

Hermitian Forms and Zeros of a Polynomial

PRANSHU GABA *

Indian Institute of Science, Bangalore
gabapranshu@ug.iisc.in

November 12, 2017

Abstract

We looked at the general properties of Hermitian (self-adjoint) matrices, and used the Schur-Cohn theorem to find the number of roots of a polynomial lying within and without the unit circle.

1 Introduction

In this paper we see the properties of Hermitian matrices, which are very interesting, as well as useful. We also see and prove the Schur-Cohn theorem to find the number of roots of a polynomial lying within the unit circle.

There are many ways to locate the roots of a polynomial. The Schur-Cohn theorem shows a surprising connection between linear algebra and roots of a polynomial. It will be used to find out how many roots of the polynomial lie inside and outside the unit circle.

2 Definitions

2.1 Inner product

A binary operator $\langle \cdot, \cdot \rangle: \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$

2.2 Adjoint

The adjoint of a matrix $A \in \mathbb{C}_n$, denoted by A^* , is the matrix that satisfies $\langle A^*x, y \rangle = \langle x, Ay \rangle$.

The adjoint can be obtained by taking its transpose, followed by taking the complex conjugate of every element. If the ij^{th} of A is a_{ij} , then the ij^{th} entry of A^* is $\overline{a_{ji}}$. Note that A^* is a linear transformation.

a_{ij} is defined as $\langle Ae_j, e_i \rangle$.

*)

2.3 Positive Definite

Definition. Any matrix $B \in \mathbb{M}_n$ that satisfies $\langle Bx, x \rangle \geq 0$ for all $x \in \mathbb{C}^n$ is called a positive semidefinite matrix.

Corollary. *All the eigenvalues of positive semidefinite matrix are non-negative.*

2.4 Hermitian Matrices

Hermitian matrices (also known as self-adjoint matrices) are matrices that satisfy $A = A^*$.

Theorem. *All the eigenvalues of a Hermitian matrix are real.*

Proof. Left as an exercise. □

Theorem. A^*A is always positive semidefinite.

Proof. $\langle A^*Ax, x \rangle = \langle Ax, Ax \rangle = \|Ax\|^2 > 0$ □

2.5 Unitary Matrices

A square matrix U is a unitary matrix if $U^*U = I$. The determinant of a unitary matrix is 1. It preserves inner product, $\langle Ux, Uy \rangle = \langle x, y \rangle$.

2.6 Matrix S

Let S be the $n \times n$ square matrix
$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Note that S is a nilpotent matrix of order n , i.e. S^n is a zero matrix.

2.7 Norm of a matrix

2.7.1 Operator norm

Given $A \in \mathbb{M}_n$, define $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|$ to be the operator norm of A . The triangle inequality $\|A + B\| \leq \|A\| + \|B\|$ is satisfied.

2.7.2 Hilbert-Schmidt norm

The Hilbert-Schmidt norm of matrix A , is defined as the square root of sum of squares of all entries in A .

$$\|A\|_2 = \left(\sum_{i,j} |a_{ij}|^2 \right)^{1/2}$$

The operator norm is always less than or equal to the Hilbert-Schmidt norm.

2.8 Trace

The trace of a matrix is the sum of the diagonal elements of the matrix.

$$\text{Tr}(A) = \sum_{i=1}^n \langle Ae_i, e_i \rangle$$

Theorem. The trace of A^*A is equal to the Hilbert-Schmidt norm of A . $\text{Tr}(A^*A) = \|A\|_2$

2.9 Diagonalization

Hermitian matrices can be diagonalized. For every Hermitian matrix A , there exists a diagonal matrix Λ such that $A = U^*\Lambda U$. Here U is some unitary matrix.

Lemma. If $A \in \mathbb{M}_n(\mathbb{C})$ and $\langle Ax, x \rangle \in \mathbb{R}$ for every x , then $A = A^*$.

Proof. Let $\alpha \in \mathbb{C}$ and $h, g \in \mathbb{C}^n$. Then \square

Corollary. Every positive semidefinite matrix is Hermitian.

2.10 Projectors

A matrix P is a projector if $P^2 = P$ and $P^* = P$

3 Schur-Cohn Theorem

Given any polynomial $p(z) = a_0z^n + a_1z^{n-1} + \dots + a_n$ with complex coefficients, we are interested in finding how many of its roots lie within the unit circle and how many roots lie outside. Without loss of generality, let $a_0 = 1$ as it does not change the roots of the polynomial.

Suppose p has roots α_i . Then $p(z) = (z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_n)$.

$$\text{Then } p(S) \text{ is } \begin{bmatrix} a_n & a_{n-1} & \ddots & \ddots & a_1 \\ 0 & a_n & a_{n-1} & \ddots & \ddots \\ 0 & 0 & a_n & \ddots & \ddots \\ 0 & 0 & 0 & \ddots & a_{n-1} \\ 0 & 0 & 0 & 0 & a_n \end{bmatrix}.$$

This can be factorized as $p(S) = (S - \alpha_1 I)(S - \alpha_2 I) \dots (S - \alpha_n I)$. Let $B_j = S - \alpha_j I$.

Next, define q as the polynomial $\overline{a_n}z^n + \overline{a_{n-1}}z^{n-1} + \dots + \overline{a_0}$. Note that its roots are $\frac{1}{\overline{\alpha_i}}$. We get $q(z) = (1 - \overline{\alpha_1}z)(1 - \overline{\alpha_2}z) \dots (1 - \overline{\alpha_n}z)$. Also, $q(S) = (I - \overline{\alpha_1}S)(I - \overline{\alpha_2}S) \dots (I - \overline{\alpha_n}S)$. Let $C_j = I - \overline{\alpha_j}S$.

Let H be equal to $\|q(S)x\|^2 - \|p(S)x\|^2$

H can also be written as $\langle (q(S)^*q(S) - p(S)^*p(S))x, x \rangle$.

We can now state the Schur-Cohn theorem:

Theorem. The polynomial p , it will have k roots inside the circle, and $n - k$ roots outside the circle iff k eigenvalues of H are positive and $n - k$ are negative.

4 Proof

We will first prove the Schur-Cohn theorem for $n = 1$, that is for linear polynomials. It will then be extended to polynomials of higher degrees with the help of the Spectral theorem and the Courant-Fischer theorem.

4.1 Linear Polynomial

Let's write $q(S)$ and $p(S)$ as a product of the linear terms. $q(S)^*q(S) - p(S)^*p(S)$
 $= (C_1C_2C_3 \dots C_n)^*(C_1C_2C_3 \dots C_n) - (B_1B_2B_3 \dots B_n)^*(B_1B_2B_3 \dots B_n)$

Let's look at $C_1^*C_1 - B_1^*B_1$ first. Substituting the values of C_1 and B_1 , we get

$$\begin{aligned} & C_1^*C_1 - B_1^*B_1 \\ &= (I - \overline{\alpha_1}S)^*(I - \overline{\alpha_1}S) - (S - \alpha_1I)^*(S - \alpha_1I) \\ &= (I - \alpha_1S^*)(I - \overline{\alpha_1}S) - (S^* - \overline{\alpha_1}I)(S - \alpha_1I) \\ &= (I - \alpha_1S^* - \overline{\alpha_1}S + |\alpha_1|^2 S^*S) - (S^*S - \alpha_1S^* - \overline{\alpha_1}S + |\alpha_1|^2 I) \\ &= I - |\alpha_1|^2 I - S^*S + |\alpha_1|^2 S^*S \\ &= (1 - |\alpha_1|^2)(I - S^*S) \end{aligned}$$

Note that $I - S^*S$ is a positive definite matrix. If $|\alpha| < 1$, then the root of the linear polynomial lies within the unit circle. Also note that H has one negative eigenvalue. Similarly, if $|\alpha| > 1$, then the root of the linear polynomial lies outside the unit circle, and the eigenvalue of H is positive. This shows that the Schur-Cohn theorem is true for $n = 1$. We will now extend the proof for all n .

4.2 Spectral theorem

Theorem. Let $A \in \mathbb{M}_n$ be a Hermitian matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n$. Then A can be written as $U\Lambda U^*$, where U is a unitary matrix, and Λ is a diagonal matrix with real entries.

Proof. Left to the reader. □

4.3 Courant-Fischer theorem

Theorem. Let $A \in \mathbb{M}_n$ be a Hermitian matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n$.

Proof. Left to the reader. □

5 Extensions

5.1 Arbitrary radius

5.2 Limitations

6 Conclusion

Thank You!

7 Acknowledgements