

ECE410: Linear Control Systems

Output Feedback Stabilization of a Cart-Pendulum Robot

PRA102

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1 Introduction

This lab builds on lab 3 where we defined the cart-pendulum system and state feedback controllers for the purpose of driving the system to its linearization equilibrium, i.e, upright position at the origin. We will now be using output feedback control using observers to estimate the internal state and then control the system based on it. We redefine the internal state $\mathbf{x} = [z \quad \dot{z} \quad \theta \quad \dot{\theta}]^T$, where z is now the position of the cart, and we are able to observe the output $\mathbf{y} = [z \quad \theta]^T$.

The state-estimate-based feedback controller $u = K\hat{\mathbf{x}}$ was designed with gain matrix K such that $\sigma(A+BK)$ was assigned to $\{-5.1, -5.2, -5.3, -5.4\}$. Furthermore, two different observers were designed. The first observer with (corrective) gain matrix L_1 was designed to assign the closed-loop spectrum $\sigma(A + L_1C)$ to the set $\{-10, -11, -12, -13\}$. Similarly, observer with gain L_2 was designed to be a relatively aggressive with $\sigma(A + L_2C)$ assigned to $\{-40, -41, -42, -43\}$.

This system in its closed-loop output-feedback form, when augmented with the state estimate error $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$, has the following dynamics.

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} A + BK & -BK \\ \mathbf{0} & (A + LC) \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \mathbf{e} \end{bmatrix}$$

2 Noiseless state estimation

We designed observers by assigning eigenvalues for the closed-loop observer subsystem that accepts the true output from the real system and its corresponding input u , to produce an estimate of the internal state $\hat{\mathbf{x}}$ as follows.

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= A\hat{\mathbf{x}} + Bu + L(\hat{\mathbf{y}} - \mathbf{y}) \\ \mathbf{y} &= C\hat{\mathbf{x}} + Du \end{aligned}$$

2.1 Observer gains L_1 and L_2

$$L_1 = \begin{bmatrix} -22.9338 & -1.0388 \\ -130.7798 & -14.0775 \\ -0.9570 & -23.0662 \\ -10.9830 & -168.2588 \end{bmatrix}$$
$$L_2 = 10^3 \begin{bmatrix} -0.0829 & -0.0011 \\ -1.7161 & -0.0464 \\ -0.0009 & -0.0831 \\ -0.0380 & -1.7629 \end{bmatrix}$$

2.2 Plots for state estimation error evolution

Figures 1 and 2 show the state estimation error evolutions for the linearized and nonlinear system respectively. The error plotted here is $\tilde{\mathbf{x}} = \hat{\mathbf{x}} - \mathbf{x}$.

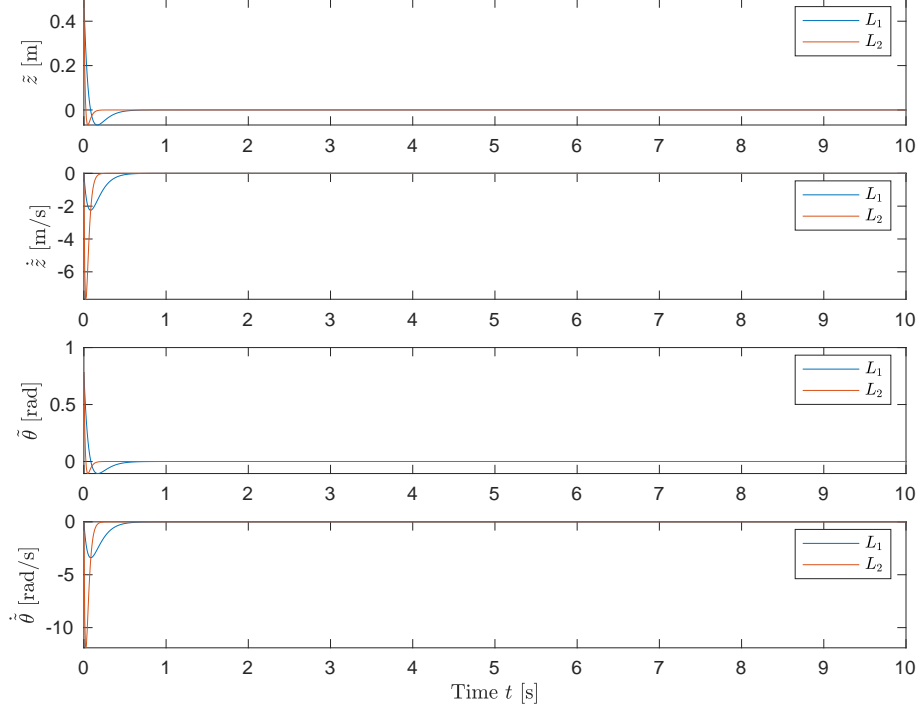


Figure 1: Linear System State Estimation Error Evolution

2.3 Evaluation of Results

Figures 1 and 2 exhibit very similar behaviour, especially for the first two states. This is because the system has been linearized at $(0, 0, 0, 0)$ and the initial conditions are not too far from the equilibrium. As seen in lab 1, the linear and nonlinear system exhibit similar behaviour closer to the equilibrium point. Recall, for the linear system, $\dot{\tilde{\mathbf{x}}} = (A + LC)\tilde{\mathbf{x}}$, whereas for the non-linear system $\dot{\tilde{\mathbf{x}}} = \dot{\hat{\mathbf{x}}} - \dot{\mathbf{x}} = A\hat{\mathbf{x}} + Bu + L(C\hat{\mathbf{x}} - \mathbf{y}) - f(\mathbf{x}, u)$. For points closer to the equilibrium, $f(\mathbf{x}, u)$ and $A\mathbf{x} + Bu$ are close, and hence the $\tilde{\mathbf{x}}_{NL} \approx \tilde{\mathbf{x}}_L$.

In both the figures, L_2 is the more aggressive observer, and tries to quickly match the outputs, which are z and θ . This results in a quicker convergence of system state estimation error (for all states) when L_2 is used. Because L_2 has a quicker convergence, the derivative states (\dot{z} and $\dot{\theta}$) witness sharper shoots than seen in L_1 . Regardless, both L_1 and L_2 reach steady-state within one second for all states.

3 State estimation with measurement noise

While the previous section compared the designed observers in a simulated environment, this section attempts to analyse the designed observers in a more realistic scenario. Observers take in the output of a system \mathbf{y} , with the help of sensors, and then estimated values of the states are produced. Sensors are prone to high noise, so it is important to test the designed observers with noise. For this section, $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$ and noise was introduced to the output, $\mathbf{y} = C\mathbf{x} + W(t)$, where W was a white jointly-Gaussian noise process with zero mean and covariance $\Sigma = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.001 \end{bmatrix}$.

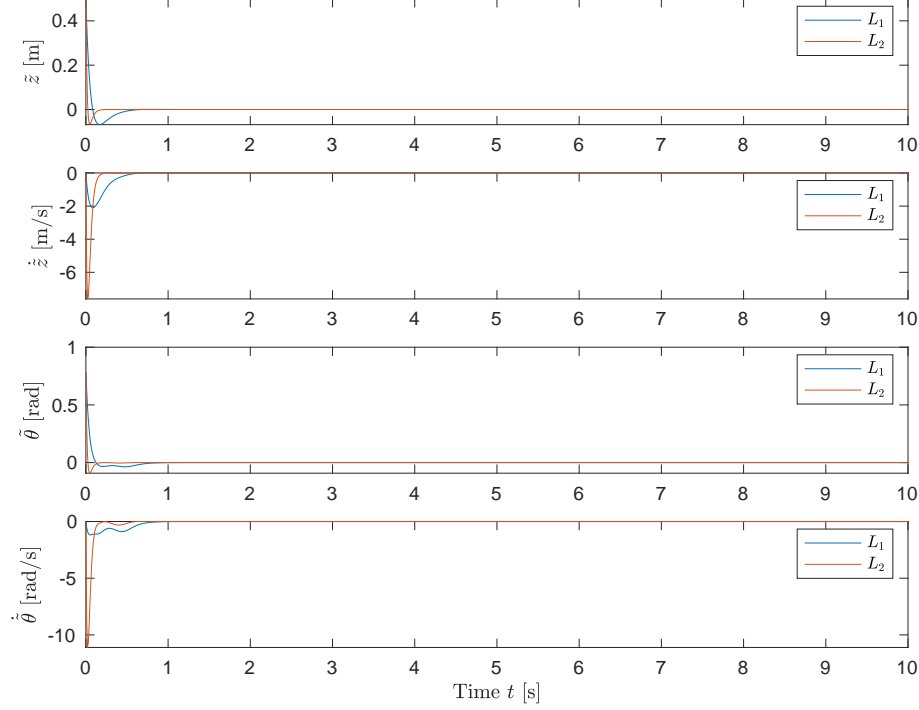


Figure 2: Nonlinear System State Estimation Error Evolution

3.1 Plots

3.2 Difference between L_1 and L_2 behaviors

L_2 places the eigenvalues of $A + L_2C$ further away to the left of the imaginary s -axis. Hence, compared to L_1 , lower eigenvalues introduce a faster convergence to the true state, as seen in the previous section. Essentially by L_2 being more aggressive, it introduces sharper corrections to sharp changes in the observed output. The same behaviour gets amplified when noise is introduced to the system. We can thus conclude that since L_2 is more sensitive to changes in the output, it ends up trying to "follow" the noise.

Since the poles of the closed-loop error subsystem ($A + LC$) are pushed towards higher frequencies with L_2 , it allows a wider band of frequencies to take part in its state evolution dynamics. Especially when corrupted with white noise (equal energy over the entire spectrum), it is consequently more negatively impacted in its noise-rejection (or correction) capability than L_1 . Essentially, because L_2 has a higher cutoff frequency than L_1 , higher frequency content is captured by L_2 and the effect is clearly visible in the plots.

3.3 Comparing MSEs associated with L_1 and L_2

The calculated MSE (mean-squared error) for L_1 :

$$\text{MSE}_{L_1} = [0.0007 \quad 0.0169 \quad 0.0002 \quad 0.0072]$$

The calculated MSE for L_2 :

$$\text{MSE}_{L_2} = [0.0020 \quad 0.6377 \quad 0.0004 \quad 0.1091]$$

MSE_{L_2} is higher than MSE_{L_1} because its faster convergence property. Additionally, x_2 and x_4 , states that are derivatives of other states (x_1 and x_3 respectively) have a higher MSE since differentiation is a high-pass

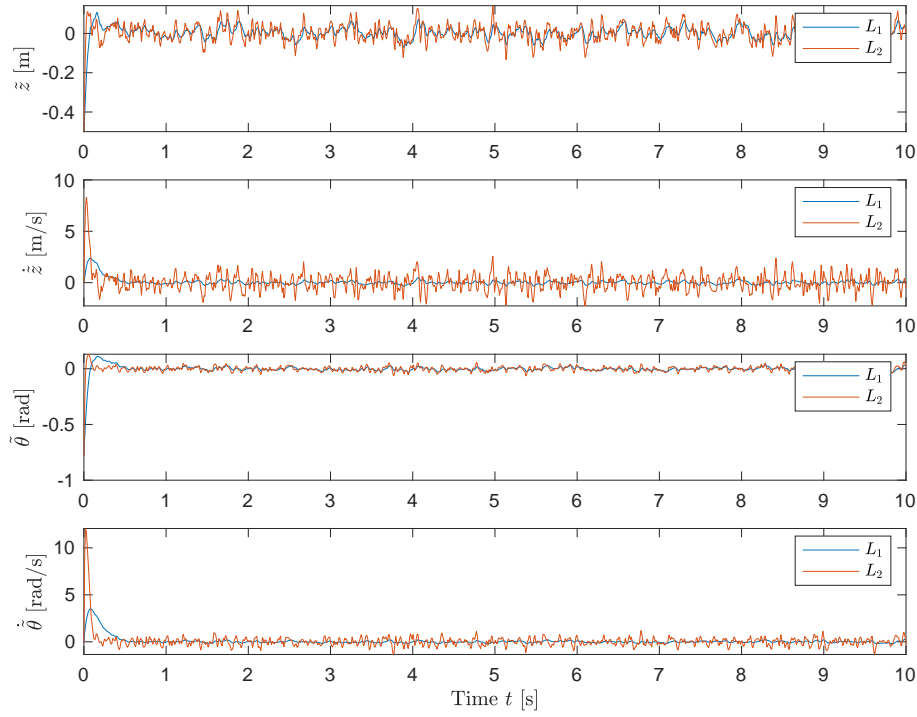


Figure 3: Linear System State Estimation Error Evolution with Noise

operation, and hence it lets more noise pass through and/or amplifies it. As noise in the system aggravates, there are more abrupt changes in estimation of internal states.

4 Noiseless output feedback control

With the analysis now performed on just the state estimates, this section now incorporates the state estimates to generate control input. This changes the combined output feedback controller:

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= (A + LC + BK)\hat{\mathbf{x}} - L\mathbf{y} \\ u &= K\hat{\mathbf{x}}\end{aligned}$$

While true states are often not available to track, this section examines the affect of state estimates $\hat{\mathbf{x}}$ have on the true states \mathbf{x} .

4.1 Plots

In figures 4 and 5, K corresponds to the true state under state feedback control, L_1 corresponds to true state under output feedback control with gain L_1 , and L_2 corresponds to true state under output feedback control with gain L_2 .

4.2 Behaviour of the linearized system

As seen in the previous section, L_2 has a faster convergence due to its convergence-rate-controlling-eigenvalues being more negative. Because $\hat{\mathbf{x}}$ converges to its true value sooner, a more accurate control signal $u = K\hat{\mathbf{x}}$ is

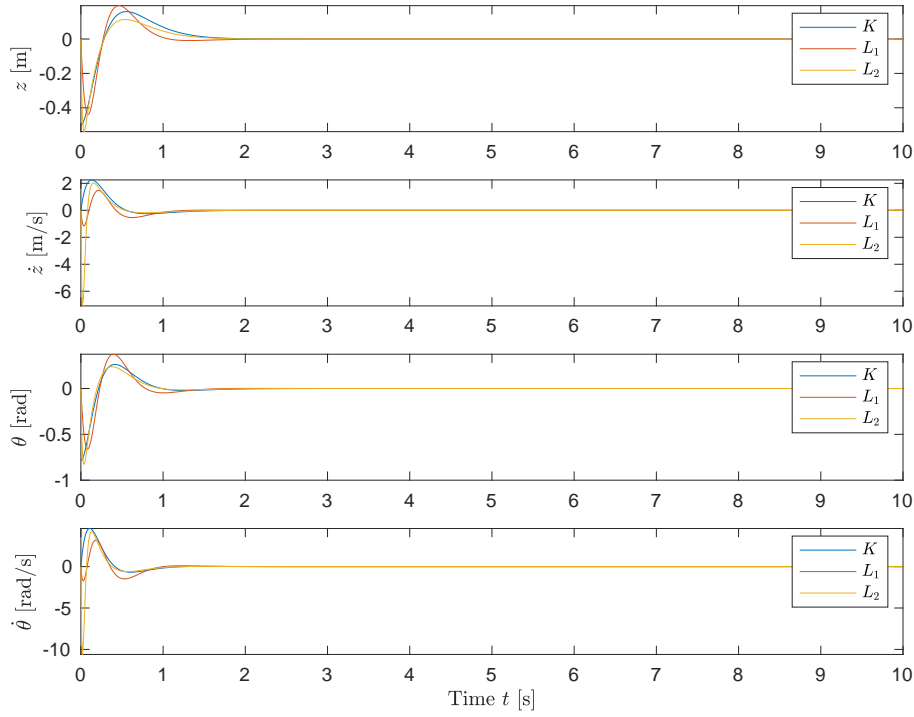


Figure 4: Linear System State Evolution with Output Feedback Control

generated, and hence the estimated states are able to follow the true states more accurately when L_2 is used over L_1 . Note, the "spiking" behaviour of the derivative states is also observed in this case. This "spiking" behaviour causes the estimated derivative states to overshoot initially, adding an undesirable characteristic to L_2 . This was the most significant difference from the state-feedback controller output.

We are only able to observe \mathbf{z} and θ , and those states are being tracked well by L_2 . Recall, the state feedback controller output starts at the true state values, i.e., the initial conditions, whereas the state estimates start at 0. Both observers are trying to compensate for that by trying to reach \mathbf{z} and θ quickly, but that comes at the price of the spikes in $\dot{\mathbf{z}}$ and $\dot{\theta}$.

4.3 Behaviour of the nonlinear system

Even in the case of a nonlinear system, L_2 is able to converge to the true state faster than L_1 . Moreover, L_2 is able to follow the true system response behaviour. On the contrary, the slower response of L_1 introduces an oscillatory behaviour in the system and results in a slower convergence as well as different behaviour from the true system response. For L_1 to perform better, it needs a more of a warm start (closer to true values).

5 Output feedback control with measurement noise

Similar to the role of noise in the previous section, the final section of this lab aims to compare the designed observers if noise is introduced in the system output.

5.1 Plot

Refer to figure 6 for this section's plots.

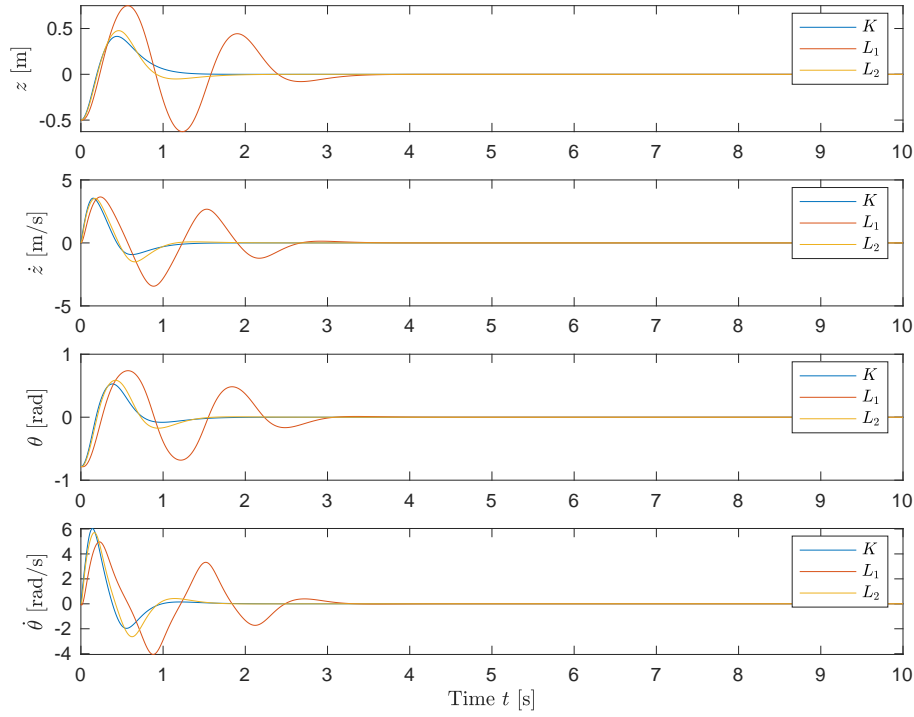


Figure 5: Nonlinear System State Evolution with Output Feedback Control

5.2 Comparison with performance in noiseless observation

As discussed in output 3, L_2 is able to detect changes faster than L_1 . Figure 6 shows that for $t < 1$ second, L_2 actually does not capture too much noise, and is able to adapt to changes in the output well. However, the closer it gets to the steady-state, the more the estimated states become prone to noise as it finally starts to "converge" to noise better.

5.3 Impact of observer speed (gains L_1 and L_2) on controller performance

While both L_2 and L_1 converge to the linearization equilibrium as steady state, L_2 keeps capturing the noise given to it. This noise is further amplified in the derivative states (\dot{z} and $\dot{\theta}$) as differentiation is a high-pass operation.

5.4 Further observations

As expected, L_2 is more sensitive to changes in the output. This results in unwanted behaviour when noise is introduced. L_1 has a smoother system response, because of its lower cutoff frequency, eliminating unnecessary noise. This was discussed when MSEs were computed, and L_1 exhibited better behaviour when subject to noise.

Essentially, L_2 fails here because it is trying to track the noise introduced. While it does converge to 0, further noise keeps showing up in the system response. Furthermore, having noise being tracked by the observer is not ideal, as the controller will have abrupt changes. Controllers are physical systems, and having these abrupt fluctuations can be difficult to realize and/or damage the physical system.

This experiment outlines one major benefit and flaw of designing observers with eigenvalues away from the imaginary axis. The benefit of such a design is that the system output is converged more quickly, but this

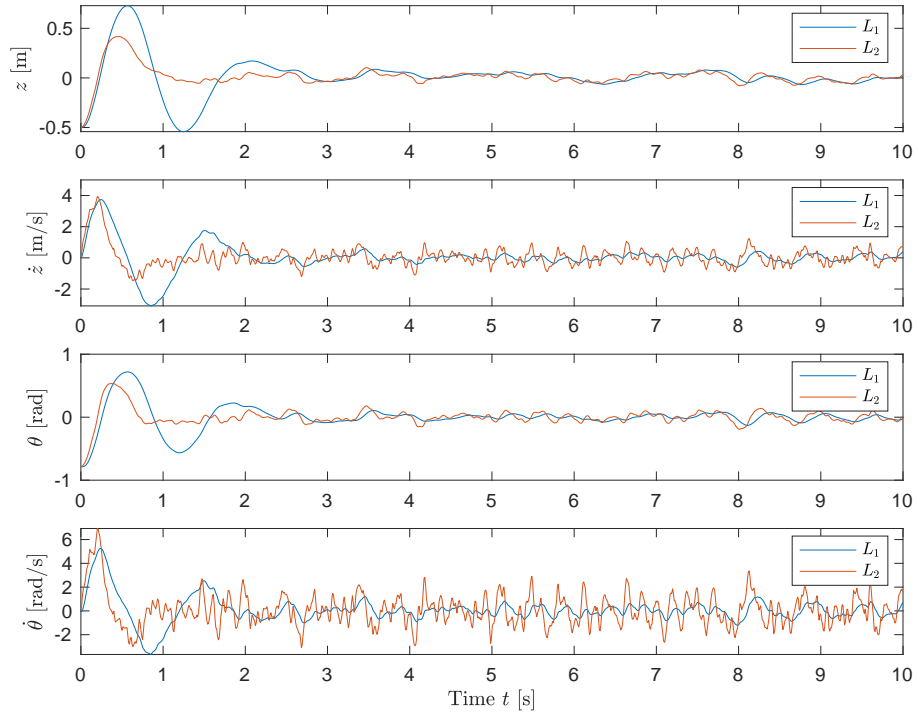


Figure 6: Nonlinear System State Evolution with Output Feedback Control and Noise

comes at the expense of tracking noise as well. The advantage is very ideal, but tracking noise is not ideal. Hence, to correct for this, the controller output could be passed through a low-pass filter.