ECE537 - Random Processes
Programming Assignment 4
Nov 20, 2021

Generating Random Processes (Low-Pass Processes)

We may use an array of i.i.d. Gaussian r.v.'s $\{X_k\}$ to generate a low-pass random process, i.e. a random process with power spectral density with a certain bandwidth B. We may think of these samples as the samples of a bandlimited white noise process with bandwidth $B = \frac{1}{2T}$, where T is the time spacing of the samples. The process is then given by

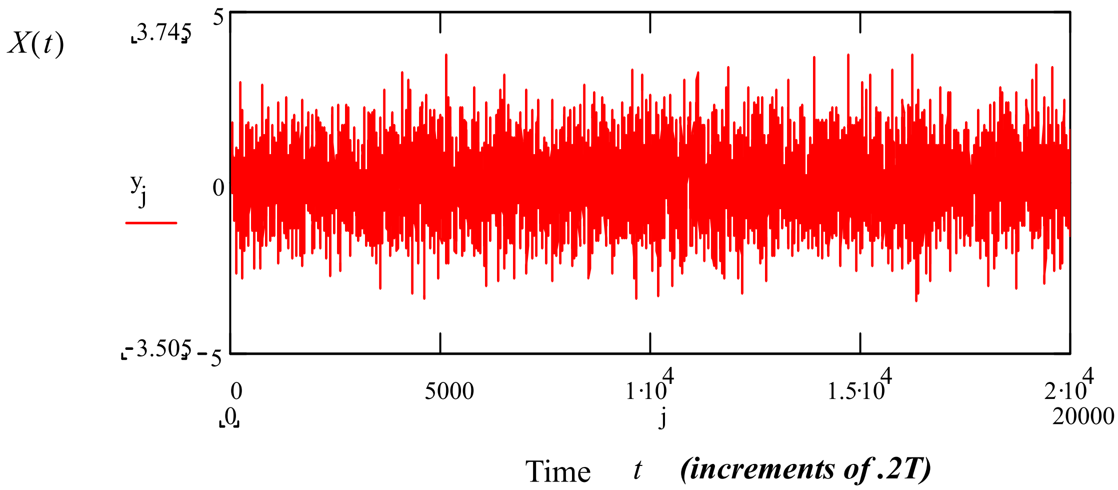
$$X(t) = \sum_k X_k \mathrm{sinc}\left(\frac{t-kT}{T}\right)$$

In other words we are using the sinc function as an interpolating function for the samples $\{X_k\}$. For computation we need to truncate the above sum. Now to evaluate the above at t we note that $k = \left\lfloor \frac{t}{T} \right\rfloor$ is the index of the sample that is closest to t on the left side. We choose 5 terms on either side of the variable t as follows:

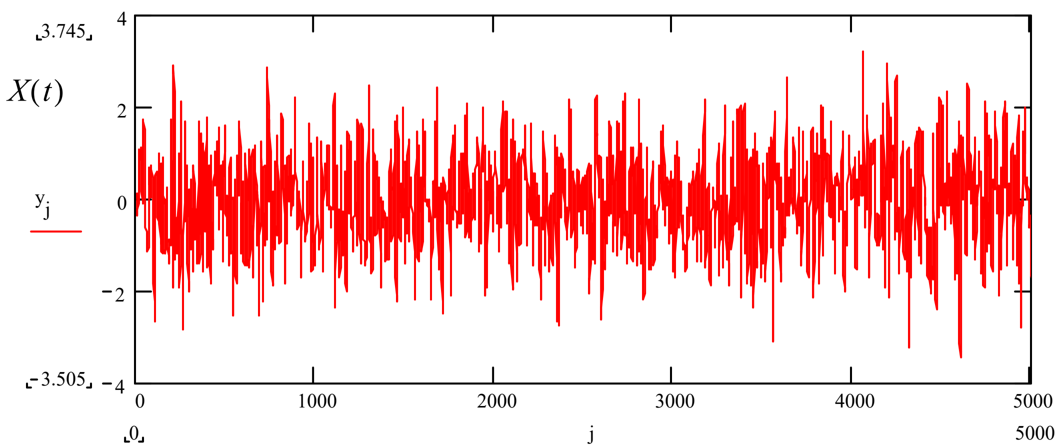
$$X(t) = \sum_{k = \left\lfloor \frac{t}{T} \right\rfloor - 5}^{\left\lfloor \frac{t}{T} \right\rfloor + 5} X_k \mathrm{sinc}\left(\frac{t-kT}{T}\right)$$

The above is a convenient formula to compute values of a sample function of the process at any time t. We may now plot the above to display a typical sample function of the process: In order to obtain a smooth curve we plot the above by sampling at a rate that is higher that $\frac{1}{T}$. We choose a rate $\frac{5}{T}$, or a sampling period equal $\Delta t = 0.2 \times T$. The samples are $y_j = X(j\Delta t)$.

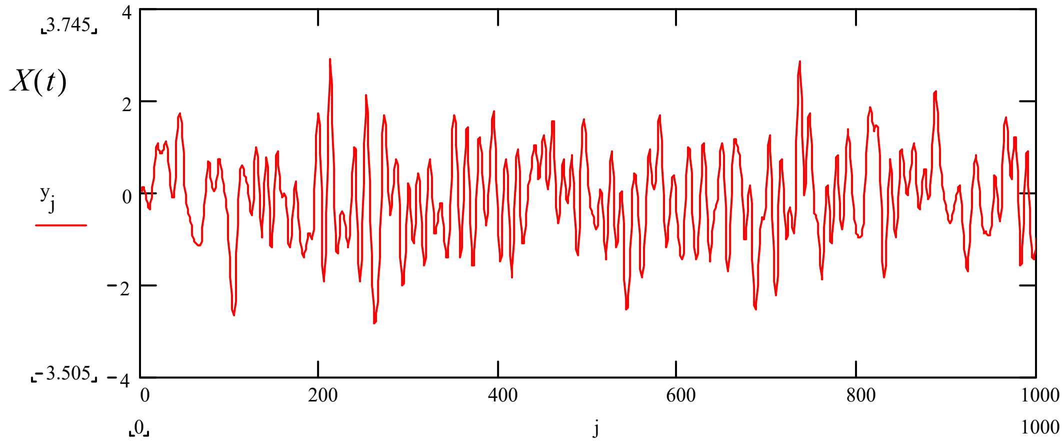
1. Generate a set of i.i.d. Gaussian r.v.'s, X_k, with zero mean and variance equal to 1. Plot $X(t)$ for the case of $t \in [0, 4000T]$. You should get something similar to the following for three different time scales:



Just like with an oscilloscope we can change the time scale. The following shows the same sample function in the time $t \in [0, 5000T]$



The following shows the same sample function in the time $t \in [0, 1000T]$



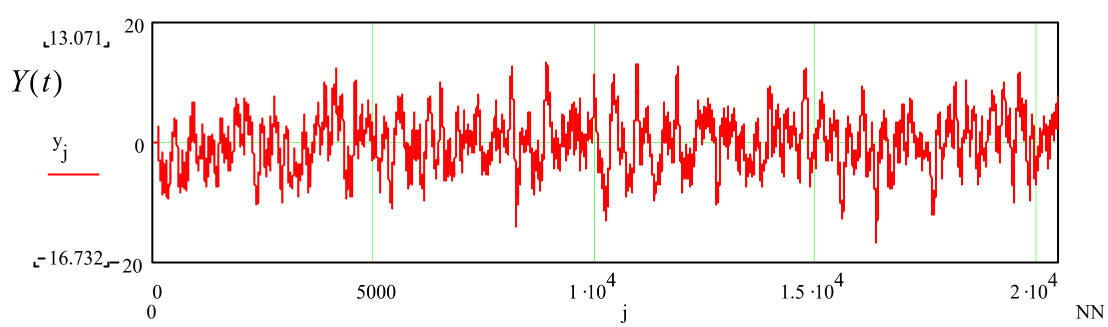
Now we will consider a more general filter $h(t)$. As an example we consider the filter $h(t) = e^{-at}$ for $0 < t < 20$ and zero elsewhere. This is a truncated impulse response for the RC low pass filter. For $a = 0$ $h(t)$ is the moving average filter. To simplify the expressions we will also assume $T = 1$. With this truncated impulse response our noise process becomes the following:

$$Y(t) = \Delta t \sum_{k=0}^{100} X(\lfloor t \rfloor - k\Delta t) e^{-a(t - \lfloor t \rfloor + k\Delta t)}$$

where we have defined $X(t) = 0$ for $t < 0$.

$Y(t)$ can be evaluated and plotted for any sequence of points. $Y(t)$ is a filtered version of the process $X(t)$.

2. ***Sample*** $Y(t)$ ***at the rate*** $\frac{5}{T} = 5$ ***(as above) and plot for the two cases*** $a = 0$ ***and*** $a = 0.2$**.** ***For the case*** $a = 0$ ***you should get something similar to the following:***

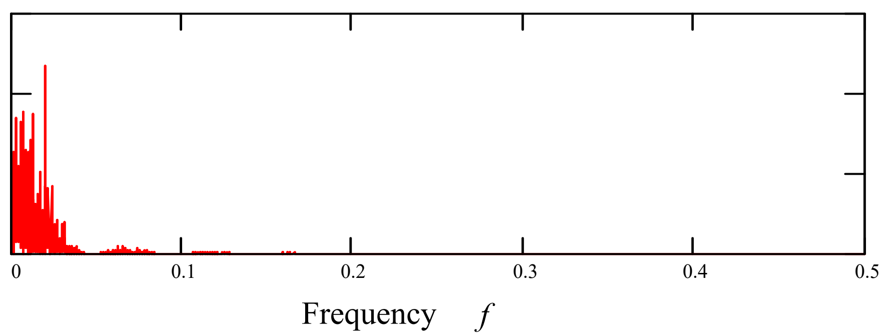


3. ***Determine the power spectral density*** $(S_Y(f))$ ***of the above process analytically. Note that the "longer" is the impulse response of*** $h(t)$ ***the narrower is the power spectral density of the process. For the above filter determine*** $|H(f)|^2$ ***for*** $a = 0$ ***and for*** $a = 0.2$ ***and the corresponding PSD's for the process*** $Y(t)$**.**

Estimate of Power Spectral Density from a Sample Function

Now we can get an estimate of $S_Y(f)$ by taking the Fourier Transform of the sample function of the process that we have generated. Do an FFT of length equal to N.

4. ***Define a set of time samples for the FFT as follows*** $y_i = Y(i)$ ***for*** $i = 0, \ldots, N-1$***, i.e. we are choosing a sampling rate equal to*** $\frac{1}{T}$***. Now take the FFT and plot the square of the absolute value of the FFT samples versus the index. Note that for the FFT we have the following relation*** $N = \frac{1}{\Delta t \Delta f}$***.. In our case we have*** $\Delta t = T = 1$ ***and*** $N = 2^{13}$***. Hence we may compute*** Δf ***and associate the frequency*** $f = k\Delta f$ ***with the k-th component of the FFT. You should get something like the following for*** $a = 0$**.**



Note that we have labelled the frequency axis as $f = k\Delta f$.

5. Repeat the above estimate of the PSD $n_t = 20$ ***times and plot the average value. What is the limiting value of the plot as*** $n_t \to \infty$***? You may think of this as the ensemble average approach to determine the power spectral density. It is the definition of PSD.***

6. Generate a very long sample function of the process $Y(t)$ ***with*** $\alpha = 0.2$ ***over the time interval*** $t \in [0, 100000T]$***. Use the ergodic property of the process to estimate (compute) the autocorrelation function of the process*** $Y(t)$ ***and plot it. Then use the Wiener Khinchin theorem to determine the power spectral density of*** $Y(t)$***.***