Random Variables, Vectors, Processes and Their Statistical Description

Lecture 3

Deterministic Signals : Amplitude described by math formula

Random Signals : Precise description difficult

Let $S = \{\zeta_1, \zeta_2, \ldots\}$ be Universal Set

$\mathcal{F}$ = Collection of subsets of 'S' with S itself is called σ field.

eg. $\mathcal{F} = \{(\zeta_1, \zeta_2), (\zeta_1, \zeta_2, \zeta_3)\}$

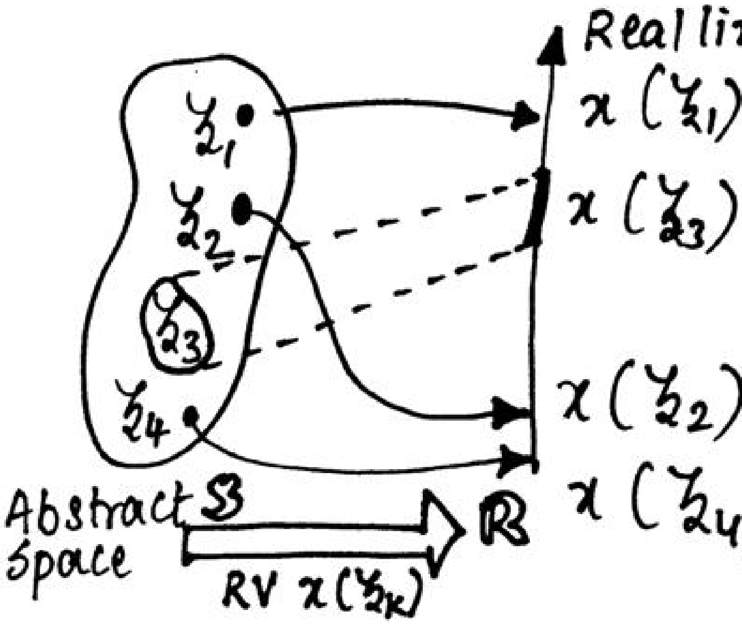
Elements of $\mathcal{F}$ called Events

$Pr\{\zeta_k\}, k = 1, 2, \cdots$ is called Probability of event ζ_k

The Probability Space

Probability space = $(S, \mathcal{F}, Pr)$

* Basic Space contains abstract events/outcomes difficult to manipulate
* $Pr\{\cdot\}$ function difficult to manipulate
  So 'RV'.



R.V. $x(\zeta_k)$ is a mapping

$$S \Longrightarrow \mathbb{R}$$

Assigns x to every ζ_k (event) which satisfies

(1) $x(\zeta) \leq x$ is an event $\forall x$

(2) $Pr(x(\zeta_k) = \infty) = 0$ and $Pr(x(\zeta_k) = -\infty) = 0$.

Complex Random Variable

Complex RV $\quad x(\zeta) = \underbrace{x_R(\zeta)}_{\text{real valued RV}} + j\,\underbrace{x_I(\zeta)}_{\text{real valued RV}}$

RV – is neither 'R' nor 'V', but a mapping

However $\quad \underbrace{x(\zeta_k)}_{\text{RV}} = \underbrace{x}_{\text{value of RV}}$

if x is discrete valued $\{x_k\}$; Discrete RV

else Continuous RV.

There are Mixed RV.

CDF, PDF, and PMF

- $\boxed{\begin{array}{c}\text{CDF}\\ \text{of } x(\zeta)\end{array}}$: $F_x(x) = Pr\{x(\zeta) < x\}$

where $Pr\{x(\zeta) \leq x\}$ is a function of the set $\{x(\zeta) \leq x\}$

$\boxed{\begin{array}{c}\text{Pdf of}\\ x(\zeta)\end{array}}$ $f_x(x) = \dfrac{d}{dx} \underbrace{F_x(x)}_{\text{CDF}}$

But $f_x(x) \underbrace{\Delta x}_{\text{interval}} = \Delta F_x(x) = F_x(x+\Delta x) - F_x(x)$

$= \underbrace{Pr\left(x < x(\zeta) \leq x + \Delta x\right)}_{\text{Probability}}$

Properties

$$f_X(x) = \frac{d}{dx} F_X(x) \text{ ; Integrate both sides}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(v)\, dv$$

For discrete RV : pmf $p_k = P_r\{X(\zeta) = x_k\}$

Properties :

$0 \le F_X(x) \le 1$; $F_X(-\infty) = 0$; $F_X(\infty) = 1$;

$f_X(x) \geq 0$; $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

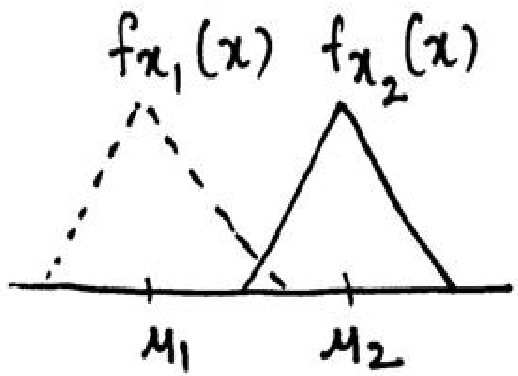
$$\therefore P_r\{x_1 < X(\zeta) \le x_2\} = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)\, dx$$

Moments of RV/PDF

RV^s characterized by Pdf
Pdf^s represented by Moments

$$E\{x(\zeta)\} = \underline{\mu_x} = \sum_k x_k p_k \quad ; \quad x(\zeta) \text{ discrete}$$

$$\int_{-\infty}^{\infty} x f_x(x)\, dx \quad ; \quad x(\zeta) \text{ Continuous}$$



$f_x(x)$ is symmetric abt. 'a'
then $\mu_x = a$

Moments

Note: $$E\left[\alpha x(\zeta) + \beta\right] = \alpha \mu(x) + \beta$$

* $$\underbrace{E\left[g(x(\zeta))\right]}_{y(\zeta)\text{ - which is a transformation of } x(\zeta)} = \int_{-\infty}^{\infty} g(x)\, f_x(x)\, dx$$

$$E\left[x^m(\zeta)\right] = \int_{-\infty}^{\infty} x^m f_x(x)\, dx = r_x^{(m)}$$

$r_x^{(m)}$ = m^{th} order moment of $x(\zeta)$

$r_x^{(2)}$ = Mean squared value. $\left[\; E(x^2(\zeta)) \neq E^2(x(\zeta))\right]$

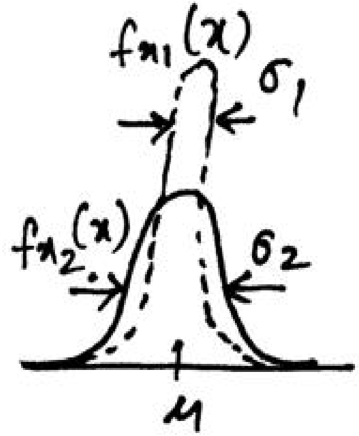
Central Moments

- Central moments : $\gamma_x^{(m)} = E\{[x(\zeta) - \mu_x]^m\}$

  m^{th} order central moment : $\gamma_x^{(m)} = (x - \mu_x)^m f_x(x)\, dx.$

  $\gamma_x^{(2)} = \sigma_x^2 = \text{var}[x(\zeta)] = E\{[x(\zeta) - \mu_x]^2\}$

  $\sigma_x = \sqrt{\gamma_x^{(2)}} = \text{Std. Deviation}$



Also $\underbrace{\gamma_x^{(m)}}_{\text{central moment}} = \sum_{k=0}^{m} \underbrace{\binom{m}{k}(-1)^k \mu_x^k\, r_x^{(n-k)}}_{\text{moments}}$

$$\sigma_x^2 = r_x^{(2)} - \mu_x^2 = E[x^2(\zeta)] - E^2[x(\zeta)]$$

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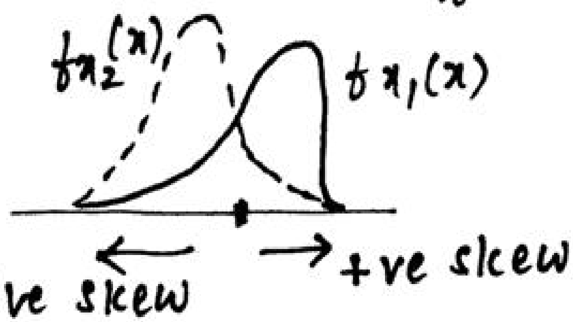
Higher Order Moments

Skewness : Degree of asymmetry around 'μ' of the [illegible]

Normalized 3^{rd} order central moment

$$\text{Skew } \tilde{\kappa}_x^{(3)} = E\left\{\left[\frac{x(\zeta)-\mu_x}{\sigma_x}\right]^3\right\} = \frac{\gamma_x^{(3)}}{\sigma_x^3}.$$



Kurtosis: Relative flatness of a distn. about μ

4^{th} order : $\bar{\kappa}_x^{(4)} = E\left\{\left[\frac{x(\zeta)-\mu_x}{\sigma_x}\right]^4\right\} - 3 = \frac{\gamma_x^{(4)}}{\sigma_x^4} - 3$

Characteristic Function

- characteristic function of $\underbrace{x(\zeta)}_{RV}$

$$\underbrace{\phi_x(\xi)}_{F.T} = E\left\{ e^{j\xi x(\zeta)} \right\} = \int_{-\infty}^{\infty} f_x(x)\, e^{j\xi x}\, dx \tag{1}$$

Replace ξ by s, we have

moment gen. fn. $\underbrace{\bar{\Phi}_x(s)}_{L.T} = E\left\{ e^{s x(\zeta)} \right\} = \int_{-\infty}^{\infty} f_x(x)\, e^{sx}\, dx$

$$\underbrace{r_x^{(m)}}_{m^{th} \text{ order moment of } x(\zeta)} = \frac{d^m}{ds^m}\left[\bar{\Phi}_x(s)\right]\Big|_{s=0} = (-j)^m \frac{d^m}{d\xi^m}\left[\phi_x(\xi)\right]\Big|_{\xi=0}$$

where $m = 1, 2, \cdots$

Cumulants of RV

Cumulant generating function

$$\bar{\Psi}_x(s) = \ln \bar{\Phi}_x(s) = \ln E\{e^{s x(\zeta)}\}$$

$\Psi_x(\xi)$ = 2nd characteristic function

Cumulants of RV $x(\zeta)$

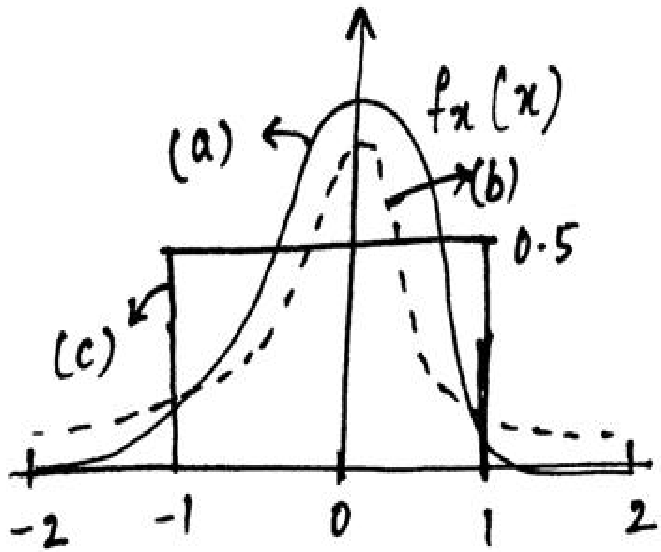
$$\kappa_x^{(m)} = \frac{d^m}{ds^m}\left[\bar{\Psi}_x(s)\right]\Big|_{s=0} = (-j)^m \frac{d^m}{d\xi^m}\left[\Psi_x(\xi)\right]\Big|_{\xi=0}$$

for $m = 1, 2, 3, \cdots$

$\underline{\kappa_x^{(0)}} = 0$; $\underline{\kappa_x^{(1)}} = r_1(x) = \mu_x = 0$; $\underline{\kappa_x^{(2)}} = r_x^{(2)} = \sigma_x^2$

$\underline{\kappa_x^{(3)}} = r_x^{(3)}$; $\underline{\kappa_x^{(4)}} = r_x^{(4)} - 3\sigma_x^4$; ....

Normal, Cauchy, Uniform RV



$N(\mu_x, \sigma_x^2)$

(a) $f_x(x) = \dfrac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right]$

$-\infty < \mu < \infty$; $\sigma \geq 0$

$\phi_x(\xi) = \exp\left(j\mu_x \xi - \frac{1}{2}\sigma_x^2 \xi^2\right)$

$\gamma_x^{(m)} = E\{[x(\zeta) - \mu_x]^m\}$

$= \begin{cases} 1 \cdot 3 \cdot 5 \cdots (m-1)\sigma_x^m & ; m \text{ even} \\ 0 & ; m \text{ odd} \end{cases}$

But $\gamma_x^{(4)} = \underline{3}\sigma_x^4$ ✓

$\therefore$ kurtosis $= 0$.

(b) Cauchy RV and (c) Uniform

Random Vectors

R.V. ✓, Now lets see Random Vector ($\vec{RV}$)

$$\vec{RV} = \vec{x}(\zeta) = [x_1(\zeta), x_2(\zeta), \cdots, x_M(\zeta)]^T$$

$\vec{RV}$ characterized by 'jt. cdf' as

$$F_x(x_1, \cdots, x_M) = P_r\{x_1(\zeta) < x_1, \cdots, x_M(\zeta) \leq x_M\}$$

'or' $F_x(x) = P_r\{x(\zeta) \leq x\}$

$\vec{RV}$ characterized by jt. pdf as

$$\underbrace{f_x(x)}_{\text{mdf}} = \frac{\partial}{\partial x_1} \cdots \cdots \frac{\partial}{\partial x_M} F_x(x)$$

Individual RV $\therefore f_{x_j}(x_j) = \underbrace{\int \cdots \int}_{(M-1)} f_x(x)\, dx_1 \cdots dx_j \cdots dx_M$

Characterization of Random Vectors

$$F_{\mathbf{x}}(\mathbf{x}) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_M} f_{\mathbf{x}}(\mathbf{v})\, dv_1 \cdots dv_m = \int_{-\infty}^{\mathbf{x}} f_{\mathbf{x}}(\mathbf{v})\, d\mathbf{v}$$

$*$ If $\Pr\{x_1(\zeta) \le x_1,\ x_2(\zeta) \le x_2\}$

$$= \Pr\{x_1(\zeta) \le x_1\} \Pr\{x_2(\zeta) \le x_2\}$$

$$\Rightarrow \quad F_{x_1 x_2}(\cancel{\mathbf{x}})\,(x_1, x_2) = F_{x_1}(x_1)\, F_{x_2}(x_2) \ * \text{ and}$$

$$f_{x_1 x_2}(x_1, x_2) = f_{x_1}(x_1)\, f_{x_2}(x_2)$$

Complex Random Vectors

Complex RV and $\vec{RV}$

Map space 'S' $\rightarrow$ complex space 'C'

$$\mu = E\left[x(\zeta)\right] = E\left[x_R(\zeta) + j\,x_I(\zeta)\right] = \mu_{x_R} + j\,\mu_{x_I}$$

$$\sigma_x^2 = E\{|x(\zeta) - \mu_x|^2\} = E\{|x(\zeta)|^2\} - |\mu_x|^2$$

complex $\vec{RV}$ $x(\zeta) = x_R(\zeta) + x_I(\zeta) = \begin{bmatrix} x_{R_1} \\ \vdots \\ x_{R_M} \end{bmatrix} + j \begin{bmatrix} x_{I_1}(\zeta) \\ \vdots \\ x_{I_M}(\zeta) \end{bmatrix}$

cdf, marginal pdf, etc for complex $\vec{RV}$ is simple extension to scalar case.

Statistical Description of Random Vectors

- statistical description of $\vec{RV}$

(a) Mean Vector $\mu_x = E\{x(\zeta)\} = \begin{bmatrix} E\{x_1(\zeta)\} \\ E\{x_2(\zeta)\} \\ \vdots \\ E\{x_M(\zeta)\} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_M \end{bmatrix}$

(b) Second order moments

Auto Correlation matrix

$$R_x = E\{x(\zeta)\, x^H(\zeta)\} = \begin{bmatrix} r_{11} & \cdots & r_{1M} \\ \vdots & & \vdots \\ r_{M1} & & r_{MM} \end{bmatrix}$$

2nd order moments : $r_{ii} = E\{|x_i(\zeta)|^2\} = r_{x_i}^{(2)}$

$$r_{ij} = E\{x_i(\zeta)\, x_j^*(\zeta)\} = r_{ji}^*, \quad i \neq j$$

- is the correlation and matrix $R_x = R_x^H$

Co-variance between RV

AutoCovariance Matrix

$$\Gamma_x = E\{[x(\zeta)-\mu_x][x(\zeta)-\mu_x]^H\} = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1M} \\ \vdots & \ddots & \vdots \\ \gamma_{M1} & \cdots & \gamma_{MM} \end{bmatrix}$$

$$\underbrace{\gamma_{ii}}_{\text{self variance } (\sigma_{x_i}^2)} = E\{|x_i(\zeta)-\mu_i|^2\} \qquad i = 1, 2, \cdots M$$

Covariance between $x_i(\zeta)$ and $x_j(\zeta)$

$$\gamma_{ij} = \{[x_i(\zeta)-\mu_i][x_j(\zeta)-\mu_j]^*\}$$

$$= E[x_i(\zeta)\, x_j^*(\zeta)] - \mu_i \mu_j^* = \gamma_{ji}^*$$

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Also
Correlation Co-eff $\rho_{ij} = \dfrac{\gamma_{ij}}{\sigma_i \sigma_j} = \rho_{ji}$ $\left[\begin{array}{l} \rho_{ij} = 1 \text{ ; perfect corr.} \\ \quad\;\; = 0 \text{ ; uncorrelated} \end{array} \right.$

Correlation between RV

$$\Gamma_x = E\left\{ [x(\zeta) - \mu_x][x(\zeta) - \mu_x]^H \right\}$$

$$\underbrace{\Gamma_x}_{\text{Auto Cov.}} = \underbrace{R_x}_{\text{Auto Corr.}} - \mu_x \mu_x^H$$

* uncorrelated does not imply independent
  But vice versa is true

$$E[x_i(\zeta)\, x_j^*(\zeta)] = E[x_i(\zeta)]\, E[x_j^*(\zeta)]$$

From ⓐ $\underline{\gamma_{ij} = 0}$

* RV '<u>Orthogonal</u>' if correlation

$$r_{ij} = E\ x_i(\zeta)\, x_j^*(\zeta) = 0;\ i \neq j$$

Linear transformation of $\vec{RV}$

$$y(\zeta) = g[x(\zeta)] = \underbrace{A}_{[L \times M]} x(\zeta)$$

$\vec{RV}$ $y(\zeta)$ is completely characterized by $f_Y(Y)$ the pdf.

Assume 'L = M' and $\underbrace{\text{'A'}}_{\text{Matrix}}$ non singular, real valued

$$\therefore \quad f_Y(Y) = \frac{f_x(g^{-1}(y))}{|J|} \; ; \; J = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_M}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_M} & \cdots & \frac{\partial y_M}{\partial y_M} \end{bmatrix} = |\det A|$$

Statistical Description of LTRV

- $$\therefore f_y(y) = \frac{f_x(A^{-1}y)}{|\det A|} \; ; \; (\text{Real valued } \vec{RV})$$

  $$= \frac{f_x(A^{-1}y)}{|\det A|} \; ; \; (\text{complex valued } \vec{RV})$$

(a) Mean : $\mu_y = E\{y(\zeta)\} = A\,E\{x(\zeta)\} = A\mu_x$

(b) AutoCorr: $R_y = E\{yy^H\} = E\{Axx^HA^H\} = AR_xA^H$.

(c) AutoCov: $\Gamma_y = A\Gamma_xA^H$

(d) Cross Correlation Matrix: $R_{xy} = E\{x(\zeta)x^H(\zeta)A^H\} = R_xA^H$

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Normal $\vec{RV}$ (Real valued)

$$f_x(x) = \frac{1}{(2\pi)^{M/2} |\Pi_x|^{1/2}} \exp\left[-\frac{1}{2} (x-\mu_x)^T \Pi_x^{-1} (x-\mu_x) \right]$$

Normal $\vec{RV}$ (complex valued)

$$f_x(x) = \frac{1}{\pi^M |\Pi_x|} \exp\left[-(x-\mu_x)^H \Pi_x^{-1} (x-\mu_x) \right]$$

μ_x : Mean ; Π_x : Covariance Matrix

Plugging in mean of RV $x\,(\frac{1}{2}) = \mu_x$ and var $= \sigma_x^2$ gives scalar case (Check!)

Properties of normal distribution of a $\vec{RV}$

(1) Pdf is completely specified by mean and Cov. matrix

(b) If Components of x ($\frac{4}{2}$) are mutually uncorrelated then they are also independent

(c) Linear transformation of a normal $\vec{RV}$ is also normal (Plug y in place of x)

(d) Fourth order moment can be expressed in terms of second order moments.

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Sums of Independent Random Variables

Sums of Independent RV ($y(\zeta)$ as $\{x_k(\zeta)\}_1^M$

$$y(\zeta) = c_1 x_1(\zeta) + c_2 x_2(\zeta) + \cdots + c_M x_M(\zeta)$$

$$y(\zeta) = \sum_{k=1}^{M} c_k x_k(\zeta)$$

Mean:

$$\mu_y = \sum_{k=1}^{M} c_k \mu_{x_k}$$

Variance:

$$\sigma_y^2 = E\left\{ \left| \sum_{k=1}^{M} c_k \left[x_k(\zeta) - \mu_{x_k} \right] \right|^2 \right\}$$

$$\sigma_y^2 = \sum_{k=1}^{M} |c_k|^2 \sigma_{x_k}^2$$ [using statistical independence] of variables

Pdf:

Let $y(\zeta) = x_1(\zeta) + x_2(\zeta)$

Characteristic Functions

- 1^{st} characteristic function : $\phi_y(\xi) = E\left[e^{j\xi y(\xi)}\right]$

$$= E\left[e^{j\xi(x_1(\xi)+x_2(\xi))}\right] \overset{\because \text{Ind.}}{=} E\left[e^{j\xi x_1(\xi)}\right] E\left[e^{j\xi x_2(\xi)}\right]$$

$\phi_y(\xi) = \phi_{x_1}(\xi)\,\phi_{x_2}(\xi)$; using Convolution property we have

$$f_y(y) = f_{x_1}(y) * f_{x_2}(y)$$

2^{nd} characteristic function : $\Psi_y(\xi) = \Psi_{x_1}(\xi) + \Psi_{x_2}(\xi)$

m^{th} Order Cumulant of $y(\xi)$

$$k_m^{(y)} = k_m^{(x_1)} + k_m^{(x_2)}$$

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- If f[redacted](x) is "PDF of a" uniform random variable $[x_k(1/2)]_{k=1}^{4}$. and $Y_M(1/2) = \sum_{k=1}^{M} x_k$; $M = 2, 3, 4$.

If we start $f_{Y_2}(y)$ and go on to $f_{Y_4}(y)$ i.e, as 'M' increases pdf gets closer to Gaussian pdf.

Stable and Infinitely divisible distributions as assignment (reading)

- ~~Stable → Means preserved~~

- Stable distn. → distn. preserved under convolution (self reproduce)

eg. Gaussian pdf has finite variance σ^2 and stable

* Central Limit theorem (CLT)

If $y(\zeta) = \sum_{k=1}^{M} c_k x_k(\zeta)$; then does the cdf converge as $M \to \infty$. If each $x(\zeta)$ is IID-Stable and $\mu_{x_k} < \infty$, $\sigma_{x_k}^2 < \infty$ then it does. Else?

CLT: $$Y_M(\zeta) = \frac{\sum_{k=1}^{M} x_k(\zeta) - \mu_{y_M}}{\sigma_{y_M}}$$ [Distn. of Normalized sum]

converges to that of a "Normal RV" with zero mean and unit SD as $M \to \infty$

Stochastic Processes

Extend concept of RV and $\vec{RV}$ to "**sequences**".

Sample space $S = \{ \zeta_1, \zeta_2, \cdots]$ occuring with a probability $P_r\{\zeta_k\}$, $k = 1, 2, \ldots$

Define a Rule that assigns each ζ_k to a sequence $x(n, \zeta_k)$, $-\infty < n < \infty$

$\{ S, P_r, x(n, \zeta) \}$ constitute a DT Stochastic process

Defn: $x(n, \zeta)$, $-\infty < n < \infty$, is a random sequence if for a fixed n_0, $x(n_0, \zeta)$ is a RV

Summary

Set of all $\{x(n,\zeta)\}$ is called ensemble

Each $x(n,\zeta_k)$ is called realization of the sample sequence.

		n	ζ
$x(n,\zeta)$	RV	fixed	variable
	SS	variable	fixed
	N	fixed	fixed
	SP	variable	variable

RV: Rand. variable

SS: Sample seq.

N: Number

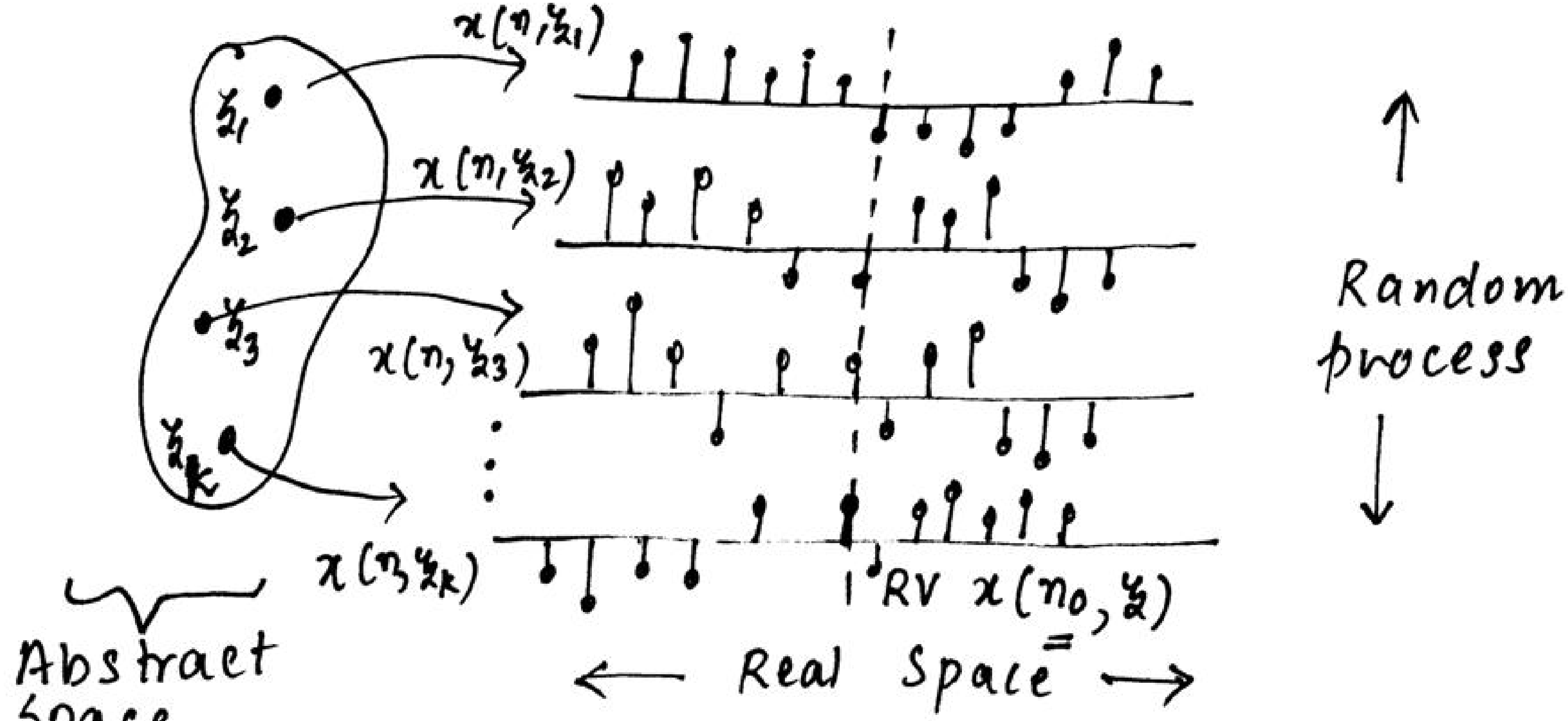
SP: Stoch. Process

Random Sequence is also called time series

Random/Stochastic process are same

Random Sequences

Realization of a Random Process = Random sequence



Random Signal – Predictable then underlying process deterministic else Regular

Description of Random Processes

- Random signal characterized by
  1) Pdf 2) Correlation 3) Signal power
  4) Histograms

Description with cdf (K^{th} order)

$$F_x(x_1, \cdots x_K; n_1, \cdots, n_K) = P_r\{x(n_1) \le x_1, \cdots, x(n_K) \le x_K\}$$

with Pdf

$$f_x(x_1 \cdots x_K; n_1, \cdots, n_K) = \frac{\partial^{2K} F_x(x_1, \cdots, x_K; n_1, \cdots, n_K)}{\partial x_{R1} \cdots \partial x}$$

However random processes described by 'averages'.

Description of Random Processes

- Statistical Description of random process
  Notation : use $\underline{x(n)}$ to describe $x(n, \zeta)$ and also single realization $x(n)$

  Second order Statistical description

$$\mu_x(n) = E[x(n)] = E\{x_R(n) + j\, x_I(n)]$$

$$\sigma_x^2(n) = E\left[|x(n) - \mu_x(n)|^2\right] = E\left[|x(n)|^2\right] - |\mu_x(n)|^2.$$

  Autocorrelation sequence $r_{xx}(n)$ described as

$$r_{xx}(n_1, n_2) = E\left[x(n_1)\, x^*(n_2)\right]$$

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Description of Random Processes

- Auto Covariance of $x(n)$

$$\gamma_{xx}(n_1, n_2) = E\left[(x(n_1) - \mu_x(n_1))(x(n_2) - \mu_x(n_2))^* \right]$$

$$\gamma_x(n_1, n_2) = r_{xx}(n_1, n_2) - \mu_x(n_1)\,\mu_x^*(n_2)$$

Cross correlation: $r_{xy}(n_1, n_2) = E\left[x(n_1)\, y^*(n_2)\right]$

Cross covariance: $\gamma_{xy}(n_1, n_2) = r_{xy}(n_1, n_2) - \mu_x(n_1)\,\mu_y^*(n_2)$

Normalized Cross correlation: $\sigma_{xy}(n_1, n_2) = \dfrac{\gamma_{xy}(n_1, n_2)}{\sigma_x(n_1)\,\sigma_y(n_2)}$

Properties of Stochastic processes

Independent process

a) $f_x(x_1, \cdots, x_k ; n_1, \cdots, n_k) = f_1(x_1; n_1), \cdots, f_k(x_k; n_k)$

$\forall k, n_i, i = 1, \cdots, k$

b) If all random variables have same pdf $f(x), \forall k$ then $x(n)$ is <u>I.I.D</u>

c) Uncorrelated process: $\gamma_x(n_1, n_2) = \begin{cases} \sigma_x^2(n_2) ; & n_1 = n_2 \\ 0 ; & n_1 \neq n_2 \end{cases}$

d) Orthogonal Process: $r_x(n_1, n_2) = \begin{cases} \sigma_x^2(n_1) + |\mu_x(n_1)|^2 ; & n_1 = n_2 \\ 0 ; & n_1 \neq n_2 \end{cases}$

Properties of Stochastic Processes

- Wide sense periodic : if $\mu_x(n) = \mu_x(n+N), \forall n$

  and $r_x(n_1, n_2) = r_x(n_1+N, n_2) = r_x(n_1, n_2+N)$
  $= r_x(n_1+N, n_2+N)$

joint stochastic Processes (Properties)

a) $f_{xy}(x, y; n_1, n_2) = f_x(x; n_1)\, f_y(y; n_2)$:→ Independent

b) $\gamma_{xy}(n_1, n_2) = 0 \; ; \forall n_1 \neq n_2$:→ Uncorrelated

c) $r_{xy}(n_1, n_2) = 0 \; ; \forall \; n_1 \neq n_2$:→ Orthogonal

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Stationarity of Random process

Statistical description of $x(n)$ = S.D. of $x(n+k)$, $\forall k$

a) Order 'N' Stationary

$$f_x(x_1, \cdots, x_N; n_1, \cdots, n_N) = f_x(x_1, \cdots, x_N; n_{1+k}, \cdots, n_{N+k})$$

If $x(n)$ is stationary for all orders $N = 1, 2, \cdots$ then "Strict sense stationary"

b) WSS (Wide Sense Stationary)

Stationary upto order $N=2$.

i)

Stationarity (Contd.)

- A random signal WSS if

  i) Mean is a constant, independent of 'n' and Var

  $$E\{x(n)\} = \mu_x .$$

  $$Var[x(n)] = \sigma_x^2 .$$

  ii) AutoCorrelation depends on the lag

  $$l = n_1 - n_2$$

  $$r_x(n_1, n_2) = r_x(n_1 - n_2) = r_x(l) = E\{x(n+l)\,x^*(n)\}$$

  $$= E\{x(n)\,x^*(n-l)\}$$

- Alternately AutoCovariance $\gamma_x(l) = r_x(l) - |\mu_x|^2 .$

Stationarity (Contd.)

Weiner Process : $x(n)$ is running sum of independent steps or Increments

Jointly WSS ($x(n)$ and $y(n)$) if

$$\gamma_{xy}(l) = E\{x(n)\,y^*(n-l)\} = r_{xy}(l) - \mu_x \mu_y^*$$

Properties of Autocorrelation sequences

(i) $\gamma_x(0) = \sigma_x^2 + |\mu_x|^2 \geq 0$

$\gamma_x(0) \geq |\gamma_x(l)|, \ \forall l$

$|\mu_x|^2$ = Av. DC Power

$\sigma_x^2 = \gamma_x(0)$; Av. AC Power

$r_x(0)$ = Total Av. power.

ii) $\gamma_x(-l) = \gamma_x(l)$: conjugate symmetric about lag.

Ergodicity

- Asymptotic Stationary : If statistics of $x(n)$ and $x(n+k)$ become stationary as $k \to \infty$

  Ergodicity : Ergodic means all statistical info can be obtained from any single representative member of the ensemble

  So we now replace operations or Expectation over an ensemble to a single realization called Time Average $= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (\cdot)$

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Ergodicity

<u>Time Averages</u> on single realizations of a random process

Mean $= \langle x(n) \rangle$; Mean Square $\langle |x(n)|^2 \rangle$

Variance $= \langle | x(n) - \langle x(n) \rangle |^2 \rangle$

AutoCorrelation $= \langle x(n) x^*(n-l) \rangle$

AutoCovariance $= \langle [x(n) - \langle x(n) \rangle] [x(n-l) - \langle x(n) \rangle]^* \rangle$

Ergodic Random Process : Random process with a single sufficient realization

$x(n)$ is Ergodic if its "Ensemble Average = Time Av."

Ergodicity

* Ergodic in <u>mean</u> : if $\langle x(n) \rangle = E\{x(n)\}$

* Ergodic in correlation : if $\langle x(n)\, x^*(n-l) \rangle = E\{x(n)\, x^*(n-l)\}$

* Joint Ergodicity : Two random signals are jointly ergodic if i) they are individually ergodic

ii) $\langle x(n)\, y^*(n-l) \rangle = E\{x(n)\, y^*(n-l)\}$

True Estimate in practice of Time Average

$$= \frac{1}{2N+1} \sum_{n=-N}^{N} (\cdot)$$

ERGODICITY :

" ONE REALIZATION OF THE RANDOM SIGNAL $x(n)$ as $n \to \infty$ TAKES ON VALUES WITH THE SAME STATISTIC as $x(n_1)$ at $n = n_1$ "

- Discuss Random signal variability with Figure 3.8 in Kogon.

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Frequency domain description of Stationary Processes

PSD : Fourier transform of its Autocorrelation sequence $r_x(l)$

Zero Mean Non periodic $R_x(e^{jw}) = \sum_{l=-\infty}^{\infty} r_x(l)\, e^{-jwl}$ [w: freq. in radians per sample]

Non Zero/Zero Mean Periodic

$$R_x(e^{jw}) = \sum_i 2\pi A_i\, \delta(w - w_i)$$

A_i : Amplitudes at frequencies at w_i

and $r_x(l) = 1/2\pi \int_{-\pi}^{\pi} R_x(e^{jw})\, e^{jwl}\, dw$

Properties of PSD

(i) $R_x(e^{jw})$ is a real valued periodic function

(ii) If $x(n)$ is real, then $R_x(e^{jw}) = R_x(e^{-jw})$

iii) $R_x(e^{jw}) \geq 0$; Non Negative

iv) Area under $R_x(e^{jw})$ = Av. Power of $x(n)$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} R_x(e^{jw})\, dw = r_x(0) = E\{|x(n)|^2\} \geq 0$$

Other Random Sequences

White Noise

$$w(n) \sim WN(\mu_w, \sigma_w^2)$$

'iff' $E\{w(n)\} = \mu_x$ and

$$r_w(l) = E\{w(n)\, w^*(n-l)\} = \sigma_w^2 \delta(l)$$

or $R_w(e^{jw}) = \sigma_w^2 \; ; \quad -\pi \le w \le \pi$

Strict White Noise

$$w(n) \sim IID(\mu_w, \sigma_w^2)$$

Harmonic Process

$$x(n) = \sum_{k=1}^{M} A_k \cos(\omega_k n + \phi_k) \; ; \quad \omega_k \neq 0$$

. A_k, $\{\omega_k\}_1^M$ are constants, $\{\phi_k\}_1^M$ is a RV in $[0, 2\pi]$

Harmonic Process (contd.)

* $E\{x(n)\} = 0$; * ACF $r_x(l) = \frac{1}{2}\sum_{k=1}^{N} A_k^2 \cos \omega_k l$;

$-\infty < l < \infty$

PSD: $R_x(e^{j\omega}) = \sum_{k=-M}^{M} 2\pi\left(\frac{A_k^2}{4}\right)\delta(\omega-\omega_k) = \sum_{k=-M}^{M} \frac{\pi}{2} A_k^2 \delta(\omega-\omega_k)$

Generalized Harmonic Process

$x(n) = \sum_{k=1}^{M} A_k e^{j(\omega_k n + \phi_k)}$

and $r_x(l) = \sum_{k=1}^{M} |A_k|^2 e^{j\omega_k l}$

Power Spectrum = M impulses at ω_k with amp. $2\pi|A_k|^2$.

- Cross PSD (ZM and jointly stationary stoch. processes)

$$R_{xy}(e^{jw}) = \sum_{\forall l} r_{xy}(l)\, e^{-jwl}$$

and $$r_{xy}(l) = \frac{1}{2\pi}\int_{-\pi}^{\pi} R_{xy}(e^{jw})\, e^{jwl}\, dw$$

Since $r_{xy}(l) = r_{yx}^{*}(-l)$; $R_{xy}(e^{jw}) = R_{yx}^{*}(e^{jw})$

Normalized Cross Spectrum (Coherence)

$$G_{xy}(e^{jw}) = \frac{R_{xy}(e^{jw})}{\sqrt{R_x(e^{jw})}\sqrt{R_y(e^{jw})}}$$

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- Cross Spectral density functions (complex)

$$R_x(z) = \sum_{\forall l} r_x(l)\, z^{-l} \quad ; \text{ complex PSD}$$

$$R_{xy}(z) = \sum_{\forall l} r_{xy}(l)\, z^{-l} \quad ; \text{ complex CPSD}$$

assuming $z = e^{j\omega}$, is within the ROC and $r_x(l)$ and $r_{xy}(l)$ are absolutely summable

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