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Solution 1: Gaussian Mixture Models (Final Exam 2018, Problem 5)

We are given a pre-trained Gaussian Mixture Model (GMM) for the marks of $K = 2$ classes: undergraduate students ($w_1 = \frac{2}{3}, \mu_1 = 70, \sigma_1^2 = 100$) and graduate students ($w_2 = \frac{1}{3}, \mu_2 = 80, \sigma_2^2 = 25$).

a) Taking a random variable X, with the probability distribution defined by the GMM, the probability $P(X \geq 80) = P(X \geq 80 \mid \text{undergraduate})P(\text{undergraduate}) + P(X \geq 80 \mid \text{graduate})P(\text{graduate})$ by the law of total probability. The weights of the GMM convey the share of total population for each class, and hence $P(\text{undergraduate}) = w_1$ and $P(\text{graduate}) = w_2$. Further using the approximation $P(|X - \mu_X| \leq \sigma_X) = \frac{2}{3}$, we get $P(X \geq 80) = (\frac{1}{2} - \frac{2}{3}\frac{1}{2})\frac{2}{3} + \frac{1}{2}\frac{1}{3} = \frac{5}{18}$.

b) The probability $P(\text{undergraduate} \mid X \geq 80) = \frac{P(\text{undergraduate}, X \geq 80)}{P(X \geq 80)} = \frac{P(X \geq 80 | \text{undergraduate})P(\text{undergraduate})}{P(X \geq 80)}$, using Bayes' theorem. Then, using the previous part, we get $P(\text{undergraduate} \mid X \geq 80) = \frac{(\frac{1}{2} - \frac{2}{3}\frac{1}{2})\frac{2}{3}}{\frac{5}{18}} = \frac{2}{5}$.