Hermitian Forms and Zeros of a Polynomial

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Abstract

We looked at the general properties of Hermitian (self-adjoint) matrices, and used the Schur-Cohn theorem to find the number of roots of a polynomial lying within and without the unit circle.

1 Introduction

In this paper we see the properties of Hermitian matrices, which are very interesting, as well as useful. We also see and prove the Schur-Cohn theorem to find the number of roots of a polynomial lying within the unit circle.

There are many ways to locate the roots of a polynomial. The Schur-Cohn theorem shows a surprising connection between linear algebra and roots of a polynomial. It will be used to find out how many roots of the polynomial lie inside and outside the unit circle.

2 Definitions

2.1 Inner product

A binary operator $\langle \cdot, \cdot \rangle \colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$

2.2 Adjoint

The adjoint of a matrix $A \in \mathbb{C}_n$, denoted by A^*, is the matrix obtained by taking its transpose, followed by taking the complex conjugate of every element. If the ij^{th} of A is a_{ij}, then the ij^{th} entry of A^* is $\overline{a_{ji}}$. Note that A^* is a linear transformation. The adjoint satisfies $\langle Ax, y \rangle = \langle x, A^*y \rangle$.

2.3 Positive Definite

Definition. Any matrix $B \in \mathbb{M}_n$ that satisfies $\langle Bx, x \rangle \geq 0$ for all $x \in \mathbb{C}^n$ is called a positive semidefinite matrix.

Corollary. *All the eigenvalues of positive semidefinite matrix are non-negative.*

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2.4 Hermitian Matrices

Hermitian matrices (also known as self-adjoint matrices) are matrices that satisfy $A = A^*$.

Theorem. *All the eigenvalues of a Hermitian matrix are real.*

Proof. Left as an exercise. □

Theorem. *A^*A is always positive semidefinite.*

Proof. $\langle A^*Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2 > 0$ □

2.5 Unitary Matrices

2.6 Matrix S

Let S be the $n \times n$ square matrix $\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$.

Note that S is a nilpotent matrix of order n, i.e. S^n is a zero matrix.

2.7 Norm of a matrix

2.7.1 Operator norm

Given $A \in \mathbb{M}_n$, define $\|A\| = \sup\limits_{x \neq 0} \dfrac{\|Ax\|}{\|x\|} = \sup\limits_{\|x\|=1} \|Ax\|$ to be the operator norm of A. The triangle inequality $\|A + B\| \leq \|A\| + \|B\|$ is satisfied.

2.7.2 Hilbert-Schmidt norm

The Hilbert-Schmidt norm of matrix A, is defined as the square root of sum of squares of all entries in A.

$$\|A\|_2 = \left(\sum_{i,j} |a_{ij}|^2 \right)^{1/2}$$

The operator norm is always less than or equal to the Hilbert-Schmidt norm.

2.8 Trace

The trace of a matrix is the sum of the diagonal elements of the matrix.
$\mathrm{Tr}(A) = \sum_{i=1}^{n} \langle Ae_i, e_i\rangle$

Theorem. $\mathrm{Tr}(A^*A) = \|A\|_2$

2.9 Diagonalization

Hermitian matrices can be diagonalized. For every Hermitian matrix A, there exists a diagonal matrix Λ such that $A = U^*\Lambda U$. Here U is some unitary matrix.

Lemma. *If $A \in \mathbb{M}_n(\mathbb{C})$ and $\langle Ax, x\rangle \in \mathbb{R}$ for every x, then $A = A^*$.*

Proof. Let $\alpha \in \mathbb{C}$ and $h, g \in \mathbb{C}^n$. Then □

Corollary. *Every positive semidefinite matrix is Hermitian.*

2.10 Projectors

A matrix P is a projector if $P^2 = P$ and $P^* = P$

3 Schur-Cohn Theorem

Given any polynomial $p(z) = a_0z^n + a_1z^{n-1} + \cdots + a_n$ with complex coefficients, we are interested in finding how many of its roots lie within the unit circle and how many roots lie outside. Without loss of generality, let $a_0 = 1$ as it does not change the roots of the polynomial.

Suppose p has roots α_i. Then $p(z) = (z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n)$.

Then $p(S)$ is $\begin{bmatrix} a_n & a_{n-1} & \ddots & \ddots & a_1 \\ 0 & a_n & a_{n-1} & \ddots & \ddots \\ 0 & 0 & a_n & \ddots & \ddots \\ 0 & 0 & 0 & \ddots & a_{n-1} \\ 0 & 0 & 0 & 0 & a_n \end{bmatrix}$.

This can be factorized as $p(S) = (S - \alpha_1 I)(S - \alpha_2 I)\cdots(S - \alpha_n I)$. Let $B_j = S - \alpha_j I$.

Next, define q as the polynomial $\overline{a_n}z^n + \overline{a_{n-1}}z^{n-1} + \cdots + \overline{a_0}$. Note that its roots are $\frac{1}{\overline{\alpha_i}}$. We get $q(z) = (1 - \overline{\alpha_1}z)(1 - \overline{\alpha_2}z)\cdots(1 - \overline{\alpha_n}z)$. Also, $q(S) = (I - \overline{\alpha_1}S)(I - \overline{\alpha_2}S)\cdots(I - \overline{\alpha_n}S)$. Let $C_j = I - \overline{\alpha_j}S$.

Let H be equal to $\|q(S)x\|^2 - \|p(S)x\|^2$

H can also be written as $\langle (q(S)^*q(S) - p(S)^*p(S))x, x\rangle$.

We can now state the Schur-Cohn theorem:

Theorem. *The polynomial p, it will have k roots inside the circle, and $n - k$ roots outside the circle iff k eigenvalues of H are positive and $n - k$ are negative.*

4 Proof

4.1 Linear Polynomial

Let's write $q(S)$ and $p(S)$ as a product of the linear terms. $q(S)^*q(S) - p(S)^*p(S)$
$= (C_1C_2C_3\ldots C_n)^*(C_1C_2C_3\ldots C_n) - (B_1B_2B_3\ldots B_n)^*(B_1B_2B_3\ldots B_n)$

Let's look at $C_1^*C_1 - B_1^*B_1$ first. Substituting the values of C_1 and B_1, we get

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\begin{aligned}
&C_1^*C_1 - B_1^*B_1 \\
&= (I - \overline{\alpha_1}S)^*(I - \overline{\alpha_1}S) - (S - \alpha_1 I)^*(S - \alpha_1 I) \\
&= (I - \alpha_1 S^*)(I - \overline{\alpha_1}S) - (S^* - \overline{\alpha_1}I)(S - \alpha_1 I) \\
&= (I - \alpha_1 S^* - \overline{\alpha_1}S + |\alpha_1|^2 S^*S) - (S^*S - \alpha_1 S^* - \overline{\alpha_1}S + |\alpha_1|^2 I) \\
&= I - |\alpha_1|^2 I - S^*S + |\alpha_1|^2 S^*S \\
&= (1 - |\alpha_1|^2)(I - S^*S)
\end{aligned}
$$

Note that $I - S^*S$ is a posititve definite matrix. If $|\alpha| < 1$, then the root of the linear polynomial lies within the unit circle. Also note that H has one negative eigenvalue. Similarly, if $|\alpha| > 1$, then the root of the linear polynomial lies outside the unit circle, and the eigenvalue of H is positive. This shows that the Schur-Cohn theorem is true for $n = 1$. We will now extend the proof for all n.

4.2 Spectral theorem and proof

4.3 CF theorem and proof

5 Extensions

5.1 Arbitrary radius

5.2 Limitations

6 Conclusion

Thank You!

7 Acknowledgements