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<u>Ans 1.]</u> $\dot{\underline{x}} = \underbrace{A}_{A \in \mathbb{R}^{3\times3} \text{ (given)}}\underline{x}$ has the system matrix (A) spectrum as: $\sigma(A) = \{-2, j, -j\}$. Since the $\sigma(A) \notin \mathbb{C}^-$ (OLHP), the system is not asymptotically stable, but ~~too~~ the system is <u>stable</u> (internally) as it has 3 distinct eigenvalues and hence 3 linearly independent eigenvectors $(m_i = l_i \ \forall \lambda_i \in \sigma(A))$.

<u>Ans 2.]</u> $\dot{\underline{x}} = A\underline{x} + Bu$, $u \in \mathbb{R}$ $\underline{x} \in \mathbb{R}^3$ $A \in \mathbb{R}^{3\times3}$

$y = C\underline{x}$, $y \in \mathbb{R}$, $C \in \mathbb{R}^{1\times3}$; $u(t) = 0 \ \forall t \in \mathbb{R}$, I.C. $\underline{x}(0) = \underline{x}_0$, and

$y(t) = 1 + 3\exp(-t)\cos(t + \pi/3)$.

This tells us that the system has at least 3 modes (2 being coupled as exponentially varying oscillations). But since the system can only have 3 modes at maximum (due to $|\sigma(A)| = 3$), we now know that $\sigma(A) = \{0, -1 \pm j\}$ by reading them off from the system output. From that, we can conclude that the system is not asymptotically stable, but internally stable as it has 3 distinct eigenvalues (of the sys. matrix) & $\sigma(A) \notin \mathbb{C}^-$.

<u>Ans 3]</u> $\dot{\underline{x}} = \underbrace{\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{A} \underline{x}$. (LTI)

a) Commentary on stability of (LTI): First, we will extract the eigenvalues of A, $\sigma(A) =$ roots of $\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda+1 & -1 & -1 \\ 0 & \lambda & -1 \\ 0 & 1 & \lambda \end{bmatrix}\right)$

$= (\lambda+1)[\lambda^2 + 1] + 1(0) - 1(0) \Rightarrow (\lambda+1)\cdot(\lambda^2+1) = 0$.

$\therefore \sigma(A) = \{-1, \pm j\} \notin \mathbb{C}^-$.

Thus, the system is not asymptotically stable, but it is internally stable as it has distinct eigenvalues.

b) The set X_0 of all I.C. converging to 0 asymptotically, need the mode corresponding to $\lambda = -1$. This can be constructed with the knowledge of the P matrix in the similarity transform of A. We need eigenvectors

• $\underline{v}_1$: $\begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \underline{0} \Rightarrow v_2 + v_3 = 0$; $v_2 - v_3 = 0 \Rightarrow v_2 = v_3 = 0$.

$v_1 \in \mathbb{R}$.

$\therefore \underline{v}_1 = [1\ 0\ 0]^T$.

• $\underline{v}_2$: $\begin{bmatrix} j+1 & -1 & -1 \\ 0 & j & -1 \\ 0 & 1 & j \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \underline{0} \Rightarrow$ $v_1(j+1) - v_2 - v_3 = 0 \Big] \rightarrow v_1 j + v_1 - j - 1 = 0 \Rightarrow v_1 = 1$

$v_2 j - v_3 = 0$
$v_2 + j v_3 = 0 \Big] \Rightarrow j v_2 = v_3$ & $v_2 = -j v_3$

$v_3 = j$, $v_2 = 1$

$\underline{v}_2 = [1\ 1\ j]^T$

and $\underline{v}_3 = \underline{v}_2^* = [1\ 1\ -j]^T$

- Thus the P matrix $= [v_1 \mid Re(v_2) \mid Im(v_2)]$. For the set $\mathcal{X}_0$, we need all vectors x s.t. $z = P^{-1}x = \mu\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mu \in \mathbb{R}$. (modal coordinates) → to select/activate the e^{-t} mode only.

$\Rightarrow Pz = x$

$$Pz = \mu\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = x \quad \text{for some } \mu\in\mathbb{R}$$

$$\therefore\ x = \mu\begin{bmatrix}1+0\\0\\0\end{bmatrix} = \mu\begin{bmatrix}1\\0\\0\end{bmatrix}.$$

Thus $\mathcal{X}_0 = \{x \in \mathbb{R}^3 \mid x$ is of the form $\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\mu$, for $\mu\in\mathbb{R}\}$.

B) Similarly, to have purely sinusoidal outputs, we need to have the first mode deactivated.

$$\Rightarrow\ x = \begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0\\ *\\ *\end{bmatrix} = Pz,$$

$$\therefore\ x = \begin{bmatrix}\mu_1\\ \mu_1\\ \mu_2\end{bmatrix}, \text{ for } \mu_1, \mu_2 \in \mathbb{R}.$$

Thus, $\mathcal{X}_1 = \{x\in\mathbb{R}^3 \mid x = \begin{bmatrix}\mu_1\\ \mu_1\\ \mu_2\end{bmatrix}$ for $\mu_1,\mu_2\in\mathbb{R}\}$.

Ans 4.] $\dot{x} = \begin{bmatrix}1&3\\0&-2\end{bmatrix}x + \begin{bmatrix}1\\-1\end{bmatrix}u$, $y = [1\ 1]x$.

a) $G(s) = C(sI-A)^{-1}B + D^{0} = [1\ 1]\begin{bmatrix}s-1&-3\\0&s+2\end{bmatrix}^{-1}\begin{bmatrix}1\\-1\end{bmatrix} = [1\ 1]\dfrac{\begin{bmatrix}s+2&3\\0&s-1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}}{(s-1)(s+2)}$

$= [1\ 1]\dfrac{\begin{bmatrix}s+2-3\\-(s-1)\end{bmatrix}}{(s-1)(s+2)} \Rightarrow G(s) = \dfrac{s-1-(s-1)}{(s-1)(s+2)} = \underline{0}$

b) The system is BIBO stable, since for every input, the output is 0 (bounded). Internally though the system is unstable since it has $\sigma(A) = \{1, -2\} \not\subset \bar{\mathbb{C}}^-$ (CLHP).

Ans 5.] $A = \begin{bmatrix}1&0&-1\\2&3&1\end{bmatrix}$. $\quad C_3 \mapsto C_3 + C_2 - C_1$. ...

a) $\text{Rank}(A) = \dim(Im(A)) \Rightarrow \text{RREF}(A) \sim \left[\begin{array}{c|c|c}1&0&0-1\\2&3&3-2\end{array}\right] \sim \begin{bmatrix}1&0&0\\2&3&0\end{bmatrix}$

$\sim \boxed{\begin{bmatrix}1&0&0\\2&1&0\end{bmatrix}}$

$\therefore$ Rank(A) = 2.

$\begin{bmatrix}1&0&-1\\2&3&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \to \begin{bmatrix}x_1 - x_2\\ x_1 2 + 3x_2 + 1x_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$.

$x_1\begin{bmatrix}1\\2\end{bmatrix} + x_2\begin{bmatrix}0\\3\end{bmatrix} + x_3\begin{bmatrix}-1\\1\end{bmatrix} = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\3\end{bmatrix}\right\} = Im(A)$.

b) By the rank-nullity theorem, $\dim(\ker(A)) = \text{nullity}(A) = n - \text{Rank}(A)$
$= 3 - 2 = 1$.

c) $\begin{bmatrix}1&0&-1\\2&3&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \underline{0} \Rightarrow x_1 - x_3 = 0 \Rightarrow x_1 = x_3$.

$2x_1 + 3x_2 + x_3 = 0$

$\to 3x_3 + 3x_2 = 0 \to -x_3 = x_2$

$\therefore$ $\text{Ker}(A) = \text{span}\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$

d) $\underline{y} = A\underline{x}$ is not injective, since it maps $\mathbb{R}^3 \to \mathbb{R}^2$, and has $\dim(\ker(A)) = 1$. The function is surjective since the basis of $\mathrm{Im}(A) \in \mathbb{R}^2$ has two basis vectors.

e) Multiple solutions do exist to $\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{b}$, since we have a redundant column in A, so a solution will not be unique. We can find all solutions by selecting all possible combinations of columns & solving for scale factors yielding $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as the result.

Ans 6.] $V = \{ \underline{x} \in \mathbb{R}^3 \mid x_1 - x_2 = 0,\ x_1 - 2x_3 = 0 \} \Rightarrow x_1 = x_2,\ x_1 = 2x_3 \Rightarrow \mathrm{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1/2 \end{bmatrix}}_{\underline{v}} \right\}$

a) Since $V = \mathrm{span}\{\underline{v}\}$, we have that V is a subspace, with one basis.

b) $\dim(V) = 1$, $\text{basis}(V) = \underline{v} = \begin{bmatrix} 1 \\ 1 \\ 1/2 \end{bmatrix}$.

c) Independent complement vector space W can be the $x_1 - x_2$ plane, i.e. $W = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} = \mathbb{R}^2$, s.t. $V \oplus W = \mathbb{R}^3$.

Ans 7.] $V = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, $W = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$.

a) $\dim(V) = 2$, $\dim(W) = 2$, and for $V + W = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$, we have the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, whose rank will yield the $\dim(V + W)$ since it will tell us the # of linearly independent basis / column vectors. a_1, a_2 are mutually linearly independent; same for a_3 & a_4. So even though the column / basis vectors are pairwise linearly independent, as one group, we can still express one (any) column as a linear combination of 3. $\Rightarrow$ $\dim(V + W) = \mathrm{Rank}(A) = 3$.

b) Since $\dim(V) = 2$, $\dim(W) = 2$, & $\dim(V + W) = 3$, we have $\dim(V \cap W) = 1$.

c) Basis for $V + W$ is by the logic in (a): $\mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

d) By inspection, a basis for $V \cap W$ is: $\mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Ans 8.] $V = \{ \underline{x} \in \mathbb{R}^4 \mid x_1 - x_2 = 0,\ 2x_1 + x_2 = 0 \}$, $W = \{ \underline{x} \in \mathbb{R}^4 \mid x_2 + 3x_3 = 0,\ -2x_2 + x_3 = 0 \}$

for V: $x_1 - x_2 = 0$
$\Rightarrow x_1 = x_2$.
& $x_1 = -x_2/2 \Rightarrow x_1 = x_2 = 0$.

for W: $x_2 = -3x_3$
$x_2 = +x_3/2$
$\Rightarrow 3x_3 = \frac{x_3}{2}$.
$\Rightarrow x_3 = 0 = x_2$.

$\therefore V = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \mathrm{span}\{\underline{v}_1, \underline{v}_2\}$ → lin. indep.

$W = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \mathrm{span}\{\underline{w}_1, \underline{w}_2\}$

a) $\dim(V) = 2 = \dim(\mathrm{Im}[\underline{v}_1\ \underline{v}_2])$; similarly, $\dim(W) = 2$. $V + W = \mathrm{span}\{\underline{v}_1, \underline{v}_2, \underline{w}_1, \underline{w}_2\}$ Which shows visually that all vectors are L.I. $\Rightarrow$ $\dim(V + W) = 4$.

b) Therefore, $\dim(V \cap W) = \dim(W) + \dim(V) - \dim(V + W) = 2 + 2 - 4 = 0$.
$\Rightarrow \boxed{V \cap W = \{\underline{0}\}} \Rightarrow V$ & W are indep. complements !!

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c) From (a), we can see that the basis for $V+W = V \oplus W = \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}$.

d) From (b), we can see that since $V \cap W = \{\underline{0}\}$, there is no basis, as this is a trivial (null) subspace containing only the zero vector, & has $\dim(\cdot) = 0$.

Ans 9.] $\dot{\underline{x}} = \underbrace{\begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix}}_{A} \underline{x}$, $\mathcal{V} = \{ x \in \mathbb{R}^3 \mid x_1 = x_2 \}$.

$\Rightarrow \mathcal{V} = \text{span}\left\{ \underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{\underline{v}_1}, \underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix}}_{\underline{v}_2}, \underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\underline{v}_3} \right\}$ $\to$ span $\left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$.

a) $A\underline{v}_1 = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\-1\end{bmatrix} \in \mathcal{V}$ $\to$ $= -\underline{v}_3$

$A\underline{v}_2 = \begin{bmatrix}-1\\-1\\0\end{bmatrix} \in \mathcal{V} : -\underline{v}_2$

$A\underline{v}_3 = \begin{bmatrix}1\\1\\-1\end{bmatrix} \in \mathcal{V}$. $\to$ $\underline{v}_2 - \underline{v}_3$

$\Rightarrow$ $\boxed{A\mathcal{V} \subset \mathcal{V}}$, meaning that $\mathcal{V}$ is A-invariant subspace.

b) $\underline{z} = T(\underline{x}) = P^{-1}\underline{x}$; Let $\overbrace{P = [\underline{v}_1 | \underline{v}_2 | \underline{v}_3]}^{\text{error: not invertible!}}$. Then, we can see that we can break input vector into coordinates of $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$, in which case, the output per component only depends on the coordinates in $\mathcal{V}$ & then multiplying by the corresponding 'axis' vector in $\underline{z}$. That is, $\underline{x} = \hat{v}_1 \underline{v}_1 + \hat{v}_2 \underline{v}_2 + \hat{v}_3 \underline{v}_3$ then $A\underline{x} = \dot{\underline{x}} = \hat{v}_1 \begin{bmatrix}0\\0\\-1\end{bmatrix} + \hat{v}_2 \begin{bmatrix}0\\-1\\0\end{bmatrix} + \hat{v}_3 \begin{bmatrix}0\\1\\-1\end{bmatrix}$, which shows that z_1 is always 0.

c) Outside $\mathcal{V}$, there are vectors in span $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix} \right\} = W$. $\forall \underline{x} \in W$, $A\underline{x} = \underline{x} \begin{bmatrix}0\\-1\\-1\end{bmatrix}$.
i.e. eigenvalue = 1. Unstable $\Rightarrow$ not A.S.

d) Inside $\mathcal{V}$, eigenvalues = $\{-1\}$ ~~1}, also unstable (not A.S)~~ $\to$ since eig($\underline{v}_3$) = +1
Thus, yes, it is A.S.

* note to self: Redo Q9!!