Indian Association for the Cultivation of Science

(Deemed to be University under the *de novo* category)

BS-MS Program

End-Semester Examination-2025 (Spring Semester-II)

Subject: Mathematics II | *Subject Code(s): MAT 1201*

Full marks: 50 | *Time allotted: 3 hrs*

Answer Question No. **10** and any **six** of the remaining questions.

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1. Let $I \subseteq \mathbb{R}$ be an open interval, let $f : I \to \mathbb{R}$ be differentiable on I. Suppose that $f''(a)$ exists at $a \in I$. Show that

$$f''(a) = \lim_{h \to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}.$$

Give an example where the limit exists, but the function does not have a second derivative at a. [7]

2. If $f : [0,1] \to \mathbb{R}$ is continuous and $\int_0^x f(t)dt = \int_x^1 f(t)dt$ for all $x \in [0,1]$, show that $f(x) = 0$ for all $x \in [0,1]$. [7]

3. Suppose that f is continuous on $[a,b]$ and that g is nonnegative and integrable on $[a,b]$. Prove that

$$\int_a^b f(x)g(x)dx = f(\xi) \int_a^b g(x)dx$$

for some $\xi \in [a,b]$. [7]

4. Find the Taylor polynomial for the function $f(x) = \arctan x$. [7]

5. Let $T : V \to W$ be a linear trasformation, where V and W are vector spaces. If V is infinite dimensional, prove that at least one of $T(V)$ or $N(T)$ is infinite dimensional. [7]

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T(\mathbf{k}) = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k},\ T(\mathbf{j} + \mathbf{k}) = \mathbf{i},\ T(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \mathbf{j} - \mathbf{k}.$$

(a) Compute $T(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ and determine the nullity and rank of T.

(b) Determine the matrix of T.

[7]

7. In the real vector space $C(1, e)$, define an inner product by the equation

$$(f, g) = \int_1^e (log x) f(x) g(x) dx.$$

(a) If $f(x) = \sqrt{x}$, compute $\|f\|$.

(b) Find a linear polynomial $g(x) = a + bx$ that is orthogonal to the constant function 1.

[7]

8. Let A and B be two matrices such that

$$AB - BA = I$$

where I is the identity matrix. What can you say about A and B. [7]

9. Assume $T : V \to V$ is a hermitian operator,

(a) Prove that T^n is hermitian for every positive integer n, and that T^{-1} is hermitian if T is invertible.

(b) What can you conclude about T^n and T^{-1} if T is skew-hermitian?

[7]

10. Given an $n \times n$ matrix A with real entries such that $A^2 = -I$. Prove the following statements about A.

(a) A is nonsingular.

(b) n is even.

(c) A has no real eigenvalues.

(d) $\det A = 1$.

[8]