EE602 - Statistical Signal Processing
Lecture 2 : Review of DSP Basics for SSP

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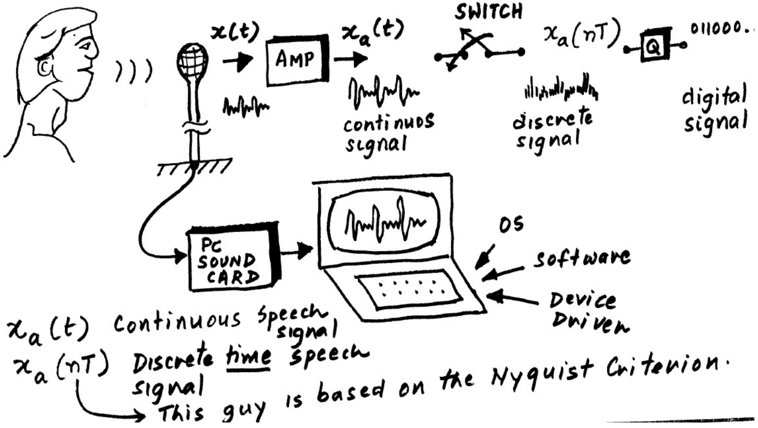
Outline

Overview of Sampling



Discrete Time Sequences

Impulse $\delta[n] = 1, \quad n = 0$
$\qquad\qquad\; = 0, \quad n \neq 0$

unit step $u[n] = 1, \quad n \geq 0$
$\qquad\qquad\;\; = 0, \quad n < 0$

$\rightarrow$ $u[n] = \sum_{k=-\infty}^{n} \delta[k]$

$\delta[n] = u[n] - u[n-1]$

Exponential $x[n] = A\alpha^n$, $x[n]$ is real, $\forall A, \alpha$ real

Sinusoidal $x[n] = A\cos(\omega n + \phi)$, A- Amplitude, ω- Frequency, ϕ- Phase offset

Note: $x[n]$ is periodic in the time variable n with period N only if $N = \frac{2\pi k}{\omega}$ where k is an integer.

contrast: $x_a(t) = A\cos(\Omega t + \phi)$ is always periodic with period $= 2\pi/\Omega$

The Complex Exponential Sequence

Complex Exponential Sequence

$$x[n] = A e^{j\omega n}$$
$$= |A| e^{j\phi} e^{j\omega n}$$
$$= |A| \cos(\omega n + \phi) + j|A| \sin(\omega n + \phi)$$

* $A = |A| e^{j\phi}$

Note : $A e^{j(\omega + \pi 2)n} = A e^{j\omega n}$

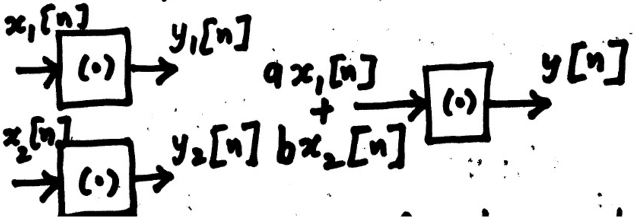
ω is the frequency variable with period 2π

Discrete Time Systems

- Discrete time systems

$$y[n] = \underbrace{T}_{\text{Transformation operator}}(x[n])$$

LTI System



Linear Discrete Time Invariant Systems

condition for linearity

$T(a x_1[n] + b x_2[n]) = aT(x_1[n]) + bT(x_2[n])$

also called 'superposition'

Condition for time Invariance

If $y[n] = T(x[n])$ then

$y[n-n_0] = T(x[n-n_0])$

Linearity + TI = LTI

An LTI system is COMPLETELY characteri
by its IMPULSE RESPONSE

Stability and Causality

system.

2 useful Properties
LTI System

1. Stability : BIBO

$|x[n]| < \infty \xrightarrow{\text{means}} |y[n]| < \infty$

Bounded Input → Bounded Output

·OR· $\sum_{\forall k} |h[n]| < \infty$

2. Causality : $h[n] = 0$, for $n < 0$ — y[n] depends on present+Past

Stability and Causality : An Example

- consider $h[n] = A \cdot a^n$, for $n \geq 0$
  $= 0$, otherwise

This is an exponentially decaying imp. res.

* $h[n] = 0$, for $n < 0$:→ system is causal
* (i) For $|a| < 1$, $\sum_{\forall n} |h[n]| = A \sum_{n=0}^{\infty} |a|^n$

From the Geom. Series relation $\sum_{n=0}^{\infty} b^n = \frac{1}{1-b}$ for $|b| < 1$

$\sum_{\forall n} |h[n]| = A / 1 - |a|$
↳ Hence stable

* But for $|a| \geq 1$, the series does not converge, hence system unstable

[If $h[n] = u[n]$ then LTI s becomes Unstable]

The DTFT Pair

Discrete Fourier transform PAIR

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(w) e^{jwn} dw$$

$$X(w) = \sum_{\forall n} x[n] e^{-jwn}$$

Recall: $e^{j(w+2\pi)n} = e^{jwn}$

* $x[n]$ = superposition of infinitesimally small exponentials $dw\, X(w) e^{jwn}$ WHERE $X(w)$ can be viewed as the "scaling" factor
* For the pair to exist $x[n]$ MUST be absolutely summable, or $x[n]$ is 'STABLE'

Properties of DTFT

i) $X(\omega) = X_r(\omega) + j X_i(\omega) = |X(\omega)| e^{j\angle X(\omega)}$

Magnitude $= \sqrt{|X_r|^2 + |X_i|^2}$, Unw. Phase $= \angle X(\omega)$

ii) $X(\omega + 2\pi) = X(\omega)$

iii) $X(\omega) = X^*(-\omega) \rightarrow$ conjugate symmetry

$|X(\omega)| = |X(-\omega)|$ & $X_r(\omega) = X_r(-\omega)$

$\angle X(\omega) = -\angle X(-\omega)$ & $X_i(\omega) = -X_i(-\omega)$

Conjugate symmetry means Mag and Real part even while phase and Imag parts are odd

. NOT CONJUGATE SYMMETRIC $\rightarrow$ SEQUENCE IS COMPLEX

Energy Density of a Signal

$$\sum_{\forall n} x[n]^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

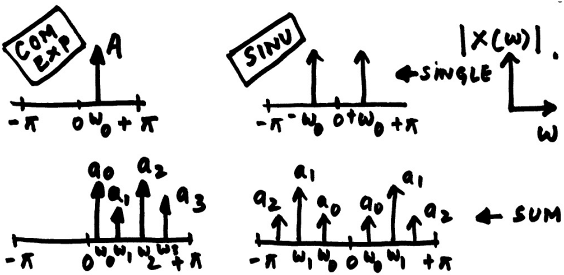
$|x[n]|^2 \rightarrow$ Energy/unit time
$|X(\omega)|^2 \rightarrow$ Energy/unit freq.

] Energy density is power at a part time/freq.

if $x[n] = \delta[n-n_0]$, $X(\omega) = e^{-j\omega n_0}$

Note that 'Energy of $x[n]$ in time domain is unity at $n = n_0$'
'Energy in the freq is distributed over $[-\pi, \pi]$ but the Average Energy = 1.

DTFT of Single and Multiple Exponential and Sinusoidal sequences



IF $x[n] \leftrightarrow X(\omega)$

$$X(\omega) = \sum_{\forall r} 2\pi \cdot A\,\delta(\omega - \omega_0 + r\,2\pi)$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\,\delta(\omega-\omega_0)\, e^{j\omega n}\, d\omega = A\, e^{j\omega_0 n}$$

Important Generalizations

a/ $$A\cos(\omega_0 n+\phi) \leftrightarrow \pi A e^{j\phi}\delta(\omega-\omega_0)+\pi A e^{-j\phi}\delta(\omega+\omega_0)$$

b/ $$\sum_{k=0}^{N} a_k e^{j\omega_k n+\phi_k} \leftrightarrow \sum_{k=0}^{N} 2\pi a_k e^{j\phi_k}\delta(\omega-\omega_k)$$

c/ $$\sum_{k=0}^{N} a_k\cos(\omega_k n+\phi_k) \leftrightarrow \sum_{k=0}^{N} \pi a_k e^{j\phi_k}\delta(\omega-\omega_k) + \pi a_k e^{-j\phi_k}\delta(\omega+\omega_k)$$

a : Sinusoidal Sequence
b : Multiple complex exponentials
c : Multiple Sinusoids

$\underbrace{\text{a, b, c}}$ → Speech is often modeled as mult. Sinus's.

The Z - Transform

$$X(z) = \sum_{\forall n} x[n] z^{-n}, \quad z = re^{jw}$$

For $r = 1$, the ZT reduces to

$$X(w) = X(z)\big|_{z = e^{jw}}$$

ZT is a generalization of the FT which makes sequences that are not absolutely summable converge

eg:

$$X(re^{jw}) = \sum_{\forall n} (x[n] r^{-n}) e^{jwn}$$

Note that $x[n]r^{-n}$ may be absolutely summable whereas $x[n]$ is NOT.

ROC - Region of Convergence

All values of r such that

$$\sum_{\forall n} |x[n] r^{-n}| < \infty \; ; \text{ Note } |z| = |re^{jw}| = r$$

The IZT and ROC

$$x[n] \longleftrightarrow X(z)$$

$$\frac{1}{2\pi j}\oint_c X(z)\, z^{n-1}\, dz \qquad \sum_{\forall n} x[n]\, z^{-n}$$

$\oint_c$ is the contour integral over a counterclk wise contour encircling the origin in the z-plane

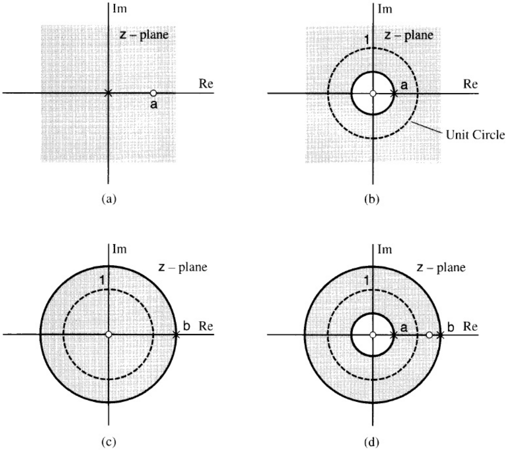
Usually $X(z) = \dfrac{P(z)}{Q(z)}$, is rational

Roots of $P(z)$ define 'zeros'
Roots of $Q(z)$ define 'poles'

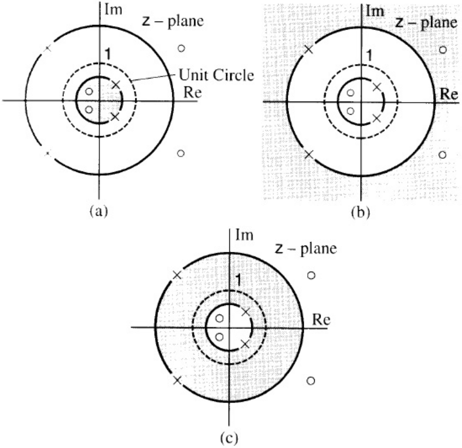
i) $\delta[n] - a\,\delta[n-1]$, ii) $a^n u[n]$, iii) $-b^n u[-n-1]$

iv) $a^n u[n] - b^n u[-n-1]$

The ROC for Four Typical Sequences



The ROC - Contd.



Transfer Function of LTI Systems

$$x[n] \rightarrow \boxed{\text{LTI S/M}} \rightarrow y(n)$$

$$y(n) = x[n] * h[n]$$

if $x[n] = e^{jwn}$

$$y[n] = x[n] * e^{jwn}$$

$$= \sum_{\forall k} h[k]\, e^{jw[n-k]}$$

$$= e^{jwn} \sum h[k]\, e^{jwk}$$

$$y[n] = e^{jwn} H(w)$$

An exponential input to an LTI S/M outputs the same fn. but scaled by $H(w)$

$\therefore$ e^{jwn} is an eigen function and $H(w)$ is the associated eigen value

Convolution and Windowing Theoroem

i) convolution theoroem

Given $x[n] \leftrightarrow X(\omega)$, $h[n] \leftrightarrow H(\omega)$

if $y[n] = x[n] * h[n]$

then $Y(\omega) = X(\omega) H(\omega)$

ii) windowing theoroem (Dual of conv. thm)

if $y[n] = x[n] w[n]$

then $$Y(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\theta) W(\omega - \theta) d\theta$$

$$= \frac{1}{2\pi} X(\omega) \circledast W(\omega)$$

$\circledast$ is circular convolution - each function defined in the interval $[-\pi, \pi]$ and is shifted mod 2π in the process

The Constant Co-efficient Linear Difference Equation

LTI systems that satisfy stability and causality are called digital filters
DIGITAL FILTERS are characterised by

$$x[n] \rightarrow \boxed{} \rightarrow y[n]$$

$$y[n] = \sum_{k=1}^{N} \alpha_k \, y[n-k] + \sum_{k=0}^{M} \beta_k \, x[n-k]$$

Taking Z transform

$$Y(z) = \sum_{k=1}^{N} \alpha_k Y(z) z^{-k} + \sum_{k=0}^{M} \beta_k X(z) z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} \beta_k z^{-k}}{1 - \sum_{k=1}^{N} \alpha_k z^{-k}}$$

Stability → All poles inside unit ole
Causality → ROC is outside outermost pole

Pole Zero concepts from the CCLDE

Factored transfer function

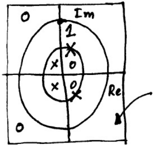
$$H(z) = A z^{-r} \frac{\prod_{k=1}^{M_i}(1-a_k z^{-1}) \prod_{k=1}^{M_o}(1-b_k z)}{\prod_{k=1}^{N_i}(1-c_k z^{-1})}$$

where $|a_k|, |b_k|, |c_k| < 1$, $M_i + M_o = M$,

$N_i = N$, $(1-a_k z^{-1}) \rightarrow$ zeros inside unit o'le

$(1-b_k z^{-1}) \rightarrow$ zeros outside ..

$(1-c_k z^{-1}) \rightarrow$ zeros inside ..



The Discrete Fourier Transform (DFT)

DFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j \frac{2\pi}{N} nk}, \quad 0 \leq n \leq N-1$$

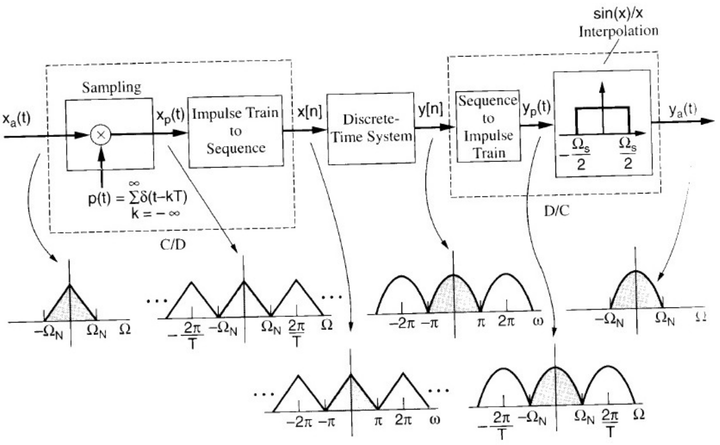
$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} nk}, \quad 0 \leq k \leq N-1$$

Note $\omega = \frac{2\pi}{N} k$

$$* \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2 \rightarrow \text{PSD}$$

NOTE : $Y(K) = X(K)\, H(K) \rightarrow y[n] = x[n] \circledast h[n]$
for $n = 0, 1, \cdots (N-1)$.

Sampling and Reconstruction of the Speech Signal



Sampling the Impulse Response

Sampling the system response

$h[n] = h_a(nT)$ (Impulse Inv. method)

Discrete ← continuous

$$\therefore H(\omega) = \frac{1}{T} H_a(\omega/T)$$

$$h_a(t) = \sum_{k=1}^{N} A_k e^{s_k t} u(t)$$

$$\therefore h[n] = h_a[nT] = \sum_{k=1}^{N} A_k e^{s_k nT} u[n]$$

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{s_k T} z^{-1}}$$

References

- T. Quatieri, *Discrete Time Speech Signal Processing* , Prentice Hall
- Rabiner and Schafer, *Discrete Time Processing of Speech Signals* , Prentice Hall