiltonian cycle in a dodecahedron.

The **icosian game** is a mathematical game invented in 1857 by William Rowan Hamilton. The game's object is finding a Hamiltonian cycle along the edges of a dodecahedron such that every vertex is visited a single time, no edge is visited twice, and the ending point is the same as the starting point. The puzzle was distributed commercially as a pegboard with holes at the nodes of the dodecahedral graph and was subsequently marketed in Europe in many forms.

The motivation for Hamilton was the problem of symmetries of an icosahedron, for which he invented **icosians**—an algebraic tool to compute the symmetries.[1] The solution of the puzzle is a cycle containing twenty (in ancient Greek *icosa*) edges (i.e. a Hamiltonian circuit on the icosahedron).

See also

- Dual polyhedron
- Seven Bridges of Königsberg

References

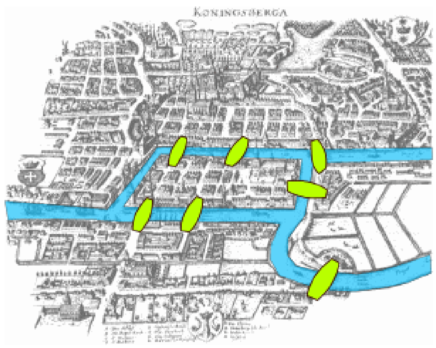
[1] "Icosian Game" (http://www.daviddarling.info/encyclopedia/I/Icosian_Game.html). . Retrieved 2008-11-28.

External links

- Puzzle Museum article with pictures (http://puzzlemuseum.com/month/picm02/200207icosian.htm)

Seven Bridges of Königsberg

The **Seven Bridges of Königsberg** is a notable historical problem in mathematics. Its negative resolution by Leonhard Euler in 1735 laid the foundations of graph theory and presaged the idea of topology.



Map of Königsberg in Euler's time showing the actual layout of the seven bridges, highlighting the river Pregel and the bridges

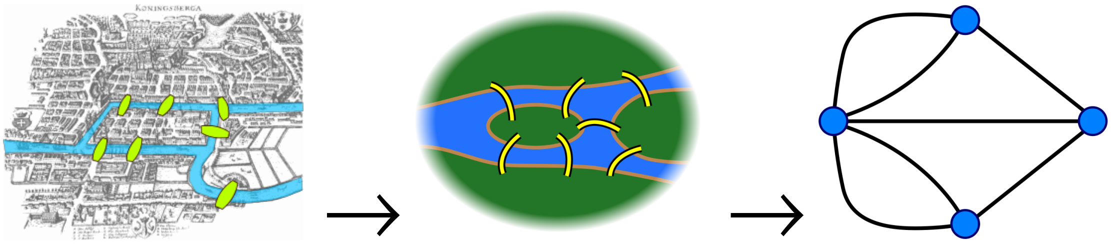
The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other and the mainland by seven bridges.

The problem was to find a walk through the city that would cross each bridge once and only once. The islands could not be reached by any route other than the bridges, and every bridge must have been crossed completely every time (one could not walk half way onto the bridge and then turn around and later cross the other half from the other side).

Euler's analysis

Euler proved that the problem has no solution.

To start with, Euler pointed out that the choice of route inside each landmass is irrelevant. The only important feature of a route is the sequence of bridges crossed. This allowed him to reformulate the problem in abstract terms (laying the foundations of graph theory), eliminating all features except the list of land masses and the bridges connecting them. In modern terms, one replaces each land mass with an abstract "vertex" or node, and each bridge with an abstract connection, an "edge", which only serves to record which pair of vertices (land masses) is connected by that bridge. The resulting mathematical structure is called a graph.



Since only the connection information is relevant, the shape of pictorial representations of a graph may be distorted in any way without changing the graph itself. Only the existence (or lack) of an edge between each pair of nodes is significant. For example, it does not matter whether the edges drawn are straight or curved, or whether one node is to the left or right of another.

Next, Euler observed that (except at the endpoints of the walk) whenever one enters a vertex by a bridge, one leaves the vertex by a bridge. In other words, during any walk in the graph, the number of times one enters a non-terminal vertex equals the number of times one leaves it. Now if every bridge is traversed exactly once it follows that for each land mass (except possibly for the ones chosen for the start and finish), the number of bridges touching that land mass is **even** (half of them, in the particular traversal, will be traversed "toward" the landmass, the other half "away" from it). However, all the four land masses in the original problem are touched by an **odd** number of bridges (one is touched by 5 bridges and the other three by 3). Since at most two land masses can serve as the endpoints of a putative walk, the existence of a walk traversing each bridge once leads to a contradiction.

In modern language, Euler shows that the existence of a walk in a graph which traverses each edge once depends on the degrees of the nodes. The degree of a node is the number of edges touching it. Euler's argument shows that a necessary condition for the walk of the desired form to exist is that the graph be connected and have exactly zero or two nodes of odd degree. This condition turns out also to be sufficient—a result stated by Euler and later proven by Carl Hierholzer. Such a walk is now called an *Eulerian path* or *Euler walk* in his honor. Further, if there are nodes of odd degree, all Eulerian paths start at one of them and end at the other. Since the graph corresponding to historical Königsberg has four nodes of odd degree, it cannot have an Eulerian path.

An alternative form of the problem asks for a path that traverses all bridges and also has the same starting and ending point. Such a walk is called an *Eulerian circuit* or an *Euler tour*. Such a circuit exists if and only if the graph is connected and there are no nodes of odd degree at all. All Eulerian circuits are also Eulerian paths, but not all paths are also circuits.

Euler's work was presented to the St. Petersburg Academy on August 26, 1735, and published as *Solutio problematis ad geometriam situs pertinentis* (The solution of a problem relating to the geometry of position) in the journal *Commentarii academiae scientiarum Petropolitanae* in 1741.[1] It is available in English in The World of Mathematics.

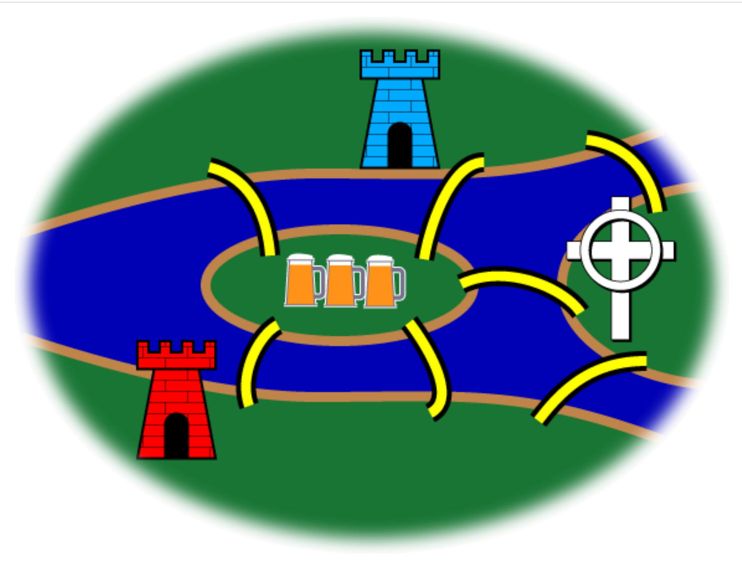
Significance in the history of mathematics

In the history of mathematics, Euler's solution of the Königsberg bridge problem is considered to be the first theorem of graph theory, a subject now generally regarded as a branch of combinatorics. Combinatorial problems of other types had been considered since antiquity.

In addition, Euler's recognition that the key information was the number of bridges and the list of their endpoints (rather than their exact positions) presaged the development of topology. The difference between the actual layout and the graph schematic is a good example of the idea that topology is not concerned with the rigid shape of objects.

Variations

The classic statement of the problem, given above, uses **unidentified** nodes—that is, they are all alike except for the way in which they are connected. There is a variation in which the nodes are **identified**—each node is given a unique name or color.



The northern bank of the river is occupied by the *Schloß*, or castle, of the Blue Prince; the southern by that of the Red Prince. The east bank is home to the Bishop's *Kirche*, or church; and on the small island in the center is a *Gasthaus*, or inn.

It is understood that the problems to follow should be taken in order, and begin with a statement of the original problem:

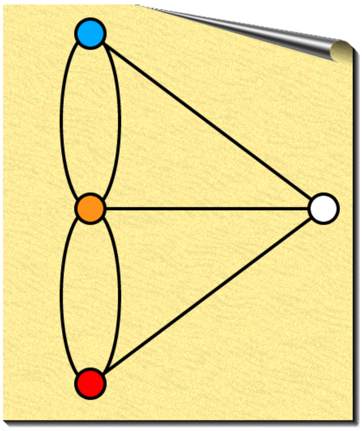
It being customary among the townsmen, after some hours in the *Gasthaus*, to attempt to **walk the bridges**, many have returned for more refreshment claiming success. However, none have been able to repeat the feat by the light of day.

8: The Blue Prince, having analyzed the town's bridge system by means of graph theory, concludes that the bridges cannot be walked. He contrives a stealthy plan to build an eighth bridge so that he can begin in the evening at his *Schloß*, walk the bridges, and end at the *Gasthaus* to brag of his victory. Of course, he wants the Red Prince to be unable to duplicate the feat from the Red Castle. *Where does the Blue Prince build the eighth bridge?*

9: The Red Prince, infuriated by his brother's Gordian solution to the problem, wants to build a ninth bridge, enabling *him* to begin at his *Schloß*, walk the bridges, and end at the *Gasthaus* to rub dirt in his brother's face. His brother should then no longer walk the bridges himself. *Where does the Red Prince build the ninth bridge?*

10: The Bishop has watched this furious bridge-building with dismay. It upsets the town's *Weltanschauung* and, worse, contributes to excessive drunkenness. He wants to build a tenth bridge that allows *all* the inhabitants to walk the bridges and return to their own beds. *Where does the Bishop build the tenth bridge?*

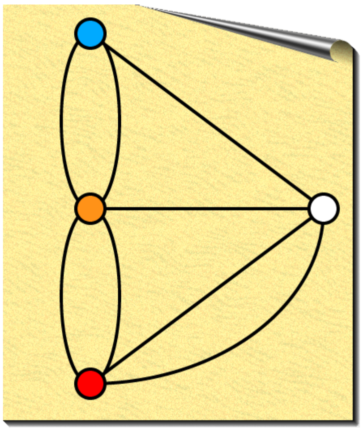
Solutions



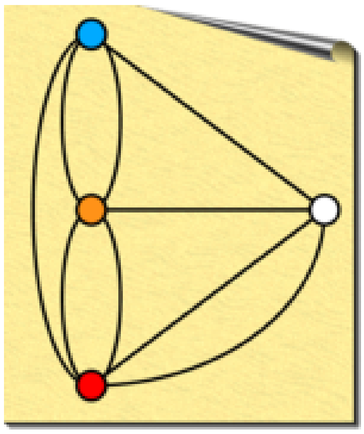
The colored graph

Reduce the city, as before, to a graph. Color each node. As in the classic problem, no Euler walk is possible; coloring does not affect this. All four nodes have an odd number of edges.

8: Euler walks are possible if 2 nodes or fewer have an odd number of edges. If we have 2 nodes with an odd number of edges, the walk must begin at one such node and end at the other. Since there are only 4 nodes in the puzzle, the solution is simple. The walk desired must begin at the blue node and end at the orange node. Thus, a new edge is drawn between the other two nodes. Since they each formerly had an odd number of edges, they must now have an even number of edges, fulfilling all conditions. This is a change in parity from an odd to even degree.



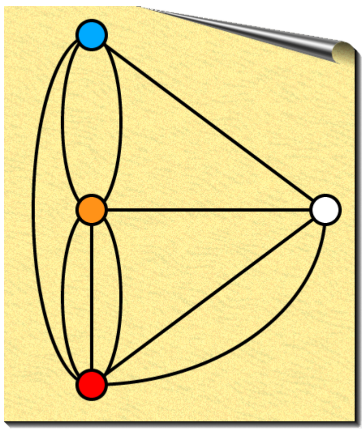
The 8th edge



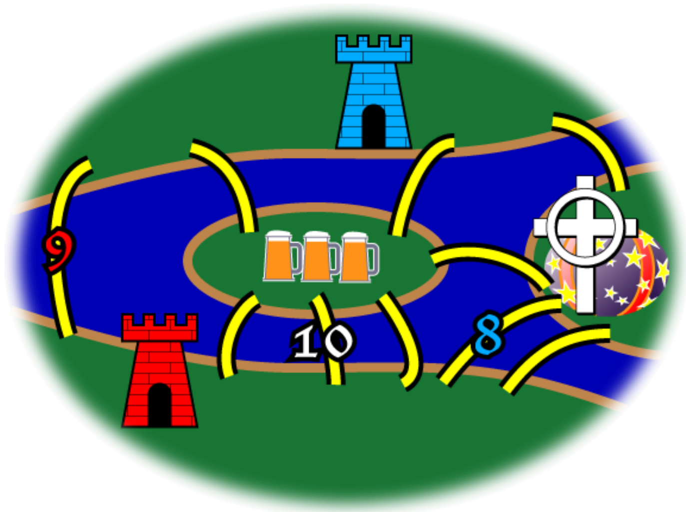
The 9th edge

9: The 9th bridge is easy once the 8th is solved. The desire is to enable the red castle and forbid the blue castle as a starting point; the orange node remains the end of the walk and the white node is unaffected. To change the parity of both red and blue nodes, draw a new edge between them.

10: The 10th bridge takes us in a slightly different direction. The Bishop wishes every citizen to return to his starting point. This is an Euler cycle and requires that all nodes be of even degree. After the solution of the 9th bridge, the red and the orange nodes have odd degree, so their parity must be changed by adding a new edge between them.



The 10th edge



8th, 9th, and 10th bridges

Present state of the bridges

Two of the seven original bridges were destroyed during the bombing of Königsberg in World War II. Two others were later demolished and replaced by a modern highway. The three other bridges remain, although only two of them are from Euler's time (one was rebuilt in 1935).[2] Thus, there are now five bridges in Königsberg.

In terms of graph theory, two of the nodes now have degree 2, and the other two have degree 3. Therefore, an Eulerian path is now possible, but since it must begin on one island and end on the other, it is impractical for tourists.[3]

At Canterbury University, in Christchurch, New Zealand, a model of the bridges is incorporated in a grass area between the old Physical Sciences Library and the Erskine Building, Housing the Departments of Mathematics, Statistics and Computer Science.[4]

See also

- Glossary of graph theory
- Icosian game
- Hamiltonian path
- Water, gas, and electricity
- Travelling salesman problem

References

[1] The Euler Archive (http://www.math.dartmouth.edu/~euler/pages/E053.html), commentary on publication, and original text, in Latin.

[2] Taylor, Peter (December 2000). "What *Ever* Happened to Those Bridges?" (http://www.amt.canberra.edu.au/koenigs.html). Australian Mathematics Trust. . Retrieved 2006-11-11.

[3] Stallmann, Matthias (July 2006). "The 7/5 Bridges of Koenigsberg/Kaliningrad" (http://www.csc.ncsu.edu/faculty/stallmann/SevenBridges/). . Retrieved 2006-11-11.

[4] "About - Mathematics and Statistics - University of Canterbury" (http://www.math.canterbury.ac.nz/php/about/). *math.canterbury.ac.nz*. . Retrieved November 4, 2010.

External links

- Kaliningrad and the Konigsberg Bridge Problem (http://mathdl.maa.org/convergence/1/?pa=content&sa=viewDocument&nodeId=1310&bodyId=1452) at Convergence (http://mathdl.maa.org/convergence/1/)
- Euler's original publication (http://math.dartmouth.edu/~euler/docs/originals/E053.pdf) (in Latin)
- The Bridges of Königsberg (http://www.jimloy.com/puzz/konigs.htm)
- How the bridges of Königsberg help to understand the brain (http://www.nonlinearbiomedphys.com/content/1/1/3)
- Euler's Königsberg's Bridges Problem (http://www.contracosta.edu/math/konig.htm) at Math Dept. Contra Costa College (http://www.contracosta.edu/math/)
- Pregel - A Google graphing tool named after this problem (http://googleresearch.blogspot.com/2009/06/large-scale-graph-computing-at-google.html)

Geographical coordinates: 54°42′12″N 20°30′56″E

Water, gas, and electricity

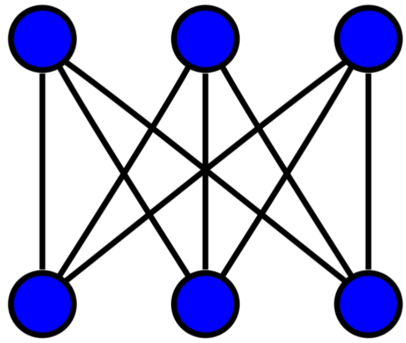
The classical mathematical puzzle known as **water, gas, and electricity**, the **(three) utilities problem**, or sometimes the **three cottage problem**, can be stated as follows:

> Suppose there are three cottages on a plane (or sphere) and each needs to be connected to the gas, water, and electric companies. Using a third dimension or sending any of the connections through another company or cottage are disallowed. Is there a way to make all nine connections without any of the lines crossing each other?

This is intended as an abstract mathematical puzzle and imposes constraints that would not be issues in a practical engineering scenario.

Solution

The answer is no; It is impossible to connect the three cottages with the three different utilities without at least one of the connections crossing another.



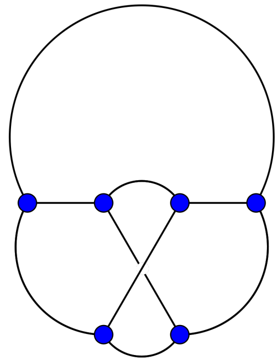
The utility graph, also known as the Thomsen graph or $K_{3,3}$

The problem is part of the mathematical field of topological graph theory which studies the embedding of graphs on surfaces. In more formal graph-theoretic terms, the problem asks whether the complete bipartite graph $K_{3,3}$ is planar. This graph is often referred to as the **utility graph** in reference to the problem.[1] The graph is equivalent to the circulant graph $Ci_6(1,3)$. Kazimierz Kuratowski proved in 1930 that $K_{3,3}$ is nonplanar, and thus that the problem has no solution.

One proof of the impossibility of finding a planar embedding of $K_{3,3}$ uses a case analysis involving the Jordan curve theorem, in which one examines different possibilities for the locations of the vertices with respect to the 4-cycles of the graph and shows that they are all inconsistent with a planar embedding. Alternatively, it is possible to show that any bridgeless bipartite planar graph with n vertices and m edges has $m \leq 2n - 4$ by combining the Euler formula $n - m + f = 2$ (where f is the number of faces of a planar embedding) with the observation that the number of faces is at most half the number of edges (because each face has at least four edges and each edge belongs to exactly two faces). In the utility graph, $m = 9$ and $2n - 4 = 8$, violating this inequality, so the utility graph cannot be planar.

Two important characterizations of planar graphs, Kuratowski's theorem that the planar graphs are exactly the graphs that contain neither $K_{3,3}$ nor the complete graph K_5 as a subdivision, and Wagner's theorem that the planar graphs are exactly the graphs that contain neither $K_{3,3}$ nor K_5 as a minor, encompass this result.

Generalizations



$K_{3,3}$ drawn with only one crossing.

$K_{3,3}$ is toroidal, which means it can be embedded on the torus. In terms of the three cottage problem this means the problem can be solved by punching two holes through the plane (or the sphere) and connecting them with a tube. This changes the topological properties of the surface and using the tube we can connect the three cottages without crossing lines. An equivalent statement is that the graph genus of the utility graph is one, and therefore it cannot be embedded in a surface of genus less than one. A surface of genus one is equivalent to a torus.

Pál Turán's "brick factory problem" asks more generally for a formula for the minimum number of crossings in a drawing of the complete bipartite graph $K_{a,b}$ in terms of the numbers of vertices a and b on the two sides of the bipartition. The utility graph $K_{3,3}$ may be drawn with only one crossing, but not with zero crossings, so its crossing number is one. A toroidal embedding of $K_{3,3}$ may be obtained by replacing the crossing by a tube, as described above, in which the two holes where the tube connects to the plane are placed along one of the crossing edges on either side of the crossing.

Notes

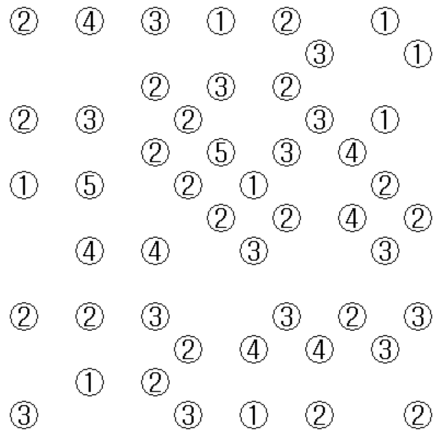
[1] Utility Graph (http://mathworld.wolfram.com/UtilityGraph.html) from *mathworld.wolfram.com*

External links

- Utility Graph (http://mathworld.wolfram.com/UtilityGraph.html) at MathWorld
- The Utilities Puzzle (http://www.archimedes-lab.org/How_to_Solve/Water_gas.html) explained and 'solved' at Archimedes-lab.org
- 3 Utilities Puzzle (http://www.cut-the-knot.org/do_you_know/3Utilities.shtml) at cut-the-knot
- Proof That the Impossible Puzzle is Impossible (http://www.jimloy.com/puzz/puzzle2.htm)

Hashiwokakero

Hashiwokakero (橋をかけろ *Hashi o kakero*; lit. "build bridges!") is a type of logic puzzle published by Nikoli. It has also been published in English under the name *Bridges* or *Chopsticks* (an understandably mistaken translation; the *hashi* of the title, 橋, means *bridge*; *hashi* written with another character, 箸, means *chopsticks*). It has also appeared in *The Times* under the name *Hashi*. In France, Denmark, the Netherlands, and Belgium it is published under the name Ai-Ki-Ai.



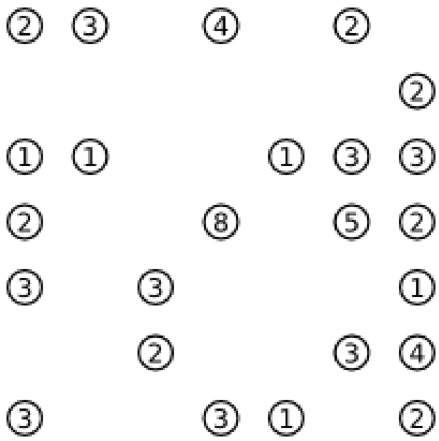
Moderately difficult *Hashiwokakero* puzzle (solution)

Rules

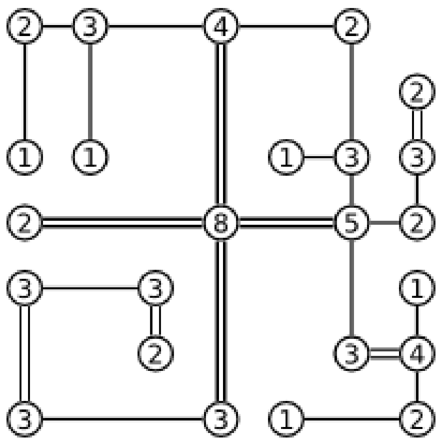
Hashiwokakero is played on a rectangular grid with no standard size, although the grid itself is not usually drawn. Some cells start out with (usually encircled) numbers from 1 to 8 inclusive; these are the *islands*. The rest of the cells are empty.

The goal is to connect all of the islands by drawing a series of bridges between the islands. The bridges must follow certain criteria:

- They must begin and end at distinct islands, travelling a straight line in between.
- They must not cross any other bridges or islands.
- They may only run perpendicularly.
- At most two bridges connect a pair of islands.
- The number of bridges connected to each island must match the number on that island.
- The bridges must connect the islands into a single connected group.



A simpler puzzle



Solution of above puzzle

Solution methods

Solving a *Hashiwokakero* puzzle is a matter of procedural force: having determined where a bridge must be placed, placing it there can eliminate other possible places for bridges, forcing the placement of another bridge, and so on.

An island showing '3' in a corner, '5' along the outside edge, or '7' anywhere must have at least one bridge radiating from it in each valid direction, for if one direction did not have a bridge, even if all other directions sported two bridges, not enough will have been placed.

Obviously, a '4' in a corner, '6' along the border, or '8' anywhere must have two bridges in each direction. This can be generalized as added bridges obstruct routes: a '3' that can only be travelled from vertically must have at least one bridge each for up and down, for example.

It is common practice to cross off islands whose bridge quota has been reached. In addition to reducing mistakes, this can also help locate potential "short circuits": keeping in mind that all islands must be connected by one network of bridges, a bridge that would create a closed network that no further bridges could be added to can only be permitted if it immediately yields the solution to the complete puzzle. The simplest example of this is two islands showing '1' aligned with each other; unless they are the only two islands in the puzzle, they cannot be connected by a bridge, as that would complete a network that cannot be added to, and would therefore force those two islands to be unreachable by any others.

There is a solution using integer linear programming in the MathProg examples included in GLPK.

History

Hashiwokakero first appeared in Puzzle Communication Nikoli in issue #31 (September 1990), although an earlier form of the puzzle appeared in issue #28 (December 1989).

See also

- List of Nikoli puzzle types

References

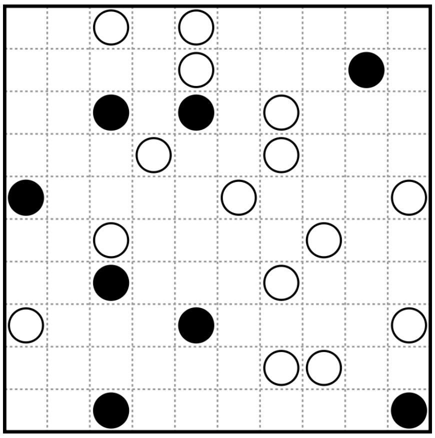
- Puzzle Cyclopedia, Nikoli, 2004. ISBN 4-89072-406-0.
- Nikoli's English page on Hashiwokakero [1]

References

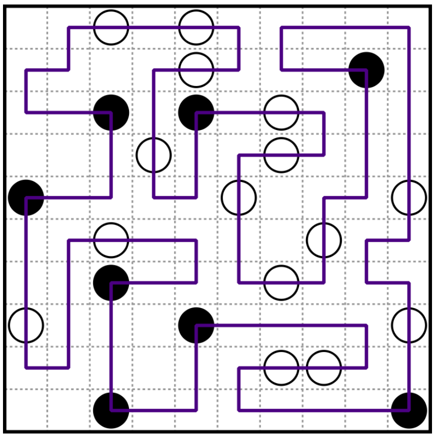
[1] http://www.nikoli.co.jp/en/puzzles/hashiwokakero/

Masyu

Masyu (Japanese: ましゅ IPA [maʃɯ]; translates as "evil influence") is a type of logic puzzle designed and published by Nikoli. The purpose of its creation was to present a puzzle that uses no numbers or letters and yet retains depth and aesthetics.



Sample puzzle



Solution to above puzzle

Rules

Masyu is played on a rectangular grid of squares, some of which contain circles; each circle is either "white" (empty) or "black" (filled). The goal is to draw a single continuous non-intersecting loop that properly passes through all circled cells. The loop must "enter" each cell it passes through from the center of one of its four sides and "exit" from a different side; all turns are therefore 90 degrees.

The two varieties of circle have differing requirements for *how* the loop must pass through them:

- White circles must be traveled straight through, but the loop must turn in the previous and/or next cell in its path;
- Black circles must be turned upon, but the loop must travel straight through the next and previous cells in its path.

History

The early version of *Masyu* first appeared in *Puzzle Communication Nikoli* #84 under the title of *Shinju no Kubikazari* (真珠の首飾り, meaning "pearl necklace"). That puzzle contains only white circles. Black circles were introduced in *Puzzle Communication Nikoli* #90, and the puzzle was renamed *Shiroshinju Kuroshinju* (白真珠□真珠, meaning "white pearls and black pearls"). This improvement deepened the puzzle and made it gain popularity. *Masyu*, which is originally a misreading by Nikoli's president of kanji 真珠 (*shinju*), and apparently became an inside joke at the Nikoli office, was adopted in *Puzzle Communication Nikoli* #103 to replace the old lengthy name.

Solution methods

Understanding the nuances of the circles and how they interact with each other is the key to solving a *Masyu* puzzle. Generally speaking, it is easiest to start along the outside border of the grid and work inwards. Here are some basic scenarios where portions of the loop can be determined:

- Any segment travelling from a black circle must travel two cells in that direction without intersecting another part of the loop or the outer border; each black cell must have two such segments at a right angle. The logical combination of those two statements is that if a segment from a black cell cannot be drawn in some orthogonal direction, a segment in the *opposite* direction *must* be drawn. For example, if one cannot legally travel up two cells from a black circle, then the loop must travel down from that black circle for two cells. This has two common results:
    - Any black circle along the outer border or one cell from the outer border must have a segment leading away from the border (and those sufficiently near a corner must lead from both walls, defining the loop's path

through the circle);
- Orthogonally adjacent black circles must have segments travelling away from each other.

- White circles along the outer border obviously need the loop to travel through them parallel to the border; if two white circles along a border are adjacent or are one cell apart, then the loop will need to turn away from the border just beyond the circles.
- If three or more white circles are orthogonally contiguous and collinear, then the loop will need to pass through each of those circles perpendicular to the line of circles.

As in other loop-construction puzzles, "short circuits" also need to be avoided: as the solution must consist of a single loop, any segment that would close a loop is forbidden unless it immediately yields the solution to the entire puzzle.

Like many other combinatory and logic puzzles, Masyu can be very difficult to solve; solving Masyu on arbitrarily large grids is an NP-complete problem.[1] However, published instances of puzzles have generally been constructed in such a way that they can be solved in a reasonable amount of time.

See also

- List of Nikoli puzzle types

References

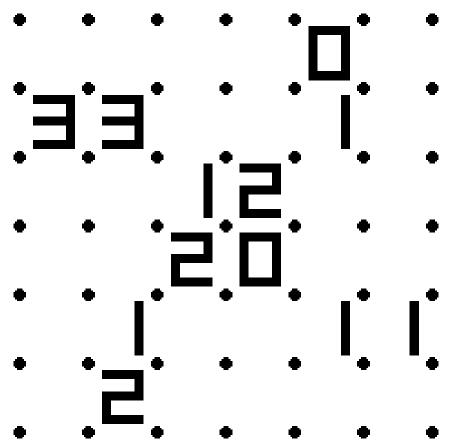
[1] Erich Friedman. "Pearl Puzzles are NP-Complete". In preparation. 2002. (http://www.stetson.edu/~efriedma/papers/pearl/pearl.html).

External links

- Masyu page at Web Nikoli (http://www.nikoli.co.jp/en/puzzles/masyu/)
- Father of Sudoku puzzles next move (http://news.bbc.co.uk/2/hi/asia-pacific/6745433.stm) BBC

Slitherlink

Slitherlink (also known as Fences, Takegaki, Loop the Loop, Loopy, Ouroboros, Suriza and Dotty Dilemma) is a logic puzzle developed by publisher Nikoli.


Moderately difficult *Slitherlink* puzzle (solution)

Rules

Slitherlink is played on a rectangular lattice of dots. Some of the squares formed by the dots have numbers inside them. The objective is to connect horizontally and vertically adjacent dots so that the lines form a simple loop with no loose ends. In addition, the number inside a square represents how many of its four sides are segments in the loop.

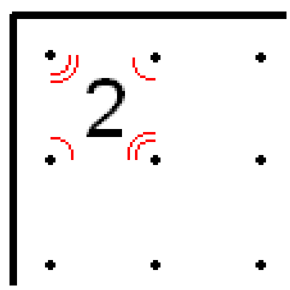
Other types of planar graphs can be used in lieu of the standard grid, with varying numbers of edges per vertex or vertices per polygon. These patterns include snowflake, Penrose, Laves and Altair tilings. These add complexity by varying the number of possible paths from an intersection, and/or the number of sides to each polygon; but similar rules apply to their solution.

Solution methods

Notation

Whenever the number of lines around a cell matches the number in the cell, the other potential lines must be eliminated. This is usually indicated by marking an X on lines known to be empty.

Another useful notation when solving Slitherlink is a seventy-five-degree arc between two adjacent lines, to indicate that *exactly one* of the two must be filled. A related notation is a double arc between adjacent lines, indicating that *both or neither* of the two must be filled. These notations are not necessary to the solution, but can be helpful in deriving it.


Arc notation around a 2 in a corner.

Many of the methods below can be broken down into two simpler steps by use of arc notation.

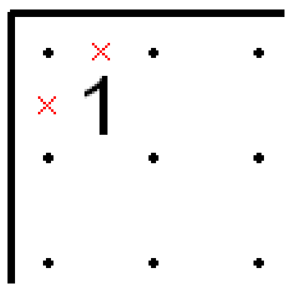
Exactly 2 or 0 lines at each point

A key to many deductions in Slitherlink is that every point has either exactly two lines connected to it, or no lines. So if a point which is in the centre of the grid, not at an edge or corner, has three incoming lines which are X'd out, the fourth must also be X'd out. This is because the point cannot have just one line - it has no exit route from that point. Similarly, if a point on the edge of the grid, not at a corner, has two incoming lines which are X'd out, the third must also be X'd out. And if a corner of the grid has one incoming line which is X'd out, the other must also be X'd out.

Application of this simple rule leads to increasingly complex deductions. Recognition of these simple patterns will help greatly in solving Slitherlink puzzles.

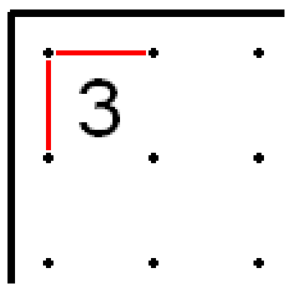
Corners

- If *a 1 is in a corner*, the actual corner's lines may be X'ed out, because a line that entered said corner could not leave it except by passing by the 1 again. This also applies if two lines leading into the 1-box at the same corner are X'ed out.



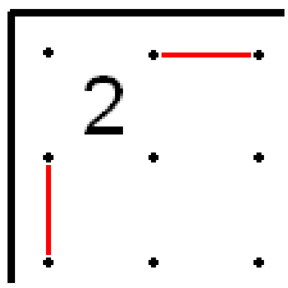
A 1 in a corner.

- If *a 3 is in a corner*, the two outside edges of that box can be filled in because otherwise the rule above would have to be broken.



A 3 in a corner.

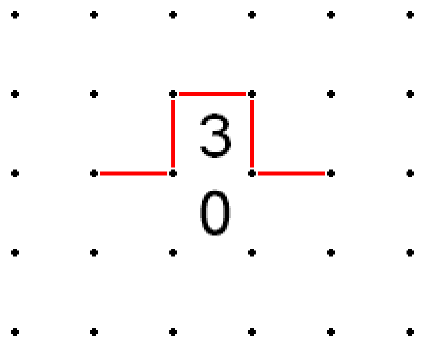
- If *a 2 is in a corner*, two lines must be going away from the 2 at the border.



A 2 in a corner.

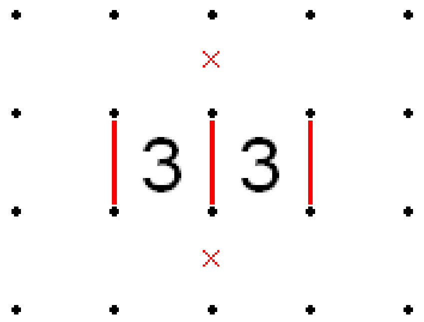
Rules for squares with 3

- If *a 3 is adjacent to a 0*, either horizontally or vertically, then all edges of that 3 can be filled except for the one touching the 0. In addition, the two lines perpendicular to the adjacent boxes can be filled.



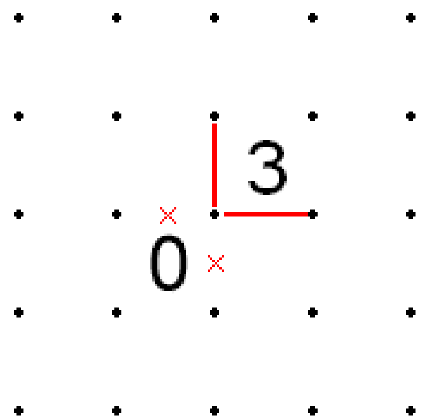
A 3 adjacent to an 0.

- If *two 3s are adjacent* to each other horizontally or vertically, their common edge must be filled in, because the only other option is a closed oval that is impossible to connect to any other line. Second, the two outer lines of the group (parallel to the common line) must be filled in. Thirdly, the line through the 3s will always wrap around in an "S" shape. Therefore, the line between the 3s cannot continue in a straight line, and those sides which are in a straight line from the middle line can be X'd out.



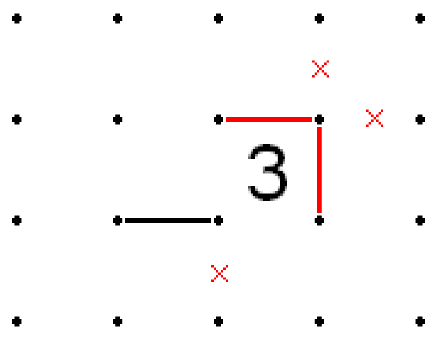
Two adjacent 3s.

- If a 3 is adjacent to a 0 diagonally, both sides of the 3 that meet the 0's corner must be filled. This is because if either of those sides were open, the line ending in the corner of the 0 would have no place to go. This is similar to the 3-in-a-corner rule.



3 diagonally next to 0.

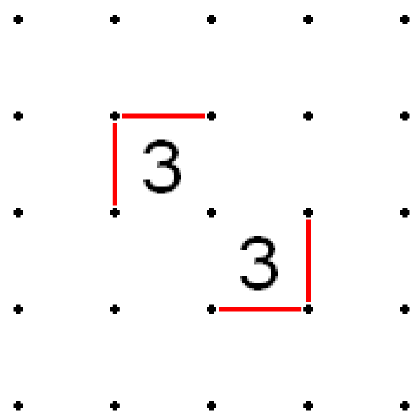
- Similarly, if a 3 has a corner with X's in both directions going away from that corner, then both sides of the 3 that meet that corner must be filled. This is because if one of those two sides of the 3 were open, the other would have to be filled (because the 3 can only have one open side) but would meet 3 Xs at that corner, which is impossible because each point on the grid must have exactly 2 or 0 lines.
- If *a line reaches a corner of a 3*, there must be lines on both sides of the 3 that said corner is not adjacent to, because if the 3's sole empty space were not adjacent to it, the corner would have three lines connected to it. Furthermore, the segment leading away from the 3 at the corner reached by the line must be empty; if it were filled, neither of the remaining 2 undetermined sides of the 3 would be able to contain a line.



A 3 next to a line.

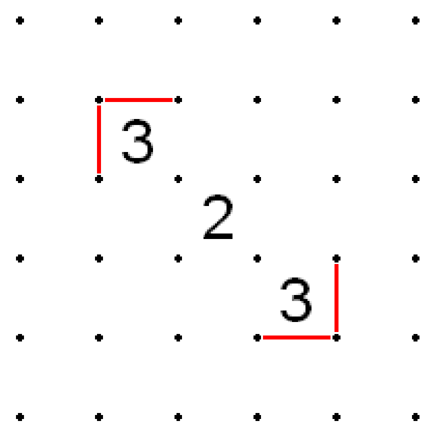
Diagonals of 3s and 2s

- If *two 3s are adjacent diagonally*, the edges which do not run into the common point must be filled in.



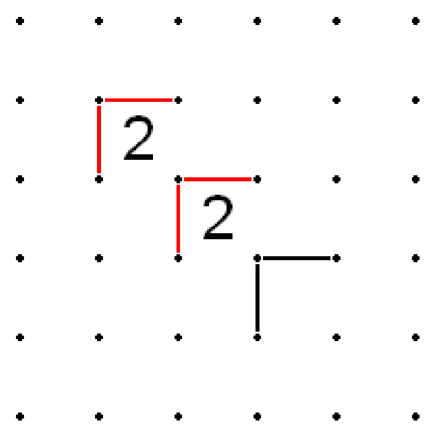
Two diagonal 3s.

- Similarly, if two 3s are in the same diagonal, but separated by any number of 2s (and only 2s) the outside edges of the 3s must be filled in, just as if they were adjacent diagonally.



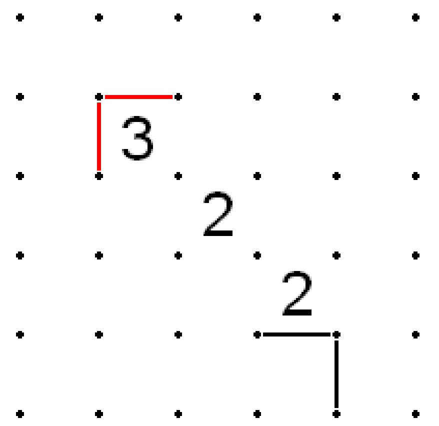
Diagonal 3s with 2 between them.

- If there is a series of 2s in a diagonal line and an angled line meets the corner of the 2 at one end of the series, a matching angled line can be drawn all the way up the series.



One or more diagonal 2s terminated by two filled lines pointing into them.

- Here there is a diagonal series of 2s ending in a 3. This example combines several of the rules illustrated above. The 2 at the end of the diagonal has an angled line which includes one (but not both) of the sides at its **outer corner**: the corner of the angle is at the 2's furthest corner from 3. This implies that both of the outer sides of the three must be filled. This is because: (i) the right-end side of the lower 2 must be empty so (ii) either the left or top line of the lower 2 must be filled so (iii) the middle 2 cannot have lines on both its right and bottom side (otherwise 3 lines would meet at its bottom right corner, which is not allowed) so (iv) it must have a line on either its top or left side so (v) the 3 must have both its top and left sides filled (see above). A similar chain of logic can be applied to any diagonal of 2s ending in a 3.



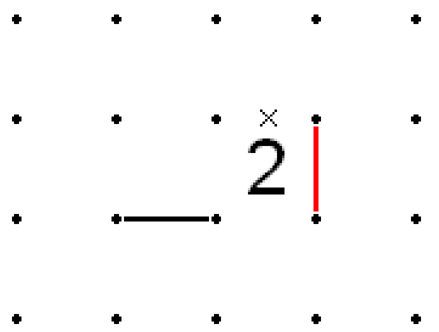
A diagonal of 2s ending in a 3.

Diagonals of a 3 and 1

- If *a 1 and a 3 are adjacent diagonally and the outer two sides of the 1 are X'd out*, then the outer two sides of the 3 must be filled in. The opposite is the same: if the outer two corners of the 3 are filled in, then the outer two corners of the 1 must be X'd out.

A rule for squares with 2

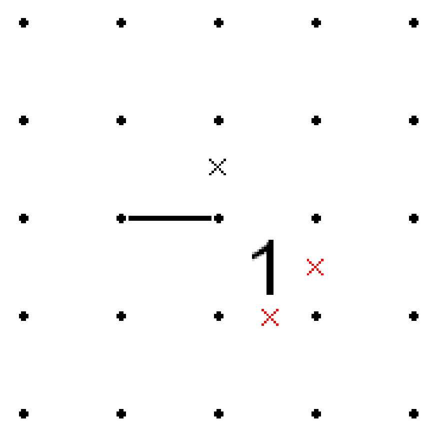
If a 2 has any surrounding line X'd, then a line coming into either of the two corners not adjacent to the X'd out line cannot immediately exit at right angles away from the 2, as then two lines around the 2 would be impossible, and can therefore be X'd. This means that the incoming line must continue on one side of the 2 or the other. This in turn means that the second line of the 2 must be on the only remaining free side, adjacent to the originally X'd line, so that can be filled in.
Conversely, if a 2 has a line on one side, and an adjacent X'd out line, then the second line must be in one of the two remaining sides, and exit from the opposite corner (in either direction). If either of those two exits is X'd out, then it must take the other route.



A 2 next to a line.

Rules for squares with 1

- If *a line comes into a corner of a 1* and if one of the three remaining directions that the line can continue, the one that is not a side of the 1 is a known blank, then the two sides of the 1 opposite that corner can be X'd out.
- This also applies in reverse. That is, if a line comes into the corner of a 1, and the two opposite edges of the 1 are already X'd out, the line cannot go away from the 1 since that would put X's around all sides of the 1.



A 1 next to a line.

- If two 1s are diagonally adjacent, then of the eight segments around those two cells, either the "inner" set of four segments sharing a common endpoint (the point shared by the 1s) or the other "outer" set of four segments must all be X'd out. Thus if any two inner or outer segments in one 1 are X'd, the respective inner or outer segments of the other 1 must also be X'd.

An even number of ends in a closed region

In a closed-off region of the lattice (from which there is no path for any lines to "escape"), there cannot exist an odd number of unconnected segment-ends, since all of the segment-ends must connect to something. Often, this will rule out one or more otherwise feasible options.

Jordan curve theorem

In an exceptionally difficult puzzle, one may use the Jordan curve theorem, which states that any open curve that starts and ends outside of a closed curve must intersect the closed curve an even number of times. In particular, this means that any row of the grid must have an even number of vertical lines and any column must have an even number of horizontal lines. When only one potential line segment in one of these groups is unknown, you can determine whether it is part of the loop or not with this theorem.

A simple strategy to assist in using this theorem is to "paint" (sometimes called "shade") the outside and the inside areas. When you see two outside cells, or two inside cells next to each other, then you know that there is not a line between them. The converse is also true: if you know there is no line between two cells, then those cells must be the same "color" (both inside or both outside). Similarly, if an outside cell and an inside cell are adjacent, you know there must be a filled line between them; and again the converse is true.

History

Slitherlink is an original puzzle of Nikoli; it first appeared in Puzzle Communication Nikoli #26 (June 1989). The editor combined two original puzzles contributed there. At first, every square contained a number.

Videogames

Slitherlink video games have been featured for the Nintendo DS handheld game console, with Hudson Soft releasing *Puzzle Series Vol. 5: Slitherlink* in Japan on November 16, 2006, and Agetec including Slitherlink in its Nikoli puzzle compilation, *Brain Buster Puzzle Pak*, released in North America on June 17, 2007.[1]

See also

- List of Nikoli puzzle types
- Category:Logic puzzles

References

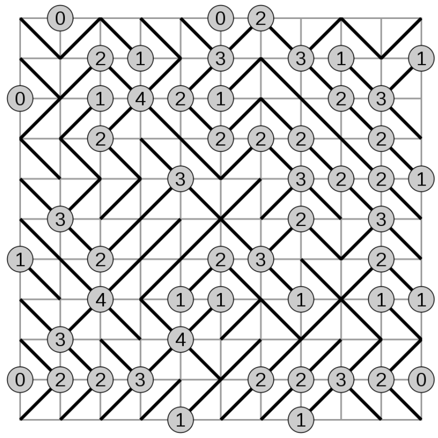
[1] Puzzle - Brain Buster Puzzle Pak - Agetec, Inc (http://www.agetec.com/catalog/product_info.php?products_id=37)

External links

- Nikoli's English page on *Slitherlink* (http://www.nikoli.co.jp/en/puzzles/slitherlink/)
- On the NP-completeness of the Slitherlink Puzzle (http://fano.ics.uci.edu/cites/Document/On-the-NP-completeness-of-the-Slither-Link-Puzzle.html) - Slitherlink is NP-complete
- Site discussing non-grid forms of Slitherlink including snowflake, penrose, laves and altair (http://www.krazydad.com/slitherlink/)

Gokigen Naname

Gokigen Naname is a binary-determination logic puzzle published by Nikoli.



A solved Gokigen Naname puzzle

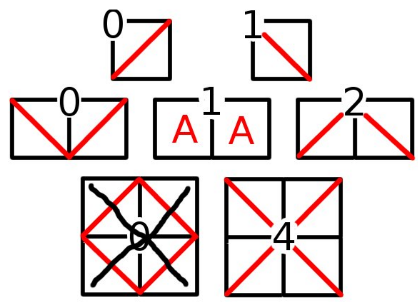
Rules

Gokigen Naname is played on a rectangular grid in which numbers in circles appear at some of the intersections on the grid.

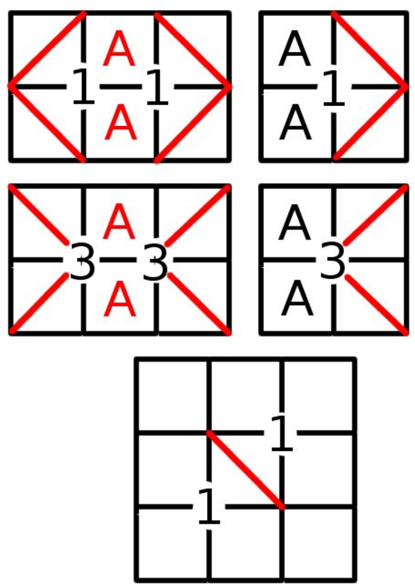
The object is to draw diagonal lines in each cell of the grid, such that the number in each circle equals the number of lines extending from that circle. Additionally, it is forbidden for the diagonal lines to form an enclosed loop. Unlike many of Nikoli's similar puzzles, such as Hashiwokakero, a single network of lines is not required.

Solution methods

Every cell must contain a diagonal line. Numbers in the corners or along the edges of the grid will have more obvious solutions. A circle with a 4 will always have the same solution, as will a circle with a zero.



Immediate deductions from single numbers.
(top) Either 0 or 1 in a corner specifies that cell.
(middle row) A number at an edge determines both adjacent cells unless it is 1, in which case it is known that the two share a common value which is not yet determined.
(bottom) A lone number away from the edge gives no information unless it is 4. (0 cannot occur away from an edge, because it would force a closed loop)



Deductions from two numbers. When immediately adjacent and not at the edge, either two 1s or two 3s require the intervening two cells to share the same value. When it is known that a 1 adjoins two cells with the same value, one must contact that position; therefore the opposite two cells must form a semicircle around the 1. When a 3 adjoins two cells with the same value, one does not contact it, so the opposite two cells must both meet that position. A cell away from the edge cannot connect two 1s at opposite corners, because that would force a closed loop.

`              /         \`
`(1)  (1)  =>   (1)  (1)`
`              \         /`
	not valid, if a (1) is on the outer rim

`                   /              \`
`(1)  (2)  (1)  =>   (1)  (2)  (1)`
`                   \              /`
	not valid, if a (1) is on the outer rim

`                             /                        \`
`(1)  (2)  ...  (2)  (1)  =>   (1)  (2)  ...  (2)  (1)`
`                             \                        /`
	not valid, if a (1) is on the outer rim

`     (1)          (1)`
`          =>      \`
`(1)           (1)`	

`              /    \    \`
`(1)  (1)       (1)  (1)`
`          =>  \    \    /`
`(1)            (1)`
`              \    /`	

`              \         /`
`(3)  (3)  =>   (3)  (3)`
`              /         \`	

<pre> \ / (4) => (4) / \</pre>	

See also

- List of Nikoli puzzle types

External links

- Play Gokigen Naname online [1]
- A free Gokigen Naname game to download [2]

References

[1] http://www.janko.at/Raetsel/Gokigen/index.htm
[2] http://www.mulm.at/downloads.html

Tiling puzzles

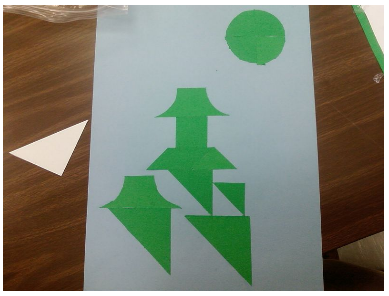
Tiling puzzle

Tiling puzzles are puzzles involving two-dimensional packing problems in which a number of flat shapes have to be assembled into a larger given shape without overlaps (and often without gaps). Some tiling puzzles ask you to dissect a given shape first and then rearrange the pieces into another shape. Other tiling puzzles ask you to dissect a given shape while fulfilling certain conditions. The two latter types of tiling puzzles are also called dissection puzzles.

Tiling puzzles may be made from wood, metal, cardboard, plastic or any other sheet-material. Many tiling puzzles are now available as computer games.

Tiling puzzles have a long history. Some of the oldest and most famous are jigsaw puzzles and the Tangram puzzle.

Other examples of tiling puzzles include:



Example of a tiling puzzle that utilizes negative space.

- Conway puzzle
- Domino tiling, of which the mutilated chessboard problem is one example
- Eternity puzzle
- Puzz-3D
- Squaring the square
- Tantrix
- T puzzle
- Pentominoes

Many three-dimensional mechanical puzzles can be regarded as three-dimensional tiling puzzles.

See also

- Dissection puzzle
- Polyforms
- Sliding puzzle
- Tessellation
- Wang tile

Dissection puzzle

A **dissection puzzle**, also called a **transformation puzzle** or *Richter Puzzle*[1] , is a tiling puzzle where a solver is given a set of pieces that can be assembled in different ways to produce two or more distinct geometric shapes. The creation of new dissection puzzles is also considered to be a type of dissection puzzle. Puzzles may include various restraints, such as hinged pieces, pieces that can fold, or pieces that can twist. Creators of new dissection puzzles emphasize using a minimum number of pieces, or creating novel situations, such as ensuring that every piece connects to another with a hinge.

History

Dissection puzzles are an early form of geometric puzzle. The earliest known descriptions of dissection puzzles are from the time of Plato (427–347 BCE) in Ancient Greece, and involve the challenge of turning two equal squares into one larger square using four pieces. Other ancient dissection puzzles were used as graphic depictions of the Pythagorean theorem. A famous ancient Greek dissection puzzle is the Ostomachion, a mathematical treatise attributed to Archimedes.

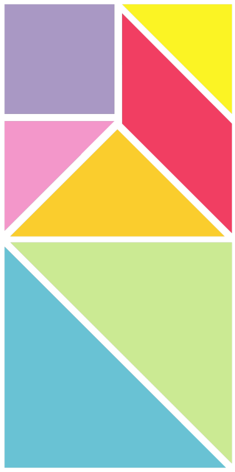
Ostomachion is a dissection puzzle attributed to Archimedes

In the 10th century, Arabic mathematicians used geometric dissections in their commentaries on Euclid's *Elements*. In the 18th century, Chinese scholar Tai Chen described an elegant dissection for approximating the value of π.

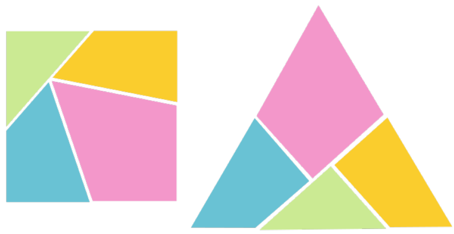
The puzzles saw a major increase in general popularity in the late 19th century when newspapers and magazines began running dissection puzzles. Puzzle creators Sam Loyd in the United States and Henry Dudeney in the United Kingdom were among the most published. Since then, dissection puzzles have been used for entertainment and maths education, and creation of complex dissection puzzles is considered an entertaining use of geometric principles by mathematicians and maths students.

Types of dissection puzzle

Some types of dissection puzzle are intended to create a large number of different geometric shapes. Tangram is a popular dissection puzzle of this type. The seven pieces can be configured into one of a few home shapes, such as the large square and rectangle that the pieces are often stored in, to any number of smaller squares, triangles, parallelograms, or esoteric shapes and figures. Some geometric forms are easy to create, while others present an extreme challenge. This variability has ensured the puzzle's popularity.



A Tangram puzzle, with its pieces in the rectangular "storage" configuration.



The haberdasher's problem, created by Henry Dudeney.

Other dissections are intended to move between a pair of geometric shapes, such as a triangle to a square, or a square to a five-pointed star. A dissection puzzle of this description is the **haberdasher's problem**, proposed by in 1907 by Henry Dudeney. The puzzle is a dissection of a triangle to a square, with only three cuts. It is one of the simplest regular polygon to square dissections known, and is now a classic example.

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[1] Forbrush, William Byron (1914). *Manual of Play* (http://books.google.com/books?id=FpoWAAAAIAAJ&pg=PA315&dq="The+Anchor+Puzzle"&hl=en&ei=7CO2TKLqLMPflgfU1pnuBQ&sa=X&oi=book_result&ct=result&resnum=3&ved=0CDIQ6AEwAjgK#v=onepage&q="The Anchor Puzzle"&f=false). Jacobs. p. 315. . Retrieved 2010-10-13.

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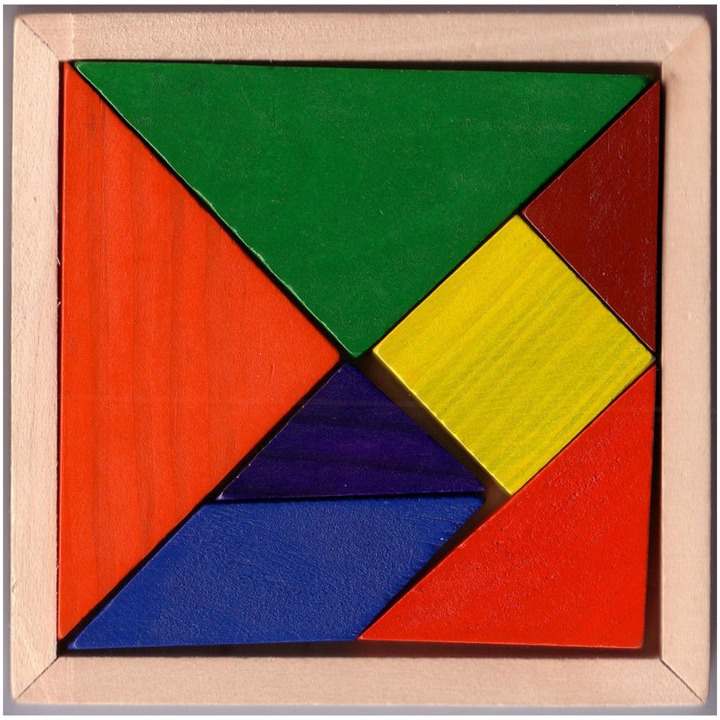
See also

- Ostomachion
- Pizza theorem
- Puzzle
- Tiling puzzles

Tangram

The **tangram** (Chinese: 七巧板; pinyin: *qī qiǎo bǎn*; literally "seven boards of skill") is a dissection puzzle consisting of seven flat shapes, called *tans*, which are put together to form shapes. The objective of the puzzle is to form a specific shape (given only an outline or silhouette) using all seven pieces, which may not overlap. It was originally invented in China at some unknown point in history, and then carried over to Europe by trading ships in the early 19th century. It became very popular in Europe for a time then, and then again during World War I. It is one of the most popular dissection puzzles in the world.[1] [2]



Like most modern sets, this wooden tangram is stored in the square configuration.

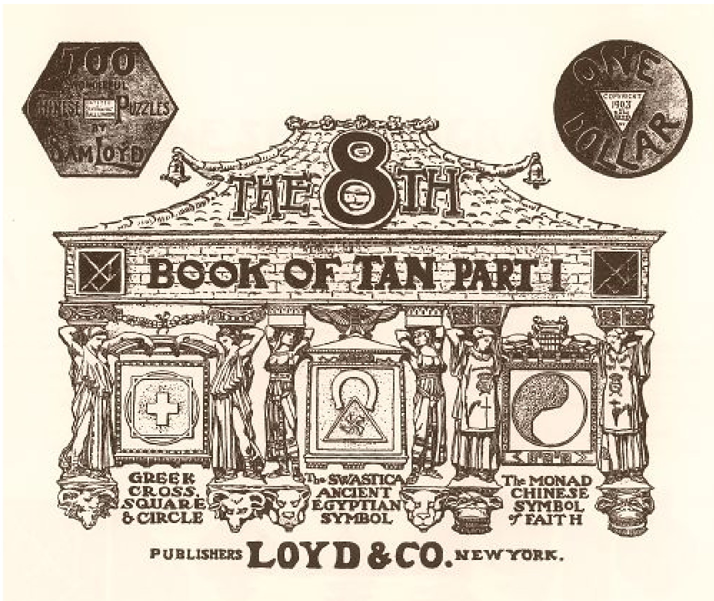
History

Reaching The Western World (1815-1820s)

A caricature published in France in 1818, when the Tangram craze was at its peak. The caption reads: " 'Take care of yourself, you're not made of steel. The fire has almost gone out and it is winter.' 'It kept me busy all night. Excuse me, I will explain it to you. You play this game, which is said to hail from China. And I tell you that what Paris needs right now is to welcome that which comes from far away.' "

The tangram had already been around in China for a long time when it was first brought to America by Captain M. Donnaldson, on his ship, *Trader*, in 1815. When it docked in Canton, the captain was given a pair of Sang-hsia-k'o's Tangram books from 1815.[3] They were then brought with the ship to Philadelphia, where it docked in February 1816. The first Tangram book to be published in America was based on the pair brought by Donnaldson.

The puzzle was originally popularized by *The Eighth Book Of Tan*, a fictitious history of Tangram, which claimed that the game was invented 4,000 years prior by a god named Tan. The book included 700 shapes, some of which are impossible to solve.[4]



Cover art from "The 8th Book of Tan", by Sam Loyd, a spoof of the puzzle's history that began the Tangram Craze in the Western World

The puzzle eventually reached England, where it became very fashionable indeed.[3] The craze quickly spread to other European countries.[3] This was mostly due to a pair of British Tangram books, *The Fashionable Chinese Puzzle*, and the accompanying solution book, *Key*.[5] Soon, tangram sets were being exported in great number from China, made of various materials, from glass, to wood, to tortoise shell.[6]

Many of these unusual and exquisite tangram sets made their way to Denmark. Danish interest in tangrams skyrocketed around 1818, when two books on the puzzle were published, to much enthusiasm.[7] The first of these was *Mandarinen* (About the Chinese Game). This was written by a student at Copenhagen University, which was a non-fictional work about the history and popularity of tangrams. The second, *Der nye chinesisre Saadespil* (The new Chinese Puzzle Game), consisted of 339 puzzles copied from The 8th Book of Tan, as well as one original.[7]

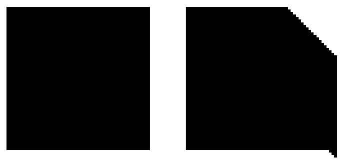
One contributing factor in the popularity of the game in Europe was that although the Catholic Church forbade many forms of recreation on the sabbath, they made no objection to puzzle games such as the tangram.[8]

The Second Craze in Germany and America (1891-1920s)

Tangrams were first introduced to the German public by industrialist Freidrich Richter around 1891.[9] The sets were made out of stone or false earthenware,[10] and marketed under the name "The Anchor Puzzle".[9]

More internationally, the First World War saw a great resurgence of interest in Tangrams, on the homefront and trenches of both sides. During this time, it occasionally went under the name of "The Sphinx", an alternate title for the "Anchor Puzzle" sets.[11] [12]

Paradoxes



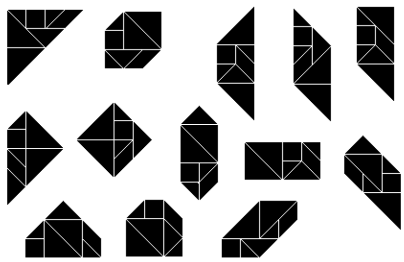
Loyd's paradox

A tangram paradox is an apparent dissection fallacy: Two figures composed with the same set of pieces, one of which seems to be a proper subset of the other.[13] One famous paradox is that of the two monks, attributed to Dudeney, which consists of two similar shapes, one with and the other missing a foot.[14] Another is proposed by Sam Loyd in *The Eighth Book Of Tan*:

> The seventh and eighth figures represent the mysterious square, built with seven pieces: then with a corner clipped off, and still the same seven pieces employed.[15]

Other similar, but possible, apparent paradoxes are in fact fallacious. For example, in the case of the two monks mentioned above, the foot is actually compensated for in the second figure by a subtly larger body.[13]

Number of configurations



The 13 convex shapes matched with Tangram set

Over 5,900 different tangram problems have been compiled from 19th century texts alone, and the current number is ever-growing.[16] The number is finite, however. Fu Traing Wang and Chuan-chin Hsiung proved in 1942 that there are only thirteen convex tangram configurations (configurations such that a line segment drawn between any two points on the configuration's edge always pass through the configuration's interior, i.e., configurations with no recesses in the outline).[17] [18]

Pieces

Choosing a unit of measurement so that the seven pieces can be assembled to form a square of side one unit and having area one square unit, the seven pieces are:

- 2 large right triangles (hypotenuse 1, sides $\sqrt{2}/2$, area $1/4$)
- 1 medium right triangle (hypotenuse $\sqrt{2}/2$, sides $1/2$, area $1/8$)
- 2 small right triangle (hypotenuse $1/2$, sides $\sqrt{2}/4$, area $1/16$)
- 1 square (sides $\sqrt{2}/4$, area $1/8$)
- 1 parallelogram (sides of $1/2$and $\sqrt{2}/4$, area $1/8$)

Of these seven pieces, the parallelogram is unique in that it has no reflection symmetry but only rotational symmetry, and so its mirror image can only be obtained by flipping it over. Thus, it is the only piece that may need to be flipped when forming certain shapes.

See also

- Tiling puzzle
- Ostomachion
- Mathematical puzzle

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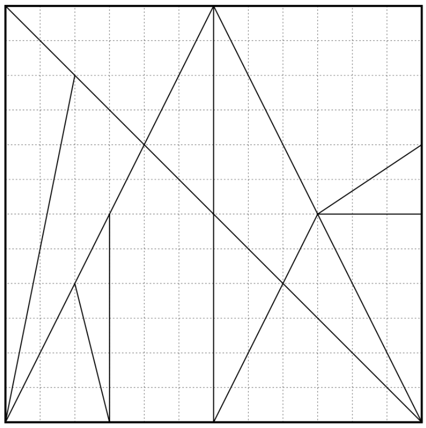
External links

- " Tangram (http://demonstrations.wolfram.com/Tangram/)" by Enrique Zeleny, Wolfram Demonstrations Project

Ostomachion



Ostomachion (after Suter; this version requires a lateral stretch by a factor of two to match that in the Archimedes Palimpsest)

Ostomachion, also known as ***loculus Archimedius*** (Archimedes' box in Latin) and also as ***syntomachion***, is a mathematical treatise attributed to Archimedes. This work has survived fragmentarily in an Arabic version and in a copy of the original ancient Greek text made in Byzantine times.[1] The word Ostomachion has as its roots in the Greek "Ὀστομάχιον",[2] which means "bone-fight", from "ὀστέον" (*osteon*), "bone"[3] + "μάχη" (*mache*), "fight, battle, combat".[4] Note that the manuscripts refer to the word as "**Stomachion**", an apparent corruption of the original Greek. Ausonius gives us the correct name "Ostomachion" (*quod Graeci ostomachion vocavere*). The Ostomachion which he describes was a puzzle similar to tangrams and was played perhaps by several persons with pieces made of bone. It is not known which is older, Archimedes' geometrical investigation of the figure, or the game.

Game

Ostomachion (after Suter): square reformed with some pieces turned over

The game is a 14-piece dissection puzzle forming a square. One form of play to which classical texts attest is the creation of different objects, animals, plants etc. by rearranging the pieces: an elephant, a tree, a barking dog, a ship, a sword, a tower etc. Another suggestion is that it exercised and developed memory skills in the young. James Gow, in his Short History of Greek Mathematics (1884), footnotes that the purpose was to put the pieces back in their box, and this was also a view expressed by W. W. Rouse Ball in some intermediate editions of Mathematical Essays and Recreations, but edited out from 1939.

Mathematical problem

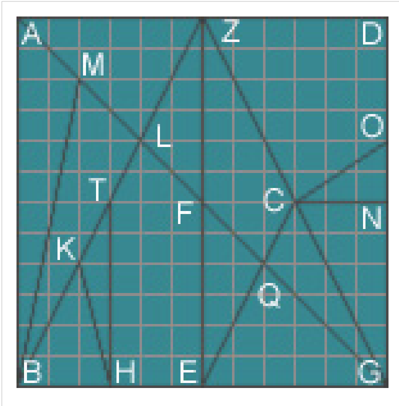
In the fragment in Arabic translated by Heinrich Suter it is shown that each of the pieces has an area that is a rational fraction of the total area of the dissected parallelogram. In the Greek version of the treatise, some investigation is made as to the sizes of the angles of the pieces to see which could go together to make a straight line. This might have been preparatory to consideration of the number of ways the pieces might reform some prescribed shape, for example the box in which the pieces were contained, although there is not enough of the Greek text remaining to be sure. If this is the case, then Archimedes anticipated aspects of combinatorics. Combinatorics often involves finding the number of ways a given problem can be solved, subject to well-defined constraints. For example, the number of ways of reforming a square using the pieces as proposed by Suter is 536 without distinguishing the result up to rotations and reflections of the square, but allowing the pieces to be turned over.[1] One such solution is shown in colour to the right. However, if pieces are not allowed to be turned over, for example, if obverse and reverse can be distinguished or, in the case of Suter's pieces, on account of the sharpness of some angles, the corresponding number is 4. If rotations and reflections are treated as distinct, these numbers rise to 17,152 and 64, respectively. The counts of 4 and 64 may be verified easily as lower bounds by elementary group theory, but were confirmed as exact by Bill Cutler shortly after his determination of the count 536 and by the same computer program. So, there are at least four different answers that we might give just considering Suter's proposal. Clearly, to count, you have to know what counts. When, as here, the number of outcomes is so sensitive to the assumptions made, it helps to state them explicitly. Put another way, combinatorics can help sharpen our awareness of tacit assumptions. If, say, answers like 4 or 64 are unacceptable for some reason, we have to re-examine our presumptions, possibly questioning whether Suter's pieces can be turned over in reforming their square. As emerges below, there is also some objection to Suter's proposal which would render this combinatorial discussion of the Suter board academic.

The Greek text of the fragment can also be found in the Bibliotheca Augustana [5] website.

This is an English translation of the text of the fragmentary Arabic manuscript (translated from Heinrich Suter's German translation in: *Archimedis opera omnia*, vol. 2, p. 420 sqq., ed. J. L. Heiberg, Leipzig 1881, as published in the Bibliotheca Augustana [6] website):

" We draw a [rectangular] parallelogram ABGD, we bisect BG in E and draw EZ perpendicular to BG, we draw the diagonals AG, BZ, and ZG, we also bisect BE in H, and draw HT perpendicular to BE, then we put the ruler at point H and - looking to point A - we draw HK, then bisect AL in M, and draw BM. So the A-E rectangle is divided into seven parts.

Now we bisect DG in N, ZG in C, we draw EC and attaching the ruler to the points B and C we draw CO, furthermore CN. Thus the rectangle ZG is also divided in seven parts, but in another way than the first one. Therefore, the whole square has fourteen parts.



We now demonstrate that each of the fourteen parts is in rational relationship to the whole square.

Because ZG is the diagonal of the rectangle Z-G, the triangle DZG is half of this rectangle, that means 1/4 of the square. But the triangle GNC is 1/4 of triangle DZG, because, if we extend the line EC, it comes to point D, and that means triangle GDC has half area of the triangle DZG and is equal to the two triangles GNC and DNC taken together; that means triangle GNC is 1/16 of the square. If we presume that line OC is orientated to point B, as we have drawn it before, so the line NC is parallel to BG, which is the side of the square and of the triangle OBG, so we get the proportion

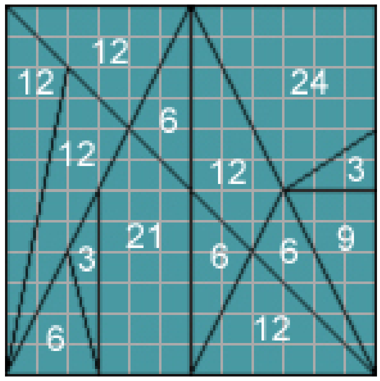
BG : NC = GO : NO.

But BG is four times NC, and in the same way GO four times NO; therefore is GN three times NO, and triangle GNC = 3 ONC. However, as we have shown, triangle GNC is 1/16 of the square, that means triangle ONC = 1/48 of the square. Furthermore, as triangle GDZ = 1/4 of the square, and therefore GNC = 1/16 of that triangle and NCO = 1/48 of that, it remains for the rectangle DOCZ = 1/6 of the square's area. According to the proposition that line NC intersects point F, and GE is parallel to CF, we get the proportion

EC : CF = EQ : CQ = GQ : FQ.

Because EQ = 2 CQ and GQ = 2 FQ, triangle EQG is double to the two triangles GCQ and EFQ. It is clear, that triangle EGZ = 2 times triangle EFG, because ZE = 2 FE. Otherwise, the triangle EGZ = 1/4 of the square, that means triangle EFG = 1/8 of the square. This triangle is three times as big as each of the two triangles EFQ and GCQ, so each of these two triangles = 1/24 of the square A-G. And the triangle EGQ is *double to each of the two triangles EFQ and GCQ, so it is = 1/12 of the square. Furthermore because ZF = EF, triangle ZFG = triangle EFG. If we now take away triangle GCQ (= triangle EFQ), it remains rectangle FQCZ (= triangle EGQ), therefore rectangle FQCZ = 1/12 of the square A-G.*

We have now divided the rectangle Z-G in 7 parts, and go on to divide the other rectangle.



If an Ostomachion were to be imposed onto a 12-unit square, this diagram shows the area of each piece.

Because BZ and EC are two parallel diagonals, and ZF = EF, therefore triangle ZLF = EFQ, and also triangle ZLF = 1/24 of the square A-G. Because BH = HE, triangle BEZ is four times the triangle BHT, because each of them is rectangular. As triangle BEZ = 1/4 of the square ABGD, triangle BHT = 1/16 of that. According to our proposition the line HK intersects point A, so we get the proportion

AB: HT = BK: KT.

Because AB = 2 HT, including BK = 2 KT and BT = 3 KT, triangle BHT is three times the triangle KHT. However, because triangle BHT = 1/16 of the whole square, triangle KHT = 1/48 of that. Triangle BKH is double the triangle KHT, so = 1/24 of the square. As further BL = 2 ZL, and AL = 2 LF, triangle ABL is twice the triangle ALZ, and ALZ double the triangle ZLF. However, because triangle ZLF = 1/24 of the whole square, triangle ALZ = 1/12 of that, so triangle ABL = 1/6. But triangle ABM = triangle BML, so each of these two triangles = 1/12 of the square. It's left the pentagon LFEHT = 7/48 of the entire square.

We have now also divided the square AE into 7 sections, therefore, the whole figure ABGD in 14 parts. Each of these fourteen parts is in rational relationship to the whole, and that is what we wanted."

However, Suter was translating unpointed Arabic, in which, as he concedes, equals and twice are easily confused. At a crucial point in his translation he ignores this in making his figure a square; instead he makes a typographical error, equating, not the sides, but a side and a diagonal, in which case the figure cannot be a rectangle. Suter does know that, for the areal results discussed in the text, it suffices that the figure be a parallelogram. Suter may have been under the impression that the figure had to be a square, so made it one.

The dissection lines given by Archimedes' Stomachion from the Archimedes Codex, as agreed by Heiberg and Dijksterhuis, are a subset of those of Suter's square board when the latter is subjected to a lateral stretch by a factor of two. This was recognised by Richard Dixon Oldham, FRS, in a letter to Nature in March, 1926. The letter triggered a wave of interest in the dissection puzzle, with kits for sale and a feature article in The New York Times that August; Popular Science Monthly had related items in its issues for November, 1926 and February, March, May and June, 1927, including Stomachion competitions with cash prizes. As each of the two unit squares is cut diagonally, the pieces can be arranged in a single square of side the square root of two, after the manner deployed by Socrates in Plato's Meno. Thus, the 14 pieces still form a square, even when the dissected board is not a square. Naturally, the count of solutions also changes, but here we now have three possible boxes: the two unit squares side by side; the two unit squares one on top of the other; and this single square of side the square root of two. The single square can be formed from four congruent right isosceles triangles, but this can also be accomplished with three right isosceles triangles, two congruent and the third double their area, as in Tangram, as well as in a third way where only one right isosceles triangle is formed. (Bill Cutler has made a comprehensive study of the associated counts comparable to those mentioned for the Suter board.) With two exceptions, the pieces can be formed from right triangles with legs in the ratios 1:1, 1:2 and 1:3, with some pieces similar to others, possibly at different scales; this observation provides a comparatively easy means to determine the proportional areas of the pieces.

Suter's translation can now be checked against the fragment of Archimedes' Stomachion in the Archimedes Codex, as was presumably not open to Suter himself when he made the translation. Heiberg, followed by Dijksterhius, seem to have thought that the two texts, the Greek discussing angles, the Arabic areas, were so different there was no relation between the two, so may have felt there was nothing to check. But, in Suter's square, the diagonals cross at right angles, whence the nature the angles examined in the first proposition of Archimedes' Stomachion, whether acute or obtuse, is immediate, making it puzzling why Archimedes would prove this as a proposition. Instead, the first proposition, if it is seen to relate to the underlying figure of the dissected Stomachion board, sets up two squares side by side, in which setting the nature of the angles is more sensitive, but still tractable. The two squares separately suggest two frames of an iterative process. If this process is continued, rational approximations of the square root of two are obtained; these are the side-diameter numbers familiar to the Greeks. At each step of the process, the nature of the angles switches back and forth, highlighting the significance of the first proposition. Whether or not this has any bearing on Archimedes' Stomachion is another matter, but it does provide a geometrical means to handle recurrence relations if you do not have algebraic notation, such as subscripts, and the geometry is fully within Archimedes' capabilities. The same technique readily produces the bounds for the square root of three that Archimedes assumes without comment in Measurement of a Circle; if it were applied to the dissection of a square proposed by Suter, analogous bounds for the square root of five are obtained. The geometry can be worked independently of knowledge of Pell's equation or of the properties of convergents of a continued fraction, but to similar effect.

The dissection of the two squares side by side can also be seen as a layered composite or collage of instances of the diagrams for Elements II.9, 10. Viewed numerically, these propositions provide the link between the legs of right triangles in the unit square grid and those of right triangles in the overlaid diagonal square grid where the right triangles share their hypotenuse, although Euclid proves them in terms of geometrical lines. Thus, pairs of successive side numbers in the overlaid grid are associated with pairs of successive diameter numbers in the unit square grid, as remarked by Proclus, rather than side and diameter numbers being paired. The ratios within these pairs are then rational approximations for the tangent of $\pi/8$, in agreement with the iterative process of the previous paragraph, which is, in effect, an angle bisection algorithm. This reminds us that a natural setting for the side-diameter numbers is a nest of regular octagons. The analogue for the square root of three is not an angle bisection algorithm, but it does produce rational approximations to the tangent of $\pi/12$, with the regular dodecagon as a setting. Instead of simply approximating the square roots in each case, we can approximate the corresponding regular polygons. But, again, whether this was actually Archimedes' agenda is another matter.

Irrespective of the parallelogram adopted for the Stomachion board, the presence of several centroids of triangles on the board as further points of it might seem another Archimedean theme, not least as they facilitate the computation of the relative areas of the pieces. Most obviously, Q is the centroid of the right triangle GZE, so the pieces in that triangle are either one sixth or one third of it. But similarly K is the centroid of right triangle EAB, so triangle HKB has area one sixth of it. In Equilibrium of Planes, Archimedes makes use of the concurrence of the three medians of a triangle in the centroid, but does not prove it. As it happens, for both K and Q, we see two medians intersecting in them, but the pairings are not the same, although the triangles GZE and EAB are translations of one another.

Notes

[1] Darling, David (2004). *The universal book of mathematics: from Abracadabra to Zeno's paradoxes*. John Wiley and Sons, p. 188. ISBN 0471270474

[2] ὀστομάχιον (http://www.perseus.tufts.edu/hopper/text?doc=Perseus:text:1999.04.0057:entry=o)stoma/xion), Henry George Liddell, Robert Scott, *A Greek-English Lexicon*, on Perseus Digital Library

[3] ὀστέον (http://www.perseus.tufts.edu/hopper/text?doc=Perseus:text:1999.04.0057:entry=o)ste/on), Henry George Liddell, Robert Scott, *A Greek-English Lexicon*, on Perseus Digital Library

[4] μάχη (http://www.perseus.tufts.edu/hopper/text?doc=Perseus:text:1999.04.0057:entry=ma/xh), Henry George Liddell, Robert Scott, *A Greek-English Lexicon*, on Perseus Digital Library

[5] http://www.fh-augsburg.de/~harsch/graeca/Chronologia/S_ante03/Archimedes/arc_ost0.html

[6] http://www.fh-augsburg.de/~harsch/graeca/Chronologia/S_ante03/Archimedes/arc_ost2.html

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- J. Väterlein, Roma ludens (Heuremata - Studien zu Literatur, Sprachen und Kultur der Antike, Bd. 5), Amsterdam: Verlag B. R. Grüner bv 1976

External links

- Heinrich Suter, Loculus (http://quod.lib.umich.edu/u/umhistmath/)
- James Gow, Short History (http://quod.lib.umich.edu/u/umhistmath/)
- W. W. R. Ball, Recreations and Essays (http://www.archive.org/details/mathematicalrecr00ballrich/)
- Ostomachion, a Graeco-Roman puzzle (http://www.archimedes-lab.org/latin.html#archimede)
- Professor Chris Rorres (http://www.mcs.drexel.edu/~crorres/Archimedes/Stomachion/intro.html)
- Kolata, Gina. "In Archimedes' Puzzle, a New Eureka Moment." *The New York Times*. December 14, 2003 (http://query.nytimes.com/gst/fullpage.html?res=9D00E6DD133CF937A25751C1A9659C8B63)
- A tour of Archimedes' Stomachion (http://math.ucsd.edu/~fan/stomach/), by Fan Chung and Ronald Graham.

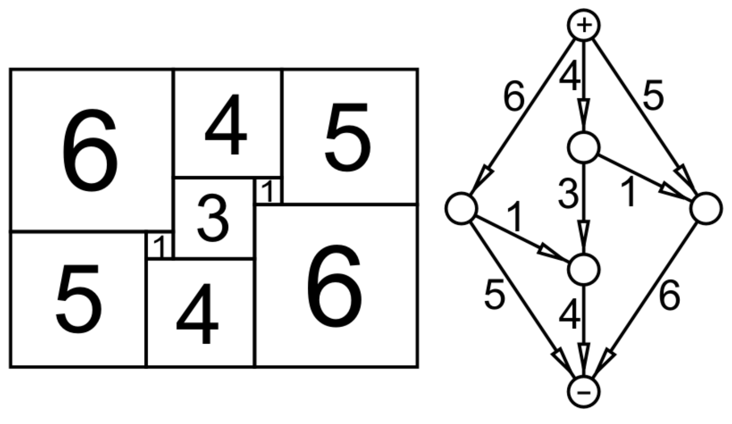
Squaring the square

Squaring the square is the problem of tiling an integral square using only other integral squares. (An **integral square** is a square whose sides have integer length.) The name was coined in a humorous analogy with squaring the circle. Squaring the square is an easy task unless additional conditions are set. The most studied restriction is that the squaring be **perfect**, meaning that the sizes of the smaller squares are all different. A related problem is **squaring the plane**, which can be done even with the restriction that each natural number occurs exactly once as a size of a square in the tiling.

Perfect squared squares

A "perfect" squared square is a square such that each of the smaller squares has a different size.

It is first recorded as being studied by R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte, at Cambridge University. They transformed the square tiling into an equivalent electrical circuit — they called it "Smith diagram" — by considering the squares as resistors that connected to their neighbors at their top and bottom edges, and then applied Kirchhoff's circuit laws and circuit decomposition techniques to that circuit.



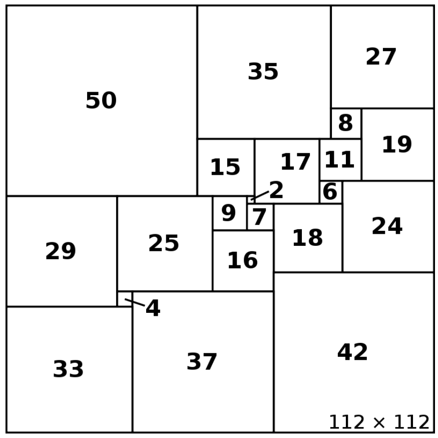
Smith diagram of a rectangle

The first perfect squared square was found by Roland Sprague in 1939.

If we take such a tiling and enlarge it so that the formerly smallest tile now has the size of the square S we started out from, then we see that we obtain from this a tiling of the plane with integral squares, each having a different size.

Martin Gardner has published an extensive [1] article written by W. T. Tutte about the early history of squaring the square.

Simple squared squares



Lowest-order perfect squared square

A "simple" squared square is one where no subset of the squares forms a rectangle or square, otherwise it is "compound". The smallest simple perfect squared square was discovered by A. J. W. Duijvestijn using a computer search. His tiling uses 21 squares, and has been proved to be minimal. The smallest perfect compound squared square was discovered by T.H. Willcocks in 1946 and has 24 squares; however, it was not until 1982 that Duijvestijn, Pasquale Joseph Federico and P. Leeuw mathematically proved it to be the lowest-order example.[2]

The smallest simple squared square forms the logo of the Trinity Mathematical Society.

Mrs. Perkins's quilt

When the constraint of all the squares being different sizes is relaxed, a squared square such that the side lengths of the smaller squares do not have a common divisor larger than 1 is called a "Mrs. Perkins's quilt". In other words, the greatest common divisor of all the smaller side lengths should be 1.

The **Mrs. Perkins's quilt problem** is to find a Mrs. Perkins's quilt with the fewest pieces for a given $n \times n$ square.

Squaring the plane

In 1975, Solomon Golomb raised the question whether the whole plane can be tiled by squares whose sizes are all natural numbers without repetitions, which he called the **heterogeneous tiling conjecture**. This problem was later publicized by Martin Gardner in his Scientific American column and appeared in several books, but it defied solution for over 30 years. In *Tilings and Patterns*, published in 1987, Branko Grünbaum and G. C. Shephard stated that in all perfect integral tilings of the plane known at that time, the sizes of the squares grew exponentially.

Recently, James Henle and Frederick Henle proved that this, in fact, can be done. Their proof is constructive and proceeds by "puffing up" an ell-shaped region formed by two side-by-side and horizontally flush squares of different sizes to a perfect tiling of a larger rectangular region, then adjoining the square of the smallest size not yet used to get another, larger ell-shaped region. The squares added during the puffing up procedure have sizes that have not yet appeared in the construction and the procedure is set up so that the resulting rectangular regions are expanding in all four directions, which leads to a tiling of the whole plane.

Cubing the cube

Cubing the cube is the analogue in three dimensions of squaring the square: that is, given a cube C, the problem of dividing it into finitely many smaller cubes, no two congruent.

Unlike the case of squaring the square, a hard but solvable problem, cubing the cube is impossible. This can be shown by a relatively simple argument. Consider a hypothetical cubed cube. The bottom face of this cube is a squared square; lift off the rest of the cube, so you have a square region of the plane covered with a collection of cubes

Consider the smallest cube in this collection, with side *c* (call it *S*). Since the smallest square of a squared square cannot be on its edge, its neighbours will all tower over it, meaning that there isn't space to put a cube of side larger than *c* on top of it. Since the construction is a cubed cube, you're not allowed to use a cube of side equal to *c*; so only smaller cubes may stand upon *S*. This means that the top face of S must be a squared square, and the argument continues by infinite descent. Thus it is not possible to dissect a cube into finitely many smaller cubes of different sizes.

Similarly, it is impossible to hypercube a hypercube, because each cell of the hypercube would need to be a cubed cube, and so on into the higher dimensions.

Notes

[1] http://www.squaring.net/history_theory/brooks_smith_stone_tutte_II.html

[2] "Compound Perfect Squares", By A. J. W. Duijvestijn, P. J. Federico, and P. Leeuw, Published in American Mathematical Monthly (http://www.maa.org/pubs/monthly.html) Volume 89 (1982) pp 15-32

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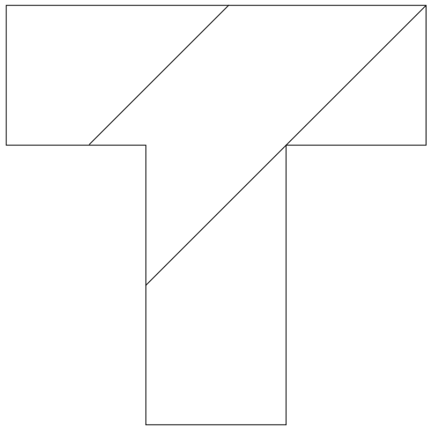
- Martin Gardner, "Squaring the square," in *The 2nd Scientific American Book of Mathematical Puzzles and Diversions.*
- Frederick V. Henle and James M. Henle, "Squaring the plane" (http://maven.smith.edu/~jhenle/stp/stp.pdf), *American Mathematical Monthly*, 115, January 2008, 3–12

External links

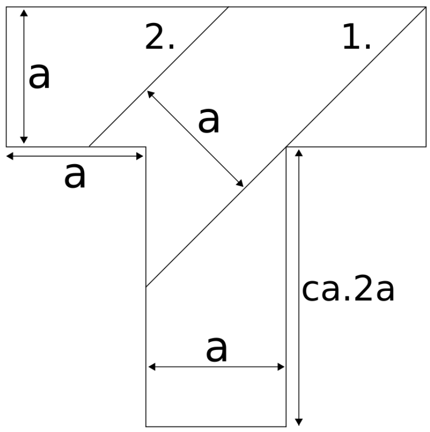
- Perfect squared squares:
  - http://www.squaring.net/
  - http://www.maa.org/editorial/mathgames/mathgames_12_01_03.html
  - http://www.math.uwaterloo.ca/navigation/ideas/articles/honsberger2/index.shtml
  - http://www.math.niu.edu/~rusin/known-math/98/square_dissect
  - http://www.stat.ualberta.ca/people/schmu/preprints/sq.pdf
- Nowhere-neat squared squares:
  - http://karl.kiwi.gen.nz/prosqtre.html
- Mrs. Perkins's quilt:
  - Mrs. Perkins's Quilt (http://mathworld.wolfram.com/MrsPerkinssQuilt.html) on MathWorld

T puzzle

The **T puzzle** is a puzzle the object of which is to assemble four shapes into the form of a letter T.



Solution of the T puzzle



Construction

External links

- T-Puzzle [1] at MathWorld
- T-Puzzle solutions [2]

References

[1] http://mathworld.wolfram.com/T-Puzzle.html
[2] http://tpuzzle.info/index.html

Eternity puzzle

The **eternity puzzle** is a geometric puzzle with a million-pound prize, created by Christopher Monckton, who put up half the money himself, the other half being put up by underwriters in the London insurance market. The puzzle was distributed by the Ertl Company.

The puzzle consists of filling a large almost regular dodecagon with 209 irregularly shaped smaller polygons. All the polygons had the same color and had between seven and eleven sides.[1] It was launched in June 1999, by Ertl Toys, marketed to amateur puzzle solvers and 500,000 copies were sold worldwide, with the game becoming a craze at one point. Eternity was the best-selling puzzle or game in the UK at its price-point of £35 in its launch month. It was voted Puzzle of the Year in Australia.

Before marketing the puzzle, Monckton had thought that it would take at least three years before anyone could crack the puzzle.[1] One estimate made at the time stated that the puzzle had 10^{500} possible attempts at a solution, and it would take longer than the lifetime of the Universe to calculate all of them even if you had a million computers.[2] According to Eternity's rules, possible solutions to the puzzle would be received by mail on September 21, 2000. If no correct solutions were opened, the mail for the next year would be kept until September 30, 2001, the process being repeated every year until 2003, after which no entries would be accepted.

Solution

The puzzle was solved on May 15, 2000, before the first deadline by two Cambridge mathematicians, Alex Selby and Oliver Riordan, who had used an ingenious technique to vastly accelerate their solution.[3] They realised that it was trivial to fill the board almost completely, to an "end-game position" where an irregularly-shaped void had to be filled with only a few pieces, at which point the pieces left would be the "wrong shapes" to fill the remaining space. The hope of solving the end-game depended vitally on having pieces that were easy to tile together in a variety of shapes.

They started a computer search to find which pieces tiled well or badly, and then used these data to alter their otherwise-standard backtracking search program to use the bad pieces first, in the hope of being left with only good pieces in the hard final part of the search. This heuristic approach paid off rapidly, with a complete solution being obtained within seven months of brute-force search on two domestic PCs.

The puzzle's inventor jokingly claimed in 2000 that the earlier-than-expected discovery had forced him to sell his 67-room house to pay the prize.[1] [4] Unprompted, in 2006, he disclosed that the claim had been a PR stunt to boost sales over Christmas, that the house's sale was unrelated, and that he was going to sell it anyway.[4]

Future puzzles

Eternity II was launched in Summer 2007 with a prize of $2 million.[4]

References

[1] "£1m Eternity jackpot scooped" (http://news.bbc.co.uk/2/hi/entertainment/992393.stm), *BBC News*, 2000-10-26,

[2] Duncan Richer (July 1999), *The Eternity Puzzle* (http://nrich.maths.org/public/viewer.php?obj_id=1354&part=index&refpage=articles.php), NRICH,

[3] "Description of (Eternity solver) method" (http://www.archduke.org/eternity/method/desc.html). *Alex Selby (and Oliver Riordan)*. 2007-06-16. . Retrieved 2007-06-16.

[4] "Aristocrat admits tale of lost home was stunt to boost puzzle sales" (http://thescotsman.scotsman.com/scotland/Aristocrat-admits-tale-of-lost.3340554.jp). *The Scotsman*. 2007-01-24. . Retrieved 2007-01-24.

External links

- A detailed article on how the puzzle was solved (http://plus.maths.org/issue13/features/eternity/index.html)
- Discussion of the Eternity puzzle and related problems (http://mathpuzzle.com/eternity.html)
- Alex Selby's page on the Eternity puzzle (http://www.archduke.org/eternity/)
- Wolfram MathWorld article (http://mathworld.wolfram.com/Eternity.html)

Edge-matching puzzle

An **edge-matching puzzle** is a type of tiling puzzle involving tiling an area with (typically regular) polygons whose edges are distinguished with colours or patterns, in such a way that the edges of adjacent tiles match.

Edge-matching puzzles are known to be NP-complete, and capable of conversion to and from equivalent jigsaw puzzles and polyomino packing puzzle.[1]

The first edge-matching puzzles were patented in the U.S. by E. L. Thurston in 1892.[2]

Current examples of commercial edge-matching puzzles include the Eternity II puzzle, TetraVex, Dodek Duo and Kadon Enterprises' range of edge-matching puzzles.

See also

- Domino tiling
- Three-Dimensional Edge-Matching Puzzle

References

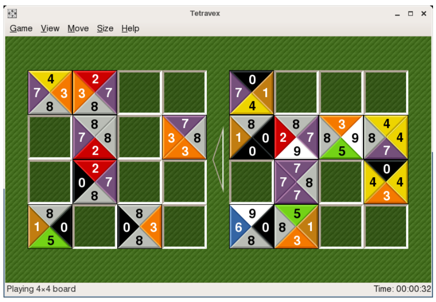
[1] Erik D. Demaine, Martin L. Demaine. "Jigsaw Puzzles, Edge Matching, and Polyomino Packing: Connections and Complexity" (http://theory.lcs.mit.edu/~edemaine/papers/Jigsaw_GC/paper.pdf). . Retrieved 2007-08-12.
[2] "Rob's puzzle page: Edge Matching" (http://home.comcast.net/~stegmann/pattern.htm#edgematch). . Retrieved 2007-08-12.

External links

- Erich's Matching Puzzles Collection (http://www.stetson.edu/~efriedma/rubik/match/index.html)
- Color- and Edge-Matching Polygons (http://mitglied.lycos.de/polyforms/coloredpolygons/index.html) by Peter Esser
- Rob's puzzle page (http://home.comcast.net/~stegmann/pattern.htm) by Rob Stegmann

TetraVex

TetraVex is a puzzle computer game, available for Windows and Linux systems.



Tetravex running on Ubuntu

Gameplay

TetraVex is an edge-matching puzzle. The player is presented with a grid (by default, 3x3) and nine square tiles, each with a number on each edge. The objective of the game is to place the tiles in the grid in the proper position as fast as possible. Two tiles can only be placed next to each other if the numbers on adjacent faces match.

Availability

TetraVex was originally available for Windows in Windows Entertainment Pack 3. It was later re-released as part of the Best of Windows Entertainment Pack. It is also available as an open source game on the GNOME desktop.

Origins

The original version of TetraVex (for the Windows Entertainment Pack 3) was written (and named) by Scott Ferguson who was also the Development Lead and an architect of the first version of Visual Basic[1] . TetraVex was inspired by "the problem of tiling the plane" as described by Donald Knuth on page 382 of *Volume 1: Fundamental Algorithms*, the first book in his The Art of Computer Programming series.

In the TetraVex version for Windows, the Microsoft *Blibbet* logo is displayed if the player solves a 6 by 6 puzzle.

The tiles are also known as McMahon Squares, named for Percy McMahon who explored their possibilities in the 1920s.[2]

Counting the possible number of TetraVex

Since the game is simple in its definition, it is easy to count how many possible TetraVex are for each grid of size $n \times n$. For instance, if $n = 1$, there are 10^4 possible squares with a number from zero to nine on each edge. Therefore for $n = 1$ there are 10^4 possible TetraVex puzzles.

Proposition: There are $10^{2n(n+1)}$ possible TetraVex in a grid of size $n \times n$.

Proof sketch: For $n = 1$ this is true. We can proceed with mathematical induction.

Take a grid of $(n+1) \times (n+1)$. The first n rows and the first n columns form a grid of $n \times n$ and by the hypothesis of induction, there are $10^{2n(n+1)}$ possible TetraVex on that subgrid. Now, for each possible TetraVex on the subgrid, we can choose 10^3 squares to be placed in the position $(1, n+1)$ of the grid (first row, last column), because only one side is determined. Once this square is chosen, there are 10^2 squares available to be placed in the position $(2, n+1)$, and fixing that square we have 10^2 possibilities for the square on position $(3, n+1)$. We can go on until the square on position $(n, n+1)$ is fixed.
The same is true for the last row: there are 10^3 possibilities for the square on the first column and last row, and 10^2 for all the other.

This gives us . Then the proposition is proven by induction. □

It is easy to see that if the edges of the squares are allowed to take m possible numbers, then there are $m^{2n(n+1)}$ possible TetraVex puzzles.

See also

- Eternity II

References

[1] "The Birth of Visual Basic" (http://www.forestmoon.com/BIRTHofVB/BIRTHofVB.html). Forestmoon.com. . Retrieved 2010-05-11.
[2] "MacMahon squares" (http://www.daviddarling.info/encyclopedia/M/MacMahon_squares.html). Daviddarling.info. 2007-02-01. . Retrieved 2010-07-30.

Eternity II puzzle

The **Eternity II puzzle**, aka E2 or E II, is a puzzle competition which was released on 28 July 2007.[1] It was invented by Christopher Monckton, and is marketed and copyrighted by TOMY UK Ltd. A $2 million prize is being offered to the first complete solution.

Puzzle mechanics

The Eternity II puzzle is an edge-matching puzzle which involves placing 256 square puzzle pieces into a 16 by 16 grid, constrained by the requirement to match adjacent edges. It has been designed to be difficult to solve by brute-force computer search.

Each puzzle piece has its edges on one side marked with different shape/colour combinations (collectively called "colours" here), each of which must match precisely with its neighbouring side on each adjacent piece when the puzzle is complete. The other side of each piece is blank apart from an identifying number, and is not used in the puzzle. Thus, each piece can be used in only 4 orientations. There are 22 colours, not including the gray edges. Five of those can only be found on border and corner pieces and 17 only on so called inner pieces and the side of the border piece across from the gray colour. This puzzle differs from the first Eternity puzzle in that there is a starter piece which must be placed near the center of the board. (See PDF rulebook on official website.[2])

Two Clue Puzzles were available with the launch of the product, which, if solved, each give a piece position on the main 256-piece puzzle. Clue Puzzle 1 is 6 by 6, with 36 pieces and Clue Puzzle 2 is 12 by 6, with 72 pieces. Two further puzzles were made available in 2008. Clue Puzzle 3 is 6 by 6, with 36 pieces, and Clue Puzzle 4 is 12 by 6, with 72 pieces.

The number of possible configurations for the Eternity II puzzle, assuming all the pieces are distinct, and ignoring the fixed pieces with pre-determined positions, is $256! \times 4^{256}$, roughly 1.15×10^{661}. A tighter upper bound to the possible number of configurations can be achieved by taking into account the fixed piece in the center and the restrictions set on the pieces on the edge: $1 \times 4! \times 56! \times 195! \times 4^{195}$, roughly 1.115×10^{557}.

Solution submissions

After the first scrutiny date on 31 December 2008 it was announced that no complete solution had been found. A prize of $10,000 was awarded to Anna Karlsson from Lund in Sweden for a partial solution with 467 matching edges out of 480.[3] The full prize money is still available to whoever matches all the edges first.

The second scrutiny date was noon GMT on 31 December 2009. A communication from Tomy Webcare stated:

> "I can now confirm that although we received many excellent entries we have not received any complete entries therefore, Eternity II still remains unsolved and the clock is now ticking to claim the $2m prize. All solutions received this year will be locked away in a vault until the final scrutiny date, 31st December 2010. On that day, all solutions will be opened in date order received and the first person with a complete solution wins $2million."

The official website, however, as of October, 2010, does not indicate that the contest will continue, stating that the "Scrutiny date" is "31.12.2009". In addition, some vendor links to buy the puzzle no longer work (Tomy themselves still sell the product). The most recent "Latest News" links on the website are dated January, 2010.[4]

History and puzzle construction

The original Eternity puzzle was a tiling puzzle with a million-pound prize, created by Christopher Monckton. Launched in June 1999, it was solved by an ingenious computer search algorithm designed by Alex Selby and Oliver Riordan, which exploited combinatorial weaknesses of the original puzzle design.[5] The prize money was paid out in full to Selby and Riordan.

The Eternity II puzzle was designed by Monckton in 2005, this time in collaboration with Selby and Riordan, who designed a computer program that generated the final Eternity II design.[6] According to the mathematical game enthusiast Brendan Owen, the Eternity II puzzle appears to have been designed to avoid the combinatorial flaws of the previous puzzle, with design parameters which appear to have been chosen to make the puzzle as difficult as possible to solve. In particular, unlike the original Eternity puzzle, there are likely only to be a very small number of possible solutions to the problem.[7] Owen estimates that a brute-force backtracking search might take around 2×10^{47} steps to complete. [8]

Monckton was quoted by *The Times* in 2005 as saying:

> "Our calculations are that if you used the world's most powerful computer and let it run from now until the projected end of the universe, it might not stumble across one of the solutions."[6]

Although it has been demonstrated that the class of edge-matching puzzles, of which Eternity II is a special case, is in general NP-complete,[9] the same can be said of the general class of polygon packing problems, of which the original Eternity puzzle was a special case.

Like the original Eternity puzzle, it is easy to find large numbers of ways to place substantial numbers of pieces on the board whose edges all match, making it seem that the puzzle is easy. However, given the low expected number of possible solutions, it is presumably astronomically unlikely that any given partial solution will lead to a complete solution.

See also

- TetraVex
- Satisfiability problem

References

[1] "Description of Eternity II release" (http://www.prnewswire.co.uk/cgi/news/release?id=188486). *PR*. 2007-02-16. . Retrieved 2007-02-16.

[2] Download PDF rule book (http://uk.eternityii.com/about-eternity2/download/) from official site.

[3] http://sydsvenskan.se/lund/article407252/Lundafamilj-bast-i-varlden-pa-svarknackt-pussel.html Link in Swedish

[4] http://uk.eternityii.com/

[5] "Description of Selby and Riordan's Eternity I solver method" (http://www.archduke.org/eternity/method/desc.html). *Alex Selby (and Oliver Riordan)*. 2007-06-16. . Retrieved 2007-06-16.

[6] Elliott, John (2005-12-04). "£1m says this really is the hardest jigsaw" (http://www.timesonline.co.uk/tol/news/uk/article745506.ece). London: Times Online. . Retrieved 2007-11-09.

[7] ""Design" page on Brendan Owen's Eternity II website" (http://eternityii.mrowen.net/design.html). . Retrieved 2007-11-09.

[8] ""Solving" page on Brendan Owen's Eternity II website" (http://eternityii.mrowen.net/solving.html). . Retrieved 2007-11-09.

[9] Erik D. Demaine, Martin L. Demaine. "Jigsaw Puzzles, Edge Matching, and Polyomino Packing: Connections and Complexity" (http://theory.lcs.mit.edu/~edemaine/papers/Jigsaw_GC/paper.pdf) (PDF). . Retrieved 2007-08-12.

External links

- The homepage for Eternity II (http://www.eternityii.com/)
- Open Source Eternity II Editor/Solver software (http://sourceforge.net/projects/eternityii/)
- Open Source Eternity II puzzle software (http://sourceforge.net/projects/e2manual/)
- E2Lab : Free Eternity II Editor/Solver software (http://eternityii.free.fr/)
- Description of Eternity II and discussion of solvers (http://grokcode.com/10/e2-the-np-complete-kids-game-with-the-2-million-prize/)
- Description of the Eternity II solver used by Anna Karlsson (Site seems closed) (http://www.fingerboys.se/eii/eii_details.html/)

Three-dimensional edge-matching puzzle

A **three-dimensional edge-matching puzzle** is a type of edge-matching puzzle or tiling puzzle involving tiling a three-dimensional area with (typically regular) polygonal pieces whose edges are distinguished with colors or patterns, in such a way that the edges of adjacent pieces match. Edge-matching puzzles are known to be NP-complete, and capable of conversion to and from equivalent jigsaw puzzles and polyomino packing puzzle.[1]

Three-dimensional edge-matching puzzles are not currently under direct U.S. patent protection, since the 1892 patent by E. L. Thurston has expired.[2]

Current examples of commercial three-dimensional edge-matching puzzles include the Dodek Duo, The Enigma, Mental Misery,[3] and Kadon Enterprises' range of three-dimensional edge-matching puzzles[4] .

See also

- Edge-matching puzzle
- Domino tiling

References

External links

- Erich's 3-D Matching Puzzles (http://www2.stetson.edu/~efriedma/rubik/3match/index.html)
- Color- and Edge-Matching Polygons (http://mitglied.lycos.de/polyforms/coloredpolygons/index.html) by Peter Esser
- Rob's puzzle page (http://home.comcast.net/~stegmann/pattern.htm) by Rob Stegmann
- More about edgematching (http://www.gamepuzzles.com/moredge.htm)

Packing problem

Packing problems are a class of optimization problems in recreational mathematics which involve attempting to pack objects together (often inside a container), as densely as possible. Many of these problems can be related to real life storage and transportation issues. Each packing problem has a dual covering problem, which asks how many of the same objects are required to completely cover every region of the container, where objects are allowed to overlap.

In a packing problem, you are given:

- 'containers' (usually a single two- or three-dimensional convex region, or an infinite space)
- 'goods' (usually a single type of shape), some or all of which must be packed into this container

Usually the packing must be without overlaps between goods and other goods or the container walls. The aim is to find the configuration with the maximal density. In some variants the overlapping (of goods with each other and/or with the boundary of the container) is allowed but should be minimised.

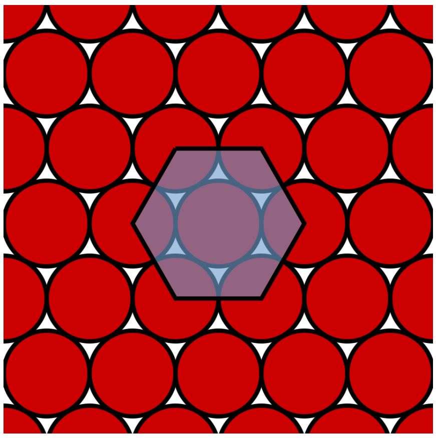
Packing infinite space

Many of these problems, when the container size is increased in all directions, become equivalent to the problem of packing objects as densely as possible in infinite Euclidean space. This problem is relevant to a number of scientific disciplines, and has received significant attention. The Kepler conjecture postulated an optimal solution for packing spheres hundreds of years before it was proven correct by Thomas Callister Hales. Many other shapes have received attention, including ellipsoids, tetrahedra, icosahedra, and unequal-sphere dimers.

Hexagonal packing of circles

These problems are mathematically distinct from the ideas in the circle packing theorem. The related circle packing problem deals with packing circles, possibly of different sizes, on a surface, for instance the plane or a sphere.

Circles (and their counterparts in other dimensions) can never be packed with 100% efficiency in dimensions larger than one (in a one dimensional universe, circles merely consist of two points). That is, there will always be unused space if you are only packing circles. The most efficient way of packing circles, hexagonal packing produces approximately 90% efficiency. [1]



The hexagonal packing of circles on a 2-dimensional Euclidean plane.

Sphere packings in higher dimensions

In three dimensions, the face-centered cubic lattice offers the best *lattice* packing of spheres, and is believed to be the optimal of all packings. The 8-dimensional E8 lattice and 24-dimensional Leech lattice are also believed to be optimal.

Packings of Platonic solids in three dimensions

Cubes can easily be arranged to fill three-dimensional space completely, the most natural packing being the cubic honeycomb. No other Platonic solid can tile space on its own, but some preliminary results are known. Tetrahedra can achieve a packing of at least 85%. One of the best packings of regular dodecahedra is based on the aforementioned face-centered cubic (FCC) lattice.

Tetrahedra and octahedra together can fill all of space in an arrangement known as the tetrahedral-octahedral honeycomb.

Packing in 3-dimensional containers

Spheres into a Euclidean ball

The problem of finding the smallest ball such that k disjoint open unit balls may be packed inside it has a simple and complete answer in n-dimensional Euclidean space if $k \leq n+1$, and in an infinite dimensional Hilbert space with no restrictions. It is worth describing in detail here, to give a flavor of the general problem. In this case, a configuration of k pairwise tangent unit balls is available. Place the centers at the vertices $a_1, .., a_k$ of a regular $(k-1)$ dimensional simplex with edge 2; this is easily realized starting from an orthonormal basis. A small computation shows that the distance of each vertex from the barycenter is $\sqrt{2\left(1-\frac{1}{k}\right)}$. Moreover, any other point of the space necessarily has a larger distance from *at least* one of the k vertices. In terms of inclusions of balls, the k open unit balls centered at $a_1, .., a_k$ are included in a ball of radius $r_k := 1+\sqrt{2\left(1-\frac{1}{k}\right)}$, which is minimal for this configuration.

To show that this configuration is optimal, let $x_1, ..., x_k$ be the centers of k disjoint open unit balls contained in a ball of radius r centered at a point x_0. Consider the map from the finite set $\{x_1, .. x_k\}$ into $\{a_1, .. a_k\}$ taking x_j in the corresponding a_j for each $1 \leq j \leq k$. Since for all $1 \leq i < j \leq k$, $\|a_i - a_j\| = 2 \leq \|x_i - x_j\|$ this map is 1-Lipschitz and by the Kirszbraun theorem it extends to a 1-Lipschitz map that is globally defined; in particular, there exists a point a_0 such that for all $1 \leq j \leq k$ one has $\|a_0 - a_j\| \leq \|x_0 - x_j\|$, so that also $r_k \leq 1 + \|a_0 - a_j\| \leq 1 + \|x_0 - x_j\| \leq r$. This shows that there are k disjoint unit open balls in a ball of radius r if and only if $r \geq r_k$. Notice that in an infinite dimensional Hilbert space this implies that there are infinitely many disjoint open unit balls inside a ball of radius r if and only if $r \geq 1+\sqrt{2}$. For instance, the unit balls centered at $\sqrt{2}e_j$, where $\{e_j\}_j$ is an orthonormal basis, are disjoint and included in a ball of radius $1+\sqrt{2}$ centered at the origin. Moreover, for $r < 1+\sqrt{2}$, the maximum number of disjoint open unit balls inside a ball of radius r is $\left\lfloor \frac{2}{2-(r-1)^2} \right\rfloor$.

Spheres in a cuboid

Determine the number of spherical objects of given diameter *d* can be packed into a cuboid of size *a* × *b* × *c*.

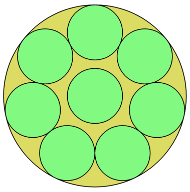
Packing in 2-dimensional containers

Packing circles

Circles in circle

Some of the more non-trivial circle packing problems are packing unit circles into the smallest possible larger circle.

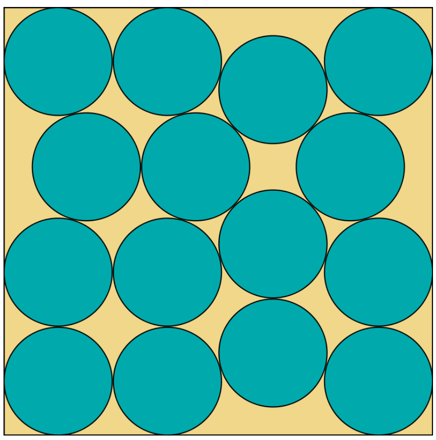
Minimum solutions:

Number of circles	Circle radius
1	1
2	2
3	2.154...
4	2.414...
5	2.701...
6	3
7	3
8	 3.304...
9	3.613...
10	3.813...
11	3.923...
12	4.029...
13	4.236...
14	4.328...
15	4.521...
16	4.615...
17	4.792...
18	4.863...
19	4.863...
20	5.122...

Circles in square

Pack n unit circles into the smallest possible square. This is closely related to spreading points in a unit square with the objective of finding the greatest minimal separation, d_n, between points[2] . To convert between these two formulations of the problem, the square side for unit circles will be $L=2+2/d_n$.

Optimal solutions have been proven for $n\leq30$.[3]

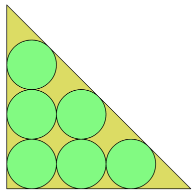


The optimal packing of 15 circles in a square.

Circles in isosceles right triangle

Pack n unit circles into the smallest possible isosceles right triangle (lengths shown are length of leg)

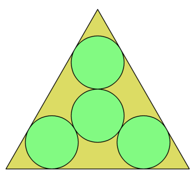
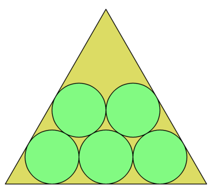
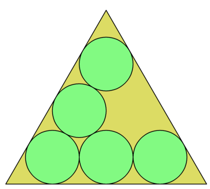
Minimum solutions:

Number of circles	Length
1	3.414...
2	4.828...
3	5.414...
4	6.242...
5	7.146...
6	7.414... 
7	8.181...
8	8.692...
9	9.071...
10	9.414...
11	10.059...
12	10.422...
13	10.798...
14	11.141...
15	11.414...

Circles in equilateral triangle

Pack *n* unit circles into the smallest possible equilateral triangle (lengths shown are side length).

Minimum solutions:

Number of circles	Length
1	3.464...
2	5.464...
3	5.464...
4	6.928... 
5	7.464...  
6	7.464...
7	8.928...
8	9.293...

9	9.464...
10	9.464...
11	10.730...
12	10.928...
13	11.406...
14	11.464...
15	11.464...

Circles in regular hexagon

Pack *n* unit circles into the smallest possible regular hexagon (lengths shown are side length).

Minimum solutions:

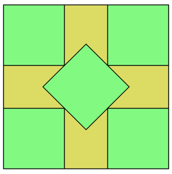
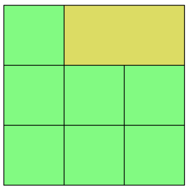
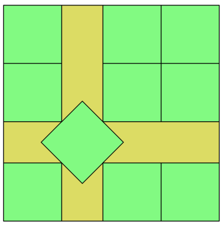
Number of circles	Length
1	1.154...
2	2.154...
3	2.309...
4	2.666...
5	2.999...
6	3.154...
7	3.154...
8	3.709...
9	4.011...
10	4.119...
11	4.309...
12	4.309...
13	4.618...
14	4.666...
15	4.961...

Packing squares

Squares in square

A problem is the **square packing** problem, where one must determine how many squares of side 1 you can pack into a square of side a. Obviously, if a is an integer, the answer is a^2, but the precise, or even asymptotic, amount of wasted space for a a non-integer is open.

Proven minimum solutions:[4]

Number of squares	Square size
1	1
2	2
3	2
4	2
5	2.707 ($2 + 2^{-1/2}$) 
6	3
7	 3
8	3
9	3
10	3.707 ($3 + 2^{-1/2}$) 

Other results:

- If you can pack $n^2 - 2$ squares in a square of side a, then $a \geq n$.[5]
- The naive approach (side matches side) leaves wasted space of less than $2a + 1$.[4]
- The wasted space is asymptotically $o(a^{7/11})$.[6]
- The wasted space is **not** asymptotically $o(a^{1/2})$.[7]
- 11 unit squares cannot be packed in a square of side less than $2 + 2\sqrt{4/5}$.[8]

Squares in circle

Pack *n* squares in the smallest possible circle.

Minimum solutions:

Number of squares	Circle radius
1	0.707...
2	1.118...
3	1.288...
4	1.414...
5	1.581...
6	1.688...
7	1.802...
8	1.978...
9	2.077...
10	2.121...
11	2.215...
12	2.236...

Packing rectangles

Identical rectangles in a rectangle

The problem of packing multiple instances of a single rectangle of size (l,w), allowing for 90^{o} rotation, in a bigger rectangle of size (L,W) has some applications such as loading of boxes on pallets and, specifically, woodpulp stowage.

For example, it is possible to pack 147 rectangles of size (137,95) in a rectangle of size (1600,1230)[9] .

Related fields

In tiling or tesselation problems, there are to be no gaps, nor overlaps. Many of the puzzles of this type involve packing rectangles or polyominoes into a larger rectangle or other square-like shape.

There are significant theorems on tiling rectangles (and cuboids) in rectangles (cuboids) with no gaps or overlaps:

> Klarner's theorem: An $a \times b$ rectangle can be packed with $1 \times n$ strips iff $n \mid a$ or $n \mid b$.[10]
>
> de Bruijn's theorem: A box can be packed with a harmonic brick $a \times a\,b \times a\,b\,c$ if the box has dimensions $a\,p \times a\,b\,q \times a\,b\,c\,r$ for some natural numbers p, q, r (i.e., the box is a multiple of the brick.)

The study of polyomino tilings largely concerns two classes of problems: to tile a rectangle with congruent tiles, and to pack one of each n-omino into a rectangle.

A classic puzzle of the second kind is to arrange all twelve pentominoes into rectangles sized 3×20, 4×15, 5×12 or 6×10.

See also

- Set packing
- Bin packing problem
- Slothouber-Graatsma puzzle
- Conway puzzle
- Tetris
- Covering problem
- Knapsack problem
- Tetrahedron packing
- Cutting stock problem
- Kissing number problem

Notes

[1] http://mathworld.wolfram.com/CirclePacking.html

[2] Croft, Hallard T.; Falconer, Kenneth J.; Guy, Richard K. (1991). *Unsolved Problems in Geometry*. New York: Springer-Verlag. pp. 108–110. ISBN 0-387-97506-3.

[3] Eckard Specht (20 May 2010). "The best known packings of equal circles in a square" (http://hydra.nat.uni-magdeburg.de/packing/csq/csq.html). . Retrieved 25 May 2010.

[4] Erich Friedman, "Packing unit squares in squares: a survey and new results" (http://www.combinatorics.org/Surveys/ds7.html), *The Electronic Journal of Combinatorics* **DS7** (2005).

[5] M. Kearney and P. Shiu, "Efficient packing of unit squares in a square" (http://www.combinatorics.org/Volume_9/Abstracts/v9i1r14.html), *The Electronic Journal of Combinatorics* **9**:1 #R14 (2002).

[6] P. Erdős and R. L. Graham, "On packing squares with equal squares" (http://www.math.ucsd.edu/~sbutler/ron/75_06_squares.pdf), *Journal of Combinatorial Theory, Series A* **19** (1975), pp. 119–123.

[7] K. F. Roth and R. C. Vaughan, "Inefficiency in packing squares with unit squares", *Journal of Combinatorial Theory, Series A* **24** (1978), pp. 170-186.

[8] W. Stromquist, "Packing 10 or 11 unit squares in a square" (http://www.combinatorics.org/Volume_10/Abstracts/v10i1r8.html), *The Electronic Journal of Combinatorics* **10** #R8 (2003).

[9] E G Birgin, R D Lobato, R Morabito, "An effective recursive partitioning approach for the packing of identical rectangles in a rectangle", *Journal of the Operational Research Society*, 2010, **61**, pp. 306-320.

[10] Wagon, Stan (August–September 1987). "Fourteen Proofs of a Result About Tiling a Rectangle" (http://mathdl.maa.org/images/upload_library/22/Ford/Wagon601-617.pdf). *The American Mathematical Monthly* **94** (7): 601–617. . Retrieved 6 Jan 2010.

References

- Weisstein, Eric W., " Klarner's Theorem (http://mathworld.wolfram.com/KlarnersTheorem.html)" from MathWorld.
- Weisstein, Eric W., " de Bruijn's Theorem (http://mathworld.wolfram.com/deBruijnsTheorem.html)" from MathWorld.

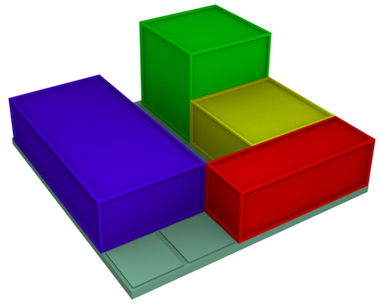
External links

Many puzzle books as well as mathematical journals contain articles on packing problems.

- Links to various MathWorld articles on packing (http://mathworld.wolfram.com/Packing.html)
- MathWorld notes on packing squares. (http://mathworld.wolfram.com/SquarePacking.html)
- Erich's Packing Center (http://www.stetson.edu/~efriedma/packing.html)
- www.packomania.com (http://www.packomania.com/) A site with tables, graphs, calculators, references, etc.
- "Box Packing" (http://demonstrations.wolfram.com/BoxPacking/) by Ed Pegg, Jr., the Wolfram Demonstrations Project, 2007.

Conway puzzle

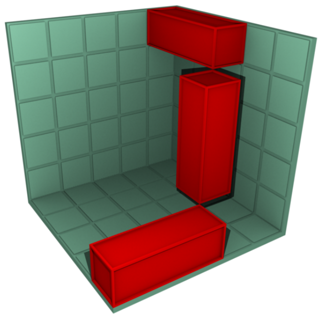
Conway's puzzle is a packing problem using rectangular blocks, named after its inventor, mathematician John Conway. It calls for packing thirteen $1 \times 2 \times 4$ blocks, one $2 \times 2 \times 2$ block, one $1 \times 2 \times 2$ block, and three $1 \times 1 \times 3$ blocks into a $5 \times 5 \times 5$ box.[1]



Pieces used in the Conway puzzle, one of each kind.

Solution

The solution of the Conway puzzle is straightforward when one realizes that the three $1 \times 1 \times 3$ blocks need to be placed so that precisely one of them appears in each $5 \times 5 \times 1$ slice of the cube.[2]



A possible placement for the three 1×1×3 blocks.

See also

- Slothouber–Graatsma puzzle

External links

- The Conway puzzle in Stewart Coffin's "The Puzzling World of Polyhedral Dissections" [3]

References

[1] "Conway Puzzle" (http://mathworld.wolfram.com/ConwayPuzzle.html). *Wolfram MathWorld*. . Retrieved 2007-03-14.
[2] Elwyn R. Berlekamp, John H. Conway and Richard K. Guy: Wining ways for your mathematical plays, 2nd ed, vol. 4, 2004.
[3] http://www.johnrausch.com/PuzzlingWorld/chap03f.htm

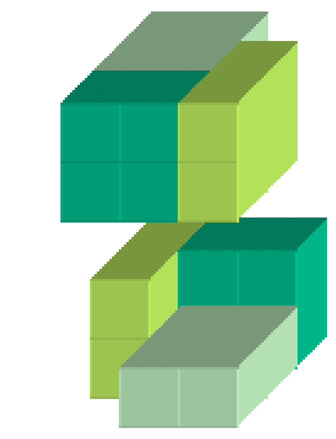
Slothouber–Graatsma puzzle

The **Slothouber–Graatsma puzzle** is a packing problem that calls for packing six 1 × 2 × 2 blocks and three 1 × 1 × 1 blocks into a 3 × 3 × 3 box. The solution to this puzzle is unique (up to mirror reflections and rotations).

The puzzle is essentially the same if the three 1 × 1 × 1 blocks are left out, so that the task is to pack six 1 × 2 × 2 blocks into a cubic box with volume 27. The Slothouber–Graatsma puzzle is regarded as the smallest nontrivial 3D packing problem.

Solution

The solution of the Slothouber–Graatsma puzzle is straightforward when one realizes that the three 1 × 1 × 1 blocks (or the three holes) need to be placed along a body diagonal of the box, as each of the 3 x 3 layers in the various directions needs to contain such a unit block. This follows because the larger blocks can only fill an even number of the 9 cells in each 3 x 3 layer.[1]



Solution of Slothouber-Graatsma puzzle showing the six 1 x 2 x 2 pieces in exploded view.

Variations

The Slothouber–Graatsma puzzle is an example of a cube-packing puzzle using convex rectangular blocks. More complex puzzles involving the packing of convex rectangular blocks have been designed. The best known example is the Conway puzzle which asks for the packing of eighteen convex rectangular blocks into a 5 x 5 x 5 box. A harder convex rectangular block packing problem is to pack forty-one 1 x 2 x 4 blocks into a 7 x 7 x 7 box (thereby leaving 15 holes).[1]

See also

- Conway puzzle
- Soma cube
- Bedlam cube

External links

- The Slothouber-Graatsma puzzle in Stewart Coffin's "The Puzzling World of Polyhedral Dissections" [3]
- Jan Slothouber and William Graatsma: Cubic constructs [2]
- William Graatsma and Jan Slothouber: Dutch mathematical art [3]

References

[1] Elwyn R. Berlekamp, John H. Conway and Richard K. Guy: Wining ways for your mathematical plays, 2nd ed, vol. 4, 2004.
[2] http://www.designws.com/pagina/1ccceng.htm
[3] http://www.genicap.com/Site/Components/SitePageCP/ShowPage.aspx?ItemID=f4e85ecc-81d0-48f8-b972-d606e9dcd60a&SelectedMenuItemID=b0947550-80c1-4fa7-af5b-e26d883ff532

Polyforms

Polyform

In recreational mathematics, a **polyform** is a plane figure constructed by joining together identical basic polygons. The basic polygon is often (but not necessarily) a convex plane-filling polygon, such as a square or a triangle. More specific names have been given to polyforms resulting from specific basic polygons, as detailed in the table below. For example, a square basic polygon results in the well-known polyominoes.

Construction rules

The rules for joining the polygons together may vary, and must therefore be stated for each distinct type of polyform. Generally, however, the following rules apply:

1. Two basic polygons may be joined only along a common edge.
2. No two basic polygons may overlap.
3. A polyform must be connected (that is, all one piece; see connected graph, connected space). Configurations of disconnected basic polygons do not qualify as polyforms.
4. The mirror image of an asymmetric polyform is not considered a distinct polyform (polyforms are "double sided").

Generalizations

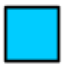
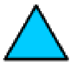
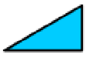
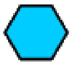
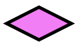
Polyforms can also be considered in higher dimensions. In 3-dimensional space, basic polyhedra can be joined along congruent faces. Joining cubes in this way produces the polycubes.

One can allow more than one basic polygon. The possibilities are so numerous that the exercise seems pointless, unless extra requirements are brought in. For example, the Penrose tiles define extra rules for joining edges, resulting in interesting polyforms with a kind of pentagonal symmetry.

When the base form is a polygon that tiles the plane, rule 1 may be broken. For instance, squares may be joined orthogonally at vertices, as well as at edges, to form *polyplets* or *polykings*.[1]

Types and applications

Polyforms are a rich source of problems, puzzles and games. The basic combinatorial problem is counting the number of different polyforms, given the basic polygon and the construction rules, as a function of *n*, the number of basic polygons in the polyform. Well-known puzzles include the pentomino puzzle and the Soma cube.

Basic polygon (monoform)		Polyform	Applications
	line segment	polystick	
	square	polyomino	pentomino puzzle, Lonpos puzzle, Fillomino, Tentai Show, Ripple Effect (puzzle), LITS, Nurikabe, Sudoku
	equilateral triangle	polyiamond	
	30°-60°-90° triangle	polydrafter	Eternity puzzle
	right isosceles (45°-45°-90°) triangle	polyabolo	
	regular hexagon	polyhex	
	cube	polycube	Soma cube
	square (in three dimensions)	polyominoid	

References

[1] Weisstein, Eric W., " Polyplet (http://mathworld.wolfram.com/Polyplet.html)" from MathWorld.

External links

- Weisstein, Eric W., " Polyform (http://mathworld.wolfram.com/Polyform.html)" from MathWorld.
- *The Poly Pages* at RecMath.org (http://www.recmath.org/PolyPages/index.htm), illustrations and information on many kinds of polyforms.

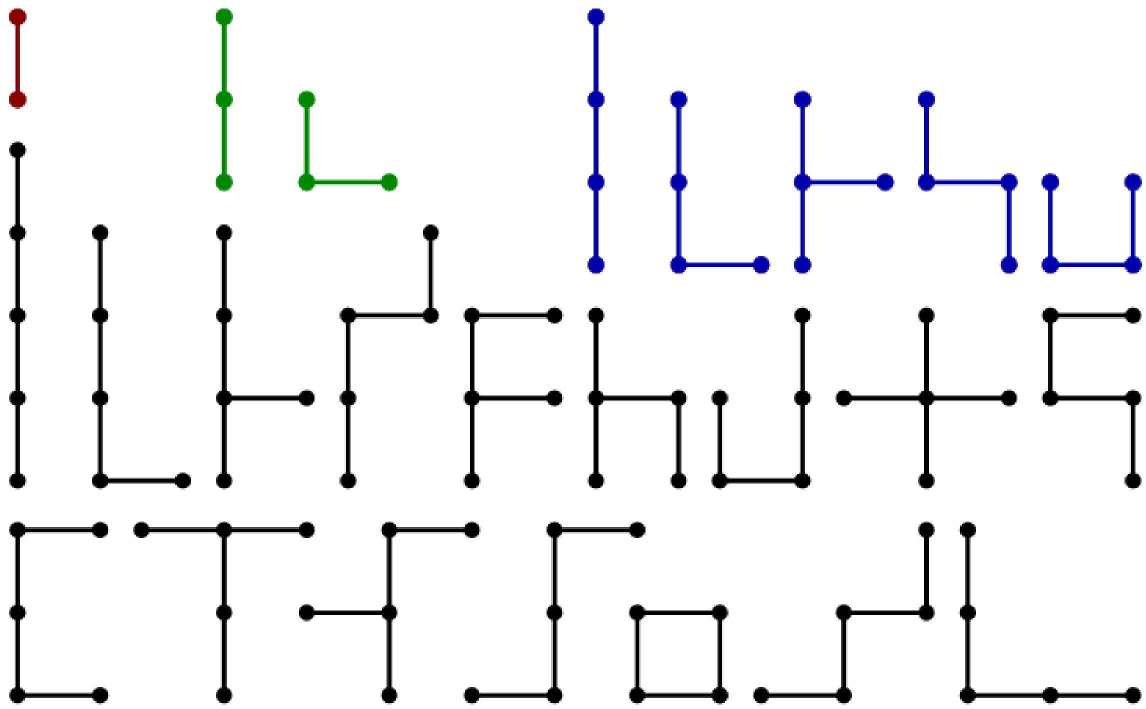
Polystick

In recreational mathematics, a **polystick** (or **polyedge**) is a polyform with a line segment (a 'stick') as the basic shape. Polysticks result when identical line segments are joined together end-to-end at 0° or 90° angles.

The possible different polysticks, not including rotations and reflections, include:

- 1 monostick,
- 2 disticks,
- 5 tristicks, and
- 16 tetrasticks.

The following diagram shows the polysticks of sizes 1 through 4, including the 1 monostick (red), 2 disticks (green), 5 tristicks (blue), and 16 tetrasticks (black).



References

- *Polysticks Puzzles & Solutions*, at Polyforms Puzzler [1]
- *Counting polyforms*, at the Solitaire Laboratory [2]
- A019988 [3] *Number of ways of embedding a connected graph with n edges in the square lattice*, at the On-Line Encyclopedia of Integer Sequences
- *Covering the Aztec Diamond with One-sided Tetrasticks*, Alfred Wassermann, University of Bayreuth, Germany [4]
- *Polypolylines* [5], at Math Magic [6]

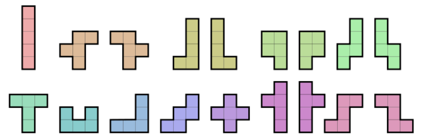
References

[1] http://puzzler.sourceforge.net/docs/polysticks.html
[2] http://www.solitairelaboratory.com/polyenum.html
[3] http://en.wikipedia.org/wiki/Oeis%3Aa019988
[4] http://did.mat.uni-bayreuth.de/wassermann/tetrastick.pdf
[5] http://www2.stetson.edu/~efriedma/mathmagic/0806.html
[6] http://www2.stetson.edu/~efriedma/mathmagic/

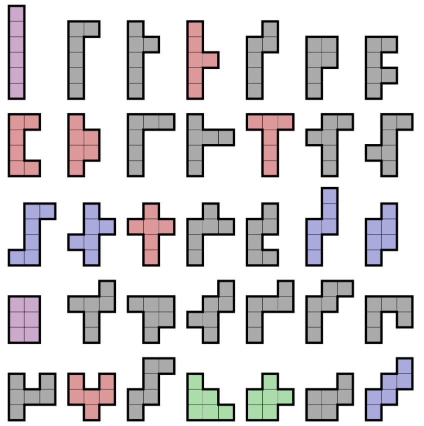
Polyomino

A **polyomino** is a plane geometric figure formed by joining one or more equal squares edge to edge. It is a polyform whose cells are squares. It may be regarded as a finite subset of the regular square tiling with a connected interior.

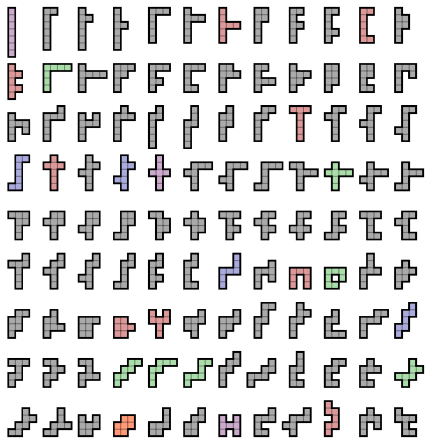
Polyominoes are classified according to how many cells they have:



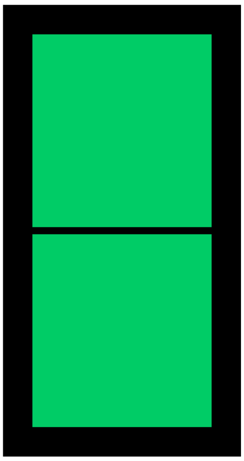
The 18 one-sided pentominoes, including 6 mirrored pairs



The 35 free hexominoes, coloured according to their symmetry.



The 108 free heptominoes.



The single free domino.

Number of cells	Name
1	monomino
2	domino
3	tromino or triomino
4	tetromino
5	pentomino or pentamino
6	hexomino
7	heptomino
8	octomino
9	nonomino or enneomino
10	decomino
11	undecomino or hendecomino
12	dodecomino

Polyominoes have been used in popular puzzles since at least 1907, and the enumeration of pentominoes is dated to antiquity.[1] Many results with the pieces of 1 to 6 squares were first published in Fairy Chess Review between the years 1937 to 1957, under the name of "dissection problems." The name *polyomino* was invented by Solomon W. Golomb in 1953 and it was popularized by Martin Gardner.[2]

Related to polyominoes are polyiamonds, formed from equilateral triangles; polyhexes, formed from regular hexagons; and other plane polyforms. Polyominoes have been generalised to higher dimensions by joining cubes to form polycubes, or hypercubes to form polyhypercubes.

Like many puzzles in recreational mathematics, polyominoes raise many combinatorial problems. The most basic is enumerating polyominoes of a given size. No formula has been found except for special classes of polyominoes. A number of estimates are known, and there are algorithms for calculating them.

Polyominoes with holes are inconvenient for some purposes, such as tiling problems. In some contexts polyominoes with holes are excluded, allowing only simply connected polyominoes.[3]

Enumeration of polyominoes

Free, one-sided, and fixed polyominoes

There are three common ways of distinguishing polyominoes for enumeration:[4] [5]

- *free* polyominoes are distinct when none is a rigid transformation (translation, rotation, reflection or glide reflection) of another. (Think of pieces that can be picked up and flipped over.)
- *one-sided polyominoes* are distinct when none is a translation or rotation of another. (Think of pieces that cannot be flipped over.)
- *fixed* polyominoes are distinct when none is a translation of another. (Think of pieces that can be neither flipped nor rotated.)

The following table shows the numbers of polyominoes of various types with *n* cells. Let A_n denote the number of fixed polyominoes with *n* cells (possibly with holes).

n	name	free (sequence A000105 [6] in OEIS)	free with holes (A001419 [7])	free without holes (A000104 [8])	one-sided (A000988 [9])	fixed (A_n, A001168 [10])
1	monomino	1	0	1	1	1
2	domino	1	0	1	1	2
3	tromino or triomino	2	0	2	2	6
4	tetromino	5	0	5	7	19
5	pentomino	12	0	12	18	63
6	hexomino	35	0	35	60	216
7	heptomino	108	1	107	196	760
8	octomino	369	6	363	704	2,725
9	nonomino or enneomino	1,285	37	1,248	2,500	9,910
10	decomino	4,655	195	4,460	9,189	36,446
11	undecomino	17,073	979	16,094	33,896	135,268
12	dodecomino	63,600	4,663	58,937	126,759	505,861

As of 2004, Iwan Jensen has enumerated the fixed polyominoes up to *n*=56: $A_{56}$is approximately 6.915×10^{31}.[11] Free polyominoes have been enumerated up to *n*=28 by Tomás Oliveira e Silva.[12]

Symmetries of polyominoes

The dihedral group $D_4$is the group of symmetries (symmetry group) of a square. This group contains four rotations and four reflections. It is generated by alternating reflections about the *x*-axis and about a diagonal. One free polyomino corresponds to at most 8 fixed polyominoes, which are its images under the symmetries of D_4. However, those images are not necessarily distinct: the more symmetry a free polyomino has, the fewer distinct fixed counterparts it has. Therefore, a free polyomino which is invariant under some or all non-trivial symmetries of D_4 may correspond to only 4, 2 or 1 fixed polyominoes. Mathematically, free polyominoes are equivalence classes of fixed polyominoes under the group D_4.

Polyominoes have the following possible symmetries;[13] the least number of squares needed in a polyomino with that symmetry is given in each case:

- 8 fixed polyominoes for each free polyomino:

- no symmetry (4)
- 4 fixed polyominoes for each free polyomino:
  - mirror symmetry with respect to one of the grid line directions (4)
  - mirror symmetry with respect to a diagonal line (3)
  - 2-fold rotational symmetry: C_2(4)
- 2 fixed polyominoes for each free polyomino:
  - symmetry with respect to both grid line directions, and hence also 2-fold rotational symmetry: D_2(2)
  - symmetry with respect to both diagonal directions, and hence also 2-fold rotational symmetry: D_2(7)
  - 4-fold rotational symmetry: C_4(8)
- 1 fixed polyomino for each free polyomino:
  - all symmetry of the square: D_4(1).

The following table shows the numbers of polyominoes with *n* squares, sorted by symmetry groups.

n	none (A006749 [14])	mirror (90°) (A006746 [15])	mirror (45°) (A006748 [16])	C_2(A006747 [17])	D_2 (90°) (A056877 [18])	D_2 (45°) (A056878 [19])	C_4(A144553 [20])	D_4(A142886 [21])
1	0	0	0	0	0	0	0	1
2	0	0	0	0	1	0	0	0
3	0	0	1	0	1	0	0	0
4	1	1	0	1	1	0	0	1
5	5	2	2	1	1	0	0	1
6	20	6	2	5	2	0	0	0
7	84	9	7	4	3	1	0	0
8	316	23	5	18	4	1	1	1
9	1,196	38	26	19	4	0	0	2
10	4,461	90	22	73	8	1	0	0
11	16,750	147	91	73	10	2	0	0
12	62,878	341	79	278	15	3	3	3

Algorithms for enumeration of fixed polyominoes

Inductive algorithms

Each polyomino of order *n*+1 can be obtained by adding a square to a polyomino of order *n*. This leads to algorithms for generating polyominoes inductively.

Most simply, given a list of polyominoes of order *n*, squares may be added next to each polyomino in each possible position, and the resulting polyomino of order *n*+1 added to the list if not a duplicate of one already found; refinements in ordering the enumeration and marking adjacent squares that should not be considered reduce the number of cases that need to be checked for duplicates.[22] This method may be used to enumerate either free or fixed polyominoes.

A more sophisticated method, described by Redelmeier, has been used by many authors as a way of not only counting polyominoes (without requiring that all polyominoes of order *n* be stored in order to enumerate those of order *n*+1), but also proving upper bounds on their number. The basic idea is that we begin with a single square, and from there, recursively add squares. Depending on the details, it may count each *n*-omino *n* times, once from starting from each of its *n* squares, or may be arranged to count each once only.

The simplest implementation involves adding one square at a time. Beginning with an initial square, number the adjacent squares, clockwise from the top, 1, 2, 3, and 4. Now pick a number between 1 and 4, and add a square at that location. Number the unnumbered adjacent squares, starting with 5. Then, pick a number larger than the previously picked number, and add that square. Continue picking a number larger than the number of the current square, adding that square, and then numbering the new adjacent squares. When *n* squares have been created, an *n*-omino has been created.

This method ensures that each fixed polyomino is counted exactly *n* times, once for each starting square. It can be optimized so that it counts each polyomino only once, rather than *n* times. Starting with the initial square, declare it to be the lower-left square of the polyomino. Simply do not number any square which is on a lower row, or left of the square on the same row. This is the version described by Redelmeier.

If one wishes to count free polyominoes instead, then one may check for symmetries after creating each *n*-omino. However, it is faster[23] to generate symmetric polyominoes separately (by a variation of this method)[24] and so determine the number of free polyominoes by Burnside's lemma.

Transfer-matrix method

The most modern algorithm for enumerating the fixed polyominoes was discovered by Iwan Jensen.[25] An improvement on Andrew Conway's method,[26] it is exponentially faster than the previous methods (however, its running time is still exponential in *n*).

Both Conway's and Jensen's versions of the transfer-matrix method involve counting the number of polyominoes that have a certain width. Computing the number for all widths gives the total number of polyominoes. The basic idea behind the method is that possible beginning rows are considered, and then to determine the minimum number of squares needed to complete the polyomino of the given width. Combined with the use of generating functions, this technique is able to count many polyominoes at once, thus enabling it to run many times faster than methods that have to generate every polyomino.

Although it has excellent running time, the tradeoff is that this algorithm uses exponential amounts of memory (many gigabytes of memory are needed for *n* above 50), is much harder to program than the other methods, and can't currently be used to count free polyominoes.

Asymptotic growth of the number of polyominoes

Fixed polyominoes

Theoretical arguments and numerical calculations support the estimate

$$A_n \sim \frac{c\lambda^n}{n}$$

where $\lambda = 4.0626$ and $c = 0.3169$.[27] However, it should be emphasized that this result is not proven and the values of λ and *c* are only estimates.

The known theoretical results are not nearly as specific as this estimate. It has been proven that

$$\lim_{n\to\infty} (A_n)^{1/n} = \lambda$$

exists. In other words, A_n grows exponentially. The best known lower bound for λ is 3.980137.[28] The best known upper bound, not improved since the 1970s, is $\lambda < 4.65$.[29]

To establish a lower bound, a simple but highly effective method is concatenation of polyominoes. Define the upper-right square to be the rightmost square in the uppermost row of the polyomino. Define the bottom-left square similarly. Then, the upper-right square of any polyomino of size *n* can be attached to the bottom-left square of any polyomino of size *m* to produce a unique (*n*+*m*)-omino. This proves $A_nA_m \leq A_{n+m}$. Using this equation, one can show $\lambda \geq (A_n)^{1/n}$ for all *n*. Refinements of this procedure combined with data for A_n produce the lower bound given above.

The upper bound is attained by generalizing the inductive method of enumerating polyominoes. Instead of adding one square at a time, one adds a cluster of squares at a time. This is often described as adding *twigs*. By proving that every n-omino is a sequence of twigs, and by proving limits on the combinations of possible twigs, one obtains an upper bound on the number of n-ominoes. For example, in the algorithm outlined above, at each step we must choose a larger number, and at most three new numbers are added (since at most three unnumbered squares are adjacent to any numbered square). This can be used to obtain an upper bound of 6.75. Using 2.8 million twigs, Klarner and Rivest obtained an upper bound of 4.65.

Free polyominoes

Approximations for the number of fixed polyominoes and free polyominoes are related in a simple way. A free polyomino with no symmetries (rotation or reflection) corresponds to 8 distinct fixed polyominoes, and for large n, most n-ominoes have no symmetries. Therefore, the number of fixed n-ominoes is approximately 8 times the number of free n-ominoes. Moreover, this approximation is exponentially more accurate as n increases.[13]

Special classes of polyominoes

Exact formulas are known for enumerating polyominoes of special classes, such as the class of *convex* polyominoes and the class of *directed* polyominoes.

The definition of a *convex* polyomino is different from the usual definition of convexity. A polyomino is said to be *column convex* if its intersection with any vertical line is convex (in other words, each column has no holes). Similarly, a polyomino is said to be *row convex* if its intersection with any horizontal line is convex. A polyomino is said to be *convex* if it is row and column convex.

A polyomino is said to be *directed* if it contains a square, known as the *root*, such that every other square can be reached by movements of up or right one square, without leaving the polyomino.

Directed polyominoes,[30] column (or row) convex polyominoes,[31] and convex polyominoes[32] have been effectively enumerated by area n, as well as by some other parameters such as perimeter, using generating functions.

Uses of polyominoes

Polyominoes have fostered significant research in mathematics[33] and are a fertile subject for logic puzzles and recreational mathematics.[34] Challenges are often posed for covering (tiling) a prescribed region, or the entire plane, with polyominoes,[35] or folding a polyomino to create other shapes. Gardner proposed several simple games with a set of free pentominoes and a chessboard. Some variants of the Sudoku puzzle use polyomino-shaped regions on the grid. The game Tetris is based on the seven one-sided tetrominoes, and the board game Blokus uses all of the free polyominoes up to pentominoes.

Tiling regions with sets of polyominoes

Puzzles commonly ask for tiling a given region with a given set of polyominoes, such as the 12 pentominoes. Golomb's and Gardner's books have many examples. A typical puzzle is to tile a 6×10 rectangle with the twelve pentominoes; the 2339 solutions to this were found in 1960.[36] Where multiple copies of the polyominoes in the set are allowed, Golomb defines a hierarchy of different regions that a set may be able to tile, such as rectangles, strips, and the whole plane, and shows that whether polyominoes from a given set can tile the plane is undecidable, by mapping sets of Wang tiles to sets of polyominoes.[37]

Tiling regions with copies of a single polyomino

Another class of problems asks whether copies of a given polyomino can tile a rectangle, and if so, what rectangles they can tile.[38] These problems have been extensively studied for particular polyominoes,[39] and tables of results for individual polyominoes are available.[40] Klarner and Göbel showed that for any polyomino there is a finite set of *prime* rectangles it tiles, such that all other rectangles it tiles can be tiled by those prime rectangles.[41] [42]

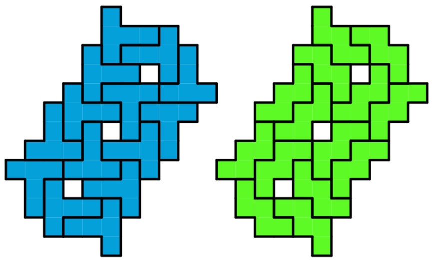
Beyond rectangles, Golomb gave his hierarchy for single polyominoes: a polyomino may tile a rectangle, a half strip, a bent strip, an enlarged copy of itself, a quadrant, a strip, a half plane, the whole plane, certain combinations, or none of these. There are certain implications among these, both obvious (for example, if a polyomino tiles the half plane then it tiles the whole plane) and less so (for example, if a polyomino tiles an enlarged copy of itself, then it tiles the quadrant). Polyominoes of orders up to 6 are characterised in this hierarchy (with the status of one hexomino, later found to tile a rectangle, unresolved at that time).[43]

Tiling the plane with copies of a single polyomino

Tiling the plane with copies of a single polyomino has also been much discussed. It was noted in 1965 that all polyominoes of orders 1 through 6 tile the plane,[44] and then that all but four heptominoes will do so.[45] It was then established by David Bird that all but 26 octominoes tile the plane.[46] Rawsthorne found that all but 235 polyominoes of order 9 tile,[47] and such results have been extended to higher orders by Rhoads (to order 14)[48] and others. Polyominoes tiling the plane have been classified by the symmetries of their tilings and by the number of aspects (orientations) in which the tiles appear in them.[49] [50] [51]

Tiling a common figure with various polyominoes

The *compatibility problem* is to take two or more polyominoes and find a figure that can be tiled with each. Polyomino compatibility has been widely studied since the 1990s, and Jorge Luis Mireles and Giovanni Resta have published websites of systematic results.[52] [53] The general problem can be hard. The first compatibility figure for the L and X pentominoes was published in 2005 and had 80 tiles of each kind.[54] Many pairs of polyominoes have been proved incompatible by systematic exhaustion. No algorithm is known for deciding whether two arbitrary polyominoes are compatible.



A minimal compatibility figure for the T and W pentominoes.

Etymology

The word *polyomino* and the names of the various orders of polyomino are all back-formations from the word *domino*, a common game piece consisting of two squares, with the first letter *d-* fancifully interpreted as a version of the prefix *di-* meaning "two". The name *domino* for the game piece is believed to come from the spotted masquerade garment *domino,* from Latin *dominus*.[55]

Most of the numerical prefixes are Greek. Polyominoes of order 9 and 11 more often take the Latin prefixes *nona-* (nonomino) and *undeca-* (undecomino) than the Greek prefixes *ennea-* (enneomino) and *hendeca-* (hendecomino).

See also

- Percolation theory, the mathematical study of random subsets of integer grids. The finite connected components of these subsets form polyominoes.
- Young diagram, a special kind of polyomino used in number theory to describe integer partitions and in mathematical physics to describe representations of the symmetric group.
- Blokus, a board game using polyominoes.
- Squaregraph, a kind of undirected graph including as a special case the graphs of vertices and edges of polyominoes.
- Commons:Permutomino

Notes

[1] Golomb (*Polyominoes*, Preface to the First Edition) writes “the observation that there are twelve distinctive patterns (the pentominoes) that can be formed by five connected stones on a Go board … is attributed to an ancient master of that game”.

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[5] Golomb, chapter 6

[6] http://en.wikipedia.org/wiki/Oeis%3Aa000105

[7] http://en.wikipedia.org/wiki/Oeis%3Aa001419

[8] http://en.wikipedia.org/wiki/Oeis%3Aa000104

[9] http://en.wikipedia.org/wiki/Oeis%3Aa000988

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[12] Tomás Oliveira e Silva. "Animal enumerations on the {4,4} Euclidean tiling" (http://www.ieeta.pt/~tos/animals/a44.html). . Retrieved 2007-05-06.

[13] Redelmeier, section 3

[14] http://en.wikipedia.org/wiki/Oeis%3Aa006749

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[16] http://en.wikipedia.org/wiki/Oeis%3Aa006748

[17] http://en.wikipedia.org/wiki/Oeis%3Aa006747

[18] http://en.wikipedia.org/wiki/Oeis%3Aa056877

[19] http://en.wikipedia.org/wiki/Oeis%3Aa056878

[20] http://en.wikipedia.org/wiki/Oeis%3Aa144553

[21] http://en.wikipedia.org/wiki/Oeis%3Aa142886

[22] Golomb, pp. 73–79

[23] Redelmeier, section 4

[24] Redelmeier, section 6

[25] Jensen, Iwan (February 2001). "Enumerations of Lattice Animals and Trees" (http://arxiv.org/abs/cond-mat/0007239) (preprint). *Journal of Statistical Physics* **102** (3–4): 865–881. doi:10.1023/A:1004855020556. . Retrieved 2007-05-10.

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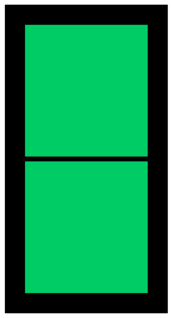
External links

- An interactive polyomino-tiling application (http://gfredericks.com/main/sandbox/polyominoes)
- Karl Dahlke's polyomino finite-rectangle tilings (http://www.eklhad.net/polyomino/)
- An implementation and description of Jensen's method (http://www-cs-faculty.stanford.edu/~knuth/programs.html#polyominoes)
- A paper describing modern estimates (PS) (http://www.statslab.cam.ac.uk/~grg/books/hammfest/19-sgw.ps)
- Weisstein, Eric W., " Polyomino (http://mathworld.wolfram.com/Polyomino.html)" from MathWorld.
- MathPages – Notes on enumeration of polyominoes with various symmetries (http://www.mathpages.com/home/kmath039.htm)
- List of dissection problems in Fairy Chess Review (http://www.mayhematics.com/d/db.htm)
- *Tetrads* (http://demonstrations.wolfram.com/Tetrads/) by Karl Scherer, Wolfram Demonstrations Project.

Domino

In mathematics, a **domino** is a polyomino of order 2, that is, a polygon in the plane made of two equal-sized squares connected edge-to-edge.[1] When rotations and reflections are not considered to be distinct shapes, there is only one *free* domino.



The single free domino

Since it has reflection symmetry, it is also the only *one-sided* domino (with reflections considered distinct). When rotations are also considered distinct, there are two *fixed* dominoes: The second one can be created by rotating the one above by 90°.[2] [3]

In a wider sense, the term *domino* is often understood to simply mean a tile of any shape.[4]

See also

- Domino tiling, a covering of a geometric figure with dominoes; these figure in several celebrated problems:
  - Aztec diamond problem
  - Mutilated chessboard problem, a domino tiling problem concerning an 8×8 chessboard with two deleted squares
- Dominoes, a set of domino-shaped gaming pieces

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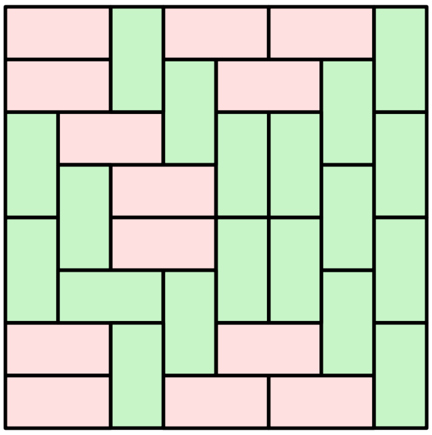
Domino tiling



Domino tiling of a square

A **domino tiling** of a region in the Euclidean plane is a tessellation of the region by dominos, shapes formed by the union of two unit squares meeting edge-to-edge. Equivalently, it is a matching in the grid graph formed by placing a vertex at the center of each square of the region and connecting two vertices when they correspond to adjacent squares.

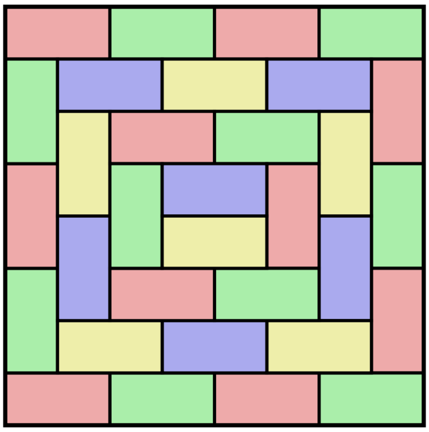
Height functions

For some classes of tilings on a regular grid in two dimensions, it is possible to define a height function associating an integer to the nodes of the grid. For instance, draw a chessboard, fix a node A_0 with height 0, then for any node there is a path from A_0 to it. On this path define the height of each node A_{n+1} (i.e. corners of the squares) to be the height of the previous node A_n plus one if the square on the right of the path from A_n to A_{n+1} is black, and minus one else.
More details can be found in Kenyon & Okounkov (2005).

Thurston's height condition

William Thurston (1990) describes a test for determining whether a simply-connected region, formed as the union of unit squares in the plane, has a domino tiling. He forms an undirected graph that has as its vertices the points (x,y,z) in the three-dimensional integer lattice, where each such point is connected to four neighbors: if x+y is even, then (x,y,z) is connected to (x+1,y,z+1), (x-1,y,z+1), (x,y+1,z-1), and (x,y-1,z-1), while if x+y is odd, then (x,y,z) is connected to (x+1,y,z-1), (x-1,y,z-1), (x,y+1,z+1), and (x,y-1,z+1). The boundary of the region, viewed as a sequence of integer points in the (x,y) plane, lifts uniquely (once a starting height is chosen) to a path in this three-dimensional graph. A necessary condition for this region to be tileable is that this path must close up to form a simple closed curve in three dimensions, however, this condition is not sufficient. Using more careful analysis of the boundary path, Thurston gave a criterion for tileability of a region that was sufficient as well as necessary.

Counting tilings of regions



Domino tiling of an 8×8 square using the minimum number of long-edge-to-long-edge pairs (1 pair in the center).

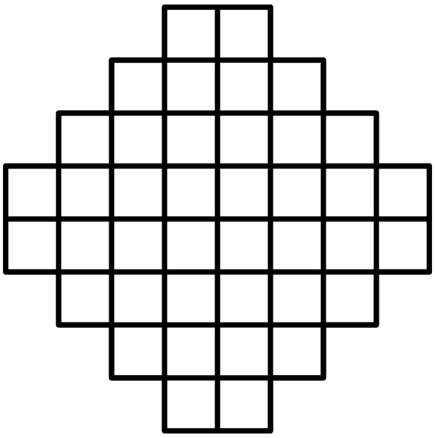
The number of ways to cover a m -by- n rectangle with $mn/2$ dominoes, calculated independently by Temperley & Fisher (1961) and Kasteleyn (1961), is given by

$$\prod_{j=1}^{m}\prod_{k=1}^{n}\left(4\cos^2\frac{\pi j}{m+1}+4\cos^2\frac{\pi k}{n+1}\right)^{1/4}.$$

The sequence of values generated by this formula for squares with $m = n = 0, 2, 4, 6, 8, 10, 12, \ldots$ is

1, 2, 36, 6728, 12988816, 258584046368, 53060477521960000, ... (sequence A004003 [1] in OEIS).

These numbers can be found by writing them as the Pfaffian of an *mn* by *mn* antisymmetric matrix whose eigenvalues can be found explicitly. This technique may be applied in many mathematics-related subjects, for example, in the classical, 2-dimensional computation of the dimer-dimer correlator function in statistical mechanics.



An Aztec diamond of order 4, with 1024 domino tilings

The number of tilings of a region is very sensitive to boundary conditions, and can change dramatically with apparently insignificant changes in the shape of the region. This is illustrated by the number of tilings of an Aztec diamond of order n, where the number of tilings is $2^{(n+1)n/2}$. If this is replaced by the "augmented Aztec diamond" of order n with 3 long rows in the middle rather than 2, the number of tilings drops to the much smaller number $D(n,n)$, a Delannoy number, which has only exponential rather than super-exponential growth in n. For the "reduced Aztec diamond" of order n with only one long middle row, there is only one tiling.

See also

- Statistical mechanics
- Gaussian free field, the scaling limit of the height function in the generic situation (e.g., inside the inscribed disk of a large aztec diamond)
- Mutilated chessboard problem, a puzzle concerning domino tiling of a 62-square subset of the chessboard
- Tatami, floor mats in the shape of a domino that are used to tile the floors of Japanese rooms, with certain rules about how they may be placed

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[4] http://www.emis.de/journals/JIS/VOL5/Sellers/sellers4.pdf
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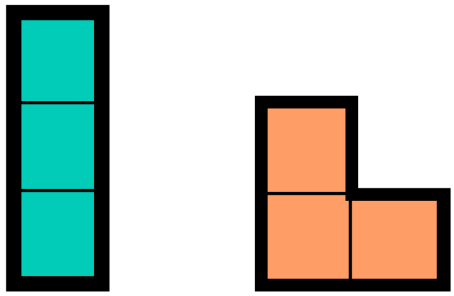
Tromino



All possible free trominos

A **tromino** (or **triomino**) is a polyomino of order 3, that is, a polygon in the plane made of three equal-sized squares connected edge-to-edge.[1] When rotations and reflections are not considered to be distinct shapes, there are only two different *free* trominoes: "I" and "L" (the "L" shape is also called "V").

Since both free trominoes have reflection symmetry, they are also the only two *one-sided* trominoes (trominoes with reflections considered distinct). When rotations are also considered distinct, there are six *fixed* trominoes: two I and four L shapes. They can be obtained by rotating the above forms by 90°, 180° and 270°.[2] [3]

See also

- Triominoes, a game like dominoes but using triangular pieces

References

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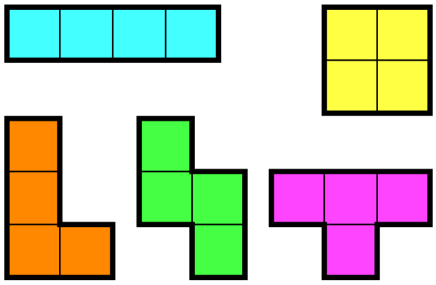
[2] Weisstein, Eric W. "Triomino" (http://mathworld.wolfram.com/Triomino.html). From MathWorld – A Wolfram Web Resource. . Retrieved 2009-12-05.

[3] Redelmeier, D. Hugh (1981). "Counting polyominoes: yet another attack". *Discrete Mathematics* **36**: 191–203. doi:10.1016/0012-365X(81)90237-5.

External links

- Golomb's inductive proof of a tromino theorem (http://www.cut-the-knot.org/Curriculum/Geometry/Tromino.shtml) at cut-the-knot
- Tromino Puzzle (http://www.cut-the-knot.org/Curriculum/Games/TrominoPuzzle.shtml) at cut-the-knot
- Interactive Tromino Puzzle (http://www.amherst.edu/~nstarr/puzzle.html) at Amherst College

Tetromino



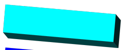
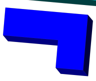
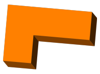
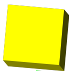
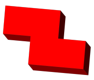
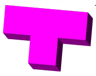
The 5 free tetrominoes

A **tetromino**, also called a tetramino or tetrimino, is a geometric shape composed of four squares, connected orthogonally.[1] [2] This, like dominoes and pentominoes, is a particular type of polyomino. The corresponding polycube, called a **tetracube**, is a geometric shape composed of four cubes connected orthogonally.

A popular use of tetrominoes is in the video game *Tetris*.

The seven tetrominoes

Ordinarily, polyominoes are discussed in their *free* forms, which treat rotations and reflections in two dimensions as congruent. In that case, there are five unique tetrominoes. However, due to the overwhelming association of tetrominoes with *Tetris*, which uses *one-sided* tetrominoes (making reflections distinct but all rotations congruent), people recognize seven distinct tetrominoes:

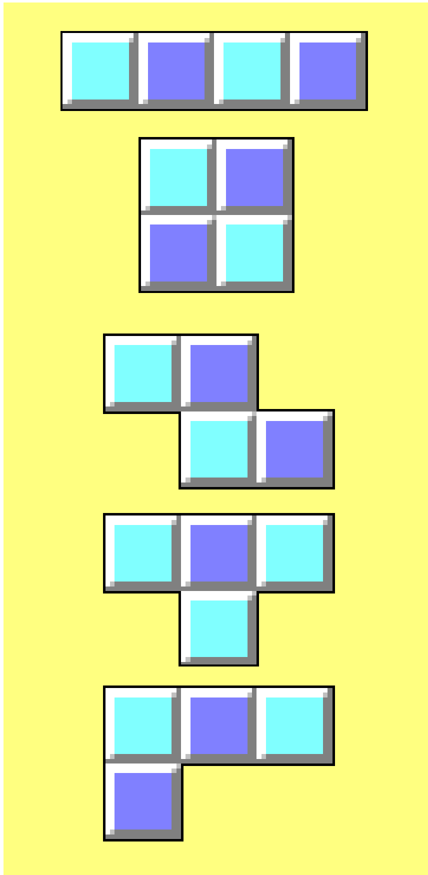
-  I (also called "stick", "straight", "long", "line"): four blocks in a straight line
-  J (also called "inverted L" or "Gamma"): a row of three blocks with one added below the right side.
  -  L (also called "gun"[3]): a row of three blocks with one added below the left side. This piece is a reflection of J but cannot be rotated into J in two dimensions; this is an example of chirality. However, in three dimensions, this piece is identical to J.
-  O (also called "square",[3] "package", "block"): four blocks in a 2×2 square.
-  S (also called "inverted N", "reverse squiggly", "s-zigzag"): two stacked horizontal dominoes with the top one offset to the right
  -  Z (also called "N", "skew", "snake",[3] "squiggly", "z-zigzag"): two stacked horizontal dominoes with the top one offset to the left. The same symmetry properties as with L and J apply with S and Z.
-  T: a row of three blocks with one added below the center.

The *free tetrominoes* additionally treat reflection (rotation in the third dimension) as equivalent. This eliminates J and Z, leaving five free tetrominoes: I, L, O, S (also called N or Z), and T.

The *fixed tetrominoes* do not allow rotation or reflection. There are 2 distinct fixed I tetrominoes, four J, four L, one O, two S, four T, and two Z, for a total of 19 fixed tetrominoes.

Tiling the rectangle and filling the box with 2D pieces

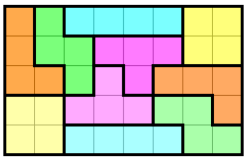
Although a complete set of free tetrominoes has a total of 20 squares, and a complete set of one-sided tetrominoes has 28 squares, it is not possible to pack them into a rectangle, like hexominoes and unlike pentominoes. The proof is that a rectangle covered with a checkerboard pattern will have 10 or 14 each of light and dark squares, while a complete set of free tetrominoes (pictured) has 11 light squares and 9 dark squares, and a complete set of one-sided tetrominoes has 15 light squares and 13 dark squares.



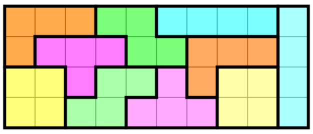
The five free tetrominoes, top to bottom I, O, Z, T, L, marked with light and dark squares.

A bag including two of each free tetromino, which has a total area of 40 squares, can fit in 4×10 and 5×8 cell rectangles. Likewise, two sets of one-sided tetrominoes can be fit to a rectangle in more than one way. The corresponding tetracubes can also fit in 2×4×5 and 2×2×10 boxes.

5×8 rectangle



4×10 rectangle



2×4×5 box

```
 layer 1     :     layer 2

Z Z T t I    :    l T T T i
L Z Z t I    :    l l l t i
L z z t I    :    o o z z i
L L O O I    :    o o O O i
```

2×2×10 box

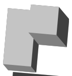
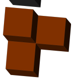
```
      layer 1            :           layer 2

L L L z z Z Z T O O    :    o o z z Z Z T T T l
L I I I I t t t O O    :    o o i i i i t l l l
```

Etymology

The name "tetromino" is a combination of the prefix *tetra-* "four" (from Ancient Greek τετρα-), and "domino".

Tetracubes

Each tetromino has a corresponding tetracube, which is the tetromino extruded by one unit. Three more tetracubes are possible, all created by placing a unit cube on the bent tricube:

-  Left screw: unit cube placed on top of anticlockwise side. Chiral in 3D.
-  Right screw: unit cube placed on top of clockwise side. Chiral in 3D.
-  Branch: unit cube placed on bend. Not chiral in 3D.

However, going to three dimensions means that rotation is allowed in three dimensions. Thus, the two L-shaped pieces are now equivalent, as are the two S-shaped pieces.

Filling the box with 3D pieces

In 3D, these eight tetracubes (suppose each piece consists of 4 cubes, L and J are the same, Z and S are the same) can fit in a 4×4×2 or 8×2×2 box. The following is one of the solutions. D, S and B represent right screw, left screw and branch point, respectively:

4×4×2 box

```
layer 1  :  layer 2

S T T T  :  S Z Z B
S S T B  :  Z Z B B
O O L D  :  L L L D
O O D D  :  I I I I
```

8×2×2 box

```
    layer 1     :     layer 2

D Z Z L O T T T : D L L L O B S S
D D Z Z O B T S : I I I I O B B S
```

If chiral pairs (D and S) are considered as identical, remaining 7 pieces can fill 7×2×2 box. (C represents D or S.)

```
   layer 1    :    layer 2

L L L Z Z B B : L C O O Z Z B
C I I I I T B : C C O O T T T
```

See also

- Soma cube
- Tetris

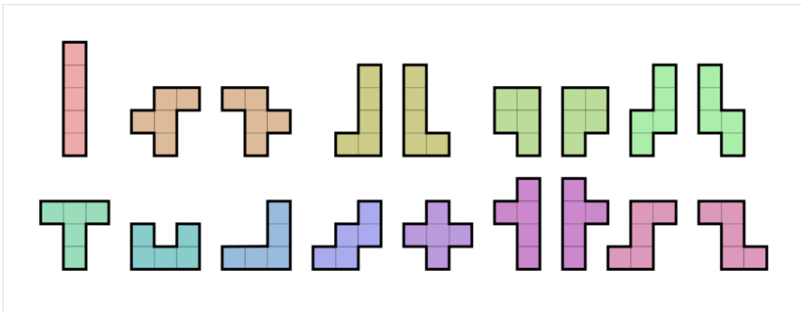
References

[1] Golomb, Solomon W. (1994). *Polyominoes* (2nd ed.). Princeton, New Jersey: Princeton University Press. ISBN 0–691–02444–8.
[2] Redelmeier, D. Hugh (1981). "Counting polyominoes: yet another attack". *Discrete Mathematics* **36**: 191–203. doi:10.1016/0012-365X(81)90237-5.
[3] Demaine, Hohenberger, and Liben-Nowell. Tetris Is Hard, Even to Approximate (http://www.cs.carleton.edu/faculty/dlibenno/papers/tetris/tetris-short.pdf).

External links

- Vadim Gerasimov, "Tetris: the story."; The story of Tetris (http://vadim.oversigma.com/Tetris.htm)
- The Father of Tetris (http://www.tetris-today.com/story/original-tetris0.shtml) (Web Archive copy of the page here (http://web.archive.org/web/20061202094148/http://www.tetris-today.com/story/original-tetris0.shtml))
- Open-source Tetrominoes game (http://code.google.com/p/tetrominoes/)

Pentomino



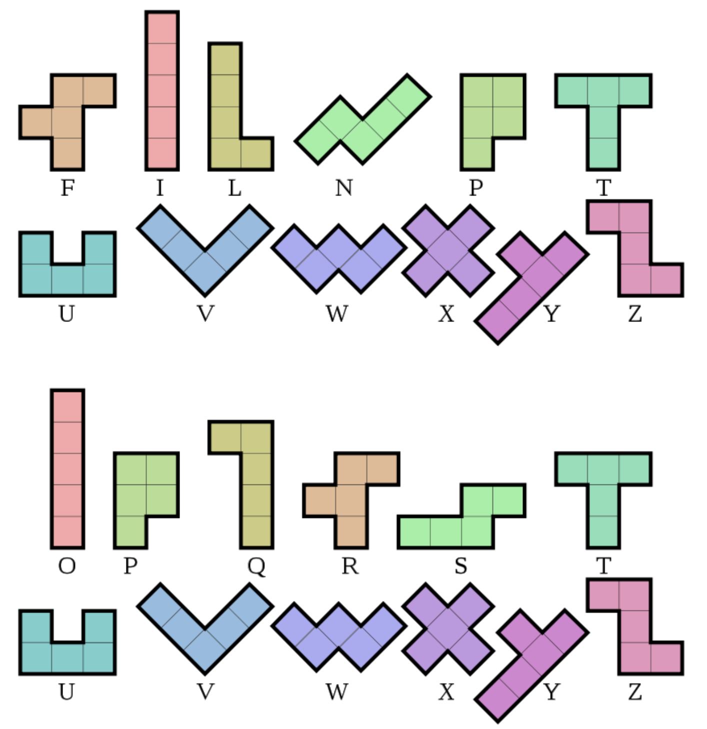
A **pentomino** is a polyomino composed of five (Ancient Greek πέντε / *pénte*) congruent squares, connected along their edges (which sometimes is said to be an orthogonal connection).

There are 12 different *free* pentominoes, often named after the letters of the Latin alphabet that they vaguely resemble. Ordinarily, the pentomino obtained by reflection or rotation of a pentomino does not count as a different pentomino.

The F, L, N, P, Y, and Z pentominoes are chiral in two dimensions; adding their reflections (F', J, N', Q, Y', S) brings the number of *one-sided* pentominoes to 18. The others, lettered I, T, U, V, W, and X, are equivalent to some rotation of their mirror images. This matters in some computer games, where mirror image moves are not allowed, such as Tetris-clones and Rampart.

Each of the twelve pentominoes can be tiled to fill the plane. In addition, each chiral pentomino can be tiled without using its reflection.

John Horton Conway proposed an alternate labeling scheme. He uses O instead of I, Q instead of L, R instead of F, and S instead of N. The resemblance to the letters is a bit more strained (most notably that the "O," a straight line, bears no resemblance to an actual letter O), but this scheme has the advantage that it uses 12 consecutive letters of the alphabet. This scheme is used in connection with Conway's Game of Life, so it talks about the R-pentomino instead of the F-pentomino.



Comparison of pentomino labeling schemes. The first naming convention is the one used in this article. The second method is Conway's.

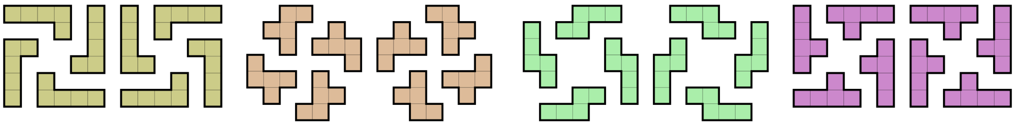
Symmetry

Considering rotations of multiples of 90 degrees only, there are the following symmetry categories:

- L, N, P, F and Y can be oriented in 8 ways: 4 by rotation, and 4 more for the mirror image. Their symmetry group consists only of the identity mapping.
- T, and U can be oriented in 4 ways by rotation. They have an axis of reflection symmetry aligned with the gridlines. Their symmetry group has two elements, the identity and the reflection in a line parallel to the sides of the squares.
- V and W also can be oriented in 4 ways by rotation. They have an axis of reflection symmetry at 45° to the gridlines. Their symmetry group has two elements, the identity and a diagonal reflection.
- Z can be oriented in 4 ways: 2 by rotation, and 2 more for the mirror image. It has point symmetry, also known as rotational symmetry of order 2. Its symmetry group has two elements, the identity and the 180° rotation.
- I can be oriented in 2 ways by rotation. It has two axes of reflection symmetry, both aligned with the gridlines. Its symmetry group has four elements, the identity, two reflections and the 180° rotation. It is the dihedral group of order 2, also known as the Klein four-group.
- X can be oriented in only one way. It has four axes of reflection symmetry, aligned with the gridlines and the diagonals, and rotational symmetry of order 4. Its symmetry group, the dihedral group of order 4, has eight elements.

If reflections of a pentomino are considered distinct, as they are with one-sided pentominoes, then the first and fourth categories above double in size, resulting in an extra 6 pentominoes for a total of 18. If rotations are also considered distinct, then the pentominoes from the first category count eightfold, the ones from the next three categories (T, U, V, W, Z) count fourfold, I counts twice, and X counts only once. This results in 5×8 + 5×4 + 2 + 1 = 63 *fixed* pentominoes.

For example, the eight possible orientations of the L, F, N, and Y pentominoes are as follows:

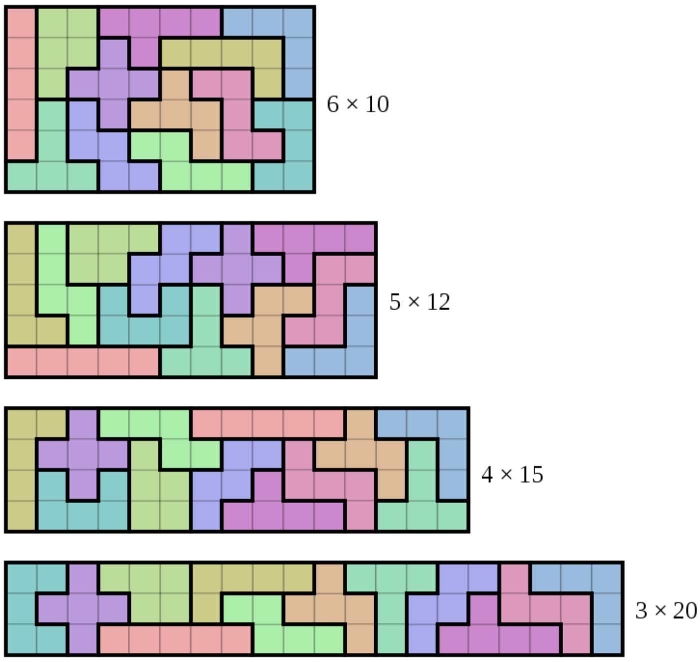


For 2D figures in general there are two more categories:

- Being orientable in 2 ways by a rotation of 90°, with two axes of reflection symmetry, both aligned with the diagonals. This type of symmetry requires at least a heptomino.
- Being orientable in 2 ways, which are each other's mirror images, for example a swastika. This type of symmetry requires at least an octomino.

Tiling rectangles

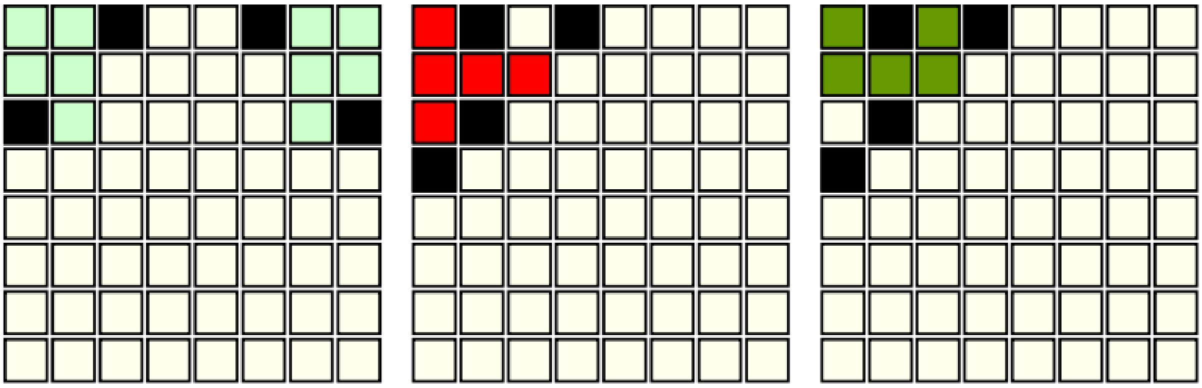
A standard **pentomino puzzle** is to tile a rectangular box with the pentominoes, i.e. cover it without overlap and without gaps. Each of the 12 pentominoes has an area of 5 unit squares, so the box must have an area of 60 units. Possible sizes are 6×10, 5×12, 4×15 and 3×20. The avid puzzler can probably solve these problems by hand within a few hours. A more challenging task, typically requiring a computer search, is to count the total number of solutions in each case.



The 6×10 case was first solved in 1960 by Colin Brian and Jenifer Haselgrove.[1] There are exactly 2339 solutions, excluding trivial variations obtained by rotation and reflection of the whole rectangle, but including rotation and reflection of a subset of pentominoes (sometimes this is possible and provides in a simple way an additional solution; e.g., with the 3×20 solution shown, the other one is obtained by rotating a set of seven pentominoes. (Leave the U, X, P, I and V as they are and rotate the middle section, as a whole, around.))

The 5×12 box has 1010 solutions, the 4×15 box has 368 solutions, and the 3×20 box has just 2 solutions (one is shown in the figure and the other one explained in the text.)

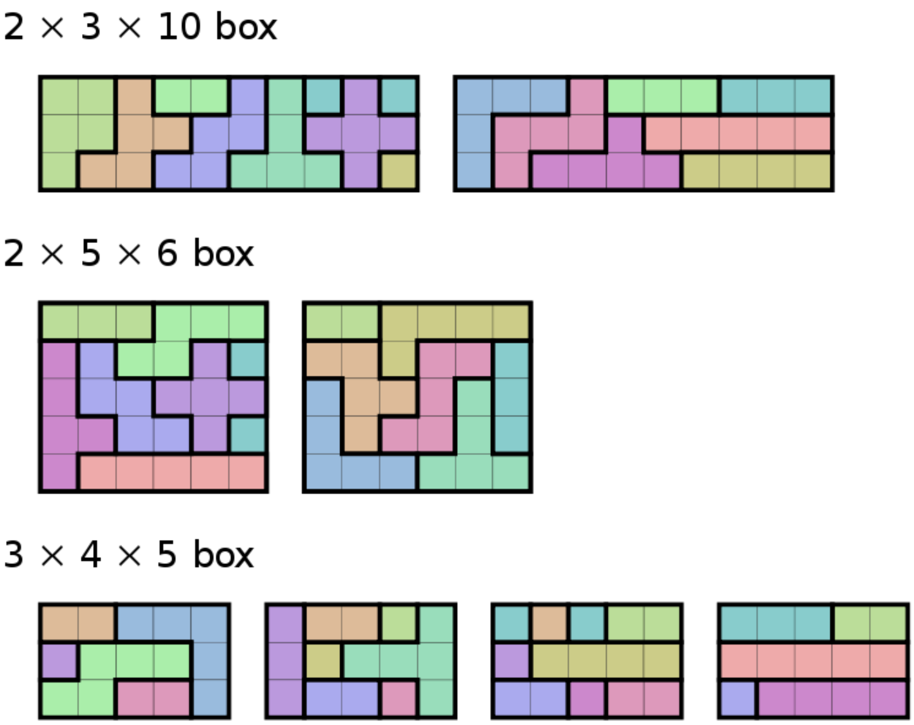
A somewhat easier (more symmetrical) puzzle, the 8×8 rectangle with a 2×2 hole in the center, was solved by Dana Scott as far back as 1958[2] . There are 65 solutions. Scott's algorithm was one of the first applications of a backtracking computer program. Variations of this puzzle allow the four holes to be placed in any position. One of the external links uses this rule. Most such patterns are solvable, with the exceptions of placing each pair of holes near two corners of the board in such a way that both corners could only be fitted by a P-pentomino, or forcing a T-pentomino or U-pentomino in a corner such that another hole is created.



Efficient algorithms have been described to solve such problems, for instance by Donald Knuth.[3] Running on modern hardware, these pentomino puzzles can now be solved in mere seconds.

Filling boxes

A **pentacube** is a polycube of five cubes. Twelve of the 29 pentacubes correspond to the twelve pentominoes extruded to a depth of one square. A **pentacube puzzle** or 3D **pentomino puzzle**, amounts to filling a 3-dimensional box with these 1-layer pentacubes, i.e. cover it without overlap and without gaps. Each of the 12 pentacubes consists of 5 unit cubes, and are like 2D pentominoes but with unit thickness. Clearly the box must have a volume of 60 units. Possible sizes are 2×3×10, 2×5×6 and 3×4×5. Following are several solutions.



Alternatively one could also consider combinations of five cubes which are themselves 3D, i.e., are not part of one layer of cubes. However, in addition to the 12 extruded pentominoes, 6 sets of chiral pairs and 5 pieces make total 29 pieces, resulting 145 cubes, which will not make a 3D box.

Board game

There are board games of skill based entirely on pentominoes, called **pentominoes**.

One of the games is played on an 8×8 grid by two or three players. Players take turns in placing pentominoes on the board so that they do not overlap with existing tiles and no tile is used more than once. The objective is to be the last player to place a tile on the board.

The two-player version has been weakly solved in 1996 by Hilarie Orman. It was proved to be a first-player win by examining around 22 billion board positions [4]

Pentominoes, and similar shapes, are also the basis of a number of other tiling games, patterns and puzzles. For example, a French board game called Blokus is played with 4 opposing color sets of polyominoes. In Blokus, each color begins with every pentomino (12), as well as every tetromino (5), every triomino (2), every domino (1) , and every monomino (1). Like the game Pentominoes, the goal is to use all of your tiles, and a bonus is given if the monomino is played on the very last move. The player with the fewest blocks remaining wins.

Parker Brothers released a multi-player pentomino board game called Universe in 1966. Its theme is based on an outtake from the movie 2001: A Space Odyssey in which the astronaut (seen playing chess in the final version) is playing a two-player pentomino game against a computer. The front of the board game box features scenes from the movie as well as a caption describing it as the "game of the future". The game comes with 4 sets of pentominoes (in red, yellow, blue, and white). The board has two playable areas: a base 10x10 area for two players with an additional 25 squares (two more rows of 10 and one offset row of 5) on each side for more than two players.

The second manufacturer of a Pentomino based game is Lonpos. Lonpos has a number of games that uses the same Pentominoes, but on different game planes. The so-called 101 game has a 5 x 11 plane. By changing the shape of the plane, thousands of puzzles can be played (although only a relatively small selection of these puzzles are available in print).

Literature

Pentominoes were featured in a prominent subplot of Arthur C. Clarke's novel *Imperial Earth*, published in 1975. They were also featured in Blue Balliett's *Chasing Vermeer*, which was published in 2003 and illustrated by Brett Helquist, as well as its sequels, *The Wright 3* and *The Calder Game*.

Video games

- Lojix on the *ZX Spectrum* is clearly derived from pentomino, though it uses a non-standard set of 20 blocks and a 10*10 box. Released in late 1983, the game was marketed via the announcement of a cash prize for the first person to solve the puzzle.
- *Tetris* was inspired by pentomino puzzles, although it uses four-block tetrominoes. Some Tetris clones and variants, like the `games/5s` of Plan 9 from Bell Labs, and *Magical Tetris Challenge*, do use pentominoes.
- *Daedalian Opus* uses pentomino puzzles throughout the game.
- *Yohoho! Puzzle Pirates* carpentry minigame is based on pentomino puzzles.
- *Chime* uses pentominoes for its pieces.

See also

- Lonpos
- Tiling puzzle

Notes

[1] C. B. Haselgrove; Jenifer Haselgrove (October 1960). "A Computer Program for Pentominoes". *Eureka* **23**: 16–18.
[2] Dana S. Scott (1958). "Programming a combinatorial puzzle". Technical Report No. 1, Department of Electrical Engineering, Princeton University.
[3] Donald E. Knuth. "Dancing links" (http://www-cs-faculty.stanford.edu/~knuth/papers/dancing-color.ps.gz) (Postscript, 1.6 megabytes). Includes a summary of Scott's and Fletcher's articles.
[4] Hilarie K. Orman. Pentominoes: A First Player Win (http://www.msri.org/publications/books/Book29/files/orman.pdf) (Pdf).

References

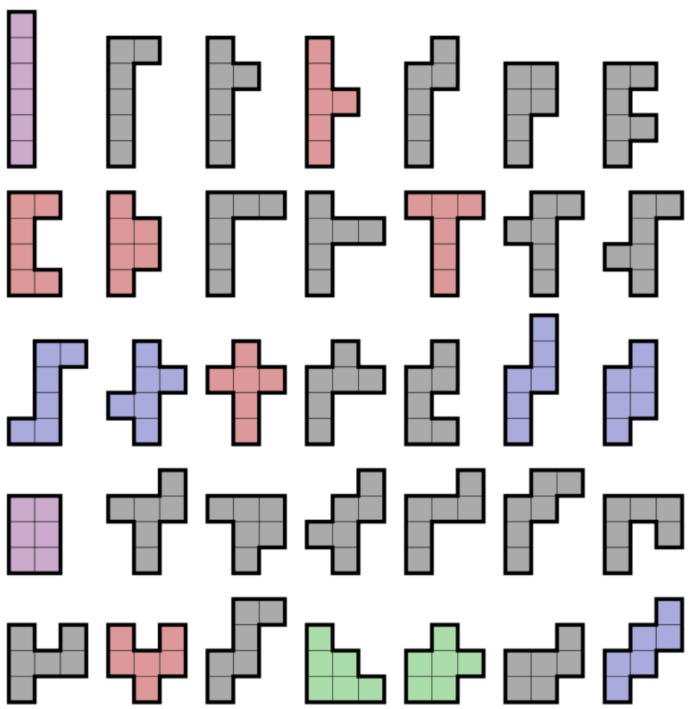
- Chasing Vermeer (http://www.scholastic.com/chasingvermeer), with information about the book Chasing Vermeer and a click-and-drag pentomino board.

External links

- Pentomino configurations and solutions (http://isomerdesign.com/Pentomino) An exhaustive listing of solutions to many of the classic problems showing how each solution relates to the others.

Hexomino

A **hexomino** (or **6-omino**) is a polyomino of order 6, that is, a polygon in the plane made of 6 equal-sized squares connected edge-to-edge.[1] The name of this type of figure is formed with the prefix hex(a)-. When rotations and reflections are not considered to be distinct shapes, there are 35 different *free* hexominoes. When reflections are considered distinct, there are 60 *one-sided* hexominoes. When rotations are also considered distinct, there are 216 *fixed* hexominoes.[2] [3]



The 35 free hexominoes

Symmetry

The figure shows all possible free hexominoes, coloured according to their symmetry groups:

- 20 hexominoes (coloured grey) have no symmetry. Their symmetry group consists only of the identity mapping.
- 6 hexominoes (coloured red) have an axis of mirror symmetry aligned with the gridlines. Their symmetry group has two elements, the identity and a reflection in a line parallel to the sides of the squares.

- 2 hexominoes (coloured green) have an axis of mirror symmetry at 45° to the gridlines. Their symmetry group has two elements, the identity and a diagonal reflection.
- 5 hexominoes (coloured blue) have point symmetry, also known as rotational symmetry of order 2. Their symmetry group has two elements, the identity and the 180° rotation.
- 2 hexominoes (coloured purple) have two axes of mirror symmetry, both aligned with the gridlines. Their symmetry group has four elements. It is the dihedral group of order 2, also known as the Klein four-group.

If reflections of a hexomino are considered distinct, as they are with one-sided hexominoes, then the first and fourth categories above would each double in size, resulting in an extra 25 hexominoes for a total of 60. If rotations are also considered distinct, then the hexominoes from the first category count eightfold, the ones from the next three categories count fourfold, and the ones from the last category count twice. This results in 20 × 8 + (6+2+5) × 4 + 2 × 2 = 216 fixed hexominoes.

Packing and tiling

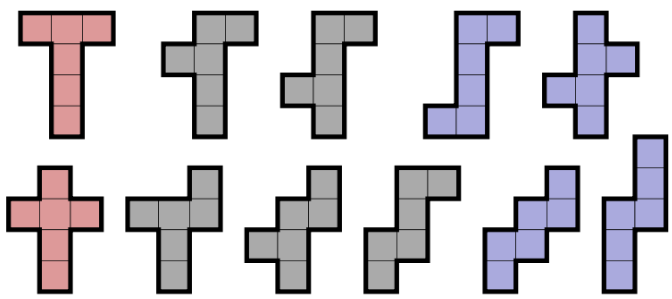
Although a complete set of 35 hexominoes has a total of 210 squares, it is not possible to pack them into a rectangle. (Such an arrangement is possible with the 12 pentominoes which can be packed into any of the rectangles 3 × 20, 4 × 15, 5 × 12 and 6 × 10.) A simple way to demonstrate that such a packing of hexominoes is not possible is via a parity argument. If the hexominoes are placed on a checkerboard pattern, then 11 of the hexominoes will cover an even number of black squares (either 2 white and 4 black or vice-versa) and 24 of the hexominoes will cover an odd number of black squares (3 white and 3 black). Overall, an even number of black squares will be covered in any arrangement. However, any rectangle of 210 squares will have 105 black squares and 105 white squares.

However, there are other simple figures of 210 squares that can be packed with the hexominoes. For example, a 15 × 15 square with a 3 × 5 rectangle removed from the centre has 210 squares. With checkerboard colouring, it has 106 white and 104 black squares (or vice versa), so parity does not prevent a packing, and a packing is indeed possible.[4] Also, it is possible for two sets of pieces to fit a rectangle of size 420.

Each of the 35 hexominoes is capable of tiling the plane.

Polyhedral nets for the cube

A polyhedral net for the cube is necessarily a hexomino, with 11 hexominoes actually being nets. They appear on the right, again coloured according to their symmetry groups.



All 11 unfoldings of the cube

References

[1] Golomb, Solomon W. (1994). *Polyominoes* (2nd ed.). Princeton, New Jersey: Princeton University Press. ISBN 0–691–02444–8.

[2] Weisstein, Eric W. "Hexomino" (http://mathworld.wolfram.com/Hexomino.html). From MathWorld – A Wolfram Web Resource. . Retrieved 2008-07-22.

[3] Redelmeier, D. Hugh (1981). "Counting polyominoes: yet another attack". *Discrete Mathematics* **36**: 191–203. doi:10.1016/0012-365X(81)90237-5.

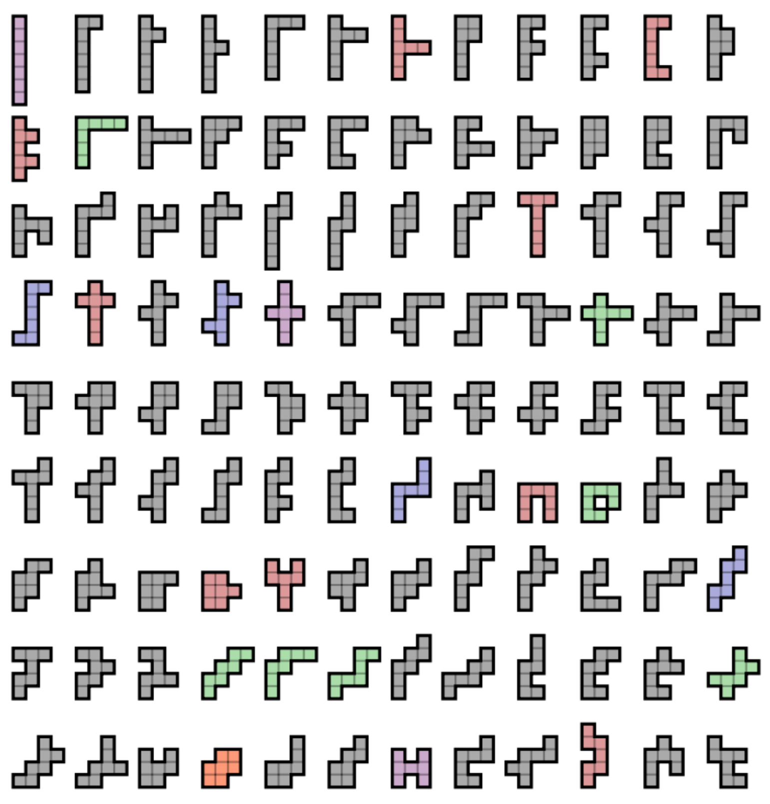
[4] Mathematische Basteleien: Hexominos (http://www.mathematische-basteleien.de/hexominos.htm) **(English)**

External links

- Page by Jürgen Köller on hexominoes, including symmetry, packing and other aspects (http://www.mathematische-basteleien.de/hexominos.htm)
- Polyomino page (http://www.ics.uci.edu/~eppstein/junkyard/polyomino.html) of David Eppstein's *Geometry Junkyard* (http://www.ics.uci.edu/~eppstein/junkyard/)
- Eleven animations showing the patterns of the cube (http://perso.wanadoo.fr/therese.eveilleau/pages/truc_mat/textes/cube_patrons.htm) **(French)**
- Polypolygon tilings (http://www.uwgb.edu/dutchs/symmetry/polypoly.htm), Steven Dutch.

Heptomino

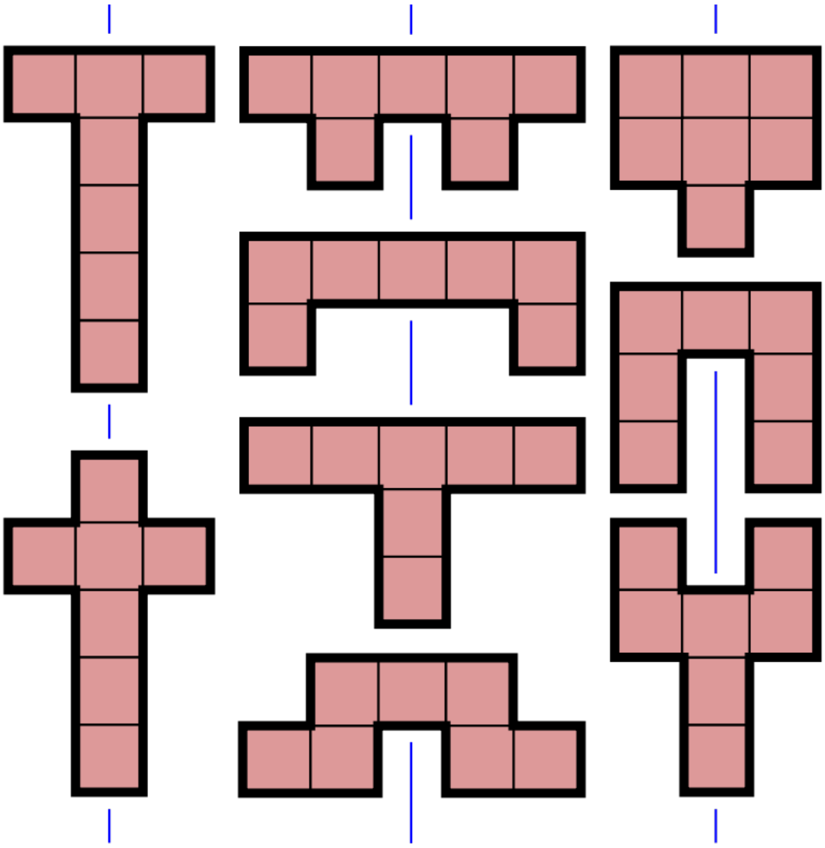
A **heptomino** (or **7-omino**) is a polyomino of order 7, that is, a polygon in the plane made of 7 equal-sized squares connected edge-to-edge.[1] The name of this type of figure is formed with the prefix hept(a)-. When rotations and reflections are not considered to be distinct shapes, there are 108 different *free* heptominoes. When reflections are considered distinct, there are 196 *one-sided* heptominoes. When rotations are also considered distinct, there are 760 *fixed* heptominoes.[2] [3]


The 108 free heptominoes

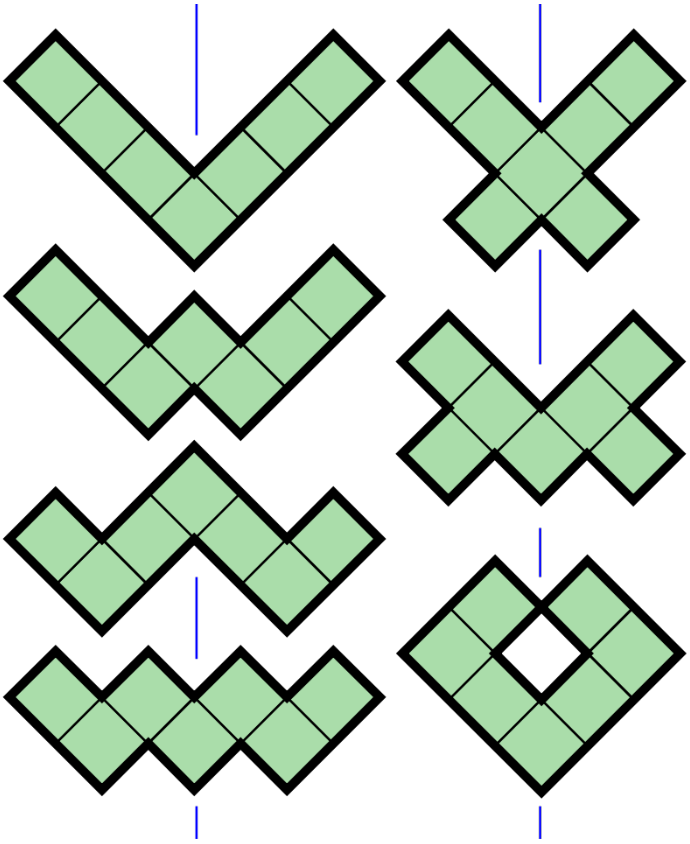
Symmetry

The figure shows all possible free heptominoes, coloured according to their symmetry groups:

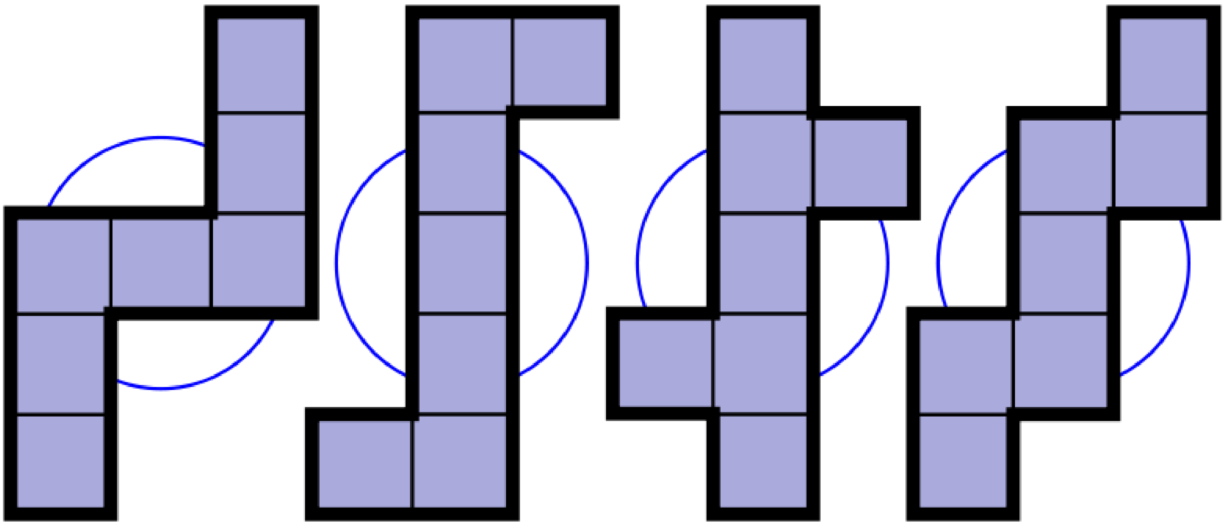
- 84 heptominoes (coloured grey) have no symmetry. Their symmetry group consists only of the identity mapping.
- 9 heptominoes (coloured red) have an axis of reflection symmetry aligned with the gridlines. Their symmetry group has two elements, the identity and the reflection in a line parallel to the sides of the squares.



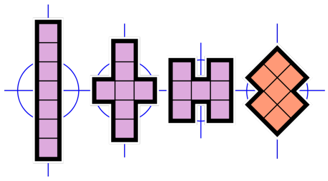
- 7 heptominoes (coloured green) have an axis of reflection symmetry at 45° to the gridlines. Their symmetry group has two elements, the identity and a diagonal reflection.



- 4 heptominoes (coloured blue) have point symmetry, also known as rotational symmetry of order 2. Their symmetry group has two elements, the identity and the 180° rotation.



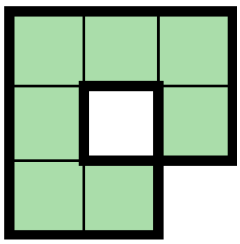
- 3 heptominoes (coloured purple) have two axes of reflection symmetry, both aligned with the gridlines. Their symmetry group has four elements, the identity, two reflections and the 180° rotation. It is the dihedral group of order 2, also known as the Klein four-group.
- 1 heptomino (coloured orange) has two axes of reflection symmetry, both aligned with the diagonals. Its symmetry group also has four elements. Its symmetry group is also the dihedral group of order 2 with four elements.



If reflections of a heptomino are considered distinct, as they are with one-sided heptominoes, then the first and fourth categories above would each double in size, resulting in an extra 88 heptominoes for a total of 196. If rotations are also considered distinct, then the heptominoes from the first category count eightfold, the ones from the next three categories count fourfold, and the ones from the last two categories count twice. This results in $84 \times 8 + (9+7+4) \times 4 + (3+1) \times 2 = 760$ fixed heptominoes.

Packing and tiling

Although a complete set of 108 heptominoes has a total of 756 squares, it is not possible to pack them into a rectangle. The proof of this is trivial, since there is one heptomino which has a hole.[4]


Heptomino with hole

All but four heptominoes are capable of tiling the plane; the one with a hole is one such example.[5] In fact, under some definitions, figures such as this are not considered to be polyominoes because they are not topological disks.

References

[1] Golomb, Solomon W. (1994). *Polyominoes* (2nd ed.). Princeton, New Jersey: Princeton University Press. ISBN 0–691–02444–8.

[2] Weisstein, Eric W. "Heptomino" (http://mathworld.wolfram.com/Heptomino.html). From MathWorld – A Wolfram Web Resource. . Retrieved 2008-07-22.

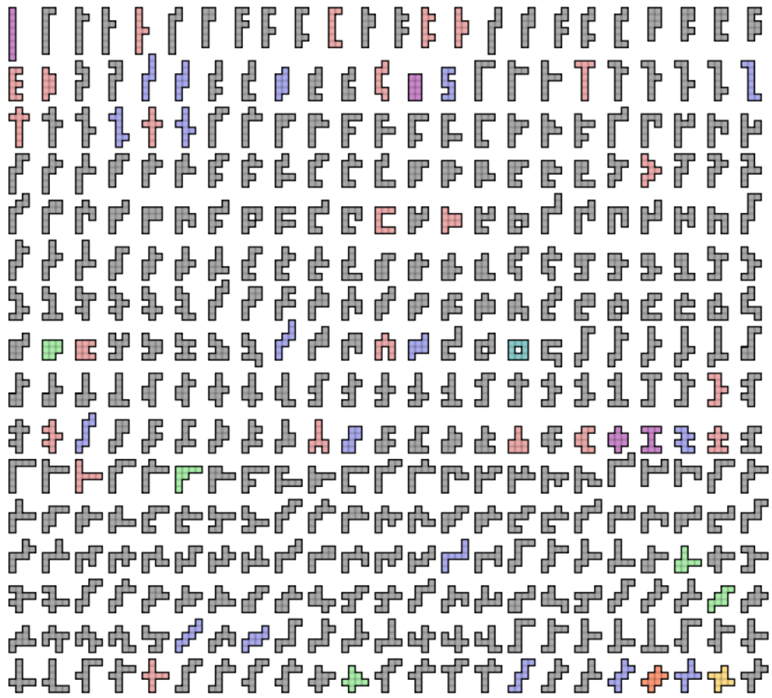
[3] Redelmeier, D. Hugh (1981). "Counting polyominoes: yet another attack". *Discrete Mathematics* **36**: 191–203. doi:10.1016/0012-365X(81)90237-5.

[4] Grünbaum, Branko; Shephard, G. C. (1987). *Tilings and Patterns*. New York: W. H. Freeman and Company. ISBN 0-7167-1193-1.

[5] Gardner, Martin (August 1965). "Thoughts on the task of communication with intelligent organisms on other worlds". *Scientific American* **213** (2): 96–100.

Octomino

An **octomino** (or **8-omino**) is a polyomino of order 8, that is, a polygon in the plane made of 8 equal-sized squares connected edge-to-edge.[1] The name of this type of figure is formed with the prefix oct(a)-. When rotations and reflections are not considered to be distinct shapes, there are 369 different *free* octominoes. When reflections are considered distinct, there are 704 *one-sided* octominoes. When rotations are also considered distinct, there are 2,725 *fixed* octominoes.[2] [3]

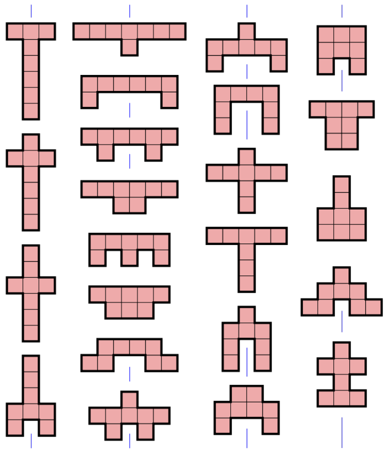


The 369 free octominoes

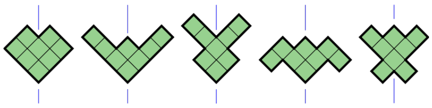
Symmetry

The figure shows all possible free octominoes, coloured according to their symmetry groups:

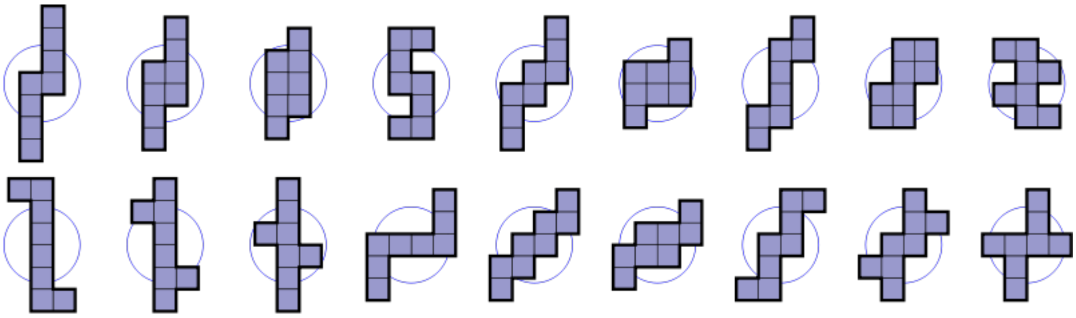
- 316 octominoes (coloured grey) have no symmetry. Their symmetry group consists only of the identity mapping.
- 23 octominoes (coloured red) have an axis of reflection symmetry aligned with the gridlines. Their symmetry group has two elements, the identity and the reflection in a line parallel to the sides of the squares.



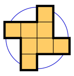
- 5 octominoes (coloured green) have an axis of reflection symmetry at 45° to the gridlines. Their symmetry group has two elements, the identity and a diagonal reflection.



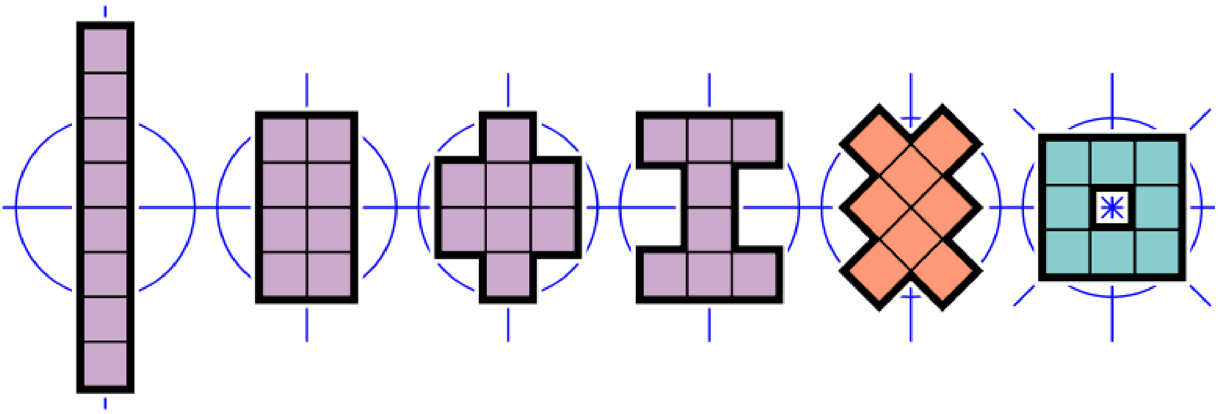
- 18 octominoes (coloured blue) have point symmetry, also known as rotational symmetry of order 2. Their symmetry group has two elements, the identity and the 180° rotation.



- 1 octomino (coloured yellow) has rotational symmetry of order 4. Its symmetry group has four elements, the identity and the 90°, 180° and 270° rotations.



- 4 octominoes (coloured purple) have two axes of reflection symmetry, both aligned with the gridlines. Their symmetry group has four elements, the identity, two reflections and the 180° rotation. It is the dihedral group of order 2, also known as the Klein four-group.
- 1 octomino (coloured orange) has two axes of reflection symmetry, both aligned with the diagonals. Its symmetry group is also the dihedral group of order 2 with four elements.
- 1 octomino (coloured blue-green) has four axes of reflection symmetry, aligned with the gridlines and the diagonals, and rotational symmetry of order 4. Its symmetry group, the dihedral group of order 4, has eight elements.

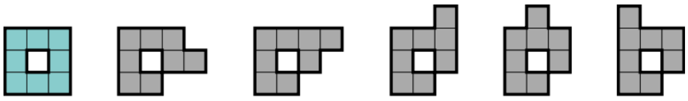


The set of octominoes is the lowest polyomino set in which all eight possible symmetries are realized. The next higher set with this property is the dodecomino (12-omino) set.[3]

If reflections of an octomino are considered distinct, as they are with one-sided octominoes, then the first, fourth and fifth categories above double in size, resulting in an extra 335 octominoes for a total of 704. If rotations are also considered distinct, then the octominoes from the first category count eightfold, the ones from the next three categories count fourfold, the ones from categories five to seven count twice, and the last octomino counts only once. This results in 316 × 8 + (23+5+18) × 4 + (1+4+1) × 2 + 1 = 2,725 fixed octominoes.

Packing and tiling

6 octominoes have a hole. This makes it trivial to prove that the complete set of octominoes cannot be packed into a rectangle, and that not all octominoes can be tiled. However, it has been proven that 343 free octominoes, or all but 26, do tile the plane.[4]



Under some definitions, figures with holes are not considered to be polyominoes because they are not topological disks.

References

[1] Golomb, Solomon W. (1994). *Polyominoes* (2nd ed.). Princeton, New Jersey: Princeton University Press. ISBN 0–691–02444–8.

[2] Weisstein, Eric W. "Octomino" (http://mathworld.wolfram.com/Octomino.html). From MathWorld – A Wolfram Web Resource. . Retrieved 2008-07-22.

[3] Redelmeier, D. Hugh (1981). "Counting polyominoes: yet another attack". *Discrete Mathematics* **36**: 191–203. doi:10.1016/0012-365X(81)90237-5.

[4] Gardner, Martin (August 1975). "More about tiling the plane: the possibilities of polyominoes, polyiamonds and polyhexes". *Scientific American* **233** (2): 112–115.

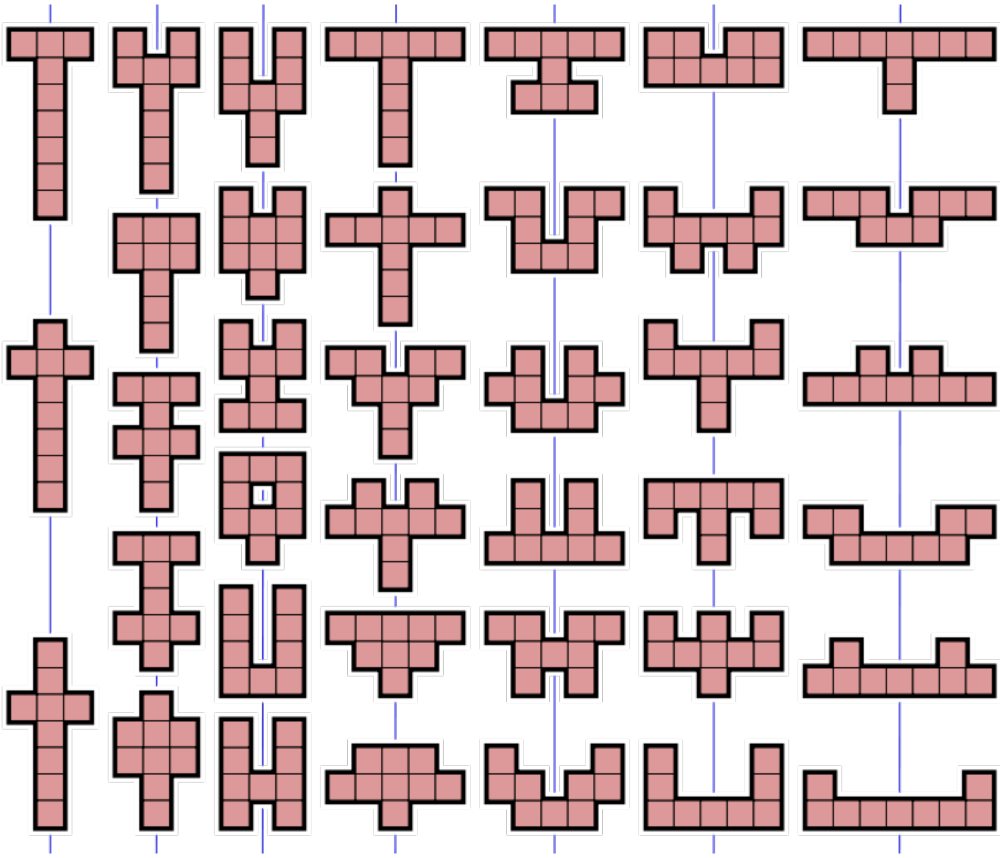
Nonomino

A **nonomino** (**enneomino** or **9-omino**) is a polyomino of order 9, that is, a polygon in the plane made of 9 equal-sized squares connected edge-to-edge.[1] The name of this type of figure is formed with the prefix non(a)-. When rotations and reflections are not considered to be distinct shapes, there are 1,285 different *free* nonominoes. When reflections are considered distinct, there are 2,500 *one-sided* nonominoes. When rotations are also considered distinct, there are 9,910 *fixed* nonominoes.[2]

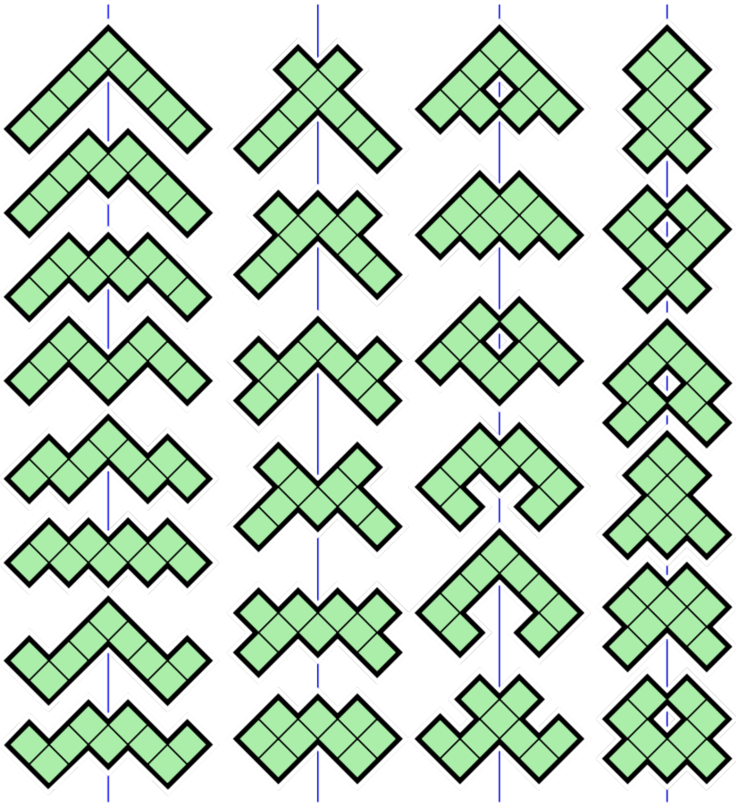
Symmetry

The 1,285 free nonominoes can be classified according to their symmetry groups:[2]

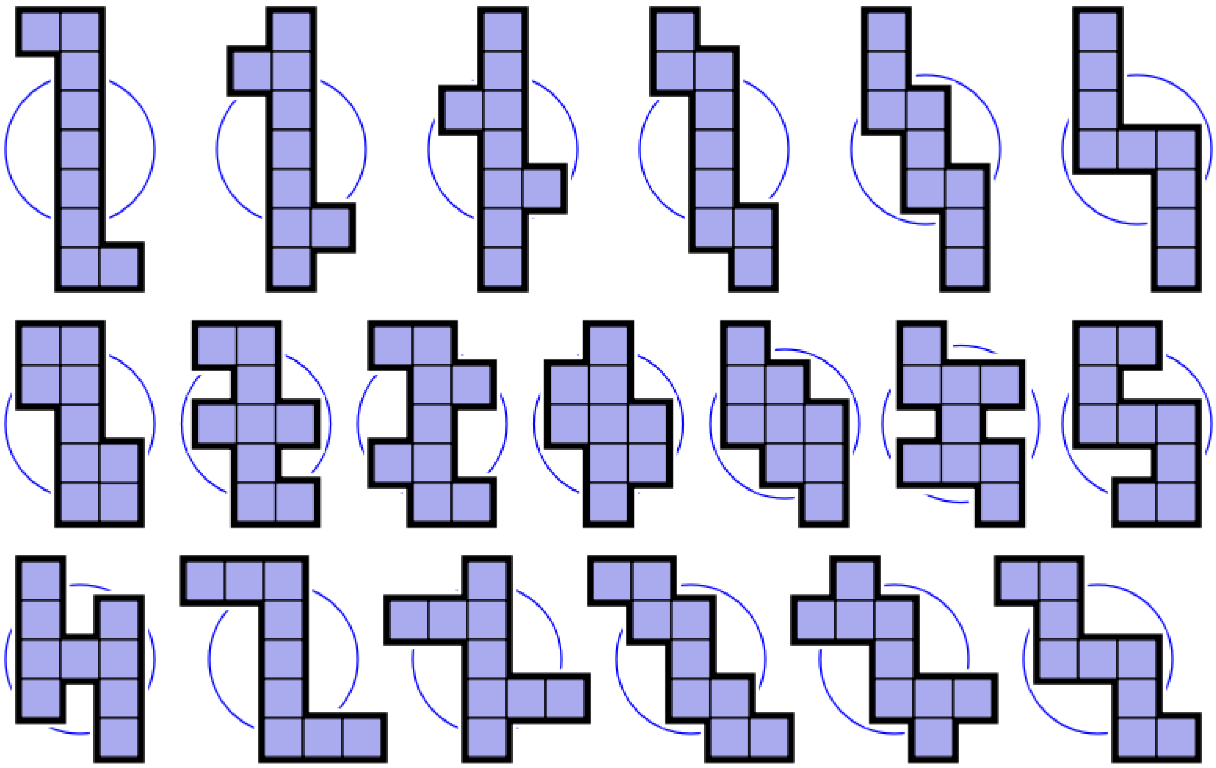
- 1,196 nonominoes have no symmetry. Their symmetry group consists only of the identity mapping.
- 38 nonominoes have an axis of reflection symmetry aligned with the gridlines. Their symmetry group has two elements, the identity and the reflection in a line parallel to the sides of the squares.



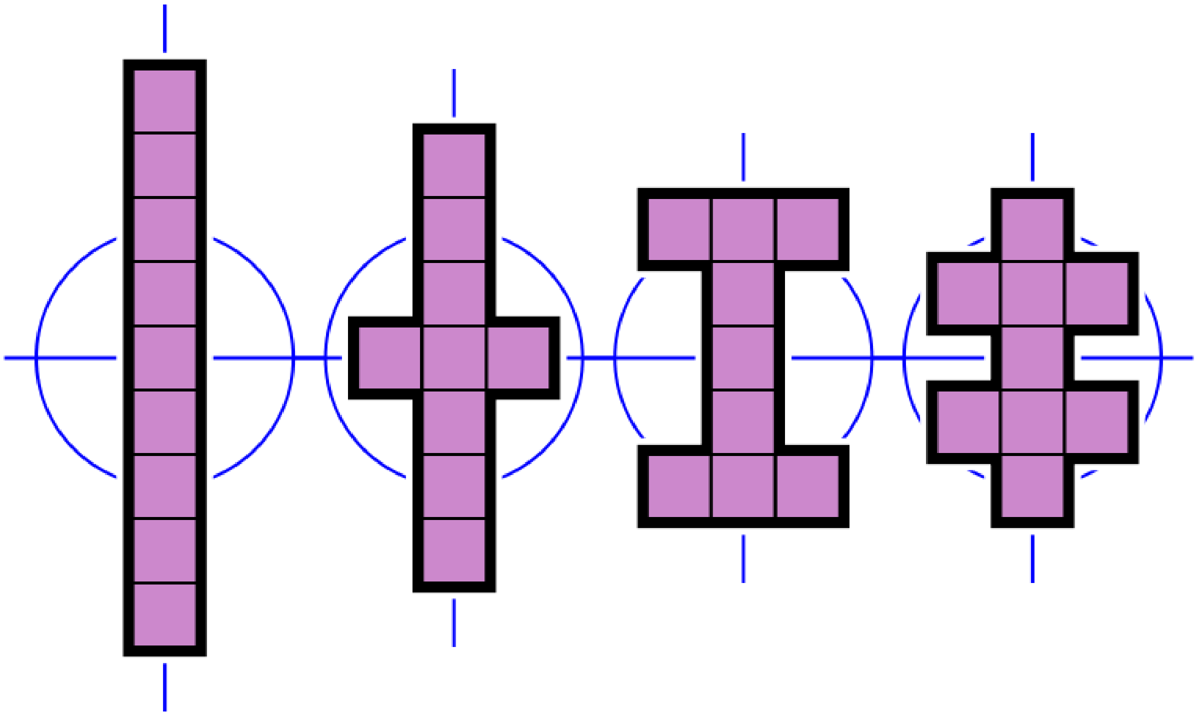
- 26 nonominoes have an axis of reflection symmetry at 45° to the gridlines. Their symmetry group has two elements, the identity and a diagonal reflection.



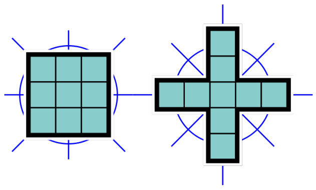
- 19 nonominoes have point symmetry, also known as rotational symmetry of order 2. Their symmetry group has two elements, the identity and the 180° rotation.



- 4 nonominoes have two axes of reflection symmetry, both aligned with the gridlines. Their symmetry group has four elements, the identity, two reflections and the 180° rotation. It is the dihedral group of order 2, also known as the Klein four-group.



- 2 nonominoes have four axes of reflection symmetry, aligned with the gridlines and the diagonals, and rotational symmetry of order 4. Their symmetry group, the dihedral group of order 4, has eight elements.

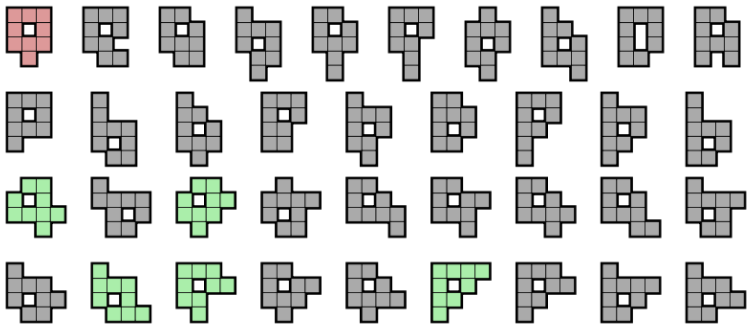


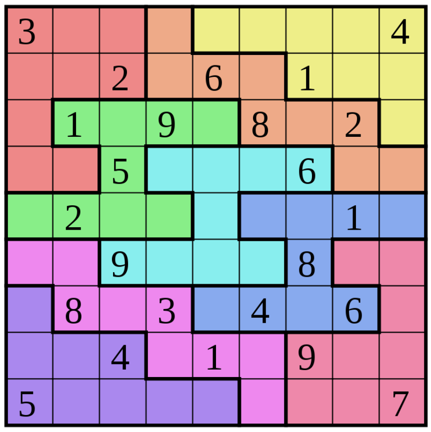
Unlike octominoes, there are no nonominoes with rotational symmetry of order 4 or with two axes of reflection symmetry aligned with the diagonals.

If reflections of a nonomino are considered distinct, as they are with one-sided nonominoes, then the first and fourth categories above double in size, resulting in an extra 1,215 nonominoes for a total of 2,500. If rotations are also considered distinct, then the nonominoes from the first category count eightfold, the ones from the next three categories count fourfold, the ones from the fifth category count twice, and the ones from the last category count only once. This results in 1,196 × 8 + (38+26+19) × 4 + 4 × 2 + 2 = 9,910 fixed nonominoes.

Packing and tiling

37 nonominoes have a hole.[3] [4] This makes it trivial to prove that the complete set of nonominoes cannot be packed into a rectangle, and that not all nonominoes can be tiled. However, it has been proven that 1,050 free nonominoes, or all but 235, do tile the plane.[5]





A nonomino or Jigsaw Sudoku puzzle, as seen in the *Sunday Telegraph*

References

[1] Golomb, Solomon W. (1994). *Polyominoes* (2nd ed.). Princeton, New Jersey: Princeton University Press. ISBN 0–691–02444–8.

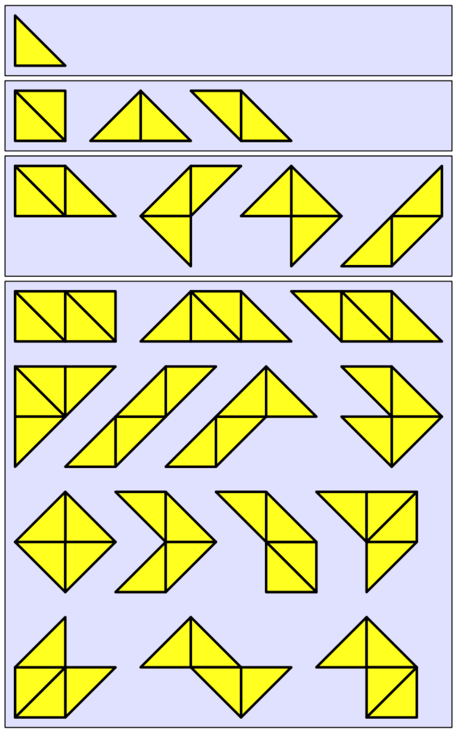
[2] Redelmeier, D. Hugh (1981). "Counting polyominoes: yet another attack". *Discrete Mathematics* **36**: 191–203. doi:10.1016/0012-365X(81)90237-5.

[3] Weisstein, Eric W. "Polyomino" (http://mathworld.wolfram.com/Polyomino.html). From MathWorld – A Wolfram Web Resource. . Retrieved 2009-12-05.

[4] Sequence A001419 (http://en.wikipedia.org/wiki/Oeis:a001419) in the On-Line Encyclopedia of Integer Sequences (OEIS)

[5] Rawsthorne, Daniel A. (1988). "Tiling complexity of small *n*-ominoes (*n*<10)". *Discrete Mathematics* **70**: 71–75. doi:10.1016/0012-365X(88)90081-7.

Polyabolo



In recreational mathematics, a **polyabolo** (also known as a **polytan**) is a polyform with an isosceles right triangle as the base form.[1]

The name is a back formation from 'diabolo' although the shape formed by joining two triangles at just one vertex is not a proper polyabolo. On a false analogy as if di- in diabolo means twice, polyaboloes with from 1 to 10 triangles are called respectively monoboloes, diaboloes, triaboloes, tetraboloes, pentaboloes, hexaboloes, heptaboloes, octaboloes, enneaboloes, and decaboloes.

There are two ways in which a square in a polyabolo can consist of two isosceles right triangles, but polyaboloes are considered equivalent if they have the same boundaries. The number of nonequivalent polyaboloes composed of 1, 2, 3, … triangles is 1, 3, 4, 14, 30, 107, 318, 1116, 3743, … (sequence A006074 [2] in OEIS).

Polyaboloes that are confined strictly to the plane and cannot be turned over may be termed one-sided. The number of one-sided polyaboloes composed of 1, 2, 3, … triangles is 1, 4, 6, 22, 56, 198, 624, 2182, 7448, … (sequence A151519 [3] in OEIS).

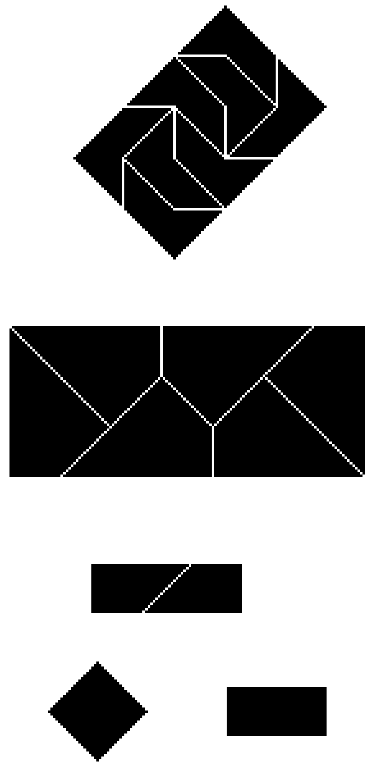
As for a polyomino, a polyabolo that can be neither turned over nor rotated may be termed fixed. A polyabolo with no symmetries (rotation or reflection) corresponds to 8 distinct fixed polyaboloes.

A non-simply connected polyabolo is one that has one or more holes in it. The smallest value of n for which an n-abolo is non-simply connected is 7.

Tiling rectangles with copies of a single polyabolo

In 1968, David A. Klarner defined the *order* of a polyomino. Similarly, the order of a polyabolo P can be defined as the minimum number of congruent copies of P that can be assembled (allowing translation, rotation, and reflection) to form a rectangle.

A polyabolo has order 1 if and only if it is itself a rectangle. Polyaboloes of order 2 are also easily recognisable. Solomon W. Golomb found polyaboloes, including a triabolo, of order 8[4] . Michael Reid found a heptabolo of order 6[5] .



Tiling rectangles with polyaboloes.

References

[1] Gardner, Martin (June 1967). "The polyhex and the polyabolo, polygonal jigsaw puzzle pieces". *Scientific American* **216** (6): 124–132.

[2] http://en.wikipedia.org/wiki/Oeis%3Aa006074

[3] http://en.wikipedia.org/wiki/Oeis%3Aa151519

[4] Golomb, Solomon W. (1994). *Polyominoes (2nd ed.)*. Princeton University Press. p. 101. ISBN 0-691-02444-8.

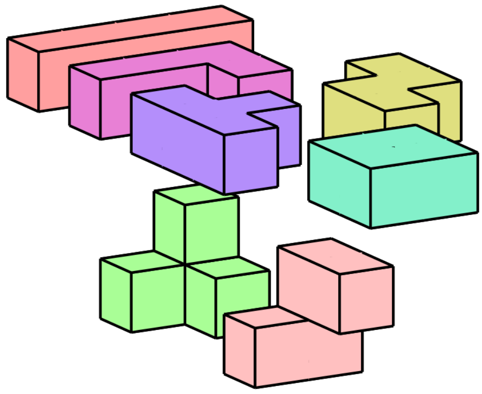
[5] Goodman, Jacob E.; O'Rourke, Joseph (2004). *Handbook of Discrete and Computational Geometry (2nd ed.)*. Chapman & Hall/CRC. p. 349. ISBN 1-58488-301-4.

External links

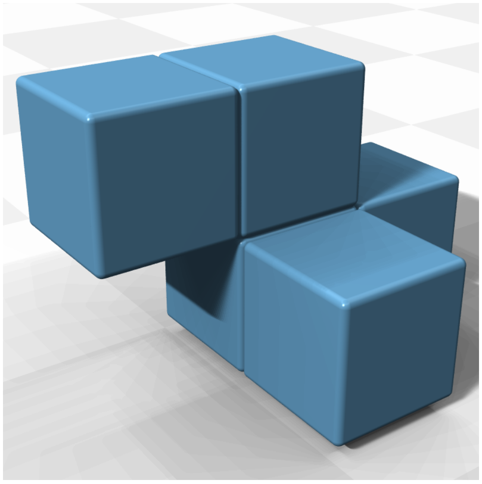
- Weisstein, Eric W., " Polyabolo (http://mathworld.wolfram.com/Polyabolo.html)" from MathWorld.
- Weisstein, Eric W., " Triabolo (http://mathworld.wolfram.com/Triabolo.html)" from MathWorld.

Polycube

A **polycube** is a solid figure formed by joining one or more equal cubes face to face. It is a polyform whose base form is a cube. Polycubes are the three-dimensional analogues of the planar polyominoes. The Soma cube and the Bedlam cube are examples of packing problems based on polycubes.



The seven free tetracubes



A chiral pentacube

Enumerating polycubes

Like polyominoes, polycubes can be enumerated in two ways, depending on whether chiral pairs of polycubes are counted as one polycube or two. For example, 6 tetracubes have mirror symmetry and one is chiral, giving a count of 7 or 8 tetracubes respectively. Unlike polyominoes, polycubes are usually counted with mirror pairs distinguished, because one cannot turn a polycube over to reflect it as one can a polyomino. In particular, the Soma cube uses both forms of the chiral tetracube.

n	Name of *n*-polycube	Number of one-sided *n*-polycubes (reflections counted as distinct) (sequence A000162 [1] in OEIS)	Number of free *n*-polycubes (reflections counted together) (sequence A038119 [2] in OEIS)
1	monocube	1	1
2	dicube	1	1
3	tricube	2	2
4	tetracube	8	7
5	pentacube	29	23
6	hexacube	166	112
7	heptacube	1023	607
8	octocube	6922	3811

Kevin Gong has enumerated polycubes up to n=16. See the external links for a table of these results.

See also

- polyform
- polyhex
- polyomino
- polyominoid
- tessellation

External links

- Kevin Gong's enumeration of polycubes [3]
- Weisstein, Eric W., "Polycube [4]" from MathWorld.
- Polycubes [5], at The Poly Pages [6]
- Polycube Symmetries [7]

References

[1] http://en.wikipedia.org/wiki/Oeis%3Aa000162
[2] http://en.wikipedia.org/wiki/Oeis%3Aa038119
[3] http://kevingong.com/Polyominoes/Enumeration.html
[4] http://mathworld.wolfram.com/Polycube.html
[5] http://recmath.org/PolyPages/PolyPages/index.htm?Polycubes.html
[6] http://recmath.org/PolyPages/index.htm
[7] http://userpages.monmouth.com/~colonel/csym/index.html

Polydrafter

In recreational mathematics, a **polydrafter** is a polyform with a triangle as the base form. The triangle has angles of 30°, 60° and 90°, like a set square—hence the name. The polydrafter was invented by Christopher Monckton, whose Eternity Puzzle was composed of 209 hexadrafters.

The term *polydrafter* was coined by Ed Pegg, Jr. The 14 tridrafters — all possible clusters of three drafters — make an excellent small puzzle that is surprisingly difficult to solve by hand.

External links

- Weisstein, Eric W., "Polydrafter [1]" from MathWorld.

References

[1] http://mathworld.wolfram.com/Polydrafter.html

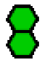
Polyhex

In recreational mathematics, a **polyhex** is a polyform with a regular hexagon (or 'hex' for short) as the base form.

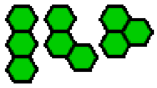
As with polyominoes, polyhexes may be enumerated as *free* polyhexes (where rotations and reflections count as the same shape), *fixed* polyhexes (where different orientations count as distinct) and *one-sided* polyhexes (where mirror images count as distinct but rotations count as identical). They may also be distinguished according to whether they may contain holes. The number of free n-hexes for n = 1, 2, 3, … is 1, 1, 3, 7, 22, 82, 333, 1448, … (sequence A000228 [1] in OEIS); the number of free polyhexes with holes is given by A038144 [2]; the number of free polyhexes without holes is given by A018190 [3]; the number of fixed polyhexes is given by A001207 [4]; the number of one-sided polyhexes is given by A006535 [5].

The Monohex:

The Dihex:

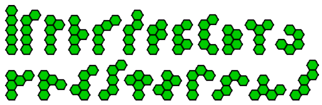


The 3 Trihexes:

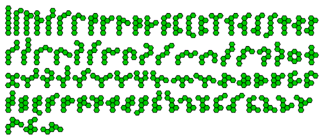


The 7 Tetrahexes:

The 22 Pentahexes:



The 82 Hexahexes:



See also

- Tessellation
- Percolation theory
- Polyiamond - tilings with equilateral triangles
- Polyomino - tilings with squares
- Polycyclic aromatic hydrocarbon - hydrocarbons whose structure is based on polyhexes

References

[1] http://en.wikipedia.org/wiki/Oeis%3Aa000228
[2] http://en.wikipedia.org/wiki/Oeis%3Aa038144
[3] http://en.wikipedia.org/wiki/Oeis%3Aa018190
[4] http://en.wikipedia.org/wiki/Oeis%3Aa001207
[5] http://en.wikipedia.org/wiki/Oeis%3Aa006535

Polyiamond

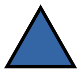
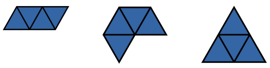
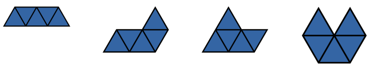
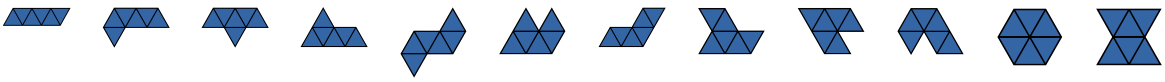
A **polyiamond** (also **polyamond** or simply **iamond**) is a polyform in which the base form is an equilateral triangle. The word *polyiamond* is a back-formation from *diamond*, because this word is often used to describe the shape of a pair of equilateral triangles placed base to base, and the initial "di-" looked like a Greek prefix meaning "two-".

Counting polyiamonds

The basic combinatorial question is how many different polyiamonds with a given number of triangles exist. If mirror images are considered identical, the number of possible n-iamonds for n = 1, 2, 3, … is (sequence A000577 [1] in OEIS):

1, 1, 1, 3, 4, 12, 24, 66, 160, …

As with polyominoes, *fixed* polyiamonds (where different orientations count as distinct) and *one-sided* polyiamonds (where mirror images count as distinct but rotations count as identical) may also be defined. The number of free polyiamonds with holes is given by A070764 [2]; the number of free polyiamonds without holes is given by A070765 [3]; the number of fixed polyiamonds is given by A001420 [4]; the number of one-sided polyiamonds is given by A006534 [5].

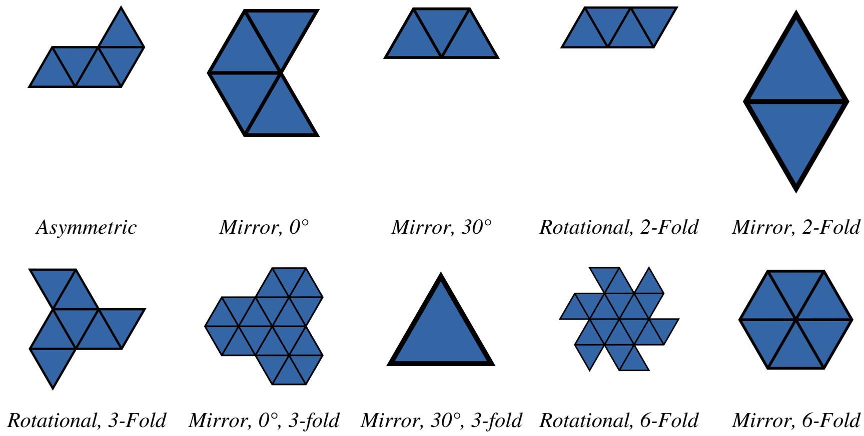
Name	Number of forms	Forms
Moniamond	1	
Diamond	1	
Triamond	1	
Tetriamond	3	
Pentiamond	4	
Hexiamond	12	

Symmetries

Possible symmetries are mirror symmetry, 2-, 3-, and 6-fold rotational symmetry, and each combined with mirror symmetry.

2-fold rotational symmetry with and without mirror symmetry requires at least 2 and 4 triangles, respectively. 6-fold rotational symmetry with and without mirror symmetry requires at least 6 and 18 triangles, respectively. Asymmetry requires at least 5 triangles. 3-fold rotational symmetry without mirror symmetry requires at least 7 triangles.

In the case of only mirror symmetry we can distinguish having the symmetry axis aligned with the grid or rotated 30° (requires at least 4 and 3 triangles, respectively); ditto for 3-fold rotational symmetry, combined with mirror symmetry (requires at least 18 and 1 triangles, respectively).



Generalizations

Like polyominoes, but unlike polyhexes, polyiamonds have three-dimensional counterparts, formed by aggregating tetrahedra. However, polytetrahedra do not tile 3-space in the way polyiamonds can tile 2-space.

Tessellations

Any polyiamond of order 6 or less can be used to tile the plane. All but one of the heptiamonds can be used to tile the plane.[6]

See also

- Tessellation
- Percolation theory

External links

- Weisstein, Eric W., "Polyiamond [7]" from MathWorld.
- VERHEXT [8] — a 1960s puzzle game by Heinz Haber based on hexiamonds

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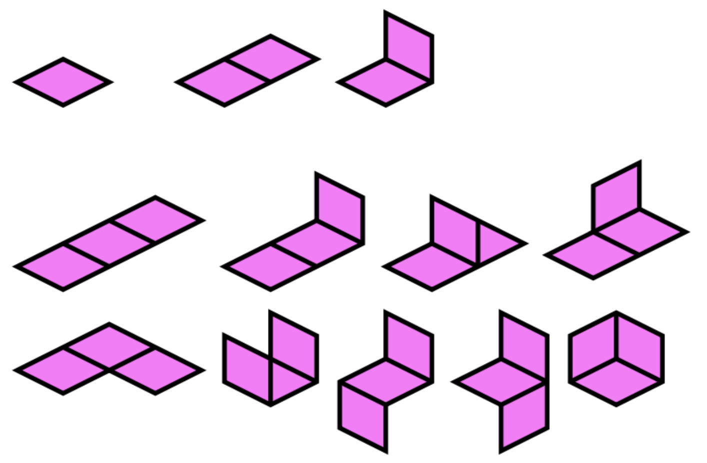
[1] http://en.wikipedia.org/wiki/Oeis%3Aa000577
[2] http://en.wikipedia.org/wiki/Oeis%3Aa070764
[3] http://en.wikipedia.org/wiki/Oeis%3Aa070765
[4] http://en.wikipedia.org/wiki/Oeis%3Aa001420
[5] http://en.wikipedia.org/wiki/Oeis%3Aa006534
[6] http://www.mathpuzzle.com/Tessel.htm
[7] http://mathworld.wolfram.com/Polyiamond.html
[8] http://home.arcor.de/mdoege/verhext/

Polyominoid


The polyominoids for $n = 1$ through $n = 3$

In geometry, a **polyominoid** (or **minoid** for short) is a set of equal squares in 3D space, joined edge to edge at 90- or 180-degree angles. The polyominoids include the polyominoes, which are just the planar polyominoids. The surface of a cube is an example of a *hexominoid,* or 6-cell polyominoid. Polyominoids appear to have been first proposed by Richard A. Epstein.[1]

90-degree connections are called *hard*; 180-degree connections are called *soft*. This is because, in manufacturing a model of the polyominoid, a hard connection would be easier to realize than a soft one.[2] Polyominoids may be classified as *hard*, *soft*, and *mixed* according to how their edges are joined, except that the unique monominoid has no connections of either kind, which makes it both hard and soft by default. The number of soft polyominoids for each n equals the number of polyominoes for the same n.

As with other polyforms, two polyominoids that are mirror images may be distinguished. *One-sided* polyominoids distinguish mirror images; *free* polyominoids do not.

The table below enumerates soft, hard and mixed free polyominoids of up to 4 cells and the total number of free and one-sided polyominoids of up to 6 cells.

	Free				One-sided
Cells	**Soft**	**Hard**	**Mixed**	**Total**	**Total**
1	0	1	0	1	1
2	1	1	0	2	2
3	2	5	2	9	11
4	5	15	34	54[3]	80
5	?	?	?	448	780
6	?	?	?	4650	8781

Generalization to higher dimensions

In general one can define an *n,k-polyominoid* as a polyform made by joining k-dimensional hypercubes at 90° or 180° angles in n-dimensional space, where $1 \le k \le n$.

- Polysticks are 2,1-polyominoids.
- Polyominoes are 2,2-polyominoids.
- The polyforms described above are 3,2-polyominoids.
- Polycubes are 3,3-polyominoids.

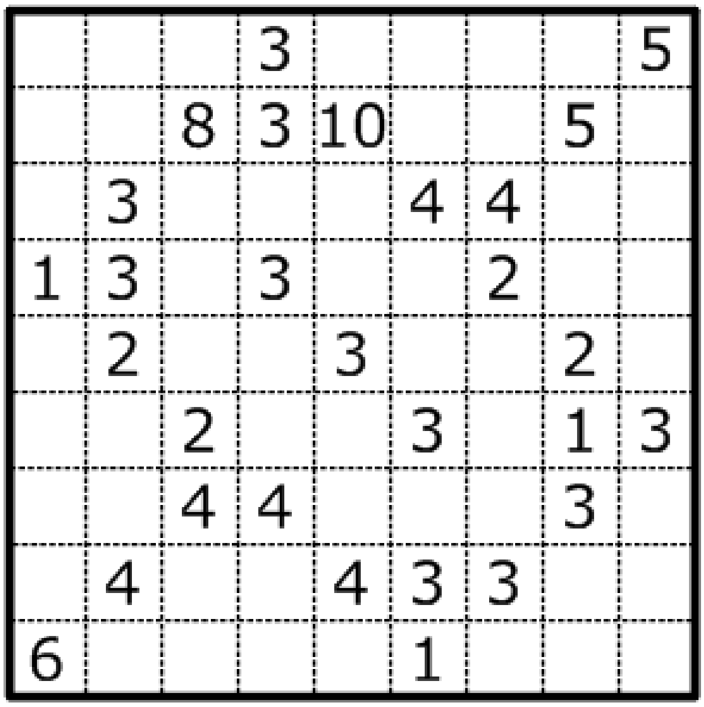
References

[1] Epstein, Richard A. (1977), *The Theory of Gambling and Statistical Logic* (rev. ed.). Academic Press. ISBN 0-12-240761-X. Page 369.

[2] The Polyominoids (http://web.archive.org/web/20091027093920/http://geocities.com/jorgeluismireles/polyominoids/) (archive of The Polyominoids (http://www.geocities.com/jorgeluismireles/polyominoids/))

[3] Galvagni Figures and Plover Figures for Tetrominoids (http://www.monmouth.com/~colonel/noid4gal/noid4gal.html)

Fillomino



Moderately difficult sample puzzle

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Fillomino (フィルオミノ) is a type of logic puzzle published by Nikoli. Other published titles for the puzzle include *Allied Occupation*. As of 2005, three books consisting entirely of *Fillomino* puzzles have been published by Nikoli.

Rules

Fillomino is played on a rectangular grid with no standard size; the internal grid lines are often dotted. (When published as *Allied Occupation* in the World Puzzle Championship, the cells of the grid are circular, but this is purely an aesthetic concern.) Some cells of the grid start containing numbers, referred to as "givens". The goal is to divide the grid into polyominoes (by filling in their boundaries) such that each given number *n* in the grid is part of an *n*-omino and that no two polyominoes of matching size (number of cells) are orthogonally adjacent (share a side).

Unlike some of its contemporaries among puzzles, there need not be a one-to-one correspondence between givens and polyominoes in the solution; it is possible for two givens with matching number to belong to the same polyomino in the solution, and for a polyomino to have no given at all.

Solution methods

It is common practice in solving a *Fillomino* puzzle to add numbers to the empty cells when it is determined what size polyomino each must belong to; these numbers are effectively treated identically to the givens. As well as making it clear where many border segments must be drawn - such as between any two differing numbers, or surrounding a region of matching numbers whose quantity is that number - it also permits the second part of the puzzle's rule to be visualized as simply "the same number cannot appear on both sides of a border", which greatly accelerates solving. A curious side effect of numbering every cell is that when the puzzle is completed, the numbers alone unambiguously define the solution, the actual borders being trivially deducible. This makes communication of a solution without a grid quite feasible; indeed, solutions for *Allied Occupation* give only the numbers. (Nikoli always publishes solutions to their *Fillomino* puzzles with both the polyomino borders drawn in and numbers given in every cell.)

The typical means of starting a *Fillomino* puzzle is to draw in the obvious borders between non-matching givens and surrounding all polyominoes completed by the givens alone ('1's, pairs of orthogonally adjacent '2's, and so on). From there, the solver searches for three things, possibly in combination:

- Potential overloads. Each polyomino in the solution, if it were completely numbered, would contain matching numbers whose quantity is that number. If there is a place in the grid where adding a particular number would result in an orthogonally contiguous region of too many copies of that number, then borders to that cell from those numbers may be drawn in. Often the givens alone provide these, most commonly a pair of *diagonally* adjacent '2's: placing a '2' in either of the cells that share a side with both givens would result in an overload, so four cell borders may be drawn in (in the shape of a plus sign) separating the '2's.
- Limited domains. Every number in the grid - whether given or deduced - must ultimately be bordered into a region with that number of cells in it. Often, a number will require other cells to be in its region due to not having any alternative location to expand into. The most obvious case is a number (other than '1') bordered on three sides; the cell sharing the fourth side must belong to the same region, and consequently can bear the same number. The same principle applies to numbers bordered on only two sides but cannot possibly expand into enough cells in only one direction, and so on.
- Defined cells. In more challenging circumstances, sometimes working with the empty cells is easier than working with numbers. The most obvious case is when a single cell without a number becomes completely surrounded; without any help from other numbers, that cell must be a monomino, and can be marked with '1'. Similarly, two orthogonally adjacent empty cells surrounded together must be a domino, as two monominos cannot share a side. Even cells in regions not completely surrounded may be defined; a common occurrence is for an empty cell as part of a small region mostly bordered by solved polyominoes to have only one legal size of polyomino available to it, with other sizes being too large or would result in matching-size polyominoes sharing a side. This is perhaps best recognized by considering what number can legally be placed in such a cell and determining that only one exists.

Variants

Fillomino adapts to different geometries; hexagonal grids can be used, with the only change in the rules being replacing all instances of *polyomino* with *polyhex*. Another variant was published by Nikoli under the name *NIKOJI*; letters are used as givens instead of numbers, where the letters and polyominoes have a one-to-one correspondence and only matching letters have matching polyominoes (in size, shape, orientation, and letter position).

See also

- List of Nikoli puzzle types

External links

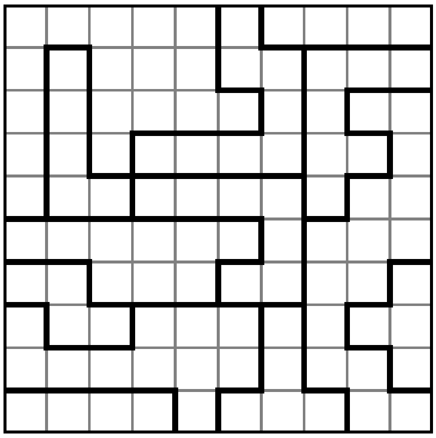
- Nikoli's English page on Fillomino [1]

References

[1] http://www.nikoli.co.jp/en/puzzles/fillomino/

LITS

LITS, formerly known as **Nuruomino** (ヌルオミノ), is a binary determination puzzle published by Nikoli.



Moderately difficult *LITS* puzzle (solution)

Rules

LITS is played on a rectangular grid, typically 10×10; the grid is divided into polyominoes, none of which have fewer than four cells. The goal is to shade in a tetromino within each pre-printed polyomino such that no two matching tetrominoes are orthogonally adjacent (with rotations and reflections counting as matching), and that the shaded cells form a valid nurikabe: they are all orthogonally contiguous (form a single polyomino) and contain no 2×2 square tetrominoes as subsets.

History

The puzzle was first printed in *Puzzle Communication Nikoli #106*; the original title is a combination of 'nuru' (Japanese: "to paint") and 'omino' (polyomino). In issue #112, the title was changed to the present one, which represents the four (of five) tetrominoes used in the puzzle: the L-shape, the straight, the T-shape, and the skew (square tetrominoes may never appear in the puzzle as they are a direct violation of the rule).

See also

- List of Nikoli puzzle types

External links

- Nikoli's English page on *LITS* [1]

References

[1] http://www.nikoli.co.jp/en/puzzles/lits/

Mechanical puzzles

Mechanical puzzle

A **mechanical puzzle** is a puzzle presented as a set of mechanically interlinked pieces.

History

The oldest known mechanical puzzle comes from Greece and appeared in the 3rd century BC. The game consists of a square divided into 14 parts, and the aim was to create different shapes from these pieces. This is not easy to do. (see Ostomachion loculus Archimedius)

In Iran "puzzle-locks" were made as early as the 17th century AD.

The next known occurrence of puzzles is in Japan. In 1742 there is a mention of a game called "Sei Shona-gon Chie No-Ita" in a book. Around the year 1800 the Tangram puzzle from China became popular, and 20 years later it had spread through Europe and America.

The company Richter from Rudolstadt began producing large amounts of Tangram-like puzzles of different shapes, the so-called "Anker-puzzles".

In 1893 professor Hoffman wrote a book called *Puzzles; Old and New*. It contained, amongst other things, more than 40 descriptions of puzzles with secret opening mechanisms. This book grew into a reference work for puzzle games and modern copies exist for those interested.

Puzzle design by W.Altekruse

The beginning of the 20th century was a time in which puzzles were greatly fashionable and the first patents for puzzles were recorded. The puzzle shown in the picture, made of 12 identical pieces by W. Altekruse in the year 1890, was an example of this.

With the invention of modern polymers manufacture of many puzzles became easier and cheaper.

Categories

Assembly puzzles

Hoffman's packing problem

In this category, the puzzle is present in component form, and the aim is to produce a certain shape. The Soma cube made by Piet Hein, the Pentomino by Solomon Golomb and the aforementioned laying puzzles Tangram and “Anker-puzzles” are all examples of this type of puzzle. Furthermore, problems in which a number of pieces have to be arranged so as to fit into a (seemingly too small) box are also classed in this category.

The image shows a variant of Hoffman's packing problem. The aim is to pack 27 cuboids with side lengths A, B, C into a box of side length A+B+C, subject to two constraints:

1) A, B, C must not be equal

2) The smallest of A, B, C must be larger than $(A + B + C)/4$

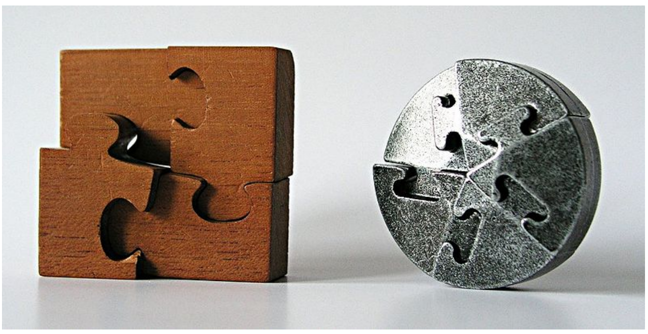
One possibility would be A=18, B=20, C=22 – the box would then have to have the dimensions 60×60×60.

Modern tools such as laser cutters allow the creation of complex 2 dimensional puzzles made of wood or acrylic plastic. In recent times this has become predominant and puzzles of extraordinarily decorative geometry have been designed. This makes use of the multitude of ways of subdividing areas into repeating shapes.

Computers aid in the design of new puzzles. A computer allows an exhaustive search for solution – with its help a puzzle may be designed in such a way that it has the fewest possible solutions, or a solution requiring the most steps possible. The consequence is that solving the puzzle can be very difficult.

The use of transparent materials enables the creation of puzzles, in which pieces have to be stacked on top of each other. The aim is to create a specific pattern, image or colour scheme in the solution. For example, one puzzle consists of several discs in which angular sections of varying sizes are differently coloured. The discs have to be stacked so as to create a colour circle (red->blue->green->red) around the discs.

Disassembly puzzles


Disassembly puzzles

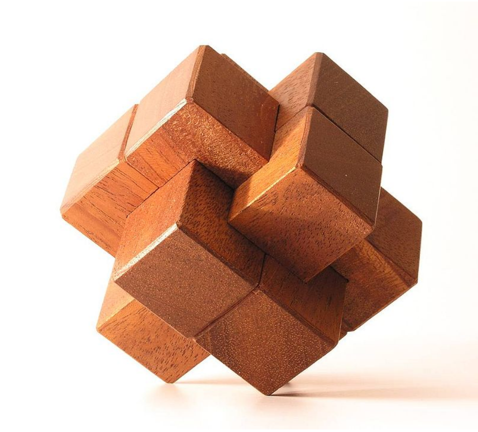
The puzzles in this category are usually solved by opening or dividing them into pieces. This includes those puzzles with secret opening mechanisms, which are to be opened by trial and error. Furthermore, puzzles consisting of several metal pieces linked together in some fashion are also considered part of this category.

The two puzzles shown in the picture are especially good for social gatherings, since they appear to be very easily taken apart, but in reality many people cannot solve this puzzle. The problem here lies in the shape of the interlocking pieces — the mating surfaces are tapered, and thus can only be removed in one direction. However, each piece has two oppositely sloping tapers mating with the two adjoining pieces so that the piece cannot be removed in either direction.

Boxes called secret boxes or puzzle boxes with secret opening mechanisms extremely popular in Japan, are included in this category. These caskets contain more or less complex, usually invisible opening mechanisms which reveal a small hollow space on opening. There is a vast variety of opening mechanisms, such as hardly visible panels which need to be shifted, inclination mechanisms, magnetic locks, movable pins which need to be rotated into a certain position up and even time locks in which an object has to be held in a given position until a liquid has filled up a certain container.

Interlocking puzzles

In an interlocking puzzle, one or more pieces hold the rest together, or the pieces are mutually self-sustaining. The aim is to completely disassemble and then reassemble the puzzle. Examples of these are the well-known Chinese wood knots.



Chinese wood knot

Both assembly and disassembly can be difficult – contrary to assembly puzzles, these puzzles usually do not just fall apart easily.

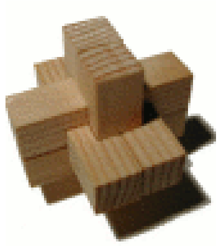
The level of difficulty is usually assessed in terms of the number of moves required to remove the first piece from the initial puzzle.

The image shows one of the most notorious representatives of this category, the Chinese wood knot. In this particular version designed by Bill Cutler, 5 moves are needed before the first piece can be removed.

The known history of these puzzles reaches back to the beginning of the 18th century. In 1803 a catalog by “Bastelmeier” contained two puzzles of this type. Professor Hoffman's puzzle book mentioned above also contained two interlocking puzzles.

At the beginning of the 19th century the Japanese took over the market for these puzzles. They developed a multitude of games in all kinds of different shapes – animals, houses and other objects - whereas the development in the western world revolved mainly around geometrical shapes.

With the help of computers, it has recently become possible to analyze complete sets of games played. This process was begun by Bill Cutler with his analysis of all the Chinese wood knots. From October 1987 to August 1990 all the 35 657 131 235 different variations were analyzed. The calculations were done by several computers in parallel and would have taken a total of 62.5 years on a single computer.



A Burr puzzle being disassembled

With shapes different from the Chinese cross the level of difficulty lately reached levels of up to 100 moves for the first piece to be removed, a scale humans would struggle to grasp. The peak of this development is a puzzle in which the addition of a few pieces doubles the number of moves.

However, computer analysis also led to another trend: since the rotation of pieces cannot, with today's software, be analyzed by computers , there has been a trend to create puzzles whose solution must include at least one rotation. These then have to be solved by hand.

Prior to the 2003 publication of the RD Design Project by Owen, Charnley and Strickland, puzzles without right angles could not be efficiently analyzed by computers. Stewart Coffin has been creating puzzles based upon the rhombic dodecahedron since the 1960s. These made use of strips with either six or three edges. These kinds of puzzles often have extremely irregular components, which come together in a regular shape only at the very last step. Furthermore, the 60° angles allow designs in which several objects have to be moved at the same time. The “Rosebud” puzzle is a prime example of this: in this puzzle 6 pieces have to be moved from one extreme position, in which they are only touching at the corners, to the center of the completed object.

Disentanglement Puzzles

A disentanglement puzzle. The object is to remove the string with the two balls from the wire construction.

For these kind of puzzles, the goal is to disentangle a metal or string loop from an object. Topology plays an important role with these puzzles.

The image shows a version of the derringer puzzle. Although simple in appearance, it is quite challenging - most puzzle sites rank it among their hardest puzzles.

Vexiers are a different sort of disentanglement puzzle - two or more metal wires, which have been intertwined, are to be untangled. They, too, spread with the general puzzle craze at the end of the 19th century. A large number of the Vexiers still available today originate in this period.

So called ring puzzles, of which the Chinese rings are part, are a different type of Vexier. In these puzzles a long wire loop must be unsnarled from a mesh of rings and wires. The number of steps required for a solution often has an exponential relationship with the number of loops in the puzzle. The common type, which connects the rings to a bar with cords (or loose metal equivalents) has a movement pattern identical to the Gray binary code, in which only one bit changes from one code word relative to its immediate neighbor.

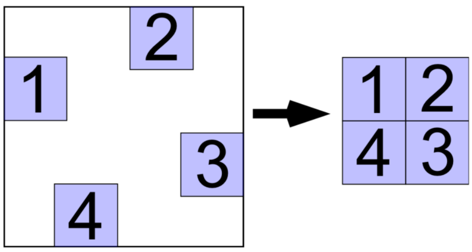
A noteworthy puzzle, known as the Chinese rings, Cardans' rings, the Baguenaudier or the Renaissance puzzle was Mentioned in circa 1500 as Problem 107 of the manuscript “De Viribus Quantitatis” by Luca Pacioli. The puzzle is again referred to by Girolamo Cardano in the 1550 edition of his book “De subtililate.” Although the puzzle is a disentanglement type Puzzle it also has mechanical puzzle attributes, and the solution can be derived as a binary mathematical procedure.

The Chinese rings are associated with the tale that in the Middle Ages, knights would give these to their wives as a present, so that in their absence they may fill their time. Tavern puzzles, made of steel, are based on forging exercises that provided good practice for blacksmith apprentices.[1]

Niels Bohr used disentanglement puzzles called Tangloids to demonstrate the properties of spin to his students.

Fold Puzzles

The aim in this particular genre of puzzles is to fold a printed piece of paper in such a way as to obtain a target picture. In principle, Rubik's Magic could be counted in this category. A better example is shown in the picture. The task is to fold the square piece of paper so that the four squares with the numbers lie next to each other without any gaps and form a square. This puzzle is pretty complex already.



Example of a fold puzzle

Another folding puzzle is folding prospectuses and city maps. Despite the often visible folding direction at the folding points it can be extraordinarily difficult to put the paper back into the form with which it originally came. The reason these maps are difficult to restore to their original state is that the folds are designed for a paper-folding machine, in which the optimum folds are not of the sort an average person would try to use.

Lock puzzle

These puzzles, also called **trick locks**, are locks (often padlocks) which have an unusual locking mechanism. The aim is to open the lock. If you are given a key, it will not open the lock in the conventional way. For some locks it may then be more difficult to restore the original situation.

Trick vessels

These are vessels “with a twist”. The aim is to either drink or pour from a container without spilling any of the liquid. Puzzle containers are an ancient form of game. The Greeks and Phoenicians made containers which had to be filled via an opening at the bottom. In the 9th century a number of different containers were described in detail in a Turkish book. In the 18th century the Chinese also produced these kinds of drinking containers.


Example of a trick vessel

One example is the puzzle jug: the neck of the container has many holes which make it possible to pour liquid into the container, but not out of it. Hidden to the puzzler's eye, there is a small tubular conduit all the way through the grip and along the upper rim of the container up to the nozzle. If one then blocks the opening at the upper end of the grip with one finger, it is possible to drink liquid from the container by sucking on the nozzle.

Other examples include Fuddling cup and Pot crown.

Impossible Objects


An "impossible" object

Impossible objects are objects which at first sight do not seem possible. The most well known impossible object is the ship in a bottle. The goal is to discover how these objects are made. Another well known puzzle is one consisting of a cube made of two pieces interlocked in 4 places by seemingly inseparable links (example [2]). The solutions to these are to be found in different places. There are all kinds of objects which fit this description – bottles in which there are objects that are far too large (see impossible bottles), Japanese hole coins with wooden arrows and rings through them, wooden spheres in a wooden frame with far too small openings and many more.

The apple and arrow in the picture are made of one piece of wood each. The hole is in effect too small to fit the arrow through it and there are no signs of gluing. How was it made?

Dexterity puzzles

The games listed in this category are not strictly puzzles as such, as dexterity and endurance are of more importance here. Often, the aim is to incline a box fitted with a transparent cover in just the right way as to cause a small ball to fall into a hole.

By tilting the box, one must try to lead the ball along the line and to the goal without dropping it in one of the many strategically-placed holes.

Sequential movement puzzle

A puzzle with the name Skewb

The puzzles in this category require a repeated manipulation of the puzzle to get the puzzle to a certain target condition. Well-known puzzles of this sort are the Rubik's Cube and the Tower of Hanoi. This category also includes those puzzles in which one or more pieces have to be slid into the right position, of which the N-puzzle is the best known. Rush Hour or Sokoban are other examples.

The Rubik's cube caused an unprecedented boom of this category. The sheer number of variants is staggering. Cubes of dimensions 2×2×2, 3×3×3, 4×4×4, 5×5×5, 6×6×6 and 7×7×7 have been made, as well as tetrahedral, octahedral, icosahedral, and dodecahedral variants and even some based on different types of cylinders. With a varying orientation of the axis of rotation a variety of puzzles with the same basic shape can be created. Furthermore, one can obtain further cuboidal puzzles by removing one layer from a cube. These cuboidal puzzles take irregular shapes when they are manipulated.

The picture shows another, less well-known example of this kind of puzzle. It is just easy enough that it can still be solved with a bit of trial and error, and a few notes, as opposed to Rubik's Cube which is too difficult to just solve by trial.

Simulated mechanical puzzles

While many computer games and computer puzzles simulate mechanical puzzles, these simulated mechanical puzzles are usually not strictly classified as mechanical puzzles.

Other notable mechanical puzzles

- Chinese Ring Puzzle: Recursive iron ring manipulation (ancient)
- Nintendo Ten Billion Barrel: manipulate mechanically connected parts of a barrel
- Pyraminx: manipulate mechanically connected parts of a pyramid

See also

- Bedlam cube
- Puzzle
- Puzzle ring
- Miguel Berrocal

References

[1] Ronald V. Morris, "Social Studies around the Blacksmith's Forge: Interdisciplinary Teaching and Learning" (http://heldref-publications.metapress.com/link.asp?id=d1781747033t1262), *The Social Studies*, **vol.98**, No.3 May–June 2007, pp.99-104, Heldref Publications doi:10.3200/TSSS.98.3.99-104.
[2] http://www.mechanicalpuzzles.org/mypuzzles/html/doubledovetail.html

- Puzzles Old & New, by Professor Hoffmann, 1893
- Puzzles Old and New, by Jerry Slocum & Jack Botermans, 1986
- New Book of Puzzles, by Jerry Slocum & Jack Botermans, 1992
- Ingenious & Diabolical Puzzles, by Jerry Slocum & Jack Botermans, 1994
- The Tangram Book, by Jerry Slocum, 2003
- The 15 Puzzle, by Jerry Slocum & Dic Sonneveld, 2006

This article draws heavily on the corresponding article in the German wikipedia.

External links

- Website of John Rausch's puzzle collection (http://www.johnrausch.com/puzzleWorld/default.htm) Contains a well-illustrated puzzle collection and numerous Java applets of sliding puzzles. Stewart Coffin's (one of the most important puzzle designers) books may be obtained, too.
- Rob's Puzzle Page (http://www.robspuzzlepage.com/) Photos of a very large collection of all types of puzzles, plus analysis, commentary, and links to many other sites of interest to puzzlers
- Puzzle designs prepared by ISHINO Keiichiro (http://www.asahi-net.or.jp/~rh5k-isn/Puzzle/index.html.en) This site contains a huge collection of accurate descriptions for assembly and interlocking puzzles. Those skilled enough can create their own creation with the designs presented.
- Puzzle iT web site by Dr. Florian Radut (http://www.puzzle.ro/) A British-Romanian puzzle collection: mechanical puzzles, 2D puzzles, 3D puzzles, logic games and anagram puzzles, some of them used in TV Game-Shows, SMS contests and online contests.
- Oskar van Deventer (http://www.gamepuzzles.com/oskar.htm) A prolific inventor of highly innovative mechanical and other puzzles.
- Archimedes' Lab Puzzle Website (http://www.archimedes-lab.org) Contains a huge quantity of puzzles to make and to solve along with various educational activities.
- The Jerry Slocum Mechanical Puzzle Collection (http://www.dlib.indiana.edu/collections/slocum/) Highlights from Jerry Slocum's extensive puzzle collection, including many puzzles of historical significance.

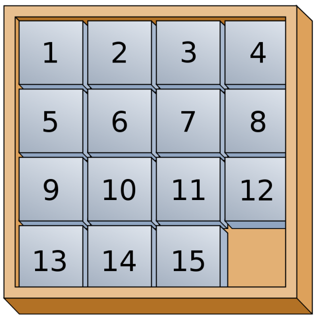
Sliding puzzle

A **sliding puzzle**, **sliding block puzzle**, or **sliding tile puzzle** challenges a player to slide usually flat pieces along certain routes (usually on a board) to establish a certain end-configuration.

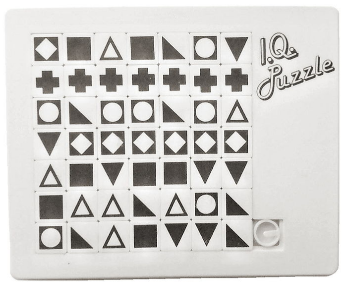
The fifteen puzzle is the oldest type of sliding block puzzle. Sam Loyd is often wrongly credited with making sliding puzzles popular based on his false claim that he invented the fifteen puzzle. The 15 Puzzle, invented by Noyes Chapman, created a puzzle craze in 1880. Unlike other tour puzzles, a sliding block puzzle prohibits lifting any piece off the board. This property separates sliding puzzles from rearrangement puzzles. Hence finding moves, and the paths opened up by each move, within the two-dimensional confines of the board, are important parts of solving sliding block puzzles.

Sliding puzzles are essentially two-dimensional in nature, even if the sliding is facilitated by mechanically interlinked pieces (like partially encaged marbles) or three-dimensional tokens. As this example shows, some sliding puzzles are mechanical puzzles. However, the mechanical fixtures are usually not essential to these puzzles; the parts could as well be tokens on a flat board which are moved according to certain rules.

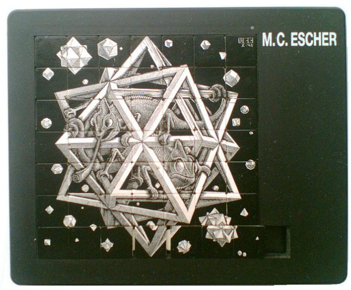
This type of puzzle has been computerized, and is available to play for free on-line from many web pages. It is a descendant of the jigsaw puzzle in that its point is to form a picture on-screen. The last square of the puzzle is then displayed automatically once the other pieces have been lined up.



A solved 15-puzzle.



A 7x7 sliding block puzzle. The task for this puzzle is to arrange it so that no tile design is repeated in any row column or diagonal. There is more than one solution to this puzzle.



A 7x7 puzzle. This one solves to a picture, like a jigsaw puzzle.

Examples of sliding puzzles

- Fifteen puzzle
- Klotski
- Minus Cube
- Jumbly

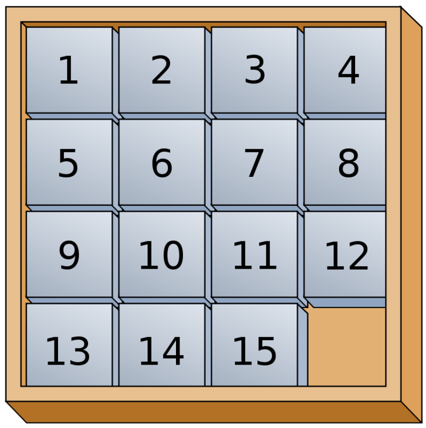
See also

- Puzzle
- Mechanical puzzles
- Combination puzzles
- Transport Puzzle
- Rush Hour
- Rubik's Cube
- RO (game) A rotational variation

References

- *Sliding Piece Puzzles* (by Edward Hordern, 1986, Oxford University Press, ISBN 0-19-853204-0) is said to be the definitive volume on this type of puzzle.
- *Winning Ways* (by Elwyn Ralph Berlekamp et al., 1982, Academic Press)
- *The 15 Puzzle* (by Jerry Slocum & Dic Sonneveld, 2006, Slocum Puzzle Foundation)
- maa.org column on Sliding-block puzzles (http://www.maa.org/editorial/mathgames/mathgames_12_13_04.html) which rebuts the claims in the Economist.
- US Patent 4872682 (http://patft.uspto.gov/netacgi/nph-Parser?patentnumber=4872682) - sliding puzzle wrapped on Rubik's Cube

Fifteen puzzle



A solved 15-puzzle

The ***n*-puzzle** is known in various versions, including the **8 puzzle**, the **15 puzzle**, and with various names (**Gem Puzzle**, **Boss Puzzle**, **Game of Fifteen**, **Mystic Square** and many others). It is a sliding puzzle that consists of a frame of numbered square tiles in random order with one tile missing. If the size is 3×3, the puzzle is called the 8-puzzle or 9-puzzle, and if 4×4, the puzzle is called the 15-puzzle or 16-puzzle. The object of the puzzle is to place the tiles in order (see diagram) by making sliding moves that use the empty space.

The n-puzzle is a classical problem for modelling algorithms involving heuristics. Commonly used heuristics for this problem include counting the number of misplaced tiles and finding the sum of the Manhattan distances between each block and its position in the goal configuration. Note that both are *admissible*, i.e., they never overestimate the number of moves left, which ensures optimality for certain search algorithms such as A*.

Solvability

Johnson & Story (1879) used a parity argument to show that half of the starting positions for the *n*-puzzle are impossible to resolve, no matter how many moves are made. This is done by considering a function of the tile configuration that is invariant under any valid move, and then using this to partition the space of all possible labeled states into two equivalence classes of reachable and unreachable states.

The invariant is the parity of permutations of all 16 squares (15 pieces plus empty square) plus the parity of the taxicab distance moved by the empty square. This is an invariant because each move changes the parity of the permutation and the parity of the taxicab distance. In particular if the empty square is not moved the permutation of the remaining pieces must be even.

Johnson & Story (1879) also showed that the converse holds on boards of size $m \times n$ provided m and n are both at least 2: all even permutations *are* solvable. This is straightforward but a little messy to prove by induction on m and n starting with $m=n=2$. Archer (1999) gave another proof, based on defining equivalence classes via a hamiltonian path.

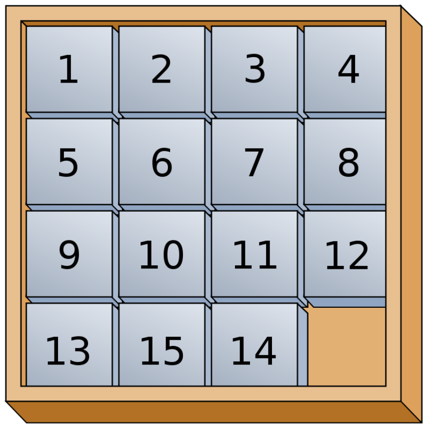
Wilson (1974) studied the analogue of the 15 puzzle on arbitrary finite connected and non-separable graphs. (A graph is called separable if removing a vertex increases the number of components.) He showed that, except for polygons, and one exceptional graph on 7 vertices, it is possible to obtain all permutations unless the graph is bipartite, in which case exactly the even permutations can be obtained. The exceptional graph is a regular hexagon with one diagonal and a vertex at the center added; only 1/6 of its permutations can be obtained.

For larger versions of the *n*-puzzle, finding a solution is easy, but the problem of finding the *shortest* solution is NP-hard.[1] For the 15-puzzle, lengths of optimal solutions range from 0 to 80 moves; the 8-puzzle can be solved in 31 moves or fewer (integer sequence A087725 [2]).

The symmetries of the fifteen puzzle form a groupoid (not a group, as not all moves can be composed);[3] [4] this groupoid acts on configurations.

History

The puzzle was "invented" by Noyes Palmer Chapman,[5] a postmaster in Canastota, New York, who is said to have shown friends, as early as 1874, a precursor puzzle consisting of 16 numbered blocks that were to be put together in rows of four, each summing to 34. Copies of the improved Fifteen Puzzle made their way to Syracuse, New York by way of Noyes' son, Frank, and from there, via sundry connections, to Watch Hill, RI, and finally to Hartford (Connecticut), where students in the American School for the Deaf started manufacturing the puzzle and, by December 1879, selling them both locally and in Boston (Massachusetts). Shown one of these, Matthias Rice, who ran a fancy woodworking business in Boston, started manufacturing the puzzle sometime in December 1879 and convinced a "Yankee Notions" fancy goods dealer to sell them under the name of "Gem Puzzle". In late-January 1880, Dr. Charles Pevey, a dentist in Worcester, Massachusetts, garnered some attention by offering a cash reward for a solution to the Fifteen Puzzle.[5]



Sam Loyd's unsolvable 15-puzzle, with tiles 14 and 15 exchanged. This puzzle is not solvable because it would require a change of the invariant.

The game became a craze in the U.S. in February 1880, Canada in March, Europe in April, but that craze had pretty much dissipated by July. Apparently the puzzle was not introduced to Japan until 1889.

Noyes Chapman had applied for a patent on his "Block Solitaire Puzzle" on February 21, 1880. However, that patent was rejected, likely because it was not sufficiently different from the August 20, 1878 "Puzzle-Blocks" patent (US 207124) granted to Ernest U. Kinsey.[5]

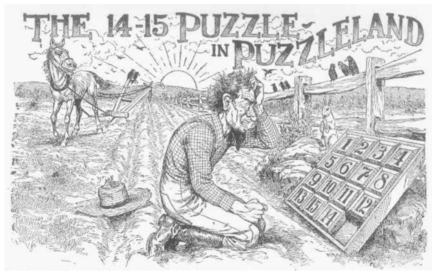
U.S. Political cartoon about finding a Republican presidential candidate in 1880

Sam Loyd

Sam Loyd claimed from 1891 until his death in 1911 that he invented the puzzle, for example writing in the *Cyclopedia of Puzzles* (published 1914), p. 235 [6]:

> *The older inhabitants of Puzzleland will remember how in the early seventies I drove the entire world crazy over a little box of movable pieces which became known as the "14-15 Puzzle".*

This is false – Loyd had nothing to do with the invention or popularity of the puzzle, and in any case the craze was in 1880, not the early 1870s:[5]



Sam Loyd's illustration

> *Sam Loyd did not invent the 15 puzzle and had nothing to do with promoting or popularizing it. The puzzle craze that was created by the 15 Puzzle began in January 1880 in the US and in April in Europe. The craze ended by July 1880 and Sam Loyd's first article about the puzzle was not published until sixteen years later, January 1896. Loyd first claimed in 1891 that he invented the puzzle, and he continued until his death a 20 year campaign to falsely take credit for the puzzle. The actual inventor was Noyes Chapman, the Postmaster of Canastota, New York, and he applied for a patent in March 1880.*

Some later interest was fuelled by Loyd offering a $1,000 prize for anyone who could provide a solution for achieving a particular combination specified by Loyd, namely reversing the 14 and 15.[7] This was impossible, as had been shown over a decade earlier by Johnson & Story (1879), as it required a transformation from an even to an odd combination.

Miscellaneous

The Minus Cube, manufactured in the USSR, is a 3D puzzle with similar operations to the 15-puzzle.

Bobby Fischer was an expert at solving the 15-Puzzle. He had been timed to be able to solve it within 25 seconds; Fischer demonstrated this on November 8, 1972 on *The Tonight Show Starring Johnny Carson*.

See also

- Rubik's Cube
- Minus Cube
- Sliding puzzle
- Combination puzzles
- Mechanical puzzles
- Jeu de taquin, an operation on skew Young tableaux similar to moves of the 15 puzzle.
- Pebble motion problems

Notes

[1] Daniel Ratner, Manfred K. Warmuth. *Finding a Shortest Solution for the N × N Extension of the 15-PUZZLE Is Intractable*. National Conference on Artificial Intelligence, 1986.

[2] http://www.research.att.com/~njas/sequences/A087725

[3] The 15-puzzle groupoid (1) (http://www.neverendingbooks.org/index.php/the-15-puzzle-groupoid-1.html), Never Ending Books

[4] The 15-puzzle groupoid (2) (http://www.neverendingbooks.org/index.php/the-15-puzzle-groupoid-2.html), Never Ending Books

[5] *The 15 Puzzle*, by Jerry Slocum & Dic Sonneveld. ISBN 1-890980-15-3

[6] http://www.mathpuzzle.com/loyd/cop234-235.jpg

[7] Korf, Richard E. (2000), "Recent progress in the design and analysis of admissible heuristic functions" (https://www.aaai.org/Papers/AAAI/2000/AAAI00-212.pdf), SARA 2000. Abstraction, reformulation, and approximation: 4th international symposium, Texas, USA. LNCS 1864, Springer, pp. 45–55, doi:10.1007/3-540-44914-0_3, ISBN 978-3-540-67839-7, , retrieved 2010-04-26

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- Archer, Aaron F. (1999), "A modern treatment of the 15 puzzle" (http://dx.doi.org/10.2307/2589612), *The American Mathematical Monthly* **106** (9): 793–799, MR1732661, ISSN 0002-9890
- Johnson, Wm. Woolsey; Story, William E. (1879), "Notes on the "15" Puzzle" (http://www.jstor.org/stable/2369492), *American Journal of Mathematics* (The Johns Hopkins University Press) **2** (4): 397–404, ISSN 0002-9327
- Wilson, Richard M. (1974), "Graph puzzles, homotopy, and the alternating group", *Journal of Combinatorial Theory. Series B* **16**: 86–96, doi:10.1016/0095-8956(74)90098-7, MR0332555, ISSN 0095-8956

External links

- The history of the 15 puzzle. (http://www.xs4all.nl/~hc11/15puzzle/15puzzen.htm) by Harry Broeders (http://www.xs4all.nl/~hc11/)

Combination puzzle

A **combination puzzle**, also known as a **sequential move puzzle**, is a puzzle which consists of a set of pieces which can be manipulated into different combinations by a group of operations. The puzzle is solved by achieving a particular combination starting from a random (scrambled) combination. Often, the solution is required to be some recognisable pattern such as 'all like colours together' or 'all numbers in order'. The most famous of these puzzles is the original Rubik's Cube, a cubic puzzle in which each of the six faces can be independently rotated. Each of the six faces is a different colour, but each of the nine pieces on a face is identical in colour, in the solved condition. In the unsolved condition colours are distributed amongst the pieces of the cube.

The mechanical construction of the puzzle will usually define the rules by which the combination of pieces can be altered. This leads to some limitations on what combinations are possible. For instance, in the case of the Rubiks Cube, there are a large number of combinations that can be achieved by randomly placing the coloured stickers on the cube, but not all of these can be achieved by manipulating the cube rotations. Similarly, not all the combinations that are mechanically possible from a disassembled cube are possible by manipulation of the puzzle. Since neither unpeeling the stickers nor disassembling the cube is an allowed operation, the possible operations of rotating various faces limit what can be achieved.

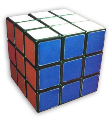
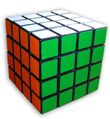
Although a mechanical realization of the puzzle is usual, it is not actually necessary. It is only necessary that the rules for the operations are defined. The puzzle can be realized entirely in virtual space or as a set of mathematical statements. In fact, there are some puzzles that can *only* be realized in virtual space. An example is the 4-dimensional 3×3×3×3 tesseract puzzle, simulated by the MagicCube4D software.

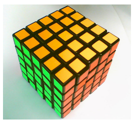
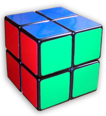
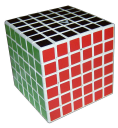
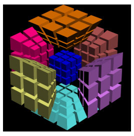
Properties

There have been many different shapes of Rubik type puzzles constructed. As well as cubes, all of the regular polyhedra and many of the semi-regular and stellated polyhedra have been made.

Regular cuboids

A cuboid is a rectilinear polyhedron. That is, all its edges form right angles. Or in other words (in the majority of cases), a box shape. A regular cuboid, in the context of this article, is a cuboid puzzle where all the pieces are the same size in edge length.

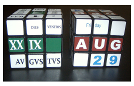
Picture	Data	Comments
	Commercial name: Rubik's Cube Geometric shape: Cube Piece configuration: 3×3×3	The original Rubik's Cube
	Commercial name: Rubik's Revenge Geometric shape: Cube Piece configuration: 4×4×4	Solution is much the same as 3×3×3 cube except additional (and relatively simple) algorithm(s) are required to unscramble the centre pieces and edges and additional parity not seen on the 3x3x3 Rubik's Cube.

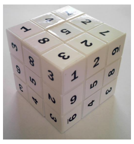
	Commercial name: Professor's Cube Geometric shape: Cube Piece configuration: 5×5×5	Solution is much the same as 3×3×3 cube except additional (and relatively simple) algorithm(s) are required to unscramble the centre pieces and edges.
	Commercial name: Pocket Cube Geometric shape: Cube Piece configuration: 2×2×2	Simpler to solve than the standard cube in that only the algorithms for the corner pieces are required. It is nevertheless surprisingly non-trivial to solve.
	Commercial name: V-CUBE Geometric shape: Cube Piece configuration: 2×2×2 to 11×11×11 *Main articles: V-Cube 6, V-Cube 7*	Panagiotis Verdes holds a patent to a method which is said to be able to make cubes up to 11×11×11. He has fully working products for 5×5×5, 6×6×6 and 7×7×7 cubes. Solution strategies are similar to those of the 4×4×4 and 5×5×5.
	4-Dimensional puzzle Geometric shape: Tesseract Piece configuration: 3×3×3×3	This is the 4-dimensional analog of a cube and thus cannot actually be constructed. However, it can be drawn or represented by a computer. Seriously more difficult to solve than the standard cube, although the techniques follow much the same principles. There are many other sizes of virtual cuboid puzzles ranging from the trivial 3×3 to the unsolved 5-dimensional 7×7×7×7×7.
[1] [2][3][4]	Non-uniform cuboids Geometric shape: Cuboid Piece configuration (1st): 2×2×3 Piece configuration (2nd): 2×3×3 Piece configuration (3rd): 3×4×4 Piece configuration (4th): 2×2×6	Most of the puzzles in this class of puzzle are generally custom made in small numbers. Most of them start with the internal mechanism of a standard puzzle. Additional cubie pieces are then added, either modified from standard puzzles or made from scratch. The four shown here are only a sample from a very large number of examples. Those with two or three different numbers of even or odd rows also have the ability to change their shape.
[5]	Siamese cubes Geometric shape: Fused cubes Piece configuration: two 3×3×3 fused 1×1×3	Siamese cubes are two or more puzzles that are fused so that some pieces are common to both cubes. The picture here shows two 3×3×3 cubes that have been fused. The largest example known to exist is in The Puzzle Museum [6] and consists of three 5×5×5 cubes that are siamese fused 2×2×5 in two places.
[7]	Extended cubes Geometric shape: Box Piece configuration: 3×3×3 extended 3×3×5	These puzzles are made by bonding additional cubies to an existing puzzle. They therefore do not add to the complexity of the puzzle configuration, they just make it look more complex. Solution strategies remain the same, though a scrambled puzzle can have a strange appearance.
[8]	Commercial name: Boob cube Geometric shape: Box Piece configuration: 1×1×2	Very possibly the simplest regular cuboid puzzle to solve. Completely trivial solution as the puzzle consists of only two cubies.

	Commercial name: Void cube Geometric shape: Menger Sponge with 1 iteration Piece configuration: 3x3x3 with a hole in the middle.	Solutions to this cube is similar to a regular 3x3x3. However, parity errors are possible in these solutions. This cube uses a special mechanism due to absence of a central core.

Pattern variations

There are many puzzles which are mechanically identical to the regular cuboids listed above but have variations in the pattern and colour of design. Some of these are custom made in very small numbers, sometimes for promotional events. The ones listed in the table below are included because the pattern in some way affects the difficulty of the solution or is notable in some other way.

Picture	Data	Comments
No Picture	Commercial name: Junior Cube Geometric shape: Cube Piece configuration: 2×2×2	Mechanically identical to the Pocket Cube. However, much easier to solve as it only uses two colours.
[9]	Commercial name: Fooler Cube Geometric shape: Cube Piece configuration: 3×3×3	Mechanically identical to the standard 3×3×3 cube but not a real puzzle since all the faces are the same colour. There are also cubes which have only three colours, either one colour per pair of opposite faces or one colour per layer.
	Commercial name: Calendar Cube Geometric shape: Cube Piece configuration: 3×3×3	Mechanically identical to the standard 3×3×3 cube, but with specially printed stickers for displaying the date. Much easier to solve since five of the six faces are ignored. Ideal produced a commercial version during the initial cube craze. Sticker sets are also available for converting a normal cube into a calendar.
[10]	Rubik's Cube for the blind Geometric shape: Cube Piece configuration: 3×3×3	Mechanically identical to the standard 3×3×3 cube. However the pieces are in some way tactile to allow operation by blind persons, or to solve blindfolded. The cube pictured is the original "Blind Man's Cube" made by Politechnika. This is coloured the same as the standard cube, but there is an embossed symbol on each square which corresponds to a colour.

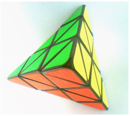
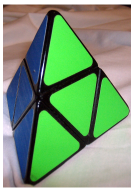
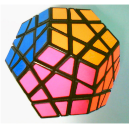
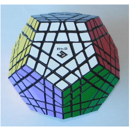
	Commercial Name: Magic Cube Geometric shape: Cube Piece configuration: 3×3×3	Mechanically identical to the standard 3×3×3 cube. However, the numbers on the centre pieces force the solver to become aware that each one can be in one of four orientations, thus hugely increasing the total number of combinations. The number of combinations of centre face orientations is 4^6. However, odd combinations (overall odd number of rotations) of the centre faces cannot be achieved with legal operations. The increase is therefore $x2^{11}$ over the original making the total approximately 10^{24} combinations. This adds to the difficulty of the puzzle but not astronomically; only one or two additional algorithms are required to effect a solution. Note that the puzzle can be treated as a number magic square puzzle on each of the six faces with the magic constant being 15 in this case.
[11]	Patterned cubes Geometric shape: Cube Piece configuration: 3×3×3	Mechanically identical to the standard 3×3×3 cube. The pattern, which is often a promotional logo or pictures of performers, will usually have the effect of making the orientation of the centre pieces 'count' in the solution. The solution is therefore the same as the 'Magic Square' cube above.
	Commercial name: Sudoku Cube Geometric shape: Cube Piece configuration: 3×3×3	Identical to the Rubik's Cube in mechanical function, it adds another layer of difficulty in that the numbers must all have the same orientation and there are no colors to follow. The name reflects its superficial resemblance to the two-dimensional Sudoku number puzzle.
Over The Top [12]	Commercial name: Over The Top Geometric shape: Cube Piece configuration: 17x17x17 Inventor: Oskar van Deventer	A remarkable extension to the basic Rubik's Cube. Experimental; not yet prototyped. Would be made by 3-D printing of plastic. Corners are much larger in proportion, and edge pieces match that larger dimension; they are narrow, and do not resemble cubes. The rest of the cubelets are 15x15 arrays on each side of the whole cube; as planned, they would be only 4 mm on a side. The mechanism is a 3x3x3 core, with thin "vanes" for the center edges; the rest of the cubelets fill in the gaps. The core has a sphere at its center. Surrounding the core are six concentric spherical shells (or more, depending on your definition). The scheme is quite different from that of Panagiotis Verdes, the inventor of the V Cubes. Mr. van Deventer is a noted inventor of puzzles.

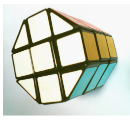
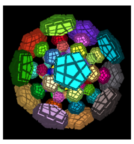
Irregular cuboids

An irregular cuboid, in the context of this article, is a cuboid puzzle where not all the pieces are the same size in edge length. This category of puzzle is often made by taking a larger regular cuboid puzzle and fusing together some of the pieces to make larger pieces. In the formulae for piece configuration, the configuration of the fused pieces is given in brackets. Thus, (as a simple regular cuboid example) a 2(2,2)x2(2,2)x2(2,2) is a 2×2×2 puzzle, but it was made by fusing a 4×4×4 puzzle. Puzzles which are constructed in this way are often called "bandaged" cubes. However, there are many irregular cuboids that have not (and often could not) be made by bandaging.

Picture	Data	Comments
	Commercial name: Skewb Geometric shape: Cube Piece configuration: -	Similar to the original Rubik's Cube, the Skewb differs in that its four axes of rotation pass through the corners of the cube rather than the centres of the faces. As a result, it is a deep-cut puzzle in which each twist scrambles all six faces.
[13]	Bandaged Cubes Geometric shape: Cube Piece configuration: various	The example shown in the link is a simple example of a large number of bandaged cubes that have been made.
	Commercial name: Square One Geometric shape: Cube	A variation on the original Rubik's Cube where it can be turned in such a manner as to distort the cubical shape of the puzzle. The Square One consists of three layers. The upper and lower layers contain kite and triangular pieces. The middle layer contains two trapezoid pieces, which together may form an irregular hexagon or a square. Square One is an example of another very large class of puzzle — cuboid puzzles which have cubies that are not themselves all cuboid.

Other polyhedra

Picture	Data	Comments
	Commercial Name: Pyraminx Geometric shape: Tetrahedron Piece configuration: -	Pyramid shaped puzzle similar to Rubik's cube in operation and solution.
	Commercial Name: Pyramorphix Geometric shape: Tetrahedron Piece configuration: 2×2×2	Pyramid shaped puzzle with a 2×2×2 cube mechanism.
	Commercial Name: Megaminx Geometric shape: Dodecahedron Piece configuration: Similar to 3×3×3	12-sided polyhedron puzzle similar to Rubik's cube in operation and solution.
	Commercial Name: Gigaminx, Teraminx, Petaminx Geometric shape: Dodecahedron	Megaminx variants with multiple layers per face. The Gigaminx has 2 layers per face, for a total of 5 layers per edge; the Teraminx has 3 layers per face, 7 layers per edge; and the Petaminx has 4 layers per face, 9 layers per edge.
	Commercial Name: Impossiball Geometric shape: Rounded icosahedron Piece configuration: -	Rounded icosahedron puzzle similar to Pocket Cube in operation and solution.

	Commercial Name: Alexander's Star Geometric shape: Great dodecahedron Piece configuration: -	12-sided Nonconvex uniform polyhedron puzzle similar to Rubik's cube in operation and solution.
	Commercial Name: BrainTwist Geometric shape: Tetrahedron Piece configuration: -	The BrainTwist is a unique tetrahedral puzzle with an ability to "flip", showing only half of the puzzle at a time.
	Commercial Name: Dogic Geometric shape: Icosahedron Piece configuration: -	The Dogic is an icosahedron cut into 60 triangular pieces around its 12 tips and 20 face centers.
	Commercial Name: Skewb Diamond Geometric shape: Octahedron Piece configuration: 3×3×3	An octahedral variation on the Skewb, it is a deep-cut puzzle very similar to the Skewb and is a dual-polyhedron transformation.
	Commercial Name: Skewb Ultimate Geometric shape: Dodecahedron Piece configuration: -	While appearing more difficult than the Skewb Diamond, it is functionally the same as the Skewb and Skewb Diamond. The puzzle is cut in a different manner but the same solutions can be used to solve it by identifying what pieces are equivalent.
	Commercial Name: Barrel Cube Geometric shape: Octagonal Prism Piece configuration: 3×3×3	Mechanically identical to the 3×3×3 cube. It does, however, have an interesting difference in its solution. The vertical corner columns are different colours to the faces and do not match the colours of the vertical face columns. The corner columns can therefore be placed in any corner. On the face of it, this makes the solution easier, however odd combinations of corner columns cannot be achieved by legal moves. The solver may unwittingly attempt an odd combination solution, but will not be aware of this until the last few pieces.
	Commercial Name: Diamond Cube Geometric shape: Rhombicuboctahedron Piece configuration: 3×3×3	Mechanically identical to the 3×3×3 cube although the example pictured is easier to solve due to the restricted colour scheme. This puzzle is a rhombicuboctahedron but not a uniform one as the edge pieces are oblong rather than square. There is in existence a similar puzzle actually called Rhombicuboctahedron which *is* uniform.
	Commercial Name: Pyraminx Crystal Geometric shape: Dodecahedron Piece configuration: -	A dodecahedron cut into 20 corner pieces and 40 edge pieces. It is similar to the Megaminx, but is deeper cut, giving edges that behave differently from the Megaminx's edges when twisted.
	Commercial Name: Magic 120-cell Geometric shape: 120-cell Piece configuration: 3×3×3×3	Virtual 4-dimensional puzzle, the 4-D analogue of the Megaminx.

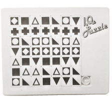
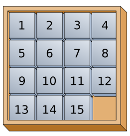
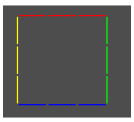
Other geometric shapes

Picture	Data	Comments
	Commercial Name: Magic Ball Geometric shape: Sphere Piece configuration: 3×3×3	Also known as Rubik's Sphere. Mechanically identical to the 3×3×3 cube in operation and solution. The only practical difference is that it is rather hard to grip. This accounts for the poor condition of this specimen, as the coloured stickers tend to get forced off in use.

Non-Rubik style three-dimensional

Picture	Data	Comments
	Commercial Name: Rubik's Clock Piece configuration: 3×3×2 12-position dials	Rubik's Clock is a two-sided puzzle, each side presenting nine clocks to the puzzler. There are four wheels, one at each corner of the puzzle, each allowing the corresponding corner clock to be rotated directly.
	Commercial Name: Rubik's Snake Piece configuration: 24 pieces	Some would not count this as a combinational puzzle despite it bearing the Rubik name. Also known as Rubik's Twist. There is no one solution to this puzzle but multiple different shapes can be made.

Two-dimensional

Picture	Data	Comments
	Sliding piece puzzle Piece configuration: 7×7	These ubiquitous puzzles come in many sizes and designs. The traditional design is with numbers and the solution forms a magic square. There have been many different designs, the example shown here uses graphic symbols instead of numbers. The solution requires that there are no repeated symbols in any row column or diagonal. The picture shows the puzzle unsolved.
	Sliding piece puzzle with picture Piece configuration: 7×7	Mechanically, no different from the puzzle above. However, the picture on the pieces gives the puzzle something of the nature of a jigsaw in addition to being a combination puzzle. Note that the picture consists of multitude of a polyhedra which have been made into Rubik puzzles.
	Fifteen puzzle Piece configuration: 4×4-1	The original sliding piece puzzle.
	Rubik's Magic	Not entirely 2D. Invoves flipping parts back onto itself.
	Commercial name:2D Magic Cube Geometric shape:Square Piece configuration: 3×3	Another virtual puzzle in the Rubik series, but this time a very simple one.

	Klotski Piece configuration: 4×5-2 with some fused pieces	A traditional sliding piece puzzle. There are now endless variations of this original puzzle implemented as computer games.

See also

- N-dimensional sequential move puzzles

References

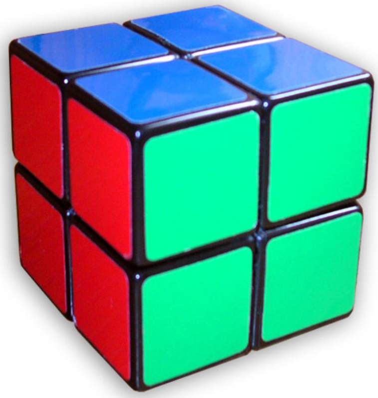
[1] http://www.twistypuzzles.com/cgi-bin/puzzle.cgi?pkey=24
[2] http://www.twistypuzzles.com/cgi-bin/puzzle.cgi?pkey=21
[3] http://www.twistypuzzles.com/cgi-bin/puzzle.cgi?pkey=1357
[4] http://www.twistypuzzles.com/cgi-bin/puzzle.cgi?pkey=1285
[5] http://www.twistypuzzles.com/cgi-bin/puzzle.cgi?pkey=1271
[6] (http://www.puzzlemuseum.com/month/picm03/200301mcu.htm) Collection of cube puzzles at The Puzzle Museum
[7] http://www.twistypuzzles.com/cgi-bin/puzzle.cgi?pkey=439
[8] http://www.twistypuzzles.com/cgi-bin/puzzle.cgi?pkey=13
[9] http://www.twistypuzzles.com/cgi-bin/puzzle.cgi?pkey=663
[10] http://www.twistypuzzles.com/cgi-bin/puzzle.cgi?pkey=63
[11] http://www.twistypuzzles.com/cgi-bin/puzzle.cgi?pkey=237
[12] http://www.shapeways.com/model/64058/over_the_top___17x17x17____3500.html
[13] http://www.twistypuzzles.com/cgi-bin/puzzle.cgi?pkey=530

External links

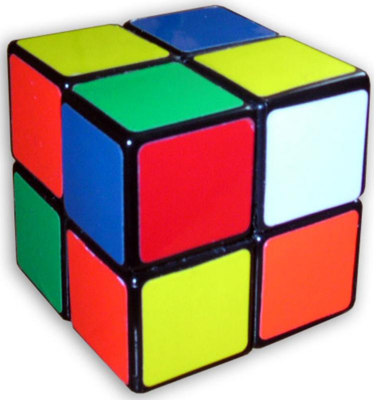
- A large database of twisty puzzles (http://www.twistypuzzles.com)
- A collection of Java applet combination puzzle simulators (http://www.mud.ca/puzzler/JPuzzler/JPuzzler.html)
- The Puzzle Museum (http://www.puzzlemuseum.com/)
- The Magic Polyhedra Patent Page (http://www.calormen.com/TwistyPuzzles/twisty.htm)

Pocket Cube

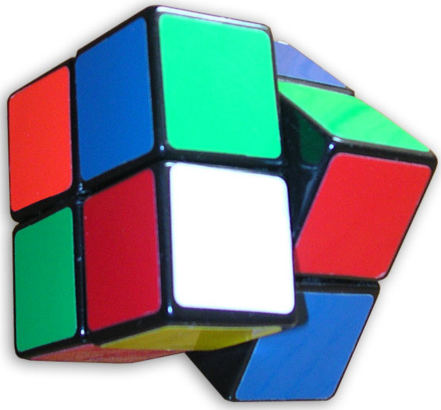
The **Pocket Cube** (also known as the **Mini Cube**) is the 2×2×2 equivalent of a Rubik's Cube. The cube consists of 8 corner pieces, and no other types of cubies.



Solved Pocket Cube.



Scrambled Pocket Cube.



Pocket Cube with one side tilted.

Permutations

Any permutation of the 8 corner cubies is possible (8! positions), and 7 of the cubies can be independently rotated (3^7 positions). There is nothing identifying the orientation of the cube in space, reducing the positions by a factor of 24. This is derived from the fact all 24 possible positions and orientations of the first corner are equivalent because of the lack of face centers. This factor does not appear when calculating the permutations of N×N×N cubes where N is odd, since those puzzles have fixed centers which identify the cube's spatial orientation. The number of possible positions of the cube is

$$\frac{8! \times 3^7}{24} = 7! \times 3^6 = 3674160.$$

The maximum number of turns required to solve the cube is up to 11 full turns, or up to 14 quarter turns.

The number *f* of positions that require *n* full twists and number *q* of positions that require *n* quarter turn twists are:

n	*f*	*q*
0	1	1
1	9	6
2	54	27
3	321	120
4	1847	534
5	9992	2256
6	50136	8969
7	227536	33058
8	870072	114149
9	1887748	360508
10	623800	930588
11	2644	1350852
12	0	782536
13	0	90280
14	0	276

Records

Erik Akkersdijk holds the current world record of solving the Pocket Cube in competition, with a time of 0.96 seconds set at the *Geneva Open 2008*. Rowe Hessler tied this record at the *US Nationals 2010*. For the best average time of 5 solves, Feliks Zemdegs holds the world record with a time of 2.12 seconds set at Melbourne Cube day 2010.[1]

Vicente Albíter of Mexico solving it in 1.55 seconds at the *Mexican Open 2008*

Variants

At Rubik's online store, an easier version of the Pocket Cube exists, dubbed the "Junior Cube". This version has only two colors, with a picture of a monkey on one face.

The Rubik's Ice Cube is a version of the Pocket Cube with transparent plastic and translucent stickers. It comes with a clear blue, ice-like display base.

The Eastsheen company produces a variation as well. It has a different, smoother-turning mechanism and is noticeably larger (5 cm) than the original.

Methods

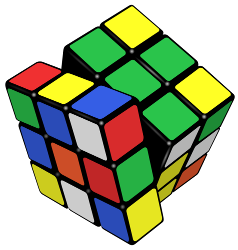
There are several commonly used speed methods on the 2×2×2. The two most popular 2×2×2 specific speed methods are the Guimond and Ortega methods. Both of these methods have been proven to have the potential to break 5 seconds on average. Extremely fast but algorithm-heavy methods include the Stern-Sun (SS), Erik-Gunnar (EG), and CLL methods.

See also

- Pyramorphix, a pyramidal puzzle that uses the same mechanism
- Rubik's Cube (3×3×3)
- Rubik's Revenge (4×4×4)
- Professor's Cube (5×5×5)
- V-Cube 6 (6×6×6)
- V-Cube 7 (7×7×7)
- Speedcubing
- Combination puzzles

[1] World Cube Association Official Results - 2×2×2 Cube (http://www.worldcubeassociation.org/results/regions.php?eventId=222).

Rubik's Cube

Rubik's Cube	
	
Type	Puzzle
Inventor	Ernő Rubik
Company	Ideal Toys
Country	Hungary
Availability	1974–present
Official website [1]	

The **Rubik's Cube** is a 3-D mechanical puzzle invented in 1974[2] by Hungarian sculptor and professor of architecture Ernő Rubik. Originally called the "Magic Cube",[3] the puzzle was licensed by Rubik to be sold by Ideal Toys in 1980[4] and won the German Game of the Year special award for Best Puzzle that year. As of January 2009, 350 million cubes have sold worldwide[5] [6] making it the world's top-selling puzzle game.[7] [8] It is widely considered to be the world's best-selling toy.[9]

In a classic Rubik's Cube, each of the six faces is covered by nine stickers, among six solid colours (traditionally white, red, blue, orange, green, and yellow).[10] A pivot mechanism enables each face to turn independently, thus mixing up the colours. For the puzzle to be solved, each face must be a solid colour. Similar puzzles have now been produced with various numbers of stickers, not all of them by Rubik. The original 3×3×3 version celebrates its thirtieth anniversary in 2010.

Conception and development

Prior attempts

In March 1970, Larry Nichols invented a 2×2×2 "Puzzle with Pieces Rotatable in Groups" and filed a Canadian patent application for it. Nichols's cube was held together with magnets. Nichols was granted U.S. Patent 3655201 [11] on April 11, 1972, two years before Rubik invented his Cube.

On April 9, 1970, Frank Fox applied to patent his "Spherical 3×3×3". He received his UK patent (1344259) on January 16, 1974.

Rubik's invention

In the mid-1970s, Ernő Rubik worked at the Department of Interior Design at the Academy of Applied Arts and Crafts in Budapest.[12] Although it is widely reported that the Cube was built as a teaching tool to help his students understand 3D objects, his actual purpose was solving the structural problem of moving the parts independently without the entire mechanism falling apart. He did not realize that he had created a puzzle until the first time he scrambled his new Cube and then tried to restore it.[13] Rubik obtained Hungarian patent HU170062 for his "Magic Cube" in 1975 but did not take out international patents at that time. The first test batches of the product were produced in late 1977 and released to Budapest toy shops. Magic Cube was held together with interlocking plastic pieces that prevented the puzzle being easily pulled apart, unlike the magnets in Nichols's design. In September 1979, a deal was signed with Ideal Toys to bring the Magic Cube to the Western world, and the puzzle made its international debut at the toy fairs of London, Paris, Nuremberg and New York in January and February 1980.

After its international debut, the progress of the Cube towards the toy shop shelves of the West was briefly halted so that it could be manufactured to Western safety and packaging specifications. A lighter Cube was produced, and Ideal Toys decided to rename it. "The Gordian Knot" and "Inca Gold" were considered, but the company finally decided on "Rubik's Cube", and the first batch was exported from Hungary in May 1980. Taking advantage of an initial shortage of Cubes, many imitations appeared.

Packaging of Rubik's Cube, Toy of the year 1980- Ideal Toy Corp 1980, Made in Hungary.

Patent disputes

Nichols assigned his patent to his employer Moleculon Research Corp., which sued Ideal Toy Company in 1982. In 1984, Ideal lost the patent infringement suit and appealed. In 1986, the appeals court affirmed the judgment that Rubik's 2×2×2 Pocket Cube infringed Nichols's patent, but overturned the judgment on Rubik's 3×3×3 Cube.[14]

Even while Rubik's patent application was being processed, Terutoshi Ishigi, a self-taught engineer and ironworks owner near Tokyo, filed for a Japanese patent for a nearly identical mechanism, which was granted in 1976 (Japanese patent publication JP55-008192). Until 1999, when an amended Japanese patent law was enforced, Japan's patent office granted Japanese patents for non-disclosed technology within Japan without requiring worldwide novelty.[15] [16] Hence, Ishigi's patent is generally accepted as an independent reinvention at that time.[17] [18] [19]

Rubik applied for another Hungarian patent on October 28, 1980, and applied for other patents. In the United States, Rubik was granted U.S. Patent 4378116 [20] on March 29, 1983, for the Cube.

Greek inventor Panagiotis Verdes patented[21] a method of creating cubes beyond the 5×5×5, up to 11×11×11, in 2003 although he claims he originally thought of the idea around 1985.[22] As of June 19, 2008, the 5×5×5, 6×6×6, and 7×7×7 models are in production in his "V-Cube" line.

Mechanics

A standard Rubik's cube measures 5.7 cm (approximately 2¼ inches) on each side. The puzzle consists of twenty-six unique miniature cubes, also called "cubies" or "cubelets". Each of these includes a concealed inward extension that interlocks with the other cubes, while permitting them to move to different locations. However, the centre cube of each of the six faces is merely a single square façade; all six are affixed to the core mechanism. These provide structure for the other pieces to fit into and rotate around. So there are twenty-one pieces: a single core piece consisting of three intersecting axes holding the six centre squares in place but letting them rotate, and twenty smaller plastic pieces which fit into it to form the assembled puzzle.

Rubik's Cube partially disassembled

Each of the six center pieces pivots on a screw (fastener) held by the center piece, a "3-D cross". A spring between each screw head and its corresponding piece tensions the piece inward, so that collectively, the whole assembly remains compact, but can still be easily manipulated. The screw can be tightened or loosened to change the "feel" of the Cube. Newer official Rubik's brand cubes have rivets instead of screws and cannot be adjusted.

The Cube can be taken apart without much difficulty, typically by rotating the top layer by 45° and then prying one of its edge cubes away from the other two layers. Consequently it is a simple process to "solve" a Cube by taking it apart and reassembling it in a solved state.

There are six central pieces which show one coloured face, twelve edge pieces which show two coloured faces, and eight corner pieces which show three coloured faces. Each piece shows a unique colour combination, but not all combinations are present (for example, if red and orange are on opposite sides of the solved Cube, there is no edge piece with both red and orange sides). The location of these cubes relative to one another can be altered by twisting an outer third of the Cube 90°, 180° or 270°, but the location of the coloured sides relative to one another in the completed state of the puzzle cannot be altered: it is fixed by the relative positions of the centre squares. However, Cubes with alternative colour arrangements also exist; for example, with the yellow face opposite the green, the blue face opposite the white, and red and orange remaining opposite each other.

Douglas Hofstadter, in the July 1982 issue of *Scientific American*, pointed out that Cubes could be coloured in such a way as to emphasise the corners or edges, rather than the faces as the standard colouring does; but neither of these alternative colourings has ever become popular.

Mathematics

Permutations

The original (3×3×3) Rubik's Cube has eight corners and twelve edges. There are 8! (40,320) ways to arrange the corner cubes. Seven can be oriented independently, and the orientation of the eighth depends on the preceding seven, giving 3^7 (2,187) possibilities. There are 12!/2 (239,500,800) ways to arrange the edges, since an odd permutation of the corners implies an odd permutation of the edges as well. Eleven edges can be flipped independently, with the flip of the twelfth depending on the preceding ones, giving 2^{11} (2,048) possibilities.[23]

$$8! \times 3^7 \times 12!/2 \times 2^{11} \approx 4.33 \times 10^{19}$$

There are exactly 43,252,003,274,489,856,000 permutations, which is approximately forty-three quintillion. The puzzle is often advertised as having only "billions" of positions, as the larger numbers could be regarded as incomprehensible to many. To put this into perspective, if one had as many 57-millimeter Rubik's Cubes as there are permutations, one could cover the Earth's surface with a layer 275 cubes thick.

The preceding figure is limited to permutations that can be reached solely by turning the sides of the cube. If one considers permutations reached through disassembly of the cube, the number becomes twelve times as large:

$$8! \times 3^8 \times 12!/2 \times 2^{13} \approx 5.19 \times 10^{20}.$$

The full number is 519,024,039,293,878,272,000 or 519 quintillion possible arrangements of the pieces that make up the Cube, but only one in twelve of these are actually solvable. This is because there is no sequence of moves that will swap a single pair of pieces or rotate a single corner or edge cube. Thus there are twelve possible sets of reachable configurations, sometimes called "universes" or "orbits", into which the Cube can be placed by dismantling and reassembling it.

Centre faces

The original Rubik's Cube had no orientation markings on the centre faces, although some carried the words "Rubik's Cube" on the centre square of the white face, and therefore solving it does not require any attention to orienting those faces correctly. However, if one has a marker pen, one could, for example, mark the central squares of an unscrambled Cube with four coloured marks on each edge, each corresponding to the colour of the adjacent face. Some Cubes have also been produced commercially with markings on all of the squares, such as the Lo Shu magic square or playing card suits. Thus one can nominally solve a Cube yet have the markings on the centres rotated; it then becomes an additional test to solve the centers as well.

Marking the Rubik's Cube increases its difficulty because this expands its set of distinguishable possible configurations. When the Cube is unscrambled apart from the orientations of the central squares, there will always be an even number of squares requiring a quarter turn. Thus there are $4^6/2$ = 2,048 possible configurations of the centre squares in the otherwise unscrambled position, increasing the total number of possible Cube permutations from 43,252,003,274,489,856,000 (4.3×10^{19}) to 88,580,102,706,155,225,088,000 (8.9×10^{22}).[24]

Algorithms

In Rubik's cubists' parlance, a memorised sequence of moves that has a desired effect on the cube is called an algorithm. This terminology is derived from the mathematical use of *algorithm*, meaning a list of well-defined instructions for performing a task from a given initial state, through well-defined successive states, to a desired end-state. Each method of solving the Rubik's Cube employs its own set of algorithms, together with descriptions of what the effect of the algorithm is, and when it can be used to bring the cube closer to being solved.

Most algorithms are designed to transform only a small part of the cube without scrambling other parts that have already been solved, so that they can be applied repeatedly to different parts of the cube until the whole is solved. For example, there are well-known algorithms for cycling three corners without changing the rest of the puzzle, or flipping the orientation of a pair of edges while leaving the others intact.

Some algorithms have a certain desired effect on the cube (for example, swapping two corners) but may also have the side-effect of changing other parts of the cube (such as permuting some edges). Such algorithms are often simpler than the ones without side-effects, and are employed early on in the solution when most of the puzzle has not yet been solved and the side-effects are not important. Towards the end of the solution, the more specific (and usually more complicated) algorithms are used instead, to prevent scrambling parts of the puzzle that have already been solved.

Solutions

Move notation

Many 3×3×3 Rubik's Cube enthusiasts use a notation developed by David Singmaster to denote a sequence of moves, referred to as "Singmaster notation".[25] Its relative nature allows algorithms to be written in such a way that they can be applied regardless of which side is designated the top or how the colours are organised on a particular cube.

- *F* (Front): the side currently facing you
- *B* (Back): the side opposite the front
- *U* (Up): the side above or on top of the front side
- *D* (Down): the side opposite the top, underneath the Cube
- *L* (Left): the side directly to the left of the front
- *R* (Right): the side directly to the right of the front
- *f* (Front two layers): the side facing you and the corresponding middle layer
- *b* (Back two layers): the side opposite the front and the corresponding middle layer
- *u* (Up two layers) : the top side and the corresponding middle layer
- *d* (Down two layers) : the bottom layer and the corresponding middle layer
- *l* (Left two layers) : the side to the left of the front and the corresponding middle layer
- *r* (Right two layers) : the side to the right of the front and the corresponding middle layer
- *x* (rotate): rotate the entire Cube on *R*
- *y* (rotate): rotate the entire Cube on *U*
- *z* (rotate): rotate the entire Cube on *F*

When a prime symbol (′) follows a letter, it denotes a face turn counter-clockwise, while a letter without a prime symbol denotes a clockwise turn. A letter followed by a 2 (occasionally a superscript ^{2}) denotes two turns, or a 180-degree turn. *R* is right side clockwise, but *R*' is right side counter-clockwise. The letters *x*, *y*, and *z* are used to indicate that the entire Cube should be turned about one of its axes. When *x*, *y* or *z* are primed, it is an indication that the cube must be rotated in the opposite direction. When they are squared, the cube must be rotated twice.

For methods using middle-layer turns (particularly corners-first methods) there is a generally accepted "MES" extension to the notation where letters *M*, *E*, and *S* denote middle layer turns. It was used e.g. in Marc Waterman's Algorithm.[26]

- *M* (Middle): the layer between L and R, turn direction as L (top-down)
- *E* (Equator): the layer between U and D, turn direction as D (left-right)
- *S* (Standing): the layer between F and B, turn direction as F

The 4×4×4 and larger cubes use an extended notation to refer to the additional middle layers. Generally speaking, uppercase letters (*F B U D L R*) refer to the outermost portions of the cube (called faces). Lowercase letters (*f b u d ℓ r*) refer to the inner portions of the cube (called slices). An asterisk (L*), a number in front of it (2L), or two layers in parenthesis (Lℓ), means to turn the two layers at the same time (both the inner and the outer left faces) For example: (*Rr*)' ℓ2 *f*' means to turn the two rightmost layers counterclockwise, then the left inner layer twice, and then the inner front layer counterclockwise.

Optimal solutions

Although there are a significant number of possible permutations for the Rubik's Cube, there have been a number of solutions developed which allow for the cube to be solved in well under 100 moves.

Many general solutions for the Rubik's Cube have been discovered independently. The most popular method was developed by David Singmaster and published in the book *Notes on Rubik's "Magic Cube"* in 1981. This solution involves solving the Cube layer by layer, in which one layer (designated the top) is solved first, followed by the middle layer, and then the final and bottom layer. After practice, solving the Cube layer by layer can be done in under one minute. Other general solutions include "corners first" methods or combinations of several other methods. In 1982, David Singmaster and Alexander Frey hypothesised that the number of moves needed to solve the Rubik's Cube, given an ideal algorithm, might be in "the low twenties". In 2007, Daniel Kunkle and Gene Cooperman used computer search methods to demonstrate that any 3×3×3 Rubik's Cube configuration can be solved in 26 moves or less.[27] [28] [29] In 2008, Tomas Rokicki lowered that number to 22 moves,[30] [31] [32] and in July 2010, a team of researchers including Rokicki, working with Google, proved the so-called "God's number" to be 20.[33] [34] This is optimal, since there exist some starting positions which require at least 20 moves to solve.

A solution commonly used by speed cubers was developed by Jessica Fridrich. It is similar to the layer-by-layer method but employs the use of a large number of algorithms, especially for orienting and permuting the last layer. The cross is done first followed by first-layer corners and second layer edges simultaneously, with each corner paired up with a second-layer edge piece. This is then followed by orienting the last layer then permuting the last layer (OLL and PLL respectively). Fridrich's solution requires learning roughly 120 algorithms but allows the Cube to be solved in only 55 moves on average.

Philip Marshall's *The Ultimate Solution to Rubik's Cube* is a modified version of Fridrich's method, averaging only 65 twists yet requiring the memorization of only *two* algorithms.[35]

A now well-known method was developed by Lars Petrus. In this method, a 2×2×2 section is solved first, followed by a 2×2×3, and then the incorrect edges are solved using a three-move algorithm, which eliminates the need for a possible 32-move algorithm later. The principle behind this is that in layer by layer you must constantly break and fix the first layer; the 2×2×2 and 2×2×3 sections allow three or two layers to be turned without ruining progress. One of the advantages of this method is that it tends to give solutions in fewer moves.

In 1997, Denny Dedmore published a solution described using diagrammatic icons representing the moves to be made, instead of the usual notation.[36]

Competitions and records

Speedcubing competitions

Speedcubing (or speedsolving) is the practice of trying to solve a Rubik's Cube in the shortest time possible. There are a number of speedcubing competitions that take place around the world.

The first world championship organised by the *Guinness Book of World Records* was held in Munich on March 13, 1981. All Cubes were moved 40 times and lubricated with petroleum jelly. The official winner, with a record of 38 seconds, was Jury Froeschl, born in Munich. The first international world championship was held in Budapest on June 5, 1982, and was won by Minh Thai, a Vietnamese student from Los Angeles, with a time of 22.95 seconds.

Since 2003, the winner of a competition is determined by taking the average time of the middle three of five attempts. However, the single best time of all tries is also recorded. The World Cube Association maintains a history of world records.[37] In 2004, the WCA made it mandatory to use a special timing device called a Stackmat timer.

In addition to official competitions, informal alternative competitions have been held which invite participants to solve the Cube in unusual situations. Some such situations include:

- Blindfolded solving[38]
- Solving the Cube with one person blindfolded and the other person saying what moves to do, known as "Team Blindfold"
- Solving the Cube underwater in a single breath[39]
- Solving the Cube using a single hand[40]
- Solving the Cube with one's feet[41]

Of these informal competitions, the World Cube Association only sanctions blindfolded, one-handed, and feet solving as official competition events.[42]

In blindfolded solving, the contestant first studies the scrambled cube (i.e., looking at it normally with no blindfold), and is then blindfolded before beginning to turn the cube's faces. Their recorded time for this event includes both the time spent examining the cube and the time spent manipulating it.

Records

The current world record for single time on a 3×3×3 Rubik's Cube was set by Feliks Zemdeg, who had a best time of 6.77 seconds at the Melbourne Cube Day 2010. The world record average solves is currently held by Feliks Zemdegs, which is 7.91 set at the same event.[43]

On March 17, 2010, 134 school boys from Dr Challoner's Grammar School, Amersham, England broke the previous Guinness World Record for most people solving a Rubik's cube at once in 12 minutes.[44] The previous record set in December 2008 in Santa Ana, CA achieved 96 completions.

Variations

There are different variations of Rubik's Cubes with up to seven layers: the 2×2×2 (Pocket/Mini Cube), the standard 3×3×3 cube, the 4×4×4 (Rubik's Revenge/Master Cube), and the 5×5×5 (Professor's Cube), the 6×6×6 (V-Cube 6), and 7×7×7 (V-Cube 7).

Variations of Rubik's Cubes, clockwise from upper left: V-Cube 7, Professor's Cube, V-Cube 6, Pocket Cube, original Rubik's Cube, Rubik's Revenge. Clicking on a cube in the picture will redirect to the respective cube's page.

CESailor Tech's E-cube is an electronic variant of the 3x3x3 cube, made with RGB LEDs and switches.[45] There are two switches on each row and column. Pressing the switches indicates the direction of rotation, which causes the LED display to change colours, simulating real rotations. The product was demonstrated at the Taiwan government show of College designs on October 30, 2008.

Another electronic variation of the 3×3×3 Cube is the Rubik's TouchCube. Sliding a finger across its faces causes its patterns of coloured lights to rotate the same way they would on a mechanical cube. The TouchCube was introduced at the American International Toy Fair in New York on February 15, 2009.[46] [47]

The Cube has inspired an entire category of similar puzzles, commonly referred to as *twisty puzzles*, which includes the cubes of different sizes mentioned above as well as various other geometric shapes. Some such shapes include the tetrahedron (Pyraminx), the octahedron (Skewb Diamond), the dodecahedron (Megaminx), the icosahedron (Dogic). There are also puzzles that change shape such as Rubik's Snake and the Square One.

Novelty Keychain

Custom-built puzzles

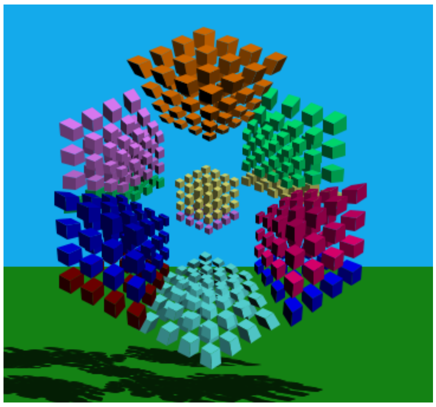
In the past, puzzles have been built resembling the Rubik's Cube or based on its inner workings. For example, a cuboid is a puzzle based on the Rubik's Cube, but with different functional dimensions, such as, 2×3×4, 3×3×5, or 2×2×4. Many cuboids are based on 4×4×4 or 5×5×5 mechanisms, via building plastic extensions or by directly modifying the mechanism itself.

Some custom puzzles are not derived from any existing mechanism, such as the Gigaminx v1.5-v2, Bevel Cube, SuperX, Toru, Rua, and 1×2×3. These puzzles usually have a set of masters 3D printed, which then are copied using molding and casting techniques to create the final puzzle.

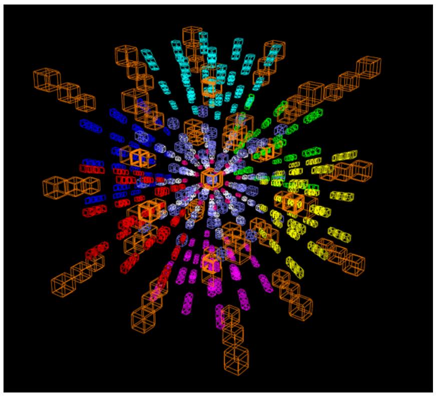
Other Rubik's Cube modifications include cubes that have been extended or truncated to form a new shape. An example of this is the Trabjer's Octahedron, which can be built by truncating and extending portions of a regular 3×3. Most shape mods can be adapted to higher-order cubes. In the case of Tony Fisher's Rhombic Dodecahedron, there are 3×3, 4×4, 5×5, and 6×6 versions of the puzzle.

Rubik's Cube software

Puzzles like the Rubik's Cube can be simulated by computer software, which provide functions such as recording of player metrics, storing scrambled Cube positions, conducting online competitions, analyzing of move sequences, and converting between different move notations. Software can also simulate very large puzzles that are impractical to build, such as 100×100×100 and 1,000×1,000×1,000 cubes, as well as virtual puzzles that cannot be physically built, such as 4- and 5-dimensional analogues of the cube.[48] [49]



Magic Cube 4D, a 4×4×4×4 virtual puzzle



Magic Cube 5D, a 3x3x3x3x3 virtual puzzle

Popular culture

Many movies and TV shows have featured characters that solve Rubik's Cubes quickly to establish their high intelligence. Rubik's cube also regularly feature as motifs in works of art.

See also

- Combination puzzles (also known as "twisty" puzzles)
- N-dimensional sequential move puzzles
- *Rubik, the Amazing Cube*
- *Rubik's 360*
- Rubik's cube group
- *Sudoku Cube*
- *Cubage (video game)*
- Octacube

- God's algorithm

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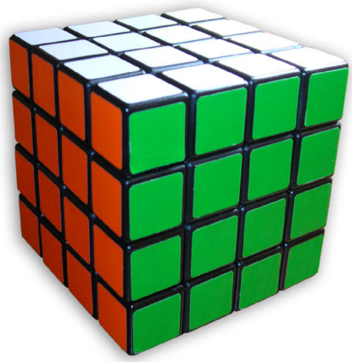
- *Unscrambling The Cube* by M. Razid Black & Herbert Taylor, Introduction by Professor Solomon W. Golomb
- *University of Michigan's endover cube decorated to look like a Rubik's cube* (http://www.ur.umich.edu/0708/Apr07_08/18.php)

External links

- Rubik's Cube (http://www.dmoz.org/Games/Puzzles/Mechanical/Rubik/) at the Open Directory Project
- Rubik's official site (http://www.rubiks.com/)
- World Cube Association (WCA) (http://www.worldcubeassociation.org)
- Speedcubing.com (http://www.speedcubing.com)
- Speedsolving.com Wiki (http://www.speedsolving.com/wiki)
- How to Solve a Rubik's Cube (Video) (http://www.youtube.com/watch?v=609nhVzg-5Q)
- How to Solve a Rubik’s Cube (http://www.wikihow.com/Solve-a-Rubik's-Cube-(Easy-Move-Notation))
- More than 2,000 cubes and cubelike puzzles (http://www.helm.lu/gallery2/main.php?g2_itemId=9236)
- Adventures in Group Theory: Rubik's Cube, Merlin's Machine, and Other Mathematical Toys (http://www.press.jhu.edu/books/title_pages/9554.html)
- A forum for twisty puzzle enthusiasts; includes commentary and advice about making custom puzzles (http://twistypuzzles.com/forum/)

Rubik's Revenge

The **Rubik's Revenge** (also known as the **Master Cube**) is the 4×4×4 version of Rubik's Cube. Invented by Péter Sebestény, the Rubik's Revenge was nearly called the **Sebestény Cube** until a somewhat last-minute decision changed the puzzle's name to attract fans of the original Rubik's Cube. Unlike the original puzzle (and the 5×5×5 cube), it has no fixed facets: the centre facets (four per face) are free to move to different positions.

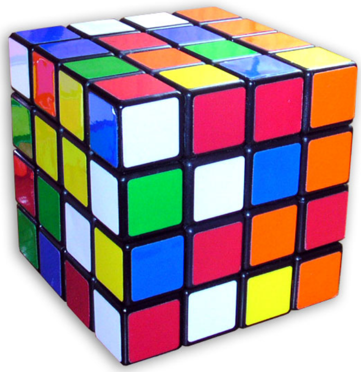


Rubik's Revenge in solved state

Methods for solving the 3×3×3 cube work for the edges and corners of the 4×4×4 cube, as long as one has correctly identified the relative positions of the colours — since the centre facets can no longer be used for identification.

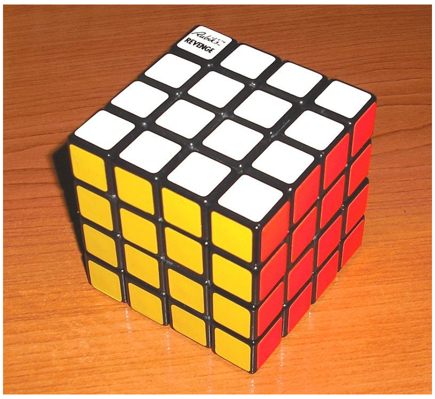
Mechanics

The puzzle consists of the 56 unique miniature cubes ("cubies") on the surface. However, the centre four cubes of each face are merely single square facades hooked into the inner mechanism of the cube. This is the largest change to the 3×3×3 cube, because the centre pieces can move in relation to each other, unlike the fixed centres on the original. The Cube can be taken apart without much difficulty, typically by turning one side through a 30° angle and prying an edge cubelet upward until it dislodges.



Rubik's Revenge in scrambled state

The original mechanism designed by Sebestény uses a grooved ball to hold the centre pieces in place. The edge pieces are held in place by the centres and the corners are held in place by the edges, much like the original cube. There are three mutually perpendicular grooves for the centre pieces to slide through. Each groove is only wide enough to allow one row of centre pieces to slide through it. The ball is shaped to prevent the centre pieces of the other row from sliding, ensuring that the ball remains aligned with the outside of the cube. Turning one of the centre layers moves either just that layer or the ball as well.[1]

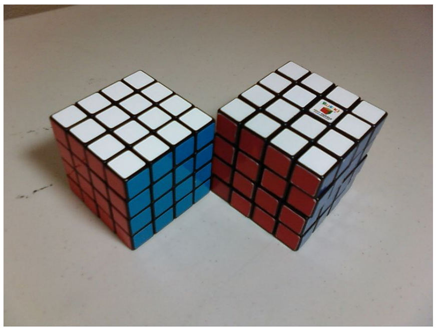


Early Rubik's Revenge cube, with white opposite blue and green opposite yellow

The Eastsheen version of the cube, which is slightly smaller at 6 cm to an edge, has a completely different mechanism. Its mechanism is very similar to Eastsheen's version of the Professor's cube, instead of the ball-core mechanism. There are 42 pieces (36 movable and six fixed) completely hidden within the cube, corresponding to the centre rows on the Professor's Cube. This design is more durable than the original and also allows for screws to be used to tighten or loosen the cube. The central spindle is specially shaped to prevent it from becoming misaligned with the exterior of the cube.[2]

There are 24 edge pieces which show two coloured sides each, and eight corner pieces which show three colours. Each corner piece or pair of edge pieces shows a unique colour combination, but not all combinations are present (for example, there is no piece with both red and orange sides, if red and orange are on opposite sides of the solved Cube). The location of these cubes relative to one another can be altered by twisting the layers of the cube, but the location of the coloured sides relative to one another in the completed state of the puzzle cannot be altered: it is fixed by the relative positions of the centre squares and the distribution of colour combinations on edge and corner pieces.

For most recent Cubes, the colours of the stickers are red opposite orange, yellow opposite white, and green opposite blue. However, there also exist Cubes with alternative colour arrangements (yellow opposite green, blue opposite white and red opposite orange). The Eastsheen version has purple (opposite red) instead of orange.



An Eastsheen cube is on the left, and an official Rubik's Revenge is on the right.

A disassembled Rubik's Revenge, showing all the pieces and central ball

A disassembled Eastsheen 4×4×4

Permutations

Rubik's Revenge with a tilted side

There are 8 corner cubelets, 24 edge cubelets and 24 centre cubelets.

Any permutation of the corner cubelets is possible, including odd permutations. Seven of the corner cubelets can be independently rotated, and the eighth cubelet's orientation depends on the other seven, giving $8!\times3^7$ combinations.

There are 24 centre cubelets, which can be arranged in 24! different ways. Assuming that the four centre cubelets of each colour are indistinguishable, the number of permutations is reduced to $24!/(4!^6)$ arrangements. The reducing factor comes from the fact that there are 4! ways to arrange the four pieces of a given colour. This is raised to the sixth power because there are six colours. An odd permutation of the corner cubelets implies an odd permutation of the centre cubelets, and vice versa; however, even and odd permutations are indistinguishable because of identically coloured centre cubelets.[3] There are several ways to make the centre pieces distinguishable, which would make an odd centre permutation visible.

The 24 edge cubelets cannot be flipped, because the internal shape of the pieces is asymmetrical. The two edge cubelets in each matching pair are distinguishable, since the colours on a cubelet are reversed relative to the other. Any permutation of the edge cubelets is possible, including odd permutations, giving 24! arrangements, independently of the corner or centre cubelets.

Assuming the cube does not have a fixed orientation in space, and that the permutations resulting from rotating the cube without twisting it are considered identical, the number of permutations is reduced by a factor of 24. This is derived from the fact that all 24 possible positions and orientations of the first corner are equivalent because of the lack of face centres. This factor does not appear when calculating the permutations of N×N×N cubes where N is odd, since those puzzles have fixed centres which identify the cube's spatial orientation.

This gives a total number of permutations of

$$\frac{8! \times 3^7 \times 24!^2}{4!^6 \times 24} \approx 7.40 \times 10^{45}.$$

The full number is 7 401 196 841 564 901 869 874 093 974 498 574 336 000 000 000 possible permutations[4] (about 7,401 septillion or 7.4 septilliard on the long scale or 7.4 quattuordecillion on the short scale).

Some versions of Rubik's Revenge have one of the centre pieces marked with a logo, distinguishing it from the other three of the same colour. This increases the number of distinguishable permutations by a factor of four to 2.96×10^{46}, although any of the four possible positions for this piece could be regarded as correct.

Solutions

There are several methods that can be used to solve a Rubik's Revenge. The layer by layer method that is often used for the 3×3×3 cube is usually used on the Rubik's Revenge. One of the most common methods is to first group the centre pieces of common colours together, then to pair edges that show the same two colours. Once this is done, turning only the outer layers of the cube allows it to be solved like a 3×3×3 cube. However, certain positions that cannot be solved on a standard 3×3×3 cube may be reached. There are two possible problems not found on the 3x3x3. The first is two edge pieces reversed on one edge, resulting in the colours for that edge not matching the rest of the cubies on either face:

Notice that these two edge pieces are swapped. The second is two edge pairs being swapped with each other:

These situations are known as parity errors. These positions are still solvable; however, special algorithms must be applied to fix the errors.

One of several approaches to solve this cube is to first pair the edges, and then the centres. This, too, is vulnerable to the parity errors described above.

Some methods are designed to avoid the parity errors described above. For instance, solving the corners and edges first and the centres last would avoid such parity errors. Once the rest of the cube is solved, any permutation of the centre pieces can be solved. Note that it is possible to apparently exchange a pair of face centres by cycling 3 face centres, two of which are visually identical.

World records

As of November 2010, the world record for 4×4×4 Cube is held by Feliks Zemdegs of Australia with a time of 31.97 seconds set at the Melbourne Cube day 2010. The best average time of 35.80 seconds is also held by Feliks Zemdegs set at the same competition. [5]

See also

- Pocket Cube (2×2×2)
- Rubik's Cube (3×3×3)
- Professor's Cube (5×5×5)
- V-Cube 6 (6×6×6)
- V-Cube 7 (7×7×7)
- Combination puzzles

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Further reading

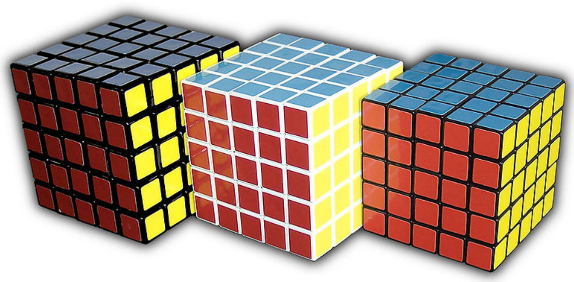
- Rubik's Revenge: The Simplest Solution by William L. Mason
- Speedsolving the Cube by Dan Harris, 'Rubik's Revenge' pages 100-120.
- The Winning Solution to Rubik's Revenge by Minh Thai, with Herbert Taylor and M. Razid Black.

External links

- Beginner/Intermediate solution to the Rubik's Revenge (http://www.speedcubing.com/chris/4-solution.html) by Chris Hardwick
- Rubik's Revenge Solution (http://www.helm.lu/cube/solutions/revenge/index.htm) good pictures, pair the edges, and then the centres solution.
- 'K4' Method (http://rxdeath.com/k4/) Advanced direct solving method.
- Patterns (http://www.randelshofer.ch/rubik/patterns_revenge.html) A collection of pretty patterns for Rubik's Revenge
- Program Rubik's Cube 3D Unlimited size (http://kubrub.googlepages.com/rubikscube)

Professor's Cube

The **Professor's Cube** is a mechanical puzzle, a 5×5×5 version of the Rubik's Cube. It has qualities in common with both the original 3×3×3 Rubik's Cube and the 4×4×4 Rubik's Revenge, and knowing the solution to either can help when working on the 5×5×5 cube.



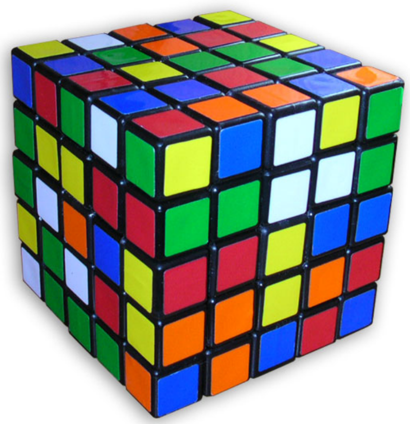
Official Professor's Cube (left) V-Cube 5 (center) and Eastsheen 5x5x5 (right).

Naming

Early versions of the 5×5×5 cube sold at Barnes and Noble were marketed under the name "Professor's Cube," but currently, Barnes and Noble sells cubes that are simply called "5×5." Mefferts.com offers a limited edition version of the 5×5×5 cube called the Professor's Cube. This version has colored tiles rather than stickers.[1] Verdes Innovations sells a version called the V-Cube 5.[2]

The V-Cube 5 in original packaging.

Workings



Scrambled.

The original Professor's Cube design by Udo Krell works by using an expanded 3×3×3 cube as a mantle with the center edge pieces and corners sticking out from the spherical center of identical mechanism to the 3×3×3 cube. The non central center pieces are fitted into spaces on the surface of the 3×3×3 mantle, and the non central edges slotted between them. All non-central pieces have extensions that fit into allotted spaces on the outer pieces of the 3×3×3, which keeps them from falling out of the cube while making a turn. The fixed centers have two sections (one visible, one hidden) which can turn independently. This feature is unique to the original design.[3]

The Eastsheen version of the puzzle uses a different mechanism. The fixed centers hold the center cubelets next to the central edges in place, which in turn hold the edge cubelets. The non-central edges hold the corners in place, and the internal sections of the corner pieces do not reach the center of the cube.[4]

The V-Cube 5 mechanism, designed by Panagiotis Verdes, has elements in common with both. The corners reach to the center of the puzzle (like the original mechanism) and the center pieces hold the central edges in place (like the Eastsheen mechanism). The middle edges and center pieces adjacent to them make up the supporting frame and these have extensions which hold rest of the pieces together. This allows smooth and fast rotation and creating arguably the fastest and most durable version of the puzzle. Unlike the original 5×5×5 design, the V-Cube 5 mechanism was designed with speedcubing in mind.[5]

A disassembled Professor's Cube.

A disassembled V-Cube 5.

A disassembled Eastsheen cube.

Durability

An original cube with a misaligned center. This cannot occur on the Eastsheen or V-Cube puzzles.

The original Professor's Cube is inherently more delicate than the 3×3×3 Rubik's Cube due to the much greater number of moving parts. It is not recommended that it be used for speedcubing. The puzzle should not be excessively forced to twist and it must be aligned properly before twisting to prevent damage.[6] It is far more likely to break due to twisting misaligned rows. If twisted while not fully aligned, it may cause the pieces diagonal to the corners to almost fully come out. It is simply fixed by turning the face back to where it was, causing the piece to go back to its original position. Excessive force may cause the colored tile to break off completely. In such a case, the puzzle will not fall apart, but a colored square would be gone. Both the Eastsheen 5×5×5 and the V-Cube 5 are designed with different mechanisms in an attempt to remedy the fragility of the original design.

Permutations

There are 98 pieces on the exterior of the cube: 8 corner cubelets, 36 edge cubelets (two types), and 54 center cubelets (48 movable of two types, 6 fixed).

Any permutation of the corner cubelets is possible, including odd permutations, giving 8! (40,320) possible arrangements. Seven of the corner cubelets can be independently rotated, and the eighth cubelet's orientation depends on the other seven, giving 3^7 combinations.

There are three types of center cubelet, corner-centers, edge-centers and inner centers. The inner centers are in a fixed permutation with the reference frame and cannot be changed. Assuming the center cubelets of each colour are indistinguishable, there are 24! ways to arrange each of the other two types, divided by $4!^6$. This reducing factor results from the fact that there are 4! ways to arrange the four cubelets of each color, raised to the sixth power because there are six colors. The total permutations for all of the movable center cubelets is the product of the permutations of corner-centers and edge centers, $(24!/(4!^6))^2$ or $24!^2/4!^{12}$.

The 24 outer edge cubelets cannot be flipped, since the interior shape of those pieces is asymmetrical. The two cubelets in each matching pair are distinguishable, since the pieces are mirror images of each other. Any permutation of the outer edge cubelets is possible, including odd permutations, giving 24! arrangements. The 12 central edge cubelets can be flipped. Eleven can be flipped and arranged independently, giving $12!/2 \times 2^{11}$ or $12! \times 2^{10}$ possibilities (an odd permutation of the corner cubelets implies an odd permutation of the central edge cubelets, and vice versa, thus the division by 2). There are $24! \times 12! \times 2^{10}$ possibilities for the inner and outer edge cubelets together.

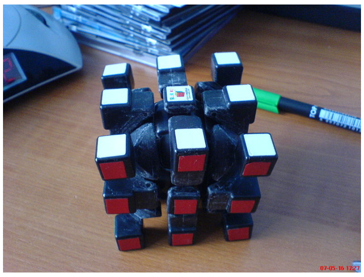
This gives a total number of permutations of

$$\frac{8! \times 3^7 \times 12! \times 2^{10} \times 24!^3}{4!^{12}} \approx 2.83 \times 10^{74}$$

The full number is precisely 282 870 942 277 741 856 536 180 333 107 150 328 293 127 731 985 672 134 721 536 000 000 000 000 000 possible permutations[7] (about 283 duodecillion on the long scale or 283 tresvigintillion on the short scale).

Some variations of the Professor's Cube have one of the center pieces marked with a logo, which can be put into four different orientations. This increases the number of permutations by a factor of four to 1.13×10^{75}, although any orientation of this piece could be regarded as correct. Other variations increase the difficulty by making the orientation of all center pieces visible. An example of this is shown below.

Solution

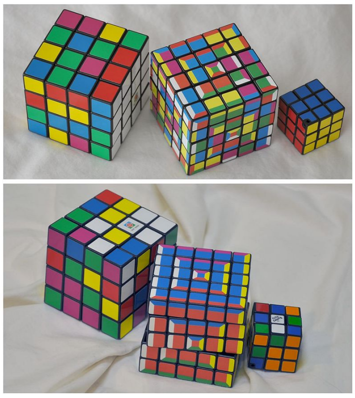


An original Professor's Cube with many of the pieces removed, showing the 3×3×3 equivalence of the remaining pieces.

People able to rapidly solve puzzles like this usually favour the strategy of grouping similar edge pieces into solid strips, and centers into one-colored blocks. This allows the cube to be quickly solved with the same methods one would use for a 3×3×3 cube. As illustrated to the right, the fixed centers, middle edges and corners can be treated as equivalent to a 3×3×3 cube. As a result, the parity errors sometimes seen on the 4×4×4 cannot occur on the 5×5×5 unless the cube has been tampered with.

Another frequently used strategy is to solve the edges of the cube first. The corners can be placed just as they are in any previous order of cube puzzle, and the centers are manipulated with an algorithm similar to the one used in the 4×4×4 cube.

A less frequently used strategy is to solve one side and one rim first, then the 2nd, 3rd and 4th rim, and finally the last side and rim. That is, like building a four floor building. First the basement, then each floor, and finally the roof. This method is among other described at [8] .



Center is an EastSheen 5x5x5 cube with multicolored stickers, which increase difficulty.

World records

As of September 2010, the current record for solving the Professor's Cube in an official competition is 1 minute 2.93 seconds, set by Feliks Zemdegs, citizen of Australia at the Australian Nationals 2010. He also holds the current world record for an average of five solves, 1:07.59, set at the same event.[9]

See also

- Pocket Cube – A 2×2×2 version of the puzzle
- Rubik's Cube – The original version of this puzzle
- Rubik's Revenge – A 4×4×4 version of the puzzle
- V-Cube 6 - A 6×6×6 version of the puzzle
- V-Cube 7 - A 7×7×7 version of the puzzle
- Combination puzzles

References

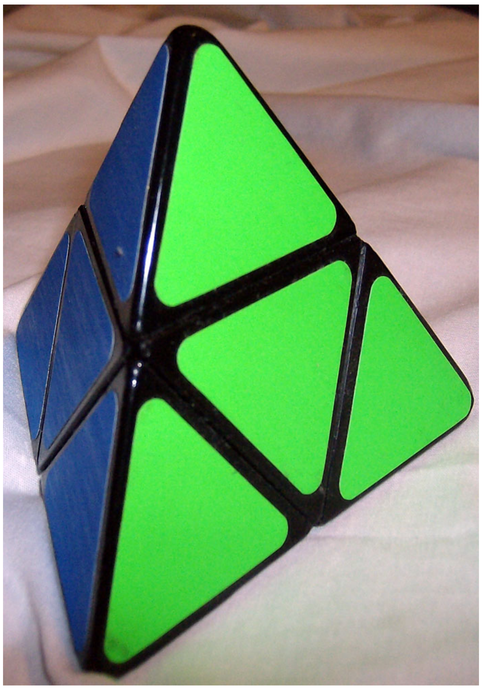
[1] Meffert's Professor's Cube (http://www.mefferts.com/products/details.php?lang=en&category=13&id=238)
[2] Verdes' Innovations V-Cube 5 page (http://www.v-cubes.com/prod_info/v-cube5.php)
[3] United States Patent 4600199 (http://www.freepatentsonline.com/4600199.html)
[4] United States Patent 6129356 (http://www.freepatentsonline.com/6129356.html)
[5] United States Patent 20070057455 (http://www.freepatentsonline.com/y2007/0057455.html)
[6] Rubi's 5x5x5 Cube notice section (http://www.rubiks.com/Shop/Products/Rubiks 5x5 Cube Hex Packaging.aspx)
[7] Cubic Circular Issues 3 & 4 (http://www.jaapsch.net/puzzles/cubic3.htm#p18) David Singmaster, 1982
[8] Rubiks-Cube.org (http://www.rubiks-cube.org)
[9] World Cube Association Official Results - 5x5x5 Cube (http://www.worldcubeassociation.org/results/e.php?i=555)

External links

- How to solve Professor's Cube (http://www.bigcubes.com/5x5x5/5x5x5.html)
- Professor's Cube text solution (http://wiki.playagaingames.com/tiki-index.php?page=5x5x5+Cube+Solution)
- Professor's Cube interactive solution (http://www.rubiks-zauberwuerfel.de)

Pyramorphix

The **Pyramorphix** is a tetrahedral puzzle similar to the Rubik's Cube. It has a total of 8 movable pieces to rearrange, compared to the 20 of the Rubik's cube. Though it looks like a simpler version of the Pyraminx, it is an edge-turning puzzle with the mechanism identical to that of the Pocket Cube.



The Pyramorphix in its solved state.

Description

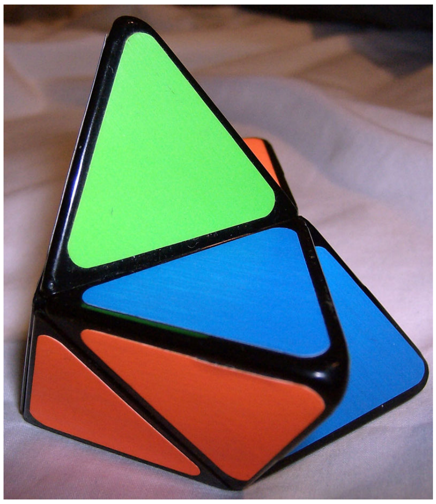
At first glance, the Pyramorphix appears to be a trivial puzzle. It resembles the Pyraminx, and its appearance would suggest that only the four corners could be rotated. In fact, the puzzle is a specially shaped 2×2×2 cube. Four of the cube's corners are reshaped into pyramids and the other four are reshaped into triangles. The result of this is a puzzle that changes shape as it is turned.

The purpose of the puzzle is to scramble the colors and the shape, and then restore it to its original state of being a tetrahedron with one color per face.

Number of combinations

The puzzle is available either with stickers or plastic tiles on the faces. Both have a ribbed appearance, giving a visible orientation to the flat pieces. This results in 3,674,160 combinations, the same as the 2×2×2 cube.

However, if there were no means of identifying the orientation of those pieces, the number of combinations would be reduced. There would be 8! ways to arrange the pieces, divided by 24 to account for the lack of center pieces, and there would be 3^4 ways to rotate the four pyramidal pieces.


The Pyramorphix, scrambled.

$$\frac{8! \times 3^4}{24} = 136080$$

The Pyramorphix can be rotated around three axes by multiples of 90°. The corners cannot rotate individually as on the Pyraminx. The Pyramorphix rotates in a way that changes the position of center pieces not only with other center pieces but also with corner pieces, leading to a variety of shapes.

Master Pyramorphix

The **Master Pyramorphix** is a more complex variant of the Pyramorphix. Like the Pyramorphix, it is an edge-turning tetrahedral puzzle capable of changing shape as it is twisted, leading to a large variety of irregular shapes. Several different variants have been made, including flat-faced custom-built puzzles by puzzle fans and Uwe Mèffert's commercially-produced pillowed variant (pictured), sold through his puzzle shop *Meffert's*.

The Master Pyramorphix

The puzzle consists of 4 corner pieces, 4 face centers, 6 edge pieces, and 12 non-center face pieces. Being an edge-turning puzzle, the edge pieces only rotate in-place, while the rest of the pieces can be permuted. The face centers and corner pieces are interchangeable, and the non-center face pieces may be flipped, leading to a wide variety of exotic shapes as the puzzle is twisted. If only 180° turns are made, it is possible to scramble only the colors while retaining the puzzle's tetrahedral shape.

In spite of superficial similarities, the puzzle is *not* related to the Pyraminx, which is a face-turning puzzle. The corner pieces are non-trivial; they cannot be simply rotated in-place to the right orientation.

Solutions

Despite its appearance, the puzzle is in fact equivalent to a shape modification of the original 3x3x3 Rubik's Cube. Its 4 corner pieces and 4 face centers together are equivalent to the 8 corner pieces of the Rubik's Cube, its 6 edge pieces are equivalent to the face centers of the Rubik's Cube, and its non-center face pieces are equivalent to the edge pieces of the Rubik's Cube. Thus, the same methods used to solve the Rubik's Cube may be used to solve the Master Pyramorphix, with a few minor differences: the edge pieces are sensitive to orientation, unlike the usual coloring scheme used for the Rubik's Cube, and the face centers are *not* sensitive to orientation. In effect, it behaves as a Rubik's Cube with a non-standard coloring scheme where face-center orientation matters, and the orientation of 4 of the 8 corner pieces do not matter.

The Master Pyramorphix, color-scrambled

The Master Pyramorphix, color- and shape- scrambled

Unlike the Square One, another shape-changing puzzle, the most straightforward solutions of the Master Pyramorphix do not involve first restoring the tetrahedral shape of the puzzle and then restoring the colors; most of the algorithms carried over from the 3x3x3 Rubik's Cube translate to shape-changing permutations of the Master Pyramorphix. Some methods, such as the equivalent of Phillip Marshal's *Ultimate Solution*, show a gradual progression in shape as the solution progresses; first the non-center face pieces are put into place, resulting in a partial restoration of the tetrahedral shape except at the face centers and corners, and then the complete restoration of tetrahedral shape as the face centers and corners are solved.

Number of combinations

There are four corners and four face centers. These may be interchanged with each other in 8! different ways. There are 3^7 ways for these pieces to be oriented, since the orientation of the last piece depends on the preceding seven, and the texture of the stickers makes the face center orientation visible. There are twelve non-central face pieces. These can be flipped in 2^{11} ways and there are 12!/2 ways to arrange them. The three pieces of a given color are distinguishable due to the texture of the stickers. There are six edge pieces which are fixed in position relative to one another, each of which has four possible orientations. If the puzzle is solved apart from these pieces, the number of edge twists will always be even, making $4^6/2$ possibilities for these pieces.

The Master Pyramorphix, partially solved

The Master Pyramorphix, with maximal face-piece flip, equivalent to the "superflip" configuration of the 3x3x3 Rubik's Cube

$$8! \times 3^7 \times 12! \times 2^9 \times 4^6 \approx 8.86 \times 10^{22}$$

The full number is 88 580 102 706 155 225 088 000.

However, if the stickers were smooth the number of combinations would be reduced. There would be 3^4 ways for the corners to be oriented, but the face centers would not have visible orientations. The three non-central face pieces of a given color would be indistinguishable. Since there are six ways to arrange the three pieces of the same color and there are four colors, there would be 2^{11}×12!/6^4 possibilities for these pieces.

$$\frac{8! \times 3^4 \times 12! \times 2^{10} \times 4^6}{6^4} \approx 5.06 \times 10^{18}$$

The full number is 5 062 877 383 753 728 000.

See also

- Rubik's Cube
- Pyraminx
- Skewb Diamond
- Pocket Cube
- Combination puzzles
- Mechanical puzzles

External links

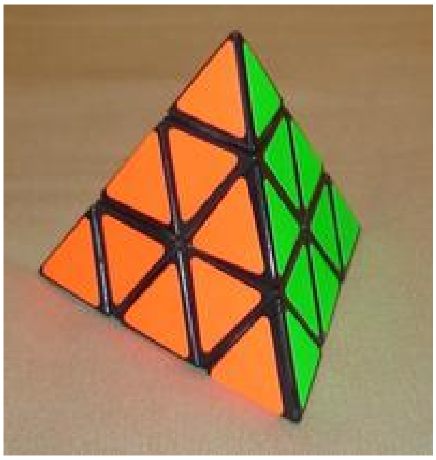
- Jaap's pyramorphix page [1]
- A Java applet which includes the Pyramorphix [2]

References

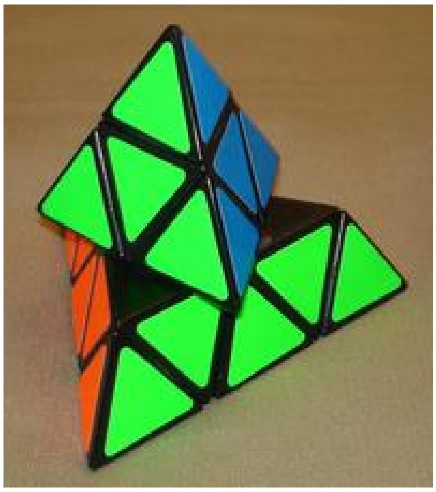
[1] http://www.jaapsch.net/puzzles/pyramorf.htm
[2] http://www.mud.ca/puzzler/JPuzzler/JPuzzler.html

Pyraminx

The **Pyraminx** is a tetrahedral puzzle similar to the Rubik's Cube. It was invented and patented by Uwe Meffert, and introduced by Tomy Toys of Japan (then the 3rd largest toy company in the world) in 1981.[1] Meffert continues to sell it in his toy shop, Meffert's [2].

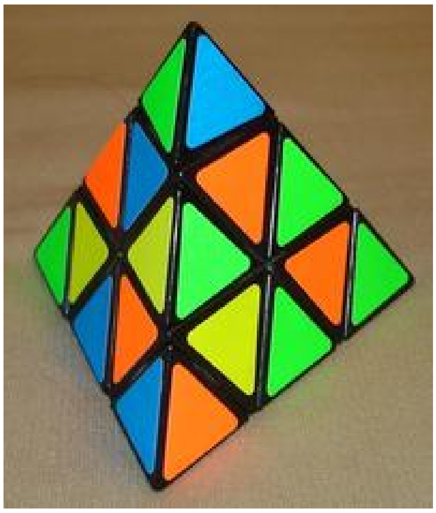

Pyraminx in its solved state

Description



Pyraminx in the middle of a twist

The Pyraminx is a puzzle in the shape of a tetrahedron, divided into 4 axial pieces, 6 edge pieces, and 4 trivial tips. It can be twisted along its cuts to permute its pieces. The axial pieces are octahedral in shape, although this is not immediately obvious, and can only rotate around the axis they are attached to. The 6 edge pieces can be freely permuted. The trivial tips are so called because they can be twisted independently of all other pieces, making them trivial to place in solved position. Meffert also produces a similar puzzle called the **Tetraminx**, which is the same as the Pyraminx except that the trivial tips are removed, turning the puzzle into a truncated tetrahedron.



Scrambled Pyraminx

The purpose of the Pyraminx is to scramble the colors, and then restore them to their original configuration.

The 4 trivial tips can be trivially rotated to line up with the axial piece which they are respectively attached to; and the axial pieces are also easily rotated so that their colors line up with each other. This leaves only the 6 edge pieces as a real challenge to the puzzle. They can be solved by repeatedly applying two 4-twist sequences, which are mirror-image versions of each other. These sequences permute 3 edge pieces at a time, and change their orientation differently, so that a combination of both sequences is sufficient to solve the puzzle. However, more efficient solutions (requiring a smaller total number of twists) are generally available (see below).

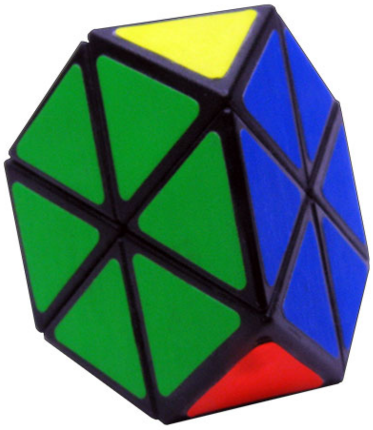
The twist of any axial piece is independent of the other three, as is the case with the tips. The six edges can be placed in 6!/2 positions and flipped in 2^5 ways, accounting for parity. Multiplying this by the 3^8 factor for the axial pieces gives 75,582,720 possible positions. However, setting the trivial tips to the right positions reduces the possibilities to 933,120, which is also the number of possible patterns on the Tetraminx. Setting the axial pieces as well reduces the figure to only 11,520, making this a rather simple puzzle to solve.

Optimal solutions

The maximum number of twists required to solve the Pyraminx is 11. There are 933,120 different positions (disregarding rotation of the trivial tips), a number that is sufficiently small to allow a computer search for optimal solutions. The table below summarizes the result of such a search, stating the number *p* of positions that require *n* twists to solve the Pyraminx:

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
p	1	8	48	288	1728	9896	51808	220111	480467	166276	2457	32

Records


A solved Tetraminx.

The current world record for a single solve of the Pyraminx stands at 2.65 seconds, set by Brúnó Bereczki of Hungary at the European Championship 2010.[3] The best average time of 3.71 seconds is held by Yohei Oka of Japan, set at the Cube Camp Kanazawa 2010.

See also

- Pyramorphix and Master Pyramorphix, two tetrahedral puzzles which resemble the Pyraminx but are mechanically very different from it
- Rubik's cube
- Skewb
- Skewb Diamond
- Megaminx
- Dogic
- Combination puzzles

References

[1] http://www.mefferts.com/puzzles-pyraminx-kokonotsu.htm
[2] http://www.mefferts.com
[3] World Cube Association: Official results (http://www.worldcubeassociation.org/results/regions.php?regionId=&eventId=pyram&years=&separate=Separate) retrieved 17th Jan 2009.

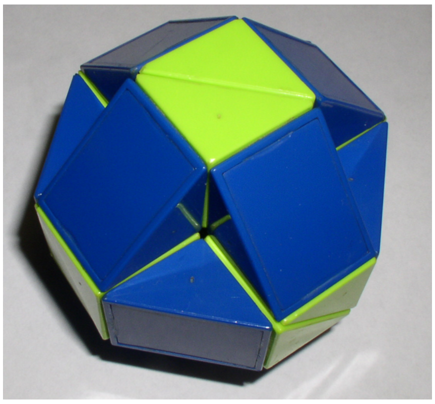
External links

- Jaap's Pyraminx and related puzzles page, with solution (http://www.jaapsch.net/puzzles/pyraminx.htm)
- Pyraminx solution (http://www.puzzlesolver.com/puzzle.php?id=28) from PuzzleSolver (http://www.puzzlesolver.com/)
- A solution to the Pyraminx (ftp://ftp.comlab.ox.ac.uk/pub/Cards/txt/Pyramix.txt) by Jonathan Bowen
- An efficient and easy to follow solution favoured by speed solvers (http://web.archive.org/web/20071231043207/http://www.geocities.com/rubiks_galaxia/PyraSol.html)
- Online Pyraminx (http://www.mud.ca/puzzler/JPuzzler/JPuzzler.html)

Rubik's Snake

A **Rubik's Snake** (also **Rubik's Twist**, Rubik's Transformable Snake, Rubik’s Snake Puzzle) is a toy with twenty-four wedges identically shaped liked prisms[1] , specifically right isosceles triangular prisms. The wedges are connected, by spring bolts[1] , such that they can be twisted, but not separated. Through this twisting the Rubik's Snake can attain positions including a straight line, a ball (technically a nonuniform concave rhombicuboctahedron), a dog, a duck, a rectangle, a snake, and many more imaginative shapes and figures.

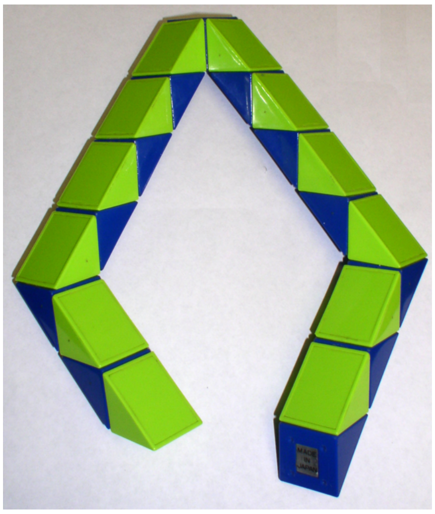
The snake was invented by Professor Ernő Rubik, better known as the inventor of the Rubik's Cube.[1]



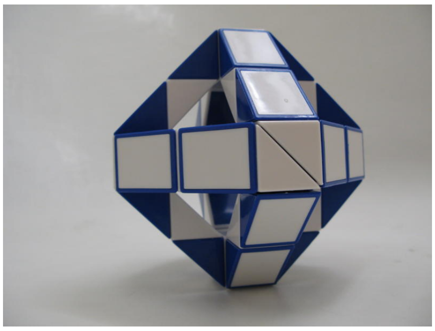
Snake in a ball solution

Structure

The 24 prisms are aligned in row with an alternating orientation (normal and upside down). Each prism can adopt 4 different positions each with an offset of 90°. Usually the prisms have an alternating color.



Snake bent in 4 sides.

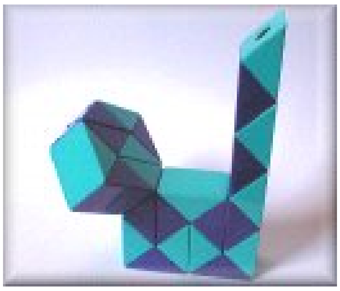


Two identically formed rubik's snakes: one octahedron

Notation

Twisting instructions

example figure: ThreePeaks



example figure: Cat

The description of an arbitrary shape or figure is based on a set of instructions of twisting the prisms. The starting point is a straight line, where the 12 prisms at the bottom are numbered from 1 to 12. The left and the right turning area of these prisms are labeled which L and R respectively. The four possible positions of the each turning area numbered with 0, 1, 2 and 3 (twist between the bottom prism and its neighbor). The numbering is based on the first clockwise turn of a prism. The position 0 is the starting position and therefore isn’t explicitly noted. A twist is described as:

1. Number of the prism: 1 to 12
2. Left or right side of the prism: L or R
3. Position of the twist: 1, 2 or 3

Before twisting make sure your snake is in a position where the last triangle to the left is facing down so there is a slope.

- for example **Three Peaks**

  6R1-6L3-5R2-5L3-4R2-4L1-1R1-3L3-3R2-7L2-7R3-8L1-8R2-9L1-9R2-10L3-12R3-11L
- for example **Cat**

  9R2-9L2-8L2-7R2-6R2-6L2-5L3-4L2-3R2-2R2-2L2

Machine processing

The position of the 23 turning areas can also be written directly after each other. Here the position 0, 1, 2 and 3 are always based on the degree of twist between the right-hand prisms relative to the left-hand prism, if you look at the axis of rotation from the right. But this notation is impractical for manual twisting, because you don’t know in which order the twists occur.

- for example **Three Peaks**

  10012321211233232123003
- for example **Cat**

  02202201022022022000000

Fiore method

Rather than numbers Albert Fiore uses letters to refer to the direction the second (rightward) section is turned in relation to the first (leftward) section: D, L, U, and R.[2] These are listed consecutively rather than numbered, so that a completely straight figure rather than being presumed as a starting point is notated DDDDDDDDDDDDDDDDDDDDDDD.[3]

Mathematics

The number of different shapes of the Rubik's Snake is at most $4^{23} = 70368744177664 \approx 7 \cdot 10^{13}$, i.e. 23 turning areas with 4 positions each. The real number of different shapes is lower and still unknown, since some configurations are spatially impossible (because they would require multiple prisms to occupy the same region of space).

See also

- Mechanical puzzles
- Combination puzzles
- Nonplanar flexagons

Sources

[1] Fiore, Albie (1981). *Shaping Rubik's Snake*, p.7. ISBN 0 14 00 6181 9.
[2] Fiore (1981), p.9.
[3] Fiore (1981), p.11.

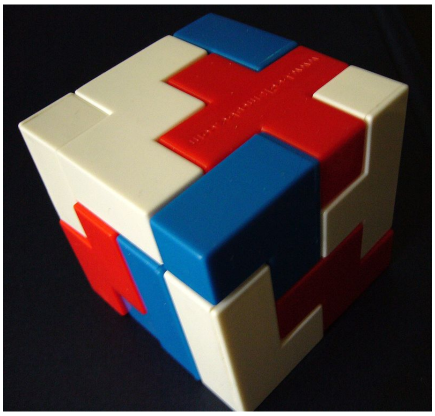
porgen

External links

- Official Rubik's Online Site (http://www.rubiks.com/)
- Collection of shapes and figures of Rubik's Snake (http://www.thomas-wolter.de/index_en.htm)
- glsnake (http://spacepants.org/src/glsnake/) - open-source cross-platform implementation of Rubik's Snake (also ported to XScreenSaver)

Bedlam cube

The **Bedlam cube** is a solid dissection puzzle invented by British puzzle expert Bruce Bedlam.[1] [2]



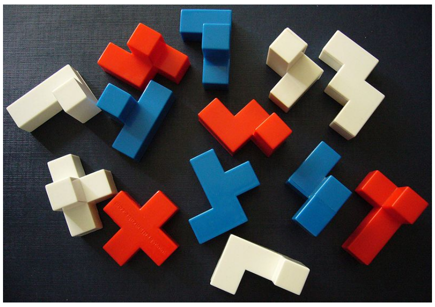
Bedlam cube

Design

The puzzle consists of thirteen polycubic pieces: twelve pentacubes and one tetracube. The objective is to assemble these pieces into a 4 x 4 x 4 cube. There are 19,186 distinct ways of doing so, excluding rotations and reflections.

Although the Bedlam cube is essentially just the logical next step up from the 3 x 3 x 3 Soma cube, it is much more difficult to solve.

History



Bedlam cube elements

Two of the BBC2 'dragons', Rachel Elnaugh and Theo Paphitis, were to invest in the The Bedlam puzzle during the second series of "Dragons' Den". They offered £100,000 for a 30% share of equity in Bedlam Puzzles. Danny Bamping (the entrepreneur behind Bedlam cube) finally chose a bank loan instead of their investment, as seen in the relevant "Where Are They Now" episode of "Dragons' Den".

Records

The official world record for assembling the Bedlam Cube is 7.77 seconds, and was set by Aleksander Iljasov (Norway) on 28 September 2007.[3]

See also

- Slothouber–Graatsma puzzle
- Conway puzzle
- Polycube

References

[1] (http://www.bedlampuzzles.com/13reasons.asp): "Bruce Bedlam the Cubes inventor is putting all his royalty fees into his company 'Stonehenge Limited'"
[2] Bruce's Theories (http://www.stonehenge.tv/intro.html) — Stonehenge Ltd website
[3] Guinness World Records

External links

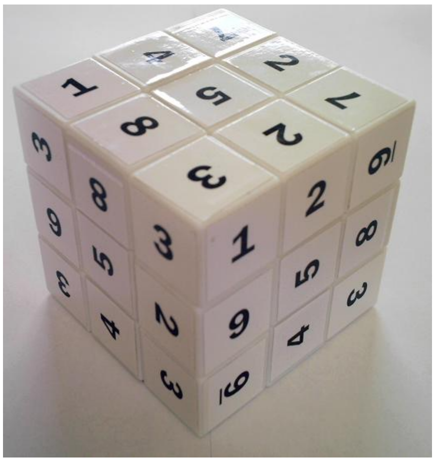
- The Official Site of Bedlam Puzzles (http://www.bedlampuzzles.com)
- Bedlam Cube solver (http://www.danieltebbutt.com/bedlam.html)
- All 19,186 Bedlam Cube Solutions (http://scottkurowski.com/bedlamcube/)

Sudoku Cube

The **Sudoku Cube** is a variation on a Rubik's Cube in which the faces have numbers one to nine on the sides instead of colours. The aim is to solve Sudoku puzzles on one or more of the sides. The toy was created in 2006 by Jay Horowitz in Sebring, Ohio.[1]

Production

The Sudoku Cube was invented by veteran toy maker Jay Horowitz after he had the idea to combine Sudoku and a Rubik's Cube. Horowitz already owned molds to produce Rubik's Cubes and was able to use them to produce his new design.[2] Mass production is completed in China by American Classic Toy Inc, a company belonging to Horowitz. The product is sold in the United States in retailers such as Barnes & Noble and FAO Schwarz. There are 12 types of Sudoku Cube, which differ in difficulty and are aimed at different age ranges.[3]



A scrambled Sudokube puzzle

Description

In a standard Rubik's Cube, the player must match up colours on each side of the cube. In the Sudoku Cube, the player must place the numbers one to nine on each side with no repetition. This is achieved by rotating the sides of the cube. Variations of the Sudoku Cube are the Sudokube and Roxdoku, as well as cubes with 4×4×4 squares instead of the normal 3×3×3.

Computer simulations

3-D programming languages such as VPython can be used to create simulations of a Sudoku Cube[4] . Such simulations can offer features such as scaling the sudokube (to create 4×4×4 or 5×5×5 puzzles), saving, resetting, undoing, and the option to design ones own sudokube patterns.

References

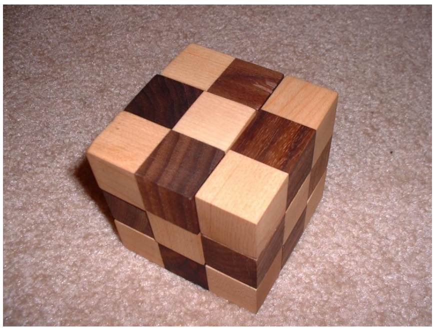
[1] "US toy maker combines Sudoku and Rubik's Cube amid popularity of brain teasers" (http://www.iht.com/articles/ap/2007/02/18/america/NA-GEN-US-Toy-Guy.php). International Herald Tribune. 2007-02-17. . Retrieved 2008-09-30.
[2] "Veteran toy maker combines Sudoku and Rubik's Cube" (http://www.cantonrep.com/index.php?ID=335451&Category=13&subCategoryID=). Canton Repository. 2007-02-10. . Retrieved 2008-09-30.
[3] Pawlyna, Andrea. "American Classic Toy, Inc" (http://info.hktdc.com/prodmag/worldent/worldent200704SupplierSpotlight01.htm). IT Figures. . Retrieved 2008-09-30.
[4] http://www.youtube.com/watch?v=fLvs5nvT2j8

See also

- Combination puzzles
- Mechanical puzzles

Soma cube

The pieces of a Soma cube (with extra coloring)



The same puzzle, assembled into a cube

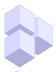
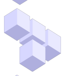
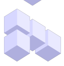
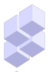
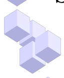
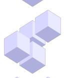
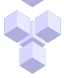
The **Soma cube** is a solid dissection puzzle invented by Piet Hein in 1936 during a lecture on quantum mechanics conducted by Werner Heisenberg. Seven pieces made out of unit cubes must be assembled into a 3x3x3 cube. The pieces can also be used to make a variety of other interesting 3D shapes.

The pieces of the Soma cube consist of all possible combinations of four or fewer unit cubes, excluding all convex shapes (i.e., the 1x1x1, 1x1x2, 1x1x3, 1x1x4 and 1x2x2 cuboids). This leaves just one three-block piece and six four-block pieces, of which two form an enantiomorphic pair. A similar puzzle consisting solely of all eight four-block pieces (including the cuboids) would contain 32 unit cubes and, thus, could not be assembled into a cube.

The Soma cube is often regarded as the 3D equivalent of polyominos. There are interesting parity properties relating to solutions of the Soma puzzle.

Soma has been discussed in detail by Martin Gardner and John Horton Conway, and the book *Winning Ways for your Mathematical Plays* contains a detailed analysis of the Soma cube problem. There are 240 distinct solutions of the Soma cube puzzle, up to rotations and reflections: these are easily generated by a simple recursive backtracking search computer program similar to that used for the eight queens puzzle.

The seven Soma pieces are all polycubes of order three or four:

-  The "L" tricube.
-  T tetracube: a row of three blocks with one added below the center.
-  L tetracube: a row of three blocks with one added below the left side.
-  S tetracube: bent triomino with block placed on outside of clockwise side.
-  Left screw tetracube: unit cube placed on top of anticlockwise side. Chiral in 3D.
-  Right screw tetracube: unit cube placed on top of clockwise side. Chiral in 3D.
-  Branch tetracube: unit cube placed on bend. Not chiral in 3D.

Trivia

Similar to Soma cube is the 3D pentomino puzzle, which can fill boxes of 2×3×10, 2×5×6 and 3×4×5 units.

Plastic Soma cube sets were commercially produced by Parker Brothers in the 1970s.

See also

- Triomino
- Tetromino
- Bedlam cube
- tangram

External links

- http://www.mathematik.uni-bielefeld.de/~sillke/POLYCUBE/SOMA/cube-secrets
- Soma Cube -- from MathWorld [1]
- Thorleif's SOMA page [2]

References

[1] http://mathworld.wolfram.com/SomaCube.html
[2] http://www.fam-bundgaard.dk/SOMA/SOMA.HTM

Think-a-Dot

The **Think-a-Dot** was a mathematical toy made by E.S.R., Inc. during the 1960s that demonstrated group theory. It had eight coloured disks on its front, and three holes on its top - left, right, and center - through which a ball bearing could be dropped. Each disk would display either a yellow or blue face, depending on whether the mechanism behind it was tipped to the right or the left. The Think-a-Dot thus had 2^8=256 internal states. When the ball fell to the bottom it would exit either to a hole on the left or the right of the device.

```
 |     |     |
 V     V     V

 O     O     O

    O     O

 O     O     O
```

As the ball passed through the Think-a-Dot, it would flip the disk mechanisms that it passed, and they in turn would determine whether the ball would be deflected to the left or to the right. Various puzzles and games were possible with the Think-a-Dot, such as flipping the colours of all cells in the minimum number of moves, or reaching a given state from a monochrome state or vice-versa.

See also

- Digi-Comp I
- Digi-Comp II
- Dr. NIM

External links

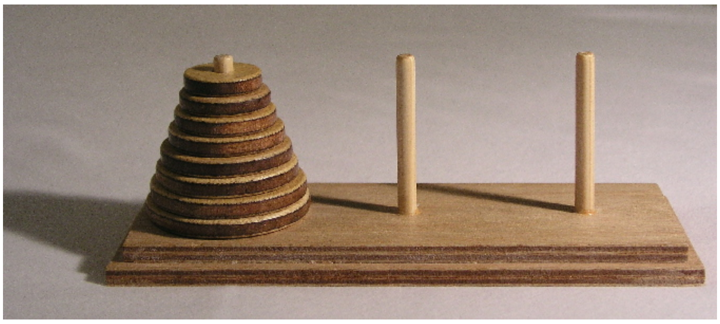
- Picture of a Think-a-Dot [1]
- Think-a-Dot instruction leaflet [2]

References

[1] http://web.archive.org/web/20091027092753/http://geocities.com/mikegleen/knex/thinkadot/pic06-ad.html
[2] http://web.archive.org/web/20091027092802/http://geocities.com/mikegleen/knex/thinkadot/pic10-leaflet.html

Tower of Hanoi

The **Tower of Hanoi** or **Towers of Hanoi** is a mathematical game or puzzle. It consists of three rods, and a number of disks of different sizes which can slide onto any rod. The puzzle starts with the disks in a neat stack in ascending order of size on one rod, the smallest at the top, thus making a conical shape.



A model set of the Towers of Hanoi (with 8 disks)

The objective of the puzzle is to move the entire stack to another rod, obeying the following rules:

- Only one disk may be moved at a time.
- Each move consists of taking the upper disk from one of the rods and sliding it onto another rod, on top of the other disks that may already be present on that rod.
- No disk may be placed on top of a smaller disk.

An animated solution of the **Tower of Hanoi** puzzle for *T(4,3)*.

Origins

The puzzle was invented by the French mathematician Édouard Lucas in 1883. There is a legend about an Indian temple which contains a large room with three time-worn posts in it surrounded by 64 golden disks. Brahmin priests, acting out the command of an ancient prophecy, have been moving these disks, in accordance with the rules of the puzzle, since that time. The puzzle is therefore also known as the Tower of Brahma puzzle. According to the legend, when the last move of the puzzle is completed, the world will end. It is not clear whether Lucas invented this legend or was inspired by it.

If the legend were true, and if the priests were able to move disks at a rate of one per second, using the smallest number of moves, it would take them $2^{64}-1$ seconds or roughly 585 billion years;[1] it would take 18,446,744,073,709,551,615 turns to finish.

There are many variations on this legend. For instance, in some tellings, the temple is a monastery and the priests are monks. The temple or monastery may be said to be in different parts of the world — including Hanoi, Vietnam, and may be associated with any religion. In some versions, other elements are introduced, such as the fact that the tower was created at the beginning of the world, or that the priests or monks may make only one move per day.

The Flag Tower of Hanoi may have served as the inspiration for the name.

Solution

The puzzle can be played with any number of disks, although many toy versions have around seven to nine of them. The game seems impossible to many novices, yet is solvable with a simple algorithm. The number of moves required to solve a Tower of Hanoi puzzle is 2^n-1, where n is the number of disks.[2]

Iterative solution

The following solution is a simple solution for the toy puzzle.

Alternate moves between the smallest piece and a non-smallest piece. When moving the smallest piece, always move it in the same direction (to the right if the starting number of pieces is even, to the left if the starting number of pieces is odd). If there is no tower in the chosen direction, move the piece to the opposite end, but then continue to move in the correct direction. For example, if you started with three pieces, you would move the smallest piece to the opposite end, then continue in the left direction after that. When the turn is to move the non-smallest piece, there is only one legal move. Doing this will complete the puzzle using the fewest number of moves to do so.[3]

It should perhaps be noted that this can be rewritten as a strikingly elegant set of rules:

Simpler statement of iterative solution

For an even number of disks:

- make the legal move between pegs A and B
- make the legal move between pegs A and C
- make the legal move between pegs B and C
- repeat until complete

For an odd number of disks:

- make the legal move between pegs A and C
- make the legal move between pegs A and B
- make the legal move between pegs B and C
- repeat until complete

In each case, a total of 2^n-1 moves are made.

Recursive solution

A key to solving this puzzle is to recognize that it can be solved by breaking the problem down into a collection of smaller problems and further breaking those problems down into even smaller problems until a solution is reached. The following procedure demonstrates this approach.

- label the pegs A, B, C—these labels may move at different steps
- let n be the total number of discs
- number the discs from 1 (smallest, topmost) to n (largest, bottommost)

To move n discs from peg A to peg C:

1. move n−1 discs from A to B. This leaves disc #n alone on peg A
2. move disc #n from A to C
3. move n−1 discs from B to C so they sit on disc #n

The above is a recursive algorithm: to carry out steps 1 and 3, apply the same algorithm again for $n-1$. The entire procedure is a finite number of steps, since at some point the algorithm will be required for $n = 1$. This step, moving a single disc from peg A to peg B, is trivial. This approach can be given a rigorous mathematical formalism with the theory of dynamic programming[4] [5]

The Tower of Hanoi is often used as an example of a simple recursive algorithm when teaching introductory programming. Implementations in various languages may be found at the Hanoimania! [6] website.

Logical analysis of the recursive solution

As in many mathematical puzzles, finding a solution is made easier by solving a slightly more general problem: how to move a tower of h (h=height) disks from a starting peg **A** (f=from) onto a destination peg **C** (t=to), **B** being the remaining third peg and assuming **t≠f**. First, observe that the problem is symmetric for permutations of the names of the pegs (symmetric group S_3). If a solution is known moving from peg **A** to peg **C**, then, by renaming the pegs, the same solution can be used for every other choice of starting and destination peg. If there is only one disk (or even none at all), the problem is trivial. If h=1, then simply move the disk from peg **A** to peg **C**. If h>1, then somewhere along the sequence of moves, the largest disk must be moved from peg **A** to another peg, preferably to peg **C**. The only situation that allows this move is when all smaller h-1 disks are on peg **B**. Hence, first all h-1 smaller disks must go from **A** to **B**. Subsequently move the largest disk and finally move the h-1 smaller disks from peg **B** to peg **C**. The presence of the largest disk does not impede any move of the h-1 smaller disks and can temporarily be ignored. Now the problem is reduced to moving h-1 disks from one peg to another one, first from **A** to **B** and subsequently from **B** to **C**, but the same method can be used both times by renaming the pegs. The same strategy can be used to reduce the h-1 problem to h-2, h-3, and so on until only one disk is left. This is called recursion. This algorithm can be schematized as follows. Identify the disks in order of increasing size by the natural numbers from 0 up to but not including h. Hence disk 0 is the smallest one and disk h-1 the largest one.

The following is a procedure for moving a tower of h disks from a peg **A** onto a peg **C**, with **B** being the remaining third peg:

- Step 1: If h>1 then first use this procedure to move the h-1 smaller disks from peg **A** to peg **B**.
- Step 2: Now the largest disk, i.e. disk h-1 can be moved from peg **A** to peg **C**.
- Step 3: If h>1 then again use this procedure to move the h-1 smaller disks from peg **B** to peg **C**.

By means of mathematical induction, it is easily proven that the above procedure requires the minimal number of moves possible, and that the produced solution is the only one with this minimal number of moves. Using recurrence relations, the exact number of moves that this solution requires can be calculated by: $2^h - 1$. This result is obtained by noting that steps 1 and 3 take T_{h-1} moves, and step 2 takes one move, giving $T_h = 2T_{h-1} + 1$.

Non-recursive solution

The list of moves for a tower being carried from one peg onto another one, as produced by the recursive algorithm has many regularities. When counting the moves starting from 1, the ordinal of the disk to be moved during move *m* is the number of times *m* can be divided by 2. Hence every odd move involves the smallest disk. It can also be observed that the smallest disk traverses the pegs f, t, r, f, t, r, etc. for odd height of the tower and traverses the pegs f, r, t, f, r, t, etc. for even height of the tower. This provides the following algorithm, which is easier, carried out by hand, than the recursive algorithm.

In alternate moves:

- move the smallest disk to the peg it has not recently come from.
- move another disk legally (there will be one possibility only)

For the very first move, the smallest disk goes to peg t if h is odd and to peg r if h is even.

Also observe that:

- Disks whose ordinals have even parity move in the same sense as the smallest disk.
- Disks whose ordinals have odd parity move in opposite sense.
- If h is even, the remaining third peg during successive moves is t, r, f, t, r, f, etc.
- If h is odd, the remaining third peg during successive moves is r, t, f, r, t, f, etc.

With this knowledge, a set of disks in the middle of an optimal solution can be recovered with no more state information than the positions of each disk:

- Call the moves detailed above a disk's 'natural' move.
- Examine the smallest top disk that is not disk 0, and note what its only (legal) move would be: (if there is no such disc, then we are either at the first or last move).
- If that move is the disk's 'natural' move, then the disc has not been moved since the last disc 0 move, and that move should be taken.
- If that move is not the disk's 'natural' move, then move disk 0.

Binary solutions

Disk positions may be determined more directly from the binary (base 2) representation of the move number (the initial state being move #0, with all digits 0, and the final state being #2^n-1, with all digits 1), using the following rules:

- There is one binary digit (bit) for each disk
- The most significant (leftmost) bit represents the largest disk. A value of 0 indicates that the largest disk is on the initial peg, while a 1 indicates that it's on the final peg.
- The bitstring is read from left to right, and each bit can be used to determine the location of the corresponding disk.
- A bit with the same value as the previous one means that the corresponding disk is stacked on top the previous disk on the same peg.
  - (That is to say: a straight sequence of 1's or 0's means that the corresponding disks are all on the same peg).
- A bit with a different value to the previous one means that the corresponding disk is one position to the left or right of the previous one. Whether it is left or right is determined by this rule:
  - Assume that the initial peg is on the left and the final peg is on the right.
  - Also assume "wrapping" - so the right peg counts as one peg "left" of the left peg, and vice versa.
  - Let n be the number of greater disks that are located on the same peg as their first greater disk and add 1 if the largest disk is on the left peg. If n is even, the disk is located one peg to the left, if n is odd, the disk located one peg to the right.

For example, in an 8-disk Hanoi:

- Move 0
  - The largest disk is 0, so it is on the left (initial) peg.
  - All other disks are 0 as well, so they are stacked on top of it. Hence all disks are on the initial peg.
- Move 2^8-1
  - The largest disk is 1, so it is on the right (final) peg.
  - All other disks are 1 as well, so they are stacked on top of it. Hence all disks are on the final peg and the puzzle is complete.
- Move 0b11011000 = 216_{10}
  - The largest disk is 1, so it is on the right (final) peg.
  - Disk two is also 1, so it is stacked on top of it, on the right peg.
  - Disk three is 0, so it is on another peg. Since n is odd(n=3), it is one peg to the right, i.e. on the left peg.
  - Disk four is 1, so it is on another peg. Since n is even(n=2), it is one peg to the left, i.e. on the right peg.
  - Disk five is also 1, so it is stacked on top of it, on the right peg.
  - Disk six is 0, so it is on another peg. Since n is odd(n=5), the disk is one peg to the right, i.e. on the left peg.
  - Disks seven and eight are also 0, so they are stacked on top of it, on the left peg.

The source and destination pegs for the *m*th move can also be found elegantly from the binary representation of *m* using bitwise operations. To use the syntax of the C programming language, the *m*th move is from peg (m&m-1)%3 to peg ((m|m-1)+1)%3, where the disks begin on peg 0 and finish on peg 1 or 2 according as whether the number of disks is even or odd. Furthermore the disk to be moved is determined by the number of times the move count (m) can be divided by 2 (i.e. the number of zero bits at the right), counting the first move as 1 and identifying the disks by the numbers 0, 1, 2 etc. in order of increasing size. This permits a very fast non-recursive computer implementation to find the positions of the disks after m moves without reference to any previous move or distribution of disks.

Gray code solution

The binary numeral system of Gray codes gives an alternative way of solving the puzzle. In the Gray system, numbers are expressed in a binary combination of 0s and 1s, but rather than being a standard positional numeral system, Gray code operates on the premise that each value differs from its predecessor by only one (and exactly one) bit changed. The number of bits present in Gray code is important, and leading zeros are not optional, unlike in positional systems.

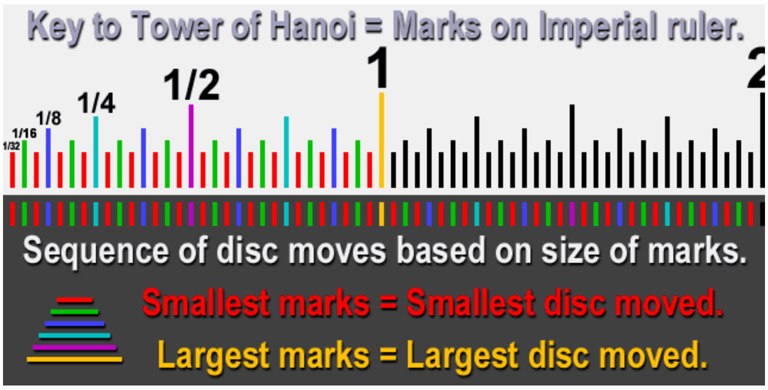
If one counts in Gray code of a bit size equal to the number of disks in a particular Tower of Hanoi, begins at zero, and counts up, then the bit changed each move corresponds to the disk to move, where the least-significant-bit is the smallest disk and the most-significant-bit is the largest.

> Counting moves from 1 and identifying the disks by numbers starting from 0 in order of increasing size, the ordinal of the disk to be moved during move m is the number of times m can be divided by 2.

This technique identifies which disk to move, but not where to move it to. For the smallest disk there are always two possibilities. For the other disks there is always one possibility, except when all disks are on the same peg, but in that case either it is the smallest disk that must be moved or the objective has already been achieved. Luckily, there is a rule which does say where to move the smallest disk to. Let f be the starting peg, t the destination peg and r the remaining third peg. If the number of disks is odd, the smallest disk cycles along the pegs in the order f->t->r->f->t->r, etc. If the number of disks is even, this must be reversed: f->r->t->f->r->t etc.[7]

Visual solution

A visual solution may be discovered by looking closely at a ruler with imperial measurements.[8] These are typically subdivided into progressively smaller marks of 1 inch, 1/2 inch, 1/4 inch, 1/8 inch, 1/16 inch and 1/32 inch divisions. These can be considered to correspond to the progressively smaller discs.



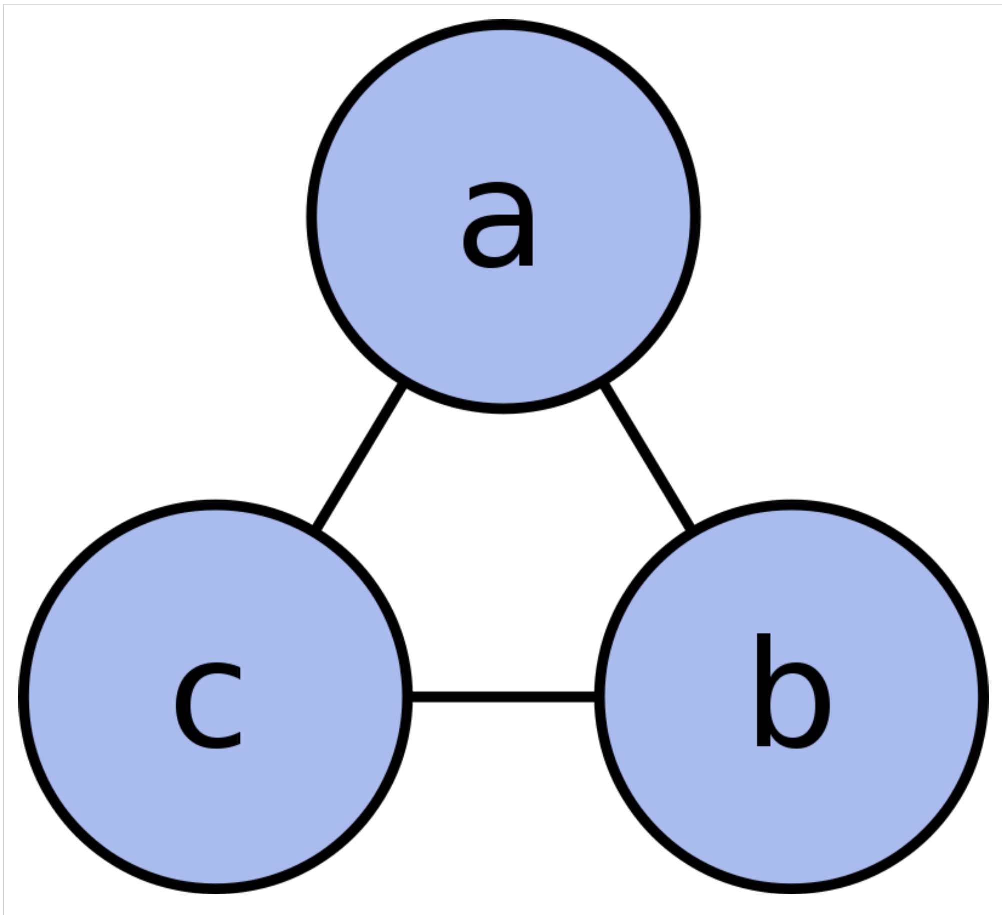
The divisions on a typical Imperial ruler can act as a key for the sequence of disc moves.

Beginning with the smallest division corresponding to the smallest disc, each step along the ruler shows which disc will be moved next. A common ruler with 1/32 inch divisions can be used to solve a Tower of Hanoi puzzle with up to six discs.

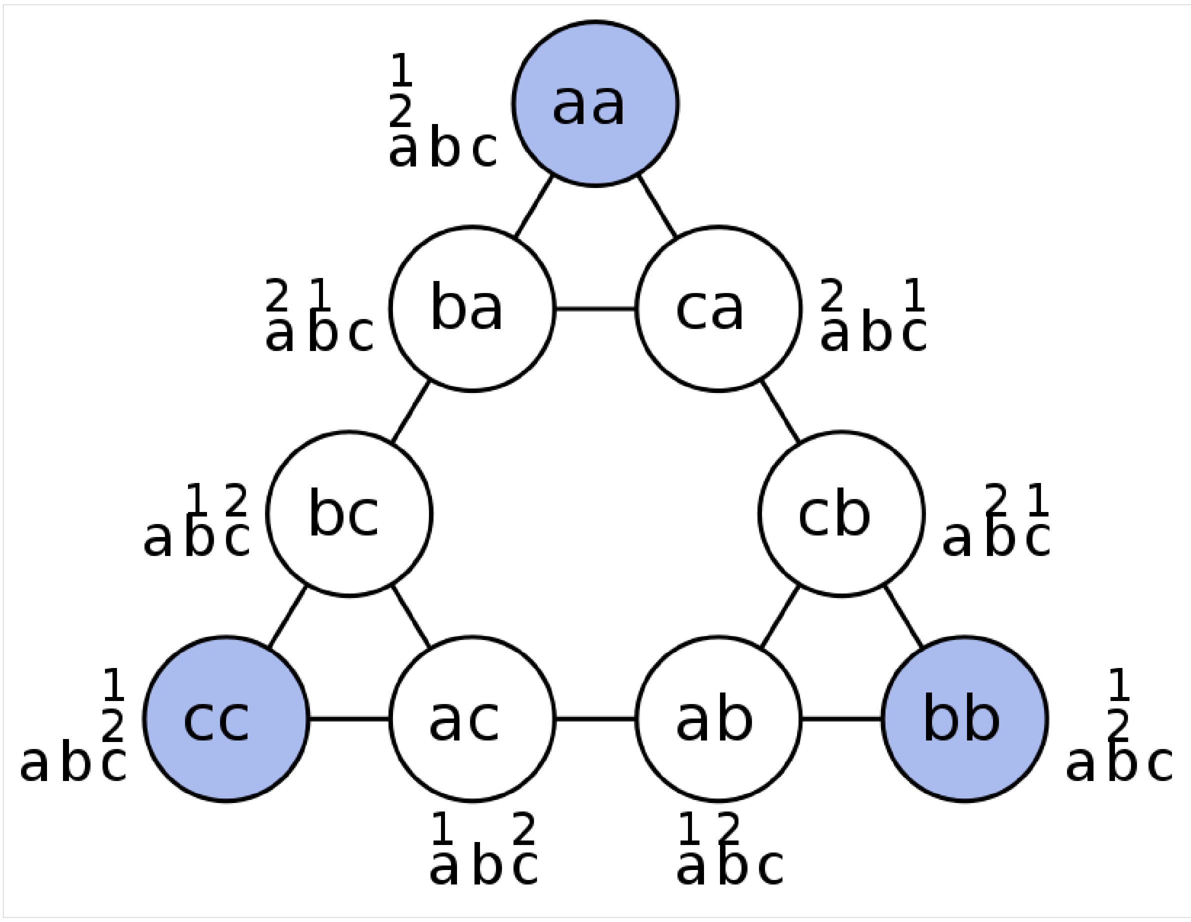
Note that this only provides a key for the sequence of disc movements, not their direction. As mentioned earlier, the smallest disc should always be cycled through pegs A, B, C, then back to A (or C, B, A, then back to C).

Graphical representation

The game can be represented by an undirected graph, the nodes representing distributions of disks and the edges representing moves. For one disk, the graph is a triangle:



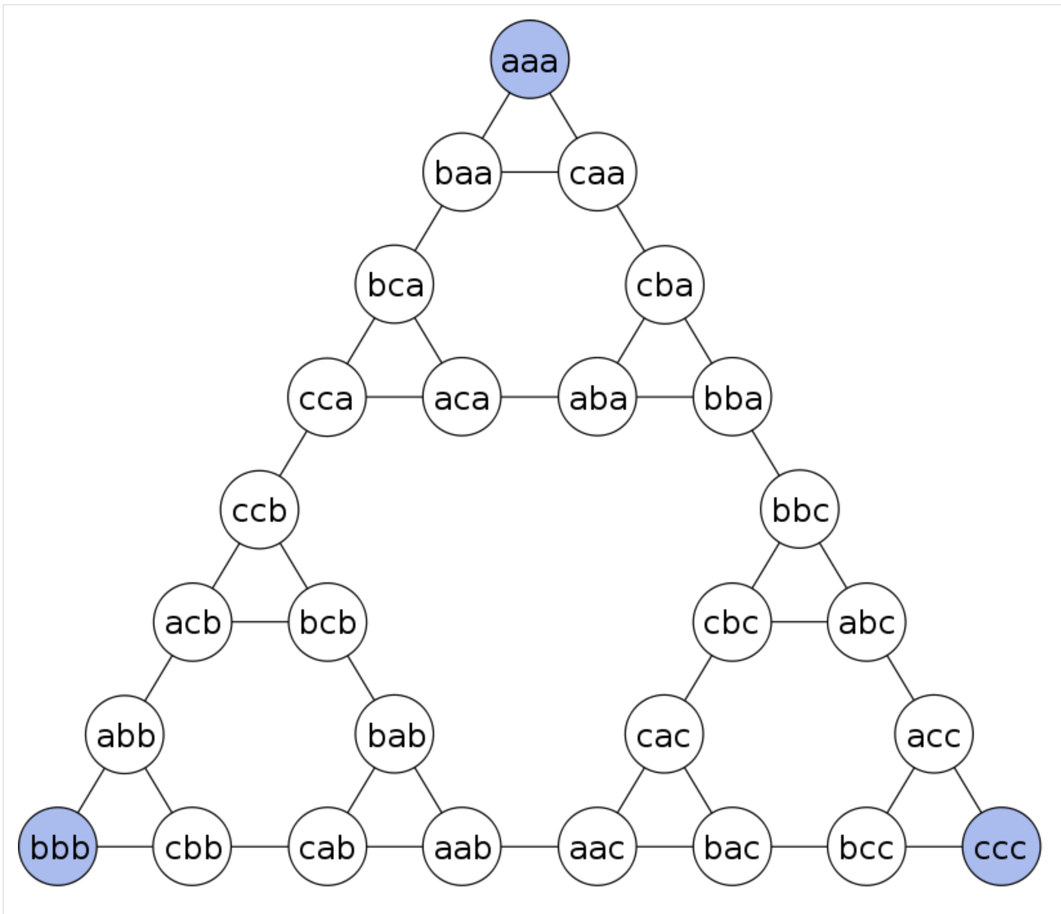
The graph for 2 disks is 3 triangles arranged in a larger triangle:



The nodes at the vertices of the outermost triangle represent distributions with all disks on the same peg.

For h+1 disks, take the graph of h disks and replace each small triangle with the graph for 2 disks.

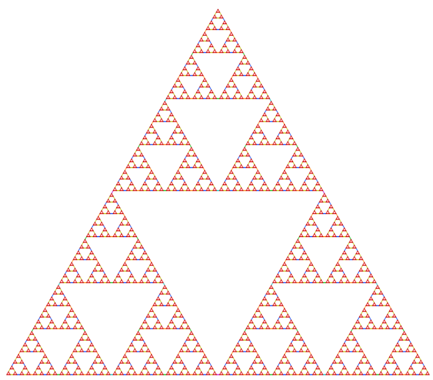
For 3 disks the graph is:



- call the pegs a, b and c
- list disk positions from left to right in order of increasing size

The sides of the outermost triangle represent the shortest ways of moving a tower from one peg to another one. The edge in the middle of the sides of the largest triangle represents a move of the largest disk. The edge in the middle of the sides of each next smaller triangle represents a move of each next smaller disk. The sides of the smallest triangles represent moves of the smallest disk.

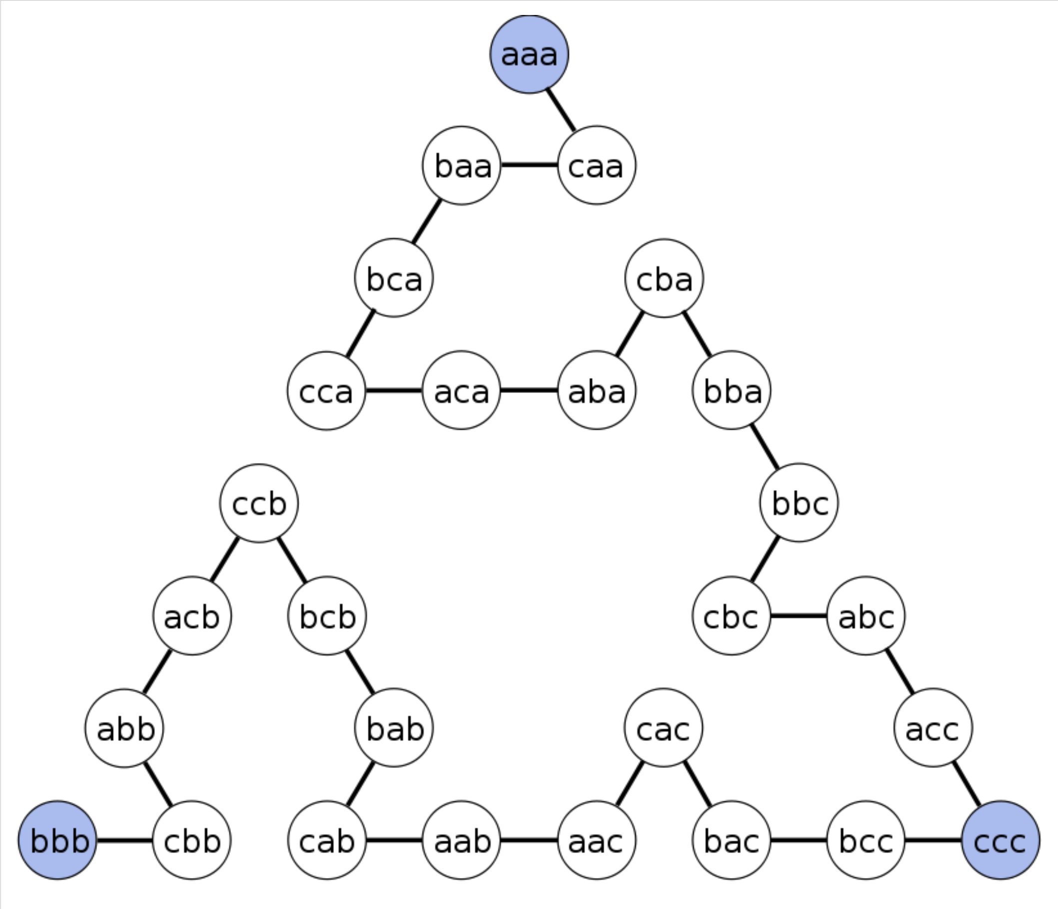
In general, for a puzzle with n disks, there are 3^n nodes in the graph; every node has three edges to other nodes, except the three corner nodes, which have two: it is always possible to move the smallest disk to the one of the two other pegs; and it is possible to move one disk between those two pegs *except* in the situation where all disks are stacked on one peg. The corner nodes represent the three cases where all the disks are stacked on one peg. The diagram for n+1 disks is obtained by taking three copies of the n-disk diagram—each one representing all the states and moves of the smaller disks for one particular position of the new largest disk—and joining them at the corners with three new edges, representing the only three opportunities to move the largest disk. The resulting figure thus has 3^{n+1} nodes and still has three corners remaining with only two edges.



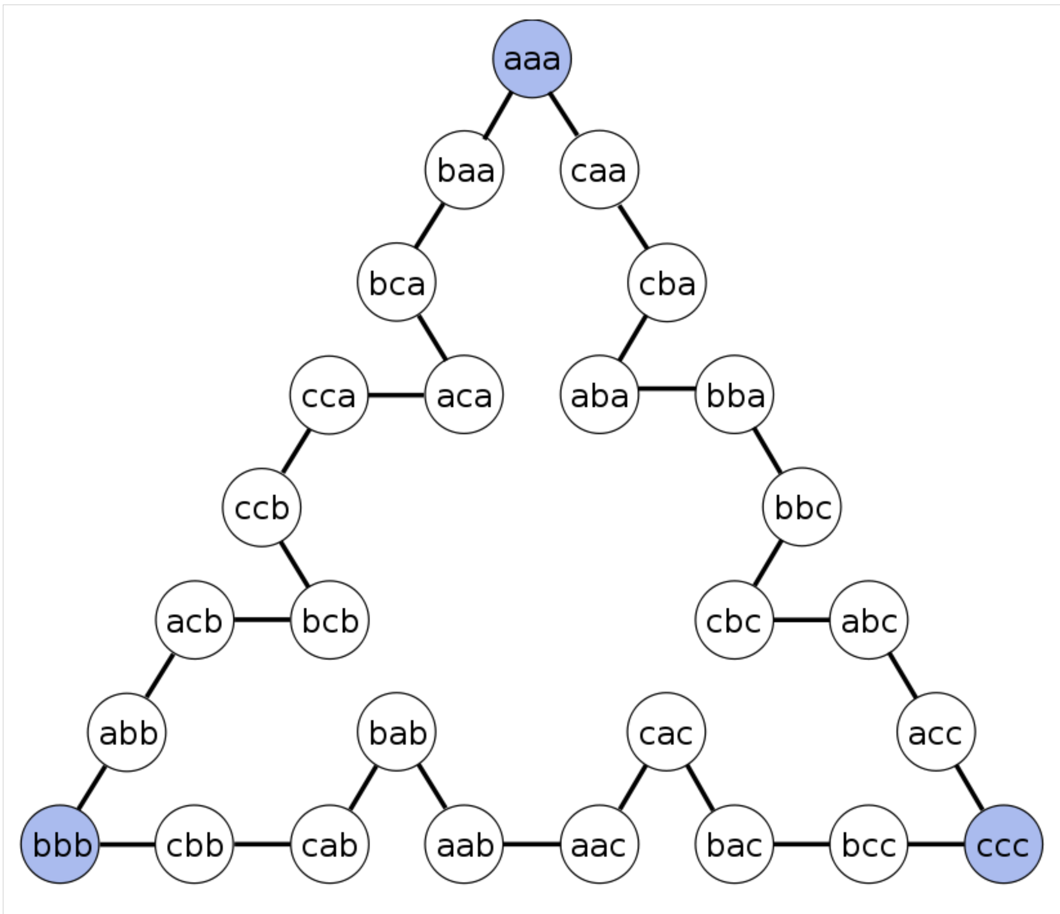
The game graph of level 7 shows the relatedness to the Sierpiński Triangle

As more disks are added, the graph representation of the game will resemble the Fractal figure, Sierpiński triangle. It is clear that the great majority of positions in the puzzle will never be reached when using the shortest possible solution; indeed, if the priests of the legend are using the longest possible solution (without re-visiting any position) it will take them 3^{64}-1 moves, or more than 10^{23} years.

The longest non-repetitive way for three disks can be visualized by erasing the unused edges:



The circular Hamiltonian path for three disks is:



The graphs clearly show that:

- From every arbitrary distribution of disks, there is exactly one shortest way to move all disks onto one of the three pegs.
- Between every pair of arbitrary distributions of disks there are one or two different shortest paths.
- From every arbitrary distribution of disks, there are one or two different longest non selfcrossing paths to move all disks to one of the three pegs.
- Between every pair of arbitrary distributions of disks there are one or two different longest non selfcrossing paths.
- Let N_h be the number of non selfcrossing paths for moving a tower of h disks from one peg to another one. Then:
  - $N_1=2$
  - $N_{h+1}=(N_h)^2+(N_h)^3$.
  - For example: $N_8 \approx 1.5456 \times 10^{795}$

Applications

The Tower of Hanoi is frequently used in psychological research on problem solving. There also exists a variant of this task called Tower of London for neuropsychological diagnosis and treatment of executive functions.

The Tower of Hanoi is also used as Backup rotation scheme when performing computer data Backups where multiple tapes/media are involved.

As mentioned above, the Tower of Hanoi is popular for teaching recursive algorithms to beginning programming students. A pictorial version of this puzzle is programmed into the emacs editor, accessed by typing M-x hanoi. There is also a sample algorithm written in Prolog.

The Tower of Hanoi is also used as a test by neuropsychologists trying to evaluate frontal lobe deficits.

General shortest paths and the number 466/885

A curious generalization of the original goal of the puzzle is to start from a given configuration of the disks where all disks are not necessarily on the same peg, and to arrive in a minimal number of moves at another given configuration. In general it can be quite difficult to compute a shortest sequence of moves to solve this problem. A solution was proposed by Andreas Hinz, and is based on the observation that in a shortest sequence of moves, the largest disk that needs to be moved (obviously one may ignore all of the largest disks that will occupy the same peg in both the initial and final configurations) will move either exactly once or exactly twice.

The mathematics related to this generalized problem becomes even more interesting when one considers the **average** number of moves in a shortest sequence of moves between two initial and final disk configurations that are chosen at random. Hinz and Chan Hat-Tung independently discovered [9] [10] (see also, [11] Chapter 1, p. 14) that the average number of moves in an n-disk Tower is exactly given by the following exact formula:

.

Note that for large enough *n*, only the first and second terms are not converging to zero, so we get an asymptotic expression: $466/885 \cdot 2^n - 1/3 + o(1)$, as $n \to \infty$. Thus, intuitively we could interpret the fraction of $466/885 \approx 52.6\%$ as representing the ratio of the labor one has to perform when going from a randomly chosen configuration to another randomly chosen configuration, relative to the difficulty of having to cross the "most difficult" path of length $2^n - 1$ which involves moving all the disks from one peg to another. An alternative explanation for the appearance of the constant 466/885 , as well as a new and somewhat improved algorithm for computing the shortest path, was given by Romik. [12]

Four pegs and beyond

Although the three-peg version has a simple recursive solution as outlined above, the *optimal* solution for the Tower of Hanoi problem with four pegs (called **Reve's puzzle**), let alone more pegs, is still an open problem. This is a good example of how a simple, solvable problem can be made dramatically more difficult by slightly loosening one of the problem constraints.

The fact that the problem with four or more pegs is an open problem does not imply that no algorithm exists for finding (all of) the optimal solutions. Simply represent the game by an undirected graph, the nodes being distributions of disks and the edges being moves and use breadth first search to find one (or all) shortest paths moving a tower from one peg onto another one. However, even smartly implemented on the fastest computer now available, this algorithm provides no way of effectively computing solutions for large numbers of disks; the program would require more time and memory than available. Hence, even having an algorithm, it remains unknown how many moves an optimal solution requires and how many optimal solutions exist for 1000 disks and 10 pegs.

Though it is not known exactly how many moves must be made, there are some asymptotic results. There is also a "presumed-optimal solution" given by the **Frame-Stewart algorithm**. The related open Frame-Stewart conjecture claims that the Frame-Stewart algorithm always gives an optimal solution. The optimality of the Frame-Stewart algorithm has been computationally verified for up to 30 disks.[13]

For other variants of the four-peg Tower of Hanoi problem, see Paul Stockmeyer's survey paper.[14]

Frame-Stewart algorithm

The Frame-Stewart algorithm, giving a *presumably-optimal solution* for four (or even more) pegs, is described below:

- Let n be the number of disks.
- Let r be the number of pegs.
- Define $T(n,r)$ to be the number of moves required to transfer n disks using r pegs

The algorithm can be described recursively:

1. For some k, $1 \leq k < n$, transfer the top k disks to a single other peg, taking $T(k,r)$ moves.
2. Without disturbing the peg that now contains the top k disks, transfer the remaining $n-k$ disks to the destination peg, using only the remaining $r-1$ pegs, taking $T(n-k,r-1)$ moves.
3. Finally, transfer the top k disks to the destination peg, taking $T(k,r)$ moves.

The entire process takes $2T(k,r)+T(n-k,r-1)$ moves. Therefore, the count k should be picked for which this quantity is minimum.
This algorithm (with the above choice for k) is presumed to be optimal, and no counterexamples are known.

Multistack Towers of Hanoi

U.S. patent number 7,566,057 issued to Victor Mascolo discloses multistack Tower of Hanoi puzzles with two or more stacks and twice as many pegs as stacks. After beginning on a particular peg, each stack displaces and is displaced by a different colored stack on another peg when the puzzle is solved. Disks of one color also have another peg that excludes all other colors, so that there are three pegs available for each color disk, two that are shared with other colors, and one that is not shared. On the shared pegs, a disk may not be placed on a different colored disk of the same size, a possibility that does not arise in the standard puzzle.

The simplest multistack game, Tower of Hanoi (2 x 4), has two stacks and four pegs, and it requires 3[T(n)] moves to solve where T(n) is the number of moves needed to solve a single stack classic of n disks. The game proceeds in seesaw fashion with longer and longer series of moves that alternate between colors. It concludes in reverse seesaw fashion with shorter and shorter such series of moves. Starting with the second series of three moves, these alternate series of moves double in length for the first half of the game, and the lengths are halved as the game concludes. The solution involves nesting an algorithm suitable for Tower of Hanoi into an algorithm that indicates when to switch between colors. When there are k stacks of n disks apiece in a game, and k > 2, it requires k[T(n)] + T(n-1) + 1 moves to relocate them.

The addition of a centrally located universal peg open to disks from all stacks converts these multistack Tower of Hanoi puzzles to multistack Reve's puzzles as described in the preceding section. In these games each stack may move among four pegs, the same combination of three in the 2 x 4 game plus the central universal peg. The simplest game of this kind (2 x 5) has two stacks and five pegs. A solution conjectured to be optimal interlocks the optimal solution of the 2 x 4 puzzle with the presumed optimal solution to Reve's puzzle. It takes R(n) + 2R(n-1) + 2 moves, where R(n) is the number of moves in the presumed optimal Reve's solution for a stack of n disks.

In popular culture

In the classic science fiction story *Now Inhale*, by Eric Frank Russell (*Astounding Science Fiction* April 1959, and in various anthologies), the human hero is a prisoner on a planet where the local custom is to make the prisoner play a game until it is won or lost, and then execution is immediate. The hero is told the game can be one of his own species', as long as it can be played in his cell with simple equipment strictly according to rules which are written down before and cannot change after play starts, and it has a finite endpoint. The game and execution are televised planet-wide, and watching the desperate prisoner try to spin the game out as long as possible is very popular entertainment; the record is sixteen days. The hero knows a rescue ship might take a year or more to arrive, so chooses to play Towers of Hanoi with 64 disks until rescue arrives. When the locals realize they've been had, they are angry, but under their own rules there is nothing they can do about it. They do change the rules, which will apply to any *future* prisoners. This story makes reference to the legend about the Buddhist monks playing the game until the end of the world, and refers to the game as **arkymalarky**. (The slang term "malarky", meaning nonsense, pre-dates this story by at least 30 years.[15])

The puzzle is featured regularly in adventure and puzzle games. Since it is easy to implement, and easily-recognised, it is well-suited to use as a puzzle in a larger graphical game. Some implementations use straight disks, but others disguise the puzzle in some other form.

See also

- Baguenaudier
- Recursion (computer science)

Notes

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Paper folding

Paper folding

Paper folding is the art of folding paper; it is known in many societies that use paper. In much of the West, the term origami is used synonymously with paper folding, though the term properly only refers to the art of paper folding in Japan.

Forms of paper folding:

- Zhezhi, Chinese paper folding
- Origami, from Japan
- Western paper folding, such as the traditional paper boats and paper planes.
- Jong-ie-jeop-gi, from Korea

The claim of the seven fold limit

A popular belief holds that it is impossible to fold a sheet of paper in half more than 7 times, folding in any direction, as the challenge had existed for many years and had never been solved. This belief was debunked by then high school student Britney Gallivan who successfully folded a piece of paper 12 times.[1] More importantly she developed the mathematical and physical explanations for the actual folding limits of incompressible materials when folding in one or two directions. After the mathematics were developed she demonstrated folding in half 12 times both by folding paper in a single direction and by folding gold foil while rotating the folding 90 degrees after each fold.

The television series *MythBusters* "busted the myth" of the 7 fold limit by folding taped-together sheets the size of a football field in half and turning 90 degrees each time, for a total of 11 folds. The first eight folds were completed by hand, while the rest were completed using both steam rollers and fork lifts.[2] This was accomplished by using not a single piece of paper but 17 large rolls of paper taped together to form a very large yet relatively thin "sheet."

Folding a piece of paper in half 100 times, if it were possible, would produce a stack of paper approximately 8×10^{22} miles in height.

See also

- Mathematics of paper folding
- Book folding
- Washi
- Regular paperfolding sequence (for example, the Dragon curve)

Notes

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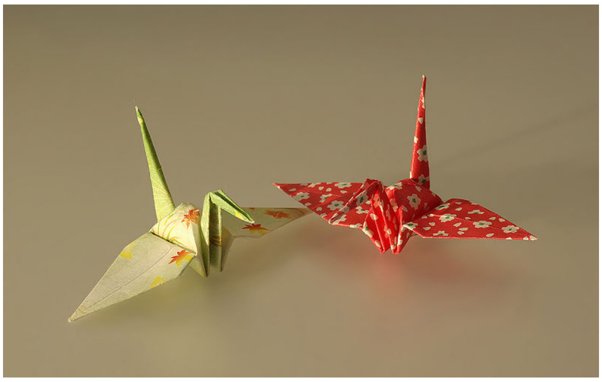
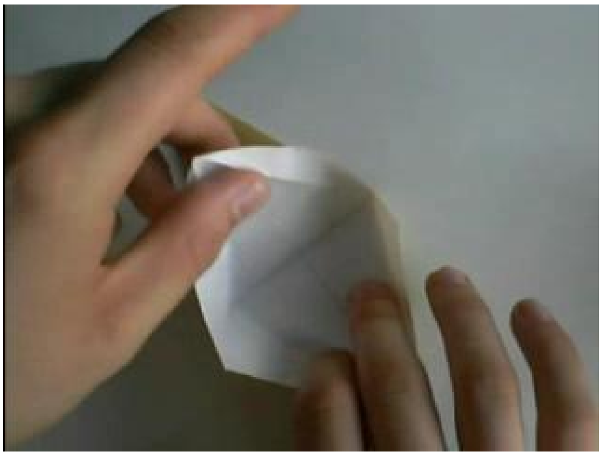
Origami


Origami cranes


the folding of an Origami crane

Origami (折り紙, from *ori* meaning "folding", and *kami* meaning "paper") is the traditional Japanese folk art of paper folding, which started in the 17th century AD and was popularized in the mid-1900s. It has since then evolved into a modern art form. The goal of this art is to transform a flat sheet of material into a finished sculpture through folding and sculpting techniques, and as such the use of cuts or glue are not considered to be origami.

The number of basic origami folds is small, but they can be combined in a variety of ways to make intricate designs. The most well known origami model is probably the Japanese paper crane. In general, these designs begin with a square sheet of paper whose sides may be different colors or prints. Traditional Japanese origami, which has been practiced since the Edo era (1603–1867), has often been less strict about these conventions, sometimes cutting the paper or using nonsquare shapes to start with.

History

Swan by Akira Yoshizawa, the father of modern origami techniques.

There is much speculation as to the origin of origami. While Japan seems to have had the most extensive tradition, there is evidence of independent paperfolding traditions in China, Germany, and Spain, among other places. However because paper decomposes rapidly, there is very little direct evidence of its age or origins, aside from references in published material.

The earliest evidence of paperfolding in Europe is a picture of a small paper boat in Tractatus de sphaera mundi from 1490. There is also evidence of a cut and folded paper box from 1440.[1] It is probable paperfolding in the west originated with the Moors much earlier,[2] it is not known if it was independently discovered or knowledge of origami came along the silk route.

Japanese school children dedicate their contribution of Thousand origami cranes at the Sadako Sasaki memorial in Hiroshima.

In Japan, the earliest unambiguous reference to a paper model is in a short poem by Ihara Saikaku in 1680 which describes paper butterflies in a dream.[3] Origami butterflies were used during the celebration of Shinto weddings to represent the bride and groom, so paperfolding already become a significant aspect of Japanese ceremony by the Heian period (794–1185) of Japanese history, enough that the reference in this poem would be recognized. Samurai warriors would exchange gifts adorned with noshi, a sort of good luck token made of folded strips of paper.

In the early 1900s, Akira Yoshizawa, Kosho Uchiyama, and others began creating and recording original origami works. Akira Yoshizawa in particular was responsible for a number of innovations, such as wet-folding and the Yoshizawa-Randlett diagramming system, and his work inspired a renaissance of the art form.[4] During the 1980s a number of folders started systematically studying the mathematical properties of folded forms, which led to a steady increase in the complexity of origami models, which continued well into the 1990s, after which some designers started returning to simpler forms.[5]

Techniques and materials

Techniques

Many origami books begin with a description of basic origami techniques which are used to construct the models. These include simple diagrams of basic folds like valley and mountain folds, pleats, reverse folds, squash folds, and sinks. There are also standard named bases which are used in a wide variety of models, for instance the bird base is an intermediate stage in the construction of the flapping bird.[6]

Origami paper

crane and papers of the same size used to fold it

Almost any laminar material can be used for folding; the only requirement is that it should hold a crease.

Origami paper, often referred to as "kami" (Japanese for paper), is sold in prepackaged squares of various sizes ranging from 2.5 cm to 25 cm or more. It is commonly colored on one side and white on the other; however, dual coloured and patterned versions exist and can be used effectively for color-changed models. Origami paper weighs slightly less than copy paper, making it suitable for a wider range of models.

Normal copy paper with weights of 70–90 g/m^2 (19-24 lb) can be used for simple folds, such as the crane and waterbomb. Heavier weight papers of 100 g/m^2 (approx. 25 lb) or more can be wet-folded. This technique allows for a more rounded sculpting of the model, which becomes rigid and sturdy when it is dry.

Foil-backed paper, just as its name implies, is a sheet of thin foil glued to a sheet of thin paper. Related to this is tissue foil, which is made by gluing a thin piece of tissue paper to kitchen aluminium foil. A second piece of tissue can be glued onto the reverse side to produce a tissue/foil/tissue sandwich. Foil-backed paper is available commercially, but not tissue foil; it must be handmade. Both types of foil materials are suitable for complex models.

Washi (和紙) is the traditional origami paper used in Japan. Washi is generally tougher than ordinary paper made from wood pulp, and is used in many traditional arts. Washi is commonly made using fibres from the bark of the gampi tree, the mitsumata shrub (*Edgeworthia papyrifera*), or the paper mulberry but also can be made using bamboo, hemp, rice, and wheat.

Artisan papers such as unryu, lokta, hanji, gampi, kozo, saa, and abaca have long fibres and are often extremely strong. As these papers are floppy to start with, they are often backcoated or resized with methylcellulose or wheat paste before folding. Also, these papers are extremely thin and compressible, allowing for thin, narrowed limbs as in the case of insect models.

Paper money from various countries is also popular to create origami with; this is known variously as Dollar Origami, Orikane, and Money Origami. Towels and toilet paper are often folded by hotel staff to indicate to guests that the bathroom has been recently cleaned.

Tools

It is common to fold using a flat surface but some folders like doing it in the air with no tools especially when displaying the folding. Many folders believe no tool should be used when folding. However a couple of tools can help especially with the more complex models. For instance a bone folder allows sharp creases to be made in the paper easily, paper clips can act as extra pairs of fingers, and tweezers can be used to make small folds. When making complex models from origami crease patterns, it can help to use a ruler and ballpoint embosser to score the creases. Completed models can be sprayed so they keep their shape better, and of course a spray is needed when wet folding.

Types of Origami

Action origami

Origami not only covers still-life, there are also moving objects; Origami can move in clever ways. Action origami includes origami that flies, requires inflation to complete, or, when complete, uses the kinetic energy of a person's hands, applied at a certain region on the model, to move another flap or limb. Some argue that, strictly speaking, only the latter is really "recognized" as action origami. Action origami, first appearing with the traditional Japanese flapping bird, is quite common. One example is Robert Lang's instrumentalists; when the figures' heads are pulled away from their bodies, their hands will move, resembling the playing of music.

Modular origami

Modular origami consists of putting a number of identical pieces together to form a complete model. Normally the individual pieces are simple but the final assembly may be tricky. Many of the modular origami models are decorative balls like kusudama, the technique differs though in that kusudama allows the pieces to be put together using thread or glue.


A stellated icosahedron made from custom papers

Chinese paper folding includes a style called 3D origami where large numbers of pieces are put together to make elaborate models. Sometimes paper money is used for the modules. This style originated from some Chinese refugees while they were detained in America and is also called Golden Venture folding from the ship they came on.

Wet-folding

Wet-folding is an origami technique for producing models with gentle curves rather than geometric straight folds and flat surfaces. The paper is dampened so it can be moulded easily, the final model keeps its shape when it dries. It can be used for instance to produce very natural looking animal models.

Pureland origami

Pureland origami is origami with the restriction that only one fold may be done at a time, more complex folds like reverse folds are not allowed, and all folds have straightforward locations. It was developed by John Smith in the 1970s to help inexperienced folders or those with limited motor skills. Some designers also like the challenge of creating good models within the very strict constraints.

Origami Tessellations

This branch of origami is one that has grown in popularity recently, but has an extensive history. Tessellations refer to the tiling of the plane where a collection of 2 dimensional figures fill a plane with no gaps or overlaps. Origami tessellations are tessellations made from a flat material, most often paper, but it can be from anything that holds a crease. The history of costuming includes tessellations done in fabric that are recorded as far back as the Egyptian Tombs.

Fujimoto was an early Japanese origami master who published books that included origami tessellations and in the 1960s there was a great exploration of tessellations by Ron Resch. Chris Palmer is an artist who has extensively explored tessellations and has found ways to create detailed origami tessellations out of silk. Robert Lang and Alex Bateman are two designers who use computer programs to design origami tessellations. The first American book on origami tessellations was just published by Eric Gjerde and the field has been expanding rapidly. There are numerous origami tessellation artists including Chris Palmer (U.S.), Eric Gjerde (U.S.), Polly Verity (Scotland), Joel Cooper (U.S.), Christine Edison (U.S.), Ray Schamp (U.S.), Roberto Gretter (Italy), Goran Konjevod (U.S.),and Christiane Bettens (Switzerland) that are showing works that are both geometric and representational.

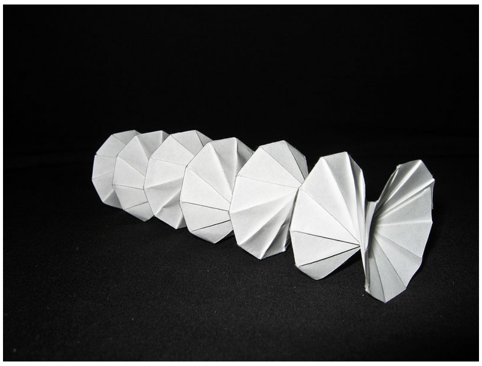
Kirigami

In Kirigami it is allowed to make cuts. In traditional Origami, there was no Kirigami. Kirigami was simply called Origami. Just in the recent century the term Kirigami developed in order to distinguish it from "pure Origami"

Mathematics and technical origami

Mathematics and practical applications

The practice and study of origami encapsulates several subjects of mathematical interest. For instance, the problem of *flat-foldability* (whether a crease pattern can be folded into a 2-dimensional model) has been a topic of considerable mathematical study.



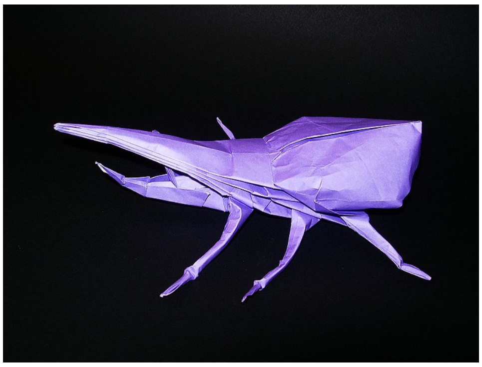
Spring Into Action, designed by Jeff Beynon, made from a single rectangular piece of paper.[7]

The problem of rigid origami ("if we replaced the paper with sheet metal and had hinges in place of the crease lines, could we still fold the model?") has great practical importance. For example, the Miura map fold is a rigid fold that has been used to deploy large solar panel arrays for space satellites.

There may soon be an origami airplane launched from space. A prototype passed a durability test in a wind tunnel on March 2008, and Japan's space agency adopted it for feasibility studies.

Technical origami

Technical origami, also known as **origami sekkei** (折り紙設計), is a field of origami that has developed almost hand-in-hand with the field of mathematical origami. In the early days of origami, development of new designs was largely a mix of trial-and-error, luck and serendipity. With advances in origami mathematics however, the basic structure of a new origami model can be theoretically plotted out on paper before any actual folding even occurs. This method of origami design was developed by Robert Lang, Meguro Toshiyuki and others, and allows for the creation of extremely complex multi-limbed models such as many-legged centipedes, human figures with a full complement of fingers and toes, and the like.



Hercules Beetle by Robert Lang.

The main starting point for such technical designs is the crease pattern (often abbreviated as CP), which is essentially the layout of the creases required to form the final model. Although not intended as a substitute for diagrams, folding from crease patterns is starting to gain in popularity, partly because of the challenge of being able to 'crack' the pattern, and also partly because the crease pattern is often the only resource available to fold a given model, should the designer choose not to produce diagrams. Still, there are many cases in which designers wish to sequence the steps of their models but lack the means to design clear diagrams. Such origamists occasionally resort to the

Sequenced Crease Pattern (abbreviated as SCP) which is a set of crease patterns showing the creases up to each respective fold. The SCP eliminates the need for diagramming programs or artistic ability while maintaining the step-by-step process for other folders to see. Another name for the Sequenced Crease Pattern is the Progressive Crease Pattern (PCP).

Paradoxically enough, when origami designers come up with a crease pattern for a new design, the majority of the smaller creases are relatively unimportant and added only towards the completion of the crease pattern. What is more important is the allocation of regions of the paper and how these are mapped to the structure of the object being designed. For a specific class of origami bases known as 'uniaxial bases', the pattern of allocations is referred to as the 'circle-packing'. Using optimization algorithms, a circle-packing figure can be computed for any uniaxial base of arbitrary complexity. Once this figure is computed, the creases which are then used to obtain the base structure can be added. This is not a unique mathematical process, hence it is possible for two designs to have the same circle-packing, and yet different crease pattern structures.

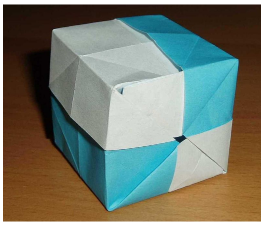
As a circle encloses the minimum amount of area for a given perimeter, circle packing allows for maximum efficiency in terms of paper usage. However, other polygonal shapes can be used to solve the packing problem as well. The use of polygonal shapes other than circles is often motivated by the desire to find easily locatable creases (such as multiples of 22.5 degrees) and hence an easier folding sequence as well. One popular offshoot of the circle packing method is box-pleating, where squares are used instead of circles. As a result, the crease pattern that arises from this method contains only 45 and 90 degree angles, which makes for easier folding.

Gallery

These pictures show examples of various types of origami.

Dollar bill elephant, an example of moneygami

Kawasaki rose, this uses the twist fold devised by Kawasaki. The calyx is made separately.



Kawasaki cube, an example of an iso-area model.

A wet-folded bull.

See also

- Card model
- Chinese paper folding
- Furoshiki
- Japanese art
- Kirigami
- List of origamists
- Origamic architecture
- Paper fortune teller
- Paper plane
- Papercraft

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Further reading

- Robert J. Lang. The Complete Book of Origami: Step-by-Step Instructions in Over 1000 Diagrams. Dover Publications, Mineola, NY. Copyright 1988 by Robert J. Lang. ISBN 0-486-25837-8 (pbk.)

  Pages 1–30 are an excellent introduction to most of these skills. Each of these 13 models is designed to let you practice one skill several times. Unfortunately, the remaining 24 models leave out lots of pre-creases. Many models are folded from non-square paper.
- Kunihiko Kasahara. Origami Omnibus: Paper Folding for Everybody. Japan Publications, inc. Tokyo. Copyright 1988 by Kunihiko Kasahara. ISBN 4-8170-9001-4

  A good book for a more advanced origamian, this book presents many more complicated ideas and theories, although the author tends to go off on long tangents about random topics. Still lots of good models though...
- Kunihiko Kasahara and Toshie Takahama. Origami for the Connoisseur. Japan Publications, inc. Tokyo. Copyright 1987 by Kunihiko Kasahara and Toshie Takahama. ISBN 0-87040-670-1

  Another good book; same comments as the previous author.
- Satoshi Kamiya. Works by Satoshi Kamiya, 1995-2003. Origami House, Tokyo. Copyright 2005 by Satoshi Kamiya.

  An extremely complex book for the elite origamian, most models take 100+ steps to complete. Includes his famous Divine Dragon Bahamut and Ancient Dragons. Instructions are in Japanese and English.
- Issei Yoshino. Issei Super Complex Origami. Origami House, Tokyo.

  Contains many complex models, notably his Samurai Helmet, Horse, and multimodular Triceratops skeleton. Instructions are in Japanese.
- *One Thousand Paper Cranes: The Story of Sadako and the Children's Peace Statue* by Takayuki Ishii, ISBN 0-440-22843-3
- *Sadako and the Thousand Paper Cranes* by Eleanor Coerr, ISBN 0-698-11802-2
- *Extreme Origami*, Kunihiko Kasahara, 2001, ISBN 0-8069-8853-3
- Ariomar Ferreira da Silva. Brincando com Origami Arquitetônico: 16 diagrams. Global Editora, São Paulo, Brazil. Copyright 1991 by Ariomar Ferreira da Silva and Leôncio de O. Carvalho. ISBN 85-260-0273-2
- *Masterworks of Paper Folding* by Michael LaFosse
- *Papercopia: Origami Designs by David Shall*, 2008 ISBN 978-0-9796487-0-0. Contains diagrams for 24 original models by the author including Claw Hammer, Daffodil, Candlestick.
- Nick Robinson. Origami For Dummies. John Wiley, Copyright 2008 by Nick Robinson. ISBN 0470758570. An excellent book for beginners, covering many aspects of origami overlooked by other books.
- Nick Robinson. Encyclopedia of Origami. Quarto, Copyright 2004 by Nick Robinson. ISBN 1-84448-025-9. An book full of stimulating designs.

- Linda Wright. Toilet Paper Origami: Delight Your Guests with Fancy Folds and Simple Surface Embellishments. Copyright 2008 by Lindaloo Enterprises, Santa Barbara, CA. ISBN 9780980092318.

External links

- Free Origami Instruction Database ! (https://sites.google.com/site/origamisite/design-instructions), a collection of links to free origami instructions, pictures and videos.
- Origami (http://www.dmoz.org/Arts/Crafts/Paper/Origami//) at the Open Directory Project
- OrigamiTube.com - Watch, Fold, Show Off! (http://www.origamitube.com) Collection of origami instructional and origami related videos.
- Origami.com (http://www.origami.com/diagram.html), collection of diagrams, suitable for beginners.
- GiladOrigami.com (http://www.giladorigami.com/Gallery_default.html), contains a large gallery.
- The Fold (http://thepaperfold.gofreeserve.com), a large collection of diagrams.
- WikiHow on How to Make Origami
- Origami.org.uk (http://www.origami.org.uk), 3D animated origami diagrams of peace crane and flapping bird.
- - Origami Surprise ! (http://www.britishorigami.info/fun/surprise.php), a brand new type of origami folding instructions.

Flexagon

In geometry, **flexagons** are flat models, usually constructed by folding strips of paper, that can be *flexed* or folded in certain ways to reveal faces besides the two that were originally on the back and front.

Flexagons are usually square or rectangular (**tetraflexagons**) or hexagonal (**hexaflexagons**). A prefix can be added to the name to indicate the number of faces that the model can display, including the two faces (back and front) that are visible before flexing. For example, a hexaflexagon with a total of six faces is called a hexahexaflexagon.

In hexaflexagon theory (that is, concerning flexagons with six sides), flexagons are usually defined in terms of *pats*.[1] [2]

Two flexagons are equivalent if one can be transformed to the other by a series of pinches and rotations. Flexagon equivalence is an equivalence relation.[1]

History

Discovery and introduction

The discovery of the first flexagon, a trihexaflexagon, is credited to the British student Arthur H. Stone who was studying at Princeton University in the USA in 1939, allegedly while he was playing with the strips he had cut off his A4 paper to convert it to letter size. Stone's colleagues Bryant Tuckerman, Richard P. Feynman and John W. Tukey became interested in the idea and formed the *Princeton Flexagon Committee*. Tuckerman worked out a topological method, called the Tuckerman traverse, for revealing all the faces of a flexagon.[3]

Flexagons were introduced to the general public by the recreational mathematician Martin Gardner, writing in 1956 in his inaugural "Mathematical Games" column for *Scientific American* magazine.[4]

Attempted commercial development

In 1955, Russell Rogers and Leonard D'Andrea of Homestead Park, Pennsylvania applied for, and in 1959 were granted, U.S. Patent #2,883,185 for the hexahexaflexagon, under the title "Changeable Amusement Devices and the Like." The patent imagined possible applications of the device "as a toy, as an advertising display device, or as an educational geometric device."[5] A few such novelties were produced by the Herbick & Held Printing Company, the Pittsburgh printing firm where Rogers worked. But the device, marketed as the "Hexmo", failed to catch on commercially.

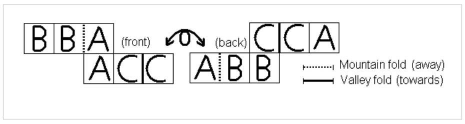
Varieties

Tetraflexagons

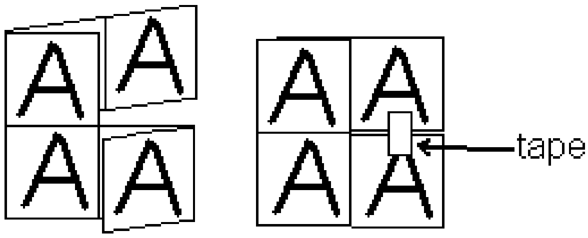
Tritetraflexagon

The tritetraflexagon is the simplest tetraflexagon (flexagon with square sides). The "tri" in the name means it has three faces, two of which are visible at any given time if the flexagon is pressed flat.

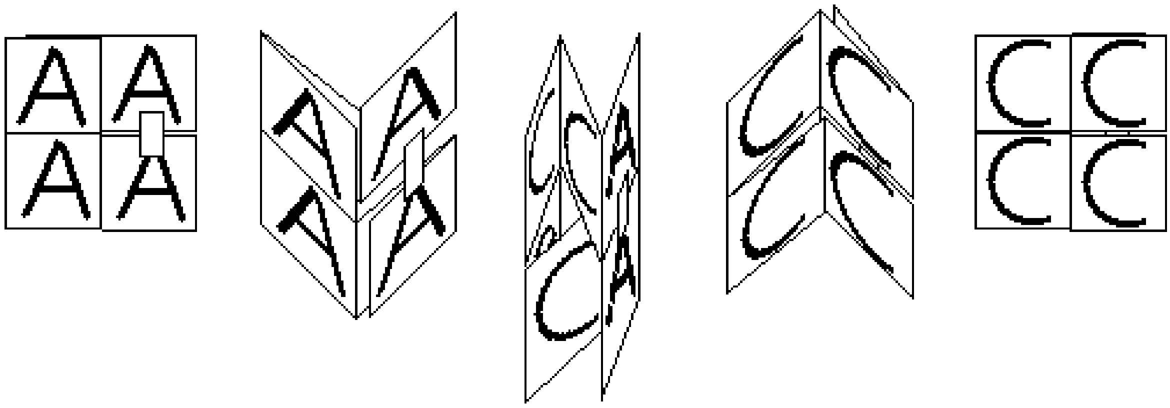
It is folded from a strip of six squares of paper like this:



To fold this shape into a tritetraflexagon, first crease each line between two squares. Then fold the mountain fold away from you and the valley fold towards you, and add a small piece of tape like this



This figure has two faces visible, built of squares marked with "A"s and "B"s. The face of "C"s is hidden inside the flexagon. To reveal it, fold the flexagon flat and then unfold it, like this



The construction of the tritetraflexagon is similar to the mechanism used in the traditional Jacob's Ladder children's toy, in Rubik's Magic and in the magic wallet trick or the Himber wallet.

Cyclic hexa-tetraflexagon

There is also a method of creating a more complicated hexatetraflexagon. To make it, take a piece of square paper and cut a square hole in the middle. Make sure all edges are straight. Then from the left hand edge, make a valley fold towards the middle. From the top, make another valley fold towards the middle. Now make a valley fold from the right hand edge towards the middle. Finally, make a valley fold from the bottom towards the middle. You now have the hexatetraflexagon. Please note that you do not need any tape or paste to make this flexagon. The most exciting thing to do with this one, is to colour both faces, then keep on flexing and colour the faces as you find them, until you get back to your starting position.

Hexaflexagons

Hexaflexagons come in great variety, distinguished by the number of faces that can be achieved by flexing the assembled figure.

Trihexaflexagon

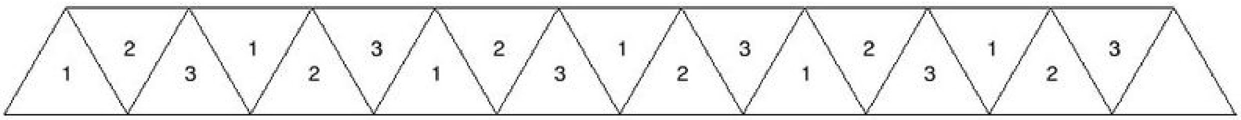
A hexaflexagon with three faces.

While this is the simplest of the hexaflexagons to make and to manage, it is a very satisfying place to begin. It is made from a single strip of paper, divided into ten equilateral triangles. Patterns are available at The Flexagon Portal [6].

It is possible to automatically section and correctly place photographs (or drawings) of your own selection onto Trihexaflexagons using the simple program Foto-TriHexaFlexagon [7].

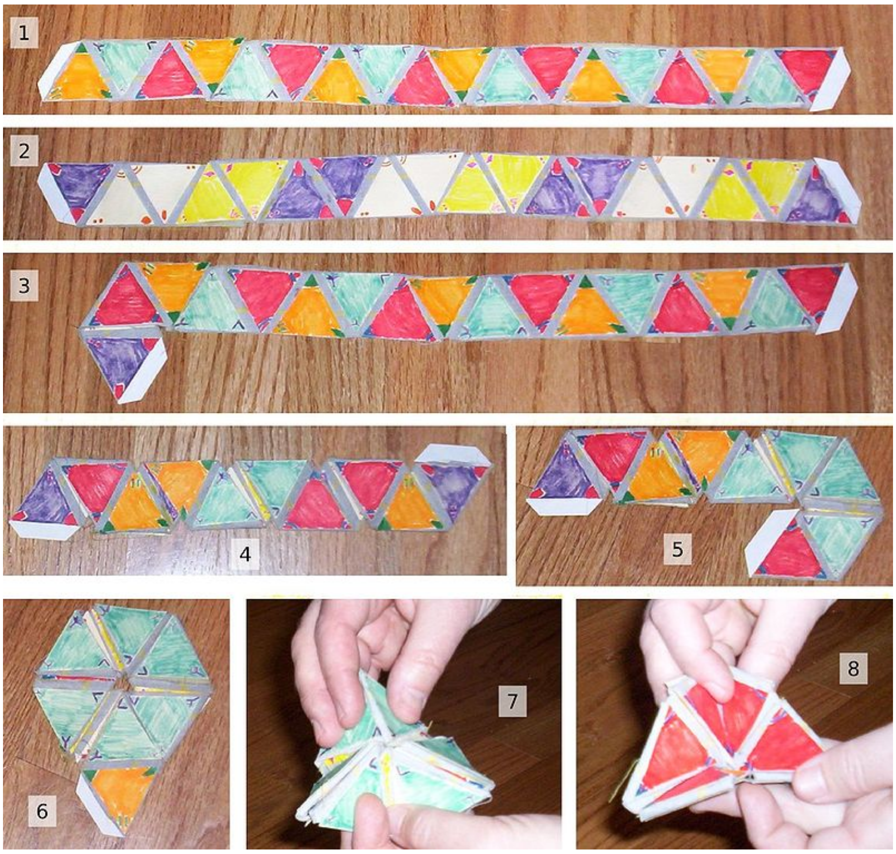
Hexahexaflexagon

This hexaflexagon has six faces.



Make a mountain fold between the first 2 and the first 3. Continue folding in a spiral fashion, for a total of nine folds. You now have a straight strip with ten triangles on each side. There are two places where 3's are next to each other; fold in both these places so as to hide the 3's, forming a hexagon with a triangular tab sticking out. Lift one end of the hexagon around the other so that the 3's near the ends are touching each other. Fold the tab over to cover the blank triangle on the other side, and glue it to the blank triangle. One side of the hexagon should be all 1's, one side should be all 2's, and all the 3's should hidden.

Photos 1-6 below show the construction of a hexaflexagon made out of cardboard triangles on a backing made from a strip of cloth. It has been decorated in six colors; orange, blue, and red in figure 1 correspond to 1, 2, and 3 in the diagram above. The opposite side, figure 2, is decorated with purple, gray, and yellow. Note the different patterns used for the colors on the two sides. Figure 3 shows the first fold, and figure 4 the result of the first nine folds, which form a spiral. Figures 5-6 show the final folding of the spiral to make a hexagon; in 5, two red faces have been hidden by a valley fold, and in 6, two red faces on the bottom side have been hidden by a mountain fold. After figure 6, the final loose triangle is folded over and attached to the other end of the original strip so that one side is all blue, and the other all orange.



Photos 7 and 8 show the process of everting the hexaflexagon to show the formerly hidden red triangles. By further manipulations, all six colors can be exposed. Faces 1, 2, and 3 are easier to find while faces 4, 5, and 6 are more difficult to find. An easy way to expose all six faces is using the *Tuckerman traverse*. It's named after Bryant Tuckerman, one of the first to investigate the properties of hexaflexagons. The *Tuckerman traverse* involves the repeated flexing by pinching one corner and flex from exactly the same corner every time. If the corner refuses to open, move to an adjacent corner and keep flexing. This procedure brings you to a 12-face cycle. During this procedure, however, 1, 2, and 3 show up three times as frequently as 4, 5, and 6. The cycle proceeds as follows:

1-3-6-1-3-2-4-3-2-1-5-2

And then back to 1 again.

Each color/face can also be exposed in more than one way. In figure 6, for example, each blue triangle has at the center its corner decorated with a wedge, but it is also possible, for example, to make the ones decorated with Y's come to the center. There are 18 such possible configurations for triangles with different colors, and they can be seen by flexing the hexahexaflexagon in all possible ways in theory, but only 15 can be flexed by the ordinary hexahexaflexagon. The 3 extra configurations are impossible due to the arrangement of the 4, 5, and 6 tiles at the back flap. (The 60-degree angles in the rhombi formed by the adjacent 4, 5, or 6 tiles will only appear on the sides and never will appear at the center because it would require one to cut the strip, which is topologically forbidden.)

The one shown is not the only hexahexaflexagon. Others can be constructed from different shaped nets of eighteen equilateral triangles. One hexahexaflexagon, constructed from an irregular paper strip, is almost identical to the one shown above, except that all 18 configurations can be flexed on this version.

Other hexaflexagons

While the most commonly seen hexaflexagons have either three or six faces, variations exist with four, five, and seven faces. Nets for these can be found here: ([8]).

Higher order flexagons

Right octaflexagon and right dodecaflexagon

In these more recently discovered flexagons, each square or equilateral triangular face of a conventional flexagon is further divided into two right triangles, permitting additional flexing modes ([9]). The division of the square faces of tetraflexagons into right isosceles triangles yields the octaflexagons ([10]), and the division of the triangular faces of the hexaflexagons into 30-60-90 right triangles yields the dodecaflexagons ([11]).

Pentaflexagon and right decaflexagon

In its flat state, the pentaflexagon looks much like the Chrysler logo: a regular pentagon divided from the center into five isosceles triangles, with angles 72-54-54. Because of its fivefold symmetry, the pentaflexagon cannot be folded in half. However, a complex series of flexes results in its transformation from displaying sides 1 and 2 on the front and back, to displaying its previously hidden sides 3 and 4. [12]

By further dividing the 72-54-54 triangles of the pentaflexagon into 36-54-90 right triangles produces one variation of the 10-sided decaflexagon ([13]).

Generalized isosceles n-flexagon

The pentaflexagon (described above) is one of an infinite sequence of flexagons based on dividing a regular n-gon into n isosceles triangles. There is the heptaflexagon ([14]), the isosceles octaflexagon ([15]), enneaflexagon ([16]), and on and on...

Nonplanar pentaflexagon and nonplanar heptaflexagon

Harold V. McIntosh also describes "nonplanar" (i.e., they can't be flexed so they lie flat) flexagons; ones folded from pentagons called *pentaflexagons* [17], and from heptagons called *heptaflexagons* [18]. These should be distinguished from the "ordinary" penta- and heptaflexagons described above, which are made out of isosceles triangles, and *can* be made to lie flat.

Bibliography

- Pook, Les (2009). *Serious Fun with Flexagons, A Compendium and Guide*. Springer. ISBN 9048125022.
- Pook, Les, *Flexagons Inside Out*, Cambridge University Press (2006), ISBN 0-521-81970-9 [19]
- Martin Gardner has written an excellent introduction to hexaflexagons in one of his *Mathematical Games* column in *Scientific American*. It also appears in:
  - *The "Scientific American" Book of Mathematical Puzzles and Diversions* (Simon & Schuster, 1959).
  - *Hexaflexagons and Other Mathematical Diversions: The First "Scientific American" Book of Puzzles and Games* (University of Chicago Press, 1988; ISBN 0226282546)
  - *Hexaflexagons, Probability Paradoxes, and the Tower of Hanoi: Martin Gardner's First Book of Mathematical Puzzles and Games* (Cambridge University Press, 2008; ISBN 0521735254)

See also

- Geometric group theory
- Cayley tree
- Octahedron: two identically formed nonplanar flexagons: one octahedron

References

[1] Oakley, C. O.; Wisner, R. J. (March 1957), "Flexagons" (http://www.jstor.org/stable/2310544), *The American Mathematical Monthly* (Mathematical Association of America) **64** (3): 143–154,

[2] Anderson, Thomas; McLean, T. Bruce; Pajoohesh, Homeira; Smith, Chasen (January 2010), "The combinatorics of all regular flexagons" (http://www.sciencedirect.com/science/article/B6WDY-4W5VD4F-1/2/e1d94639a2f71f509b049f8ab6480cb7), *European Journal of Combinatorics* **31** (1): 72–80, ISSN 0195-6698,

[3] Gardner, Martin (1988). *Hexaflexagons and Other Mathematical Diversions: The First Scientific American Book of Puzzles and Games*. University of Chicago Press. ISBN 0226282546.

[4] Jürgen Köller. "Flexagons" (http://www.mathematische-basteleien.de/flexagons.htm). . Retrieved 23 September 2009.

[5] (http://www.freepatentsonline.com/2883195.pdf)

[6] http://www.flexagon.net/

[7] http://britton.disted.camosun.bc.ca/fotothf/fotothf.htm

[8] http://www.flexagon.net

[9] http://www.eighthsquare.com/12-gon.html

[10] http://loki3.com/flex/octa.html

[11] http://loki3.com/flex/dodeca.html

[12] http://loki3.com/flex/penta.html

[13] http://loki3.com/flex/deca.html

[14] http://loki3.com/flex/hepta.html

[15] http://loki3.com/flex/octa.html#iso

[16] http://loki3.com/flex/ennea.html#iso

[17] http://delta.cs.cinvestav.mx/~mcintosh/comun/pentags/pentags.html

[18] http://delta.cs.cinvestav.mx/~mcintosh/comun/heptagon/heptagon.html

[19] http://www.cambridge.org/catalogue/catalogue.asp?isbn=0521819709

External links

Flexagons:

- My Flexagon Experiences (http://delta.cs.cinvestav.mx/~mcintosh/comun/fxgonw/fxgon.html) by Harold V. McIntosh — contains valuable historical information and theory; the author's site has several flexagon related papers listed in (http://delta.cs.cinvestav.mx/~mcintosh/oldweb/pflexagon.html) and even boasts some flexagon videos in (http://delta.cs.cinvestav.mx/~mcintosh/videos/hexaflexagonos/videosflexa.html).
- The Flexagon Portal (http://www.flexagon.net/) — Robin Moseley's site has patterns for a large variety of flexagons.
- Flexagons (http://www.mathematische-basteleien.de/flexagons.htm) is a good introduction, including a large number of links.
- Flexagons (http://loki3.com/flex/) — Scott Sherman's site, with a bewildering array of flexagons of different shapes.

Tetraflexagons:

- MathWorld's page on tetraflexagons (http://mathworld.wolfram.com/Tetraflexagon.html), including three nets
- Folding User Interfaces (http://droppingmadscience.blogspot.com/2007/02/nalu.html) - A mobile phone design concept based on a tetraflexagon; Folding the design gives access to different user interfaces.
- Flexifier (http://lab.satyr.nl/flex) - a simple online tetraflexagon generator
- Instructions for making cyclic hexa-tetraflexagon from just one piece of paper. (http://www.metacafe.com/watch/1007181/the_best_of_paper_toy_just_one_piece_of_paper_no_glue_6_face/)

Hexaflexagons:

- Flexagons (http://theory.lcs.mit.edu/~edemaine/flexagons/Conrad-Hartline-1962/flexagon.html) — 1962 paper by Antony S. Conrad and Daniel K. Hartline (RIAS)
- MathWorld entry on Hexaflexagons (http://mathworld.wolfram.com/Hexaflexagon.html)
- Hexaflexagon Toolkit (http://hexaflexagon.sourceforge.net) software for printing flexagons from your own pictures
- Hexaflexagons (http://www.coe.ufrj.br/~acmq/hexaflexagons/) — a catalog compiled by Antonio Carlos M. de Queiroz (c.1973).
  Includes a program named HexaFind that finds all the possible Tuckerman traverses for given orders of hexaflexagons.

Regular paperfolding sequence

In mathematics the **regular paperfolding sequence**, also known as the **dragon curve sequence**, is an infinite automatic sequence of 0s and 1s defined as the limit of the following process:

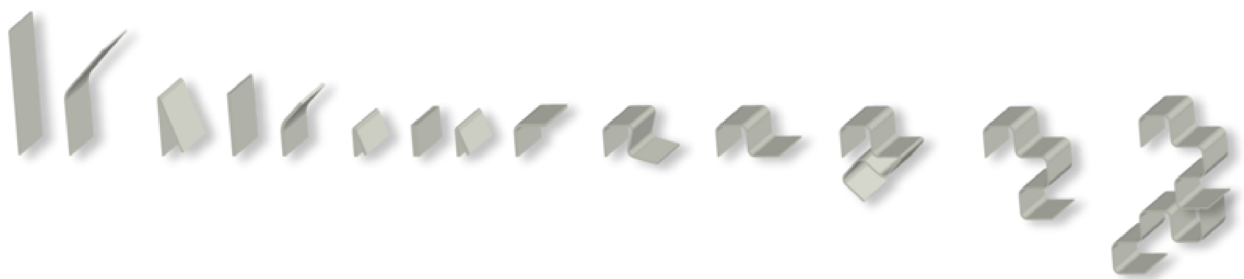
1

1 1 **0**

1 1 **0** 1 **1** 0 **0**

1 1 **0** 1 **1** 0 **0** 1 **1** 1 **0** 0 **1** 0 **0**

At each stage an alternating sequence of 1s and 0s is inserted between the terms of the previous sequence. The sequence takes its name from the fact that it represents the sequence of left and right folds along a strip of paper that is folded repeatedly in half in the same direction. If each fold is then opened out to create right angled corner, the resulting shape approaches the dragon curve fractal.[1]



Starting at $n = 1$, the first few terms of the regular paperfolding sequence are:

1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0, 1, ... (sequence A014577 [2] in OEIS)

Properties

The value of any given term t_n in the regular paperfolding sequence can be found recursively as follows. If $n = m \cdot 2^k$ where m is odd then

$$t_n = \begin{cases} 1 & \text{if } m = 1 \mod 4 \\ 0 & \text{if } m = 3 \mod 4 \end{cases}$$

Thus $t_{12} = t_3 = 0$ but $t_{12} = 1$.

The paperfolding word 1101100111001001..., which is created by concatenating the terms of the regular paperfolding sequence, is a fixed point of the morphism or string substitution rules

11 → 1101

01 → 1001

10 → 1100

00 → 1000

as follows:

11 → 1101 → 11011001 → 1101100111001001 → 11011001110010011101100011001001 ...

It can be seen from the morphism rules that the paperfolding word contains at most three consecutive 0s and at most three consecutive 1s.

The paperfolding sequence also satisfies the symmetry relation:

$$t_n = \begin{cases} 1 & \text{if } n = 2^k \\ 1 - t_{2^k - n} & \text{if } 2^{k-1} < n < 2^k \end{cases}$$

which shows that the paperfolding word can be constructed as the limit of another iterated process as follows:

1

1 **1** 0

110 **1** 100

1101100 **1** 1100100

110110011100100 **1** 110110001100100

Generating function

The generating function of the paperfolding sequence is given by

$$G(t_n; x) = \sum_{n=0}^{\infty} t_n x^n \ .$$

From the construction of the paperfolding sequence it can be seen that *G* satisfies the functional relation

$$G(t_n; x) = G(t_n; x^2) + \sum_{n=0}^{\infty} x^{4n+1} = G(t_n; x^2) + \frac{x}{1 - x^4} \ .$$

Paperfolding constant

Substituting $x = ½$ into the generating function gives a real number between 0 and 1 whose binary expansion is the paperfolding word

$$G(t_n; \frac{1}{2}) = \sum_{n=1}^{\infty} \frac{t_n}{2^n}$$

This number is known as the **paperfolding constant**[3] and has the value

$$\sum_{k=0}^{\infty} \frac{8^{2^k}}{2^{2^{k+2}} - 1} = 0.85073618820186\ldots$$ (sequence A143347 [4] in OEIS)

References

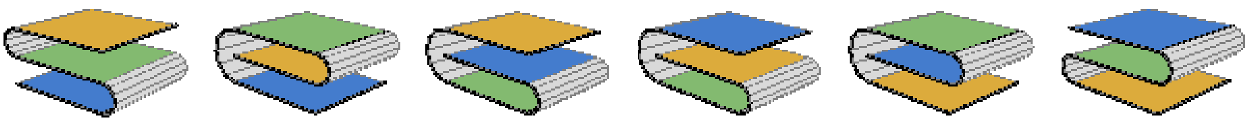
[1] Weisstein, Eric W., " Dragon Curve (http://mathworld.wolfram.com/DragonCurve.html)" from MathWorld.
[2] http://en.wikipedia.org/wiki/Oeis%3Aa014577
[3] Weisstein, Eric W., " Paper Folding Constant (http://mathworld.wolfram.com/PaperFoldingConstant.html)" from MathWorld.
[4] http://en.wikipedia.org/wiki/Oeis%3Aa143347

- Jean-Paul Allouche and Jeffrey Shallit *Automatic Sequences* Cambridge University Press 2003

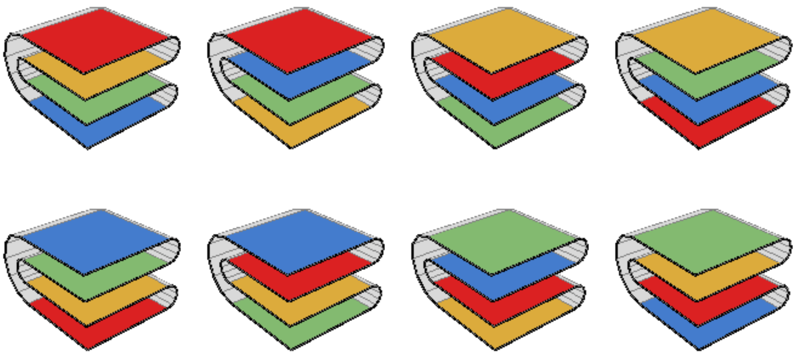
Map folding

In combinatorial mathematics the **map folding** problem is the question of how many ways there are to fold a rectangular map along its creases. A related problem called the **stamp folding** problem is how many ways there are to fold a strip of stamps.[1]

For example, there are six ways to fold a strip of three different stamps:



And there are eight ways to fold a 2×2 map along its creases:



The problem is related to a problem in the mathematics of origami of whether a square with a crease pattern can be folded to a flat figure. Some simple extensions to the problem of folding a map are NP-complete.[2]

References

[1] Weisstein, Eric W., " Map Folding (http://mathworld.wolfram.com/MapFolding.html)" from MathWorld.
[2] Esther M. Arkin, Michael A. Bender, Erik D. Demaine, Martin L. Demaine, Joseph S. B. Mitchell, Saurabh Sethia, Steven S. Skiena (September 2004). "When Can You Fold a Map?" (http://erikdemaine.org/papers/MapFolding/paper.pdf). *Computational Geometry: Theory and Applications* **29** (1): pp. 23–46. .

See also

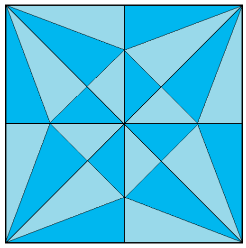
- Martin Gardner, "The Combinatorics of Paper Folding," *Wheels, Life and Other Mathematical Amusements*, New York: W. H. Freeman, 1983 pp. 60–61.
- "Folding a Strip of Labeled Stamps" from The Wolfram Demonstrations Project: http://demonstrations.wolfram.com/FoldingAStripOfLabeledStamps/

Mathematics of paper folding

The art of **paper folding**, or origami, has received a considerable amount of mathematical study. Fields of interest include a given paper model's flat-foldability (whether the model can be flattened without damaging it) and the use of paper folds to solve mathematical equations.

Pure origami

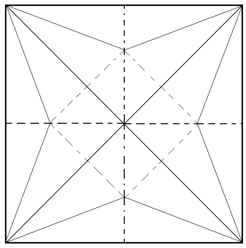
Flat folding



Two-colorability.

The construction of origami models is sometimes shown as crease patterns. The major question about such crease patterns is whether a given crease pattern can be folded to a flat model, and if so, how to fold them; this is an NP-complete problem.[1] Related problems when the creases are orthogonal are called map folding problems. There are four mathematical rules for producing flat-foldable origami crease patterns:[2]

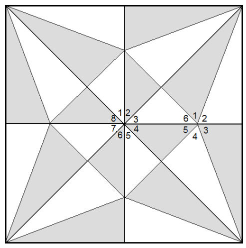
1. crease patterns are two colorable
2. Maekawa's Theorem: at any vertex the number of valley and mountain folds always differ by two in either direction
3. Kawasaki's theorem: at any vertex, the sum of all the odd angles adds up to 180 degrees, as do the even.
4. a sheet can never penetrate a fold.



Mountain-valley counting.

Paper exhibits zero Gaussian curvature at all points on its surface, and only folds naturally along lines of zero curvature. Curved surfaces which can't be flattened can be produced using a non-folded crease in the paper, as is easily done with wet paper or a fingernail.

Assigning a crease pattern mountain and valley folds in order to produce a flat model has been proven by Marshall Bern and Barry Hayes to be NP complete.[3] Further references and technical results are discussed in Part II of *Geometric Folding Algorithms*.[4]



Angles around a vertex.

Axioms

Some classical construction problems of geometry — namely trisecting an arbitrary angle, or doubling the cube — are proven to be unsolvable using compass and straightedge, but can be solved using only a few paper folds.[5] Paper fold strips can be constructed to solve equations up to degree 4. (The Huzita–Hatori axioms are one important contribution to this field of study.) Complete methods for solving all equations up to degree 4 by applying such methods are discussed in detail in *Geometric Origami*.[6]

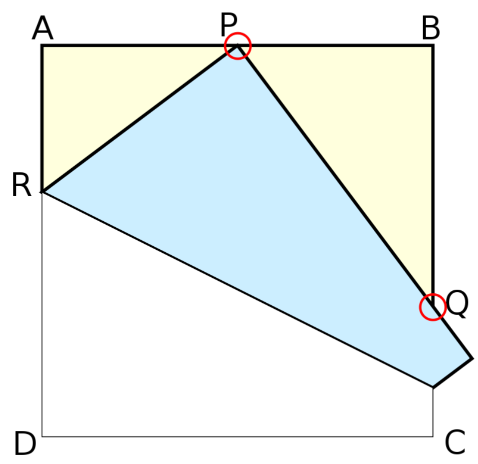
As a result of origami study through the application of geometric principles, methods such as Haga's theorem have allowed paperfolders to accurately fold the side of a square into thirds, fifths, sevenths, and ninths. Other theorems and methods have allowed paperfolders to get other shapes from a square, such as equilateral triangles, pentagons, hexagons, and special rectangles such as the golden rectangle and the silver rectangle. Methods for folding most regular polygons up to and including the regular 19-gon have been developed.[6]

Constructions

Haga's theorems

The side of a square can be divided at an arbitrary rational fraction in a variety of ways. Haga's theorems say that a particular set of constructions can be used for such divisions.[7] Surprisingly few folds are necessary to generate large odd fractions. For instance $^1⁄_5$ can be generated with three folds; first halve a side, then use Haga's theorem twice to produce first $^2⁄_3$ and then $^1⁄_5$.

The accompanying diagram shows Haga's first theorem:



BQ is always a rational if AP is.

$$BQ = \frac{2AP}{1 + AP}$$

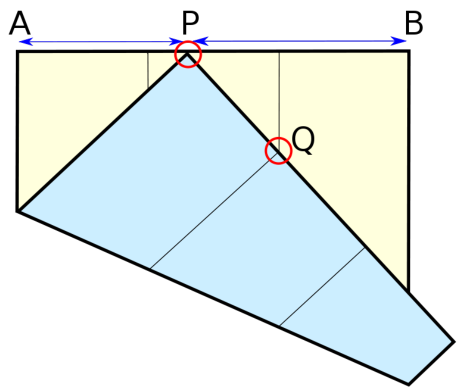
Interestingly the function changing the length *AP* to *QC* is self inverse. Let *x* be *AP* then a number of other lengths are also rational functions of *x*. For example:

Haga's first theorem

AP	BQ	QC	AR	PQ
x	$\frac{2x}{1+x}$	$\frac{1-x}{1+x}$	$\frac{1-x^2}{2}$	$\frac{1+x^2}{1+x}$
$^1⁄_2$	$^2⁄_3$	$^1⁄_3$	$^3⁄_8$	$^5⁄_6$
$^1⁄_3$	$^1⁄_2$	$^1⁄_2$	$^4⁄_9$	$^5⁄_6$
$^2⁄_3$	$^4⁄_5$	$^1⁄_5$	$^5⁄_{18}$	$^{13}⁄_{15}$
$^1⁄_5$	$^1⁄_3$	$^2⁄_3$	$^{12}⁄_{25}$	$^{13}⁄_{15}$

Doubling the cube

The classical problem of doubling the cube can be solved by first creasing a square of paper into three equal strips as shown in the diagram. Then the bottom edge is positioned so the corner point P is on the top edge and the crease mark on the edge meets the other crease mark Q. The length PB will then be the cube root of 2 times the length of AP.[8]



Doubling the cube: PB/PA = cube root of 2

The edge with the crease mark is considered a marked straightedge, something which is not allowed in compass and straightedge constructions. Using a marked straightedge in this way is called a neusis construction in geometry.

Related problems

The problem of rigid origami, treating the folds as hinges joining two flat, rigid surfaces, such as sheet metal, has great practical importance. For example, the Miura map fold is a rigid fold that has been used to deploy large solar panel arrays for space satellites.

The napkin folding problem is the problem of whether a square or rectangle of paper can be folded so the perimeter of the flat figure is greater than that of the original square.

Curved origami also poses a (very different) set of mathematical challenges.[9] Curved origami allows the paper to form developable surfaces that are not flat.

Wet-folding origami allows an even greater range of shapes.

The maximum number of times an incompressible material can be folded has been derived. With each fold a certain amount of paper is lost to potential folding. The loss function for folding paper in half in a single direction was given to be $L = \frac{\pi t}{6}(2^n + 4)(2^n - 1)$, where L is the minimum length of the paper (or other material), t is the material's thickness, and n is the number of folds possible. The distances L and t must be expessed in the same units, such as inches. This function was derived by Britney Gallivan in 2001 (then only a high school student) who then folded a sheet of paper in half 12 times, contrary to the popular belief that paper of any size could be folded at most eight times. She also derived the equation for folding in alternate directions.[10]

The fold-and-cut problem asks what shapes can be obtained by folding a piece of paper flat, and making a single straight complete cut. The solution, known as the Fold and Cut Theorem, states that any shape with straight sides can be obtained.

See also

- Napkin folding problem
- Map folding
- Regular paperfolding sequence (for example, the Dragon curve)

Notes

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[9] Siggraph: "Curved Origami" (http://www.siggraph.org/s2008/attendees/design/22.php)
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Further reading

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- Kazuo Haga (2008). Josefina C Fonacier; Masami Isoda. eds. *Origamics: Mathematical Explorations through Paper Folding*. World Scientific. ISBN 9789812834898.
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External links

- Dr. Tom Hull. "Origami Mathematics Page" (http://mars.wnec.edu/~th297133/origamimath.html)
- Origami & Math (http://www.paperfolding.com/math/) by Eric M. Andersen (http://www.paperfolding.com/email/)
- Paper Folding Geometry (http://www.cut-the-knot.org/pythagoras/PaperFolding/index.shtml) at cut-the-knot
- Dividing a Segment into Equal Parts by Paper Folding (http://www.cut-the-knot.org/pythagoras/PaperFolding/SegmentDivision.shtml) at cut-the-knot
- Britney Gallivan has solved the Paper Folding Problem (http://pomonahistorical.org/12times.htm)
- Folding Paper - Great Moments in Science - ABC (http://www.abc.net.au/science/k2/moments/s1523497.htm)
- Origami & geometry in English and in French (http://membres.lycos.fr/pliages/origamigeometry.html)
- Origami & geometry in English and in Hebrew (http://www.foldersfun.tipo.co.il)
- Origami & geometry in Spanish (http://web.archive.org/web/20091027090916/http://geocities.com/micadesa/educacion/indexeducacion.htm)

Mathematical games

Mathematical game

This article is about using mathematics to study the inner-workings of multiplayer games which, on the surface, may not appear mathematical at all. For games that directly involve mathematics in their play, see mathematical puzzle.

Mathematical Games was a column written by Martin Gardner that appeared in the Scientific American*. Information on his column and other recreational mathematics publications can be found in the recreational mathematics article.*

A **mathematical game** is a multiplayer game whose rules, strategies, and outcomes can be studied and explained by mathematics. Examples of such games are Tic-tac-toe and Dots and Boxes, to name a couple. On the surface, a game need not seem mathematical or complicated to still be a mathematical game. For example, even though the rules of Mancala are straightforward, mathematicians analyze the game using combinatorial game theory.

Mathematical games differ from mathematical puzzles in that all mathematical puzzles require math to solve them whereas mathematical games may not require a knowledge of mathematics to play them or even to win them. Thus the actual mathematics of mathematical games may not be apparent to the average player.

Some mathematical games are topics of interest in recreational mathematics.

When studying the mathematics of games, the mathematical analysis of the game is more important than actually playing the game. To analyze a game mathematically, the mathematician studies the rules of the game in order to understand the inner-workings of the game, to determine winning strategies, and to possibly to determine if a game has a solution.

Specific mathematical games and puzzles

Abstract Strategy Games (No chance involved)

Sometimes it is not immediately obvious that a particular game involves chance. Often a card game is described as "pure strategy" and such, but a game with any sort of random shuffling or face-down dealing of cards should not be considered to be "no chance".

Lattice board

- Angels and Devils
- Checkers (English draughts)
  - Checkers variants
- Chess
  - Chess variants
- Chomp
- Domineering
- Dots and boxes
- Go
  - Go variants
- Hex

- Hexapawn
- L game
- Pawn duel
- Philosopher's football
- Rhythmomachy

Non-lattice boards and other games

- Graph pebbling
- Hackenbush
- Chopsticks (Hand game)
- Nim
- Sim
- Sprouts

Chance involved or imperfect information

- 24
- Prisoner's dilemma

See also

- Solved game
- Games of skill

External links

- Historical Math Problems/Puzzles [1] at Convergence [2]

Nim

Nim is a two-player mathematical game of strategy in which players take turns removing objects from distinct heaps. On each turn, a player must remove at least one object, and may remove any number of objects provided they all come from the same heap.

Variants of Nim have been played since ancient times. The game is said to have originated in China (it closely resembles the Chinese game of "Jianshizi", or "picking stones"), but the origin is uncertain; the earliest European references to Nim are from the beginning of the 16th century. Its current name was coined by Charles L. Bouton of Harvard University, who also developed the complete theory of the game in 1901, but the origins of the name were never fully explained. The name is probably derived from German *nimm* meaning "take", or the obsolete English verb *nim* of the same meaning. It should also be noted that rotating the word *NIM* by 180 degrees results in *WIN* (see Ambigram).

Nim is usually played as a *misère* game, in which the player to take the last object loses. Nim can also be played as a *normal play* game, which means that the person who makes the last move (i.e., who takes the last object) wins. This is called normal play because most games follow this convention, even though Nim usually does not.

Normal play Nim (or more precisely the system of nimbers) is fundamental to the Sprague-Grundy theorem, which essentially says that in normal play every impartial game is equivalent to a Nim heap that yields the same outcome when played in parallel with other normal play impartial games (see disjunctive sum).

While all normal play impartial games can be assigned a nim value, that is not the case under the misère convention. Only tame games can be played using the same strategy as misère nim.

A version of Nim is played —and has symbolic importance— in the French New Wave film *Last Year at Marienbad* (1961).

Illustration

A normal play game (here between fictional players Alice and Bob) may start with heaps of 3, 4 and 5 objects:

In order to win, the player must always leave an even total number of 1's, 2's, and 4's.

```
Sizes of heaps  Moves
A B C

3 4 5           Alice takes 2 from A
1 4 5           Bob   takes 3 from C
1 4 2           Alice takes 1 from B
1 3 2           Bob   takes 1 from B
1 2 2           Alice takes entire A heap, leaving two 2s.
0 2 2           Bob   takes 1 from B
0 1 2           Alice takes 1 from C leaving two 1s. (In misère play she would take 2 from C leaving (0, 1, 0).)
0 1 1           Bob   takes 1 from B
0 0 1           Alice takes entire C heap and wins.
```

Mathematical theory

Nim has been mathematically solved for any number of initial heaps and objects; that is, there is an easily calculated way to determine which player will win and what winning moves are open to that player. In a game that starts with heaps of 3, 4, and 5, the first player will win with optimal play, whether the misère or normal play convention is followed.

The key to the theory of the game is the binary digital sum of the heap sizes, that is, the sum (in binary) neglecting all carries from one digit to another. This operation is also known as "exclusive or" (xor) or "vector addition over **GF**(2)". Within combinatorial game theory it is usually called the **nim-sum**, as will be done here. The nim-sum of x and y is written $x \oplus y$ to distinguish it from the ordinary sum, $x + y$. An example of the calculation with heaps of size 3, 4, and 5 is as follows:

```
Binary  Decimal


  011₂    3₁₀    Heap A
  100₂    4₁₀    Heap B
  101₂    5₁₀    Heap C
  ---
  010₂    2₁₀    The nim-sum of heaps A, B, and C, 3 ⊕ 4 ⊕ 5 = 2
```

An equivalent procedure, which is often easier to perform mentally, is to express the heap sizes as sums of distinct powers of 2, cancel pairs of equal powers, and then add what's left:

```
3 = 0 + 2 + 1 =     2   1      Heap A
4 = 4 + 0 + 0 = 4              Heap B
5 = 4 + 0 + 1 = 4       1      Heap C
---
2 =                 2          What's left after canceling 1s and 4s
```

In normal play, the winning strategy is to finish every move with a Nim-sum of 0. This is always possible if the Nim-sum is not zero before the move. If the Nim-sum is zero, then the next player will lose if the other player does not make a mistake. To find out which move to make, let X be the Nim-sum of all the heap sizes. Take the Nim-sum of each of the heap sizes with X, and find a heap whose size decreases. The winning strategy is to play in such a heap, reducing that heap to the Nim-sum of its original size with X. In the example above, taking the Nim-sum of the sizes is $X = 3 \oplus 4 \oplus 5 = 2$. The Nim-sums of the heap sizes A=3, B=4, and C=5 with X=2 are

$A \oplus X = 3 \oplus 2 = 1$ [Since $(011) \oplus (010) = 001$]

$B \oplus X = 4 \oplus 2 = 6$

$C \oplus X = 5 \oplus 2 = 7$

The only heap that is reduced is heap A, so the winning move is to reduce the size of heap A to 1 (by removing two objects).

As a particular simple case, if there are only two heaps left, the strategy is to reduce the number of objects in the bigger heap to make the heaps equal. After that, no matter what move your opponent makes, you can make the same move on the other heap, guaranteeing that you take the last object.

When played as a misère game, Nim strategy is different only when the normal play move would leave no heap of size 2 or larger. In that case, the correct move is to leave an odd number of heaps of size 1 (in normal play, the correct move would be to leave an even number of such heaps).

In a misère game with heaps of sizes 3, 4 and 5, the strategy would be applied like this:

```
A B C Nim-sum

3 4 5 010₂=2₁₀   I take 2 from A, leaving a sum of 000, so I will win.
1 4 5 000₂=0₁₀   You take 2 from C
1 4 3 110₂=6₁₀   I take 2 from B
1 2 3 000₂=0₁₀   You take 1 from C
1 2 2 001₂=1₁₀   I take 1 from A
0 2 2 000₂=0₁₀   You take 1 from C
0 2 1 011₂=3₁₀   The normal play strategy would be to take 1 from B, leaving an even number (2)
                  heaps of size 1.  For misère play, I take the entire B heap, to leave an odd
                  number (1) of heaps of size 1.
0 0 1 001₂=1₁₀   You take 1 from C, and lose.
```

The previous strategy for a misère game can be easily implemented in Python.

```
def nim(heaps, misere=True):
      """Computes next move for Nim in a normal or misère (default)
game, returns tuple (chosen_heap, nb_remove)"""
      X = reduce(lambda x,y: x^y, heaps)
      if X == 0: # Will lose unless all non-empty heaps have size one
            print "You will lose :("
            for i, heap in enumerate(heaps):
                  if heap > 0: # Empty any (non-empty) heap
                        chosen_heap, nb_remove = i, heap
                        break
      else:
            sums = [t^X < t for t in heaps]
            chosen_heap = sums.index(True)
            nb_remove = heaps[chosen_heap] - (heaps[chosen_heap]^X)
            heaps_twomore = 0
            for i, heap in enumerate(heaps):
                  n = heap-nb_remove if chosen_heap == i else heap
```

```
                    if n>1: heaps_twomore += 1
              # If move leaves no heap of size 2 or larger, leave an even
 (misère) or odd (normal) number of heaps of size 1
              if heaps_twomore == 0:
                    chosen_heap = heaps.index(max(heaps))
                    heaps_one = sum(t==1 for t in heaps)
                    # misère (resp. normal) strategy: if it is even
(resp. odd) make it odd (resp. even), else do not change
                    nb_remove = heaps[chosen_heap]-1 if
heaps_one%2!=misere else heaps[chosen_heap]
        return chosen_heap, nb_remove
```

Proof of the winning formula

The soundness of the optimal strategy described above was demonstrated by C. Bouton.

Theorem. In a normal Nim game, the player making the first move has a winning strategy if and only if the nim-sum of the sizes of the heaps is nonzero. Otherwise, the second player has a winning strategy.

Proof: Notice that the nim-sum (⊕) obeys the usual associative and commutative laws of addition (+), and also satisfies an additional property, $x \oplus x = 0$ (technically speaking, the nonnegative integers under ⊕ form an Abelian group of exponent 2).

Let $x_1, ..., x_n$ be the sizes of the heaps before a move, and $y_1, ..., y_n$ the corresponding sizes after a move. Let $s = x_1 \oplus ... \oplus x_n$ and $t = y_1 \oplus ... \oplus y_n$. If the move was in heap k, we have $x_i = y_i$ for all $i \neq k$, and $x_k > y_k$. By the properties of ⊕ mentioned above, we have

```
    t = 0 ⊕ t
      = s ⊕ s ⊕ t
      = s ⊕ (x_1 ⊕ ... ⊕ x_n) ⊕ (y_1 ⊕ ... ⊕ y_n)
      = s ⊕ (x_1 ⊕ y_1) ⊕ ... ⊕ (x_n ⊕ y_n)
      = s ⊕ 0 ⊕ ... ⊕ 0 ⊕ (x_k ⊕ y_k) ⊕ 0 ⊕ ... ⊕ 0
      = s ⊕ x_k ⊕ y_k

(*) t = s ⊕ x_k ⊕ y_k.
```

The theorem follows by induction on the length of the game from these two lemmas.

Lemma 1. If $s = 0$, then $t \neq 0$ no matter what move is made.

Proof: If there is no possible move, then the lemma is vacuously true (and the first player loses the normal play game by definition). Otherwise, any move in heap k will produce $t = x_k \oplus y_k$ from (*). This number is nonzero, since $x_k \neq y_k$.

Lemma 2. If $s \neq 0$, it is possible to make a move so that $t = 0$.

Proof: Let d be the position of the leftmost (most significant) nonzero bit in the binary representation of s, and choose k such that the dth bit of x_k is also nonzero. (Such a k must exist, since otherwise the dth bit of s would be 0.) Then letting $y_k = s \oplus x_k$, we claim that $y_k < x_k$: all bits to the left of d are the same in x_k and y_k, bit d decreases from 1 to 0 (decreasing the value by 2^d), and any change in the remaining bits will amount to at most 2^d-1. The first player can thus make a move by taking $x_k - y_k$ objects from heap k, then

```
t = s ⊕ x_k ⊕ y_k           (by (*))
  = s ⊕ x_k ⊕ (s ⊕ x_k)
  = 0.
```

The modification for misère play is demonstrated by noting that the modification first arises in a position that has only one heap of size 2 or more. Notice that in such a position $s \neq 0$, therefore this situation has to arise when it is the turn of the player following the winning strategy. The normal play strategy is for the player to reduce this to size 0 or 1, leaving an even number of heaps with size 1, and the misère strategy is to do the opposite. From that point on, all moves are forced.

Other variations of Nim

The subtraction game *S*(1,2,...,*k*)

In another game which is commonly known as Nim (but is better called the subtraction game *S* (1,2,...,*k*)), an upper bound is imposed on the number of objects that can be removed in a turn. Instead of removing arbitrarily many objects, a player can only remove 1 or 2 or ... or *k* at a time. This game is commonly played in practice with only one heap (for instance with $k = 3$ in the game *Thai 21* on Survivor: Thailand, where it appeared as an Immunity Challenge).

Bouton's analysis carries over easily to the general multiple-heap version of this game. The only difference is that as a first step, before computing the Nim-sums, we must reduce the sizes of the heaps modulo $k + 1$. If this makes all the heaps of size zero (in misère play), the winning move is to take *k* objects from one of the heaps. In particular, in a play from a single heap of *n* objects, the second player can win iff

$n \equiv 0 \pmod{k+1}$ (in normal play), or

$n \equiv 1 \pmod{k+1}$ (in misère play).

This follows from calculating the nim-sequence of *S*(1,2,...,*k*),

$$0.123...k0123...k0123... = \dot{0}.123\ldots\dot{k},$$

from which the strategy above follows by the Sprague-Grundy theorem.

The 21 game

The game "21" is played as a misère game with any number of players who take turns saying a number. The first player says "1" and each player in turn increases the number by 1, 2, or 3, but may not exceed 21; the player forced to say "21" loses. This can be modeled as a subtraction game with a heap of $21-n$ objects.

The winning strategy for this game is to say a multiple of 4 and after that it is guaranteed that the other player will have to say 21, barring a mistake from the first player.

This game is a zero game, i.e, it is biased in favor of the 2nd player as s/he can get to 4 first and then control the game from there, as no matter what, the 1st player will never be able to say a multiple of 4 as s/he is only allowed increments of either 1, 2 or 3.

Proof (via a sample game)-

Player Number

```
1           1
2           4
1         5,6 or 7
2           8
1         9,10 or 11
2          12
1         13,14 or 15
2          16
1         17,18 or 19
2          20
```

```
1          21
```

A multiple-heap rule

In another variation of Nim, besides removing any number of objects from a single heap, one is permitted to remove the same number of objects from each heap.

Circular Nim

Yet another variation of Nim is 'Circular Nim', where any number of objects are placed in a circle, and two players alternately remove 1, 2 or 3 adjacent objects. For example, starting with a circle of ten objects,

. . . .

three objects be taken in the first move

_ . . . . _ _

then another three

_ . _ _ _ . . . _ _

then one

_ . _ _ _ . . _ _ _

but then three objects cannot be taken out in one move.

Grundy's game

In Grundy's game, another variation of Nim, a number of objects are placed in an initial heap, and two players alternately divide a heap into two nonempty heaps of different sizes. Thus, 6 objects may be divided into piles of 5+1 or 4+2, but not 3+3. Grundy's game can be played as either misère or normal play.

Greedy Nim

See Greedy Nim.

See also

- Dr. NIM
- Fuzzy game
- Nimber
- Nimrod (computing)
- Octal games
- Solved board games
- Star (game)
- Subtract a square
- Zero game
- Pawn duel
- NIMROD
- Android Nim

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External links

- Nim Flash on Kongregate [1]
- Nim-Game in Javascript [2] – including its historical aspect at Archimedes-lab.org.
- Nim-Game in Javascript [3] IE7 and FF3 compatible
- The hot game of Nim [4] – Nim theory and connections with other games at cut-the-knot
- Nim [5] and 2-dimensional SuperNim [6] at cut-the-knot
- Pearls Before Swine [7]
- Nim-Game in Flash [8]
- iPhone Nim Game (Opens iTunes) [9]
- Play Nim with yours friends. - Brasilian site - Mathemathical games. [10]
- Ultimate Nim: The Use of Nimbers, Binary Numbers and Subpiles in the Optimal Strategy for Nim [11]
- The Game of Nim [12] at sputsoft.com [13]

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[2] http://www.archimedes-lab.org/game_nim/nim.html
[3] http://worditude.com/nimb/
[4] http://www.cut-the-knot.org/ctk/May2001.html
[5] http://www.cut-the-knot.org/nim_st.shtml
[6] http://www.cut-the-knot.org/Games/SuperNim/SNim.shtml
[7] http://www.transience.com.au/pearl.html
[8] http://www.denkwerkzeuge.ch
[9] http://itunes.apple.com/WebObjects/MZStore.woa/wa/viewSoftware?id=305390338&mt=8
[10] http://www.jogosquebracuca.com.br
[11] http://ia311518.us.archive.org/3/items/UltimateNimTheUseofNimbersBinaryNumbersandSubpilesintheOptimalStrategyforNim/Ultimate_Nim.pdf
[12] http://sputsoft.com/blog/2009/04/the-game-of-nim.html
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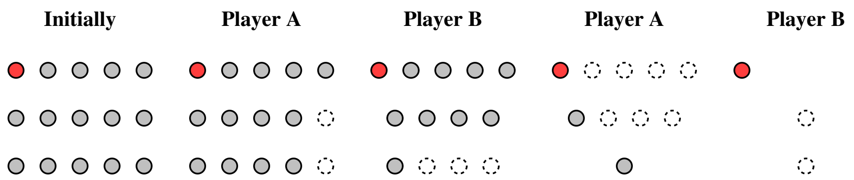
Chomp

Chomp is a 2-player game played on a rectangular "chocolate bar" made up of smaller square blocks (rectangular cells). The players take it in turns to choose one block and "eat it" (remove from the board), together with those that are below it and to its right. The top left block is "poisoned" and the player who eats this loses.

The chocolate-bar formulation of Chomp is due to David Gale, but an equivalent game expressed in terms of choosing divisors of a fixed integer was published earlier by Frederik "Fred" Schuh.

Example game

Below shows the sequence of moves in a typical game starting with a 3 × 5 bar:



Player A must eat the last block and so loses. Note that since it is provable that player A can win, at least one of A's moves is a mistake.

Who wins?

Chomp belongs to the category of impartial 2-player perfect information games.

It turns out that for any rectangular starting position bigger than 1 × 1 the 1st player can win. This can be shown using a strategy-stealing argument: assume that the 2nd player has a winning strategy against any initial 1st player move. Suppose then, that the 1st player takes only the bottom right hand square. By our assumption, the 2nd player has a response to this which will force victory. But if such a winning response exists, the 1st player could have played it as his first move and thus forced victory. The 2nd player therefore cannot have a winning strategy.

Computers can easily calculate winning moves for this game on two-dimensional boards of reasonable size.

Generalisations of Chomp

3-dimensional Chomp has an initial chocolate bar of a cuboid of blocks indexed as (i,j,k). A move is to take a block together with any block all of whose indices are greater or equal to the corresponding index of the chosen block. In the same way Chomp can be generalised to any number of dimensions.

Chomp is sometimes described numerically. An initial natural number is given, and players alternate choosing positive proper divisors of the initial number, but may not choose 1 or a multiple of a previously chosen divisor. This game models n-dimensional Chomp, where the initial natural number has n prime factors and the dimensions of the Chomp board are given by the exponents of the primes in its prime factorization.

Ordinal Chomp is played on an infinite board with some of its dimensions ordinal numbers: for example a $2 \times (\omega + 4)$ bar. A move is to pick any block and remove all blocks with both indices greater than or equal the corresponding indices of the chosen block. The case of $\omega \times \omega \times \omega$ Chomp is a notable open problem; a $100 reward has been offered[1] for finding a winning first move.

More generally, Chomp can be played on any partially ordered set with a least element. A move is to remove any element along with all larger elements. A player loses by taking the least element.

All varieties of Chomp can also be played without resorting to poison by using the misère play convention: The player who eats the final chocolate block is not poisoned, but simply loses by virtue of being the last player. This is identical to the ordinary rule when playing Chomp on its own, but differs when playing the disjunctive sum of Chomp games, where only the last final chocolate block loses.

See also

- Nim
- Hackenbush

References

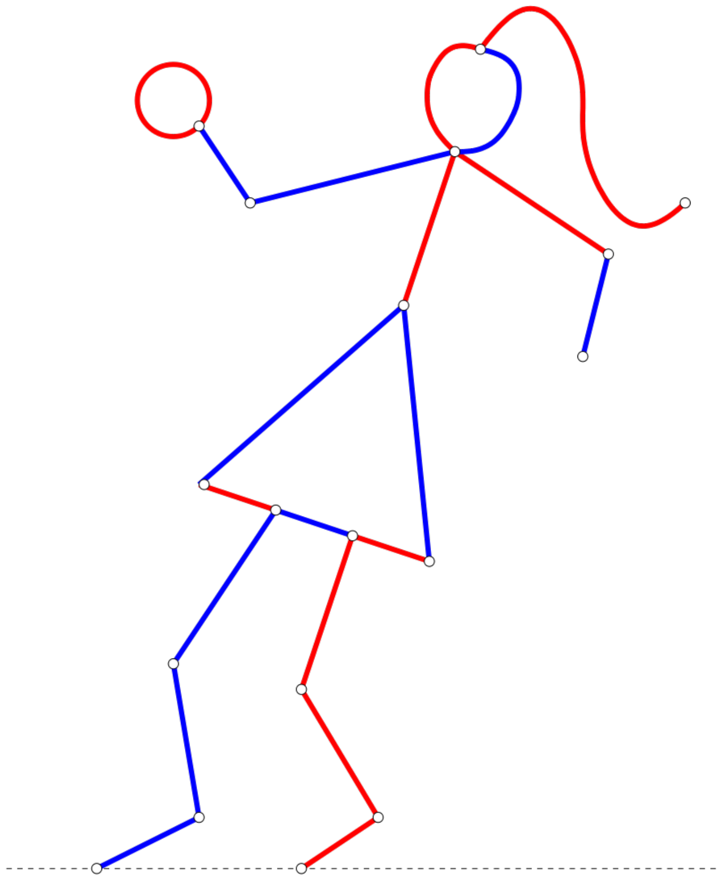
[1] p. 482 in: Games of No Chance (R. J. Nowakowski, ed.), Cambridge University Press, 1998.

External links

- More information about the game (http://www.win.tue.nl/~aeb/games/chomp.html)
- A freeware version for windows (http://www.ossiemanners.co.uk)
- Play Chomp online (http://lpcs.math.msu.su/~pentus/abacus.htm)

Hackenbush

Hackenbush is a two-player mathematical game that may be played on any configuration of colored line segments connected to one another by their endpoints and to the ground. More precisely, there is a ground (conventionally, but not necessarily, a horizontal line at the bottom of the paper or other playing area) and several line segments such that each line segment is connected to the ground, either directly at an endpoint, or indirectly, via a chain of other segments connected by endpoints. Any number of segments may meet at a point and thus there may be multiple paths to ground.



Hackenbush girl, introduced in the book *Winning Ways for your Mathematical Plays*

On his turn, a player "cuts" (erases) a line segment of his choice (from those he is allowed to select — see below). Every line segment no longer connected to the ground by any path "falls" (i.e., gets erased). According to the normal play convention of combinatorial game theory, the first player who is unable to move (because either all segments have been erased, or all those that remain belong to his opponent) loses.

Hackenbush boards can consist of finitely many (in the case of a "finite board") or infinitely many (in the case of an "infinite board") line segments. Note that the existence of an infinite number of line segments does not violate the game theory assumption that the game can be finished in a finite amount of time, provided that there are only finitely many line segments directly "touching" the ground. Even on an infinite board satisfying this condition, it may or may not be *possible* for the game to continue forever, depending on the layout of the board.

In the original folklore version of Hackenbush, any player is allowed to cut any edge: as this is an impartial game it is comparatively straightforward to give a complete analysis using the Sprague-Grundy theorem. Thus the versions of Hackenbush of interest in combinatorial game theory are more complex partisan games, meaning that the options (moves) available to one player would not necessarily be the ones available to the other player if it were his turn to move given the same position. This is achieved in one of two ways:

- ***Blue-Red Hackenbush***: Each line segment is colored either red or blue. One player (usually the first, or left, player) is only allowed to cut blue line segments, while the other player (usually the second, or right, player) is only allowed to cut red line segments.
- ***Blue-Red-Green Hackenbush***: Each line segment is colored either red, blue, or green. The rules are the same as for Blue-Red Hackenbush, with the additional stipulation that green line segments can be cut by either player.

Clearly, Blue-Red Hackenbush is merely a special case of Blue-Red-Green Hackenbush, but it is worth noting separately, as its analysis is often much simpler. This is because Blue-Red Hackenbush is a so-called *cold game*, which means, essentially, that it can never be an advantage to have the first move.

Hackenbush has often been used as an example game for demonstrating the definitions and concepts in combinatorial game theory, beginning with its use in the books *On Numbers and Games* and *Winning Ways for your Mathematical Plays* by some of the founders of the field. In particular Blue-Red Hackenbush can be used to construct surreal numbers: finite Blue-Red Hackenbush boards can construct dyadic rational numbers, while the values of infinite Blue-Red Hackenbush boards account for real numbers, ordinals, and many more general values that are neither. Blue-Red-Green Hackenbush allows for the construction of additional games whose values are not real numbers, such as star and all other nimbers.

Further analysis of the game can be done using graph theory by considering the board as a collection of vertices and edges and examining the paths to each vertex that lies on the ground (which should be considered as a distinguished vertex — it does no harm to identify all the ground points together — rather than as a line on the graph).

References

- Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy, *Winning Ways for your Mathematical Plays*, 2nd edition, A K Peters, 2001.
- John H. Conway, *On Numbers and Games*, 2nd edition, A K Peters, 2000.

External links

- Hackenstrings, and the 0.999... ?= 1 FAQ [1]

References

[1] http://www.maths.nott.ac.uk/personal/anw/Research/Hack/

Grundy's game

Grundy's game is a two-player mathematical game of strategy. The starting configuration is a single heap of objects, and the two players take turn splitting a single heap into two heaps of different sizes. The game ends when only heaps of size two and smaller remain, none of which can be split unequally. The game is usually played as a *normal play* game, which means that the last person who can make an allowed move wins.

Illustration

A normal play game starting with a single heap of 8 is a win for the first player provided he does start by splitting the heap into heaps of 7 and 1:

```
player 1: 8 → 7+1
```

Player 2 now has three choices: splitting the 7-heap into 6 + 1, 5 + 2, or 4 + 3. In each of these cases, player 1 can ensure that on the next move he hands back to his opponent a heap of size 4 plus heaps of size 2 and smaller:

```
player 2: 7+1   → 6+1+1        player 2: 7+1   → 5+2+1        player 2: 7+1   → 4+3+1
player 1: 6+1+1 → 4+2+1+1      player 1: 5+2+1 → 4+1+2+1      player 1: 4+3+1 → 4+2+1+1
```

Now player 2 has to split the 4-heap into 3 + 1, and player 1 subsequently splits the 3-heap into 2 + 1:

```
player 2: 4+2+1+1   → 3+1+2+1+1
player 1: 3+1+2+1+1 → 2+1+1+2+1+1
player 2 has no moves left and loses
```

Mathematical theory

The game can be analysed using the Sprague–Grundy theory. This requires the heap sizes in the game to be mapped onto equivalent nim heap sizes. This mapping is captured in the On-Line Encyclopedia of Integer Sequences as A002188 [1]:

```
Heap size           : 0  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 ...
Equivalent Nim heap : 0  0  0  1  0  2  1  0  2  1  0  2  1  3  2  1  3  2  4  3  0 ...
```

Using this mapping, the strategy for playing the game Nim can also be used for Grundy's game. Whether the sequence of nim-values of Grundy's game ever becomes periodic is an unsolved problem. Elwyn Berlekamp, John Horton Conway, and Richard Guy have conjectured[2] that the sequence does become periodic eventually, but despite the calculation of the first 2^{35} values by Achim Flammenkamp, the question has not been resolved.

See also

- Nim
- Sprague–Grundy theorem
- Wythoff's game
- Subtract a square

External links

- Grundy's game on MathWorld [3]
- Sprague-Grundy values for Grundy's Game by A. Flammenkamp [4]

References

[1] http://en.wikipedia.org/wiki/Oeis%3Aa002188
[2] E. Berlekamp, J. H. Conway, R. Guy. *Winning Ways for your Mathematical Plays.* Academic Press, 1982.
[3] http://mathworld.wolfram.com/GrundysGame.html
[4] http://wwwhomes.uni-bielefeld.de/achim/grundy.html

Subtract a square

Subtract-a-square (also referred to as **take-a-square**) is a two-player mathematical game of strategy starting with a positive integer and both players taking turns subtracting a non-zero square number not larger than the current value. The game is usually played as a *normal play* game, which means that the last person who can make a subtraction wins.

Illustration

A normal play game starting with the number 13 is a win for the first player provided he does start with a subtraction of 1:

```
player 1: 13 - 1*1 = 12
```

Player 2 now has three choices: subtract 1, 4 or 9. In each of these cases, player 1 can ensure that within a few moves the number 2 gets passed on to player 2:

```
player 2: 12 - 1*1 = 11       player 2:       12 - 2*2 = 8              player 2: 12 - 3*3 = 3
player 1: 11 - 3*3 =  2       player 1:        8 - 1*1 = 7              player 1:  3 - 1*1 = 2
                              player 2:  7 - 1*1 = 6 or: 7 - 2*2 = 3
                              player 1:  6 - 2*2 = 2     3 - 1*1 = 2
```

Now player 2 has to subtract 1, and player 1 subsequently does the same:

```
player 2:  2 - 1*1 = 1
player 1:  1 - 1*1 = 0
  player 2 loses
```

Mathematical theory

In the above example, the number '13' represents a winning or 'hot' position, whilst the number '2' represents a losing or 'cold' position. Given an integer list with each integer labeled 'hot' or 'cold', the strategy of the game is simple: try to pass on a 'cold' number to your opponent. This is always possible provided you are being presented a 'hot' number. Which numbers are 'hot' and which numbers are 'cold' can be determined recursively:

```
1) the number 0 is 'cold', whilst 1 is 'hot'


2) if all numbers 1 .. N have been classified as either 'hot' or 'cold', then
  2a) the number N+1 is 'cold' if only 'hot' numbers can be reached by subtracting a positive square
  2b) the number N+1 is 'hot' if at least one 'cold' number can be reached by subtracting a positive square
```

Using this algorithm, a list of cold numbers is easily derived:

```
Normal play 'cold' numbers:
0, 2, 5, 7, 10, 12, 15, 17, 20, 22, 34, 39, 44, ...
```

This sequence is captured in the On-Line Encyclopedia of Integer Sequences as A030193 [1]. Cold numbers tend to end in 0, 2, 4, 5, 7, or 9. Cold values that end with other digits are quite uncommon. This holds in particular for cold numbers ending in 6. Out of all the over 180,000 cold numbers less than 40 million, only one ends in a 6: 11,356.[2]

Extensions

The game subtract-a-square can also be played with multiple numbers. At each turn the player to make a move first selects one of the numbers, and then subtracts a square from it. Such a 'sum of normal games' can be analysed using the Sprague–Grundy theory. This requires the positions in the game subtract-a-square to be mapped onto equivalent nim heap sizes. This mapping is captured for the normal game in the On-Line Encyclopedia of Integer Sequences as A014586 [3]. Notice that all 'cold' positions get mapped onto a zero heap size.

Misère game

Subtract-a-square can also be played as a *misère* game, in which the player to make the last subtraction loses. The recursive algorithm to determine 'hot' and 'cold' numbers for the misère game is the same as that for the normal game, except that for the misère game the number 1 is 'cold' whilst 2 is 'hot'. It follows that the cold numbers for the misère variant are the cold numbers for the normal game shifted by 1:

```
Misère play 'cold' numbers:
1, 3, 6, 8, 11, 13, 16, 18, 21, 23, 35, 40, 45, ...
```

See also

- Nim
- Wythoff's game

References

[1] http://en.wikipedia.org/wiki/Oeis%3Aa030193
[2] David Bush: the uniqueness of 11,356 (http://www.ics.uci.edu/~eppstein/cgt/subsquare.html)
[3] http://en.wikipedia.org/wiki/Oeis%3Aa014586

Wythoff's game

Wythoff's game is a two-player mathematical game of strategy, played with two piles of counters. Players take turns removing counters from one or both piles; in the latter case, the numbers of counters removed from each pile must be equal. The game ends when one person removes the last counter or counters, thus winning.

Martin Gardner claims that the game was played in China under the name 減石子 *jiǎn shízǐ* ("picking stones").[1] The Dutch mathematician W. A. Wythoff published a mathematical analysis of the game in 1907.

Optimal strategy

Any position in the game can be described by a pair of integers (n, m) with $n \le m$, describing the size of both piles in the position. The strategy of the game revolves around *cold positions* and *hot positions*: in a cold position, the player whose turn it is to move will lose with best play, while in a hot position, the player whose turn it is to move will win with best play. The optimal strategy from a hot position is to move to any reachable cold position.

The classification of positions into hot and cold can be carried out recursively with the following three rules:

1. (0,0) is a cold position.
2. Any position from which a cold position can be reached in a single move is a hot position.
3. If every move leads to a hot position, then a position is cold.

For instance, all positions of the form (0, m) and (m, m) with $m > 0$ are hot, by rule 2. However, the position (1,2) is cold, because the only positions that can be reached from it, (0,1), (0,2), and (1,1), are all hot. The cold positions (n, m) with the smallest values of n and m are (0, 0), (1, 2), (3, 5), (4, 7), and (6, 10).

Formula for cold positions

Wythoff discovered that the cold positions follow a regular pattern determined by the golden ratio. Specifically, if k is any natural number and

$$n_k = \lfloor k\phi \rfloor = \lfloor m_k\phi \rfloor - m_k$$
$$m_k = \lfloor k\phi^2 \rfloor = \lceil n_k\phi \rceil = n_k + k$$

where φ is the golden ratio and we are using the floor function, then (n_k, m_k) is the k^{th} cold position. These two sequences of numbers are recorded in the Online Encyclopedia of Integer Sequences as A000201 [2] and A001950 [3], respectively.

The two sequences n_k and m_k are the Beatty sequences associated with the equation

$$\frac{1}{\phi} + \frac{1}{\phi^2} = 1\,.$$

As is true in general for pairs of Beatty sequences, these two sequences are complementary: each positive integer appears exactly once in either sequence.

See also

- Nim
- Grundy's game
- Subtract a square

References

[1] Wythoff's game at Cut-the-knot (http://www.cut-the-knot.org/pythagoras/withoff.shtml), quoting Martin Gardner's book *Penrose Tiles to Trapdoor Ciphers*
[2] http://en.wikipedia.org/wiki/Oeis%3Aa000201
[3] http://en.wikipedia.org/wiki/Oeis%3Aa001950

External links

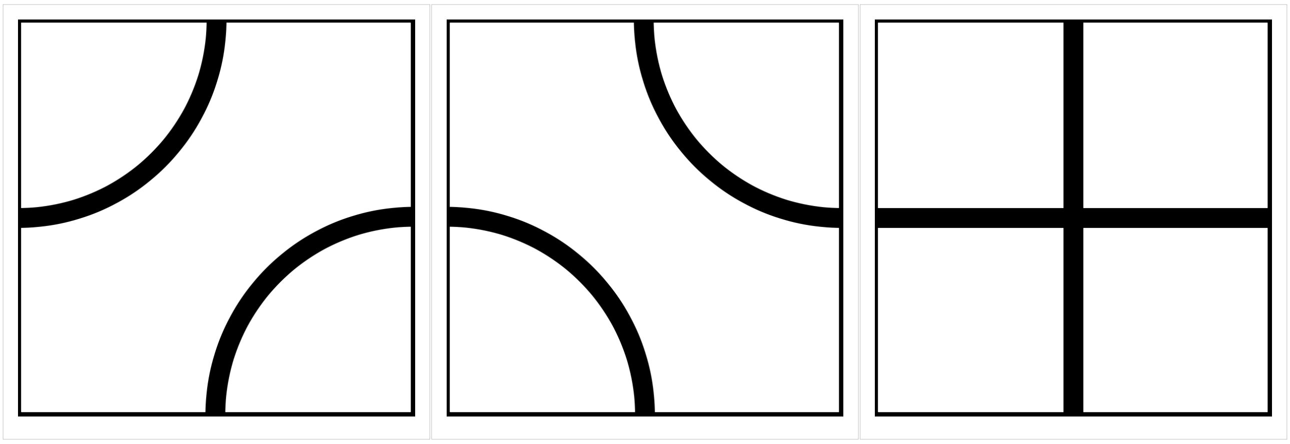
- Weisstein, Eric W., " Wythoff's Game (http://mathworld.wolfram.com/WythoffsGame.html)" from MathWorld.

Black Path Game

The **Black Path Game** (also known by various other names, such as **Brick**) is a two-player board game described and analysed in *Winning Ways for your Mathematical Plays*. It was invented by Larry Black in 1960.

Rules

The Black Path Game is played on a board ruled into squares. Any square that is not empty is filled with one of the following configurations:



These tiles are the three ways to join the sides of the square in pairs. The first two are the tiles of the Truchet tiling. One edge on the boundary of the board is designated to be the start of the path. The players alternate filling the square just after the end of the current path with one of the three configrations above, extending the path. The path may return to a previously filled square and follow the yet-unused segment on that square. The player who first causes the path to run back into the edge of the board loses the game.

Strategy

The first player has a winning strategy on any rectangular board with at least one side-length even. Imagine the board covered with dominoes. The first player should always play so that the end of the path falls on the middle of one of the dominoes. If both sides of the board are odd, the second player can instead win by using a domino tiling including every square but the one containing the first player's first move.

See also

- Trax

Cayley's mousetrap

Mousetrap is the name of a game introduced by the English mathematician Arthur Cayley. In the game, cards numbered one through n are placed in some random permutation. Then, starting with the left-most card, the player begins counting "1, 2, 3, ...", moving to the next card as they increment their count. If at any point the player's current count matches the number on the card currently being pointed to, that card is removed, and the player starts over at one on the next card. When the player reaches the end of the cards (the right-most card), they simply wrap around to the left-most card and continue counting. If the player ever removes all of the cards from the permutation in this manner, then the player wins. If the player reaches n and cards still remain, then the cards win.

External links

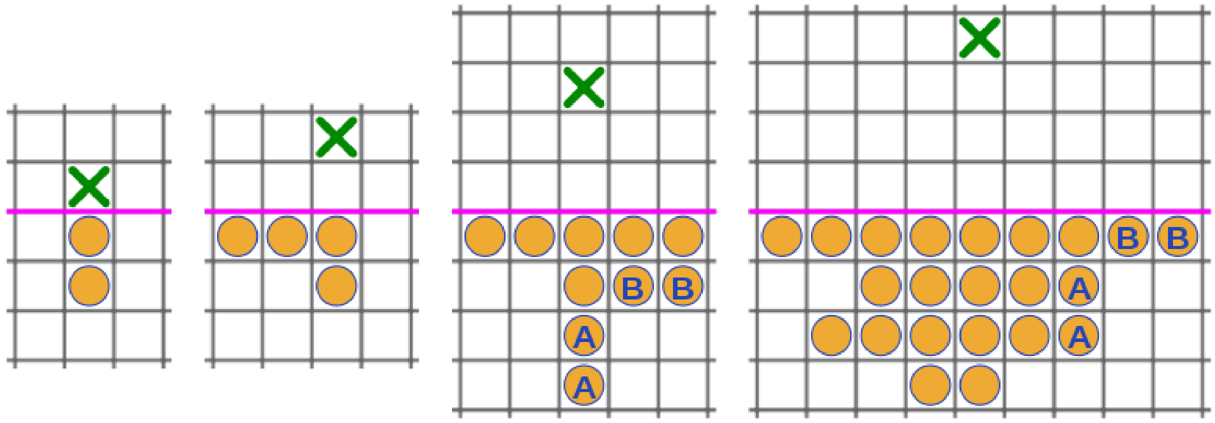
- Cayley's MouseTrap [1]
- Weisstein, Eric W., "Mousetrap [2]" from MathWorld.

References

[1] http://members.lycos.co.uk/rickycayley/CayleysMouseTrap.swf
[2] http://mathworld.wolfram.com/Mousetrap.html

Conway's Soldiers

Conway's Soldiers or the **checker-jumping problem** is a one-person mathematical game or puzzle devised and analyzed by mathematician John Horton Conway in 1961. A variant of peg solitaire, it takes place on an infinite checkerboard. The board is divided by a horizontal line that extends indefinitely. Above the line are empty cells and below the line are an arbitrary number of game pieces, or "soldiers". As in peg solitaire, a move consists of one soldier jumping over an adjacent soldier into an empty cell, vertically or horizontally (but not diagonally), and removing the soldier which was jumped over. The goal of the puzzle is to place a soldier as far above the horizontal line as possible.



Conway proved that, regardless of the strategy used, there is no finite series of moves that will allow a soldier to advance more than four rows above the horizontal line. His argument uses a carefully chosen weighting of cells (involving the golden ratio), and he proved that the total weight can only decrease or remain constant. This argument has been reproduced in a number of popular math books.

Simon Tatham and Gareth Taylor have shown that the fifth row can be reached via an *infinite* series of moves [1]; this result is also in a paper by Pieter Blue and Stephen Hartke [2]. If diagonal jumps are allowed, the 8th row can be reached but not the 9th row. It has also been shown that, in the *n*-dimensional version of the game, the highest row that can be reached is 3*n*-2. Conway's weighting argument demonstrates that the row 3*n*-1 cannot be reached. It is considerably harder to show that row 3*n*-2 can be reached (see the paper by Eriksson and Lindstrom).

Proof that the fifth row is inaccessible

Notation and definitions

Let the target square be labeled $x^0 = 1$, and all other squares be labeled x^n, where n is the number of squares away (horizontally and vertically) from the target square. If we consider the starting configuration of soldiers, below the thick red line, we can assign a score based on the sum of the values under each soldier, (e.g., $x^2 + 2x^3$etc.)

When a soldier jumps over another soldier, there are three cases to consider:

1. A Positive jump: this is when a soldier jumps **towards** the target square over another soldier. Let the value of the soldier's square be x^n, then the total change in score after a positive jump is $x^{n-2} - x^{n-1} - x^n = x^{n-2}(1 - x - x^2)$.
2. A Neutral jump; this is when a soldier jumps over another soldier but remains an equal distance from the target square after his jump (should this be necessary). In this case the change in score is - x^{n-1}.
3. A Negative jump: this is when a soldiers jumps over another into an empty square **away** from the target square. Here the change in score is $x^{n+2} - x^{n+1} - x^n = x^n(x^2 - x - 1)$.

Choosing a value of x

Let us choose a value of x such that the change in score for a positive jump is 0. Thus, we require $1 - x - x^2$= 0 (choosing $x^{n-2} = 0$gives $x = 0$which gets us nowhere). Therefore, $x^2 + x - 1 = 0$, and solving this with the quadratic equation yields $x = \frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}+1}{2}$. We choose the smaller root, $\frac{\sqrt{5}-1}{2} = \varphi \approx 0.61803\,39887\ldots$, as it is less than 1, which becomes useful later in the proof. Rearranging $\varphi^2 + \varphi - 1 = 0$, we can see that:

$\varphi^2 = 1 - \varphi$ [and multiplying by φ ;]

$\varphi^3 = \varphi - \varphi^2$

$\varphi^4 = \varphi^2 - \varphi^3$

etc...

Summing this to infinity causes all terms on the right hand side to cancel apart from the 1, i.e.,

$$\sum_{n=2}^{\infty} \varphi^n = 1$$

This can also be shown with the common ratio, where:

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \ (r < 1)$$

When r = φ :

$$\sum_{k=2}^{\infty} \varphi^k = \frac{1}{1-\varphi} - \varphi - 1 = 1$$

Solutions

Let us take the first example, where the target square is in the first row above the red line. We now consider the maximum possible initial score, that is when every square has a soldier on it. The sum of the squares on the first row below the red line, is $\varphi + 2\varphi^2 + 2\varphi^3 + \ldots = \varphi + 2(\varphi^2 + \varphi^3 + \varphi^4 + \ldots)$. [Drawing a diagram helps to visualise this]. In the next line down, every square is one further away from the target square, and so has value φ times the square above it, and so on for all the rows below the line.

Therefore, the total value of all the squares below the line is equal to:

$$S_1 = (\varphi + 2(\varphi^2 + \varphi^3 + \varphi^4 \ldots))(1 + \varphi + \varphi^2 + \varphi^3 + \ldots).$$

At this stage we observe that $\sum_{n=2}^{\infty} \varphi^n = 1$ from above, and thus the above expression simplifies to:

using $\varphi^2 = 1 - \varphi$ for the last step.

Therefore the sum of all the squares below the line when the target square is immediately above the line is $5 + 3\varphi$.

The next case we consider is when the target square is in the second row above the red line. In this case each square under the red line is one square further away from the target square than in the previous example, so the total score now is found by multiplying the total score we obtained by φ to give:

$$S_2 = \varphi(5 + 3\varphi) = 5\varphi + 3\varphi^2 = 5\varphi + 3(1 - \varphi) = 3 + 2\varphi\,.$$

Similarly:

$$S_3 = 2 + \varphi$$

$$S_4 = 1 + \varphi$$

$$S_5 = 1$$

Thus, we have shown that when the target square is five rows above the red line, the maximum possible original total score of all the soldiers is 1. In reality, with a finite number of soldiers the total will be less than one. Therefore, since a positive jump towards the target square leaves the total score unaltered, and the final score on all the soldiers must be at least 1 ($\varphi^0 = 1$ and the scores on any other soldiers left), the fifth row cannot be reached with a finite number of soldiers originally below the line.
This completes the proof.

$QED.$

References

- E. Berlekamp, J. Conway and R. Guy, Winning Ways for Your Mathematical Plays, 2nd ed., Vol. 4, Chap. 23: 803—841, A K Peters, Wellesley, MA, 2004.
- R. Honsberger, A problem in checker jumping, in Mathematical Gems II, Chap. 3: 23—28, MAA, 1976.
- H. Eriksson and B. Lindstrom, Twin jumping checkers in Z_d, Europ. J. Combinatorics, 16 (1995), 153–157.

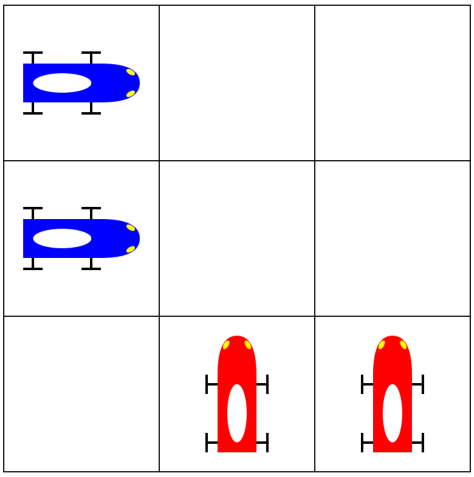
External links

- cut-the-knot.org explains the game [3]
- A page describing several variations of the game, with recent references [4]
- Weisstein, Eric W., "Conway's Soldiers [5]" from MathWorld.
- Plus online magazine describes the game [6]
- A video game based on Conway's Soldiers [7]
- Tatham and Taylor's paper [8]

References

[1] http://www.chiark.greenend.org.uk/~sgtatham/solarmy/
[2] http://www.math.unl.edu/~shartke2/
[3] http://www.cut-the-knot.org/proofs/checker.shtml
[4] http://home.comcast.net/~gibell/pegsolitaire/army/index.html
[5] http://mathworld.wolfram.com/ConwaysSoldiers.html
[6] http://plus.maths.org/issue12/xfile/
[7] http://www.yoyogames.com/games/show/6419
[8] http://tartarus.org/gareth/maths/stuff/solarmy.pdf

Dodgem



Starting position for 3×3 Dodgem

Dodgem is a simple abstract strategy game invented by Colin Vout and described in the book *Winning Ways*. It is played on an *n*×*n* board with *n-1* cars for each player—two cars each on a 3×3 board is enough for an interesting game, but larger sizes are also possible.

Play

The board is initially set up with *n-1* blue cars along the left edge and *n-1* red cars along the bottom edge, the bottom left square remaining empty. Turns alternate: player 1 ("Left")'s turn is to move any one of the blue cars one space forwards (right) or sideways (up or down). Player 2 ("Right")'s turn is to move any one of the red cars one space forwards (up) or sideways (left or right).

Cars may not move onto occupied spaces. They may leave the board, but only by a forward move. A car which leaves the board is out of the game.

The winner is the player who first has no legal move on their turn because all their cars are either off the board or blocked in by their opponent.

Theory

The 3×3 game can be completely analyzed (strongly solved) and is a win for the first player—a table showing who wins from every possible position is given on p. 686 (1st edition pagination) of *Winning Ways*, and given this information it is easy to read off a winning strategy.

David desJardins showed in 1996 (thread from rec.games.abstract [1]) that the 4×4 and 5×5 games never end with perfect play—both players get stuck shuffling their cars from side to side to prevent the other from winning. He conjectures that this is true for all larger boards.

See also

- Gardner, Martin (1987) Time Travel and Other Mathematical Bewilderments, W.H. Freeman & Company; chapter 12. ISBN 0-7167-1925-8

References

[1] http://www.ics.uci.edu/~eppstein/cgt/dodgem.html

Domineering

Domineering (also called **Stop-Gate** or **Crosscram**) is a mathematical game played on a sheet of graph paper, with any set of designs traced out. For example, it can be played on a 6×6 square, a checkerboard, an entirely irregular polygon, or any combination thereof. Two players have a collection of dominoes which they place on the grid in turn, covering up squares. One player, Left, plays tiles vertically, while the other, Right, plays horizontally. As in most games in combinatorial game theory, the first player who cannot move loses.

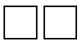
Basic examples

Single box

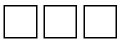
Other than the empty game, where there is no grid, the simplest game is a single box.

In this game, clearly, neither player can move. Since it is a second-player win, it is therefore a zero game.

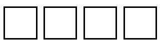
Horizontal rows



This game is a 2-by-1 grid. There is a convention of assigning the game a positive number when Left is winning and a negative one when Right is winning. In this case, Left has no moves, while Right can play a domino to cover the entire board, leaving nothing, which is clearly a zero game. Thus in surreal number notation, this game is $\{\mid 0\} = -1$. This makes sense, as this grid is a 1-move advantage for Right.



This game is also $\{\mid 0\} = -1$, because a single box is unplayable.

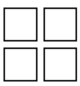


This grid is the first case of a choice. Right *could* play the left two boxes, leaving -1. The rightmost boxes leave -1 as well. He could also play the middle two boxes, leaving two single boxes. This option leaves 0+0 = 0. Thus this game can be expressed as $\{\mid 0,-1\}$. This is -2. If this game is played in conjunction with other games, this is two free moves for Right.

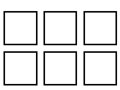
Vertical rows

Vertical columns are evaluated in the same way. If there is a row of $2n$ or $2n+1$ boxes, it counts as $-n$. A column of such size counts as $+n$.

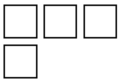
More complex grids



is a more complicated game. If Left goes first, either move leaves a 1×2 grid, which is +1. Right, on the other hand, can move to -1. Thus the surreal number notation is $\{1 \mid -1\}$. However, this is not a surreal number, because $1 > -1$. This is a Game, but not a number. The notation for this is ± 1, and it is a hot game, because each player wants to move here.



is a 2×3 grid, which is even more complex, but just like any Domineering game, it can be broken down by looking at what the various moves for Left and Right are. Left can take the left column (or equivalently, the right column), and move to ±1, but it is clearly a better idea to split the middle, leaving 2 separate games, each worth +1. Thus Left's best move is to +2. Right has four "different" moves, but they all leave the following shape in some rotation:



This game is not a hot game (also called a cold game), because each move hurts the player making it, as we can see by examining the moves. Left can move to −1, Right can move to 0 or +1. Thus this game is {−1|0,1} = {−1|0} = −½.

Our 2×3 grid, then, is {2|−½}, which can also be represented by the mean value, ¾, together with the bonus for moving (the "temperature"), 1¼, thus: $\left\{2\,\middle|\,-\frac{1}{2}\right\} = \frac{3}{4} \pm \frac{5}{4}$

High-level play

The Mathematical Sciences Research Institute held a Domineering tournament, with a $500 prize for the winner. This game was played on an 8×8 board, which proved sufficiently large to be interesting. The winner was mathematician Dan Calistrate, who defeated David Wolfe in the final. The tournament is detailed in Richard J. Nowakowski's *Games of No Chance* (p. 85).

Winning strategy

An interesting problem about Domineering is to compute the winning strategy for large boards, and particularly square boards. In 2000, Dennis Breuker, Jos Uiterwijk and Jaap van den Herik computed and published the solution for the 8x8 board[1] . The 9x9 board followed soon after some improvements of their program. Then, in 2002, Nathan Bullock solved the 10x10 board, as part of his thesis on Domineering[2] .

Interestingly, Domineering is a first-player win for the 6x6, 7x7, 8x8, 9x9 and 10x10 square boards. The other known values for rectangular boards can be found on the site of Nathan Bullock[3] .

Cram

Cram is the impartial version of Domineering. The only difference in the rules is that each player may place their dominoes in either orientation. It seems only a small variation in the rules, but it results in a completely different game, that can be analyzed with the Sprague-Grundy theorem. This game is detailed in Cram (games).

References

[1] D. Breuker, J. Uiterwijk, J. Herik Solving 8x8 domineering (http://portal.acm.org/citation.cfm?id=323507), Theoretical Computer Science, vol. 230, Jan. 2000

[2] Nathan Bullock Domineering:Solving Large Combinatorial Search Spaces (http://webdocs.cs.ualberta.ca/~games/domineering/thesis.ps) M.Sc. thesis, 2002

[3] Nathan Bullock'site : Updated Game Theoretic Values for Domineering Boards (http://webdocs.cs.ualberta.ca/~games/domineering/updated.html)

- Albert, Michael H.; Nowakowski, Richard J.; Wolfe, David (2007). *Lessons in Play: An Introduction to Combinatorial Game Theory*. A K Peters, Ltd.. ISBN 1-56881-277-9.
- Berlekamp, Elwyn R.; Conway, John H.; Guy, Richard K. (2003). *Winning Ways for Your Mathematical Plays*. A K Peters, Ltd.. ISBN 0120911507.
- Gardner, Martin (1974). "Mathematical Games: Cram, crosscram and quadraphage: new games having elusive winning strategies". *Scientific American* **230** (2): 106–108.

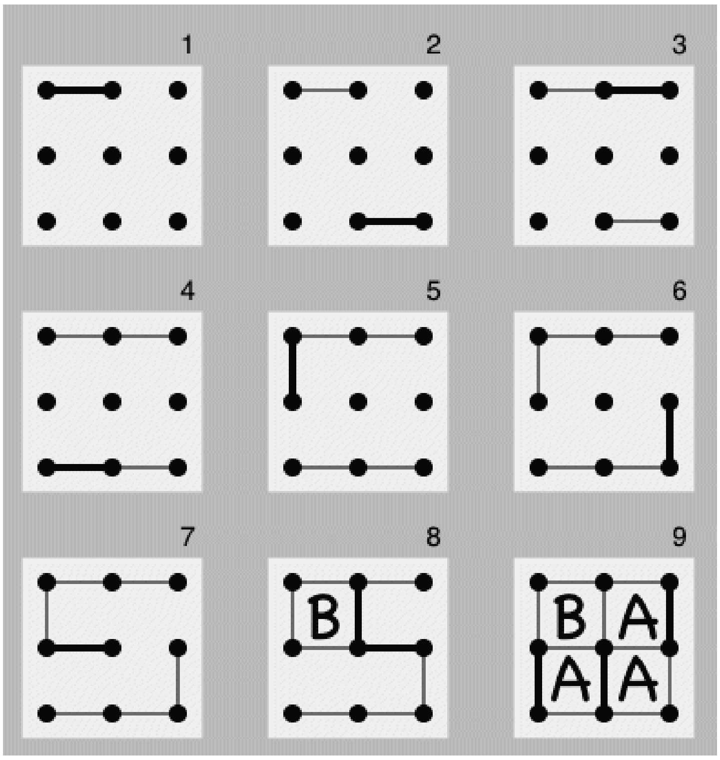
External links

- Stop-gate (http://www.boardgamegeek.com/game/7450) at BoardGameGeek
- ICGA (http://ticc.uvt.nl/icga/journal/contents/content25-2.htm#DOMINERING)

Dots and Boxes

Dots and Boxes (also known as **Boxes**, **Squares**, **Paddocks**, **Square-it**, **Dots and Dashes**, **Dots**, **Smart Dots**, **Dot Boxing**, or, simply, the **Dot Game**) is a pencil and paper game for two players (or sometimes, more than two) first published in 1889 by Édouard Lucas.

Starting with an empty grid of dots, players take turns, adding a single horizontal or vertical line between two unjoined adjacent dots. A player who completes the fourth side of a box earns one point and takes another turn. (The points are typically recorded by placing in the box an identifying mark of the player, such as an initial). The game ends when no more lines can be placed. The winner of the game is the player with the most points.

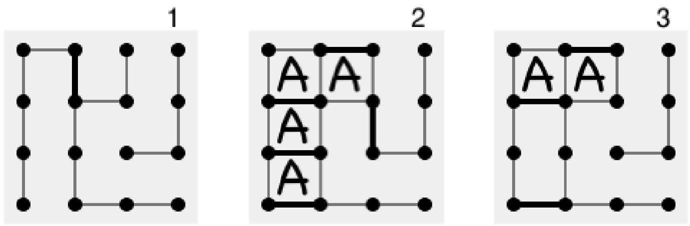


Game of dots and boxes on the 2×2 board.

The board may be of any size. When short on time, 2×2 boxes (created by a square of 9 dots) is good for beginners, and 6×6 is good for experts. In games with an even number of boxes, it is conventional that if the game is tied then the win should be awarded to the second player (this offsets the advantage of going first).

The diagram on the right shows a game being played on the 2×2 board. The second player (B) plays the mirror image of the first player's move, hoping to divide the board into two pieces and tie the game. The first player (A) makes a *sacrifice* at move 7; B accepts the sacrifice, getting one box. However, B must now add another line, and connects the center dot to the center-right dot, causing the remaining boxes to be joined together in a *chain* as shown at the end of move 8. With A's next move, A gets them all, winning 3–1.

Strategy



The double-cross strategy. Faced with position 1, a novice player would create position 2 and lose. An experienced player would create position 3 and win.

At the start of a game, play is more or less random, the only strategy is to avoid adding the third side to any box. This continues until all the remaining (potential) boxes are joined together into *chains* – groups of one or more adjacent boxes in which any move gives all the boxes in the chain to the opponent. A novice player faced with a situation like position 1 in the diagram on the left, in which some boxes can be captured, takes all the boxes in the chain, resulting in position 2. But with their last move, they have to open the next (and larger) chain, and the novice loses the game,

An experienced player faced with position 1 instead plays the *double-cross strategy*, taking all but 2 of the boxes in the chain, leaving position 3. This leaves the last two boxes in the chain for their opponent, but then the *opponent* has to open the next chain. By moving to position 3, player A wins.

The double-cross strategy applies however many long chains there are. Take all but two of the boxes in each chain, but take all the boxes in the last chain. If the chains are long enough then the player will certainly win. Therefore, when played by experts, Dots and Boxes becomes a battle for *control*: An expert player tries to force their opponent to start the first long chain. Against a player who doesn't understand the concept of a sacrifice, the expert simply has to make the correct number of sacrifices to encourage the opponent to hand him the first chain long enough to ensure a win. If the other player also knows to offer sacrifices, the expert also has to manipulate the number of available sacrifices through earlier play.

There is never any reason not to accept a sacrifice, as if it is refused, the player who offered it can always take it without penalty. Thus, the impact of refusing a sacrifice need not be considered in your strategy.

Experienced players can avoid the chaining phenomenon by making early moves to split the board. A board split into 4x4 squares is ideal. Dividing limits the size of chains- in the case of 4x4 squares, the longest possible chain is four, filling the larger square. A board with an even number of spaces will end in a draw (as the number of 4x4 squares will be equal for each player); an odd numbered board will lead to the winner winning by one square (the 4x4 squares and 2x1 half-squares will fall evenly, with one box not incorporated into the pattern falling to the winner).

A common alternate ruleset is to require all available boxes be claimed on your turn. This eliminates the double cross strategy, forcing even the experienced player to take all the boxes, and give his opponent the win.

In combinatorial game theory dots and boxes is very close to being an impartial game and many positions can be analyzed using Sprague–Grundy theory.

Unusual grids

Dots and boxes need not be played on a rectangular grid. It can be played on a triangular grid or a hexagonal grid. Investigations on a triangular variation of the game have even been carried out by Raffles Institution students. There is also a variant in Bolivia when it is played in a Chakana or Inca Cross grid, which adds more complications to the game.

Dots-and-boxes has a dual form called "strings-and-coins". This game is played on a network of coins (vertices) joined by strings (edges). Players take turns to cut a string. When a cut leaves a coin with no strings, the player pockets the coin and takes another turn. The winner is the player who pockets the most coins. Strings-and-coins can be played on an arbitrary graph. A variant played in Poland allows a player to claim a region of several squares as soon as its boundary is completed.

References

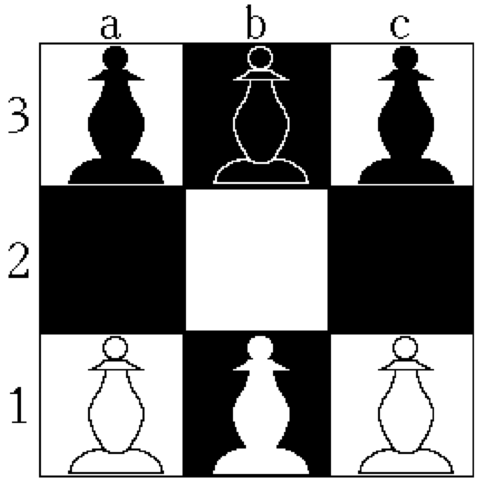
- http://cgi.cae.wisc.edu/~dwilson/play.php?moves=&opt=NB interactive dots and boxes games
- Elwyn Berlekamp (July 2000). *The Dots-and-Boxes Game: Sophisticated Child's Play*. AK Peters, Ltd. ISBN 1-56881-129-2.
- Barile, Margherita, "Dots and Boxes [1]" from MathWorld.
- David Wilson, Dots-and-Boxes Analysis [2]. Contains computer analysis of small boards.
- Ilan Vardi, Dots Strategies [3].

References

[1] http://mathworld.wolfram.com/DotsandBoxes.html
[2] http://homepages.cae.wisc.edu/~dwilson/boxes/
[3] http://web.archive.org/web/20070825035216/http://cf.geocities.com/ilanpi/dots.html

Hexapawn

Hexapawn is a deterministic two-player game invented by Martin Gardner. It is played on a rectangular board of variable size, for example on a 3×3 board or on a chessboard. On a board of size *n*×*m*, each player begins with *m* pawns, one for each square in the row closest to them. The goal of each player is to advance one of their pawns to the opposite end of the board or to prevent the other player from moving.



A 3×3 hexapawn board

Hexapawn on the 3×3 board is a solved game; if both players play well, the first player to move will always lose. Also it seems that any player cannot capture all enemy's pawns. Indeed, Gardner specifically constructed it as a game with a small game tree, in order to demonstrate how it could be played by a heuristic AI implemented by a mechanical computer. A variant of this game is octapawn.

Rules

As in chess, each pawn may be moved in two different ways: it may be moved one square forward, or it may capture a pawn one square diagonally ahead of it. A pawn may not be moved forward if there is a pawn in the next square. Unlike chess, the first move of a pawn may not advance it by two spaces. A player loses if he/she has no legal moves or the other player reaches the end of the board with a pawn.

Dawson's chess

Whenever a player advances a pawn to the penultimate rank (unless it is an isolated pawn) there is a threat to proceed to the final rank by capture. The opponent's only sensible responses are therefore either to capture the advanced pawn or to advance the threatened one, the latter only being sensible in the case that there is one threatened pawn rather than two. If one restricts 3× N hexapawn with the additional rule that the capture is always compulsory, the result is the game **Dawson's chess**.

Dawson's chess reduces to the impartial game denoted **.137** in Conway's notation. This means that it is equivalent to a Nim-like game in which:

- on a turn, the player may remove one to three objects from a heap,
- removing just one object is a legal move only if the removed object is the only object in the heap, and
- when removing three objects from a heap of five or more, the player may also split the remainder into two heaps.

The initial position is a single heap of size N . The nim-sequence for this game is

```
0.1120311033224052233011302110452740
  1120311033224455233011302110453748
  1120311033224455933011302110453748
  1120311033224455933011302110453748
  1120311033224455933011302110453748 ...,
```

where bold entries indicate the values that differ from the eventual periodic behavior of the sequence.

See also

- Pawn duel

References

- Mathematical Games, *Scientific American*, March 1962
- *Winning Ways for your Mathematical Plays*

External links

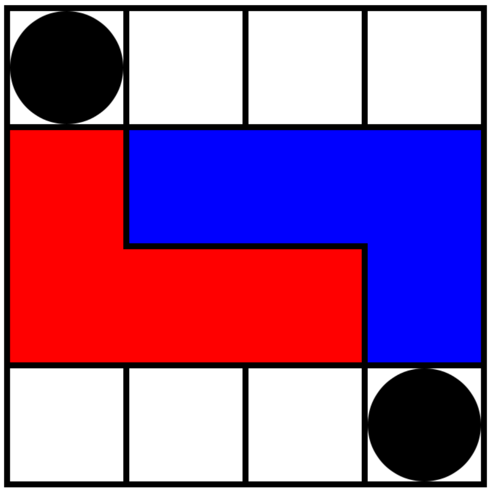
- Hexapawn [1] - an article by Robert Price.
- Hexapawn java applet [2] - source code included.

References

[1] http://www.chessvariants.org/small.dir/hexapawn.html
[2] http://www.javazoid.com/hexapawn.html

L game

The **L game** is a simple strategic game invented by Edward de Bono.



Starting positions

The L game is a two-player turn-based game played on a board of 4×4 squares. Each player has a 3×2 L-shaped piece, and there are two 1×1 neutral pieces. On each turn, a player first must move their L piece, and then may elect to move one of the single neutral pieces. Pieces cannot overlap or cover other pieces. On moving the L piece it is picked up then placed in empty squares anywhere on the board and the piece may be rotated or even flipped over in doing so, the only rule is it must be in a different position than the one it was picked up from, thus come to rest covering at least one grid square that it was not previously covering. On moving a neutral piece, which is optional, a player simply picks it up then places it in any empty square anywhere on the board (even the same square, although this is the same as not moving it). The loser is the first player unable to move their L piece to a new position.

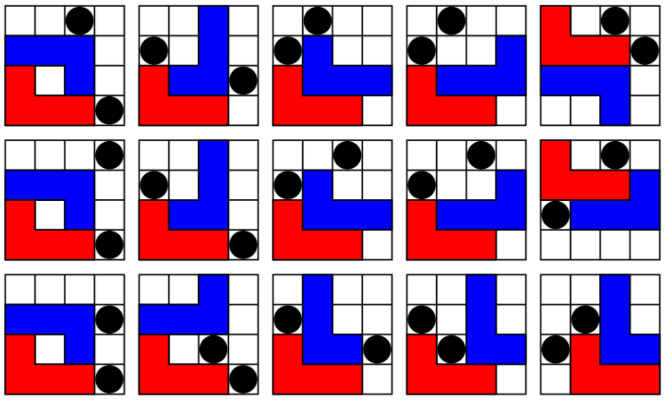
One basic strategy is to use a neutral piece and one's own piece to block a 3×3 square in one corner, and use a neutral piece to prevent the opponent's L piece from swapping to a mirror-image position. Another basic strategy is to move an L piece to block a half of the board, and use the neutral pieces to prevent the opponent's possible alternate positions.

These positions can often be achieved once a neutral piece is left in one of the eight killer spaces on the perimeter of the board. The killer spaces are the spaces on the perimeter, but not in a corner. On the next move, one either makes the previously placed killer a part of one's square, or uses it to block a perimeter position, and makes a square or half-board block with one's own L and a moved neutral piece.

L game made of wood in starting position

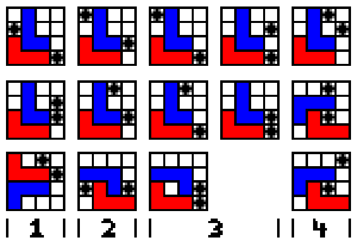
Analysis

In a game with two perfect players, neither will ever win or lose. The L game is small enough to be completely solvable. There are 2296 different possible valid ways the pieces can be arranged, not counting a rotation or mirror of an arrangement as a new arrangement, and considering the two neutral pieces to be identical. Any arrangement can be reached during the game, with it being any player's turn. Each player has lost in 15 of these arrangements, if it is that player's turn. The losing arrangements involve the losing player's L-piece touching a corner. Each player will also soon lose to a perfect player in an additional 14 arrangements. A player will be able to at least force a draw (by playing forever without losing) from the remaining 2267 positions.



All possible final positions, blue has won

Even if neither player plays perfectly, defensive play can continue indefinitely if the players are too cautious to move a neutral piece to the killer positions. If both players are at this level, a sudden-death variant of the rules permits one to move both neutral pieces after moving. A player who can look three moves ahead can defeat defensive play using the standard rules.



All positions, red to move, where red will lose to a perfect blue, and maximum number of moves remaining for red. By looking ahead one move and ensuring one never ends up in any of the above positions, one can avoid losing.

See also

- tetromino

External links

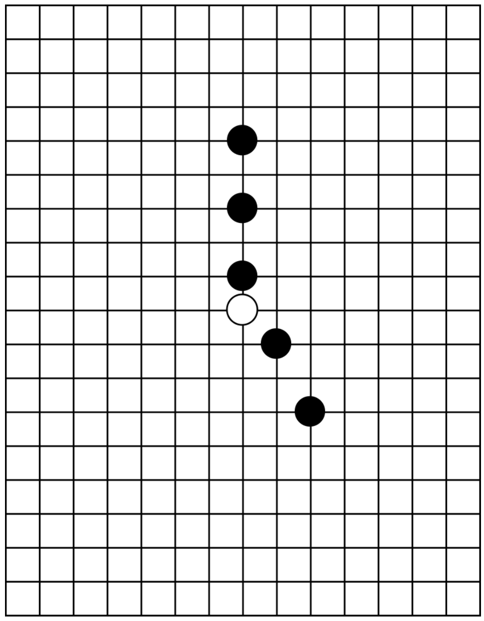
- L game on Edward de Bono's official site [1]

References

[1] http://www.edwdebono.com/debono/lgame.htm

Phutball

Phutball (short for **philosopher's football**) is a two-player board game described in Elwyn Berlekamp, John Horton Conway, and Richard Guy's *Winning Ways for your Mathematical Plays*.



A game of phutball after five men have been placed (the ball has yet to move)

Rules

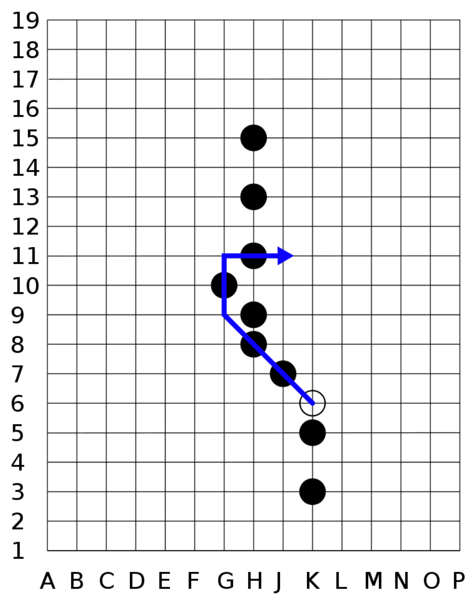
Phutball is played on the intersections of a 19×15 grid using one white stone and as many black stones as needed. In this article the two players are named Ohs (O) and Eks (X). The board is labeled A through P (omitting I) from left to right and 1 to 19 from bottom to top from Ohs' perspective. Rows 0 and 20 represent "off the board" beyond rows 1 and 19 respectively.

Given that specialized phutball boards are hard to come by, the game is usually played on a 19×19 Go board, with a white stone representing the football and black stones representing the men.

The objective is to score goals by using the men (the black stones) to move the football (the white stone) onto or over the opponent's goal line. Ohs tries to move the football to rows 19 or 20 and Eks to rows 1 or 0. At the start of the game the football is placed on the central point, unless one player gives the other a handicap, in which case the ball starts nearer one player's goal.

Players alternate making moves. A move is either to add a man to any vacant point on the board or to move the ball. There is no difference between men played by Ohs and those played by Eks.

The football is moved by a series of jumps over adjacent men. Each jump is to the first vacant point in a straight line horizontally, vertically, or diagonally over one or more men. The jumped men are then removed from the board (before any subsequent jump occurs). This process repeats for as long as there remain men available to be jumped and the player desires. Jumping is optional, there is no requirement to jump. Note that in contrast to checkers, multiple men in a row are jumped and removed as a group.



A jump

The diagram on the right illustrates a jump.

- Ohs moves the football from K6-G9-G11-J11.
- The men on J7, H8, G10, and H11 are removed.
- The jump from K6-G9-J9-G7 would not be legal, as that would jump the man on H8 twice.

If the football ends the move on or over the opponent's goal line then a goal has been scored. If the football passes through your goal line, but ends up elsewhere due to further jumps, the game continues.

Strategy

- Carefully set up sequences of jumps can be "spoiled" by extending them at critical moments.
- A jump to the left or right edge can be blocked by leaving no vacant points.
- When you jump, it is usually bad to leave an easily used return path for your opponent to "undo" your progress.

The game is sufficiently complex that checking whether there is a win in one (on an m×n board) is NP-complete. It is not known whether any player has a winning strategy or both players have a drawing strategy.

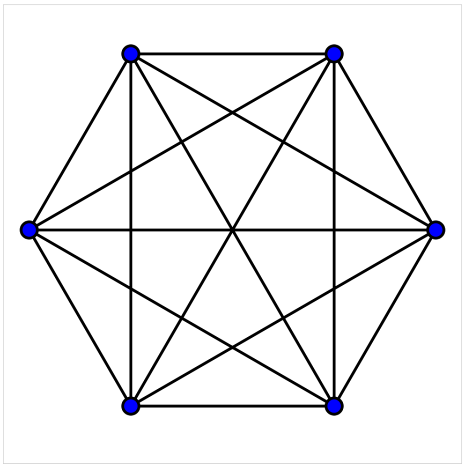
External links

- Dariusz Dereniowski: Phutball is PSPACE-hard [1]
- The game can be played on Sensei's Library [2] and on Vying Games [3].
- Web Page [4] on Richard Rognlie's Play-By-eMail Server Current and archived games can be seen here [5]
- igGameCenter [6] Play Phutball online in a browser or with an iGoogle gadget
- ConwayGo [7] - Phutball Program in Java
- Demaine, Erik D.; Demaine, Martin L. and Eppstein, David (2002). "Phutball endgames are hard" [8]. *More Games of No Chance* [8]. MSRI Publications 42, Cambridge Univ. Press. pp. 351–360.
- Grossman, J.P.; Nowakowski, Richard J. (2002). "One-Dimensional Phutball" [9]. *More Games of No Chance* [9]. MSRI Publications 42, Cambridge Univ. Press. pp. 361–367.

References

[1] http://arxiv.org/abs/0804.1777
[2] http://senseis.xmp.net/?PhilosopherSFootball
[3] http://vying.org/games/phutball
[4] http://www.gamerz.net/~pbmserv/phutball.html
[5] http://www.gamerz.net/pbmserv/List.php?Phutball
[6] http://www.iggamecenter.com
[7] http://conwaygo.sourceforge.net
[8] http://www.msri.org/publications/books/Book42/files/dephut.pdf
[9] http://www.msri.org/publications/books/Book42/files/grossman.pdf

Sim



The game of **Sim** is played by two players on a board consisting of six dots ('vertices'). Each dot is connected to every other dot by a line.

Two players take turns coloring any uncolored lines. One player colors in one color, and the other colors in another color, with each player trying to avoid the creation of a triangle made solely of their color; the player who completes such a triangle loses immediately.

Ramsey theory shows that no game of Sim can end in a tie. Specifically, since the *Ramsey number* $R(3,3)=6$, any two-coloring of the complete graph on 6 vertices (K_6) must contain a monochromatic triangle, and therefore is not a tied position. This will also apply to any super-graph of K_6.

Computer search has verified that the second player can win Sim with perfect play, but finding a perfect strategy that humans can easily memorize is an open problem.

A Java applet is available[1] for online play against a computer program. A technical report[2] by Wolfgang Slany is also available online, with many references to literature on Sim, going back to the game's introduction by Gustavus Simmons in 1969.

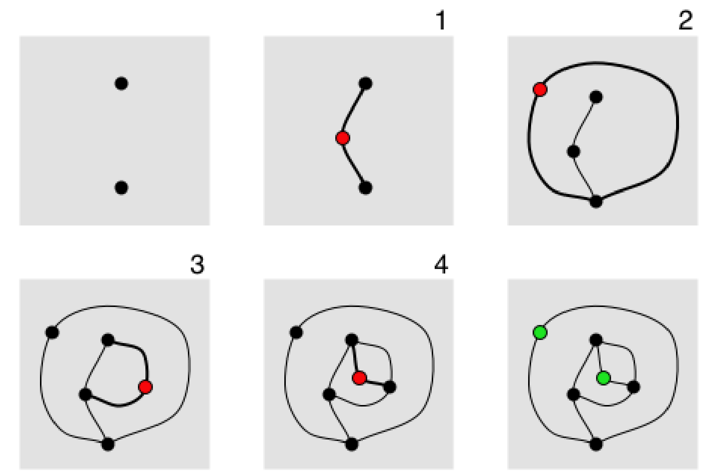
This game of Sim is one example of a Ramsey game. Other Ramsey games are possible. For instance, according to Ramsey theory any three-coloring of the complete graph on 17 vertices must contain a monochromatic triangle. A corresponding Ramsey game uses pencils of three colors. One approach can have three players compete, while another would allow two players to alternately select any of the three colors to paint an edge of the graph, until a player loses by completing a mono-chromatic triangle. It is unknown whether this latter game is a first or a second player win.

External links

[1] Java applet page (http://www.dbai.tuwien.ac.at/proj/ramsey/) on the website of the Vienna University of Technology
[2] *Graph Ramsey Games* by Wolfgang Slany (http://arxiv.org/format/cs.CC/9911004) at arXiv

Sprouts

Sprouts is a pencil-and-paper game with interesting mathematical properties. It was invented by mathematicians John Horton Conway and Michael S. Paterson at Cambridge University in 1967.



A 2-spot game of Sprouts

The game is played by two players, starting with a few spots drawn on a sheet of paper. Players take turns, where each turn consists of drawing a line between two spots (or from a spot to itself) and adding a new spot somewhere along the line. The players are constrained by the following rules.

- The line may be straight or curved, but must not touch or cross itself or any other line.
- The new spot cannot be placed on top of one of the endpoints of the new line. Thus the new spot splits the line into two shorter lines.
- No spot may have more than three lines attached to it. For the purposes of this rule, a line from the spot to itself counts as two attached lines and new spots are counted as having two lines already attached to them.

In so-called *normal play*, the player who makes the last move wins. In *misère play*, the player who makes the last move **loses**. (Misère Sprouts is perhaps the only misère combinatorial game that is played competitively in an organized forum. [1], p. 21)

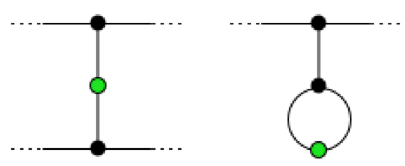
The diagram on the right shows a 2-spot game of normal-play Sprouts. After the fourth move, most of the spots are *dead*–they have three lines attached to them, so they cannot be used as endpoints for a new line. There are two spots (shown in green) that are still *alive*, having fewer than three lines attached. However, it is impossible to make another move, because a line from a live spot to itself would make four attachments, and a line from one live spot to the other would cross lines. Therefore, no fifth move is possible, and the first player loses. Live spots at the end of the game are called *survivors* and play a key role in the analysis of Sprouts.

Analysis

Suppose that a game starts with n spots and lasts for exactly m moves.

Each spot starts with three *lives* (opportunities to connect a line) and each move reduces the total number of lives in the game by one (two lives are lost at the ends of the line, but the new spot has one life). So at the end of the game there are $3n–m$ remaining lives. Each surviving spot has only one life (otherwise there would be another move joining that spot to itself), so there are exactly $3n–m$ survivors. There must be at least one survivor, namely the spot added in the final move. So $3n–m \geq 1$; hence a game can last no more than $3n–1$ moves.

At the end of the game each survivor has exactly two dead *neighbors*, in a technical sense of "neighbor"; see the diagram on the right. No dead spot can be the neighbor of two different survivors, for otherwise there would be a move joining the survivors. All other dead spots (not neighbors of a survivor) are called *pharisees* (from the Hebrew for "separated ones"). Suppose there are p pharisees. Then



Live spots (green) and their dead neighbors (black).

$$n+m = 3n-m + 2(3n-m) + p$$

since initial spots + moves = total spots at end of game = survivors + neighbors + pharisees. Rearranging gives:

$$m = 2n + p/4$$

So a game lasts for at least $2n$ moves, and the number of pharisees is divisible by 4.

Real games seem to turn into a battle over whether the number of moves will be m or m+1 with other possibilities being quite unlikely. [2] One player tries to create enclosed regions containing survivors (thus reducing the total number of moves that will be played) and the other tries to create pharisees (thus increasing the number of moves that will be played).

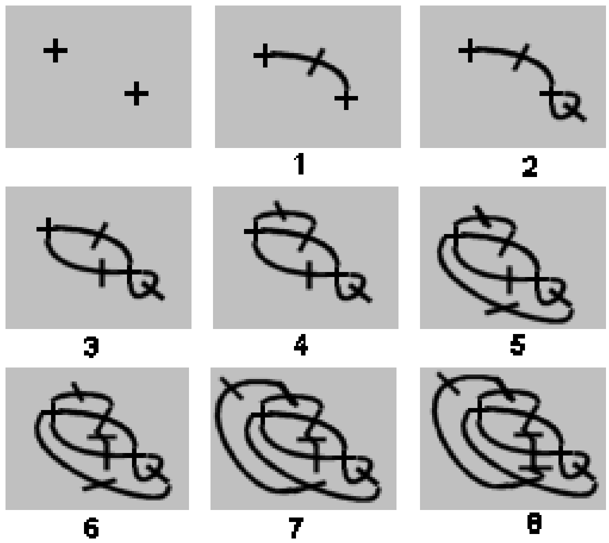
Who has the win?

By enumerating all possible moves, one can show that the first player when playing with the best possible strategy will always win in normal-play games starting with n = 3, 4, or 5 spots. The second player wins when n = 0, 1, 2, or 6.

At Bell Labs in 1990, David Applegate, Guy Jacobson, and Daniel Sleator used a lot of computer power to push the analysis out to eleven spots in normal play and nine spots in misère play. Josh Purinton and Roman Khorkov have extended this analysis to sixteen spots in misère play.[3] Julien Lemoine and Simon Viennot have calculated normal play outcomes up to thirty-two spots, plus five more games between thirty-four and forty-seven spots.[4] They have also announced a result for the seventeen-spot misère game.[5]

The normal-play results are all consistent with the pattern observed by Applegate et al. up to eleven spots and conjectured to continue indefinitely, that the first player has a winning strategy when the number of spots divided by six leaves a remainder of three, four, or five. The results for misère play do not follow as simple a pattern: up to seventeen spots, the first player wins in misère Sprouts when the remainder (mod 6) is zero, four, or five, except that the first player wins the one-spot game and loses the four-spot game.

Brussels Sprouts



A 2-cross game of Brussels Sprouts lasting eight moves

A variant of the game, called **Brussels Sprouts**, starts with a number of crosses, i.e. spots with four free ends. Each move involves joining two free ends with a curve (again not crossing any existing line) and then putting a short stroke across the line to create two new free ends.

So each move removes two free ends and introduces two more. Despite this, the game is finite, and indeed the total number of moves is predetermined by the initial number of crosses: the players cannot affect the result by their play. With *n* initial crosses, the number of moves will be $5n-2$, so a game starting with an odd number of crosses will be a first player win, while a game starting with an even number will be a second player win regardless of the moves.

To prove this (assuming that the game ends), let m denote the number of moves and v,e,f denote the number of vertices, edges, and faces of the planar graph obtained at the end of the game, respectively. We have:

- e = 2m since at each move, the player adds 2 edges.
- v = n + m since at each move, the player adds one vertex (and the game starts with n vertices).
- f = 4n since there remains exactly one free end in each face (and the number of free ends does not change during the game).

The Euler characteristic for planar graphs is 2, so 2 = f-e+v = 4n-2m+n+m = 5n-m , hence m = 5n-2.

References

- Elwyn R. Berlekamp, John Conway and Richard K. Guy, *Winning Ways for your Mathematical Plays*, 1992.
- Madras College Mathematics Department, "Mathematical Games: Sprouts." [6]
- Ivars Peterson, "Sprouts for Spring," *Science News Online.* [7]
- Danny Purvis, *World Game of Sprouts Association.* [8]
- Sprouts played an important role in the first part of Piers Anthony's novel *Macroscope*.
- The Game of Sprouts [9] at University of Utah (with an interactive applet for human-vs-human play).
- Riccardo Focardi and Flamina Luccio, *A new analysis technique for the Sprouts Game*, 2001 [10]
- David Applegate, Guy Jacobson, and Daniel Sleator, *Computer Analysis of Sprouts*, 1991 [11]
- Julien Lemoine and Simon Viennot's SproutsWiki [12]
- freeware Sprouts program (Windows) [13]

References

[1] http://www.arxiv.org/abs/math.CO/0603027
[2] http://mathforum.org/kb/message.jspa?messageID=1091005&tstart=0
[3] http://www.wgosa.org/article2009001.htm
[4] http://sprouts.tuxfamily.org/wiki/doku.php?id=records
[5] http://groups.google.com/group/sprouts-theory/msg/39c60850df4d6e8c
[6] http://www.madras.fife.sch.uk/maths/games/sprouts.html
[7] http://www.sciencenews.org/sn_arc97/4_5_97/mathland.htm
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[11] http://citeseer.csail.mit.edu/applegate91computer.html
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[13] http://www.reisz.de/3graph_en.htm

Sylver coinage

Sylver Coinage is a mathematical game for two players, invented by John H. Conway. It is discussed in chapter 18 of *Winning Ways for Your Mathematical Plays*. This article summarizes that chapter.

The two players take turns naming positive integers that are not the sum of nonnegative multiples of previously named integers. After 1 is named, all positive integers can be expressed in this way: 1 = 1, 2 = 1 + 1, 3 = 1 + 1 + 1, etc., ending the game. The player who named 1 loses. This makes Sylver Coinage a misère game, since by convention in combinatorial game theory the last player to move wins the game.

A sample game between A and B:

- A opens with 5. Now neither player can name 5, 10, 15, ...
- B names 4. Now neither player can name 4, 5, 8, 9, 10, or any number greater than 11.
- A names 11. Now the only remaining numbers are 1, 2, 3, 6, and 7.
- B names 6. Now the only remaining numbers are 1, 2, 3, and 7.
- A names 7. Now the only remaining numbers are 1, 2, and 3.
- B names 2. Now the only remaining numbers are 1 and 3.
- A names 3, leaving only 1.
- B is forced to name 1 and loses.

Each of A's moves was to a winning position.

Sylver Coinage is named after James Joseph Sylvester, who proved that if a and b are relatively prime positive integers, then $(a - 1)(b - 1) - 1$ is the largest number that is not a sum of nonnegative multiples of a and b. This is a special case of the Coin Problem.

Unlike many similar mathematical games, Sylver Coinage has not been completely solved, mainly because many positions have infinitely many possible moves. Furthermore, the main theorem that identifies a class of winning positions, due to R. L. Hutchings, is nonconstructive: it guarantees that such a position has a winning strategy but does not identify it. Hutchings's Theorem states that any of the prime numbers 5, 7, 11, 13, …, wins as a first move, but very little is known about the subsequent winning moves. Complete winning strategies are known for answering the losing openings 1, 2, 3, 4, 6, 8, 9, and 12.

References

- James J. Sylvester, *Mathematical Questions from the Educational Times* **41** (1884), 21 (question 7382).

External links

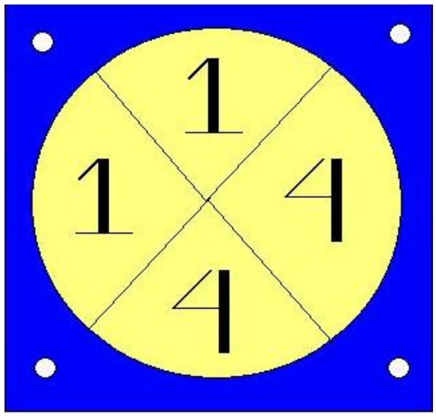
- Some recent findings about the game appear at http://www.monmouth.com/~colonel/sylver/.

24 Game

The **24 Game** is a mathematical card game in which the object is to find a way to manipulate four integers so that the end result is 24. Addition, subtraction, multiplication, or division, and sometimes other operations, may be used to make four digits from one to nine equal 24. For an example card with the numbers 4,7,8,8, a possible solution is the following: (7-8/8)*4=24.

The game has been played in Shanghai since the 1960s[1] , using ordinary playing cards. Robert Sun commercialised the game in 1988, introducing dedicated game cards bearing four numbers each, and sold it through his company, Suntex International Inc. There are nine official variations of the 24 Game. The tournament-style competition 24 Challenge is based on the game.

Original version



A Sample 24 Game Card

The original version of 24 is played with an ordinary deck of playing cards with all the face cards removed. The aces are taken to have the value 1 and the basic game proceeds by having 4 cards thrown and the first player that can achieve the number 24 exactly using only addition, subtraction, multiplication, division, and parentheses wins the hand. Some groups of players allow exponentiation, or even further operations such as roots or logarithms.

For short games of 24, once a hand is won, the cards go to the player that won. If everyone gives up, the cards are shuffled back into the deck. The game ends when the deck is exhausted and the player with the most cards wins.

Longer games of 24 proceed by first dealing the cards out to the players, each of whom contributes to each set of cards exposed. A player who solves a set takes its cards and replenishes their pile, after the fashion of War; players are eliminated when they no longer have any cards.

A slightly different version includes the face cards, Jack, Queen, and King, giving them the values 11, 12, and 13, respectively.

Overview of the commercial game

Cards are divided into three levels of difficulty. One-dot cards (with a single white dot in each corner) are often solved by simple addition, or contain three digits that can make 24, plus a 1 (in which case any other digit could be multiplied or divided by 1 to create the same digit). Two-dot cards (with two red dots) are slightly more difficult, and often require more multiplication and division than one-dot cards. Three-dot cards (with three yellow dots) are the most difficult cards, often having only one solution. In most decks of Math 24 cards, the ratio of one-dot cards to two-dot cards to three-dot cards is 1:2:1.

Variations

There have been many variations on the 24 Challenge game, as one deck of cards can be fairly easily memorized, thus creating a simple memory game instead of a skills tester. Some variations include:

- Two-digit cards — cards may have 2-digit numbers on them, leading to more difficulty as not as many students are as familiar with multiples of larger numbers.
- Variables — Cards have two wheels, each has three numbers with one number "missing." The object is to find a number (any integer 1 - 9) which, when used with the other numbers on each wheel, can make 24 on both wheels.[2]
- Fraction, Decimal, and Exponent versions are also available. They are used often in higher level tournaments.

Strategy

Mastering this game requires fast thinking, but with enough practice, the ability to spot useful patterns increases. Some experienced players can make 24 with almost any four numbers. There are several sequences that lead to 24 that players frequently try to obtain. For example, the solutions 6 X 4, 8 X 3, 2 X 12, 18 + 6, 16 + 8, and 10 + 14 often appear.

There are many common patterns that help a player to acquire a solution faster. It is beneficial to look for common multiplication patterns, such as 12 x 2, 8 x 3, and 6 x 4 in a card to equal 24. Similarly, multiplying a digit by another digit that is one more or less than the usual multiplicitive pair (for example, 4 x 5 or 4 x 7 instead of 4 x 6) then adding or subtracting the first number (20 + 4 or 28 - 4) is a common strategy for reaching 24. This strategy comes from the law of distributivity from elementary algebra. These are some more examples:

6 x 4 = ...

```
    = 6 x (3 + 1) = 6 x 3 + 6
    = 6 x (5 − 1) = 6 x 5 − 6
    = (5 + 1) x 4 = 5 x 4 + 4
    = (7 − 1) x 4 = 7 x 4 − 4
```

8 x 3 = ...

```
    = 8 x (2 + 1) = 8 x 2 + 8
    = 8 x (4 − 1) = 8 x 4 − 8
    = (7 + 1) x 3 = 7 x 3 + 3
    = (9 − 1) x 3 = 9 x 3 − 3
```

Possible solutions can be found on the back of 24 game deck boxes.

The number 1 is a powerful digit on a card. As stated above, a 1 may be used at any time to multiply or divide any number to equal itself. This is important in cards where only three digits are needed to create the value of 24. A player may simply multiply (or divide) 24 by 1 to create the final result of 24. If a 1 is not readily provided on the card, two numbers may be subtracted (such as 8-7) or divided (6/6) to make a 1.

Pencil and paper are generally not allowed during play; using such would only slow down a player anyway. Mental math is a necessary skill for playing this game.

Tournament

In Spring, there is an annual 24 tournament. The tournament happens mainly in Pennsylvania, but also is in New Jersey, California, and other states. In Pennsylvania, participating schools have a school competition. They then send one or two students to compete at a regional level, and the top four will go on to the state level. In other states, there is no state level.

Recently, some of the tournaments have not been taking place because of low funds.

In tournament play, several extra rules apply:

- A player, upon tapping the card, must state the *final* operation they used to reach 24 *first*. They then proceed to list the operations in normal order, subsequently repeating the final operation. If a player forgets to state the final operation first, they receive a penalty flag. If, in mentioning all steps they used to reach the card, they do not use the same final operation, they receive a penalty flag.
- Tapping a card with one finger, four or five fingers, or slapping with the whole hand results in a penalty flag, or a relinquishment of a card.
- Players must keep their hands (usually, only the tapping two or three fingers) on the blue region in the mat, about eight inches away from the card. Touching inside this area results in a penalty flag.
- Fake cards with no solution are slipped into tournament decks in the first round to discourage impulsive tappers. These cards are usually marked in such a way that the moderator can distinguish them from normal cards. Tapping one of these fake cards results in an automatic penalty flag.

When a player receives three penalty flags, they are disqualified from further play in that round, but they keep any points they have earned up to that point.

In order to win, the player must earn more points than everyone else in the competition. Usually there are two rounds. The players move from table to table playing other students in their same level. For each card that they receive, they get more points. The points are distributed based on the number of dots found in the corner of each card. The player at the end with the most points wins.

Platinum Series

Several variations of Math 24 cards exist, and are used in the Platinum stages of tournaments.This level of play is highest in the 24 game, and only 7/8 graders can participate .The cards used are Algebra/exponents, fractions/decimals, and integers.They can be purchased at 24 game.com (link located here [3]). These numbers are treated the same way as regular digits, and must be used in a solution once.

A special "integers" deck uses negative digits alongside positive numerals. (Cards in this version may be solved for positive or negative 24.)However, in the tournament, positive 24 must be found.

Algebra version cards contain values with a variable, such as **3y** or **2x-4**. In solving a problem, the player must state what each variable represents, then give the solution using that variable in it. Cards may contain more than one variable on a side; three-dot cards commonly use **x**, **y**, and **z** all on one card.

The "Exponent" version of Math 24 integrates roots and powers into game play.These cards have a special center marking, indicating that one digit (or result from a previous equation) must be squared, cubed, or have the square root or cube root taken. This results in a card requiring four operations, instead of the usual three. For example, a card with the digits **2**, **3**, **4**, and **8** might be solved by stating that **2**x**8**=16, the square root of 16 is 4, 4+**4**=8, and 8x**3**=24.

The last form of platinum play are the "fraction" cards. They involve fractions along with usually whole numbers, except in the case of some level three cards.

Online Game

In addition to the game produced by Suntex on small cards, there is also a 24 game online at firstinmath.com. In this game, the player finishes skill sets which earn "stickers" and include the different variations of the game, as well as bonus games that are unlocked when you complete certain versions of the game, a "Just the Facts" module where players try to complete the addition, subtraction, multiplication, and division tables in under 5 minutes, a "gym" where you learn the numbers separately (+1, +2, et cetera) as well as the tables for fractions, decimals, and negative integers, and "Measurement World," which focuses solely on measurement and includes games such as "More or Less Time" and "Equal Weight." The players can then compare their score to other players in their school and even the schools in the surrounding areas.

With the increasing popularity of smart phones, a number of 24 applications have been ported to mobile devices such as S60 phones with J2ME support, iphone and android. These games use different user interfaces due to limited screen size, yet they're able to achieve the core objective of Game 24.

See also

Planarity, an online math game in which one untangles a planar graph.

References

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[2] 24game.com - 24 Game 96 Card Decks (http://www.24game.com/s-2-24-game-96-card-decks.aspx)
[3] http://24game.com

External links

- www.24game.com The Official 24 Game website (http://www.24game.com/)
- 24 at Pagat (http://www.pagat.com/adders/24.html)
- A fast 24 game solver: both executable program and source code (http://code.google.com/p/solve24game/)
- Online 24 Game (http://www.firstinmath.com)
- iMath24s - A 24 game for S60 smartphones (http://www.handango.com/catalog/ProductDetails.jsp?storeId=2218&deviceId=1989&platformId=20&productId=269284&sectionId=7619)
- Slide 24 - The 24 game for iPod Touch and iPhone (http://itunes.apple.com/us/app/slide-24/id350488771?mt=8)
- 24.roundskylab.com the iphone version of 24 (http://24.roundskylab.com)
- Cards Math by virtuesoft.com the iphone version of 24 (http://virtuesoft.com)
- TwentyFour! - A multiplayer version of 24 for the iPhone (http://www.twentyfourgame.com)
- 24 game scorers in over 24 different programming languages on the Rosetta Code site (http://rosettacode.org/wiki/24_game)
- 24 game solvers in over 17 different programming languages on the Rosetta Code site (http://rosettacode.org/wiki/24_game/Solve)

Krypto

Krypto is a card game designed by Daniel Yovich in 1963 and published by Parker Brothers and MPH Games Co.. It is a mathematical game that promotes proficiency with basic arithmetic operations. More detailed analysis of the game can raise more complex statistical questions.

Rules of Krypto

The Krypto Deck and Home Rules

The Krypto deck consists of 56 cards: three each of numbers 1-6, 4 each of the numbers 7-10, two each of 11-17, one each of 18-25. Six cards are dealt: a common objective card at the top and five other cards below. Each player must use all five of the cards' numbers exactly once, using any combination of arithmetic operations (addition, subtraction, multiplication, and division), to form the objective card's number. The first player to come up with a correct formula is the winner.

Krypto International Tournament Rules

The official international rules for Krypto differ slightly from the house rules, and they involve a system of scorekeeping.

Five cards are dealt face up in the center of the game table. (Each player works with the same set of five cards, rather than a set exclusive to them.) Then a sixth card is dealt face up in the center of the table that becomes the Objective Card. Each player commences (mentally) to mathematically manipulate the numbers of each card so that the last solution equals the Objective Card number. Krypto International Rules specify the use of whole numbers only, using addition, subtraction, division, multiplication and/or any combination thereof ... fractions, negative numbers or square rooting are not permitted. Each of the five cards must be used once and only once. The first player to solve the problem declares "Krypto" and has 30 seconds to explain the answer. When a player "Krypto's" and cannot relate the proper solution, a new hand is dealt and the hand is replayed. The player that errored receives a minus one point in the score box for that hand and is not eligible to play for a score for the replay of that hand.

Each hand must be solved within three minutes or a new hand is dealt.

Example of Play:

```
Cards:  2, 1, 2, 2, 3 = 24 (Objective Card)
2 x   1 =  2
2 x   2 =  4
4 x   2 =  8
8 x   3 = 24 (Krypto)
```

All five cards were used once and only once to equal the Objective Card.

Another Example:

```
Cards:  1, 3, 7, 1, 8= 1 (Objective Card)
3 -  1 =  2
7 +  2 =  9
9 /  1 =  9
9 -  8 =  1 (Krypto)
```

Here is a more difficult hand:

```
       Cards:  24, 22, 23, 20, 21 = 1  (Objective Card)
       24 + 22 = 46
       46 / 23 =  2
        2 + 20 = 22
       22 - 21 =  1 (Krypto)
```

Score Keeping Rules:

Ten hands of Krypto equal one game. Players receive one point for each "Krypto". Players receive double their previous hand score each time they "Krypto" repetitively in sequence. A score returns to "1" when sequence is broken. When players "Krypto" in error, they receive a minus one (-1) in the score box for that hand. They are also eliminated from play of that hand only and the hand is re-dealt for the remaining players. All players are then eligible to score the next hand unless another error in "Kryptoing" occurs.

Example of Score Keeping

```
|----------------------------------------------------------------------|
|  Example: Krypto Score Pad               (MPH logo)                  |
|----------------------------------------------------------------------|
| Players |Hand|Hand|Hand|Hand|Hand|Hand|Hand|Hand|Hand|Hand| Highest  |
|         |  1 |  2 |  3 |  4 |  5 |  6 |  7 |  8 |  9 | 10 |Score Wins|
|----------------------------------------------------------------------|
| Debbie  |  1 |    |    |    |    |    |  1 |    |    |    |     2    |
|----------------------------------------------------------------------|
|  Mike   |    |  1 |    |    |    |    |    |    |    |    |     1    |
|----------------------------------------------------------------------|
| Shelley |    |    |  1 |  2 |  4 |    |    |    |    |  1 | 8 Winner |
|----------------------------------------------------------------------|
|  Kim    |    |    |    |    |    |  1 |    |    |    |    |     1    |
|----------------------------------------------------------------------|
|  Mary   |    |    |    |    |    |    |    |  1 |  2 |    |     3    |
|----------------------------------------------------------------------|
```

```
            * Score ties are broken by playing additional hands
```

Variations on the Game

Although the numerical distribution of the official Krypto deck tends to provide for more balanced games, it is possible to play Krypto with any six numbers. Many programs exist on the internet that can generate six numbers and allow one to manipulate them with arithmetic operations. Because of the simple nature of the game, it is easy to program krypto on most scientific calculators. Versions of Krypto that only use a smaller range of numbers (such as 1-10) are better suited for beginners, while conversely, one could play a game of Krypto with a larger range of numbers that would be more difficult.

See also

- 24 Game

References

- http://www.boardgamegeek.com/game/7694#info
- http://www.math.niu.edu/~rusin/uses-math/games/krypto/

Pirate game

The **pirate game** is a simple mathematical game. It illustrates how, if assumptions conforming to a homo economicus model of human behaviour hold, outcomes may be surprising. It is a multi-player version of the ultimatum game.

From Howard Pyle's Book of Pirates

The game

There are five rational pirates, A, B, C, D and E. They find 100 gold coins. They must decide how to distribute them.

The pirates have a strict order of seniority: A is superior to B, who is superior to C, who is superior to D, who is superior to E.

The pirate world's rules of distribution are thus: that the most senior pirate should propose a distribution of coins. The pirates, including the proposer, then vote on whether to accept this distribution. If the proposed allocation is approved by a majority or a tie vote, it happens. If not, the proposer is thrown overboard from the pirate ship and dies, and the next most senior pirate makes a new proposal to begin the system again.

Pirates base their decisions on three factors. First of all, each pirate wants to survive. Secondly, each pirate wants to maximize the number of gold coins he receives. Thirdly, each pirate would prefer to throw another overboard, if all other results would otherwise be equal.[1]

The result

It might be expected intuitively that Pirate A will have to allocate little if any to himself for fear of being voted off so that there are fewer pirates to share between. However, this is as far from the theoretical result as is possible.

This is apparent if we work backwards: if all except D and E have been thrown overboard, D proposes 100 for himself and 0 for E. He has the casting vote, and so this is the allocation.

If there are three left (C, D and E) C knows that D will offer E 0 in the next round; therefore, C has to offer E 1 coin in this round to make E vote with him, and get his allocation through. Therefore, when only three are left the allocation is C:99, D:0, E:1.

If B, C, D and E remain, B knows this when he makes his decision. To avoid being thrown overboard, he can simply offer 1 to D. Because he has the casting vote, the support only by D is sufficient. Thus he proposes B:99, C:0, D:1, E:0. One might consider proposing B:99, C:0, D:0, E:1, as E knows he won't get more, if any, if he throws B overboard. But, as each pirate is eager to throw each other overboard, E would prefer to kill B, to get the same amount of gold from C.

Assuming A knows all these things, he can count on C and E's support for the following allocation, which is the final solution:

- A: 98 coins
- B: 0 coins
- C: 1 coin
- D: 0 coins
- E: 1 coin[1]

Also, A:98, B:0, C:0, D:1, E:1 or other variants are not good enough, as D would rather throw A overboard to get the same amount of gold from B.

Extension

The game can easily be extended to up to 200 pirates (or further even if you don't increase the amount of gold). Ian Stewart extended it to an arbitrary number of pirates in the May 1999 edition of Scientific American, with further interesting results.[1]

References

[1] Stewart, Ian (1999-05), "A Puzzle for Pirates" (http://euclid.trentu.ca/math/bz/pirates_gold.pdf), *Scientific American*: 98–99,

See also

- Creative problem solving
- Lateral thinking

Chess and mathematics

Wheat and chessboard problem

The **wheat and chessboard problem** is a mathematical problem. If a chessboard were to have wheat placed upon each square such that one grain were placed on the first square, two on the second, four on the third and so on, doubling the number of grains on each subsequent square, how many grains of wheat would be on the chessboard at the finish?

	a	b	c	d	e	f	g	h	
8									8
7									7
6									6
5									5
4									4
3									3
2									2
1									1
	a	b	c	d	e	f	g	h	

Chessboard

To solve this, observe that a chess board is an 8×8 square, containing 64 squares. If the amount doubles on successive squares, then the sum of grains on all 64 squares is:

$$T_{64} = 1 + 2 + 4 + \cdots + 2^{63} = \sum_{i=0}^{63} 2^i = 2^{64} - 1$$

This equals 18,446,744,073,709,551,615.

This problem (or a variation of it) demonstrates the quick growth of exponential sequences.

The problem is sometimes expressed in terms of rice instead of wheat.

Origin of the problem

While the story behind the problem changes from person to person, the fable usually follows the same idea:

When the creator of the game of chess (in some tellings an ancient Indian mathematician, in others a legendary dravida vellalar named Sessa or Sissa) showed his invention to the ruler of the country, the ruler was so pleased that he gave the inventor the right to name his prize for the invention. The man, who was very wise, asked the king this: that for the first square of the chess board, he would receive one grain of wheat (in some tellings, rice), two for the second one, four on the third one, and so forth, doubling the amount each time. The ruler, arithmetically unaware, quickly accepted the inventor's offer, even getting offended by his perceived notion that the inventor was asking for such a low price, and ordered the treasurer to count and hand over the wheat to the inventor. However, when the treasurer took more than a week to calculate the amount of wheat, the ruler asked him for a reason for his tardiness.

The treasurer then gave him the result of the calculation, and explained that it would be impossible to give the inventor the reward. The ruler then, to get back at the inventor who tried to outsmart him, told the inventor that in order for him to receive his reward, he was to count every single grain that was given to him, in order to make sure that the ruler was not stealing from him.

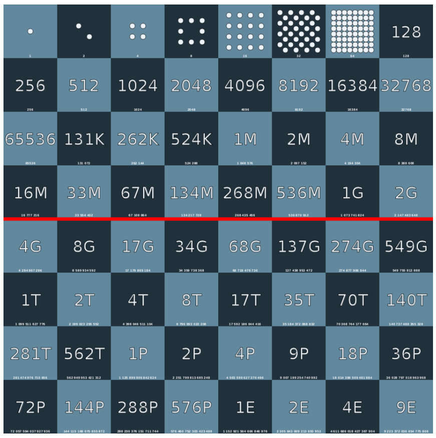
Variations

The problem also has another setting, the Roman empire. When a brave general came back to Rome, the Cæsar asked him to name a price for the services he had offered to his country. When the general asked for an exorbitant price, the Caesar, not wanting to sound cheap, or that he was going to go back on his word, made him an offer; the next day, the general was to go to the treasury, and grab a one gram gold coin, the next day, a two gram gold coin, and each day, the weight of the coin would double, and the general could take it, as long as he was able to carry it by himself. The general, seeing a good opportunity to make money quickly, agreed. However, by the end of the 18th day, the general was not able to carry any more coins. The general only received a small fraction of what he had asked the Caesar. Yakov Perelman retells the story in one of his books with brass coins instead of gold ones, starting with five gram. The general manages to take 17 coins, and the last two must be rolled in instead of being carried.

Another version places two merchants together. One merchant offers the other a deal; that for the next month, the merchant was going to give $10,000 (or, in some variants, even $100,000) to the other one, and in return, he would receive 1 cent the first day, 2 cents the second, 4 cents in the third, and so on, each time doubling the amount. The second merchant agreed, and for the first three weeks (or more, depending on the variant), he enjoyed the fortunes that the first merchant was "unwittingly" giving him, but by the end of the month, the second merchant was broke, while the first merchant was incredibly rich.

Yet another variant is about a man buying a horse and displeased with the high price. The owner offers him to buy it by paying instead one cent for the first nail in its horseshoes, two for the second nail, and so on. With each horseshoe having six nails, the results are similar to the above story.

Second half of the chessboard



An illustration of the principle.

In technology strategy, the **second half of the chessboard** is a phrase, coined by Ray Kurzweil[kurzweilbook], in reference to the point where an exponentially growing factor begins to have a significant economic impact on an organization's overall business strategy.

While the number of grains on the first half of the chessboard is large, the amount on the **second half** is vastly larger.

The number of grains of rice on the **first half** of the chessboard is 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024 + ... + 2,147,483,648, for a total of $2^{32} - 1$ = 4,294,967,295 grains of rice, or about 100,000 kg of rice, with the mass of one grain of rice taken as 25 mg [1]. This amount is about 1/1,200,000th of total rice production in India per annum (in 2005) [2].

The number of grains of rice on the **second half** of the chessboard is $2^{32} + 2^{33} + 2^{34} ... + 2^{63}$, for a total of $2^{64} - 2^{32}$ grains of rice, or the square of the number of grains on the first half of the board plus itself. (Indeed, the first square of the second half alone contains more grains than the entire first half.) On the 64th square of the chessboard alone there would be 2^{63} = 9,223,372,036,854,775,808 grains of rice, or more than two billion times as much on the first half of the chessboard.

On the entire chessboard there would be $2^{64} - 1$ = 18,446,744,073,709,551,615 grains of rice, weighing 461,168,602,000 metric tons, which would be a heap of rice larger than Mount Everest.

See also

- Moore's law
- Technology strategy
- Orders of magnitude (data)

References

1. Raymond Kurzweil (1999). *The Age of Spiritual Machines*. Viking Adult. ISBN 0-670-88217-8.

External links

- Weisstein, Eric W., "Wheat and Chessboard Problem [3]" from MathWorld.
- One telling of the fable [4]
- Salt and chessboard problem [5] - A variation on the wheat and chessboard problem with measurements of each square.

References

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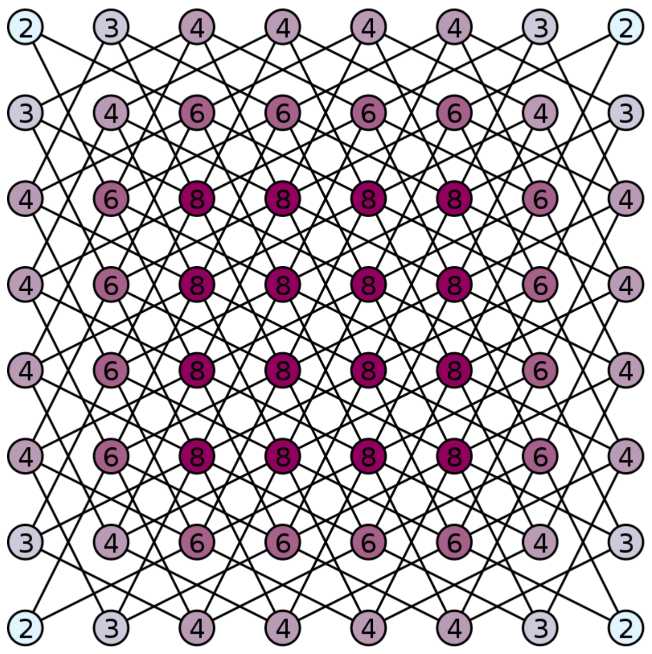
[2] http://www.irri.org/science/ricestat/
[3] http://mathworld.wolfram.com/WheatandChessboardProblem.html
[4] http://mathforum.org/~sanders/geometry/GP11Fable.html
[5] http://www.averypickford.com/Third/salt.htm

Knight's tour

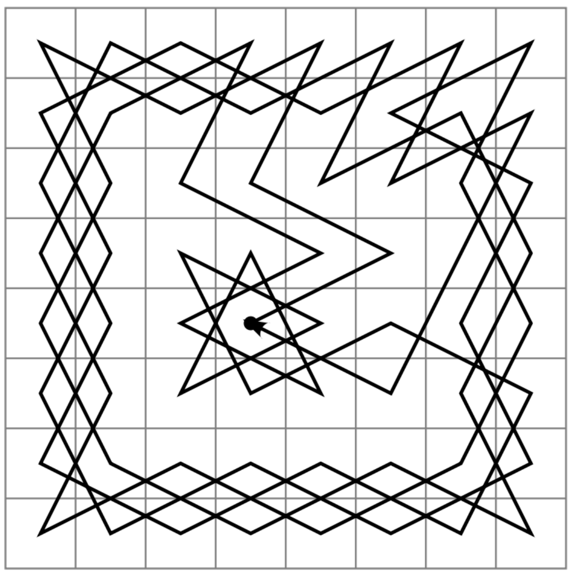
The **Knight's Tour** is a mathematical problem involving a knight on a chessboard. The knight is placed on the empty board and, moving according to the rules of chess, must visit each square exactly once. A knight's tour is called a *closed tour* if the knight ends on a square attacking the square from which it began (so that it may tour the board again immediately with the same path). Otherwise the tour is *open*. The exact number of open tours is still unknown. Creating a program to solve the knight's tour is a common problem given to computer science students.[1] Variations of the knight's tour problem involve chessboards of different sizes than the usual 8 × 8, as well as irregular (non-rectangular) boards.



Knight's graph showing all possible paths for a Knight's tour on a standard 8×8 chessboard. The numbers on each node indicate the number of possible moves that can be made from that position.

Theory

The knight's tour problem is an instance of the more general Hamiltonian path problem in graph theory. The problem of finding a closed knight's tour is similarly an instance of the hamiltonian cycle problem. Note however that, unlike the general Hamiltonian path problem, the knight's tour problem can be solved in linear time.[2]



The Knight's tour as solved by The Turk, a chess-playing machine hoax. This particular solution is closed (circular), and can be completed from any point on the board.

History

The earliest known references to the Knight's Tour problem date back to the 9th century CE. The pattern of a knight's tour on a half-board has been presented in verse form (as a literary constraint) in the highly stylized Sanskrit poem *Kavyalankara*[3] written by the 9th century Kashmiri poet Rudrata, which discusses the art of poetry, especially with relation to theater (Natyashastra). As was often the practice in ornate Sanskrit poetry, the syllabic patterns of this poem elucidate a completely different motif, in this case an open knight's tour on a half-chessboard.

One of the first mathematicians to investigate the knight's tour was Leonhard Euler. The first algorithm for completing the Knight's Tour was Warnsdorff's algorithm, first described in 1823 by H. C. Warnsdorff.

In the 20th century the Oulipo group of writers used it among many others. The most notable example is the 10 × 10 Knight's Tour which sets the order of the chapters in Georges Perec's novel *Life: A User's Manual*. The sixth game of the 2010 World Chess Championship between Viswanathan Anand and Veselin Topalov saw Anand making 13 consecutive knight moves – online commentors jested that Anand was trying to solve the Knight's Tour problem during the game.

Closed tours

On an 8 × 8 board, there are exactly 26,534,728,821,064 directed closed tours (i.e. two tours along the same path that travel in opposite directions are counted separately).[4] [5] [6] The number of *undirected* closed tours is half this number, since every tour can be traced in reverse. There are 9,862 undirected closed tours on a 6 × 6 board.[7] No closed tours exist for smaller square boards (this is a corollary of the following theorem).

Schwenk's Theorem

For any *m* × *n* board with *m* less than or equal to *n*, a ***closed*** knight's tour is always possible **unless** one or more of these three conditions are true:

1. *m* and *n* are both odd; *m* and *n* are not both 1
2. *m* = 1, 2, or 4; *m* and *n* are not both 1
3. *m* = 3 and *n* = 4, 6, or 8.

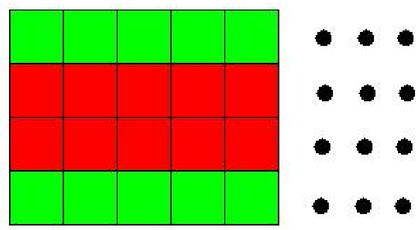
Condition 1

One can show that condition 1 prohibits closed tours by a simple argument based on parity and coloring. For the standard black-and-white coloring of the chessboard, the knight must move either from a black square to a white square or from a white square to a black square. Thus in a closed tour the knight must visit the same number of white squares as black squares, and the number of squares visited in total must therefore be even. But if *m* and *n* are both odd, the total number of squares is odd. (For example, in a 5 × 5 checkerboard there are 13 of one color and 12 of the other.) Therefore closed tours do not exist. Open tours may still exist (with the exception of the trivial case 1 × 1).

Condition 2

Condition 2 states that when the shorter side is of length 1, 2, or 4, no closed tour is possible.

When *m* = 1 or 2, no closed tour is possible because the knight would not be able to reach every square (once again, with the exception of the trivial case 1 × 1). It can be shown that a 4 × *n* board cannot have a closed tour either, by using a coloring argument.



The knight must alternate between green and red.

Start by assuming that a 4 × *n* board has a closed knight's tour. Let A_1 be the set of all squares that would be black and A_2 all the squares that would be white, if they were colored according to the alternating black-and-white checkerboard scheme. Consider the figure at right. Define B_1 to be the set of green squares and B_2 as the set of red squares. There are an equal number of red squares as green squares. Note that from a square in B_1 the knight must next jump to a square in B_2. And since the knight must visit every square once, when the knight is on a square in B_2 it must move to a square in B_1 next (otherwise the knight will need to travel to two squares in B_1 later).
If we follow the hypothetical knight's tour we will find a contradiction.

1. The first square the knight goes to will be a square of A_1 and B_1. If it is not, we switch the definitions of B_1 and B_2 so that it is true.
2. The second square must be an element of the sets A_2 and B_2.

3. The third square must be an element of A_1 and B_1.
4. The fourth square must be an element of the sets A_2 and B_2.
5. And so on.

It follows that set A_1 has the same elements as set B_1, and set A_2 has the same elements as set B_2. But this is obviously not true, as the red and green pattern shown above is not the same as a checkerboard pattern; the set of red squares is not the same as the set of black squares (or white, for that matter).

So our above assumption was false and there are no closed knight's tours for any 4 × *n* board, for any *n*.

Condition 3

Condition 3 may be proved using casework. Attempting to construct a closed knight's tour on a 3 by 4, 3 by 6, or 3 by 8 will lead to definite failure. 3 by *n* boards with *n* even and greater than 8 can be shown to have closed tours by induction (a repeating pattern).

All other cases

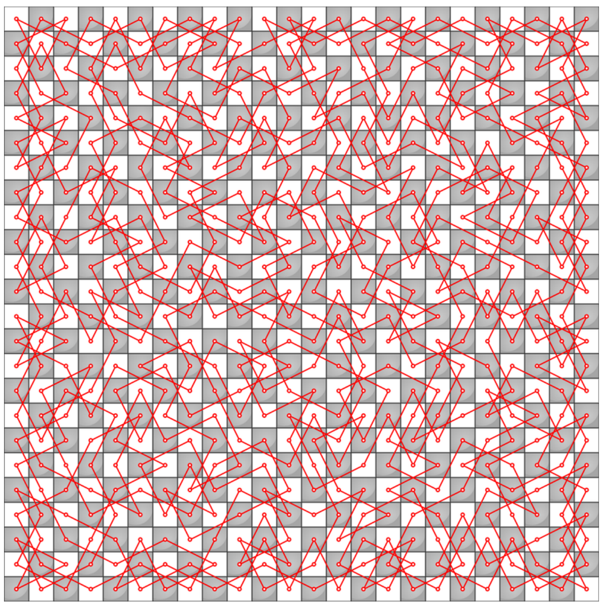
Simply proving the above three conditions does not prove the theorem; it is still required to prove that all rectangular boards that *do not* fall in one of the above three categories have closed knight's tours.[8]

Computer algorithms

Algorithms other than the simple brute-force search to find knight's tour solutions are discussed below. It is important to note that exhaustive brute force approach (one which iterates through all possible 64-long move sequences) can *never* be applied to the Knight's Tour problem. There are approximately $4{\times}10^{51}$ such sequences,[9] and it would take an unfathomable amount of time to simulate through such a large number of sequences.

Neural network solutions

The Knight's Tour problem also lends itself to being solved by a neural network implementation.[10] The network is set up such that every legal knight's move is represented by a neuron, and each neuron is initialized randomly to be either "active" or "inactive" (output of 1 or 0), with 1 implying that the neuron is part of the final solution. Each neuron also has a state function (described below) which is initialized to 0.



Closed Knight's Tour on a 24 × 24 board solved by a neural network.

When the network is allowed to run, each neuron can change its state and output based on the states and outputs of its neighbors (adjacent knight's moves) according to the following transition rules:

$$U_{t+1}(N_{i,j}) = U_t(N_{i,j}) + 2 - \sum_{N \in G(N_{i,j})} V_t(N)$$

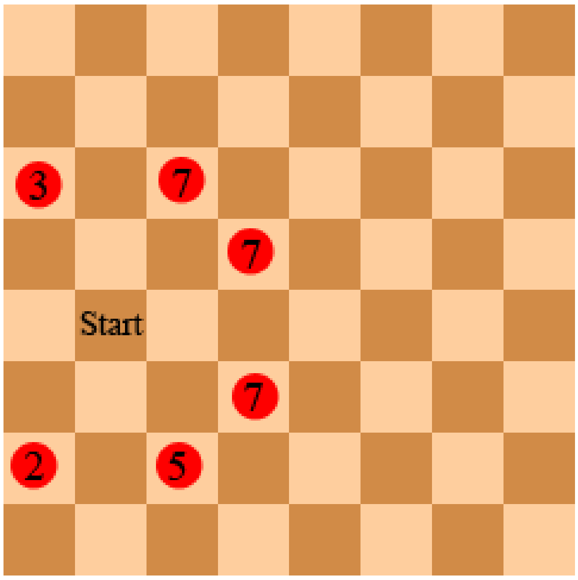
$$V_{t+1}(N_{i,j}) = \begin{cases} 1 & \text{if } U_{t+1}(N_{i,j}) > 3 \\ 0 & \text{if } U_{t+1}(N_{i,j}) < 0 \\ V_t(N_{i,j}) & \text{otherwise,} \end{cases}$$

where t represents discrete intervals of time, $U(N_{i,j})$ is the state of the neuron connecting square i to square j, $V(N_{i,j})$ is the output of the neuron from i to j, and $G(N_{i,j})$ is the set of neighbors of the neuron.
Although divergent cases are possible, the network should eventually converge, which occurs when no neuron changes its state from time t to $t+1$. When the network converges, a solution is found. The solution found by the network will be either a knight's tour, or a series of two or more independent knight's tours within the same board.

Warnsdorff's algorithm

Warnsdorff's algorithm is a heuristic method for solving the knight's tour, based on the rule of choosing the square among those immediately accessible by the knight move that would give the fewest possible knight's moves following the move to that square. In graph-theoretic terms, each move is made to the adjacent vertex with the least degree. (Pohl has devised a rule for breaking ties.) This algorithm may also more generally be applied to any graph. Although the Hamiltonian path problem is NP-hard in general, on many graphs that occur in practice this heuristic is able to successfully locate a solution in linear time.[11] The knight's tour is a special case.[12]

The algorithm was first described in "Des Rösselsprungs einfachste und allgemeinste Lösung" by H. C. Warnsdorff in 1823.[12]



A graphical representation of the Knight's Tour implemented using Warnsdorff's Algorithm. The numbers in red circles denote the number of next possible moves that the knight could make if it moves from the start position to the one with the circle on it.

Algorithm

Warnsdorff's algorithm will do for any initial position of the knight on the board. All the possible squares which may be reached in one move from this position are located, and the number of moves that the knight would be able to make from each of these squares is found.[13] Then, the algorithm dictates that the knight move to the square that has the least number of possible following moves. The process is then repeated until each square has been visited.[12] [14]

Some definitions:

- A position Q is accessible from a position P if P can move to Q by a single knight's move, and Q has not yet been visited.
- The accessibility of a position P is the number of positions accessible from P.

Algorithm:

1. set P to be a random initial position on the board
2. mark the board at P with the move number "1"
3. for each move number from 2 to the number of squares on the board:
    1. let S be the set of positions accessible from the input position
    2. set P to be the position in S with minimum accessibility
    3. mark the board at P with the current move number

4. return the marked board – each square will be marked with the move number on which it is visited.

See also

- Abu-Bakr Muhammad ben Yahya as-Suli
- George Koltanowski
- Longest uncrossed knight's path

Notes

[1] H. M. Deitel, P. J. Deitel. "Java How To Program Fifth Edition." *Prentice Hall*, Upper Saddle River, New Jersey, pp. 326-328. 2003.

[2] A. Conrad, T. Hindrichs, H. Morsy, and I. Wegener. "Solution of the Knight's Hamiltonian Path Problem on Chessboards." *Discrete Applied Math*, volume 50, no.2, pp.125-134. 1994.

[3] Satyadev, Chaudhary. *Kavyalankara of Rudrata (Sanskrit Text, with Hindi translation);*. Delhi: Parimal Sanskrit Series No. 30.

[4] Martin Loebbing; Ingo Wegener (1996). "The Number of Knight's Tours Equals 33,439,123,484,294 — Counting with Binary Decision Diagrams" (http://www.combinatorics.org/Volume_3/Abstracts/v3i1r5.html). *The Electronic Journal of Combinatorics* **3** (1): R5. . **Remark:** The authors later admitted (http://www.combinatorics.org/Volume_3/Comments/v3i1r5.html) that the announced number is incorrect. According to McKay's report, the correct number is 13,267,364,410,532 and this number is repeated in Wegener's 2000 book.

[5] Brendan McKay (1997). "Knight's Tours on an 8x8 Chessboard" (http://www.combinatorics.org/Volume_3/Comments/v3i1r5.01.ps). *Technical Report TR-CS-97-03* (Department of Computer Science, Australian National University). .

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[7] Weisstein, Eric W., " Knight's Tour (http://mathworld.wolfram.com/KnightsTour.html)" from MathWorld.

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[10] Y. Takefuji, K. C. Lee. "Neural network computing for knight's tour problems." *Neurocomputing*, 4(5):249-254, 1992.

[11] Pohl, Ira (July 1967). "A method for finding Hamilton paths and Knight's tours" (http://portal.acm.org/citation.cfm?id=363463). *Communications of the ACM* **10** (7): 446–449. doi:10.1145/363427.363463. .

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[13] Törnberg, Gunno (1999). "Warnsdorff's rule" (http://web.telia.com/~u85905224/knight/eWarnsd.htm). . Retrieved 2008-09-25.

[14] Cancela, Hector; Mordecki, Ernesto (2006-08-31) (PDF). *Counting Knight's Tours through the Randomized Warnsdorff Rule* (http://www.cmat.edu.uy/~mordecki/articles/warnsdorff.pdf). . Retrieved 2008-10-18.

External links

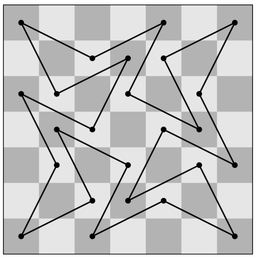
- Richard W. Henderson, Eugene Roger Apodaca A Knight of Egodeth: Zen Raptured Quietude (http://www.amazon.com/dp/0979763002), 240 Solutions to the Knights Tour in form of Game Book
- Warnsdorff's Rule (http://web.telia.com/~u85905224/knight/eWarnsd.htm) and its efficiency (http://web.telia.com/~u85905224/knight/bWarnsd.htm) from Warnsdorff's Rule Web Page
- Mario Velucchi The ultimate Knight's Tour page of Links (http://www.velucchi.it/mathchess/knight.htm)
- The knight's tour (http://www.borderschess.org/KnightTour.htm)
- Knight's tour notes (http://www.ktn.freeuk.com/sitemap.htm)
- Sloane's Integer Sequence A001230 (http://en.wikipedia.org/wiki/Oeis:a001230)
- Preprint of Pohl's paper (freely accessible) (http://www.slac.stanford.edu/cgi-wrap/getdoc/slac-pub-0261.pdf)
- SilverKnight (http://www.papnkukn.net/silverknight/) - online knight's tour game

Implementations

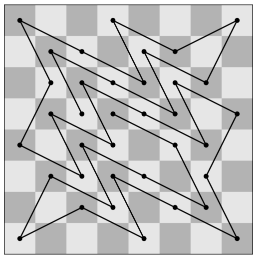
- The Knight's Tour (http://demonstrations.wolfram.com/TheKnightsTour/) by Jay Warendorff, Wolfram Demonstrations Project
- A Simple backtracking implementation in C++ (http://www.compgeom.com/~piyush/teach/3330/homeworks/knightour.cpp)
- A Simple implementation in standard Prolog (http://junk.kymhorsell.com/knight.html)
- An implementation in C# (http://www.knightstour.co.uk)
- Knight's Tours Using a Neural Network (http://dmitrybrant.com/knights-tour) Program that creates tours using a neural network, plus gallery of images.

Longest uncrossed knight's path

The **longest uncrossed** (or **nonintersecting**) **knight's path** is a mathematical problem involving a knight on a standard 8 × 8 chessboard or, more generally, on a square $n \times n$ board. The problem is to find the longest path the knight can take on the given board, such that the path does not intersect itself. A further distinction can be made between a **closed** path, which ends on the same field as where it begins, and an **open** path, which ends on a different field from where it begins.



A closed path for $n = 7$ of length 24.



An open path for $n = 8$ of length 35.

Solutions are known only up to $n = 9$. The length of the longest path, whether open or closed (OEIS sequence A003192), for $n = 3 \ldots 9$ is:

$$2, 5, 10, 17, 24, 35, 47.$$

The problem can be further generalized to rectangular $n \times m$ boards, or even to boards in the shape of any polyomino. Other standard chess pieces than the knight are less interesting, but fairy chess pieces like camel, giraffe and zebra lead to problems of comparable complexity.

See also

- A knight's tour is a self-intersecting knight's path visiting all fields of the board.
- TwixT, a board game based on uncrossed knight's paths.

References

- L. D. Yarbrough, Uncrossed knight's tours, *Journal of Recreational Mathematics* 1 (1969), no. 3, pp. 140–142.

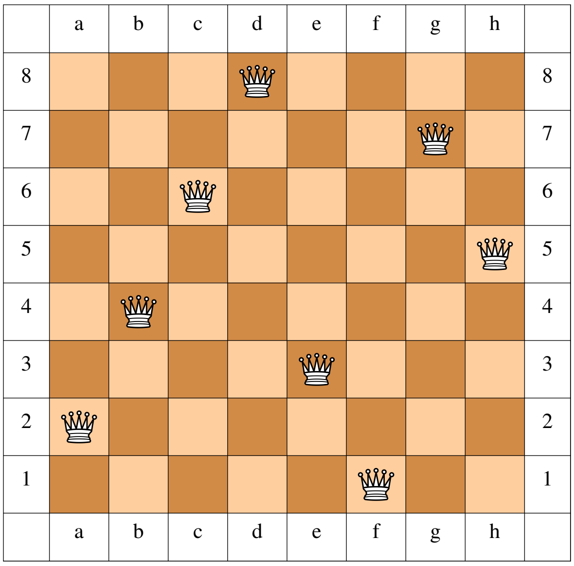
External links

- Non-Intersecting Paths by Leapers [1], by George Jelliss.

References

[1] http://www.ktn.freeuk.com/2b.htm

Eight queens puzzle



One solution.

The **eight queens puzzle** is the problem of placing eight chess queens on an 8×8 chessboard so that none of them is able to capture any other using the standard chess queen's moves. The queens must be placed in such a way that no two queens attack each other. Thus, a solution requires that no two queens share the same row, column, or diagonal. The eight queens puzzle is an example of the more general ***n*-queens problem** of placing n queens on an $n \times n$ chessboard, where solutions exist only for $n = 1$ or $n \geq 4$.

History

The puzzle was originally proposed in 1848 by the chess player Max Bezzel, and over the years, many mathematicians, including Gauss, have worked on this puzzle and its generalized n-queens problem. The first solutions were provided by Franz Nauck in 1850. Nauck also extended the puzzle to n-queens problem (on an n×n board—a chessboard of arbitrary size). In 1874, S. Günther proposed a method of finding solutions by using determinants, and J.W.L. Glaisher refined this approach.

Edsger Dijkstra used this problem in 1972 to illustrate the power of what he called structured programming. He published a highly detailed description of the development of a depth-first backtracking algorithm.[2]

Constructing a solution

The problem can be quite computationally expensive as there are 4,426,165,368 (*i.e.*, 64 choose 8) possible arrangements of eight queens on a 8×8 board, but only 92 solutions. It is possible to use shortcuts that reduce computational requirements or rules of thumb that avoids brute force computational techniques. For example, just by applying a simple rule that constrains each queen to a single column (or row), though still considered brute force, it is possible to reduce the number of possibilities to just 16,777,216 (that is, 8^8) possible combinations. Generating the permutations that are solutions of the eight rooks puzzle[1] and then checking for diagonal attacks further reduces the possibilities to just 40,320 (that is, 8!). These are computationally manageable for n = 8, but would be intractable for problems of n ≥ 20, as 20! = 2.433 * 10^{18}. Extremely interesting advancements for this and other toy problems is the development and application of heuristics (rules of thumb) that yield solutions to the *n* queens puzzle at an astounding fraction of the computational requirements.

This heuristic solves *N* queens for any *N* ≥ 4. It forms the list of numbers for vertical positions (rows) of queens with horizontal position (column) simply increasing. *N* is 8 for eight queens puzzle.

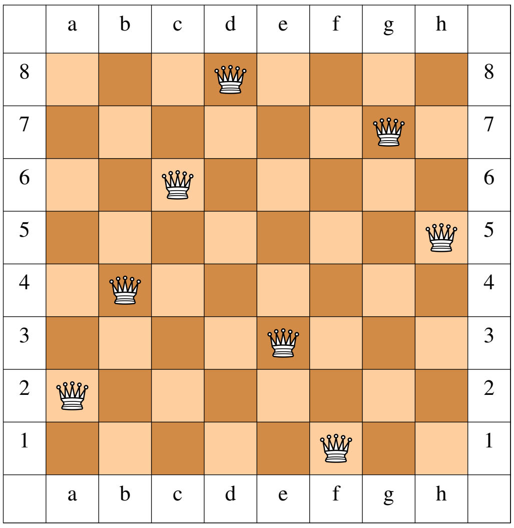
1. If the remainder from dividing *N* by 6 is not 2 or 3 then the list is simply all even numbers followed by all odd numbers ≤ N
2. Otherwise, write separate lists of even and odd numbers (i.e. 2,4,6,8 - 1,3,5,7)
3. If the remainder is 2, swap 1 and 3 in odd list and move 5 to the end (i.e. **3,1**,7,**5**)
4. If the remainder is 3, move 2 to the end of even list and 1,3 to the end of odd list (i.e. 4,6,8,**2** - 5,7,9,**1,3**)
5. Append odd list to the even list and place queens in the rows given by these numbers, from left to right (i.e. a2, b4, c6, d8, e3, f1, g7, h5)

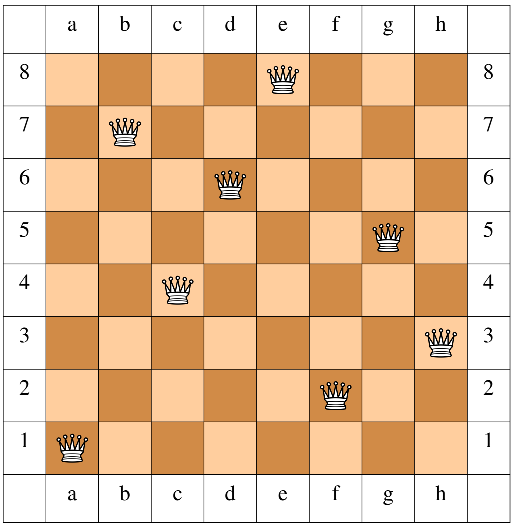
For *N* = 8 this results in the solution shown above. A few more examples follow.

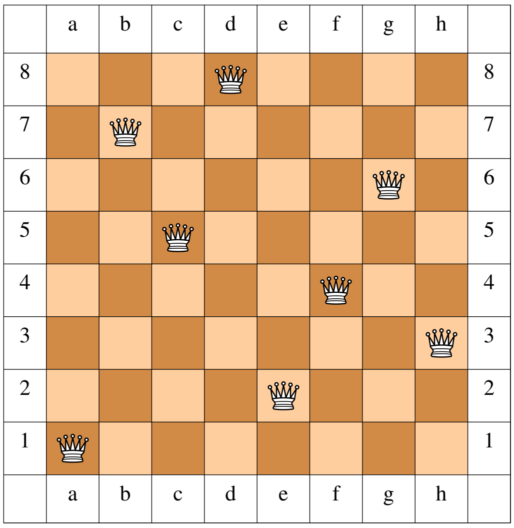
- 14 queens (remainder 2): 2, 4, 6, 8, 10, 12, 14, 3, 1, 7, 9, 11, 13, 5.
- 15 queens (remainder 3): 4, 6, 8, 10, 12, 14, 2, 5, 7, 9, 11, 13, 15, 1, 3.
- 20 queens (remainder 2): 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 3, 1, 7, 9, 11, 13, 15, 17, 19, 5.

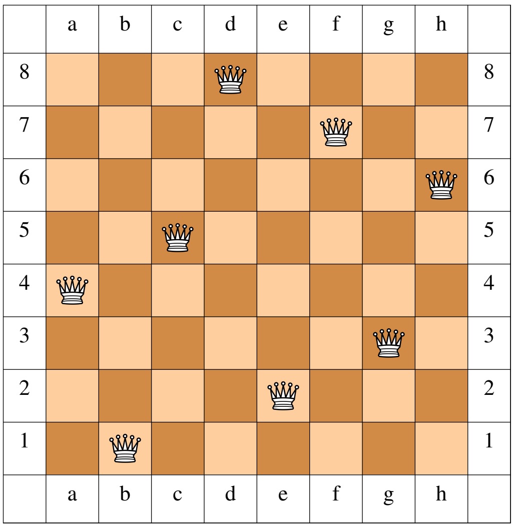
Solutions to the eight queens puzzle

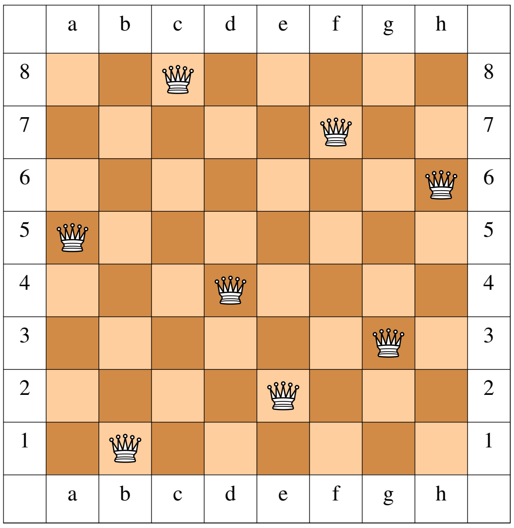
The eight queens puzzle has 92 **distinct** solutions. If solutions that differ only by symmetry operations (rotations and reflections) of the board are counted as one, the puzzle has 12 **unique** (or **fundamental**) solutions, which are presented below:

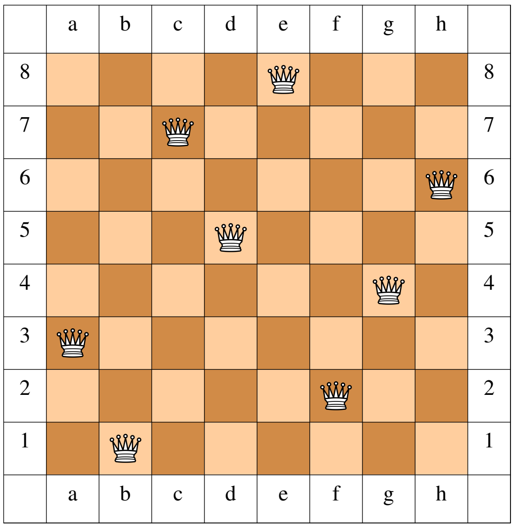

Unique solution 1

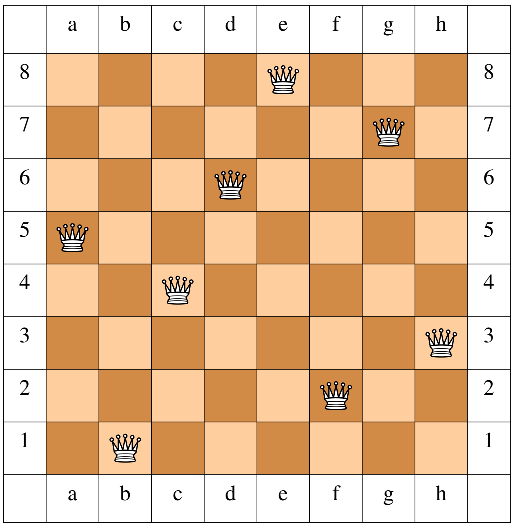

Unique solution 2


Unique solution 3

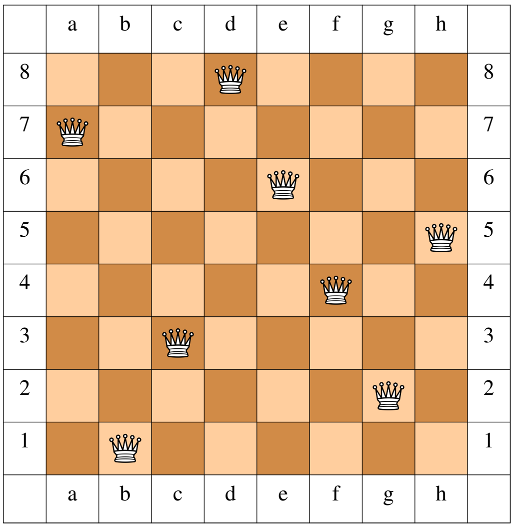

Unique solution 4


Unique solution 5

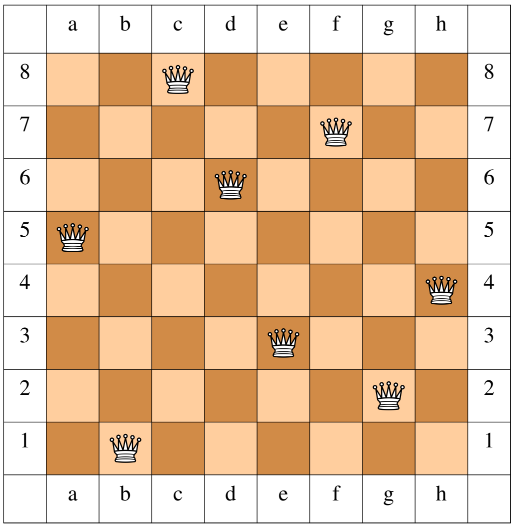

Unique solution 6



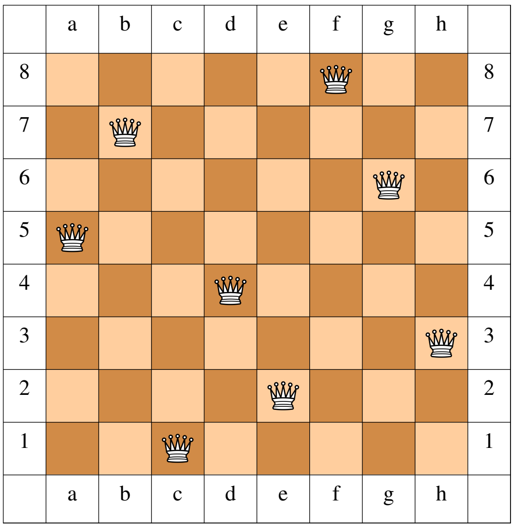
Unique solution 7



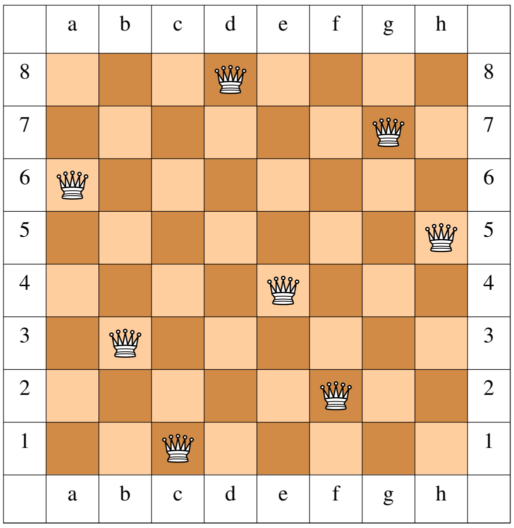
Unique solution 8



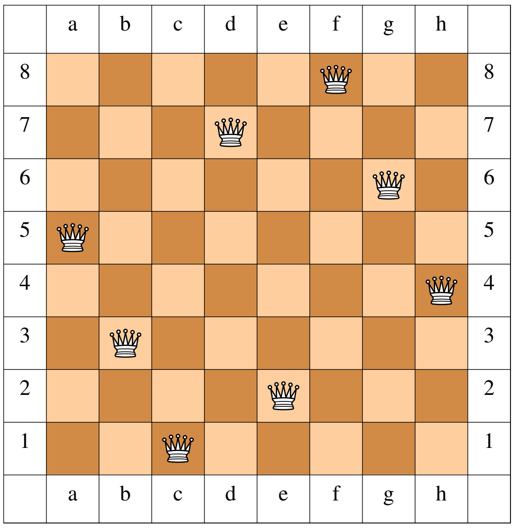
Unique solution 9



Unique solution 10



Unique solution 11



Unique solution 12

Counting solutions

The following table gives the number of solutions for placing *n* queens on an $n \times n$ board, both unique (sequence A002562 [2] in OEIS) and distinct (sequence A000170 [3] in OEIS), for n=1–14, 24–26.

***n*:**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**..**	**24**	**25**	**26**
unique:	1	0	0	1	2	1	6	12	46	92	341	1,787	9,233	45,752	..	28,439,272,956,934	275,986,683,743,434	2,789,712,466,510,289
distinct:	1	0	0	2	10	4	40	92	352	724	2,680	14,200	73,712	365,596	..	227,514,171,973,736	2,207,893,435,808,352	22,317,699,616,364,044

Note that the six queens puzzle has fewer solutions than the five queens puzzle.

There is currently no known formula for the exact number of solutions.

Related problems

Using pieces other than queens

On an 8×8 board one can place 32 knights, or 14 bishops, 16 kings or eight rooks, so that no two pieces attack each other. Fairy chess pieces have also been substituted for queens. In the case of knights, an easy solution is to place one on each square of a given color, since they move only to the opposite color.

Nonstandard boards

Pólya studied the *n* queens problem on a toroidal ("donut-shaped") board and showed that there is a solution on an n×n board if and only if n is not divisible by 2 or 3 [4] . In 2009 Pearson and Pearson algorithmically populated three-dimensional boards [5] *(n×n×n)* with n^2 queens, and proposed that multiples of these can yield solutions for a four-dimensional version of the puzzle.

Domination

Given an *n*×*n* board, the **domination number** is the minimum number of queens (or other pieces) needed to attack or occupy every square. For n=8 the queen's domination number is 5.

Nine queens problem [6]

Place nine queens and one pawn on an 8×8 board in such a way that queens don't attack each other. Further generalization of the problem (complete solution is currently unknown): given an *n*×*n* chess board and $m > n$ queens, find the minimum number of pawns, so that the *m* queens and the pawns can be set up on the board in such a way that no two queens attack each other.

Queens and knights problem [7]

Place *m* queens and *m* knights on an *n*×*n* board so that no piece attacks another.

Magic squares

In 1992, Demirörs, Rafraf, and Tanik published a method for converting some magic squares into *n* queens solutions, and vice versa.[8]

Latin squares

In an *n*×*n* matrix, place each digit 1 through *n* in *n* locations in the matrix so that no two instances of the same digit are in the same row or column.

Exact cover

Consider a matrix with one primary column for each of the *n* ranks of the board, one primary column for each of the *n* files, and one secondary column for each of the 4*n*-6 nontrivial diagonals of the board. The matrix has n^2 rows: one for each possible queen placement, and each row has a 1 in the columns corresponding to that square's rank, file, and diagonals and a 0 in all the other columns. Then the *n* queens problem is equivalent to choosing a subset of the rows of this matrix such that every primary column has a 1 in precisely one of the chosen rows and every secondary column has a 1 in at most one of the chosen rows; this is an example of a generalized exact cover problem, of which sudoku is another example.

The eight queens puzzle as an exercise in algorithm design

Finding all solutions to the eight queens puzzle is a good example of a simple but nontrivial problem. For this reason, it is often used as an example problem for various programming techniques, including nontraditional approaches such as constraint programming, logic programming or genetic algorithms. Most often, it is used as an example of a problem which can be solved with a recursive algorithm, by phrasing the *n* queens problem inductively in terms of adding a single queen to any solution to the problem of placing $n-1$ queens on an n-by-n chessboard. The induction bottoms out with the solution to the 'problem' of placing 0 queens on an n-by-n chessboard, which is the empty chessboard.

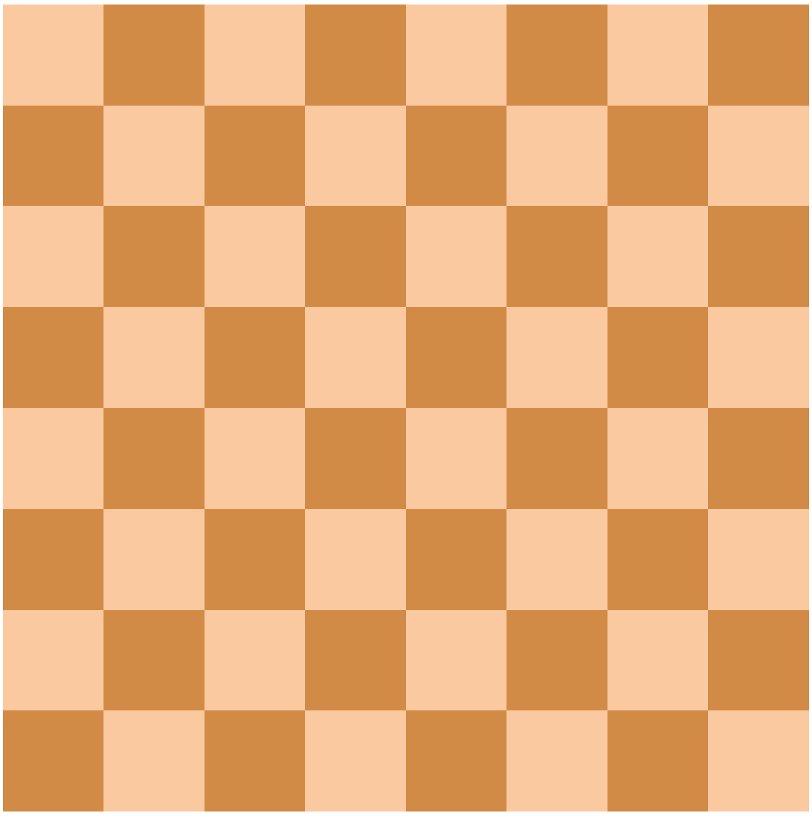
This technique is much more efficient than the naïve brute-force search algorithm, which considers all $64^8 = 2^{48} =$ 281,474,976,710,656 possible blind placements of eight queens, and then filters these to remove all placements that place two queens either on the same square (leaving only 64!/56! = 178,462,987,637,760 possible placements) or in mutually attacking positions. This very poor algorithm will, among other things, produce the same results over and over again in all the different permutations of the assignments of the eight queens, as well as repeating the same computations over and over again for the different sub-sets of each solution. A better brute-force algorithm places a single queen on each row, leading to only $8^8 = 2^{24} =$ 16,777,216 blind placements.

It is possible to do much better than this. One algorithm solves the eight rooks puzzle by generating the permutations of the numbers 1 through 8 (of which there are 8! = 40,320), and uses the elements of each permutation as indices to place a queen on each row. Then it rejects those boards with diagonal attacking positions. The backtracking depth-first search program, a slight improvement on the permutation method, constructs the search tree by considering one row of the board at a time, eliminating most nonsolution board positions at a very early stage in their construction. Because it rejects rook and diagonal attacks even on incomplete boards, it examines only 15,720 possible queen placements. A further improvement which examines only 5,508 possible queen placements is to combine the permutation based method with the early pruning method: the permutations are generated depth-first, and the search space is pruned if the partial permutation produces a diagonal attack. Constraint programming can also be very effective on this problem.

An alternative to exhaustive search is an 'iterative repair' algorithm, which typically starts with all queens on the board, for example with one queen per column. It then counts the number of conflicts (attacks), and uses a heuristic to determine how to improve the placement of the queens. The 'minimum-conflicts' heuristic — moving the piece with the largest number of conflicts to the square in the same column where the number of conflicts is smallest — is particularly effective: it finds a solution to the 1,000,000 queen problem in less than 50 steps on average. This assumes that the initial configuration is 'reasonably good' — if a million queens all start in the same row, it will obviously take at least 999,999 steps to fix it. A 'reasonably good' starting point can for instance be found by putting each queen in its own row and column so that it conflicts with the smallest number of queens already on the board.

Note that 'iterative repair', unlike the 'backtracking' search outlined above, does not guarantee a solution: like all hillclimbing (i.e., greedy) procedures, it may get stuck on a local optimum (in which case the algorithm may be restarted with a different initial configuration). On the other hand, it can solve problem sizes that are several orders of magnitude beyond the scope of a depth-first search.

An animated version of the recursive solution



This animation uses backtracking to solve the problem. A queen is placed in a column that is known not to cause conflict. If a column is not found the program returns to the last good state and then tries a different column.

See also

- Functional programming
- Mathematical game
- Mathematical puzzle
- No-three-in-line problem
- Rook polynomial
- Distributed computing
- BOINC

References

[1] Rooks Problem from Wolfram MathWorld (http://mathworld.wolfram.com/RooksProblem.html)
[2] http://en.wikipedia.org/wiki/Oeis%3Aa002562
[3] http://en.wikipedia.org/wiki/Oeis%3Aa000170
[4] G. Polya, Uber die “doppelt-periodischen” Losungen des n-Damen-Problems, George Polya: Collected papers Vol. IV, G-C. Rota, ed., MIT Press, Cambridge, London, 1984, pp. 237–247
[5] http://queens.lyndenlea.info/beyond2d.php
[6] http://www.chessvariants.org/problems.dir/9queens.html
[7] http://www.vector.org.uk/archive/v213/hui213.htm
[8] O. Demirörs, N. Rafraf, and M.M. Tanik. Obtaining n-queens solutions from magic squares and constructing magic squares from n-queens solutions. Journal of Recreational Mathematics, 24:272–280, 1992

- Bell, Jordan and Stevens, Brett, A survey of known results and research areas for **n**-queens, *Discrete Mathematics*, Vol. 309, Issue 1, 6 January 2009, 1-31.

- Watkins, John J. (2004). *Across the Board: The Mathematics of Chess Problems*. Princeton: Princeton University Press. ISBN 0-691-11503-6.
- O.-J. Dahl, E. W. Dijkstra, C. A. R. Hoare *Structured Programming*, Academic Press, London, 1972 ISBN 0-12-200550-3 see pp 72–82 for Dijkstra's solution of the 8 Queens problem.
- *Three Dimensional NxN-Queens Problems* (http://www.csse.monash.edu.au/~lloyd/tildeAlgDS/Recn/Queens3D/), L.Allison, C.N.Yee, & M.McGaughey (1988), Department of Computer Science, Monash University, Australia.
- S. Nudelman, *The Modular N-Queens Problem in Higher Dimensions*, Discrete Mathematics, vol 146, iss. 1-3, 15 November 1995, pp. 159–167.
- *On The Modular N-Queen Problem in Higher Dimensions* (http://www.liacs.nl/~kosters/nqueens/papers/gomez2004.pdf), Ricardo Gomez, Juan Jose Montellano and Ricardo Strausz (2004), Instituto de Matematicas, Area de la Investigacion Cientifica, Circuito Exterior, Ciudad Universitaria, Mexico.
- J. Barr and S. Rao, *The n-Queens Problem in Higher Dimensions*, Elemente der Mathematik, vol 61 (2006), pp. 133–137.

External links

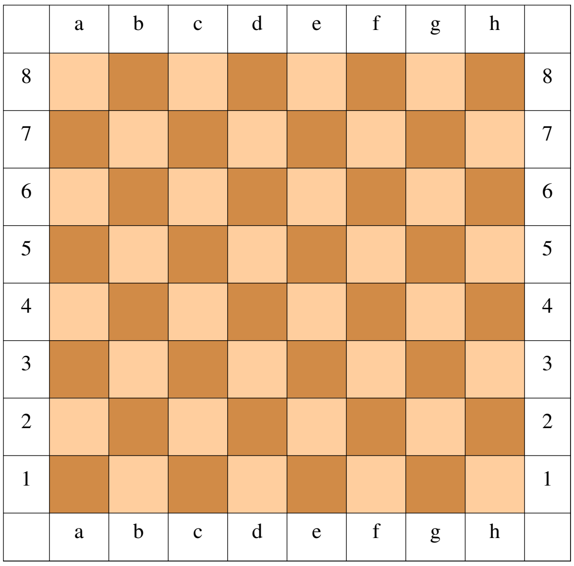
- Two-Player game based on Eight Queen problem (http://www.deduce.us)
- A visualization of N-Queens solution algorithms by Yuval Baror (http://yuval.bar-or.org/index.php?item=9)
- An Applet simulating the random-greedy solution for the n-queen problem (http://firefang.net/english/n-queens)
- MathWorld article (http://mathworld.wolfram.com/QueensProblem.html)
- Solutions to the 8-Queens Problem (http://bridges.canterbury.ac.nz/features/eight.html)
- Walter Kosters's N-Queens Page (http://www.liacs.nl/home/kosters/nqueens.html)
- Durango Bill's N-Queens Page (http://www.durangobill.com/N_Queens.html)
- On-line Guide to Constraint Programming (http://kti.ms.mff.cuni.cz/~bartak/constraints/index.html)
- NQueen@home Boinc project (http://nqueens.ing.udec.cl/)
- Ideas and algorithms for the N-Queens problem (https://sites.google.com/site/nqueensolver/)
- Queens@TUD project (uses FPGA-based solvers) (http://queens.inf.tu-dresden.de/)
- Eight queens puzzle: Flash game (http://ilovemedia.es/en/games/eight_queens_puzzle.html)
- Eight Queens Puzzle: iPhone free game (http://itunes.apple.com/us/app/8-queens/id397685941)

Links to solutions

- Takaken's N-Queen code (around 4 times faster than Jeff Somer's code because of symmetry considerations) (http://www.ic-net.or.jp/home/takaken/e/queen/index.html)
- Jeff Somers' N-Queen code (http://www.jsomers.com/nqueen_demo/nqueens.html)
- A000170 (http://en.wikipedia.org/wiki/Oeis:a000170) N Queens solutions on Sloane's On-Line Encyclopedia of Integer Sequences
- N Queens solutions achieved on the NQueen@home Boinc project (code is modified from Jeff Somers' code) (http://nqueens.ing.udec.cl/resultados.php)
- Usefull algorithms for solving the N-Queens problem (https://sites.google.com/site/nqueensolver/home/algorithm-results)
- C++ implementation of 8-queen problem all permutations (http://www.planet-source-code.com/vb/scripts/ShowCode.asp?txtCodeId=13265&lngWId=3)
- Find your own solution (http://www.hbmeyer.de/backtrack/achtdamen/eight.htm)
- Atari BASIC (http://www.atarimagazines.com/v3n12/Queens8.html)
- Atari Action! (http://www.atarimagazines.com/v4n5/8queens.html)

- Genetic algorithms (http://www.dossier-andreas.net/ai/ga.html)
- Haskell/Java hybrid (http://www.scdi.org/~avernet/projects/jaskell/queens/)
- Java (http://files.nyu.edu/eb1167/html/works/programs/queens/queens.html), mirror (http://homepages.nyu.edu/~eb1167/works/programs/queens/queens.html), solves by backtracking, code under GPL.
- Python (http://code.activestate.com/recipes/576647/)
- With out Permutations (Python) (http://code.activestate.com/recipes/577325-eight-queens-with-out-permutations/)
- Standard ML (http://www.dcs.ed.ac.uk/home/mlj/demos/queens/)
- Integer Sequences (http://www.muljadi.org/EightQueens.htm)
- Quirkasaurus' 8 Queens Solution (http://www.webcitation.org/query?url=http://www.geocities.com/quirkasaurus/queens8/index.html&date=2009-10-26+01:10:12)
- LISP solution for N-Queens Problem (http://www.obereed.net/queens/)
- ANSI C (recursive, congruence-free NxN-size queens problem solver with conflict heuristics) (http://www.0xe3.com/src/chess/)
- javascript solution for 8-Queens Problem (http://www.webcitation.org/query?url=http://www.geocities.com/ndjapic/OSIGKStanisic/Dame.html&date=2009-10-26+02:43:41)
- Brute-force solution for eight queens in a web based interactive classic BASIC environment (http://www.quitebasic.com/prj/puzzle/eight-queens-brute/)
- solution for eight queens in C++ (http://ananyamallik.blogspot.com/2010/10/solution-of-8-queen-problem.html)
- Conflict heuristics solution for the eight queens in a web based interactive classic BASIC environment (http://www.quitebasic.com/prj/puzzle/eight-queens-heuristic/)
- A Simple PHP Solution (http://www.citytalks.gr/8q/index.php)
- A table of **first-possible** N-Queens solutions, for N = 4 to 49 (less N=46 and N=48, which are work-in-progress) (http://queens.cspea.co.uk/csp-q-1stsols.html)
- Visual Prolog: N-Queen Puzzle (http://wiki.visual-prolog.com/index.php?title=N-queen_puzzle) (wiki)
- Solutions in various languages (http://rosettacode.org/wiki/N-Queens) on Rosetta Code
- N-Queens in X10 (http://dist.codehaus.org/x10/applications/samples/NQueensDist.x10)
- An Overview of Miranda (http://www.cs.kent.ac.uk/people/staff/dat/miranda/overview.pdf), including 4-line Miranda solution (*SigPlan Notices*, 21(12):158-166, Dec 1986)
- A Case Study: The Eight Queens Puzzle (http://web.engr.oregonstate.edu/~budd/Books/oopintro3e/info/chap06.pdf), including solutions in multiple languages
- ANS-Forth solution (http://www.forth.org/novice.html) Non-recursive with backtracking (demonstrates how Forth functions can take variable numbers of parameters on the stack). Also, an implementation of the heuristic algorithm described earlier in this Wikipedia article (demonstrates linked lists).
- N-Queens in Java (http://github.com/kapild/Permutations/tree/master/src/)

Mutilated chessboard problem



Mutilated chessboard problem.

The **mutilated chessboard** problem is a tiling puzzle introduced by Gamow & Stern (1958) and discussed by Martin Gardner in his *Scientific American* column "Mathematical Games." The problem is as follows:

> Suppose a standard 8x8 chessboard has two diagonally opposite corners removed, leaving 62 squares. Is it possible to place 31 dominoes of size 2x1 so as to cover all of these squares?

Most considerations of this problem in literature provide solutions "in the conceptual sense" without proofs.[1] John McCarthy proposed it[2] as a hard problem for automated proof systems.[3] In fact, its solution using the resolution system of inference is exponentially hard.[4]

Solution

The puzzle is impossible. Any way you would place a domino would cover one white square and one black square. A group of 31 dominoes would cover 31 white and 31 black squares of an unmutilated chessboard, leaving one white and one black square uncovered. The directions had you remove diagonally opposite corner squares, and such squares are always either both black or both white.[5]

Gomory's theorem

The same impossibility proof shows that no domino tiling exists whenever any two white squares are removed from the chessboard. However, if two squares of opposite colors are removed, then it is always possible to tile the remaining board with dominos; this result is called **Gomory's theorem**,[6] and is named after mathematician Ralph E. Gomory, whose proof was published in 1973.[7] Gomory's theorem can be proven using a Hamiltonian cycle of the grid graph formed by the chessboard squares; the removal of two oppositely-colored squares splits this cycle into two paths with an even number of squares each, both of which are easy to partition into dominos.

Notes

[1] Andrews, Peter B.; Bishop, Matthew (1996), "On Sets, Types, Fixed Points, and Checkerboards" (http://citeseer.ist.psu.edu/87819.html), *Theorem Proving With Analytic Tableaux and Related Methods: 5th International Workshop, Tableaux '96, Terrasini, Palermo, Italy, 15-17th, 1996, Proceedings*, Lecture Notes in Computer Science, Springer-Verlag, , "most treatments of the problem in the literature solve it in the conceptual sense, but do not actually provide proofs of the theorem in either of McCarthy's original formulations."

[2] Bancerek, Grzegorz (1995), "The Mutilated Chessboard Problem—checked by Mizar" (http://citeseer.ist.psu.edu/87819.html), in Boyer, Robert; Trybulec, Andrzej, *QED Workshop, II*, Warsaw University, pp. 25–26, , "The problem presented by John McCarthy during his lecture "Heavy duty set theory"[1] has been resolved here."

[3] Arthan, R. D. (2005) (PDF), *The Mutilated Chessboard Theorem in Z* (http://www.lemma-one.com/ProofPower/examples/wrk071.pdf), , retrieved 2007-05-06, "The mutilated chessboard theorem was proposed over 40 years ago by John McCarthy as a “tough nut to crack” for automated reasoning."

[4] Alekhnovich, Michael (2004), "Mutilated chessboard problem is exponentially hard for resolution", *Theoretical Computer Science* **310** (1-3): 513–525, doi:10.1016/S0304-3975(03)00395-5.

[5] McCarthy, John (1999), "Creative Solutions to Problems" (http://www-formal.stanford.edu/jmc/creative/creative.html), *AISB Workshop on Artificial Intelligence and Creativity*, , retrieved 2007-04-27

[6] Watkins, John J. (2004), *Across the board: the mathematics of chessboard problems*, Princeton University Press, pp. 12–14, ISBN 9780691115030.

[7] According to Mendelsohn, the original publication is in Honsberger's book. Mendelsohn, N. S. (2004), "Tiling with dominoes" (http://www.jstor.org/stable/4146865), *The College Mathematics Journal* (Mathematical Association of America) **35** (2): 115–120, doi:10.2307/4146865, ; Honsberger, R. (1973), *Mathematical Gems I*, Mathematical Association of America.

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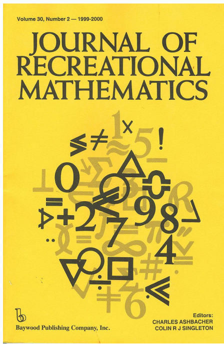
- Gardner, Martin (1994), *My Best Mathematical and Logic Puzzles*, Dover, ISBN 0486281523

External links

- Dominoes on a Checker Board by Jim Loy (http://www.jimloy.com/puzz/dominos.htm)
- Dominoes on a Checker Board (http://explorepdx.com/dominos.html)
- Gomory's Theorem (http://demonstrations.wolfram.com/GomorysTheorem/) by Jay Warendorff, The Wolfram Demonstrations Project.

Publications

Journal of Recreational Mathematics

Journal of Recreational Mathematics	
	
Discipline	Recreational mathematics
Language	English
Edited by	Charles Ashbacher, Lamarr Widmer
Publication details	
Publisher	Baywood Publishing Company (USA)
Publication history	1968-present
Frequency	Quarterly
Indexing	
ISSN	0022-412X [1]
Links	
• Journal homepage [2]	

The ***Journal of Recreational Mathematics*** is an American journal dedicated to recreational mathematics, started in 1968. It is published quarterly by the Baywood Publishing Company.

Harry Nelson was editor for five years and Joseph Madachy was the editor for many years. Charles Ashbacher and Colin Singleton took over as editors when Madachy retired. The current editors are Ashbacher and Lamarr Widmer.

The journal contains:

1. Original articles
2. Book reviews
3. Alphametics And Solutions To Alphametics
4. Problems And Conjectures
5. Solutions To Problems And Conjectures
6. Proposer's And Solver's List For Problems And Conjectures

Indexing

The journal is indexed in:

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- Book Review Index
- International Bibliography of Periodical Literature
- International Bibliography of Book Reviews
- Readers' Guide to Periodical Literature
- The Gale Group

References

[1] http://www.worldcat.org/issn/0022-412X
[2] http://www.baywood.com/journals/PreviewJournals.asp?Id=0022-412x

Winning Ways for your Mathematical Plays

Winning Ways for your Mathematical Plays (Academic Press, 1982) by Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy is a compendium of information on mathematical games. It was first published in 1982 in two volumes.

The first volume introduces combinatorial game theory and its foundation in the surreal numbers; partizan and impartial games; Sprague-Grundy theory and misère games. The second volume applies the theorems of the first volume to many games, including nim, sprouts, dots and boxes, Sylver coinage, philosopher's football, fox and geese. A final section on puzzles analyzes the Soma cube, Rubik's Cube, peg solitaire, and Conway's game of life.

A republication of the work by A K Peters splits the content into four volumes.

Editions

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- 2nd edition, Wellesley, Massachusetts: A. K. Peters Ltd., 4 vols., 2001–2004; vol. 1: ISBN 1-56881-130-6; vol. 2: ISBN 1-56881-142-X; vol. 3: ISBN 1-56881-143-8; vol. 4: ISBN 1-56881-144-6.

Games mentioned in the book

This is a partial list of the games mentioned in the book.

Note: Misere games not included

- Hackenbush
  - Blue-Red Hackenbush
  - Blue-Red-Green Hackenbush (Introduced as *Hackenbush Hotchpotch* in the book)
  - Childish Hackenbush
- Ski-Jumps
- Toads-and-Frogs
- Cutcake
  - Maundy Cake
  - (2nd Unnamed Cutcake variant by Dean Hickerson)
  - Hotcake

  - Coolcakes
    - Baked Alaska
- Eatcake
  - Turn-and_Eatcake
- Col
- Snort
- Nim (Green Hackenbush)
  - Prim
  - Dim
  - Lasker's Nim
- Seating Couples
- Northcott's Game (Poker-Nim)
- The White Knight
- Wyt Queens (Wythoff's Game)
- Kayles
  - Double Kayles
  - Quadruple Kayles
- Dawson's Chess
- Dawson's Kayles
- Treblecross
- Grundy's Game
  - Mrs. Grundy
- Domineering
- No Highway
- De Bono's *L*-Game
- Snakes-and-Ladders (Adders-and-Ladders)
- Jelly Bean Game
- Dividing Rulers

See also

- *On Numbers and Games* by John H. Conway, one of the three coauthors of *Winning Ways*

External links

- Descriptions of games from the book [1]

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[1] http://www.madras.fife.sch.uk/maths/games/index.html

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