Quant Preparation - Sample Problems

1. You are given a black-box which returns a random number between 0 and 1. How would you use two instances of the random number given by black box to generate a uniformly random point inside a circle of radius 1.

2. Given k sorted arrays with total number of elements in k arrays equal to n (number in each array may not be equal), give an algorithm to sort all the combined elements in $O(n \log k)$ time.

3. You are given black-box which returns a random number between 0 and 1. You keep generating random numbers X_1, X_2 and so on and store the sum of all those random numbers. You stop as soon as the sum exceeds 1. What is the expected number of random variables used in the process? (**Hint:** The answer is not 2)

4. The king has 500 barrels of wine, but one of them is poisoned. Anyone drinking the poisoned wine will die within 12 hours. The king has four prisoners whom he is willing to sacrifice in order to find the poisoned barrel. Can this be done within 48 hours?

5. You are given a balance (that is, a weighing machine with two trays) and a positive integer N. You are then to request a number of weights. You pick which denominations of weights you want and how many of each you want. After you receive the weights you requested, you are given a thing whose weight is an integer between 1 and N, inclusive. Using the balance and the weights you requested, you must now determine the exact weight of the thing. As a function of N, how few weights can you get by requesting?
   Hint: The number of weights required are less than $\log_2 N$. In fact, the answer is $\log_3(2N + 1)$. How will you choose weights ? Proving that this number is optimal is a very interesting problem.

6. The people in a country are partitioned into clans(families). You can assume that each person belongs to only one clan. In order to estimate the average size of a clan, a survey is conducted where 1000 randomly selected people are asked to state the size of the clan to which they belong. How does one compute an estimate average clan size from the data collected?
   Note: The problem is not as trivial as it looks.

7. An airplane has 50 seats, and its 50 passengers have their own assigned seats. The first person to enter the plane ignores his seat assignment and instead picks a seat on random. Each subsequent person to enter the plane takes her assigned seat, if available, and otherwise chooses a seat on random. What is the probability that the last passenger gets to sit in her assigned seat?

8. You are initially located at origin in the x-axis. You start a random work with equal probability of moving left or right one step at a time. What is the probability that you will reach point a before reaching point $-b$ where $a, b \in N$.

9. Prove that there exists no integer solution to the equation $x^y = y^x$ except $x = 4$ and $y = 2$ for $x, y \in Z$, $x, y > 0$ and $x > y$.

10. Consider a random walk around the edges of a square. From any vertex, the probability of moving to any adjacent vertex is 1/2. Suppose the walk stops as soon as after all traversing through all the vertices, you return to your starting vertex. What is the expected path length?

11. For how many integers x, $x^2 - 3x - 19$ is divisible by 289?

12. For how many positive integral values of n, the expression $n(4n + 1)$ is a perfect square?

13. Suppose f is a function such that for all real x and for some relatively prime natural numbers a, b:
$f(x + a) \geq f(x) + a$ and
$f(x + b) \leq f(x) + b$
Show that $f(x + 1) = f(x) + 1$ for all x.

14. Let I be a 10 X 10 identity matrix and J be a 10 X 10 matrix of all ones. $A = 2I + 3J$. Find A^{-1}.

15. Prove that the following SHUFFLE algorithm does not generate a uniform random permutation of first n natural numbers.
SHUFFLE($A[1 \ldots n]$)
for $i = 1$ to n Swap $A[i]$ with $A[Rand(n)]$
end
Here $Rand(n)$ returns a random number between 1 to n.
Describe a correct SHUFFLE algorithm whose expected running time is $O(n)$.

16. Assume a stick is broken at random into three pieces. What is the probability that the pieces can form a triangle? Solve it in following cases:

    - Two break points are selected randomly (and distributed uniformly).
    - The stick is first broken into two pieces. The longest is then broken into two.
    - The stick is first broken into two pieces. A piece randomly selected with probability 1/2 is then broken into two.
    - The stick is first broken into two pieces. A piece randomly selected with the probability proportional to its length is then broken into two.