Hermitian Forms and Zeros of a Polynomial

Pranshu Gaba *

Indian Institute of Science, Bangalore
pranshu@ug.iisc.in

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Abstract

We looked at the general properties of Hermitian (self-adjoint) matrices, and used the Schur-Cohn theorem to find the number of roots of a polynomial lying within and without the unit circle.

I. Introduction

In this paper we see the properties of Hermitian matrices, which are very interesting, as well as useful. We also see and prove the Schur-Cohn theorem to find the number of roots of a polynomial lying within the unit circle.

There are many ways to locate the roots of a polynomial. The Schur-Cohn theorem shows a surprising connection between linear algebra and roots of a polynomial to help find out how many roots lie inside and outside the unit circle.

II. Hermitian Matrices

The adjoint of a matrix $A \in \mathbb{C}_n$ is the matrix obtained by taking its transpose, followed by taking the complex conjugate of every element. The adjoint of matrix A is denoted by A^*. If the ij^{th} of A is a_{ij}, then the ij^{th} entry of A^* is $\overline{a_{ji}}$. We see that A^* is a linear transformation. The adjoint satisfies $\langle Ax, y\rangle = \langle x, A^*y\rangle$.

Hermitian matrices (also known as self-adjoint matrices) are matrices that satisfy $A = A^*$. All the eigenvalues of a Hermitian matrix are real.

Definition. Any matrix $B \in \mathbb{M}_n$ that satisfies $\langle Bx, x\rangle \geq 0$ for all $x \in \mathbb{C}^n$ is called a positive semidefinite matrix.

Corollary. *All the eigenvalues of positive semidefinite matrix are non-negative.*

A^*A is always positive semidefinite.

Hermitian matrices can be diagonalized. For every Hermitian matrix A, there exists a diagonal matrix Λ such that $A = U^*\Lambda U$. Here U is some unitary matrix.

Lemma. *If $A \in \mathbb{M}_n(\mathbb{C})$ and $\langle Ax, x\rangle \in \mathbb{R}$ for every x, then $A = A^*$.*

Proof. Let $\alpha \in \mathbb{C}$ and $h, g \in \mathbb{C}^n$. Then □

Corollary. *Every positive semidefinite matrix is Hermitian.*

III. Schur-Cohn Theorem

Given any polynomial $p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ with complex coefficients, we are interested in finding how many of its roots lie within the unit circle and how many roots lie outside. Without loss of generality, let $a_0 = 1$ as it does not change the roots of the polynomial.

Suppose p has roots α_i. Then $p(z) = (z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n)$.

Let S be the $n \times n$ square matrix $\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$. Note that S is a nilpotent matrix of order n, i.e. S^n is a zero matrix. Then

*:)

$p(S)$ is $\begin{bmatrix} a_n & a_{n-1} & \ddots & \ddots & a_1 \\ 0 & a_n & a_{n-1} & \ddots & \ddots \\ 0 & 0 & a_n & \ddots & \ddots \\ 0 & 0 & 0 & \ddots & a_{n-1} \\ 0 & 0 & 0 & 0 & a_n \end{bmatrix}$.

This can be factorized as $p(S) = (S - \alpha_1 I)(S - \alpha_2 I)\cdots(S - \alpha_n I)$. Let $B_j = S - \alpha_j I$.

Next, define q as the polynomial $\overline{a_n} z^n + \overline{a_{n-1}} z^{n-1} + \cdots + \overline{a_0}$. Note that its roots are $\frac{1}{\overline{\alpha_i}}$. We get $q(z) = (1 - \overline{\alpha_1} z)(1 - \overline{\alpha_2} z)\cdots(1 - \overline{\alpha_n z})$. Also, $q(S) = (I - \overline{\alpha_1} S)(I - \overline{\alpha_2} S)\cdots(I - \overline{\alpha_n} S)$. Let $C_j = I - \overline{\alpha_j} S$.

Let H be equal to $\|q(S)x\|^2 - \|p(S)x\|^2$

H can also be written as $\langle (q(S)^* q(S) - p(S)^* p(S))x, x \rangle$.

We can now state the Schur-Cohn theorem:

Theorem. *The polynomial p, it will have k roots inside the circle, and $n-k$ roots outside the circle iff k eigenvalues of H are positive and $n-k$ are negative.*

IV. Proof

$q(S)^* q(S) - p(S)^* p(S)$
$= (C_1 C_2 C_3 \ldots C_n)^* (C_1 C_2 C_3 \ldots C_n) - (B_1 B_2 B_3 \ldots B_n)^* (B_1 B_2 B_3 \ldots B_n)$

Let's look at $C_1^* C_1 - B_1^* B_1$ first. Substituting the values of C_1 and B_1, we get

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\begin{aligned}
& C_1^* C_1 - B_1^* B_1 \\
&= (I - \overline{\alpha_1} S)^* (I - \overline{\alpha_1} S) - (S - \alpha_1 I)^* (S - \alpha_1 I) \\
&= (I - \alpha_1 S^*)(I - \overline{\alpha_1} S) - (S^* - \overline{\alpha_1} I)(S - \alpha_1 I) \\
&= (I - \alpha_1 S^* - \overline{\alpha_1} S + |\alpha_1|^2 S^* S) - (S^* S - \alpha_1 S^* - \overline{\alpha_1} S + |\alpha_1|^2 I) \\
&= I - |\alpha_1|^2 I - S^* S + |\alpha_1|^2 S^* S \\
&= (1 - |\alpha_1|^2)(I - S^* S)
\end{aligned}
$$

Note that $I - S^* S$ is a positive definite matrix. If $|\alpha| < 1$, then the root of the linear polynomial lies within the unit circle. Also note that H has one negative eigenvalue. Similarly, if $|\alpha| > 1$, then the root of the linear polynomial lies outside the unit circle, and the eigenvalue of H is positive. This shows that the Schur-Cohn theorem is true for $n = 1$. We will now extend the proof for all n.

V. Extensions

VI. Conclusion