Mathematics of Kalman Filtering for IMU-based State Estimation

Pranjay Yelkotwar

1 Introduction

This document outlines the mathematical framework underlying Kalman filtering for state estimation using data from an Inertial Measurement Unit (IMU). A Kalman filter is a method that combines predictions based on previous states with noisy measurements to estimate the state of a moving object, such as its position, velocity, and orientation.

2 What is the "State"?

The **state** of a system is a complete description of its essential characteristics at a given moment. In the context of tracking a moving object, such as a car, the state includes:

- **Position**: $\mathbf{p} \in \mathbb{R}^3$ (X, Y, Z coordinates)
- **Velocity**: $\mathbf{v} \in \mathbb{R}^3$ (speed in the X, Y, and Z directions)
- **Orientation**: $\mathbf{q} \in \mathbb{R}^4$ (represented as a quaternion)

The quaternion is crucial for accurately representing orientation in 3D space while avoiding singularities and inefficiencies.

3 Inertial Measurement Unit (IMU)

An IMU provides key measurements that help in state estimation:

- **Specific force**: $\mathbf{f} \in \mathbb{R}^3$ (acceleration excluding gravity)
- **Angular velocity**: $\boldsymbol{\omega} \in \mathbb{R}^3$ (rate of change of orientation)

These measurements allow us to estimate how the object's position and orientation change over time.

4 Kalman Filter Process

The Kalman filter operates in two main steps: prediction and update.

4.1 Prediction Step

In the prediction step, we estimate the new state of the system based on the previous state and IMU measurements:

$$
\begin{aligned}
\mathbf{p}_k &= \mathbf{p}_{k-1} + \Delta t \cdot \mathbf{v}_{k-1} + \frac{1}{2}\Delta t^2 (\mathbf{R}(\mathbf{q}_{k-1})\mathbf{f}_{k-1} + \mathbf{g}) \\
\mathbf{v}_k &= \mathbf{v}_{k-1} + \Delta t \cdot (\mathbf{R}(\mathbf{q}_{k-1})\mathbf{f}_{k-1} + \mathbf{g}) \\
\mathbf{q}_k &= \mathbf{q}_{k-1} \otimes \begin{bmatrix} 1 \\ \frac{1}{2}\boldsymbol{\omega}_{k-1}\Delta t \end{bmatrix}
\end{aligned}
$$

Here: - Δt is the time step. - $\mathbf{R}(\mathbf{q})$ is the rotation matrix derived from the quaternion $\mathbf{q}$. - $\mathbf{g}$ is the gravity vector. - $\otimes$ denotes quaternion multiplication.

The predicted state reflects the object's expected position, velocity, and orientation based on its previous state and the forces acting on it.

4.2 Covariance Propagation

The uncertainty associated with the state estimate is represented by a covariance matrix $\mathbf{P}_k$. To propagate this uncertainty, we use:

$$
\mathbf{P}_k = \mathbf{F}_k \mathbf{P}_{k-1} \mathbf{F}_k^T + \mathbf{L}_k \mathbf{Q}_k \mathbf{L}_k^T \tag{1}
$$

Where:

- $\mathbf{P}_k$ is the predicted covariance matrix.
- $\mathbf{F}_k$ is the state transition matrix, modeling how the state evolves from time step $k-1$ to k.
- $\mathbf{Q}_k$ is the process noise covariance matrix, capturing uncertainty from unmodeled dynamics.
- $\mathbf{L}_k$ is the noise Jacobian, relating the process noise to the state variables.

This equation helps track how uncertainty grows over time, considering the nature of the forces acting on the system.

4.3 Update Step

When new measurements become available, we update our state estimate:

$$
\begin{aligned}
\mathbf{K}_k &= \mathbf{P}_k \mathbf{H}^T (\mathbf{H}\mathbf{P}_k\mathbf{H}^T + \mathbf{R})^{-1} \\
\hat{\mathbf{x}}_k &= \hat{\mathbf{x}}_k + \mathbf{K}_k(\mathbf{z}_k - \mathbf{H}\hat{\mathbf{x}}_k) \\
\mathbf{P}_k &= (\mathbf{I} - \mathbf{K}_k\mathbf{H})\mathbf{P}_k
\end{aligned}
$$

Where:

- $\mathbf{K}_k$ is the Kalman gain, balancing the contributions of the prediction and measurement.
- $\mathbf{H}$ is the measurement matrix, relating the state vector to the observed quantities.
- $\mathbf{R}$ is the measurement noise covariance matrix, capturing uncertainty in the measurements.

- $\mathbf{z}_k$ is the measurement vector at time step k.
- $\hat{\mathbf{x}}_k$ is the updated state estimate.
- $\mathbf{I}$ is the identity matrix.

The Kalman gain $\mathbf{K}_k$ determines how much trust to place in the new measurement versus the prediction based on past data. The updated state estimate and covariance reflect both the prediction and the measurement's uncertainty.

5 Conclusion

Kalman filtering is a robust technique for fusing IMU data with other sensor measurements, enabling accurate state estimation over time. By continuously predicting and updating, the Kalman filter maintains a reliable estimate of the position, velocity, and orientation of a moving object while accounting for uncertainties in both the system model and measurements.