Hermitian Forms and Zeros of a Polynomial

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September 5, 2017

Abstract

We looked at the general properties of Hermitian (self-adjoint) matrices, and used the Schur-Cohn theorem to find the number of roots of a polynomial lying within and without the unit circle.

I. Introduction

In this paper we see the properties of Hermitian matrices, which are very interesting, as well as useful. We also see and prove the Schur-Cohn theorem to find the number of roots of a polynomial lying within the unit circle.

There are many ways to locate the roots of a polynomial. Using the Schur-Cohn theorem gives a nice estimate on how many roots lie inside the unit circle.

II. Hermitian Matrices

The adjoint of a matrix $A \in \mathbb{C}_n$ is the matrix obtained by taking its transpose, followed by taking the complex conjugate of every element. The adjoint of matrix A is denoted by A^*. If the ij^{th} of A is a_{ij}, then the ij^{th} entry of A^* is $\overline{a_{ji}}$. We see that A^* is a linear transformation. The adjoint satisfies $\langle Ax, y\rangle = \langle x, A^*y\rangle$.

Hermitian matrices (also known as self-adjoint matrices) are matrices that satisfy $A^* = A$. All the eigenvalues of a Hermitian matrix are real.

Definition. Any matrix $B \in \mathbb{M}_n$ that satisfies $\langle Bx, x\rangle \geq 0$ for all $x \in \mathbb{C}^n$ is called a positive semidefinite matrix.

Corollary. *All the eigenvalues of positive semidefinite matrix are non-negative.*

A^*A is always positive semidefinite.

Hermitian matrices can be diagonalized. For every Hermitian matrix A, there exists a diagonal matrix Λ such that $A = U^*\Lambda U$. Here U is some unitary matrix.

Lemma. *If $A \in \mathbb{M}_n(\mathbb{C})$ and $\langle Ax, x\rangle \in \mathbb{R}$ for every x, then $A = A^*$.*

Proof. Let $\alpha \in \mathbb{C}$ and $h, g \in \mathbb{C}^n$. Then □

Corollary. *Every positive semidefinite matrix is Hermitian.*

III. Schur-Cohn Theorem

Given a polynomial $p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$. Suppose p has roots α_i. Then $p(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$.

Without loss of generality, let $a_0 = 1$ as it does not change the roots of the polynomial.

Let S be the $n \times n$ square matrix $\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$. Note that it is nilpotent of order n, i.e. S^n is a zero matrix. Then $p(S)$ is $\begin{bmatrix} a_n & a_{n-1} & \ddots & \ddots & a_1 \\ 0 & a_n & a_{n-1} & \ddots & \ddots \\ 0 & 0 & a_n & \ddots & \ddots \\ 0 & 0 & 0 & \ddots & a_{n-1} \\ 0 & 0 & 0 & 0 & a_n \end{bmatrix}$.

This can be factorized as $p(S) = (S - \alpha_1 I)(S - \alpha_2 I)\cdots(S - \alpha_n I)$.

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Then define q as the polynomial with roots $\frac{1}{\overline{\alpha_i}}$. We get $q(z) = (1 - \overline{\alpha_1} z)(1 - \overline{\alpha_2} z) \cdots (1 - \overline{\alpha_n z})$

Let H be equal to $\|q(S)x\|^2 - \|p(S)x\|^2$

Theorem. *The polynomial p, it will have k roots inside the circle, and $n - k$ roots outside the circle iff k eigenvalues of H are positive and $n - k$ are negative.*

IV. Proof

$q(S)^* q(S) - p(S)^* p(S)$
$= (C_1 C_2 C_3 \ldots C_n)^* (C_1 C_2 C_3 \ldots C_n) - (B_1 B_2 B_3 \ldots B_n)^* (B_1 B_2 B_3 \ldots B_n)$

V. Extensions

VI. Conclusion