TABLE I

Model Order m	This Paper U_m	V_m	[3] U_m	V_m
1	1.766	—	1.662	—
3	0.058	30.45	0.306	5.43
5	0.05	1.16	0.193	1.59
6	0.04	1.25	0.108	1.79
7	0.02	2.00	0.03	3.60

TABLE II

Model Order m	This Paper U_m	V_m
1	1.766	—
3	1.275	1.385
5	0.05	25.50
6	0.04	1.25
7	0.02	2.00

TABLE III

Model Order m	This Paper U_m	V_m	[5] U_m	V_m
1	40	—	68.988	—
2	20	2	32.361	2.132
3	0.2	100	0.3	107.870
4	0.1	2	0.137	2.196
5	0.002	50	0.002	56.586
6	0.001	2	0.001	2.414

pointed out that the use of V_m alone to decide on the order can be misleading as the value of V_m may become very small, indicating that little improvement is to be gained by increasing the order, while U_m itself may be large. It seems to us to be equally important to compare U_m with a prescribed error threshold to ensure that the reduced-order model is able to provide a satisfactory approximation to the original system.

III. Numerical Examples

Example 1: Consider the eighth-order system with the following eigenvalues:

$$-0.2, -0.6\pm j10, -30.\pm j300, -40, -50, -100.$$

The values of U_m and V_m for the criterion presented in this paper along with the corresponding values of the criterion presented in [3] are presented in Table I.

If we choose the order according to V_m, both criteria will give $m=3$ as a good order. However, we can see that our criterion gives better results if we base our choice on U_m.

Example 2: This is the same as Example 1 except that the fourth and fifth eigenvalues changed to $-0.8\pm j300$. One should expect the criterion to respond to that change and it gives $m=5$ as the best order. The criterion given in [3] does not detect such a change and still gives the same values of U_m and V_m as in Table I. The values from our criterion are shown in Table II.

The results of this example show clearly the advantage of our criterion and a deficiency of the criterion given in [3].

Example 3: Consider a seventh-order system with real eigenvalues given by

$$0, -0.05, -0.1, -10, -20, -1000, -2000.$$

The values of U_m, V_m for our criterion and for the criterion given in [5] are shown in Table III.

The table shows that for real eigenvalues, the criterion given in [5] is very close to the criterion proposed in this paper, although it is more restrictive. The example shows that we would choose $m=3$ if we only consider V_m, although the corresponding value of U_m is quite large. On the other hand, if we consider V_m and require that $\|e\|_2/K<0.01$, the choice will be $m=5$, which is more appropriate for this problem.

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A Generalization of the Zero Location Theorem of Schur and Cohn

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Abstract—**The well-known Schur–Cohn test for the number of zeros of a polynomial inside the unit circle is deduced from a simple matrix identity. The present proof is not only shorter than earlier ones, but also more general: it produces an infinity of Hermitian forms which can be used in the same way as Schur's form.**

In the course of a study of analytic functions, Schur [6] found a method of deciding whether a given complex polynomial

$$p(z)=a_0z^n+a_1z^{n-1}+\cdots+a_n, \qquad a_0\neq 0 \tag{1}$$

had all its zeros in the open unit disk $\{z\in C: |z|<1\}$: it does if and only if the Hermitian form

$$\underline{H}(x)=\sum_{i=1}^{n}|\bar{a}_0x_i+\bar{a}_1x_{i+1}+\cdots+\bar{a}_{n-i}x_n|^2 - \sum_{i=1}^{n}|a_nx_i+a_{n-1}x_{i+1}+\cdots+a_ix_n|^2 \tag{2}$$

is positive definite. At Schur's suggestion, Cohn [1] generalized this result: if $\underline{H}$ is nonsingular and has π plus and ν minus signs in its canonical form, then π and ν are the numbers of zeros of p inside and outside the unit circle, respectively. This conclusion has since been proved by other methods (e.g., [4]) and there is extensive literature on the

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subject [2], [3]. Earlier proofs have all relied to a considerable extent on algebraic manipulations which have not brought out clearly what special features of the form $\underline{H}$ make it work. In this correspondence we show that the theorem is a consequence of a simple algebraic identity [(4) below]. This makes it possible to give a very easy proof and at the same time to obtain a substantial generalization.

Before stating our results let us show how $\underline{H}$ can be expressed neatly in matrix terms. Let S be the $n\times n$ "shift matrix"

$$S=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}.$$

If p is given by (1), then

$$p(S)=\begin{bmatrix} a_n & a_{n-1} & \cdots & a_1\\ 0 & a_n & \cdots & a_2\\ \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & \cdots & a_n\end{bmatrix}$$

so that, if $x=[x_1,x_2,\cdots,x_n]^T$,

$$p(S)x=[a_nx_1+\cdots+a_1x_n,a_nx_2+\cdots+a_2x_n,\cdots,a_nx_n]^T.$$

Thus, if $\|\cdot\|$ denotes the Euclidean norm on C^n, the second term in the definition (2) of $\underline{H}$ can be written $-\|p(S)x\|^2$. On performing a similar calculation for the polynomial q given by

$$\begin{aligned} q(z)&=z^np(1/\bar{z})^-\\ &=\bar{a}_nz^n+\bar{a}_{n-1}z^{n-1}+\cdots+\bar{a}_0, \end{aligned}\qquad (3)$$

we find that

$$\underline{H}(x)=\|q(S)x\|^2-\|p(S)x\|^2.$$

The only properties of S which we need to know to prove Cohn's theorem are that $I-S^*S$ has rank one and is positive semi-definite (S^* denotes the conjugate transpose of S).

Theorem: Let p be a polynomial of degree n, let q be given by (3) and let A be an $n\times n$ matrix for which $I-A^*A$ is positive semi-definite and of rank 1. If the Hermitian form

$$\underline{H}(x)=\|q(A)x\|^2-\|p(A)x\|^2$$

on C^n has canonical form

$$|y_1|^2+|y_2|^2+\cdots+|y_k|^2-|y_{k+1}|^2-\cdots-|y_n|^2$$

then p has k zeros inside the unit circle and $n-k$ outside it (counting multiplicities).

The corresponding generalization of the Schur theorem was established by a more complicated method in [8].

Proof: We can suppose that $a_o=1$ since dividing by a_o affects neither the zeros of p nor the canonical form of $\underline{H}$. Let us write

$$p(z)=(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$$

so that

$$q(z)=(1-\bar{\alpha}_1z)(1-\bar{\alpha}_2z)\cdots(1-\bar{\alpha}_nz),$$

and let

$$B_j=A-\alpha_jI,\quad C_j=I-\bar{\alpha}_jA,\qquad 1<j<n.$$

Then

$$q(A)=C_1C_2\cdots C_n,\quad p(A)=B_1B_2\cdots B_n,$$

and the matrix H of the Hermitian form $\underline{H}$ can be written

$$\begin{aligned} H&=q(A)^*q(A)-p(A)^*p(A)\\ &=C_n^*\cdots C_1^*C_1\cdots C_n-B_n^*\cdots B_1^*B_1\cdots B_n. \end{aligned}$$

We wish to express this difference as a sum of n terms, each of which is itself a difference of two products. The case $n=2$ serves to illustrate the idea.

$$\begin{aligned} C_2^*C_1^*C_1C_2-B_2^*B_1^*B_1B_2&=C_2^*C_1^*C_1C_2-C_2^*B_1^*B_1C_2\\ &\quad+B_1^*C_2^*C_2B_1-B_1^*B_2^*B_2B_1\\ &=C_2^*(C_1^*C_1-B_1^*B_1)C_2+B_1^*(C_2^*C_2-B_2^*B_2)B_1 \end{aligned}$$

(note that B_i commutes with B_j and C_j, and consequently B_i^* commutes with B_j^* and C_j^*). This extends readily to general n. In what follows $C_n^*\cdots C_{j+1}^*$, $B_{j-1}^*\cdots B_1^*$ are to be understood as the identity matrix in the cases $j=n$, $j=1$ respectively. We have

$$\begin{aligned} &\sum_{j=1}^{n}C_n^*\cdots C_{j+1}^*B_{j-1}^*\cdots B_1^*(C_j^*C_j-B_j^*B_j)B_1\cdots B_{j-1}C_{j+1}\cdots C_n\\ &\quad=\sum_{j=1}^{n}C_n^*\cdots C_j^*B_{j-1}^*\cdots B_1^*B_1\cdots B_{j-1}C_j\cdots C_n\\ &\quad\quad-\sum_{j=1}^{n}C_n^*\cdots C_{j+1}^*B_j^*\cdots B_1^*B_1\cdots B_jC_{j+1}\cdots C_n, \end{aligned}$$

on using the distributive law for matrix multiplication over addition and the commutation properties noted above. In the last two sums the $j+1$ term of the first cancels with the j term of the second, $j=1,2,\cdots,n-1$, so that the right-hand side reduces to

$$C_n^*\cdots C_1^*C_1\cdots C_n-B_n^*\cdots B_1^*B_1\cdots B_n,$$

which is H. Thus

$$H=\sum_{j=1}^{n}C_n^*\cdots C_{j+1}^*B_{j-1}^*\cdots B_1^*(C_j^*C_j-B_j^*B_j)B_1\cdots B_{j-1}C_{j+1}\cdots C_n.\qquad (4)$$

One may easily verify that

$$C_j^*C_j-B_j^*B_j=(1-|\alpha_j|^2)(I-A^*A).$$

Since $I-A^*A$ is positive semi-definite and of rank one, it is of the form uu^* for some nonzero column vector u. Hence, if we let

$$v_j=C_n^*\cdots C_{j+1}^*B_{j-1}^*\cdots B_1^*u,$$

then we can write (4) in the form

$$H=\sum_{j=1}^{n}(1-|\alpha_j|^2)v_jv_j^*$$

or, in matrix notation,

$$H=VDV^*\qquad (5)$$

where

$$\begin{aligned} D&=\mathrm{diag}\{1-|\alpha_1|^2,\cdots,1-|\alpha_n|^2\},\\ V&=[v_1v_2\cdots v_n]. \end{aligned}$$

Since $\underline{H}$ has rank n, by hypothesis, H is nonsingular, and the same is therefore true of V. Thus H is congruent to D. Since Hermitian forms with congruent matrices have the same canonical forms, this clearly implies the desired assertion.

Remark 1: In the case of Schur's Hermitian form (that is, when $A=S$), a congruence of the form (5) can be given explicitly (see [9]).

Remark 2: The hypotheses of the theorem can only be satisfied if all the eigenvalues of A have moduli less than one. Indeed, if $I-A^*A$ is positive semi-definite, then the spectral norm of A is at most one [7, I, Section 3], and hence $|\lambda|\leq 1$ for each eigenvalue λ of A. Suppose A has an eigenvalue λ of unit modulus, and let x be a corresponding eigenvector. Since the spectral norm of A is one, x is also an eigenvector of A^*, with eigenvalue $\bar{\lambda}$ (see, for example, [7, Proposition I.3.1]). We therefore have

$$p(A)x=p(\lambda)x, \quad p(A)^*x=p(\lambda)^- x,$$

$$q(A)x=q(\lambda)x, \quad q(A)^*x=q(\lambda)^- x,$$

and hence

$$(q(A)^*q(A)-p(A)^*p(A))x=(|q(\lambda)|^2-|p(\lambda)|^2)x.$$

Since $|q(z)|=|p(z)|$ whenever $|z|=1$, the right-hand side is zero, and $\underline{H}$ is therefore singular.

Remark 3: The results of [5] enable us to give a description of all candidates for the matrix A. In fact, a matrix A satisfies the three conditions

i) $I-A^*A$ is positive semi-definite,
ii) $I-A^*A$ has rank one, and
iii) every eigenvalue of A has modulus less than one,

if and only if A is unitarily equivalent to a matrix of the form

$$\begin{bmatrix} \beta_1 & s_1s_2 & -s_1\bar{\beta}_2s_2 & \cdots & (-1)^n s_1\bar{\beta}_2\cdots\bar{\beta}_{n-1}s_n \\ 0 & \beta_2 & s_2s_3 & \cdots & (-1)^{n-1}s_2\bar{\beta}_3\cdots\bar{\beta}_{n-1}s_n \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \beta_n \end{bmatrix} \tag{6}$$

where β_i is a complex number of modulus less than one and $s_i=(1-|\beta_i|^2)^{1/2}$, $1\leq i\leq n$.

Schur's form is obtained when all the β_i are taken to be zero.

The results of [5] explain the significance of the β_i. If $\lambda_o=\inf\{\lambda: \lambda q(A)^*q(A)-p(A)^*p(A)\geq 0\}$, where A is the matrix (6) then $\sqrt{\lambda_o}$ is the infimum of $\|f\|_\infty \triangleq \sup_{|z|<1}|f(z)|$ over all analytic functions $f(z)$ for which $f-(p/q)$ is divisible by $(z-\beta_1)\cdots(z-\beta_n)$ (this is the solution to the "Nevanlinna–Pick problem," see [5, Section 7]). In particular, when all $\beta_i=0$, $\sqrt{\lambda_o}$ is the infimum of $\|f\|_\infty$ over all analytic functions $f(z)$ in the open unit disk whose Maclaurin series coincides with that of p/q up to the term in z^{n-1}.

Remark 4: The above method can be adapted to give a test for zeros in any circle or half-plane. For example:

Let p be a polynomial of degree n, let

$$q(z)=p(-\bar{z})^-$$

and let A be an $n\times n$ matrix such that $A+A^*$ is positive semi-definite and of rank 1. If the Hermitian form

$$\underline{K}(x)=\|q(A)x\|^2-\|p(A)x\|^2$$

on C^n has canonical form

$$|y_1|^2+\cdots+|y_k|^2-|y_{k+1}|^2-\cdots-|y_n|^2$$

then p has k zeros in the right half-plane $\operatorname{Re} z>0$ and $n-k$ zeros in the left half-plane.

For the proof one proceeds as before, except that one writes

$$C_j=A+\bar{\alpha}_jI, \quad B_j=A-\alpha_jI$$

and uses the identity

$$C_j^*C_j-B_j^*B_j=(\alpha_j+\bar{\alpha}_j)(A+A^*).$$

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Structural Zeros in the Modal Matrix and Its Inverse

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Abstract—**It is shown that the positions of the zeros in the modal matrix of a system, and its inverse, which result from zeros in the state matrix, can be predicted by computing the corresponding reachability matrix.**

In the analysis of linear time-invariant systems, it is of great value to be able to determine from the occurrence of zeros in the state matrix the positions of zeros in the modal matrix and its inverse. When the state matrix is nondefective, i.e., has as many linearly independent eigenvectors as its order, this can be done by a simple but apparently not well-known method.

Definition 1: A *structural zero* of the modal matrix M of a matrix A is an element of M which is zero because of the occurrence of zeros in A, regardless of the values of the nonzero elements of A.

Definition 2: The *reachability matrix* $R(A)$ of a square matrix A is the matrix of ones and zeros obtained by the following process.

i) Replace each nonzero element and every principal-diagonal element of A by 1, forming a binary matrix $B_1(A)$; set k to 1.

ii) Premultiply or postmultiply $B_k(A)$ by $B_1(A)$ and replace each nonzero element of the result by 1, forming $B_{k+1}(A)$.

iii) If $B_{k+1}(A)$ differs from $B_k(A)$, increase k by 1 and return to ii); if not, $B_k(A)$ is $R(A)$.

An equivalent definition in graph-theory terms is given by Harary *et al.* [1].

Theorem: Structural zeros occur in the suitably ordered modal matrix M of a nondefective matrix A in the same positions as those in the reachability matrix $R(A)$ of A. Structural zeros also occur in these positions in the inverse N of the modal matrix and in the transition matrix e^{At}, where t is a scalar.

Proof: Since all principal-diagonal elements of $B_1(A)$ are nonzero, for element (i,j) of $B_{k+1}(A)$ to be zero it is necessary that element (i,j) of $B_k(A)$ be zero. Thus, if element (i,j) of $R(A)$ is zero, element (i,j) of $B_k(A)$ is zero for any positive integer k. If element (i,j) of $B_k(A)$ is zero, element (i,j) of $AB_{k-1}(A)$ is zero, and so is element (i,j) of A^k, by induction and noting that $B_0(A)$, which when multiplied by $B_1(A)$,

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