Hermitian Forms and Zeros of a Polynomial

Pranshu Gaba *

Indian Institute of Science, Bangalore
pranshu@ug.iisc.in

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Abstract

We looked at the general properties of Hermitian (self-adjoint) matrices, and used the Schur-Cohn theorem to find the number of roots of a polynomial lying within and without the unit circle.

I. Introduction

In this paper we see the properties of Hermitian matrices, which are very interesting, as well as useful. We also see and prove the Schur-Cohn theorem to find the number of roots of a polynomial lying within the unit circle.

There are many ways to locate the roots of a polynomial. Using the Schur-Cohn theorem gives a nice estimate on how many roots lie inside the unit circle.

II. Hermitian Matrices

The adjoint of a matrix is its conjugate transpose. The ijth entry of A^* is $\overline{a_{ji}}$

Hermitian matrices (also known as self-adjoint matrices) are matrices that satisfy $A^* = A$. All the eigenvalues of a Hermitian matrix are real.

Definition. *Any matrix $B \in \mathbb{M}_n$ that satisfies $\langle Bx, x\rangle \geq 0$ for all $x \in \mathbb{C}^n$ is called a positive semidefinite matrix.*

Corollary. *All the eigenvalues of positive semidefinite matrix are non-negative.*

Corollary. *Every positive semidefinite matrix is Hermitian.*

Hermitian matrices can be diagonalized. For every Hermitian matrix A, there exists a diagonal matrix Λ such that $A = U^*\Lambda U$. Here U is some unitary matrix.

III. Schur-Cohn Theorem

Given a polynomial $p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$. Suppose p has roots α_i. Then $p(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$.

Without loss of generality, let $a_0 = 1$ as it does not change the roots of the polynomial.

Let S be the $n \times n$ square matrix $\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$. Note that it is nilpotent of order n, i.e. S^n is a zero matrix. Then $p(S)$ is $\begin{bmatrix} a_n & a_{n-1} & \ddots & \ddots & a_1 \\ 0 & a_n & a_{n-1} & \ddots & \ddots \\ 0 & 0 & a_n & \ddots & \ddots \\ 0 & 0 & 0 & \ddots & a_{n-1} \\ 0 & 0 & 0 & 0 & a_n \end{bmatrix}$.

This can be factorized as $p(S) = (S - \alpha_1 I)(S - \alpha_2 I)\cdots(S-\alpha_n I)$.

Then define q as the polynomial with roots $\frac{1}{\overline{\alpha_i}}$. We get $q(z) = (1-\overline{\alpha_1}z)(1-\overline{\alpha_2}z)\cdots(1-\overline{\alpha_n}z)$

Let H be equal to $\|q(S)x\|^2 - \|p(S)x\|^2$

Theorem. *The polynomial p, it will have k roots inside the circle, and $n-k$ roots outside the circle iff k eigenvalues of H are positive and $n-k$ are negative.*

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IV. Proof

$q(S)^*q(S) - p(S)^*p(S)$
$= (C_1C_2C_3 \dots C_n)^*(C_1C_2C_3 \dots C_n) - (B_1B_2B_3 \dots B_n)^*(B_1B_2B_3 \dots B_n)$

V. Extensions

VI. Conclusion