Complexity Classification in Infinite-Domain Constraint Satisfaction

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CHAPTER 1

Introduction



Constraint satisfaction problems (CSPs) appear in almost every area of theoretical computer science, for instance in artificial intelligence, scheduling, computational linguistics, computational biology, verification, and algebraic computation. Many computational problems studied in those areas can be modeled by appropriately choosing a set of constraint types, the *constraint language*, that are allowed in the input instance of a CSP. In the last decade, huge progress was made to find general criteria for constraint languages that imply that the corresponding CSP can be solved efficiently [12,61,63,64,66,95,123].

Lately, the complexity of the CSP became a topic that vitalizes the field of universal algebra, since it turned out that questions about the computational complexity of CSPs translate to important universal-algebraic questions about algebras that can be associated to CSPs. This approach is now known as the *algebraic approach* to constraint satisfaction complexity. The algebraic approach has raised questions that are of central importance in universal algebra.

Another reason why the complexity of CSPs attracts attention is an exciting conjecture due to Feder and Vardi [95], which is still unresolved, and which is known as the *dichotomy conjecture*. This conjecture says that every CSP with a finite domain is either polynomial-time tractable (i.e., in P) or NP-complete. According to a well-known result by Ladner, it is known that there are NP-intermediate computational problems, i.e., problems in NP that are neither tractable nor NP-complete (unless P=NP). But the known NP-intermediate problems are extremely artificial. It would be interesting from a complexity theoretic perspective to discover more natural candidates for NP-intermediate problems. Unlike many questions in computational

complexity that are wide open, the dichotomy conjecture allows many promising partial results and different approaches (see the collection of survey articles in [80]), and therefore is an attractive research topic.

Any outcome of the dichotomy conjecture is significant: a negative answer might provide relatively natural NP-intermediate problems, which would be interesting for complexity theorists. A positive answer probably comes with a criterion which describes the NP-hard CSPs (and it would probably even provide algorithms for the polynomial-time tractable CSPs). But then we would have a fascinatingly rich catalogue of computational problems where the computational complexity is known. Such a catalogue would be a valuable tool for deciding the complexity of computational problems: since CSPs are abundant, one might derive algorithmic results by reducing the problem of interest to a known tractable CSP, and one might derive hardness results by reducing a known NP-hard CSP to the problem of interest.

Even though very powerful partial results on the dichotomy conjecture have been obtained in recent years, the impact of constraint satisfaction complexity theory on other fields in theoretical computer science has so far been modest. A reason might be that the range of problems in the literature that can be described by specifying a constraint language over a finite domain, and that have been studied independently from the CSP framework, is quite limited, and mostly focussed on specialized graph theoretic problems or Boolean satisfiability problems.

If we consider the class of all problems that can be formulated by specifying a constraint language over an *infinite* domain, the situation changes drastically. Many problems that have been studied independently in temporal reasoning, spatial reasoning, phylogenetic reconstruction, and computational linguistics can be directly formulated as CSPs. Also feasibility problems in linear (and also non-linear) programming (over the rationals, the integers, or other domains) can be cast as CSPs.

The goal of this thesis is to generalize the universal-algebraic approach to infinite domains. It turns out that this is possible when the constraint language, viewed as a relational structure $\mathfrak B$ with an infinite domain, is ω -categorical. Many of the CSPs in the mentioned application areas can be formulated with ω -categorical constraint languages — in particular, problems coming from so-called qualitative calculi in artificial intelligence tend to have formulations with ω -categorical constraint languages. While ω -categoricity is a quite strong assumption from a model-theoretic point of view (and, for example, constraint languages for linear programming cannot be ω -categorical), the class of computational problems that can be formulated with ω -categorical constraint languages is still a very large generalization of the class of CSPs that can be formulated with a constraint language over a finite domain. This will be amply demonstrated by examples of ω -categorical constraint languages from many different areas in computer science in Chapter 4.

There are several general results for ω -categorical structures that are relevant when studying the computational complexity of the respective CSPs. Every ω -categorical structure is homomorphically equivalent to an ω -categorical structure which is model-complete and a core. Model-complete cores have many good properties: for example, those structures have quantifier elimination once expanded by all primitive positive definable relations; this is treated in Chapter 3. Since homomorphically equivalent structures have the same CSP, we can therefore focus on constraint languages that have those properties.

Moreover, it can be shown that the so-called *polymorphism clone* of an ω -categorical structure \mathfrak{B} fully captures the computational complexity of the corresponding CSP (Chapter 5). By this observation, universal-algebraic techniques can be used to analyze the computational complexity of the CSP for \mathfrak{B} . Indeed, the study of CSPs has

triggered questions that are of central interest in universal algebra, and that have led to considerable new activity (see e.g. [13,22,159,187]).

Another tool that becomes useful specifically for polymorphisms over infinite domains is $Ramsey\ theory\ (Chapter\ 8)$. The basic idea here is to apply Ramsey theory to show that polymorphisms must act $canonically\$ on large parts of their domain. Typically there are only finitely many possibilities for canonical behavior, and so this technique allows to perform combinatorial analysis when proving classification results. With this approach we can also show that, under further assumptions on \mathfrak{B} , many questions about the expressive power of \mathfrak{B} become decidable, such as the question whether a given quantifier-free first-order formula is in \mathfrak{B} equivalent to a primitive positive formula.

An important feature of the universal-algebraic approach is that tractability of a CSP can be linked to the existence of polymorphisms of the constraint language. This link can be exploited in several directions: first, when we already know that a constraint language of interest has a polymorphism satisfying good properties, then this polymorphism can guide the search for an efficient algorithm for the corresponding CSP. Another direction is that we already have an algorithm (or an algorithmic technique), and that we want to know for which CSPs the algorithm is a correct decision procedure: again, polymorphisms are the key tool for this task. Finally, we might use the absence of polymorphisms with good properties to prove that a CSP is NP-hard. There are several instances where these three directions of the algebraic approach have been used very successfully for CSPs with finite domain constraint languages [12, 64, 73, 123] or ω -categorical constraint languages [41, 52].

In Chapter 9 and Chapter 10 we use polymorphisms to classify the computational complexity in some large families of constraint satisfaction problems. In Chapter 9, we study constraint languages definable over the random graph, and in Chapter 10 constraint languages definable over $(\mathbb{Q}; <)$. Even though the two underlying structures are very different from a model-theoretic point of view, and even though the classification proofs are very different in both cases, we can give a common formulation of the two classification results that delineates also the border between polynomial-time solvable and NP-complete CSPs.

Chapter outline. Constraint satisfaction problems can appear in several different forms, because there are several ways how CSPs can be formalized. The differences in formalizing constraint satisfaction problems are related to the way how instances are coded and to how the problem itself is described. In the next sections we present four formalisms; each of those formalisms is attached to a different line of research. In later sections some arguments are more natural from one perspective than from the other, so it will be convenient to have them all discussed here. See Figure 1.1 for an illustration how the four perspectives we discuss can be put into relationship to each other.

Perspective	Instance	Problem Description
Homomorphism	Structure	Structure
Sentence Evaluation	Sentence	Structure
Satisfiability	Sentence	Sentences
Existential Second-Order	Structure	Sentence

FIGURE 1.1. The four perspectives on the definition of CSPs.

1.1. The Homomorphism Perspective

A relational signature τ is a set of relation symbols R_i , each of which has an associated finite arity k_i . A relational structure \mathfrak{A} over the signature τ (also called τ -structure) consists of a set A (the domain or base set) together with a relation $R^{\mathfrak{A}} \subseteq A^k$ for each relation symbol R of arity k from τ . It causes no harm to allow structures whose domain is empty.

A homomorphism h from a structure \mathfrak{A} with domain A to a structure \mathfrak{B} with domain B and the same signature τ is a mapping from A to B that preserves each relation for the symbols in τ ; that is, if (a_1, \ldots, a_k) is in $R^{\mathfrak{A}}$, then $(h(a_1), \ldots, h(a_k))$ must be in $R^{\mathfrak{B}}$. An isomorphism is a bijective homomorphism h such that the inverse mapping $h^{-1}: B \to A$ that sends h(x) to x is a homomorphism, too.

In this thesis, a *(non-uniform)* constraint satisfaction problem *(CSP)* is a computational problem that is specified by a single structure with a finite relational signature, called the *template* (or the *constraint language*; the name 'constraint language' is typically used in the context of the second perspective on CSPs that we present in Section 1.2).

DEFINITION 1.1.1 (CSP(\mathfrak{B})). Let \mathfrak{B} be a (possible infinite) structure with a finite relational signature τ . Then CSP(\mathfrak{B}) is the computational problem to decide whether a given finite τ -structure \mathfrak{A} homomorphically maps to \mathfrak{B} .

 $CSP(\mathfrak{B})$ can be considered to be a class — the class of all finite τ -structures that homomorphically map to \mathfrak{B} .

A homomorphism from a given τ -structure $\mathfrak A$ to $\mathfrak B$ is called a *solution* of $\mathfrak A$ for $\mathrm{CSP}(\mathfrak B)$. It is in general not clear how to represent solutions for $\mathrm{CSP}(\mathfrak B)$ on a computer; however, for the definition of the problem $\mathrm{CSP}(\mathfrak B)$ we do not need to represent solutions, since we only have to decide the *existence* of solutions. To represent an input structure $\mathfrak A$ of $\mathrm{CSP}(\mathfrak B)$ we can fix any representation of the relation symbols in the signature τ , due to the assumption that τ is *finite*. Thus, $\mathrm{CSP}(\mathfrak B)$ is a well-defined computational problem for *any* infinite structure $\mathfrak B$ with finite relational signature.

EXAMPLE 1.1.2 (Digraph acyclicity). Next, consider the problem $CSP((\mathbb{Z};<))$. Here, the relation < denotes the strict linear order of the integers \mathbb{Z} . An instance \mathfrak{A} of this problem can be viewed as a directed graph (also called digraph), potentially with loops. It is easy to see that \mathfrak{A} homomorphically maps to $(\mathbb{Z};<)$ if and only if there is no directed cycle in \mathfrak{A} (loops are considered to be directed cycles, too). It is easy to see and well-known that this can be tested in linear time, for example by performing a depth-first search on the digraph \mathfrak{A} .

EXAMPLE 1.1.3 (Betweenness). The so-called betweenness problem [170] can be modeled as $CSP((\mathbb{Z}; Betw))$ where Betw is the ternary relation

$$\{(x, y, z) \in \mathbb{Z}^3 \mid (x < y < z) \lor (z < y < x)\}$$
.

This problem is one of the NP-complete problems listed in the book of Garey and Johnson [101].

EXAMPLE 1.1.4 (Cyclic-Ordering). The *Cyclic-order problem* [99] can be modeled as $CSP((\mathbb{Z}; Cycl))$ where Cycl is the ternary relation

$$\{(x,y,z) \in \mathbb{Z}^3 \mid (x < y < z) \lor (y < z < x) \lor (z < x < y)\} \ .$$

This problem is again NP-complete and can be found in [101].

EXAMPLE 1.1.5 (\mathfrak{H} -coloring problems). Let \mathfrak{H} be an (undirected) graph. We view undirected graphs as τ -structures where τ contains a single binary relation symbol E,

which denotes a symmetric and anti-reflexive relation. Then the \mathfrak{H} -coloring problem is the computational problem to decide for a given finite graph \mathfrak{G} whether there exists a homomorphism from \mathfrak{G} to \mathfrak{H} . For instance, if \mathfrak{H} is the graph K_3 (the complete graph on three vertices), then the \mathfrak{H} -coloring problem is the famous 3-colorability problem (see e.g. [101]). Similarly, for every fixed k, the k-colorability problem can be modeled as $\mathrm{CSP}(\mathfrak{H})$, for an appropriate graph \mathfrak{H} .

The next lemma (Lemma 1.1.7) is a useful test to determine whether a computational problem can be formulated as $CSP(\mathfrak{B})$ for an infinite relational structure \mathfrak{B} . An (induced) substructure of a τ -structure \mathfrak{A} is a τ -structure \mathfrak{B} with $B \subseteq A$ and $R^{\mathfrak{B}} = R^{\mathfrak{A}} \cap B^n$ for each n-ary $R \in \tau$; we also say that \mathfrak{B} is induced by B in \mathfrak{A} , and write $\mathfrak{A}[B]$ for \mathfrak{B} . The union of two τ -structures $\mathfrak{A}, \mathfrak{B}$ is the τ -structure $\mathfrak{A} \cup \mathfrak{B}$ with domain $A \cup B$ and relations $R^{\mathfrak{A} \cup \mathfrak{B}} = R^{\mathfrak{A}} \cup R^{\mathfrak{B}}$ for all $R \in \tau$. The intersection $\mathfrak{A} \cap \mathfrak{B}$ of \mathfrak{A} and \mathfrak{B} is defined analogously. A disjoint union of \mathfrak{A} and \mathfrak{B} is the union of isomorphic copies of \mathfrak{A} and \mathfrak{B} with disjoint domains. As disjoint unions are unique up to isomorphism, we usually speak of the disjoint union of \mathfrak{A} and \mathfrak{B} , and denote it by $\mathfrak{A} \oplus \mathfrak{B}$. The disjoint union of a set of τ -structures \mathcal{C} is defined analogously (and the disjoint union of an empty set of structures is the τ -structure with empty domain). A structure is called connected if it is not the disjoint union of two non-empty structures. A maximal connected substructure of \mathfrak{B} is called a connected component of \mathfrak{B} .

Definition 1.1.6. We say that a class C of relational structures is

- closed under homomorphisms iff whenever $\mathfrak{A} \in \mathcal{C}$ and \mathfrak{A} homomorphically maps to \mathfrak{B} then $\mathfrak{B} \in \mathcal{C}$;
- closed under inverse homomorphisms iff whenever $\mathfrak{B} \in \mathcal{C}$ and \mathfrak{A} homomorphically maps to \mathfrak{B} then $\mathfrak{A} \in \mathcal{C}$;
- closed under (finite) disjoint unions iff whenever $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ then the disjoint union of \mathfrak{A} and \mathfrak{B} is also in \mathcal{C} .

Note that a class \mathcal{C} of τ -structures is closed under inverse homomorphisms if and only if its complement in the class of all τ -structures is closed under homomorphisms. When a class is closed under inverse homomorphisms, or closed under homomorphisms, it is in particular closed under isomorphisms. The following is a simple, but fundamental lemma for CSPs. When \mathcal{N} is a class of τ -structures, we say that a structure \mathfrak{A} is \mathcal{N} -free if no $\mathfrak{B} \in \mathcal{N}$ homomorphically maps to \mathfrak{A} . The class of all finite \mathcal{N} -free structures we denote by $\operatorname{Forb}(\mathcal{N})$.

LEMMA 1.1.7. Let τ be a finite relational signature, and \mathcal{C} a class of finite τ -structures. Then the following are equivalent.

- (1) $C = CSP(\mathfrak{B})$ for some τ -structure \mathfrak{B} .
- (2) $C = \text{Forb}(\mathcal{N})$ for a class of finite connected τ -structures \mathcal{N} .
- (3) C is closed under disjoint unions and inverse homomorphisms.
- (4) $C = CSP(\mathfrak{B})$ for a countably infinite τ -structure \mathfrak{B} .

PROOF. It suffices to prove the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$. For the implication from (1) to (2), let \mathcal{N} be the class of all finite connected τ -structures that do not homomorphically map to \mathfrak{B} . Then by transitivity of the homomorphism relation, a τ -structure \mathfrak{A} homomorphically maps to \mathfrak{B} if and only if no $\mathfrak{C} \in \mathcal{N}$ homomorphically maps to \mathfrak{A} .

(2) implies (3). Suppose (2), and let \mathfrak{A}_1 and \mathfrak{A}_2 be two structures from Forb(\mathcal{N}). If there were a homomorphism from one of the structures $\mathfrak{C} \in \mathcal{N}$ into $\mathfrak{A}_1 \uplus \mathfrak{A}_2$, then because \mathfrak{C} is connected, it must already be a homomorphism into \mathfrak{A}_1 or \mathfrak{A}_2 , which is impossible. Hence, Forb(\mathcal{N}) is closed under disjoint unions. Closure under inverse

Triangle-Freeness

INSTANCE: An undirected graph & QUESTION: Is & triangle-free?

Acyclic-Bipartition

INSTANCE: A digraph &

QUESTION: Is there a partition $V = V_1 \uplus V_2$ of the vertices V of \mathfrak{G} such that

both $\mathfrak{G}[V_1]$ and $\mathfrak{G}[V_2]$ are acyclic?

No-Mono-Tri

INSTANCE: An undirected graph &

QUESTION: Is there a partition $V = V_1 \uplus V_2$ of the vertices V of $\mathfrak G$ such that

both $\mathfrak{G}[V_1]$ and $\mathfrak{G}[V_2]$ are triangle-free?

FIGURE 1.2. Three computational problems that are closed under disjoint unions and inverse homomorphisms.

homomorphisms follows straightforwardly from transitivity of the homomorphism relation.

(3) implies (4). Suppose that \mathcal{C} is a class of finite relational structures that is closed under disjoint unions and inverse homomorphisms. Let \mathcal{C}' be a subclass of \mathcal{C} where we select one structure from each isomorphism class of structures in \mathcal{C} . Let \mathfrak{B} be the (countably infinite) disjoint union over all structures in \mathcal{C}' (if \mathcal{C} is empty then \mathfrak{B} is by definition the empty structure¹). Clearly, every structure in \mathcal{C} homomorphically maps to \mathfrak{B} . Now, let \mathfrak{A} be a finite structure with a homomorphism h to \mathfrak{B} . By construction of \mathfrak{B} , the set h(A) is contained in the disjoint union \mathfrak{C} of a finite set of structures from \mathcal{C} . Since \mathcal{C} is closed under disjoint unions, \mathfrak{C} is in \mathcal{C} . Clearly, \mathfrak{A} homomorphically maps to \mathfrak{C} , and because \mathcal{C} is closed under inverse homomorphisms, \mathfrak{A} is in \mathcal{C} as well.

EXAMPLE 1.1.8. The computational problems in Figure 1.2 are closed under disjoint unions and inverse homomorphisms. Hence, Lemma 1.1.7 shows that they can be formulated as $CSP(\mathfrak{B})$ for some relational structure \mathfrak{B} . It is easy to see that none of those three problems can be formulated as $CSP(\mathfrak{B})$ for a *finite* structure \mathfrak{B} .

We verify this for the problem of Triangle-freeness. For a fixed n, consider the graph that contains vertices x_1, \ldots, x_n , and that contains for every pair i, j with $1 \le i < j \le n$ two additional vertices $u_{i,j}, v_{i,j}$ and the edges $(x_i, u_{i,j}), (u_{i,j}, v_{i,j}), (v_{i,j}, x_j)$. The resulting graph is clearly triangle-free. But note that every homomorphism f from this graph to a graph \mathfrak{H} with strictly less than n vertices must identify at least two of the vertices x_1, \ldots, x_n . So suppose that $f(x_i) = f(x_j)$. Because f is a homomorphism, we have that $(f(x_i), f(u_{i,j})), (f(u_{i,j}, f(v_{i,j})), (f(v_{i,j}), f(x_j))$ are edges in \mathfrak{H} . Hence, \mathfrak{H} either contains a triangle or a loop. In both cases, \mathfrak{H} cannot be the template for Triangle-Freeness. Hence we have ruled out all templates of size n-1. This concludes the proof since n was chosen arbitrarily.

We close with an important concept for finite structures \mathfrak{B} , the notion of *core structures*; generalizations to infinite structures \mathfrak{B} are presented in Section 3.6.3. Two structures \mathfrak{A} and \mathfrak{B} are called *homomorphically equivalent* if there exists a homomorphism from \mathfrak{A} to \mathfrak{B} and vice versa. An *embedding* of \mathfrak{A} into \mathfrak{B} is an injective map

¹Structures with an empty domain are often forbidden in model theory. Lemma 1.1.7 is one of the places that motivates our decision to allow them in this text.

 $f: A \to B$ such that (a_1, \ldots, a_k) is in $R^{\mathfrak{A}}$ if and only if $(f(a_1), \ldots, f(a_k))$ is in $R^{\mathfrak{B}}$. An *endomorphism* of a structure \mathfrak{B} is a homomorphism from \mathfrak{B} to \mathfrak{B} .

DEFINITION 1.1.9. A structure \mathfrak{B} is a core if all its endomorphisms are embeddings². For structures $\mathfrak{A}, \mathfrak{B}$ of the same signature, the structure \mathfrak{B} is called a core of \mathfrak{A} if \mathfrak{B} is a core and homomorphically equivalent to \mathfrak{A} .

In fact, we speak of *the core* of a finite structure \mathfrak{A} , due to the following fact, whose proof is easy and left to the reader.

Proposition 1.1.10. Every finite structure $\mathfrak A$ has a core. All cores of $\mathfrak A$ are isomorphic.

Core structures \mathfrak{B} have many pleasant properties when it comes to studying the computational complexity of $\mathrm{CSP}(\mathfrak{B})$ (see for instance Proposition 1.2.9 below). Clearly, when \mathfrak{A} and \mathfrak{B} are homomorphically equivalent, then $\mathrm{CSP}(\mathfrak{A}) = \mathrm{CSP}(\mathfrak{B})$. Therefore, and because of Proposition 1.1.10, we can assume without loss of generality that a finite structure \mathfrak{B} is a core when studying $\mathrm{CSP}(\mathfrak{B})$. We finally remark that structures with a one-element core have a trivial CSP.

PROPOSITION 1.1.11. Let \mathfrak{B} be a relational structure with a finite relational signature and a one-element core. Then $CSP(\mathfrak{B})$ is in P.

PROOF. Let \mathfrak{C} be the core of \mathfrak{B} , and let c be the unique element of \mathfrak{C} . The problem $\mathrm{CSP}(\mathfrak{B})$ can be solved as follows. Let \mathfrak{A} be an input structure of $\mathrm{CSP}(\mathfrak{B})$. If there is $(t_1,\ldots,t_n)\in R^{\mathfrak{A}}$ such that $(c,\ldots,c)\notin R^{\mathfrak{C}}$, then reject. Otherwise accept. \square

1.2. The Sentence Evaluation Perspective

Let τ be a relational signature. A first-order τ -formula $\phi(x_1, \ldots, x_n)$ is called *primitive positive* if it is of the form

$$\exists x_{n+1}, \ldots, x_m(\psi_1 \land \cdots \land \psi_l)$$

where ψ_1, \ldots, ψ_l are atomic τ -formulas, i.e., formulas of the form $R(y_1, \ldots, y_k)$ with $R \in \tau$ and $y_i \in \{x_1, \ldots, x_m\}$, of the form y = y' for $y, y' \in \{x_1, \ldots, x_m\}$, of the form \bot or \top for false and true, respectively. Note that if the domain is non-empty then we do not need a symbol \top for true, since we can use the primitive positive sentence $\exists x. x = x$ to express it. As usual, formulas without free variables are called sentences.

From a model-checking perspective, CSPs are defined as follows. We will see (in Propositions 1.2.4 and 1.2.5) that this definition is essentially the same definition as Definition 1.1.1, and that the differences are a matter of formalization³.

DEFINITION 1.2.1. Let \mathfrak{B} be a (possibly infinite) structure with a finite relational signature τ . Then $CSP(\mathfrak{B})$ is the computational problem to decide whether a given primitive positive τ -sentence ϕ is true in \mathfrak{B} .

 $^{^2}$ For finite structures \mathfrak{B} , injective self-maps must be bijective, and in fact every injective homomorphism of a structure \mathfrak{B} must be an isomorphism. For infinite structures, however, this need not be true, and for reasons that become clear in Chapter 3 we chose the present definition.

³A small difference between the homomorphism perspective and the sentence evaluation problem results from the fact that we *do* allow equality in primitive positive formulas; as we will see, adding equality to the constraint language does not affect the complexity of the CSP up to log-space reductions. There are articles, though, that study the complexity of CSPs at an even finer level than logspace-reducibility, and in those papers equality is not automatically allowed in the input to a constraint satisfaction problem.

3SAT

INSTANCE: A propositional formula in conjunctive normal form (CNF) with at most three literals per clause

QUESTION: Is there a Boolean assignment for the variables such that in each clause at least one literal is true?

Positive 1-in-3-3SAT

INSTANCE: A propositional 3SAT formula with only positive literals QUESTION: Is there a Boolean assignment for the variables such that in each clause exactly one literal is true?

Positive Not-All-Equal-3SAT

INSTANCE: A propositional 3SAT formula with only positive literals QUESTION: Is there a Boolean assignment for the variables such that in each clause neither all three literals are true nor all three are false?

FIGURE 1.3. Three Boolean satisfiability problems from the list of NP-complete problems of [101] that can be formulated as $CSP(\mathfrak{B})$ for appropriate \mathfrak{B} .

The given primitive positive τ -sentence ϕ is also called an *instance* of CSP(\mathfrak{B}). The conjuncts of an instance ϕ are called the *constraints* of ϕ . A mapping from the variables of ϕ to the elements of B that is a satisfying assignment for the quantifier-free part of ϕ is also called a *solution* to ϕ .

Some authors omit the (existential) quantifier-prefix in instances ϕ of CSP(\mathfrak{B}), and the question is then whether ϕ is *satisfiable* over \mathfrak{B} . Clearly, this is just rephrasing the problem above, but it explains the terminology of *satisfiable* and *unsatisfiable* (rather than *true* and *false*) instances of CSP(\mathfrak{B}).

EXAMPLE 1.2.2 (Boolean satisfiability problems). There are many Boolean satisfiability problems that can be cast as CSPs. Well-known examples are 3SAT (see Figure 1.3), and the restricted versions of 3SAT called 1-in-3-3SAT and NOT-ALL-EQUAL-3SAT [101]. These three problems are NP-complete. An interesting feature of the last two problems is that they remain NP-complete even when all clauses in the input only contain positive literals. With this additional restriction, the problems are called positive 1-in-3-3SAT and positive NOT-ALL-EQUAL-3SAT, and their definition can be found in Figure 1.3.

All of these problems can be formulated as $CSP(\mathfrak{B})$, for an appropriate 2-element structure \mathfrak{B} . Positive 1-in-3-3SAT can be formulated as $CSP(\mathfrak{B})$ for the template

$$\mathfrak{B} = (\{0,1\}; 1IN3)$$
 where $1IN3 = \{(0,0,1), (0,1,0), (1,0,0)\}$,

and Positive-Not-All-Equal-3SAT as CSP(B) for the template

$$\mathfrak{B} = (\{0,1\}, \text{NAE}) \text{ where } \text{NAE} = \{0,1\}^3 \setminus \{(0,0,0), (1,1,1)\}.$$

These problems can also be formulated as CSPs if we do *not* impose the restriction that all literals are positive; the corresponding problems are then called 1-in-3-3SAT and Not-All-Equal-3SAT, respectively. The idea is to use a different ternary relation for each of the eight ways how three distinct variables in a clause with three literals might be negated. In this way, we can also model the classical problem of 3SAT (again, see Figure 1.3) as a CSP. Clauses of the type $x \lor y \lor \neg z$ in the 3SAT problem will then be viewed as constraints $R^{++-}(x,y,z)$, where $R^{++-}=\{0,1\}^3\setminus\{(0,0,1)\}$

(here, x, y, z are not necessarily distinct variables). Similarly, the well-known 2SAT problem can be viewed as $CSP((\{0,1\}; R^{++}, R^{--}, R^{-+}, R^{--}))$ where

$$R^{++} = \{(0,1), (1,0), (1,1)\},$$

$$R^{+-} = \{(0,0), (1,1), (1,0)\},$$

$$R^{-+} = \{(1,1), (0,0), (0,1)\}, \text{ and }$$

$$R^{--} = \{(1,0), (0,1), (0,0)\}.$$

EXAMPLE 1.2.3 (Disequality constraints). Consider the problem $CSP((\mathbb{N};=,\neq))$. An instance of this problem can be viewed as an (existentially quantified) set of variables, some linked by equality, some by disequality⁴ constraints. Such an instance is false in $(\mathbb{N};=,\neq)$ if and only if there is a path x_1,\ldots,x_n from a variable x_1 to a variable x_n that uses only equality edges, i.e., ' $x_i = x_{i+1}$ ' is a constraint in the instance for each $1 \leq i \leq n-1$, and additionally ' $x_1 \neq x_n$ ' is a constraint in the instance. Clearly, it can be tested in linear time in the size of the input instance whether the instance contains such a path.

1.2.1. Canonical conjunctive queries. To every finite relational τ -structure \mathfrak{A} we can associate a τ -sentence, called the *canonical conjunctive query* of \mathfrak{A} , and denoted by $Q(\mathfrak{A})$. The variables of this sentence are the elements of \mathfrak{A} , all of which are existentially quantified in the quantifier prefix of the formula, which is followed by the conjunction of all formulas of the form $R(a_1, \ldots, a_k)$ for $R \in \tau$ and tuples $(a_1, \ldots, a_k) \in R^{\mathfrak{A}}$.

For example, the canonical conjunctive query $Q(K_3)$ of the complete graph on three vertices K_3 is the formula

$$\exists u \exists v \exists w \ (E(u,v) \land E(v,u) \land E(v,w) \land E(w,v) \land E(u,w) \land E(w,u)) \ .$$

The proof of the following proposition is straightforward.

PROPOSITION 1.2.4. Let \mathfrak{B} be a structure with finite relational signature τ , and let \mathfrak{A} be a finite τ -structure. Then there is a homomorphism from \mathfrak{A} to \mathfrak{B} if and only if $Q(\mathfrak{A})$ is true in \mathfrak{B} .

1.2.2. Canonical databases. To present a converse of Proposition 1.2.4, we define the *canonical database* $D(\phi)$ of a primitive positive τ -formula, which is a relational τ -structure defined as follows. We require that ϕ does not contain \bot . If ϕ contains an atomic formula of the form x = y, we remove it from ϕ , and replace all occurrences of x in ϕ by y. Repeating this step if necessary, we may assume that ϕ does not contain atomic formulas of the form x = y.

Then the domain of $D(\phi)$ is the set of variables (both the free variables and the existentially quantified variables) that occur in ϕ . There is a tuple (v_1, \ldots, v_k) in a relation R of $D(\phi)$ iff ϕ contains the conjunct $R(v_1, \ldots, v_k)$. The following is similarly straightforward as Proposition 1.2.4.

PROPOSITION 1.2.5. Let \mathfrak{B} be a structure with signature τ , and let ϕ be a primitive positive τ -sentence other than \bot . Then ϕ is true in \mathfrak{B} if and only if $D(\phi)$ homomorphically maps to \mathfrak{B} .

Due to Proposition 1.2.5 and Proposition 1.2.4, we may freely switch between the homomorphism and the logic perspective whenever this is convenient. In particular,

⁴We deliberately use the word disequality instead of inequality, since we reserve the word inequality for the relation $x \leq y$.

instances of CSP(\mathfrak{B}) can from now on be either finite structures \mathfrak{A} or primitive positive sentences ϕ .

1.2.3. Expansions. Let \mathfrak{A} be a τ -structure, and let \mathfrak{A}' be a τ' -structure with $\tau \subseteq \tau'$. If \mathfrak{A} and \mathfrak{A}' have the same domain and $R^{\mathfrak{A}} = R^{\mathfrak{A}'}$ for all $R \in \tau$, then \mathfrak{A} is called the τ -reduct (or simply reduct) of \mathfrak{A}' , and \mathfrak{A}' is called a τ' -expansion (or simply expansion) of \mathfrak{A} . When \mathfrak{A} is a structure, and R is a relation over the domain of \mathfrak{A} , then we denote the expansion of \mathfrak{A} by R by (\mathfrak{A}, R) .

The following lemma says that we can expand structures by primitive positive definable relations without changing the complexity of the corresponding CSP. Hence, primitive positive definitions are an important tool to prove NP-hardness: to show that $CSP(\mathfrak{B})$ is NP-hard, it suffices to show that there is a primitive positive definition of a relation R such that $CSP((\mathfrak{B}, R))$ is already known to be NP-hard. Stronger tools to prove NP-hardness of CSPs will be introduced in Section 5.5.

LEMMA 1.2.6. Let \mathfrak{B} be a structure with finite relational signature, and let R be a relation that has a primitive positive definition in \mathfrak{B} . Then $\mathrm{CSP}(\mathfrak{B})$ and $\mathrm{CSP}((\mathfrak{B},R))$ are linear-time equivalent. They are also equivalent under deterministic log-space reductions.

PROOF. It is clear that $CSP(\mathfrak{B})$ reduces to the new problem. So suppose that ϕ is an instance of $CSP((\mathfrak{B},R))$. Replace each conjunct $R(x_1,\ldots,x_l)$ of ϕ by its primitive positive definition $\psi(x_1,\ldots,x_l)$. Move all quantifiers to the front, such that the resulting formula is in *prenex normal form* and hence primitive positive. Finally, equalities can be eliminated one by one: for equality x=y, remove y from the quantifier prefix, and replace all remaining occurrences of y by x. Let ψ be the formula obtained in this way.

It is straightforward to verify that ϕ is true in (\mathfrak{B}, R) if and only if ψ is true in \mathfrak{B} , and it is also clear that ψ can be constructed in linear time in the representation size of ϕ . For the observation that the reduction is deterministic log-space, we need the recent result that undirected reachability can be decided in deterministic log-space [178].

EXAMPLE 1.2.7. The relation NAE (x_1, x_2, x_3) has the following primitive positive definition in $(\{0, 1\}; 1IN3)$.

```
\exists u_1, u_2, u_3, v_1, v_2, v_3, z_1, z_2, z_3 \big( \text{IIN3}(x_1, u_1, v_1) \land \text{IIN3}(x_2, u_2, v_2) \land \text{IIN3}(x_3, u_3, v_3) \\ \land \text{IIN3}(v_1, u_2, z_1) \land \text{IIN3}(v_2, u_3, z_2) \land \text{IIN3}(v_3, u_1, z_3) \land \text{IIN3}(z_1, z_2, z_3) \big)
```

To see that this works, note that when $x_1 = x_2 = x_3 = 1$, then the first three conjuncts imply that $u_1 = v_1 = u_2 = v_2 = u_3 = v_3 = 0$, and the next three conjuncts imply that $z_1 = z_2 = z_3 = 1$, and hence the last conjunct is violated. When $x_1 = x_2 = x_3 = 0$, then the first conjunct implies that $u_1 = 0$ and $v_1 = 1$, or $u_1 = 1$ and $v_1 = 0$. In both cases, the fourth conjunct implies that $z_1 = 0$. Similarly, we can infer that $z_2 = z_3 = 0$. Whence, the last conjunct is violated.

Now consider the case when exactly one out of x_1, x_2, x_3 is 0. Since the formula is symmetric with respect to x_1, x_2, x_3 , we assume without loss of generality that $x_1 = 0, x_2 = 1, x_3 = 1$. Then we can set $u_1 = z_1 = z_2 = 1$, and $v_1 = u_2 = v_2 = u_3 = v_3 = z_3 = 0$ and satisfy all conjuncts. Similarly, when exactly two out of x_1, x_2, x_3 are 0, we assume without loss of generality that $x_1 = 1, x_2 = x_3 = 0$. Then we can set $u_1 = v_1 = u_2 = u_3 = z_2 = z_3 = 0$ and $z_1 = v_2 = v_3 = 1$ and satisfy all conjuncts. \square

An automorphism of a structure \mathfrak{B} with domain B is an isomorphism between \mathfrak{B} and itself. When applying an automorphism α to an element b from B we omit brackets, that is, we write αb instead of $\alpha(b)$. The set of all automorphisms α of \mathfrak{B}

is denoted by $\operatorname{Aut}(\mathfrak{B})$, and α^{-1} denotes the inverse map of α . Let (b_1, \ldots, b_k) be a k-tuple of elements of \mathfrak{B} . A set of the form $S = \{(\alpha b_1, \ldots, \alpha b_k) \mid \alpha \in \operatorname{Aut}(\mathfrak{B})\}$ is called an *orbit of* k-tuples (the *orbit of* (b_1, \ldots, b_k)).

LEMMA 1.2.8. Let \mathfrak{B} be a structure with a finite relational signature and domain B, and let $R = \{(b_1, \ldots, b_k)\}$ be a k-ary relation that only contains one tuple $(b_1, \ldots, b_k) \in B^k$. If the orbit of (b_1, \ldots, b_k) in \mathfrak{B} is primitive positive definable, then there is a polynomial-time reduction from $CSP((\mathfrak{B}, R))$ to $CSP(\mathfrak{B})$.

PROOF. Let ϕ be an instance of $\mathrm{CSP}((\mathfrak{B},R))$ with variable set V. If ϕ contains two constraints $R(x_1,\ldots,x_k)$ and $R(y_1,\ldots,y_k)$, then replace each occurrence of y_1 by x_1 , then each occurrence of y_2 by x_2 , and so on, and finally each occurrence of y_k by x_k . We repeat this step until all constrains that involve R are imposed on the same tuple of variables (x_1,\ldots,x_k) . Replace $R(x_1,\ldots,x_k)$ by the primitive positive definition θ of its orbits in \mathfrak{B} . Finally, move all quantifiers to the front, such that the resulting formula ψ is in prenex normal form and thus an instance of $\mathrm{CSP}(\mathfrak{B})$. Clearly, ψ can be computed from ϕ in polynomial time. We claim that ϕ is true in (\mathfrak{B},R) if and only if ψ is true in \mathfrak{B} .

Suppose ϕ has a solution $s \colon V \to B$. Let s' be the restriction of s to the variables of V that also appear in ϕ . Since (b_1, \ldots, b_n) satisfies θ , we can extend s' to the existentially quantified variables of θ to obtain a solution for ψ . In the opposite direction, suppose that s' is a solution to ψ over \mathfrak{B} . Let s be the restriction of s' to V. Because $(s(x_1), \ldots, s(x_k))$ satisfies θ it lies in the same orbit as (b_1, \ldots, b_k) . Thus, there exists an automorphism α of \mathfrak{B} that maps $(s(x_1), \ldots, s(x_k))$ to (b_1, \ldots, b_k) . Then the extension of the map $x \mapsto \alpha s(x)$ that maps variables y_i of ϕ that have been replaced by x_i in ψ to the value b_i is a solution to ϕ over (\mathfrak{B}, R) .

Recall from Section 1.1 that every finite structure \mathfrak{C} is homomorphically equivalent to a core structure \mathfrak{B} , which is unique up to isomorphism. For core structures, all orbits are primitive positive definable. This fact has a simple proof for finite structures \mathfrak{B} ; however, the same fact is true for a large class of infinite structures, and presented in Chapter 3, Theorem 3.6.11. Since Theorem 3.6.11 implies the following proposition, we omit the proof at this point.

PROPOSITION 1.2.9. Let \mathfrak{B} be a finite core structure. Then orbits of k-tuples of \mathfrak{B} are primitive positive definable.

Proposition 1.2.9 and Lemma 1.2.8 have the following well-known consequence.

COROLLARY 1.2.10. Let \mathfrak{B} be a finite core structure with elements b_1, \ldots, b_n and finite signature. Then $CSP(\mathfrak{B})$ and $CSP((\mathfrak{B}, \{b_1\}, \ldots, \{b_n\}))$ are polynomial time equivalent.

1.3. The Satisfiability Perspective

Yet another perspective on the constraint satisfaction problem translates not only the instances, but also the template of the CSP into logic. This leads to a natural perspective for various model-theoretic considerations in Chapter 3. Moreover, this perspective is convenient when discussing the literature that uses *relation algebras* in the context of constraint satisfaction [88, 142]; the connection will be described in Section 1.3.2 and Section 1.3.3.

We use the opportunity to introduce some inevitable terminology from logic. We assume that the reader is already familiar with basic terminology of first-order logic; a highly recommendable text-book is Hodges [120].

1.3.1. Theories. A (first-order) theory is a set of first-order sentences. When the first-order sentences are over the signature τ , we also say that T is a τ -theory. A model of a τ -theory T is a τ -structure $\mathfrak B$ such that $\mathfrak B$ satisfies all sentences in T. Theories that have a model are called satisfiable.

DEFINITION 1.3.1. Let τ be a finite relational signature, and let T be a τ -theory. Then $\mathrm{CSP}(T)$ is the computational problem to decide for a given primitive positive τ -sentence ϕ whether $T \cup \{\phi\}$ is satisfiable.

The satisfiability perspective on CSPs stresses the fact that the problem $CSP(\mathfrak{B})$ is fully determined by the first-order theory of \mathfrak{B} , that is, by the theory that contains exactly those sentences that are true in \mathfrak{B} . In fact, it is already determined by the primitive positive sentences that are false in \mathfrak{B} .

Example 1.3.2. Let T be the theory that consists of the following sentences.

$$\forall x, y, z \; ((x < y \land y < z) \rightarrow x < z) \qquad \text{(transitivity)}$$

$$\forall x, y \; \neg (x < x) \qquad \text{(irreflexivity)}$$

$$\forall x, y, z \; ((x < y) \lor (y < x) \lor (x = y)) \qquad \text{(totality)}$$

It is straightforward to verify that CSP(T) equals $CSP((\mathbb{Z};<))$ (Example 1.1.2). \square

When T is a theory and ϕ a sentence, we say that T entails ϕ , in symbols $T \models \phi$, if every model of T satisfies ϕ . The following is clear from the definitions.

PROPOSITION 1.3.3. Let τ be a finite relational signature, and let T be a τ -theory. Suppose that T entails exactly those negations of primitive positive sentences ϕ such that $\mathfrak{B} \models \phi$. Then $\mathrm{CSP}(T)$ and $\mathrm{CSP}(\mathfrak{B})$ are the same problem.

We have already seen that two structures that are homomorphically equivalent have the same CSP; the following provides a necessary and sufficient condition that describes when two *theories* have the same CSP. Its proof is simple once the relevant notions from logic are introduced, and will be given in Section 2.1.3.

PROPOSITION 1.3.4. Let T and T' be two first-order theories. Then the following are equivalent.

- CSP(T) equals CSP(T').
- Every model of T' has a homomorphism to some model of T, and every model of T has a homomorphism to some model of T'.
- T and T' entail the same negations of primitive positive sentences.

We now present a couple of basic observations relating the definition of CSP(T) for a theory T with the definition of $CSP(\mathfrak{B})$ for a relational structure \mathfrak{B} . We start with the observation that there are theories T such that CSP(T) cannot be formulated as $CSP(\mathfrak{B})$.

EXAMPLE 1.3.5. Let τ be the signature $\{R,G\}$, where R and G are unary relation symbols, and let T be the τ -theory $\{\forall x,y \ \neg (R(x) \land G(y))\}$. There is no structure $\mathfrak B$ such that $\mathrm{CSP}(\mathfrak B)$ equals $\mathrm{CSP}(T)$. To see this, observe that $T \cup \{\exists x.R(x)\}$ is satisfiable, and $T \cup \{\exists x.G(x)\}$ is satisfiable. But any structure $\mathfrak B$ that satisfies both $\exists x.R(x)$ and $\exists x.G(x)$ also satisfies $\exists x,y(R(x) \land R(y))$, which shows that $\mathrm{CSP}(\mathfrak B)$ and $\mathrm{CSP}(T)$ are different.

We next characterize those satisfiable theories T that have a model \mathfrak{B} such that $\mathrm{CSP}(\mathfrak{B})$ and $\mathrm{CSP}(T)$ are the same problem.

PROPOSITION 1.3.6. Let τ be a finite relational signature, and let T be a satisfiable first-order τ -theory. The following are equivalent.

- (1) There is a structure \mathfrak{B} such that $CSP(\mathfrak{B})$ and CSP(T) are the same problem.
- (2) There is a model \mathfrak{B} of T such that $CSP(\mathfrak{B})$ and CSP(T) are the same problem.
- (3) For all primitive positive τ -sentences ϕ_1 and ϕ_2 , if $T \cup \{\phi_1\}$ is satisfiable and $T \cup \{\phi_2\}$ is satisfiable then $T \cup \{\phi_1, \phi_2\}$ is satisfiable as well.
- (4) T has the Joint Homomorphism Property (JHP), that is, when T has models \mathfrak{A} and \mathfrak{B} , then it also has a model \mathfrak{C} such that both \mathfrak{A} and \mathfrak{B} homomorphically map to \mathfrak{C} .

We defer the proof of this fact to Section 2.1.3 when we have some more concepts from logic available.

1.3.2. Relation Algebras. Many interesting infinite-domain CSPs, in particular in spatial and temporal reasoning, have been studied in the context of relation algebras (many examples will be given in Section 1.5 and Chapter 4). In Artificial Intelligence, relation algebras are used as a framework to formalize and study qualitative reasoning problems [88, 117, 142]. From the perspective of this thesis, the relation algebra approach does not bring substantially new tools, and Section 1.3.2 and Section 1.3.3 can be safely skipped. Here we nonetheless give a quick introduction in order to link the relation algebra terminology with the satisfiability perspective on the CSP (Section 1.3.3).

Relation algebras are designed to handle binary relations in an algebraic way; we follow the presentation in [117].

DEFINITION 1.3.7. A proper relation algebra is a domain D together with a set \mathcal{B} of binary relations over D such that

- (1) $Id := \{(x, x) \mid x \in D\} \in \mathcal{B};$
- (2) If B_1 and B_2 are from \mathcal{B} , then $B_1 \vee B_2 := B_1 \cup B_2 \in \mathcal{B}$;
- (3) $1 := \bigcup_{B \in \mathcal{B}} B \in \mathcal{B};$
- $(4) \ 0 := \emptyset \in \widetilde{\mathcal{B}};$
- (5) If $B \in \mathcal{B}$, then $-B := 1 \setminus B \in \mathcal{B}$;
- (6) If $B \in \mathcal{B}$, then $B^{\smile} := \{(x,y) \mid (y,x) \in B\} \in \mathcal{B}$;
- (7) If B_1 and B_2 are from \mathcal{B} , then $B_1 \circ B_2 \in \mathcal{B}$; where

$$B_1 \circ B_2 := \{(x, z) \mid \exists y ((x, y) \in B_1 \land (y, z) \in B_2)\}$$
.

We want to point out that in this standard definition of proper relation algebras it is not required that 1 denotes D^2 (and this will be used for instance in the proof of Proposition 1.3.16). However, in most examples that we encounter, 1 indeed denotes D^2 . The minimal non-empty elements of \mathcal{B} with respect to set-wise inclusion are called the basic relations of the relation algebra.

Example 1.3.8 (The Point Algebra). Let $D=\mathbb{Q}$ be the set of rational numbers, and consider

$$\mathcal{B} = \{\emptyset, =, <, >, \leq, \geq, \neq, \mathbb{Q}^2\} .$$

Those relations form a proper relation algebra (with atoms <,>,=, and where 1 denotes \mathbb{Q}^2) which is one of the most fundamental relation algebras and known under the name *point algebra*.

When \mathcal{B} is finite, every relation in \mathcal{B} can be written as a finite union of basic relations, and we abuse notation and sometimes write $R = \{B_1, \ldots, B_k\}$ when B_1, \ldots, B_k are basic relations, $R \in \mathcal{B}$, and $R = B_1 \cup \cdots \cup B_k$. Note that composition of basic relations determines the composition of all relations in the relation algebra, since

$$R_1 \circ R_2 = \bigcup_{B_1 \in R_1, B_2 \in R_2} B_1 \circ B_2 .$$

0	=	<	>
	=	<	>
<	<	<	1
>	>	1	>

FIGURE 1.4. The composition table for the basic relations in the point algebra.

An abstract relation algebra (Definition 1.3.9 below) is an algebra with signature $\{\lor, -, 0, 1, \circ, \check{\ }, \mathrm{Id}\}$ that satisfies laws that we expect from those operators in a proper relation algebra.

DEFINITION 1.3.9 (Compare [88,117,142]). An (abstract) relation algebra **A** is an algebra with domain A and signature $\{\lor, -, 0, 1, \circ, \lor, \mathsf{Id}\}$ such that

- the structure $(A; \vee, \wedge, -, 0, 1)$ is a Boolean algebra where \wedge is defined by $(x, y) \mapsto -(-x \vee -y)$ from and \vee ;
- \circ is an associative binary operation on A;
- $(a^{\smile})^{\smile} = a \text{ for all } a \in A;$
- $a \circ (b \vee c) = a \circ b \vee a \circ c;$
- $(a \lor b)^{\smile} = a^{\smile} \lor b^{\smile};$
- $(-a)^{\smile} = -(a^{\smile});$
- $(a \circ b)^{\smile} = b^{\smile} \circ a^{\smile}$;
- $(a \circ b) \wedge c = 0 \Leftrightarrow (b \circ c) \wedge a = 0.$

We define $x \leq y$ by $x \wedge y = x$. A subalgebra **B** of a relation algebra **A** with domain A is a relation algebra with domain $B \subset A$ such that for every function f of **A**, the element obtained by applying f to elements from B is again in B.

A representation (D, i) of **A** consists of a set D and a mapping i from the domain A of **A** to binary relations over D such that the image of i induces a proper relation algebra \mathcal{B} , and i is an isomorphism with respect to the functions (and constants) $\{\vee, -, 0, 1, \circ, \check{\ }, \operatorname{Id}\}$. In this case, we also say that **A** is the abstract relation algebra of \mathcal{B} .

There are finite relation algebras that do not have a representation [151]. Note that when (D, i) is a representation of \mathbf{A} , then i(a) is a basic relation of the induced proper relation algebra if and only if $a \neq 0$, and for every $b \leq a$ we have b = a or b = 0; we call a an atom of \mathbf{A} . Using the axioms of relation algebras, it can be shown that the composition operator is uniquely determined by the composition operator on the atoms. Similarly, the inverse of an element $a \in A$ is the disjunction of the inverses of all the atoms below a.

Example 1.3.10. The (abstract) point algebra is a relation algebra with 8 elements and 3 atoms, =, <, and >, and can be described as follows. The composition operator of the basic relations of the point algebra is shown in the table of Figure 1.4. By the observation we just made, this table determines the full composition table. The inverse of < is >, and Id denotes = which is its own inverse. This fully determines the relation algebra.

We can obtain a representation of the abstract point algebra from the point algebra with domain \mathbb{Q} presented in Example 1.3.8 in the obvious way.

1.3.3. Network Satisfaction Problems. The central computational problems that have been studied for relation algebras are network satisfaction problems [88,117, 142]. Let **A** be a finite relation algebra with domain A. An (A-) network N=(V;f) consists of a finite set of nodes V and a partial function $f: V^2 \to A$. Here, we slightly

deviate from the definition given in the papers listed above in that we allow f to be undefined on some pairs of nodes.

Two types of network satisfaction problems have been studied for \mathbf{A} -networks. The first is the network satisfaction problem for a (fixed) representation of \mathbf{A} , defined as follows.

DEFINITION 1.3.11. Let (D,i) be a representation of a finite relation algebra A. Then the network satisfaction problem for (D,i) is the computational problem to decide whether a given A-network N=(V;f) is satisfiable with respect to (D,i), that is, whether there exists a mapping $s:V\to D$ such that $(s(u),s(v))\in i(f(u,v))$ for all $u,v\in V$ where f is defined.

The second problem is the (general) network satisfaction problem for A.

DEFINITION 1.3.12. Let **A** be a finite relation algebra. Then the network satisfaction problem for **A** is the computational problem to decide whether a given **A**-network N is satisfiable, i.e., whether there exists a representation (D,i) of **A** such that N is satisfiable with respect to (D,i).

It is not surprising that every network satisfaction problem for a fixed representation is closely related to a corresponding constraint satisfaction problem; this correspondence will be described in the following. It is maybe less obvious that the same also applies to the *general* network satisfaction problem: every finite relation algebra $\bf A$ that has a representation also has a representation (D,i) such that the general network satisfaction problem for $\bf A$ and the network satisfaction problem for (D,i) are one and the same problem (Proposition 1.3.16).

To present the link between network satisfaction problems and CSPs as defined earlier we need the following notation. Let $\tau_{\mathbf{A}}$ be a signature consisting of a binary relation symbol R_a for each element $a \in A$. When (D,i) is a representation of $\tau_{\mathbf{A}}$, then this gives rise to a $\tau_{\mathbf{A}}$ -structure $\mathfrak{B}_{D,i}$ in a natural way: the domain of the structure is D, and the relation symbol R_a is interpreted by i(a). We can associate to each \mathbf{A} -network N=(V;f) a primitive positive $\tau_{\mathbf{A}}$ -sentence ϕ_N , in the following straightforward way: the variables of ϕ_N are V, and ϕ contains the conjunct $R_a(u,v)$ iff f(u,v)=a. Conversely, we can associate to each primitive positive $\tau_{\mathbf{A}}$ -sentence ϕ with variables V a network N_{ϕ} as follows. The domain of N_{ϕ} is V. Let $u,v\in V$, and list by a_1,\ldots,a_k all those elements a of A such that ϕ contains the conjunct $R_a(u,v)$. Then define f(u,v)=a for $a=(a_1 \wedge a_2 \wedge \cdots \wedge a_k)$; if k=0, then f(u,v) is undefined.

The following link between the network satisfaction problem for a fixed representation (D,i) of \mathbf{A} , and the constraint satisfaction problem for $\mathfrak{B}_{D,i}$ is straightforward from the definitions.

PROPOSITION 1.3.13. Let **A** be a finite relation algebra with representation (D, i). Then an **A**-network N is satisfiable with respect to (D, i) if and only if $\mathfrak{B}_{D,i} \models \phi_N$. Conversely, $\mathfrak{B}_{D,i}$ satisfies a primitive positive $\tau_{\mathbf{A}}$ -sentence ϕ if and only if N_{ϕ} is satisfiable with respect to (D, i).

Proposition 1.3.13 shows that network satisfaction problems for fixed representations essentially *are* constraint satisfaction problems, and that the differences are only a matter of formalization. To also relate the *general* network satisfaction problem for a finite relation algebra $\bf A$ to a constraint satisfaction problem, we define in Figure 1.5 the first-order $\tau_{\bf A}$ -theory $T_{\bf A}$ (as in [117], Section 2.3). The models of $T_{\bf A}$ correspond to the representations of $\bf A$, as described in the following.

PROPOSITION 1.3.14. Let **A** be a finite relation algebra. When \mathfrak{B} models $T_{\mathbf{A}}$, then (B,i) where B is the domain of \mathfrak{B} and i is given by $i(a) = R_a^{\mathfrak{B}}$ is a representation of

$$T_{\mathbf{A}} := \{ \forall x, y (\neg 0(x, y) \land (\mathrm{Id}(x, y) \Leftrightarrow x = y)) \}$$
 (1)

$$\cup \{ \forall x, y (1(x, y) \Leftrightarrow \bigvee_{a \in A} R_a(x, y)) \}$$
 (2)

$$\bigcup_{a \in A} \left\{ \forall x, y (R_{a^{\smile}}(x, y) \Leftrightarrow R_a(y, x) \land (R_{-a}(x, y) \Leftrightarrow \neg R_a(x, y)) \right\}$$
 (3)

$$\bigcup_{a \in A} \left\{ \forall x, y (R_{a^{\smile}}(x, y) \Leftrightarrow R_a(y, x) \land (R_{-a}(x, y) \Leftrightarrow \neg R_a(x, y)) \right\} \\
\cup \bigcup_{a, b \in A} \left\{ \forall x, y (R_{a \lor b}(x, y) \Leftrightarrow (R_a(x, y) \lor R_b(x, y))) \right\} \tag{4}$$

$$\bigcup_{a,b \in A} \left\{ \forall x, z (R_{a \circ b}(x, z) \Leftrightarrow \exists y (R_a(x, y) \land R_b(y, z))) \right\} \tag{5}$$

FIGURE 1.5. The definition of the $\tau_{\mathbf{A}}$ -theory $T_{\mathbf{A}}$.

A. Conversely, for every representation (D,i) of **A** the $\tau_{\mathbf{A}}$ -structure $\mathfrak{B}_{D,i}$ is a model of $T_{\mathbf{A}}$.

PROOF. The proof is straightforward by matching the sentences in $T_{\mathbf{A}}$ with the items of Definition 1.3.7.

Corollary 1.3.15. Let A be a finite relation algebra. Then an A-network Nis satisfiable if and only if $\phi_N \cup T_{\mathbf{A}}$ is satisfiable. Conversely, when ϕ is a primitive positive $\tau_{\mathbf{A}}$ -sentence, then the \mathbf{A} -network N_{ϕ} is satisfiable if and only if $\phi \cup T_{\mathbf{A}}$ is

It is easy to see that $T_{\mathbf{A}}$ has the Joint Homomorphism Property (JHP, introduced in Proposition 2.4.6); in fact, the disjoint union of two models of $T_{\mathbf{A}}$ is again a model

Proposition 1.3.16. Every finite relation algebra A that has a representation also has a representation (D; i) whose network satisfaction problem is the same problem as the general network satisfaction problem for A.

PROOF. Since A has a representation, and by Proposition 1.3.14, the theory $T_{\mathbf{A}}$ is satisfiable. Since $T_{\mathbf{A}}$ also has the JHP, we can apply Proposition 2.4.6 to obtain a model \mathfrak{B} of $T_{\mathbf{A}}$ with domain B be such that $\mathrm{CSP}(\mathfrak{B})$ and $\mathrm{CSP}(T_{\mathbf{A}})$ are the same problem. Then by Proposition 1.3.14, for i given by $i(a) = R_a^{\mathfrak{B}}$, the relation algebra **A** has the representation (B, i).

We then have for all \mathbf{A} -networks N the following equivalences.

$$N$$
 is satisfiable $\Leftrightarrow \phi_N \cup T_{\mathbf{A}}$ is satisfiable (Corollary 1.3.15)
 $\Leftrightarrow \mathfrak{B} \models \phi_N$ (by the properties of \mathfrak{B})
 $\Leftrightarrow \phi_N$ is satisfiable wrt. (B, i) (Proposition 1.3.13)

This concludes the proof that the representation (B, i) of **A** has a network satisfaction problem that equals the general network satisfaction problem for **A**.

In combination with Proposition 1.3.13, this implies that also every general network satisfiability problem is essentially the same problem as a CSP for an infinite template.

We close this section by discussing the weaknesses of the relation algebra approach to constraint satisfaction. First of all, the class of problems that can be formulated as a network satisfiability problem for finite relation algebra A is severely restricted. The relations that we allow in the input network are closed under unions; this introduces a sort of restricted disjunction that quickly leads to NP-hardness, and indeed only a few exceptional situations have a polynomial-time tractable network satisfiability problem [117]. The typical work-around here is to introduce another parameter, which is a subset B of the domain of \mathbf{A} , and to study the network satisfaction problem for networks N=(V;f) where the image of f is contained in B. Such subsets B are often called a *fragment* of \mathbf{A} . Note that such an additional parameter is not necessary for CSPs as studied in this thesis: with the techniques of this section, we can also formulate the network satisfaction problems for fragments of \mathbf{A} as CSPs.

Also note that the network satisfaction problem is restricted to binary relations, whereas many important CSPs can only be formulated in a natural way with relations of higher arity (see e.g. Section 1.5.2 or Section 1.5.8). As we have seen in Proposition 1.3.16, every network satisfaction problem can be formulated as $CSP(\mathfrak{B})$ for an appropriate infinite structure \mathfrak{B} ; but as the above remarks show, only a very small fraction of CSPs can be formulated as a network satisfaction problem. Even though only very specific CSPs can be formulated as the network satisfaction problem for a finite relation algebra \mathbf{A} , there are hardly any additional techniques available for studying network satisfaction problems. The tools we have for network satisfaction usually also apply to constraint satisfaction problems.

The study of composition of relations in the context of the network satisfiability problem is usually justified by the fact that a network with constraints over the relation $R \circ S$ can be simulated by networks that only have constraints over the relation R and over the relation S. To study the computational complexity of the network satisfaction problem for a fragment B of a relation algebra A, one therefore typically computes the closure of B under the operations of the relation algebra. But note that every binary relation in the closure of B is also primitive positive definable in any representation of A, and that the converse of this statement is false. Since the computational complexity is preserved also for expansions by primitive positive definable relations (see Lemma 1.2.6), primitive positive definitions therefore appear to be the more appropriate tool for studying network satisfaction problems. Apart from being more powerful, primitive positive definability has another advantage in comparison to closure in relation algebras: while the latter is intricate and not well-understood, we can offer a powerful Galois theory to study primitive positive definability of relations (see Chapter 5).

1.4. The Existential Second-Order Perspective

By a famous result of Fagin, which will be reviewed below, the complexity class NP corresponds exactly to those problems that can be formulated in existential second-order logic (ESO). An important fragment of ESO that is particularly natural when it comes to the formulation of CSPs is the logic called SNP (for $strict\ NP$; see [172] and [95]), introduced by Kolaitis and Vardi under the name $strict\ \Sigma_1^1$ [135]. An existential second-order sentence is in SNP if its first-order part is universal. There are many links between constraint satisfaction and the complexity class SNP; many of those go back to [95] and [96], some others that we present here are new.

SNP is often a convenient way to specify CSPs. However, not every problem in SNP is a CSP. In this section we present a syntactic condition that implies that an SNP sentence describes a problem of the form $CSP(\mathfrak{B})$ for an infinite structure \mathfrak{B} . Conversely, if an SNP sentence describes a CSP, then there is an equivalent SNP sentence that satisfies the syntactic condition.

The special case in which all existentially quantified relations are unary, known as *monadic SNP*, deserves special attention, and will be discussed at the end of this section.

1.4.1. Fagin's theorem. We start by reviewing Fagin's theorem (see e.g. [90]). Fix a finite relational signature τ . Let \mathcal{C} be a class of finite τ -structures that is closed under isomorphisms (that is, if $\mathfrak{B} \in \mathcal{C}$, and \mathfrak{A} is isomorphic to \mathfrak{B} , then $\mathfrak{A} \in \mathcal{C}$). We also fix some standard way to code relational structures as finite strings so that they can be given as an input to a Turing machine, see again [90]. We say that \mathcal{C} is in NP when there exists a non-deterministic polynomial time algorithm that accepts exactly the structures from \mathcal{C} under this representation.

A sentence of the form $\exists R_1, \ldots, R_m. \phi$ where ϕ is a first-order sentence with signature $\tau \cup \{R_1, \ldots, R_m\}$ is called an *existential second-order sentence*. When a structure \mathfrak{A} satisfies Φ (and this is defined in the obvious way, see e.g. [90]), we write $\mathfrak{A} \models \Phi$.

THEOREM 1.4.1 (Fagin's Theorem, see e.g. [90]). An isomorphism-closed class of finite τ -structures is in NP if and only if there exists an existential second-order sentence Φ that describes C in the sense that

$$\mathfrak{A} \in \mathcal{C}$$
 if and only if $\mathfrak{A} \models \Phi$.

1.4.2. SNP. An *SNP sentence* is an existential second-order sentence with a universal first-order part, i.e., a sentence of the form

$$\exists R_1, \ldots, R_k. \ \forall x_1, \ldots, x_n. \ \phi$$

where ϕ is quantifier-free and over the signature $\tau \cup \{R_1, \dots, R_k\}$. The class of problems that can be described by SNP sentences is called SNP, too.

EXAMPLE 1.4.2. The problem $\mathrm{CSP}((\mathbb{Z};<))$ can be described by the following SNP sentence.

$$\exists T \, \forall x, y, z \big((x < y \Rightarrow T(x, y)) \\ \wedge \big((T(x, y) \wedge T(y, z)) \Rightarrow T(x, z) \big) \wedge \neg T(x, x) \big)$$

EXAMPLE 1.4.3. The Betweenness problem $\mathrm{CSP}((\mathbb{Z};\mathrm{Betw}))$ (Example 1.1.3) can be described by the following SNP sentence.

$$\exists T \, \forall x, y, z \big(\neg T(x, x) \land \big((T(x, y) \land T(y, z)) \Rightarrow T(x, z) \big) \\ \land \big(\operatorname{Betw}(x, y, z) \Rightarrow \big((T(x, y) \land T(y, z)) \lor (T(z, y) \land T(y, x)) \big) \big)$$

EXAMPLE 1.4.4. The problem whether a given undirected graph can be partitioned into two triangle-free graphs (this problem has been called No-Mono-Tri in Example 1.1.8) can be described by the SNP sentence.

$$\exists M \, \forall x, y, z \, \left(\neg \left(M(x) \wedge M(y) \wedge M(z) \wedge E(x, y) \wedge E(y, z) \wedge E(z, x) \right) \right. \\ \left. \wedge \neg \left(\neg M(x) \wedge \neg M(y) \wedge \neg M(z) \wedge E(x, y) \wedge E(y, z) \wedge E(z, x) \right) \right)$$

The following fundamental lemma for SNP sentences is due to Feder and Vardi [96], and can be shown by a simple compactness argument (Theorem 2.3.1).

LEMMA 1.4.5 (from [96]). Let \mathfrak{A} be an infinite structure, and Φ an SNP sentence. Then $\mathfrak{A} \models \Phi$ if and only if $\mathfrak{A}' \models \Phi$ for all finite induced substructures \mathfrak{A}' of \mathfrak{A} .

Since every finite induced substructure of \mathfrak{B} homomorphically maps to \mathfrak{B} , and therefore satisfies Φ , we have the following consequence.

COROLLARY 1.4.6. Let Φ be an SNP sentence that describes $CSP(\mathfrak{B})$ for a structure \mathfrak{B} . Then \mathfrak{B} itself satisfies Φ .

1.4.3. SNP and **CSPs.** We say that two SNP sentences Φ and Ψ are *equivalent* if for all structures (equivalently: all finite structures) \mathfrak{A} we have $\mathfrak{A} \models \Phi$ if and only if $\mathfrak{A} \models \Psi$. We assume in the following that the first-order part ϕ of Φ is written in conjunctive normal form.

DEFINITION 1.4.7. Let Φ be an SNP sentence whose unquantified relation symbols are from the signature τ . Then Φ is called monotone if each literal of Φ with a symbol from $\tau \cup \{=\}$ is negative, that is, of the form $\neg R(\bar{x})$, for $R \in (\tau \cup \{=\})$.

In particular, monotone SNP sentences do not contain literals of the form x = y (hence, in the terminology of Feder and Vardi [95], we work here with monotone SNP without inequality; the reason why Feder and Vardi add the attribute without inequalities is that for them, SNP sentences are written in negation normal form, so forbidding literals of the form x = y amounts to forbidding inequalities in negation normal form).

We also assume that monotone SNP sentences do not contain literals of the form $x \neq y$. This is without loss of generality, since every monotone SNP sentence is equivalent to one which does not contain literals of the form $x \neq y$. To obtain the equivalent sentence, we remove literals of the form $x \neq y$ and replace all occurrences of y in the same clause by x. Note that the SNP sentences given in Example 1.4.2, 1.4.3, and 1.4.4 can be easily re-written into equivalent monotone SNP sentences.

The class of structures that satisfy a given monotone SNP sentences is clearly closed under inverse homomorphisms. The converse is a result by Feder and Vardi [96]; it shows that for SNP, the semantic restriction of closure under inverse homomorphisms and the syntactic restriction of monotonicity match.

THEOREM 1.4.8 (from [96]). Let Φ be an SNP sentence. Then the class of structures that satisfy Φ is closed under inverse homomorphisms if and only if Φ is equivalent to a monotone SNP sentence.

DEFINITION 1.4.9 (Connected SNP). When ψ is a clause of a first-order σ -formula ϕ in conjunctive normal form, let \mathfrak{C} be the σ -structure whose vertices are the variables of ψ , and where $(x_1, \ldots, x_n) \in R^{\mathfrak{C}}$ if and only if ψ contains a negative literal of the form $\neg R(x_1, \ldots, x_n)$. We say that ψ is connected if \mathfrak{C} is connected. We say that an SNP sentence Φ is connected if all clauses of the first-order part ϕ of Φ are connected.

THEOREM 1.4.10. Let Φ be an SNP sentence. Then the class of structures that satisfy Φ is closed under disjoint unions if and only if Φ is equivalent to a connected SNP sentence.

PROOF. Let Φ be of the form $\exists R_1, \dots, R_k \, \forall x_1, \dots, x_l. \, \phi$ where ϕ is a quantifier-free first-order formula over the signature $\sigma = (\tau \cup \{R_1, \dots, R_k\})$.

Suppose first that Φ is connected, and that \mathfrak{A}_1 and \mathfrak{A}_2 both satisfy Φ . In other words, there is a σ -expansion \mathfrak{A}_1^* of \mathfrak{A}_1 and a σ -expansion \mathfrak{A}_2^* of \mathfrak{A}_2 such that those expansions satisfy $\forall \bar{x}.\phi$. We claim that the disjoint union \mathfrak{A}^* of \mathfrak{A}_1^* and \mathfrak{A}_2^* also satisfies $\forall \bar{x}.\phi$; otherwise, there would be a clause ψ in ϕ and elements a_1, \ldots, a_q of $A_1 \cup A_2$ such that $\psi(a_1, \ldots, a_q)$ is false in \mathfrak{A}^* . Since \mathfrak{A}_1^* and \mathfrak{A}_2^* satisfy $\forall \bar{x}.\psi$, there must be i,j such that $a_i \in A_1$ and $a_j \in A_2$. But then the canonical database for ψ is disconnected, a contradiction.

For the opposite direction of the statement, assume that the class of structures that satisfy Φ is closed under disjoint unions. Consider the SNP sentence $\Psi = \exists R_1, \dots, R_k, E. \forall x_1, \dots, x_l. \psi$ where ψ is the conjunction of the following clauses (we assume without loss of generality that $l \geq 3$).

- For each relation symbol $R \in \tau$, say of arity p, and each $i < j \le p$, add the conjunct $\neg R(x_1, \ldots, x_p) \lor E(x_i, x_j)$ to ψ .
- Add the conjunct $\neg E(x_1, x_2) \lor \neg E(x_2, x_3) \lor E(x_1, x_3)$ to ψ .
- Add the conjunct $\neg E(x_1, x_2) \lor E(x_2, x_1)$ to ψ .
- For each clause ϕ' of ϕ with variables $y_1, \ldots, y_q \subseteq \{x_1, \ldots, x_l\}$, add to ψ the conjunct

$$\phi' \vee \bigvee_{i < j < q} \neg E(y_i, y_j) .$$

We claim that the connected monotone SNP sentence Ψ is equivalent to Φ . Suppose first that \mathfrak{A} is a finite structure that satisfies Φ . Then there is a σ -expansion \mathfrak{A}' of \mathfrak{A} that satisfies $\forall \bar{x}.\phi$. The expansion of \mathfrak{A}' by the relation $E = A^2$ shows that \mathfrak{A} also satisfies $\forall \bar{x}.\psi$.

Now suppose that \mathfrak{A} is a finite structure with domain A that satisfies Ψ . Then there is a $(\sigma \cup \{E\})$ -expansion \mathfrak{A}' of \mathfrak{A} that satisfies $\forall \bar{x}.\psi$. Write $\mathfrak{A}' = \mathfrak{A}'_1 \uplus \cdots \uplus \mathfrak{A}'_l$ for connected σ -structures $\mathfrak{A}'_1, \ldots, \mathfrak{A}'_l$. Note that the clauses of ψ force that the relation E denotes A_i^2 in the structure \mathfrak{A}'_i , for each $i \leq l$. Let \mathfrak{A}_i be the σ -reduct of \mathfrak{A}'_i . Then \mathfrak{A}_i satisfies $\forall \bar{x}.\phi$, because if there was a clause ϕ' from ϕ violated in \mathfrak{A}_i then the corresponding clause in ψ would be violated in \mathfrak{A}'_i . Hence, $\mathfrak{A}_i \models \Phi$ for all $i \leq l$, and since Φ is closed under disjoint unions, we also have that $\mathfrak{A} \models \Phi$.

Theorem 1.4.8 combined with the previous result shows the following.

COROLLARY 1.4.11. An SNP sentence Φ describes a problem of the form $CSP(\mathfrak{B})$ for an infinite structure \mathfrak{B} if and only if Φ is equivalent to a monotone and connected SNP sentence Ψ .

PROOF. Suppose first that Φ is a monotone SNP sentence with connected clauses. To show that Φ describes a problem of the form CSP(\mathfrak{B}) we can use Lemma 1.1.7. It thus suffices to show that the class of structures that satisfy Φ is closed under disjoint unions and inverse homomorphisms. But this has already been observed in Theorem 1.4.8 and Theorem 1.4.10.

For the implication in the opposite direction, suppose that Φ describes a problem of the form CSP(\mathfrak{B}) for some infinite structure \mathfrak{B} . In particular, the class of structures that satisfy Φ is closed under inverse homomorphisms. By Theorem 1.4.8, Φ is equivalent to a monotone SNP sentence. Moreover, the class of structures that satisfy Φ is closed under disjoint unions, and hence Φ is also equivalent to a connected SNP sentence. By inspection of the proof of Theorem 1.4.10, we see that when Φ is already monotone, then the connected SNP sentence in the proof of Theorem 1.4.10 will also be monotone. It follows that Φ is also equivalent to a connected monotone SNP sentence.

1.4.4. Monadic SNP. When we further restrict monotone SNP by only allowing unary existentially quantified relations, the corresponding class of problems, called montone monadic SNP (or, short, MMSNP), gets very close to finite domain constraint satisfaction problems. Indeed, Feder and Vardi showed that the class MM-SNP exhibits a complexity dichotomy if and only if the class of all finite domain CSPs exhibits a complexity dichotomy (that is, if the dichotomy conjecture mentioned in the introduction is true). In one direction, this is obvious since MMSNP obviously contains CSP(\mathfrak{B}) for all finite structures \mathfrak{B} (we may use a unary relation symbol for each element of \mathfrak{B}). In the other direction, Feder and Vardi showed that every problem in MMSNP is equivalent under randomized Turing-reductions to a finite domain constraint satisfaction problem. The reduction has subsequently been derandomized by Kun [141].

Theorem 1.4.12 (of [95] and [141]; see [155] for a formalization). Every problem in monotone monadic SNP is polynomial-time Turing equivalent to $CSP(\mathfrak{B})$ for a finite structure \mathfrak{B} .

Similarly as in the previous section, we might ask for a syntactic characterization of those monadic SNP sentences that describe a CSP. Note that this does not directly follow from Corollary 1.4.11, since the reductions used there introduce additional existentially quantified relations that are not monadic. However, we have the following monadic version of Theorem 1.4.8.

THEOREM 1.4.13 (Theorem 3 in [96]). Let Φ be a monadic SNP sentence. Then the class of structures that satisfy Φ is closed under inverse homomorphisms if and only if Φ is equivalent to a monotone monadic SNP sentence.

Moreover, one can show the following monadic version of Proposition 1.4.10.

PROPOSITION 1.4.14. Let Φ be a monadic SNP sentence. Then the class of structures that satisfy Φ is closed under disjoint unions if and only if Φ is equivalent to a connected monadic SNP sentence.

PROOF. Let V be the set of variables of the first-order part ϕ of Φ , let P_1, \ldots, P_k be the existential monadic predicates in Φ , and let τ be the input signature so that ϕ has signature $\{P_1, \ldots, P_k\} \cup \tau$. If Φ is connected, then it describes a problem that is closed under disjoint unions; this follows from Theorem 1.4.10.

For the opposite direction, suppose that Φ describes a problem that is closed under disjoint unions. We can assume without loss of generality that Φ is minimal in the sense that if we remove literals from some of the clauses the resulting SNP sentence is inequivalent. We shall show that then Φ must be connected. Let us suppose that this is not the case, and that there is a clause ψ in ϕ that is not connected. The clause ψ can be written as $\psi_1 \vee \psi_2$ where the set of variables $X \subset V$ of ψ_1 and the set of variables $Y \subset V$ of ψ_2 are non-empty and disjoint. Consider the formulas Φ_X and Φ_Y obtained from Φ by replacing ψ by ψ_1 and ψ by ψ_2 , respectively. By minimality of Φ there is a τ -structure \mathfrak{A}_1 that satisfies Φ but not Φ_X , and similarly there exists a τ -structure \mathfrak{A}_2 that satisfies Φ but not Φ_Y . By assumption, the disjoint union \mathfrak{A} of \mathfrak{A}_1 and \mathfrak{A}_2 satisfies Φ . So there exists a $\tau \cup \{P_1, \ldots, P_k\}$ -expansion \mathfrak{A}' of $\mathfrak{A} = \mathfrak{A}_1 \uplus \mathfrak{A}_2$ that satisfies the first-order part of Φ . Consider the substructures \mathfrak{A}'_1 and \mathfrak{A}'_2 of \mathfrak{A}' induced by A_1 and A_2 , respectively. We have that \mathfrak{A}'_1 does not satisfy ψ_1 (otherwise \mathfrak{A}_1 would satisfy Φ_X). Consequently, there is an assignment $s_1: V \to A_1$ of the universal variables that falsifies ψ_1 . By similar reasoning we can infer that there is an assignment $s_2 \colon V \to A_2$ that falsifies ψ_2 . Finally, fix any assignment $s: V \to A_1 \cup A_2$ that coincides with s_1 over X and with s_2 over Y (such an assignment exists because X and Y are disjoint). Clearly, s falsifies ψ and $\mathfrak A$ does not satisfy Φ , a contradiction.

Similarly as in Corollary 1.4.11 for SNP, we can combine the conditions of closure under inverse homomorphisms and closure under disjoint unions, and arrive at the following.

COROLLARY 1.4.15. A monadic SNP sentence Φ describes a problem of the form $CSP(\mathfrak{B})$ for an infinite structure \mathfrak{B} if and only if Φ is equivalent to a connected monotone monadic SNP sentence.

We want to remark that the problems that can be described by connected monotone monadic SNP sentences are exactly the problems called *forbidden patterns problems* in the sense of Madelaine [154]. Clearly, for every finite \mathfrak{B} the problem CSP(\mathfrak{B}) is a forbidden patterns problem. In [157] is has been shown that the problems in

MMSNP are exactly finite unions of forbidden patterns problems (going back to ideas from [95]).

We summarize the landscape of classes of computational problems from this section in Figure 1.6.

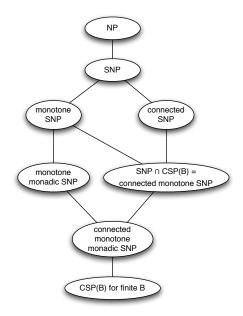


FIGURE 1.6. Fragments of SNP.

1.5. Examples

We present computational problems that have been studied in various areas of theoretical computer science, and that can be formulated as constraint satisfaction problems in the sense of Section 1.1, 1.2, 1.3, or 1.4. We describe each problem from the perspective in which the computational problem has appeared first in the literature.

Our list is by far not exhaustive; computational problems that can be exactly formulated as $CSP(\mathfrak{B})$ for an infinite structure \mathfrak{B} are abundant in almost every area of theoretical computer science.

1.5.1. Allen's interval algebra. Allen's interval algebra [5] is a formalism that is famous in artificial intelligence, and which has been introduced to reason about intervals and about the relationships between intervals.

Formally, Allen's interval algebra is a proper relation algebra (see Section 1.3.2); we can also view it as a structure with a binary relational signature. The domain is the set \mathbb{I} of all closed intervals [a,b] of rational numbers, where $a,b\in\mathbb{Q}, a< b$. When x=[a,b] is an interval, then -x denotes the interval [-b,-a]. For $R\subseteq\mathbb{Q}^2$, R^- denotes the relation $\{(-x,-y)\mid (x,y)\in R\}$. Recall that in proper relation algebras, R^{\sim} denotes the relation $\{(y,x)\mid (x,y)\in R\}$.

The basic relations of Allen's interval algebra are the 13 relations P, M, O, S, D, E (defined in Figure 1.7), P^-, M^-, O^-, S^- , and the inverse of S, D, and S^- , denoted by S^- , D^- , and $(S^-)^-$, respectively. Note that those 13 relations are pairwise disjoint, and that their union equals \mathbb{I}^2 . Recall our convention that when \mathcal{R} is a subset of the

Relation Symbol	Definition	Explanation
P	$\{([a,b],[c,d]) \mid b < c\}$	[a,b] preceds $[c,d]$
M	$\{([a,b],[c,d]) \mid b=c\}$	[a,b] meets $[c,d]$
O	$\{([a,b],[c,d]) \mid a < c < b < d\}$	[a, b] overlaps with $[c, d]$
$\mid S \mid$	$\{([a,b],[c,d]) \mid a = c \text{ and } b < d\}$	[a,b] starts $[c,d]$
D	$\{([a,b],[c,d]) \mid c < a < b < d\}$	[a,b] is during $[c,d]$
$\mid E \mid$	$\{([a,b],[c,d]) \mid a=c,b=d\}$	[a,b] equals $[c,d]$

FIGURE 1.7. The definitions for the basic relations of Allen's interval algebra.

basic relations, we write $x\mathcal{R}y$ if $(x,y) \in \bigcup_{R \in \mathcal{R}} R$. For example, $x\{P,P^-\}y$ signifies that the intervals x and y are disjoint. The 2^{13} relations that arise in this way will be called the relations of Allen's interval algebra.

An important computational problem for Allen's interval algebra is the network satisfaction problem for Allen's interval algebra. This problem can be viewed as $CSP(\mathfrak{A})$ where \mathfrak{A} is a structure with domain \mathbb{I} and a signature containing 2^{13} binary relation symbols (see Section 1.3.2). More on this structure can be found in Chapter 3, Example 3.1.11. We are sometimes sloppy and write *Allen's interval algebra* when we mean \mathfrak{A} (rather than \mathbf{A}).

The problem $CSP(\mathfrak{A})$ is NP-complete [5]. The complexity of the CSP for (binary) reducts of Allen's interval algebra has been completely classified in [140].

1.5.2. Phylogenetic reconstruction problems. In modern biology it is believed that the species in the evolution of life on earth developed in a mostly tree-like fashion: at certain time periods, species separated into sub-species. The goal of phylogenetic reconstruction is to determine the evolutionary tree from given partial information about the tree. This motivates the computational problem of rooted triple satisfiability (also called rooted triple consistency), defined below. In 1981, Aho, Sagiv, Szymanski, and Ullman [4] presented a quadratic time algorithm to this problem, motivated independently from computational biology by questions in database theory.

Let \mathfrak{T} be a tree with vertex set T and with a distinguished vertex r, the root of \mathfrak{T} . For $u,v\in T$, we say that u lies below v if the path from u to r passes through v. We say that u lies strictly below v if u lies below v and $u\neq v$. The youngest common ancestor (yca) of two vertices $u,v\in T$ is the node w such that both u and v lies below v and v has maximal distance from v.

Rooted-Triple Satisfiability

INSTANCE: A finite set of variables V, and a set of triples xy|z for $x, y, z \in V$. QUESTION: Is there a rooted tree $\mathfrak T$ with leaves L and a mapping $s\colon V\to L$ such that for every triple xy|z the yea of s(x) and s(y) lies strictly below the yea of s(x) and s(z) in $\mathfrak T$?

Another famous problem that has been studied in this context is the quartet satisfiability problem, which is NP-complete [189].

Quartet Satisfiability

INSTANCE: A finite set of variables V, and a set of quartets xy:uv with $x, y, u, v \in V$. QUESTION: Is there a tree $\mathfrak T$ with leaves L and a mapping $s\colon V\to L$ such that for every quartet $xy:uv\in R^{yca}$ the shortest path from x to y is disjoint to the shortest path from u to v?

0	=	<	>	
	=	<	>	
<	<	<	{<,=,>}	{<, }
>	>	1	>	
			{>, }	1

FIGURE 1.8. The composition table for the basic relations in the left-linear point algebra.

It is straightforward to check that the class of positive instances (viewed as relational structures) of each of those two computational problems is closed under disjoint unions and inverse homomorphisms. By Lemma 1.1.7, both the rooted triple satisfaction problem and the quartet satisfaction problem can be formulated as $CSP(\mathfrak{B})$ for an infinite structure \mathfrak{B} . We come back to those CSPs in Chapter 4.

1.5.3. Branching-time constraints. An important model in temporal reasoning is *branching time*, where for every time point the past is linearly ordered, but the future is only partially ordered.

This motivates the so-called *left-linear point algebra* [88,117], which is a relation algebra with four basic relations, denoted by =, <, >, and |. Here we imagine that 'x < y' signifies that x is *earlier in time than y*, and to the *left of y* when we draw points in the plane, and this motivates the name *left linear point algebra*. The composition operator on those four basic relations is given in Figure 1.8. The inverse of < is >, Id denotes =, and | is its own inverse, and the relation algebra is uniquely given by this data.

As explained in Section 1.3.2, the network consistency problem for the left-linear point algebra can be viewed as $CSP(\mathfrak{B})$ for an appropriate infinite structure with $16=2^4$ binary relations (one for each subset of $\{=,<,>,|\}$). In this structure, for every x the set $\{y\mid y< x\}$ is linearly ordered by by <. An explicit example of such a structure is given in Section 4.2. The network consistency problem of the left-linear point algebra is polynomial-time equivalent to the following problem, which we call branching-time satisfiability problem.

Branching-Time Satisfiability

INSTANCE: A finite relational structure $\mathfrak{A}=(A;\leq,\parallel,\neq)$ where \leq,\parallel , and \neq are binary relations.

QUESTION: Is there a rooted tree \mathfrak{T} and a mapping $s: A \to T$ such that in \mathfrak{T} the following is satisfied: a) If $(x,y) \in \leq^{\mathfrak{A}}$, then s(x) lies above s(y); b) If $(x,y) \in ||^{\mathfrak{A}}$, then neither s(x) lies strictly above s(y), nor s(y) strictly above s(x); c) If $(x,y) \in \neq^{\mathfrak{A}}$, then $s(x) \neq s(y)$.

The idea why this problem is polynomial-time equivalent to the network satisfaction problem of the left-linear point algebra is the observation that in any representation $\mathfrak B$ of the left-linear point algebra, the relation $x\{<,>,=\}y$ has the primitive positive definition

$$\exists z \ (x\{<,=\}z \land y\{<,=\}z) \ ,$$

and the relation $x\{<,|,=\}y$ has the primitive positive definition

$$\exists z \ (x\{<,=\}z \land z\{|,=\}y) ;$$

we can then use Theorem 1.2.6 (for details, see [44]).

The branching-time satisfiability problem can be formulated as $CSP(\mathfrak{C})$ for the structure with domain $C := \{0,1\}^*$ and relations \leq , \parallel , and \neq , where \leq denotes the

0	=	<	>	\prec	>
	=	<	>	\prec	>
<	<	<	{<,>}	{<,≺}	{<,≻}
>	>	1	>	\prec	\succ
\prec	\prec	\prec	{>,≺}	\prec	1
\vdash	>	>	{>,≻}	1	>

Figure 1.9. The composition table for the basic relations of Cornell's tree algebra ${\bf C}$.

relation

$$\{(u,v)\in C^2\mid u \text{ is a prefix of } v\}$$
.

The relation \neq is the disequality relation, and $u \parallel v$ holds if u and v are equal or incomparable with respect to \leq . Let < denote the intersection of \leq and \neq . Note that the structure \mathfrak{C} can *not* be used to obtain a representation of the left-linear point algebra, since $(<) \circ (<)$ does not equal <.

The first polynomial-time algorithm for the branching-time consistency problem (and therefore also for the network satisfaction problem of the left-linear point algebra) is due to Hirsch [117], and has a worst-case running time in $O(n^5)$. This has been improved by Broxvall and Jonsson [59], who presented an algorithm running in $O(n^{3.376})$ (this algorithm uses an $O(n^{2.376})$ algorithm for fast integer matrix multiplication). A simpler algorithm which does not use fast matrix multiplication and runs in O(nm) has been found in [43].

1.5.4. Cornell's tree description constraints. Motivated by problems in computational linguistics, Cornell [78] introduced the following computational problem⁵. It is a strictly more expressive problem than the branching time satisfaction problem from the previous section, but has been introduced independently from [117] and [59]. There are many equivalent formulations of this problem. One is as the general network satisfaction problem for the relation algebra \mathbf{C} with atoms =, <, >, <, and \succ which is given by the composition table in Figure 1.9. The idea is that < denotes a dense semilinear order (see Section 1.5.3), and $\prec \cup <$ denotes a linear order. The idea how to use this in natural language grammar formalisms like dependency grammars is that < represents the syntactic structure of a natural language sentence whereas $\prec \cup <$ stands for the word order.

Similarly as in Section 1.5.3, all 2^5 relations of \mathbb{C} can be obtained by repeated compositions and intersections of the four relations $\{<,=\}$, $\{\prec,=\}$, $\{\prec,-\}$, and $\{<,>,\prec,\succ\}$; for details, see [44]. The algorithm presented for the general network satisfaction problem for \mathbb{C} in [78] is not complete. A polynomial-time algorithm has been found in [44].

1.5.5. Set constraints. Many fundamental problems in artificial intelligence, knowledge representation, and verification involve reasoning about sets and relations between sets and can be modeled as constraint satisfaction problems. One of the most fundamental problems of this type is the following. We denote the set of all subsets of \mathbb{N} by $\mathcal{P}(\mathbb{N})$.

Basic Set Constraint Satisfiability

INSTANCE: A finite set of variables V, and a set ϕ of constraints of the form $x \subseteq y$, $x \mid\mid y$, or $x \neq y$, for $x, y \in V$.

 $^{^5} I$ feel personally committed to Cornell's problem since it was the first CSP with an $\omega\text{-categorical}$ template I met.

QUESTION: Is there a mapping $s: V \to \mathcal{P}(\mathbb{N})$ such that

- a) If $x \subseteq y$ is in ϕ , then s(x) is contained in s(y);
- b) If $x \mid\mid y$ is in ϕ , then s(x) and s(y) are disjoint sets;
- c) If $x \neq y$ is in ϕ , then s(x) and s(y) are distinct sets.

This problem has the shorter description $\mathrm{CSP}((\mathcal{P}(\mathbb{N});\subseteq,||,\neq))$ where $\subseteq,||,\neq)$ are binary relations over $\mathcal{P}(\mathbb{N})$, standing for the binary relations containment, disjointness, and inequality between sets. Drakengren and Jonsson [85] showed that basic set constraint satisfiability can be decided in polynomial time. They also showed that the generalization of the problem can be solved in polynomial time where each constraint has the form

$$x_1 \neq y_1 \vee \cdots \vee x_k \neq y_k \vee x_0 R y_0$$

where R is either \subseteq , ||, or \neq , and where $x_0, \ldots, x_k, y_0, \ldots, y_k$ are not necessarily distinct variables.

1.5.6. Spatial reasoning. Qualitative spatial reasoning (QSR) is concerned with representation formalisms that are considered close to conceptual schemata used by humans for reasoning about their physical environment—in particular, about processes or events and about the spatial environment in which they are situated. The approach in qualitative reasoning is to develop relational schemas that abstract from concrete metrical data of entities (for example time points, coordinate positions, or distances) by subsuming similar metric or topological configurations of entities into one qualitative representation.

There are many formalisms for qualitative spatial reasoning. In particular, several relation algebras (see Section 1.3.2) have been studied in this context. A basic example is the RCC5 relation algebra (with 5 atoms; the RCC5 relation algebra is also known under the name containment algebra [21,88]), and the RCC8 relation algebra (with 8 atoms). In both formalisms, the variables denote 'non-empty regions'. In RCC5, the five atoms are denoted by DR, PO, PP, PPI, EQ, and they stand for disjointness, proper overlap, proper containment (proper-part-or), its inverse, and equality, respectively. In RCC8, we further distinguish how the 'boundaries' of two regions relate to each other. We do not further discuss RCC8, for details, see [55,88].

There are many equivalent ways to formally define RCC5. Often, this is done by specifying the composition table for atomic relations, but we find this tedious. Here, we rather define RCC5 as the proper relation algebra whose domain are all open (or all closed) disks in \mathbb{R}^2 , and where the basic relations are disjointness (empty intersection), proper overlap, containment, the inverse of containment, and equality of disks. Then RCC5 is the abstract relation algebra of the proper relation algebra of closed disks (see Section 2.1.5 in [88]).

The network satisfaction problem for RCC5 is NP-complete; the computational complexity of the CSP for the (binary) reducts of \mathfrak{B} has been classified in [127,179]. A polynomial-time tractable case of particular interest is the *network satisfaction* problem for the basic relations of RCC5 [179], i.e., the network satisfaction problem for RCC5 when the input is restricted to networks N = (V; f) where f maps to 1 or the atoms in RCC5 only.

In any representation of RCC5, the atomic relations satisfy the following set of axioms T. We use P(x, y) as a shortcut for $PP(x, y) \vee EQ(x, y)$.

$$T := \left\{ \forall x, y, z \left(DC(x, y) \land P(z, y) \to DC(x, z) \right) \\ \forall x, y, z \left(PO(x, y) \land P(y, z) \to \left(PO(x, z) \lor PP(x, z) \right) \right) \\ \forall x, y, z \left(PP(x, y) \land PP(y, z) \to PP(x, z) \right) \\ \forall x, y, z \left(P^{-1}(x, y) \land P(y, z) \to \neg DC(x, y) \right) \right\}$$

It is easy to see that the network satisfaction problem for the basic relations of RCC5 is essentially the same problem as CSP(T), where T is the first-order theory defined as above. It can be checked easily that T satisfies item (2) in the statement of Proposition 1.3.6, and hence there exists an infinite structure $\mathfrak B$ such that $CSP(\mathfrak B)$ equals the satisfiability problem for RCC5. We will give more explicit descriptions of such an infinite structure $\mathfrak B$ in Chapter 4 (and it turns out that there are close links with the problem from Section 1.5.5).

1.5.7. Horn-SAT. The following problem is an important P-complete problem [67]. It can be solved in linear time in the size of the input [84].

Horn-SAT

INSTANCE: A propositional formula in conjunctive normal form (CNF) with at most one positive literal per clause.

QUESTION: Is there a Boolean assignment for the variables such that in each clause at least one literal is true?

We cannot model this problem as $CSP(\mathfrak{B})$ for a finite signature structure; however, note that a clause $\neg x_1 \lor \cdots \lor \neg x_k \lor x_0$ is equivalent to

$$\exists y_1, \dots, y_{k-1} \left((\neg x_1 \vee \neg x_2 \vee y_1) \wedge (\neg y_1 \vee \neg x_3 \vee y_2) \wedge \dots \wedge (\neg y_{k-1} \vee \neg x_k \vee x_0) \right).$$

Hence, by introducing new variables, there is a straightforward reduction of Horn-SAT to the restriction of Horn-SAT where every clause has at most three literals. This restricted problem, which we call Horn-3SAT, can be formulated as $CSP(\mathfrak{B})$ for

$$\mathfrak{B} = (\{0,1\}; \{(x,y,z) \mid (x \land y) \Rightarrow z\}, \{(x,y,z) \mid \neg x \lor \neg y \lor \neg z\}, \{((x,y) \mid x \Rightarrow y\}, \{((x,y) \mid \neg x \lor \neg y\}, \{0\}, \{1\}).$$

1.5.8. Precedence constraints in scheduling. The following problem has been studied in [164] in scheduling: given is a finite set of variables V, and a finite set of and/or precedence constraints, i.e., constraints of the form

$$x_0 > x_1 \lor \dots \lor x_0 > x_k \tag{6}$$

for $x_0, x_1, \ldots, x_k \in V$. The question is whether there exists an assignment $V \to \mathbb{Q}$ (equivalently, we can replace \mathbb{Q} by \mathbb{Z} , or any other infinite linearly ordered set).

As in the case of Horn-SAT, we cannot directly model this problem as $CSP(\mathfrak{B})$ for a finite signature structure \mathfrak{B} . However, note that Formula (6) is equivalent to

$$\exists y_1, \dots, y_{k-1} \big((x_0 > x_1 \lor x_0 > y_1) \land (y_1 > x_2 \lor y_1 > y_2) \land \dots \land (y_{k-1} > x_{k-1} \lor y_{k-1} > x_k) \big) .$$

This shows that and/or precedence constraints can be translated into conjunctions of constraints of the form $x_0 > x_1 \lor x_0 > x_2$ by introducing new existentially quantified variables. Hence, the problem whether a given set of and/or precedence constraints is satisfiable reduces naturally to $\mathrm{CSP}((\mathbb{Q};R^{\min}))$ where R^{\min} is the ternary relation $\{(a,b,c) \mid a>b\lor a>c\}$. Note that R^{\min} holds on exactly those triples (a,b,c) where a is larger than the minimum of b and c. The problem $\mathrm{CSP}((\mathbb{Q};R^{\min}))$ can be solved in polynomial time; this is essentially due to [164]. For more expressive

constraint languages over \mathbb{Q} that contain the relation R^{min} and whose CSP can still be solved in polynomial time, see Section 10.5.2 or Section 10.4.1.

1.5.9. Ord-Horn constraints. In this section we work with first-order formulas over the signature $\{<\}$. We write $x \leq y$ as a shortcut for $(x < y) \lor (x = y)$ (recall our convention that equality is part of first-order logic). A formula over the signature $\{<\}$ and with variables V is called *Ord-Horn* if it is a conjunction of disjunctions of the form

$$(x_1 = y_1) \vee \cdots \vee (x_k = y_k) \vee (x_0 R y_0)$$

where $x_0, x_1, \ldots, x_k, y_0, y_1, \ldots, y_k \in V$, and R is either $\leq, <, \neq$, or =.

Ord-Horn Satisfiability

INSTANCE: A finite set of variables V, and a finite set of Ord-Horn formulas with variables from V.

QUESTION: Is there an assignment $V \to \mathbb{Q}$ that satisfies all the given formulas over $(\mathbb{Q};<)$?

Nebel and Bürckert [165] showed that Ord-Horn Satisfiability can be solved in polynomial time. A relation $R \subseteq \mathbb{Q}^k$ is called *Ord-Horn* if it is definable by an Ord-Horn formula over $(\mathbb{Q}; <)$. As in the case of Horn-SAT and of and/or precedence constraints, there are structures 3 with finitely many Ord-Horn relations such that all Ord-Horn relations have a primitive positive definition in \mathfrak{B} . It can be shown that the following structure has this property (see Chapter 6).

$$\left(\mathbb{Q};\leq,\neq,\{(x,y,u,v)\mid(x=y)\Rightarrow(u=v)\}\right)$$

In Section 10.4.1 we see that constraint languages that contain and/or precedence constraints and Ord-Horn constraints can still be solved in polynomial time.

- 1.5.10. Ord-Horn interval constraints. For some (binary) reducts \mathfrak{B} of Allen's interval algebra the problem $CSP(\mathfrak{B})$ can be solved in polynomial time. The most important of these reducts is the class of Ord-Horn interval constraints, which has been introduced by Nebel and Bürkert [165]. It consists of all the relations R of Allen's interval algebra such that the relation $\{(x,y,u,v) \mid ([x,y],[u,v]) \in R\}$ is Ord-Horn (see Section 1.5.9). Now it is not hard to see that satisfiability for Ord-Horn interval constraints has a polynomial-time reduction to Ord-Horn satisfiability. This type of reduction will be studied in Section 5.5.
- 1.5.11. Linear program feasibility. Linear Programming is a computational problem of outstanding theoretical and practical importance (see e.g. [185]). It is known to be computationally equivalent to the problem to decide whether a given set of linear (non-strict) inequalities is feasible, i.e., defines a non-empty set.

Linear Program Feasibility

INSTANCE: A finite set of variables V; a finite set of linear inequalities of the form $a_1x_1 + \cdots + a_kx_k \leq a_0$ where $x_1, \ldots, x_k \in V$ and a_0, \ldots, a_k are rational numbers where numerator and denominator are represented in binary. QUESTION: Does there exist an $x \in \mathbb{R}^{|V|}$ that satisfies all inequalities?

Kachyian showed in [134] that Linear Program Feasibility can be solved in polynomial time. It is clearly not possible to formulate this problem as $CSP(\mathfrak{B})$ for a structure \mathfrak{B} with a *finite* relational signature. However, we show below that it is polynomial-time equivalent to CSP ((\mathbb{R} ; { $(x,y,z) \mid x+y=z$ }, {1}, \leq)). For this, we need the following lemma.

LEMMA 1.5.1 (from [38]). Let $n_0, \ldots, n_l \in \mathbb{Q}$ be arbitrary rational numbers. Then the relation $\{(x_1, \ldots, x_l) \mid n_1x_1 + \ldots + n_lx_l = n_0\}$ is primitive positive definable in $(\mathbb{R}; \{(x, y, z) \mid x + y = z\}, \{1\})$. Furthermore, the primitive positive formula that defines the relation can be computed in polynomial time.

The idea to prove this is to use iterated doubling to define large numbers with small primitive positive formulas. By extending the previous result to inequalities, one can prove the following.

PROPOSITION 1.5.2 (from [38]). The linear program feasibility problem for linear programs is polynomial-time equivalent to $CSP((\mathbb{R}; \{(x, y, z) \mid x + y = z\}, \{1\}, \leq))$.

1.5.12. The max-atoms problem. In our list of problems from the literature that can be formulated as $CSP(\mathfrak{B})$, we also want to include one problem in NP where it is not known whether $CSP(\mathfrak{B})$ is in P or NP-hard. Our problem is closely related to the following problem, which has been introduced in [23] and, independently, in [164].

The Max-Atoms Problem

INSTANCE: A finite set of variables V; a finite set of constraints of the form $x_0 \le max(a_1x_1,\ldots,a_kx_k)$ where $x_1,\ldots,x_k \in V$ and a_0,\ldots,a_k (the *coefficients*) are integers represented in binary.

QUESTION: Does there exist an $x \in \mathbb{Q}^{|V|}$ that satisfies all inequalities?

It is known that the Max-atoms problem is computationally equivalent to meanpayoff games [164], and therefore it is contained in NP \cap coNP. It also follows that deciding the winner in Parity games and satisfiability of the propositional μ -calculus can be reduced to the max-atoms problem. Bezem, Nieuwenhuis and Rodríguez-Carbonell [23] give an alternative proof that the problem is in NP \cap coNP. In the same paper, they also shown that a certain hypergraph reachability problem, and an intensively studied problem in max/+ algebra are equivalent to the max-atoms problem. Moreover, they show that the problem is in P when the coefficients in the input are represented in unary.

Similarly as for linear program feasibility, the Max-atoms problem cannot be formulated as $CSP(\mathfrak{B})$ for a structure \mathfrak{B} with a finite relational signature. The problem we introduce instead is

$$\mathrm{CSP}((\mathbb{Q};\{(x,y)\mid y=x+1\},\{(x,y)\mid y=2x\},R_\leq^{min}))$$

where $R_{\leq}^{min} = \{(x,y,z) \mid x \geq y \vee x \geq z\}$ is a variant of the relation R^{min} from Section 1.5.8. The max-atoms problem can be reduced to this problem: we replace expressions of the form $x_i + a_i$ by a new variable y_i , and add a primitive positive formula $\phi(x,y)$ that defines $y_i = x_i + a_i$ and can be computed in polynomial time in the input size of the max-atoms problem. We do not know how to prove hardness for the CSP above, and rather think that the problem might well be in P.

1.5.13. Unification. Unification (and unification modulo equational theories) is an proper field in computational logic, and the complexity of the unification problem has been studied in numerous variants [10]. Many unification problems can be viewed as $CSP(\mathfrak{B})$, for an appropriate infinite structure \mathfrak{B} , as we will see in the following. We start with the most fundamental unification problem.

Let $\sigma := \{f_1, \dots, f_k\}$ be a finite set function symbols, and let x be a variable symbol. Then $\mathcal{F}(x)$ denotes the set of all terms that can be constructed from τ and the variable x. The unnested unification problem over τ is the following problem⁶.

⁶This problem is known to be equivalent to the standard unification problem where the input is a single equation $t_1 \approx t_2$ for 'nested' terms $t_1, t_2 \in \mathcal{F}(x)$.

Unnested Unification Problem over τ

INSTANCE: a finite set of variables V, and a finite set of 'un-nested' term equations, i.e., expressions of the form $y_0 \approx f(y_1, \ldots, y_k)$ for $y_0, y_1, \ldots, y_k \in V$ and $f \in \tau$. QUESTION: is there an assignment $s \colon V \to \mathcal{F}(x)$ such that for every expression $y_0 = f(y_1, \ldots, y_k)$ in the input we have $s(y_0) = f(s(y_1), \ldots, s(y_k))$?

For fixed τ as above, let $\mathfrak{T} = (\mathcal{F}(x); F_1, \ldots, F_k)$ be the structure where F_i is the relation $\{(t_0, t_1, \ldots, t_r) \in (\mathcal{F}(x))^{r+1} \mid t_0 = f_i(t_1, \ldots, t_r)\}$ (here, r is the arity of f_i). It is clear that the unnested unification problem over τ can be described as $\mathrm{CSP}(\mathfrak{T})$. In a similar way, equational unification problems (see [10]) can be viewed as CSPs .

1.6. Overview

This text develops the universal-algebraic approach for complexity analysis of constraint satisfaction problems with countably infinite ω -categorical templates. Parts of the corresponding theory follow or generalize the universal-algebraic approach for CSPs with finite templates, whereas other parts are specific to infinite domains, such as the way in which we apply Ramsey theory. We then present two complexity classification results that have been obtained using this approach: the classification of temporal CSPs in Chapter 10, and Schaefer's theorem for graphs in Chapter 9. We close with Chapter 11 on classes of computational problems that probably do not exhibit a complexity dichotomy.

Publication Note. My PhD-thesis [24] also treated constraint satisfaction with ω -categorical templates, and already hinted at the relevance of polymorphisms and universal algebra. But it is only here that we fully present the universal-algebraic approach and its applications for classification projects of large classes of infinite-domain constraint satisfaction problems. The self-contained presentation in this thesis is collecting and re-combining results that have been fragmented over various publications, and often come with new proofs.

Parts of the content of this thesis have been published by co-authors and myself in conferences or journals. The example sections in Chapter 1 and Chapter 4 present CSPs studied in [25, 31, 34, 37, 44, 45, 55]. Chapter 3 contains a new proof, to be published in the journal version of [35], of the main result in [25]. Chapter 5 covers original results from [47], [27], and the survey [26]. Many results in Chapter 8 are from [54] and the survey [50].

The classification for equality constraint satisfaction problems has first been obtained in [40], but the proof presented here is new and borrows results and ideas from [29] and [51]. The classification for temporal constraint satisfaction problems in Chapter 10 is based on [41] and [42], with some additions from the survey [50]. Schaefer's theorem for graphs is based on [52] and [51], again with additions from [50] Finally, Chapter 11 also contains some results from [33].

1.7. Uncovered Topics

When choosing the material to be included in this thesis, certain restrictive choices had to be made. We comment on related lines of research or facets of the area that we had to skip.

1.7.1. Infinite Signatures. Several natural computational problems could be formulated in the form $CSP(\mathfrak{B})$ when we would allow that the structure \mathfrak{B} has a countably infinite signature. For example, we might want to view the feasibility problem for linear programs (Section 1.5.11) as $CSP(\mathfrak{B})$ where \mathfrak{B} contains all relations of the form $\{(x_1,\ldots,x_k) \mid a_1x_1+\cdots+a_kx_k \leq a_0\}$, for all rational numbers a_0,a_1,\ldots,a_k .

Indeed, several general results for constraint satisfaction that we present in this thesis would carry over to infinite signatures with no problems.

If we wanted to extend the present definition of $CSP(\mathfrak{B})$ to structures \mathfrak{B} with an infinite signature, we are faced with the difficulty to specify how the constraints in input instances of $CSP(\mathfrak{B})$ are represented. When \mathfrak{B} has a finite signature, this causes no problems, since we can fix any representation for the finite number of relation symbols; since \mathfrak{B} is considered to be fixed, the precise choice of the representation is irrelevant. When \mathfrak{B} has an infinite signature, a good choice how to code the constraints in the input very much depends on the structure \mathfrak{B} . In the example of linear programming feasibility, for instance, we might want to represent the constraint $a_1x_1 + \cdots + a_kx_k \leq a_0$ by specifying the coefficients a_0, a_1, \ldots, a_k in binary.

Note that the issue of finite versus infinite constraint languages is not specific to infinite domains, but becomes relevant already for finite domains. Typically, for infinite constraint languages over a finite domain each constraint in the input is represented by listing all tuples of the corresponding relation in the constraint language. But this is not the only, and sometimes not even the most natural way to represent the constraints. For instance for the Horn-SAT problem (see Section 1.5.7), the most natural way to present the constraint is by writing them as conjunctions of Horn-clauses. In the general setting, several representations have been proposed, some of which are more concise than listing all tuples [74], and some of which are less concise [161].

It turns out that typically when a constraint satisfaction problem with an infinite constraint language is computationally hard, then there is a finite set of relations in this language such that the CSP for this sub-language is already NP-hard. For infinite constraint languages over a finite domain, and when the constraints are represented by explicitly listing all satisfying assignments to the variables of the constraint, it has even been conjectured [65] that this might be true in general; that is, when $CSP(\mathfrak{B})$ is NP-hard under this representation, then \mathfrak{B} has a finite signature reduct with a hard CSP. This conjecture is still open. We also want to mention a conditional non-dichotomy result for infinite constraint languages from [33].

We have decided to keep in this thesis the focus on CSPs for *finite* constraint languages, for the following reasons. This allows to work with one and the same definition of the computational problem $CSP(\mathfrak{B})$ for all infinite structures \mathfrak{B} with a finite signature. Moreover, for all of the algorithms presented in this thesis it will be immediately clear under which input assumption they might be generalized to deal with an infinite constraint language. This does not prevent us from stating relevant mathematical facts in full generality when they also hold for structures with an infinite signature; only when it comes to statements about $CSP(\mathfrak{B})$ we insist that \mathfrak{B} has finite relational signature.

1.7.2. Complexity classes below P. Besides the mentioned progress on the dichotomy conjecture for finite domain CSPs, there has been considerable research activity to localize the exact complexity of CSPs inside the complexity class P, or with respect to definability in certain logics. By definability of CSP(\mathfrak{B}) we mean that there exists a sentence Φ is some logic (typically extensions of first-order logic and restrictions of least fixed point logics) such that $\mathfrak{A} \models \Phi$ if and only if \mathfrak{A} homomorphically maps to \mathfrak{B} (that is, in this case it is most natural to consider the definitions of the CSP presented in Section 1.1 and in Section 1.4).

One motivation for studying computational complexity within P is the question whether it is possible to solve problems faster in parallel models of computation. Another motivation, in particular for definability of CSPs in certain logics, is the goal to better understand the scope of existing algorithmic techniques to solve CSPs (such as Datalog, or restrictions of Datalog).

In this line of research, the computational complexity of $CSP(\mathfrak{B})$ has been completely classified when \mathfrak{B} is a two-element structure [6]. Each problem in this class is complete for one of the complexity classes NP, P, $\oplus L$, NL, L, and AC^0 under AC^0 isomorphisms. For general finite domains, several universal-algebraic conditions are known that imply hardness for various complexity classes [9, 145]. Concerning definability of CSPs, there are precise characterizations of those CSPs that are definable by a first-order sentence [8, 144, 180]. Moreover, if $CSP(\mathfrak{B})$ is not first-order definable, then it is L-complete under C^0 -reductions [145] (also see [82, 92]). For infinite domain constraint satisfaction, it appears that there are no general results about localizing the complexity of CSPs within the complexity class P yet. However, we would like to remark that already in some concrete and model-theoretically well-behaved structures $\mathfrak B$ the precise complexity of $CSP(\mathfrak B)$ within P is open. We give one example.

EXAMPLE 1.7.1. Consider the problem

$$\mathrm{CSP}((\mathbb{Q};\neq,\{(x,y,z)\mid (x=y\Rightarrow y\leq z)\wedge x\leq y))\;.$$

This problem is NL-hard since there is an easy reduction from directed reachability to this problem, and directed reachability is an NL-complete problem. By careful inspection of Tarjan's linear-time algorithm for strongly connected components [191] we see that the problem can be solved in linear time. However, the precise complexity of this problem is not known; it might be that the problem is contained in NL, but it might also be P-hard.

1.7.3. Quantified CSPs. Let \mathfrak{B} be a structure with a finite relational signature. Then the quantified constraint satisfaction problem for \mathfrak{B} , denoted QCSP(\mathfrak{B}), is the computational problem to decide for a given first-order sentence ϕ in prenex normal form and without disjunction and negation symbols whether ϕ is true in \mathfrak{B} . The difference of QCSP(\mathfrak{B}) from CSP(\mathfrak{B}) as we have presented it in Section 1.1 is that universal quantification is permitted in the input sentences ϕ .

The additional expressiveness often comes at the prize of higher computational complexity; whereas for finite structures \mathfrak{B} , the CSP for \mathfrak{B} is always in NP, there are finite structures $\mathfrak B$ where QCSP($\mathfrak B$) is PSPACE-complete. But quite surprisingly, several constraint languages with a polynomial-time tractable CSP also have a polynomial-time QCSP. This is for instance the case for 2SAT [7] (see Example 1.2.2), or for Horn-3SAT [130] (see Section 1.5.7). Similarly, it can be shown that the temporal constraint languages presented in Section 10.5.2 and Section 10.5.5 are not only tractable for the CSP, but also for the QCSP. These are attractive results, since they assert that we can solve an even more expressive computational problem than the CSP for the same constraint language without loosing polynomial-time tractability. From a methodological point of view, we remark that the universal-algebraic approach can also be applied to study the complexity of the QCSP [57]; as in the case of the CSP, the computational complexity of $QCSP(\mathfrak{B})$ is captured by the polymorphisms of $\mathfrak B$ (see Chapter 5). Classifications of the QCSP typically rely on the corresponding classification for the CSP. In particular, any hardness result for the CSP immediately translates into a hardness result for the QCSP. Moreover, in the cases where the $CSP(\mathfrak{B})$ is tractable, the algorithmic insight is often the starting point for further investigations of QCSP(\mathfrak{B}).

However, complexity classifications for QCSPs are typically harder to obtain than the corresponding complexity classifications for CSPs. One of the reasons is that several relevant universal-algebraic facts require the assumption that the algebra be *idempotent* (see Section 5.6.1). The complexity of the QCSP, however, is not preserved by homomorphic equivalence, and when we study $QCSP(\mathfrak{B})$ we can thus not

pass to the core of \mathfrak{B} . Hence, we can in general not make the assumption that the polymorphism clone of \mathfrak{B} is idempotent.

For $CSP(\mathfrak{B})$, a powerful way of proving NP-hardness is to give a primitive positive interpretation (see Section 5.5) of a Boolean template with a hard CSP. This is no longer possible for the QCSP. There are for example 3-element templates \mathfrak{B} that are preserved by a semi-lattice operation (and hence no hard Boolean CSP can be interpreted in \mathfrak{B}) where QCSP(\mathfrak{B}) is PSPACE-complete [57]. Finally, we would like to mention that PSPACE-hardness proofs for the QCSP are often much harder than NP-hardness proofs for the CSP [28,57].

From the above is not surprising that a full classification of the QCSP complexity for three-element structures is still open. Similarly, there is no classification of the QCSP for the class of temporal constraint languages presented in Chapter 10. There are concrete temporal constraint languages where the QCSP is of unknown computational complexity, for instance the QCSP for

$$(\mathbb{Q}; \{(x, y, z) \mid x = y \Rightarrow y \ge z\}).$$

For this problem, we do not know hardness for any complexity class above P, and do not know containment in any complexity class below PSPACE.

1.7.4. Non ω -categorical templates. Most methods presented in this thesis crucially rely on the assumption that the constraint languages are ω -categorical. However, systematic complexity classification is also possible for large classes of constraint languages that are not ω -categorical.

1.7.4.1. Distance CSPs. The structure $(\mathbb{Z}; succ)$ of the integers with the sucessor relation $succ = \{(x,y) \mid x=y+1\}$ constitutes one of the simplest infinite structures with a finite signature that is not ω -categorical. Structures with a first-order definition in $(\mathbb{Z}; succ)$ are particularly well-behaved from a model-theoretic perspective: they are strongly minimal [120,158], and therefore uncountably categorical (but usually not ω -categorical). In [32], the complexity of CSP(\mathfrak{B}) has been studied when \mathfrak{B} is first-order definable in $(\mathbb{Z}; succ)$. As an example, consider the directed graph with vertex set \mathbb{Z} which has an edge between x and y if the difference between x and y is either 1 or 3. This graph can be viewed as the structure $(\mathbb{Z}; R_{\{1,3\}})$ where $R_{\{1,3\}} = \{(x,y) \mid x-y \in \{1,3\}\}$, which has a first-order definition over $(\mathbb{Z}; succ)$ since $R_{\{1,3\}}(x,y)$ iff

$$succ(x, y) \vee \exists u, v \left(succ(x, u) \wedge succ(u, v) \wedge succ(v, y) \right)$$
.

Another example is the undirected graph with vertex set \mathbb{Z} where two integers x, y are linked if the *distance* between x and y is one or two.

The corresponding class of CSPs contains many natural combinatorial problems. For instance, the CSP for the structure $(\mathbb{Z}; R_{\{1,3\}})$ is the computational problem to label the vertices of a given directed graph G such that if (x,y) is an arc in G, then the difference between the label for x and the label for y is one or three. It follows from the results in [32] that this problem is in P. The CSP for the undirected graph $(\mathbb{Z}; \{(x,y) \mid |x-y| \in \{1,2\}\})$ mentioned above is exactly the 3-coloring problem, and therefore NP-complete. In general, those problems have the flavor of assignment problems where the task is to map the variables to integers such that various given constraints on differences and distances (and Boolean combinations thereof) are satisfied. Therefore, CSPs whose template is definable over $(\mathbb{Z}; succ)$ are called distance CSPs

The complexity classification for distance CSPs presented in [32] is incomplete in two respects. First, it might be the case that a structure \mathfrak{B} with a first-order definition in (\mathbb{Z} ; succ) has a finite core \mathfrak{A} . Those cores have the property that they have a single

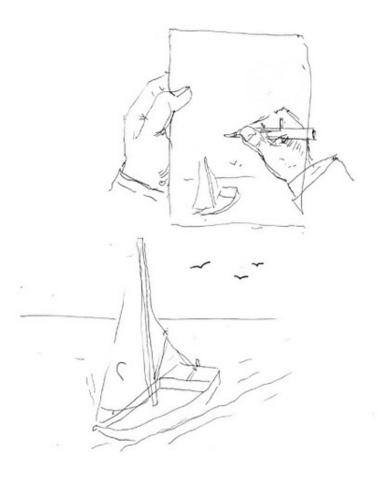
orbit (see Section 1.2.3); this follows easily from the fact that the structure (\mathbb{Z} ; succ) also has only one orbit. But the dichotomy conjecture for finite domain CSPs has not yet been established for the particular case of finite templates that have only one orbit. The second point in which the complexity classification for distance CSPs given in [32] is incomplete is that it only studies templates \mathfrak{B} that are locally finite, an additional finiteness condition, defined as follows. A graph is called locally finite if every vertex is contained in a finite number of edges; a relational structure is called locally finite if its Gaifman graph (definition given in Section 2) is locally finite. The method that is applied in [32] is relational, and not universal-algebraic.

1.7.4.2. Tractable Expansions of Linear Program Feasibility. Linear Programming is a computational problem of outstanding theoretical and practical importance. As we have seen, it is computationally equivalent to the CSP we have presented in Section 1.5.11. It seems to be interesting to investigate how far the constraint language of all linear inequalities can be expanded such that the corresponding constraint satisfaction problem remains polynomial-time solvable. It has been shown in [38] that every first-order expansion of linear programming is contained in a class called Horn-DLR from [126] and polynomial-time tractable, or otherwise NP-hard.

An important class of relations over the real numbers that generalizes the class of relations defined by linear inequalities is the class of all semi-algebraic relations, i.e., relations with a first-order definition in $(\mathbb{R}; +, *)$. By the fundamental theorem of Tarski and Seidenberg it is known that a relation $S \subseteq \mathbb{R}^n$ is semi-algebraic if and only if it has a quantifier-free first-order definition in $(\mathbb{R}; +, *, 0, 1, \leq)$. Geometrically, we can view semi-algebraic sets as unions of intersections of the solution sets of strict and non-strict polynomial inequalities. The classification of the computational complexity of CSPs for real-valued semi-algebraic constraint languages is an ambitious research project, and has important links to semidefinite programming: every semidefinite representable set is semi-algebraic and convex. Surprisingly, there are many fundamental questions in this area that are wide open, for instance the complexity of semi-linear programming feasibility (see e.g. Section 6.4.4 in [197], or [177]), or the conjecture that all convex semi-algebraic set are semidefinite representable [115], i.e., primitive positive definable over the structure that has as its relations all the solution spaces of semi-definite programs. These important questions from real algebraic geometry are out of the scope of this thesis.

CHAPTER 2

Preliminaries in Logic



This chapter collects some basic terminology and facts from logic. The notation mostly follows Hodges' text book [120], so many readers may safely skip this chapter; they can consult it later, if needed, for particular concepts that we introduce here.

2.1. Structures

In Section 1.1 we have already defined relational structures; we now give the general definition of structures that might also contain functions, since we need those later. One occasion where we need functions rather than relations is in Chapter 5 when we consider algebras (by which we mean structures with a purely functional signature) that arise from the set of polymorphisms of a structure. Moreover, several templates are most naturally defined over a structure having a functional signature,

see e.g. Section 4.3. Most definitions go parallel for functional and relational signatures, so we give them together in this section.

A signature τ is a set of relation and function symbols, each equipped with an arity. A τ -structure $\mathfrak A$ is a set A (the domain of $\mathfrak A$) together with a relation $R^{\mathfrak A} \subseteq A^k$ for each k-ary relation symbol in τ and a function $f^{\mathfrak A} \colon A^k \to A$ for each k-ary function symbol in τ ; here we allow the case k=0 to model constant symbols. Unless stated otherwise, A, B, C, \ldots denote the domains of the structures $\mathfrak A, \mathfrak B, \mathfrak C, \ldots$, respectively. We sometimes write $(A; R_1^{\mathfrak A}, R_2^{\mathfrak A}, \ldots, f_1^{\mathfrak A}, f_2^{\mathfrak A}, \ldots)$ for the relational structure $\mathfrak A$ with relations $R_1^{\mathfrak A}, R_2^{\mathfrak A}, \ldots$ and functions $f_1^{\mathfrak A}, f_2^{\mathfrak A}, \ldots$ When there is no danger of confusion, we use the same symbol for a function and its function symbol, and for a relation and its relation symbol. We say that a structure is infinite if its domain is infinite. The most important special cases of structures that appear in this thesis are relational structures, that is, structures with a purely relational signature, and algebras, that is, structures with a purely functional signature. Algebras with domain A, B, C, \ldots are typically denoted by A, B, C, \ldots

EXAMPLE 2.1.1. A group is an algebra \mathbf{G} with a binary function symbol \cdot for composition, a unary function symbol $^{-1}$ for the inverse, and a constant e for the identity element of \mathbf{G} , satisfying the sentences $\forall x, y, z. x \cdot (y \cdot z) = (x \cdot y) \cdot z, \forall x. x \cdot x^{-1} = e, \forall x. e \cdot x = x$, and $\forall x. x \cdot e = x$. In this signature, the subgroups of \mathbf{G} are precisely the subalgebras of \mathbf{G} as defined below. We typically omit the function symbol \cdot and write fg for the product of elements f, g of \mathbf{G} . Such groups will also be called abstract groups to distinguish them from permutation groups; a permutation group (over a set X) is a set of permutations of X closed under composition and inverse, and containing the identity.

When working with function symbols, it is sometimes convenient to work with multi-sorted structures, where we have distinguished unary predicates, called sorts, that define a partition of the domain, and where function symbols might only be defined on some of the sorts (that is, the function symbols might not be defined on some of the elements). We are sloppy with the formal details since they can always be worked out easily. In all our applications, the multi-sorted structures will in fact be two-sorted, in which case we denote them by $(\mathfrak{A}, \mathfrak{B})$ — here one sort induces the structure \mathfrak{A} , and the other sort induces the structure \mathfrak{B} .

2.1.1. Expansions and reducts. Let σ, τ be signatures with $\sigma \subseteq \tau$. When \mathfrak{A} is a σ -structure and \mathfrak{B} is a τ -structure, both with the same domain, such that $R^{\mathfrak{A}} = R^{\mathfrak{B}}$ for all relations $R \in \sigma$ and $f^{\mathfrak{A}} = f^{\mathfrak{B}}$ for all functions and constants $f \in \sigma$, then \mathfrak{A} is called a *reduct* of \mathfrak{B} , and \mathfrak{B} is called an *expansion* of \mathfrak{A} . An expansion \mathfrak{B} of \mathfrak{A} is called *first-order* if all new relations in \mathfrak{B} are first-order definable over \mathfrak{A} . A structure \mathfrak{A} is called a *finite reduct* of \mathfrak{B} if \mathfrak{A} is a reduct of \mathfrak{B} with a finite signature. We also write (\mathfrak{A}, R) (and, similarly, (\mathfrak{A}, f)) for the expansion of \mathfrak{A} by a new relation R (a new function or constant f, respectively).

If $\mathfrak A$ is a τ -structure and $(a_i)_{i\in I}$ a sequence of elements of A indexed by I, then $(\mathfrak A; (a_i)_{i\in I})$ is the natural $(\tau \cup \{c_i|i\in I\})$ -expansion of $\mathfrak A$ with |I| new constants, where c_i is interpreted by a_i for all $i\in I$.

- **2.1.2. Extensions and substructures.** A τ -structure $\mathfrak A$ is a *substructure* of a τ -structure $\mathfrak B$ iff
 - $A \subseteq B$,
 - for each $R \in \tau$ and for all tuples \bar{a} from $A, \bar{a} \in R^{\mathfrak{A}}$ iff $\bar{a} \in R^{\mathfrak{B}}$, and
 - for each $f \in \tau$ we have that $f^{\mathfrak{A}}(\bar{a}) = f^{\mathfrak{B}}(\bar{a})$.

In this case, we also say that \mathfrak{B} is an extension of \mathfrak{A} . Substructures \mathfrak{A} of \mathfrak{B} and extensions \mathfrak{B} of \mathfrak{A} are called *proper* if the domains of \mathfrak{A} and \mathfrak{B} are distinct. Note that for every subset S of the domain of \mathfrak{B} there is a unique smallest substructure of \mathfrak{B} whose domain contains S, which is called the substructure of \mathfrak{B} generated by S, and which is denoted by $\mathfrak{B}[S]$. We say that \mathfrak{B} is finitely generated if $\mathfrak{B}=\mathfrak{B}[S]$ for a finite set S of elements. A subalgebra of an algebra \mathbf{B} (induced by S) is simply a substructure of \mathbf{B} (generated by S) – recall that we have defined algebras as functions with a purely functional signature.

The following is a concept that we only define for relational structures.

Definition 2.1.2. The Gaifman-graph of a relational structure \mathfrak{B} with domain B is the following undirected graph: the vertex set is B, and there is an edge between distinct elements $x, y \in B$ when there is a tuple in one of the relations of \mathfrak{B} that has both x and y as entries.

A relational structure $\mathfrak B$ is readily seen to be connected (in the sense of Section 1.1) if and only if its Gaifman graph is connected (in the usual graph-theoretic sense).

2.1.3. Products. Let \mathfrak{A} and \mathfrak{B} be two structures with domain A and B, and the same signature τ . Then the (direct, or categorical) product $\mathfrak{C} = \mathfrak{A} \times \mathfrak{B}$ is the τ -structure with domain $A \times B$, which has for each k-ary $R \in \tau$ the relation that contains a tuple $((a_1,b_1),\ldots,(a_k,b_k))$ if and only if $R(a_1,\ldots,a_k)$ holds in $\mathfrak A$ and $R(b_1,\ldots,b_k)$ holds in \mathfrak{B} . For each k-ary $f\in\tau$ the structure \mathfrak{C} has the operation that maps $((a_1,b_1),\ldots,(a_k,b_k))$ to $(f(a_1,\ldots,a_k),f(b_1,\ldots,b_k))$. The direct product $\mathfrak{A} \times \mathfrak{A}$ is also denoted by \mathfrak{A}^2 , and the k-fold product $\mathfrak{A} \times \cdots \times \mathfrak{A}$, defined analogously,

We generalize the definition of products in the obvious way to infinite products. For a sequence of τ -structures $(\mathfrak{A}_i)_{i\in I}$, the direct product $\mathfrak{B}=\prod_{i\in I}\mathfrak{A}_i$ is the τ structure on the domain $\prod_{i \in I} A_i$ such that for $R \in \tau$ of arity k

$$((a_i^1)_{i\in I},\ldots,(a_i^k)_{i\in I})\in R^{\mathfrak{B}} \text{ iff } (a_i^1,\ldots,a_i^k)\in R^{\mathfrak{A}_i} \text{ for each } i\in I\;,$$

and for $f \in \tau$ of arity k, we have

$$f^{\mathfrak{B}}((a_i^1)_{i\in I},\ldots,(a_i^k)_{i\in I})=(f^{\mathfrak{A}_i}(a_i^1,\ldots,a_i^k))_{i\in I}$$
.

2.2. Mappings

Throughout the text, we use the following conventions. When $f: A \to B$ is a function, and S is a subset of A, then f(S) denotes the set $\{f(s) \mid s \in S\} \subseteq B$. When $t = (t_1, \ldots, t_k)$ is a k-tuple of elements of B, then f(t) denotes the tuple $(f(t_1),\ldots,f(t_k))$. Moreover, we use the same convention for higher-ary functions $f: B^m \to B$: when t^1, \ldots, t^m are k-tuples of elements of B, then $f(t^1, \ldots, t^m)$ denotes the k-tuple $(f(t^1_1, \ldots, t^m_1), \ldots, f(t^1_k, \ldots, t^m_k))$ (that is, the k-tuple is computed componentwise).

In the following, let \mathfrak{A} be a τ -structure with domain A and \mathfrak{B} a τ -structure with domain B. A homomorphism h from \mathfrak{A} to \mathfrak{B} is a mapping from A to B that preserves each function and each relation for the symbols in τ ; that is,

- if (a_1, \ldots, a_k) is in $\mathbb{R}^{\mathfrak{A}}$, then $(h(a_1), \ldots, h(a_k))$ must be in $\mathbb{R}^{\mathfrak{B}}$; $f^{\mathfrak{B}}(h(a_1), \ldots, h(a_k)) = h(f^{\mathfrak{A}}(a_1, \ldots, a_k))$.

When \mathbf{A}, \mathbf{B} are algebras with the same signature and domain A, B, respectively, and f is a homomorphism from **A** to **B**, then f(A) induces a subalgebra of **B**, and this subalgebra is called a homomorphic image of A.

When a mapping h preserves a relation R, we also say that R is *invariant* under h. If h does not preserve R, we also say that h violates R. A homomorphism from $\mathfrak A$ to $\mathfrak B$ is called a strong homomorphism if it also preserves the complements of the relations from $\mathfrak A$. Injective strong homomorphisms are called *embeddings*. Surjective embeddings are called isomorphisms. A homomorphism from a substructure of $\mathfrak A$ to $\mathfrak B$ is called a partial homomorphism from $\mathfrak A$ to $\mathfrak B$. An embedding from a substructure of $\mathfrak A$ into $\mathfrak B$ is called a partial isomorphism between $\mathfrak A$ and $\mathfrak B$.

Homomorphisms and isomorphisms from \mathfrak{B} to itself are called *endomorphisms* and *automorphisms*, respectively. Structures where the identity is the only automorphism are called *rigid*. When $f \colon A \to B$ and $g \colon B \to C$, then $g \circ f$ denotes the composed function $x \mapsto g(f(x))$. Clearly, the composition of two homomorphisms (embeddings, automorphisms) is again a homomorphism (embedding, automorphism). Let $\operatorname{Aut}(\mathfrak{A})$ and $\operatorname{End}(\mathfrak{A})$ be the sets of automorphisms and endomorphisms, respectively, of \mathfrak{A} . The set $\operatorname{Aut}(\mathfrak{A})$ can be viewed as a group, and $\operatorname{End}(\mathfrak{A})$ as a monoid with respect to composition; more on that can be found in Section 3.3 and Section 3.4.2.

2.3. Formulas and Theories

We assume familiarity with basic concepts of classical first-order logic; see for example [91]. In particular, we will use the concepts of *conjunctive normal form* (CNF), free and bound variables, terms and subterms, clauses, (positive and negative) literals, and atomic formulas.

We always allow the first-order formula x=y (for equality) and \bot (for 'false'), independently of the signature, unless stated otherwise. A formula without free variables will be called a *sentence*. A (*first-order*) theory is a set of (first-order) sentences. A structure \mathfrak{B} is a model of a sentence ϕ (or a theory T) if ϕ (all sentences in T, respectively) holds true in \mathfrak{B} ; in this case we write $\mathfrak{B} \models \phi$ ($\mathfrak{B} \models T$). The set of all first-order sentences that are true in a given structure \mathfrak{B} is called the *first-order theory of* \mathfrak{B} , and denoted Th(\mathfrak{B}). If a sentence or a theory has a model, we call it satisfiable. We state two basic facts that will be used later.

Theorem 2.3.1 (Compactness; see Theorem 5.1.1 in [120]). Let T be a first-order theory. If every finite subset of T is satisfiable then T is satisfiable.

When T is a theory and ϕ a sentence, we say that T entails ϕ , in symbols $T \models \phi$, if every model of T satisfies ϕ . Two theories T_1 , T_2 are said to be equivalent if $T_1 \models T_2$ and $T_2 \models T_1$.

LEMMA 2.3.2 (see Lemma 2.3.2 in [120]). Let T be a first-order τ -theory, and ϕ a first-order τ -formula with free variables x_1, \ldots, x_n . Let c_1, \ldots, c_n be distinct constants that are not in τ . Then $T \models \phi(c_1, \ldots, c_k)$ if and only if $T \models \forall x_1, \ldots, x_n.\phi$.

Let \mathfrak{B} be a τ -structure. When ϕ is a first-order τ -formula, and when x_1, \ldots, x_n is an ordered list that enumerates all the free variables, then $\phi(x_1, \ldots, x_n)$ defines over \mathfrak{B} the relation $\{(b_1, \ldots, b_n) \mid \mathfrak{B} \models \phi(b_1, \ldots, b_n)\}$. When ϕ is a τ -formula with free variables x_1, \ldots, x_n , and h is a k-ary function then h preserves ϕ if h preserves the n-ary relation that is defined by $\phi(x_1, \ldots, x_n)$. We say that a structure \mathfrak{A} is (first-order) definable in \mathfrak{B} if \mathfrak{A} and \mathfrak{B} have the same domain, and every relation from \mathfrak{A} has a first-order definition in \mathfrak{B} . Two structures \mathfrak{A} , \mathfrak{B} are (first-order) interdefinable if \mathfrak{A} is definable in \mathfrak{B} and vice versa.

A first-order τ -formula ϕ is said to be

• quantifier-free if it does not contain any quantifiers; that is, it is built from the logical connectives \land, \lor, \neg , the binary relation =, the (free) variables, and the symbols from τ only (also see Section 3.6.1);

- in prenex normal form if it is of the form $Q_1x_1 \dots Q_nx_n.\psi$ where $Q_i \in \{\forall, \exists\}$ and ψ is quantifier-free;
- *Horn* if it is written in conjunctive normal form and every clause has at most one positive literal (those formulas appear e.g. in Section 6.3);
- positive quantifier-free if ϕ is quantifier-free, and if in addition ϕ does not contain negation symbols \neg .
- existential if it is of the form $\exists x_1, \dots, x_n$. ψ where ψ is quantifier-free (those formulas appear e.g. in Section 3.6.2);
- universal if is of the form $\forall x_1, \dots, x_n$. ψ where ψ is quantifier-free;
- \exists^+ (existential positive) if it existential and if the quantifier-free part of ϕ does not contain any negation symbols (those formulas appear e.g. in Section 3.6.4);
- \forall (universal negative) if it is of the form $\forall x_1, \ldots, x_n$. $\neg \psi$ where ψ is positive quantifier-free;
- universal conjunctive if it is universal and if the quantifier-free part of ϕ does not contain any negation or disjunction symbols (those formulas appear e.g. in Section 5.6);
- $\forall \exists \ (forall\text{-}exists) \text{ if it is of the form } \forall y_1, \ldots, y_m. \ \psi \text{ where } \psi \text{ is existential } \text{ (those formulas appear e.g. in Section 3.6.2);}$
- $\forall \exists^+$ (positively restricted forall-exists) if it is of the form $\forall \bar{y}.\phi(\bar{y})$, where $\phi(\bar{y})$ is a positive boolean combination of quantifier-free formulas and existential positive formulas (those formulas appear throughout Section 3.6)
- primitive positive if it is of the form $\exists x_1, \ldots, x_n . \ \psi_1 \wedge \cdots \wedge \psi_m$, where ψ_1, \ldots, ψ_m are atomic (they are of central importance in this thesis).

We could have equivalently defined positively restricted forall-exists formulas as conjunctions of universally quantified disjunctions of primitive positive formulas and negated atomic formulas. It is easy to see that every $\forall \exists^+$ -formula can be re-written into such a formula.

Note that homomorphisms preserve all existential positive formulas. An important property of primitive positive sentences ϕ is that $\mathfrak{A} \times \mathfrak{B} \models \phi$ iff $\mathfrak{A} \models \phi$ and $\mathfrak{B} \models \phi$. Also note that partial isomorphisms preserve quantifier-free formulas, embeddings preserve existential formulas, and isomorphisms preserve first-order formulas. Embeddings that preserve all first-order formulas are called *elementary*. When \mathfrak{B} is an extension of \mathfrak{A} such that the identity map from \mathfrak{B} to \mathfrak{A} is an elementary embedding, we say that \mathfrak{B} is an *elementary extension* of \mathfrak{A} , and that \mathfrak{A} is an *elementary substructure* of \mathfrak{B} .

THEOREM 2.3.3 (Löwenheim-Skolem; see Corollary 3.1.4 in [120]). Let \mathfrak{A} be a τ -structure, X a set of elements of \mathfrak{A} , and λ a cardinal such that $|\tau| + |X| \le \lambda \le |A|$. Then \mathfrak{A} has an elementary substructure \mathfrak{B} of cardinality λ with $X \subseteq B$.

A first-order theory T is said to be *existential* if all sentences in T are existential, and the set of all existential τ -sentences that is true in a τ -structure \mathfrak{B} is called the *existential theory of* \mathfrak{B} . Analogously, we define $\forall \exists^+, \forall \exists$, universal, existential positive, and universal negative theories.

2.4. Diagrams

We need the concept of a *diagram* of a structure, in various variants. The idea is to transform a structure into a formula, similarly as in the definition of the *canonical query* given in Section 1.2.

DEFINITION 2.4.1. Let $\mathfrak A$ be a au-structure so that in $\mathfrak A$ every element is named by a constant. Then

- the set of all positive quantifier-free sentences that hold on 𝔄 is denoted by diag₊(𝔄),
- the set of all quantifier-free sentences that hold on $\mathfrak A$ is denoted by diag($\mathfrak A$),
- the set of all universal negative sentences that hold on \mathfrak{A} , is denoted by $diag_{\forall}$ (\mathfrak{A}),
- the elementary diagram of \mathfrak{A} is the set of all first-order sentences true in \mathfrak{A} , and is denoted by $diag_{f_0}(\mathfrak{A})$.

The following is straightforward from the definitions.

LEMMA 2.4.2 (Diagram lemma; Lemma 1.4.2. in [120]). Let $\mathfrak A$ and $\mathfrak B$ be relational τ -structures, and let $\mathfrak A'$ be $(\tau \cup \rho)$ -expansion of $\mathfrak A$ by constant symbols ρ such that every element of $\mathfrak A'$ is named by a constant. Then the following are equivalent.

- (1) There is a $(\tau \cup \sigma)$ -expansion \mathfrak{B}' of \mathfrak{B} such that $\mathfrak{B}' \models diag_+(\mathfrak{A}')$;
- (2) There is a homomorphism from \mathfrak{A} to \mathfrak{B} .

Diagrams are useful in the proof of the following elementary, but important lemma, which has been called the *existential amalgamation theorem* in [120]. The proof is analogous to the proof of Lemma 2.4.4 given in full detail below, and we therefore omit it.

PROPOSITION 2.4.3 (Theorem 5.4.1 in [120]). Let $\mathfrak A$ and $\mathfrak B$ be τ -structures with domain A and B, respectively. Suppose that $\bar a$ lists a subset S of A, and let $h \colon S \to B$ be a partial homomorphism from $\mathfrak A$ to $\mathfrak B$ such that every existential sentence true in $(\mathfrak B, h(\bar a))$ is also true in $(\mathfrak A, \bar a)$. Then there exists an elementary extension $\mathfrak C$ of $\mathfrak A$ and an embedding $g \colon \mathfrak B \to \mathfrak C$ such that $g(h(\bar a)) = \bar a$.

We will need several times a positive variant of Proposition 2.4.3, which is explicitly given in [120], but without proof.

LEMMA 2.4.4 (Theorem 5.4.7 in [120]). Let $\mathfrak A$ and $\mathfrak B$ be τ -structures with domain A and B, respectively. Suppose that $\bar a$ lists a subset S of A, and let $h \colon S \to B$ be a partial homomorphism from $\mathfrak A$ to $\mathfrak B$ such that every existential positive sentence true in $(\mathfrak B, h(\bar a))$ is also true in $(\mathfrak A, \bar a)$. Then there exists an elementary extension $\mathfrak C$ of $\mathfrak A$ and a homomorphism $q \colon \mathfrak B \to \mathfrak C$ such that $q(h(\bar a)) = \bar a$.

PROOF. Similar to the proof of 5.4.1 in [120]. Let σ be a set of constant symbols and \mathfrak{A}' be a $(\tau \cup \sigma)$ -expansion of \mathfrak{A} such that every element of \mathfrak{A}' is denoted by some constant symbol from σ . Let \mathfrak{B}' be an expansion of \mathfrak{B} by constant symbols such that

- \bullet every element of \mathfrak{B}' is denoted by some constant, and
- if $c \in \sigma$ is such that $c^{\mathfrak{A}'} \in S$ in \mathfrak{A}' , then $c^{\mathfrak{B}'} = h(c^{\mathfrak{A}'})$.

It suffices to show that the theory $T := \operatorname{diag}_{fo}(\mathfrak{A}') \cup \operatorname{diag}_{+}(\mathfrak{B}')$ has a model \mathfrak{C}' , since Lemma 2.4.2 then asserts the existence of a homomorphism from \mathfrak{B} to the τ -reduct \mathfrak{C} of \mathfrak{C}' , which will be an elementary extension of \mathfrak{A} . Moreover, such a homomorphism must map $h(\bar{a})$ to \bar{a} .

If T has no model, then by the compactness theorem there is a τ -formula ϕ such that $\phi(\bar{c}) \in \text{diag}_+(\mathfrak{B}')$ and $\mathfrak{A} \models \neg \exists \bar{y}.\phi(\bar{y})$. Since $\exists \bar{y}.\phi(\bar{y})$ is existential positive, the assumptions imply that $\mathfrak{B} \models \neg \exists \bar{y}.\phi(\bar{a},\bar{y})$. This contradicts that $\mathfrak{B} \models \phi(\bar{c})$.

We can now prove a generalization of the condition given in Proposition 1.3.4 from Section 1.3 that characterizes when two theories have the same CSP.

PROPOSITION 2.4.5. Let T and T' be τ -theories. The following are equivalent.

- (1) Every model of T has a homomorphism to a model of T', and every model of T' has a homomorphism to a model of T.
- (2) T and T' imply the same universal negative sentences.

PROOF. To prove the implication from (1) to (2), assume (1), and let ϕ be a universal negative sentence implied by T', and let \mathfrak{C} be a model of T. By (1), there is a homomorphism from \mathfrak{C} to a model \mathfrak{B} of T'. Since $\neg \phi$ is equivalent to an existential positive sentence, it is preserved by homomorphisms, and hence we have a contradiction to the assumption that $T' \models \phi$.

For the implication from (2) to (1), assume (2), and let \mathfrak{B} be a model of T. Let S be the existential positive theory of \mathfrak{B} . We claim that $S \cup T'$ is satisfiable. If not, then by compactness (Theorem 2.3.1) there is some finite subset $\{\phi_1, \ldots, \phi_k\}$ of S such that $T' \vdash (\neg \phi_1 \lor \cdots \lor \neg \phi_k)$. The formula $\neg \phi_1 \lor \cdots \lor \neg \phi_k$ is equivalent to a universal negative sentence ψ , and $T' \vdash \psi$, so by (2) we have that $T \vdash \psi$, and hence $\mathfrak{B} \models \psi$. We have reached a contradiction, since $\mathfrak{B} \models \phi_i$ for all $i \leq k$. So there indeed exists a model \mathfrak{A} of $S \cup T'$. Lemma 2.4.4 applied to \mathfrak{A} and \mathfrak{B} for the empty sequence \bar{a} gives a model \mathfrak{C} of $T' \cup S$ and a homomorphism from \mathfrak{B} to \mathfrak{C} .

This indeed proves Proposition 1.3.4, since theories that imply the same universal negative sentences have obviously the same CSP. It is now also easy to prove Proposition 1.3.6 from Section 1.3, characterizing those theories T for which there exists a structure \mathfrak{B} such that $CSP(T) = CSP(\mathfrak{B})$. We first show the following.

Proposition 2.4.6. For any satisfiable theory T, the following are equivalent.

- (1) there exists a structure \mathfrak{B} that satisfies an existential positive sentence ϕ if and only if $T \cup \{\phi\}$ is satisfiable.
- (2) T has a model \mathfrak{B} that satisfies every existential positive sentence ϕ where $T \cup \{\phi\}$ is satisfiable.
- (3) For all existential positive sentences ϕ_1 and ϕ_2 , if $T \cup \{\phi_1\}$ is satisfiable and $T \cup \{\phi_2\}$ is satisfiable, then $T \cup \{\phi_1, \phi_2\}$ is satisfiable as well.
- (4) T has the Joint Homomorphism Property (JHP confer Proposition 1.3.6).

PROOF. We prove $(1) \Leftrightarrow (2)$, $(2) \Leftrightarrow (3)$, $(3) \Leftrightarrow (4)$. For the implication from (1) to (2), let T' be the universal negative theory of \mathfrak{B} . By assumption, T' and T imply the same universal negative sentences, and hence by Proposition 2.4.5 there is a homomorphism from \mathfrak{B} to a model \mathfrak{C} of T. This model \mathfrak{C} has the desired property: if ϕ is existential positive such that $T \cup \{\phi\}$ is satisfiable, then \mathfrak{B} satisfies ϕ and since homomorphisms preserve existential positive formulas, \mathfrak{C} satisfies ϕ as well.

The implication from (2) to (1) and the implication from (2) to (3) are obvious. To show that (3) implies (2), assume (3). Let P be the set of all existential positive sentences ϕ such that $T \cup \phi$ is satisfiable. By the assumption that T is satisfiable, and by (3), all finite subsets of $T \cup P$ are satisfiable, so by compactness of first-order logic (Theorem 2.3.1) we have that $T \cup P$ has a model \mathfrak{B} .

Now we show that (3) implies (4). Assume (3), and let τ be the signature of T. Let \mathfrak{A}_1 and \mathfrak{A}_2 be models of T. We have to show that there exists a model \mathfrak{B} of T that admits homomorphisms from \mathfrak{A}_1 and \mathfrak{A}_2 . Let \mathfrak{A}'_1 and \mathfrak{A}'_2 be expansions of \mathfrak{A}_1 and \mathfrak{A}_2 , respectively, where every element is denoted by a distinct constant symbol. Consider the theory $T' := T \cup \operatorname{diag}_+(\mathfrak{A}'_1) \cup \operatorname{diag}_+(\mathfrak{A}'_2)$; we claim that T' is satisfiable. By compactness (Theorem 2.3.1), it suffices to show that every finite subset S of T' is satisfiable. Let $S_1 := S \cap \operatorname{diag}_+(\mathfrak{A})$ and $S_2 := S \cap \operatorname{diag}_+(\mathfrak{A}_2)$. By forming a finite conjunction, we see that S_1 and S_2 are logically equivalent to single sentences ϕ_1 and ϕ_2 , respectively. Certainly $T \cup \{\phi_1\}$ and $T \cup \{\phi_2\}$ are satisfiable since \mathfrak{A}'_1 and \mathfrak{A}'_2 are expansions of models of T and therefore satisfy all sentences from T. By (3), the

theory $T \cup \{\phi_1, \phi_2\}$ is satisfiable as well. Therefore the claim is true, and there exists a model \mathfrak{B}' of T'. Let \mathfrak{B} be the τ -reduct of \mathfrak{B}' . Finally, Lemma 2.4.2 asserts the existence of a homomorphism from \mathfrak{A}_1 to \mathfrak{B} and from \mathfrak{A}_2 to \mathfrak{B} , which proves (4).

For the implication from (4) to (3), suppose that T has the JHP, and that ϕ_1 and ϕ_2 are existential positive sentences such that $T \cup \{\phi_1\}$ has a model \mathfrak{A}_1 and $T \cup \{\phi_2\}$ has a model \mathfrak{A}_2 . By (4), there exists a model \mathfrak{B} of T such that \mathfrak{A}_1 and \mathfrak{A}_2 homomorphically map to \mathfrak{B} . Then \mathfrak{B} clearly satisfies $T \cup \{\phi_1, \phi_2\}$ since homomorphisms preserve existential positive sentences.

Note that in the statement and the proof above, the phrase existential positive can be used interchangeably with the phrase primitive positive. With the additional assumption that T has a finite relational signature, item (1) in Proposition 2.4.6 becomes the statement that there exists a structure \mathfrak{B} such that $CSP(\mathfrak{B})$ and CSP(T) are the same problem, so we indeed proved in particular Proposition 1.3.6.

2.5. Chains and Direct Limits

Chains and direct limits of sequences of τ -structures are an important method to construct models of first-order theories, and will be used for instance in Section 3.4 and Section 3.5.

Let $(\mathfrak{A}_i)_{i<\kappa}$ be a sequence of τ -structures for a relational signature τ . Then $(\mathfrak{A}_i)_{i<\kappa}$ is called a *chain* if $\mathfrak{A}_i\subseteq\mathfrak{A}_j$ for all $i< j<\kappa$. A chain is called an *elementary chain* when for all $i,j<\kappa$, the extension \mathfrak{A}_j of \mathfrak{A}_i is elementary.

DEFINITION 2.5.1. The union of the chain $(\mathfrak{A}_i)_{i<\gamma}$ is a τ -structure \mathfrak{B} defined as follows. The domain of \mathfrak{B} is $\bigcup_{i\leq\gamma}A_i$, and for each relation symbol $R\in\tau$ we put $\bar{a}\in R^{\mathfrak{B}}$ if $\bar{a}\in R^{\mathfrak{A}_i}$ for some (or all) \mathfrak{A}_i containing \bar{a} .

THEOREM 2.5.2 (Tarski-Vaught; Theorem 2.5.2 in [120]). Let $(\mathfrak{A}_i)_{i<\kappa}$ be an elementary chain. Then $\bigcup_{i<\kappa} \mathfrak{A}_i$ is an elementary extension of \mathfrak{A}_i for each $i<\kappa$.

We say that a formula ϕ is preserved in chains (of models of T) if for all chains $(\mathfrak{A}_i)_{0 \leq i < \kappa}$ of τ -structures (where all the \mathfrak{A}_i and $\mathfrak{A} := \bigcup_{i < \kappa} \mathfrak{A}_i$ are models of T), and every $\bar{a} \in A^n$ we have $\mathfrak{A} \models \phi(\bar{a})$ whenever $\mathfrak{A}_i \models \phi(\bar{a})$ for every $i < \kappa$.

Proposition 2.5.3 (Theorem 2.4.4 in [120]). Every $\forall \exists$ -formula is preserved in unions of chains.

Direct limits can be seen as a positive variant of the notion of a union of chains; we essentially replace embeddings in chains by homomorphisms. Let τ be a relational signature, and let $\mathfrak{A}_0, \mathfrak{A}_1, \ldots$ be a sequence of τ -structures such that there are homomorphisms $f_{ij} \colon \mathfrak{A}_i \to \mathfrak{A}_j$. Those homomorphisms are called *coherent* if $f_{jk} \circ f_{ij} = f_{ik}$ for every $i \leq j \leq k$.

DEFINITION 2.5.4. Let $\mathfrak{A}_0, \mathfrak{A}_1, \ldots$ be a sequence of τ -structures with coherent homomorphisms $f_{ij} \colon \mathfrak{A}_i \to \mathfrak{A}_j$. Then the direct limit $\lim_{i < \omega} \mathfrak{A}_i$ is the τ -structure \mathfrak{A} defined as follows. The domain A of \mathfrak{A} comprises the equivalence classes of the equivalence relation \sim defined on $\bigcup_{i < \omega} A_i$ by setting $x_i \sim x_j$ for $x_i \in A_i, x_j \in A_j$ iff there is a k such that $f_{ik}(x_i) = f_{jk}(x_j)$. Let $g_i \colon A_i \to A$ be the function that maps $a \in A_i$ to the equivalence class of a in A. For $R \in \tau$ and a tuple \bar{a} over A, define $\mathfrak{A} \models R(\bar{a})$ iff there is a k and a tuple \bar{b} over A_k such that $\mathfrak{A}_k \models R(\bar{b})$ and $\bar{a} = g_k(\bar{b})$.

Note that the definition of $\lim_{i < \omega} \mathfrak{A}_i$ also depends on the coherent family f_{ij} , but this is left implicit and will be clear from the context. Also note that g_i defines a homomorphism from \mathfrak{A}_i to \mathfrak{A} ; this function is called the *limit homomorphism* from \mathfrak{A}_i to the direct limit \mathfrak{A} . Note that $g_i = g_j \circ f_{ij}$ for all $i < j < \omega$.

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We have seen that unions of chains preserve $\forall \exists$ -formulas; the analogous statement for direct limits is as follows. We say that a first-order formula $\phi(x_1,\ldots,x_n)$ is preserved in direct limits (of models of T) if for all sequences $(\mathfrak{A}_i)_{0\leq i<\kappa}$ (where all the \mathfrak{A}_i and $\mathfrak{A}:=\lim_{i<\kappa}\mathfrak{A}_i$ are models of T), and every $\bar{a}\in A^n$ we have $\overline{\mathfrak{A}}\models\phi(\bar{a})$ whenever there is for every $i<\kappa$ an n-tuples \bar{a}^i where the j-th entry is a representative of the j-th entry of \bar{a} , and $\mathfrak{A}_i\models\phi(\bar{a}^i)$. The following is Theorem 2.4.6 in [119]; since it is given there without proof, we present it here for completeness.

PROPOSITION 2.5.5 (Theorem 2.4.6 in [119]). Every $\forall \exists^+$ -formula is preserved in direct limits of models of T.

PROOF. Let $(\mathfrak{A}_i)_{i<\kappa}$ be a sequence of models of T with coherent homomorphisms $h_{ij}\colon \mathfrak{A}_i\to \mathfrak{A}_j$, for $i,j<\kappa$, such that $\mathfrak{A}:=\lim_{i<\kappa}\mathfrak{A}_i$ is a model of T. Let g_i be the limit homomorphism from \mathfrak{A}_i to \mathfrak{A} . We have to show that if $\bar{a}\in A^n$ is such that for all $i<\kappa$ there is \bar{a}_i with $g_i(\bar{a}_i)=\bar{a}$ and $\mathfrak{A}_i\models\phi(\bar{a}_i)$, then $\mathfrak{A}\models\phi(\bar{a})$. Since ϕ is $\forall \exists^+$, we can assume that $\phi(\bar{x})$ is of the form $\forall \bar{y}.\phi'(\bar{x},\bar{y})$ where ϕ' is a disjunction of negated atomic formulas and existential positive formulas. Suppose that $\phi'(\bar{a},\bar{b})$ is false in \mathfrak{A} for some tuple \bar{b} of elements of \mathfrak{A} . Every disjunct ψ of $\phi'(\bar{a},\bar{b})$ is false in \mathfrak{A} . Then there exists an $i<\kappa$ such that all entries of \bar{b} have representatives in \mathfrak{A}_i , and the negated atomic disjuncts of ϕ' are already false in \mathfrak{A}_i , by definition of direct limits. Let \bar{b}_i be a tuple of elements of \mathfrak{A}_i where the j-th entry is a representative of the j-th entry in \bar{b} . Since $\mathfrak{A}_i \models \phi(\bar{a}_i)$, there must exist a disjunct ψ of ϕ' such that $\psi(\bar{a}_i,\bar{b}_i)$ holds in \mathfrak{A}_i . The limit homomorphism g_i maps (\bar{a}_i,\bar{b}_i) to (\bar{a},\bar{b}) and is a homomorphism from \mathfrak{A}_i to \mathfrak{A} , and therefore preserves existential positive formulas, contradicting the assumption that $\phi'(\bar{a},\bar{b})$ is false in \mathfrak{A} .

2.6. Types

A set Φ of formulas with free variables x_1, \ldots, x_n is called *satisfiable* over a structure \mathfrak{B} if there are elements b_1, \ldots, b_n of \mathfrak{B} such that for all sentences $\phi \in \Phi$ we have $\mathfrak{B} \models \phi(b_1, \ldots, b_n)$. We say that Φ is *satisfiable* if there exists a structure \mathfrak{B} such that Φ is satisfiable over \mathfrak{B} . For $n \geq 0$, an n-type of a theory T is a set p of formulas with free variables x_1, \ldots, x_n such that $p \cup T$ is satisfiable. An n-type p of T is maximal if $T \cup p \cup \{\phi(x_1, \ldots, x_n)\}$ is unsatisfiable for any formula $\phi \notin p$. The set of all complete n-types of $Th(\mathfrak{A})$ is denoted by $S_n^{\mathfrak{A}}$. An n-type of a structure \mathfrak{B} is an n-type of the first-order theory of \mathfrak{B} .

In a similar manner, an existential positive n-type (ep-n-type) of a theory T is a set of existential positive formulas p with free variables x_1, \ldots, x_n such that $p \cup T$ is satisfiable. A ep-n-type p of T is maximal if $T \cup p \cup \{\phi(x_1, \ldots, x_n)\}$ is unsatisfiable for any existential positive formula $\phi \notin p$. A (ep-) n-type of a structure $\mathfrak A$ is a (ep-) n-type of the theory $\mathrm{Th}(\mathfrak A)$. When S is the universal negative theory of $\mathfrak A$, then note that $p \cup \mathrm{Th}(\mathfrak A)$ is satisfiable if and only if $p \cup S$ is satisfiable; thus we could equivalently have defined ep-n-type with respect to the latter theory.

When p is an n-type, and $I \subseteq \{1, \ldots, n\}$ with |I| = k, then the *subtype of p induced by I* is the k-type obtained from p by existentially quantifying in all formulas in p the variables x_i for $i \in \{1, \ldots, n\} \setminus I$, and then renaming the variables in the resulting set of formulas to x_1, \ldots, x_k in a way that preserves the order on the indices of the variables.

An *n*-type p of \mathfrak{A} is realized in \mathfrak{A} if there exist $a_1, \ldots, a_n \in A$ such that $\mathfrak{A} \models \phi(a_1, \ldots, a_n)$ for each $\phi \in p$. The set of all first-order formulas with free variables x_1, \ldots, x_n satisfied by an *n*-tuple $\bar{a} = (a_1, \ldots, a_n)$ in \mathfrak{A} is a maximal type of \mathfrak{A} , and called the type of \bar{a} , and denoted by $\operatorname{tp}^{\mathfrak{A}}(\bar{a})$.

For an infinite cardinal κ , a structure $\mathfrak A$ is κ -saturated if, for all $\beta < \kappa$ and expansions $\mathfrak A'$ of $\mathfrak A$ by at most β constants, every 1-type of $\mathfrak A'$ is realized in $\mathfrak A'$. We say that an infinite $\mathfrak A$ is saturated when it is |A|-saturated. Realization of pp-types and pp- (κ) -saturation are defined analogously.

THEOREM 2.6.1 (Corollary 8.2.2 in [120]). Let τ be a signature and $\lambda \geq |\tau|$. Then every τ -structure \mathfrak{B} has an λ^+ -saturated elementary extension of cardinality $\leq |B|^{\lambda}$.

CHAPTER 3

Model Theory



Hodges [120] writes that "model theory is about the classification of mathematical structures, maps and sets by means of logical formulas". This text is about the computational complexity of constraint satisfaction problems for infinite structures $\mathfrak B$ —and since the constraint satisfaction problem of $\mathfrak B$ (and in particular its complexity) is fully determined by the first-order theory of $\mathfrak B$, it is not surprising that model theory has a great deal to say about constraint satisfaction problems.

Many important infinite-domain constraint satisfaction problems can be formulated with templates that are ω -categorical. The concept of ω -categoricity is of central importance in model theory, and for reasons that will become clear in Section 3.1 of this chapter, also in permutation group theory. From a model-theoretic perspective, ω -categoricity is a very strong assumption – but still many problems that have been studied in the literature, in particular constraint satisfaction problems for qualitative reasoning formalisms in artificial intelligence¹, can be formulated as CSPs with ω -categorical templates. We will also see that every connected monotone monadic SNP sentence (the corresponding problems have been called forbidden patterns problems and studied in [154–156]) describes a constraint satisfaction problem of an ω -categorical structure (Section 4.5).

In this section we present general results about ω -categorical structures: for example how to construct them (Sections 3.1 and 3.2, and 3.5), and how to algebraically

¹The question which reasoning formalisms in Artificial Intelligence should be called *qualitative* has been the topic of scientific discussion [150]. My own response to this question is: it is qualitative if and only if it can be formulated with an ω -categorical template.

characterize syntactically restricted definability of relations over ω -categorical structures (Sections 3.3, 3.4, and 3.6). Section 3.5.2 gives an exact characterization of those constraint satisfaction problems that can be formulated with ω -categorical templates.

There is already an excellent literature on ω -categoricty: most notably, the book by Cameron [70], the recent survey by Macpherson [153], and the collection [131]. Moreover, classical text-books on model theory, such as [120,133,158], always treat ω -categority, and use ω -categorical structures as a rich source of examples. The present chapter is different from those in that it focusses on techniques and facts that will be relevant for complexity classification for the corresponding constraint satisfaction problems. It contains many results that are not contained in any of the sources mentioned above (and which have been published in [35, 39–41, 47, 51, 52, 114]). Some parts can be derived from proofs in Hodges' book [120] and its original longer version [119]; our citation policy is to give the reference to the shorter version, whenever this is possible, since it is more widely accessible. When this is not possible, we quote [119]; for what is relevant in this text, the long version subsumes the short version.

3.1. ω -categorical Structures

"Every statement about all ω -categorical structures is either trivial, or false." (Martin Ziegler, 2005)

A satisfiable first-order theory T is called ω -categorical if all countable models of T are isomorphic. A structure is called ω -categorical if its first-order theory is ω -categorical. Since almost all ω -categorical structures that appear in this text will be countably infinite, we make the convention that ω -categorical structures are countably infinite. One of the first structures that were found to be ω -categorical (by Cantor [71]) is the linear order of the rational numbers (\mathbb{Q} ; <), which we will use as a running example in this section. We will see many more examples of ω -categorical structures in this section, in Section 3.2, and in Chapter 4. One of the standard approaches to verify that a structure is ω -categorical is via a so-called back-and-forth argument. To illustrate, we give the back-and-forth argument that shows that (\mathbb{Q} ; <) is ω -categorical; much more about this important concept in model theory can be found in [120, 175].

Proposition 3.1.1. The structure $(\mathbb{Q}; <)$ is ω -categorical.

PROOF. Let \mathfrak{A} be a countable model of the first-order theory T of $(\mathbb{Q};<)$. It is easy to verify that T contains (and, as this argument will show, is uniquely given by)

- $\exists x. \ x = x \text{ (no empty model)}$
- $\forall x, y, z \ ((x < y \land y < z) \Rightarrow x < z) \ (transitivity)$
- $\forall x, y. \ \neg(x < x) \ (irreflexivity)$
- $\forall x, y \ (x < y \lor y < x \lor x = y) \ (totality)$
- $\forall x \,\exists y. \, x < y \text{ (no largest element)}$
- $\forall x \, \exists y. \, y < x \, (\text{no smallest element})$
- $\forall x, z \exists y \ (x < y \land y < z) \ (density).$

An isomorphism between \mathfrak{A} and $(\mathbb{Q}; <)$ can be defined inductively as follows. Suppose that we have already defined f on a finite subset S of \mathbb{Q} and that f is an embedding of the structure induced by S in $(\mathbb{Q}; <)$ into \mathfrak{A} . Since $<^{\mathfrak{A}}$ is dense and unbounded, we can extend f to any other element of \mathbb{Q} such that the extension is still an embedding from a substructure of \mathbb{Q} into \mathfrak{A} (going forth). Symmetrically, for every element v of \mathfrak{A} we can find an element $u \in \mathbb{Q}$ such that the extension of f that

maps u to v is also an embedding (going back). We now alternate between going forth and going back; when going forth, we extend the domain of f by the next element of \mathbb{Q} , according to some fixed enumeration of the elements in \mathbb{Q} . When going back, we extend f such that the image of A contains the next element of \mathfrak{A} , according to some fixed enumeration of the elements of \mathfrak{A} . If we continue in this way, we have defined the value of f on all elements of \mathbb{Q} . Moreover, f will be surjective, and an embedding, and hence an isomorphism between \mathfrak{A} and $(\mathbb{Q};<)$.

A second important running example of this section is the random graph $(\mathbb{V}; E)$, which is a (simple and undirected) graph with a countably infinite set of vertices \mathbb{V} that is defined uniquely up to isomorphism by the following extension property: for all finite disjoint subsets U, U' of \mathbb{V} there exists a vertex $v \in \mathbb{V} \setminus (U \cup U')$ such that v is adjacent to all vertices in U and to no vertex in U'. We will see in Section 3.2 that such a graph indeed exists (Theorem 3.2.2).

Proposition 3.1.2. The random graph (V; E) is ω -categorical.

PROOF. Note that the defining property of (V; E) given above is a first-order property; a simple back-and-forth argument shows that every countably infinite graph with this property is isomorphic to (V; E).

The theorem of Ryll-Nardzewski. There are many equivalent characterizations of ω -categoricity; the most important one is in terms of the automorphism group of \mathfrak{B} . In the following, let \mathscr{G} be a set of permutations of a set X. We say that \mathscr{G} is a permutation group if \mathscr{G} contains the identity id_X , and for $g, f \in \mathscr{G}$ the functions $x \mapsto g(f(x))$ and $x \mapsto g^{-1}(x)$ are also in \mathscr{G} . For $n \geq 1$ the orbit of $(t_1, \ldots, t_n) \in X^n$ under \mathscr{G} is the set $\{(\alpha(t_1), \ldots, \alpha(t_n)) \mid \alpha \in \mathscr{G}\}$.

Definition 3.1.3. A permutation group \mathscr{G} over a countably infinite set X is oligomorphic if \mathscr{G} has only finitely many orbits of n-tuples for each $n \geq 1$.

An accessible proof of the following theorem can be found in Hodges' book (Theorem 6.3.1 in [120]).

Theorem 3.1.4 (Engeler, Ryll-Nardzewski, Svenonius). For a countably infinite structure \mathfrak{B} with countable signature, the following are equivalent:

- (1) \mathfrak{B} is ω -categorical;
- (2) the automorphism group $\operatorname{Aut}(\mathfrak{B})$ of \mathfrak{B} is oligomorphic;
- (3) for each $n \ge 1$, there are finitely many inequivalent formulas with free variables x_1, \ldots, x_n over \mathfrak{B} ;
- (4) for all $n \ge 1$, every set of n-tuples that is preserved by all automorphisms of \mathfrak{B} is first-order definable in \mathfrak{B} .

The fourth of those conditions is missing in Theorem 6.3.1 of [120]; but the implication from (3) to (4) follows from the proof given there. Conversely, suppose that $Aut(\mathfrak{B})$ are infinitely many orbits of n-tuples, for some n. Then the union of any subset of the set of all orbits of n-tuples is preserved by all automorphisms of \mathfrak{B} ; but there are only countably many first-order formulas over a countable language, so not all the invariant sets of n-tuples can be first-order definable in \mathfrak{B} .

The second condition in Theorem 3.1.4 provides another possibility to verify that a structure is ω -categorical. We again illustrate this with the structure $(\mathbb{Q}; <)$. It is not difficult but a good exercise to verify that the orbit of an n-tuple (t_1, \ldots, t_n) from \mathbb{Q}^n with respect to the automorphism group of $(\mathbb{Q}; <)$ is determined by the weak linear order induced by (t_1, \ldots, t_n) in $(\mathbb{Q}; <)$. We write weak linear order, and not linear order, because some of the elements t_1, \ldots, t_n might be equal. That is, a weak

linear order is a total quasiorder. There are less than n^n such orders, and hence the automorphism group of $(\mathbb{Q}; <)$ has a finite number of orbits of n-tuples, for all $n \ge 1$.

Lemma 3.1.5 below states a useful property that ω -categorical structures have in common with finite structures, and is an easy consequence of Königs tree lemma.

LEMMA 3.1.5. Let \mathfrak{B} be a finite or an infinite ω -categorical structure with relational signature τ , and let \mathfrak{A} be a countable τ -structure. If there is no homomorphism (embedding) from \mathfrak{A} to \mathfrak{B} , then there is a finite substructure of \mathfrak{A} that does not homomorphically map (embed) to \mathfrak{B} .

Proof. We present the proof for homomorphisms; the proof for embeddings is analogous. Suppose every finite substructure of \mathfrak{A} homomorphically maps to \mathfrak{B} . We show the contraposition of the lemma, and prove the existence of a homomorphism from \mathfrak{A} to \mathfrak{B} . Let a_1, a_2, \ldots be an enumeration of \mathfrak{A} . We construct a rooted tree with finite out-degree, where each node lies on some level $n \geq 0$. The nodes on level n are equivalence classes of homomorphisms from the substructure of $\mathfrak A$ induced by a_1, \ldots, a_n to \mathfrak{B} . Hence, there is only one vertex on level 0, which will be the root of the tree. Two such homomorphisms f and g are equivalent if there is an automorphism α of \mathfrak{B} such that $\alpha f = g$. Two equivalence classes of homomorphisms on level n and n+1 are adjacent, if there are representatives of the classes such that one is a restriction of the other. Theorem 3.1.4 asserts that $\mathfrak A$ has only finitely many orbits of k-tuples, for all $k \geq 0$ (clearly, this also holds if \mathfrak{B} is finite). Hence, the constructed tree has finite out-degree. By assumption, there is a homomorphism from the structure induced by a_1, a_2, \ldots, a_n to \mathfrak{B} for all $n \geq 0$, and hence the tree has vertices on all levels. König's lemma asserts the existence of an infinite path in the tree, which can be used to inductively to define a homomorphism h from $\mathfrak A$ to $\mathfrak B$

The restriction of h to $\{a_1, \ldots, a_n\}$ will be an element from the n-th node of the infinite path. Initially, this is trivially true if h is restricted to the empty set. Suppose h is already defined on a_1, \ldots, a_n , for $n \geq 0$. By construction of the infinite path, we find representatives h_n and h_{n+1} of the n-th and the (n+1)-st element on the path such that h_n is a restriction of h_{n+1} . The inductive assumption gives us an automorphism α of $\mathfrak A$ such that $\alpha(h_n(x)) = h(x)$ for all $x \in \{a_1, \ldots, a_n\}$. We set $h(a_{n+1})$ to be $\alpha(h_{n+1}(a_{n+1}))$. The restriction of h to a_1, \ldots, a_{n+1} will therefore be a member of the (n+1)-st element of the infinite path. The operation h defined in this way is indeed a homomorphism from $\mathfrak A$ to $\mathfrak B$.

The assumption that \mathfrak{A} is countable is necessary in Lemma 3.1.5; consider for example $\mathfrak{A} := (\mathbb{R}; <)$, which does not admit a homomorphism to $\mathfrak{B} := (\mathbb{Q}; <)$ for cardinality reasons, even though any finite substructure of \mathfrak{A} does.

COROLLARY 3.1.6. For any structure \mathfrak{C} , there is a finite structure \mathfrak{B} with the same CSP as \mathfrak{C} if and only if \mathfrak{C} has a finite core.

PROOF. If there exists a finite structure \mathfrak{B} with the same CSP as \mathfrak{C} , then every finite substructure of \mathfrak{C} homomorphically maps to the core \mathfrak{B}' of \mathfrak{B} , and by Lemma 3.1.5 there exists a homomorphism from \mathfrak{C} to the finite core structure \mathfrak{B}' (which is unique up to isomorphism); since \mathfrak{B}' also maps to \mathfrak{C} , it is a core of \mathfrak{C} . The converse is trivial.

COROLLARY 3.1.7. Two countable ω -categorical relational τ -structures $\mathfrak A$ and $\mathfrak B$ have the same CSP if and only if there is a homomorphism from $\mathfrak A$ to $\mathfrak B$ and a homomorphism from $\mathfrak B$ to $\mathfrak A$.

Corollary 3.1.7 is false for general countable relational structures. Consider for example the structure $(\mathbb{Z}; \{(x,y) \mid y=x+1\})$ — the 'infinite line', and the structure $(\mathbb{N}; \{(x,y) \mid y=x+1\})$ — the 'infinite ray'. Clearly, these two structures give rise to the same CSP, but there is no homomorphism from the line to the ray.

Several times we need variants of Lemma 3.1.5 that can be proved in the same way. For instance, we can replace homomorphism in the statement and the proof by strong homomorphism, or injective homomorphism, or mappings satisfying universal identities such as $\forall x, y$. f(x, y) = f(y, x). What is common for all those statements is that the respective property of the function can be expressed by universal first-order sentences. We make this more precise and derive the following generalization of Lemma 3.1.5 based on the compactness theorem.

LEMMA 3.1.8. Let \mathfrak{B} be a countable ω -categorical or finite structure with countable relational signature τ , let \mathfrak{A} be a countably infinite τ -structure, and let σ be a countable set of function symbols. Then for any universal $(\tau \cup \sigma)$ -theory T the following are equivalent.

- (1) The two-sorted structure $(\mathfrak{A},\mathfrak{B})$ has a $(\tau \cup \sigma)$ -expansion that satisfies T such that every $f \in \sigma$ denotes a function from \mathfrak{A} to \mathfrak{B} .
- (2) For every finite induced substructure \mathfrak{C} of \mathfrak{A} the two-sorted structure $(\mathfrak{C}, \mathfrak{B})$ has a $(\tau \cup \sigma)$ -expansion that satisfies T such that every $f \in \sigma$ denotes a function from \mathfrak{C} to \mathfrak{B} .

PROOF. Any substructure of a model of a universal theory is again a model of the theory, so (1) implies (2). Conversely, we prove the existence of a homomorphism from $\mathfrak A$ to $\mathfrak B$ by a compactness argument as follows. Let P be a unary relation symbol not contained in τ . Let $\mathfrak A'$ be an expansion of $\mathfrak A$ by countably many constants such that every element of $\mathfrak A$ is named in $\mathfrak A'$ by a constant symbol; let τ' be the (countable) signature of $\mathfrak A'$. Let D be the diagram of $\mathfrak A'$, and let S be a set of universal first-order sentences that

- forces that all function symbols from σ denote functions from the elements in P to the elements that are not in P, and
- expresses that the τ -reduct of the structure induced by the elements not in P has the same first-order theory as \mathfrak{B} .

We first prove that $D \cup S \cup T$ is satisfiable. By compactness, it suffices to prove satisfiability of $D' \cup S \cup T$ for all finite subsets D' of D. Let c_1, \ldots, c_n be the constant symbols mentioned in D'. Let \mathfrak{C}' be the structure induced by $\{c_1, \ldots, c_n\}$ in \mathfrak{A}' . Clearly, $\mathfrak{C}' \models D'$. Let \mathfrak{C} be the τ -reduct of \mathfrak{C}' . By assumption, the two-sorted structure $(\mathfrak{C}, \mathfrak{B})$ can be expanded to a two-sorted $(\tau \cup \sigma)$ -structure \mathfrak{D} that satisfies T; this structure also satisfies S. When we additionally denote the constants c_1, \ldots, c_n as in \mathfrak{A}' , then the expansion satisfies also D', and so we have found a model of $D' \cup S \cup T$.

By compactness, there exists an (infinite) model of $D \cup S \cup T$, and by Theorem 2.3.3 and since $\tau' \cup \sigma$ is countable there is also a countably infinite model \mathfrak{M} of $D \cup S \cup T$. Consider the substructure of \mathfrak{M} generated by the constants from τ' , and all the elements in P_B . It can be checked that the resulting structure \mathfrak{M}' still satisfies D and S, and since universal sentences are preserved by taking substructures, \mathfrak{M}' also satisfies T. Note that in \mathfrak{M}' , the elements from P induce a copy of \mathfrak{A} , and the complement induces a structure that is isomorphic to \mathfrak{B} , since \mathfrak{B} is finite or ω -categorical. So the functions of \mathfrak{M}' denoted by the function symbols from σ provide the required $(\tau \cup \sigma)$ -expansion of $(\mathfrak{A}, \mathfrak{B})$.

Lemma 3.1.8 is indeed a generalization of Lemma 3.1.5: to make sure that f is a homomorphism, T contains for every relation symbol $R \in \tau$ the sentence $\forall \bar{x}.(R(\bar{x}) \Rightarrow R(f(\bar{x}))$.

First-Order Interpretations. Many ω -categorical structures can be derived from other ω -categorical structures via first-order interpretations (our definition follows [120]).

If $\delta(x_1, \ldots, x_k)$ is a first-order τ -formula with k free variables x_1, \ldots, x_k , and \mathfrak{A} is a τ -structure, we write $\delta(\mathfrak{A}^k)$ for the k-ary relation that is defined by δ on \mathfrak{A} .

DEFINITION 3.1.9. A relational σ -structure \mathfrak{B} has a (first-order) interpretation I in a τ -structure \mathfrak{A} if there exists a natural number d, called the dimension of I, and

- $a \tau$ -formula $\delta_I(x_1, \ldots, x_d)$ called domain formula,
- for each atomic σ -formula $\phi(y_1, \ldots, y_k)$ a τ -formula $\phi_I(\overline{x}_1, \ldots, \overline{x}_k)$ where the \overline{x}_i denote disjoint d-tuples of distinct variables called the defining formulas,
- a surjective map $h: \delta_I(\mathfrak{A}^d) \to B$ called coordinate map,

such that for all atomic σ -formulas ϕ and all tuples $\overline{a}_i \in \delta_I(\mathfrak{A}^d)$

$$\mathfrak{B} \models \phi(h(\overline{a}_1), \dots, h(\overline{a}_k)) \Leftrightarrow \mathfrak{A} \models \phi_I(\overline{a}_1, \dots, \overline{a}_k)$$
.

If the formulas δ_I and ϕ_I are all primitive positive, we say that the interpretation I is primitive positive. Note that the dimension d, the set $S := \delta(\mathfrak{A}^k)$, and the coordinate map h determine the defining formulas up to logical equivalence; hence, we sometimes denote an interpretation by I = (d, S, h). Note that the kernel of h coincides with the relation defined by $(x = y)_I$, for which we also write $=_I$, the defining formula for equality.

We say that \mathfrak{B} is interpretable in \mathfrak{A} with finitely many parameters if there are $c_1, \ldots, c_n \in A$ such that \mathfrak{B} is interpretable in the expansion of \mathfrak{A} by the singleton relations $\{c_i\}$ for all $1 \leq i \leq n$. A first-order definition of one structure in another is in model theory often the special case of an interpretation I where $=_I$ is simply the equality relation; in this text, however, we say that a structure \mathfrak{B} is (first-order) definable in \mathfrak{A} if \mathfrak{B} has a 1-dimensional interpretation I in \mathfrak{B} where $=_I$ is the equality relation and the domain formula δ_I is logically equivalent to true.

LEMMA 3.1.10 (see Theorem 7.3.8 in [119]). Let \mathfrak{A} be an ω -categorical structure. Then every structure \mathfrak{B} that is first-order interpretable in \mathfrak{A} with finitely many parameters is ω -categorical or finite.

Note that in particular all reducts of an ω -categorical structure and all expansions of an ω -categorical structure by finitely many constants are again ω -categorical.

EXAMPLE 3.1.11. In Section 1.5 we have described Allen's Interval Algebra for temporal reasoning in Artificial Intelligence [5], and the corresponding CSP. Formally, it is easiest to describe the template $\mathfrak A$ for this CSP by a first-order interpretation I in $(\mathbb Q;<)$. The dimension of the interpretation is two, and the domain formula $\delta_I(x,y)$ is x < y. Hence, the elements of $\mathfrak A$ can indeed be viewed as non-empty closed intervals [x,y] over $\mathbb Q$. The template $\mathfrak A$ contains for each inequivalent $\{<\}$ -formula ϕ with four variables a binary relation R such that (a_1,a_2,a_3,a_4) satisfies ϕ if and only if $((a_1,a_2),(a_3,a_4)) \in R$. In particular, $\mathfrak A$ has relations for equality of intervals, containment of intervals, and so forth. By Lemma 3.1.10, $\mathfrak A$ is ω -categorical. \square

3.2. Fraïssé Amalgamation

A versatile tool to construct ω -categorical structures is Fraïssé-amalgamation. We present it here for the special case of relational structures; this is all that is needed in

the examples we are going to present. For a stronger version of Fraïssé-amalgamation for classes of structures that might involve function symbols, see [120].

In the following, let τ be a countable relational signature. The age of a τ -structure \mathfrak{A} is the class of all finite τ -structures that embed into \mathfrak{A} . Let $\mathfrak{B}_1, \mathfrak{B}_2$ be τ -structures such that \mathfrak{A} is an induced substructure of both \mathfrak{B}_1 and \mathfrak{B}_2 and all common elements of \mathfrak{B}_1 and \mathfrak{B}_2 are elements of \mathfrak{A} ; note that in this case $\mathfrak{A} = \mathfrak{B}_1 \cap \mathfrak{B}_2$. Then we call $\mathfrak{B}_1 \cup \mathfrak{B}_2$ the free amalgam of $\mathfrak{B}_1, \mathfrak{B}_2$ over \mathfrak{A} . More generally, a τ -structure \mathfrak{C} is an amalgam of \mathfrak{B}_1 and \mathfrak{B}_2 over \mathfrak{A} if for i = 1, 2 there are embeddings f_i of \mathfrak{B}_i to \mathfrak{C} such that $f_1(a) = f_2(a)$ for all $a \in \mathfrak{A}$. A strong amalgam of $\mathfrak{B}_1, \mathfrak{B}_2$ over \mathfrak{A} is an amalgam of $\mathfrak{B}_1, \mathfrak{B}_2$ over \mathfrak{A} where $f_1(B_1) \cap f_2(B_2) = f_1(A) (= f_2(A))$.

DEFINITION 3.2.1. An isomorphism-closed class C of τ -structures has the amalgamation property if for all $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2 \in C$ with $\mathfrak{A} = \mathfrak{B}_1 \cap \mathfrak{B}_2$ there is a $\mathfrak{C} \in C$ that is an amalgam of \mathfrak{B}_1 and \mathfrak{B}_2 over A. A class of finite τ -structures that contains at most countably many non-isomorphic structures, has the amalgamation property, and is closed under taking induced substructures and isomorphisms is called an amalgamation class.

Analogously, an isomorphism-closed class \mathcal{C} of τ -structures has the *free amalga-mation property* if for all $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{C}$ with $\mathfrak{A} = \mathfrak{B}_1 \cap \mathfrak{B}_2$ the free amalgam of $\mathfrak{B}_1, \mathfrak{B}_2$ over \mathfrak{A} is in \mathcal{C} . The *strong* amalgamation property is defined analogously. Note that since we only look at relational structures here (and since we allow structures to have an empty domain), the amalgamation property of \mathcal{C} implies the *joint embedding property (JEP)* for \mathcal{C} , which says that for any two structures $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{C}$ there exists a structure $\mathfrak{C} \in \mathcal{C}$ that embeds both \mathfrak{B}_1 and \mathfrak{B}_2 .

A structure \mathfrak{A} is homogeneous (sometimes also called *ultra-homogeneous* [120]) if every isomorphism between finitely generated substructures of \mathfrak{A} can be extended to an automorphism of \mathfrak{A} .

Theorem 3.2.2 (Fraïssé [97,98]; see [120]). Let τ be a countable relational signature and let \mathcal{C} be an amalgamation class of τ -structures. Then there is a homogeneous and at most countable τ -structure \mathfrak{C} whose age equals \mathcal{C} . The structure \mathfrak{C} is unique up to isomorphism, and called the Fraïssé-limit of \mathcal{C} .

When \mathcal{C} is a strong amalgamation class, then the Fraïssé-limit of \mathcal{C} has a remarkable property. Let Γ be a structure, and A a finite set of elements of Γ . Then $\operatorname{acl}_{\Gamma}(A)$ denotes the model-theoretic algebraic closure of A in Γ , i.e., the elements of Γ that lie in finite sets that are first-order definable over Γ with parameters from A. In ω -categorical structures, this is precisely the set of elements of Γ that lie in finite orbits in $\operatorname{Aut}(\Gamma)_{(A)}$. We say that a structure Γ has no algebraicity if $\operatorname{acl}_{\Gamma}(A) = A$ for all finite sets of parameters A.

THEOREM 3.2.3 (See (2.15) in [70]). A homogeneous ω -categorical structure Γ has no algebraicity if and only if the age of Γ has strong amalgamation.

EXAMPLE 3.2.4. Let \mathcal{C} be the class of all linear orders. Then \mathcal{C} is clearly closed under isomorphisms and induced substructures, and has countably many isomorphism types. To show that it also has the amalgamation property, let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{C}$, and let \mathfrak{A} be an induced substructure of both \mathfrak{B}_1 and \mathfrak{B}_2 . Let \mathfrak{C} be the free amalgam of \mathfrak{B}_1 and \mathfrak{B}_2 over \mathfrak{A} . Then \mathfrak{C} is an acyclic finite graph; therefore, any depth-first traversal of \mathfrak{C} leads to a linear ordering of the elements that is an amalgam (but not a free amalgam) in \mathcal{C} of \mathfrak{B}_1 and \mathfrak{B}_2 over \mathfrak{A} . It follows that \mathcal{C} is an amalgamation class. By homogeneity, the Fraïssé-limit of \mathcal{C} is unbounded and dense, and hence isomorphic to $(\mathbb{Q}; <)$ by Proposition 3.1.1.

EXAMPLE 3.2.5. Let \mathcal{C} be the class of all finite partially ordered sets. Amalgamation can be shown by computing the transitive closure: when \mathfrak{C} is the free amalgam of \mathfrak{B}_1 and \mathfrak{B}_2 over \mathfrak{A} , then the transitive closure of \mathfrak{C} gives an amalgam in \mathcal{C} . The Fraïssé-limit of \mathcal{C} is called the *homogeneous universal partial order*.

EXAMPLE 3.2.6. Let \mathcal{C} be the class of all finite graphs. It is even easier than in the previous examples to verify that \mathcal{C} is an amalgamation class, since here the free amalgam itself shows the amalgamation property. We can use homogeneity to verify that the Fraïssé-limit of \mathcal{C} satisfies the defining property of the random graph $(\mathbb{V}; E)$ (the existence of the random graph was left open in Section 3.1).

EXAMPLE 3.2.7. Henson [116] used Fraïssé limits to construct 2^{ω} many ω -categorical directed graphs. A tournament is a directed graph without self-loops such that for all pairs x, y of distinct vertices exactly one of the pairs (x, y), (y, x) is an arc in the graph. Note that for all classes \mathcal{N} of finite tournaments, $\operatorname{Forb}(\mathcal{N})$ is an amalgamation class, because if \mathfrak{A}_1 and \mathfrak{A}_2 are directed graphs in $\operatorname{Forb}(\mathcal{N})$ such that $\mathfrak{A} = \mathfrak{A}_1 \cap \mathfrak{A}_2$ is an induced substructure of both \mathfrak{A}_1 and \mathfrak{A}_2 , then the free amalgam $\mathfrak{A}_1 \cup \mathfrak{A}_2$ is also in $\operatorname{Forb}(\mathcal{N})$.

Henson in his proof specified an infinite set \mathcal{T} of tournaments $\mathfrak{T}_1, \mathfrak{T}_2, \ldots$ with the property that \mathfrak{T}_i does not embed into \mathfrak{T}_j if $i \neq j$; the set \mathcal{T} will be described in Section 11.3. Note that this property implies that for two distinct subsets \mathcal{N}_1 and \mathcal{N}_2 of \mathcal{T} the two sets $\text{Forb}(\mathcal{N}_1)$ and $\text{Forb}(\mathcal{N}_2)$ are distinct as well. Since there are 2^{ω} many subsets of the infinite set \mathcal{T} , there are also that many distinct homogeneous (and therefore ω -categorical) directed graphs; they are often referred to as Henson digraphs.

The structures from Example 3.2.7 can be used to prove various negative results about homogeneous structures with finite signature, for instance in Section 8.4 and in Section 11.3. A better behaved class of structures are homogeneous structures whose age is *finitely bounded* (this is the same terminology as in [153]).

DEFINITION 3.2.8. We say that a class C of finite relational τ -structures (or a structure with age C) is finitely bounded if τ is finite and there exists a finite set of finite τ -structures N such that C = Forb(N).

PROPOSITION 3.2.9. When \mathfrak{B} is finitely bounded, then $CSP(\mathfrak{B})$ is in NP.

PROOF. The problem $CSP(\mathfrak{B})$ is in monotone SNP (Section 1.4).

Fraïssé's theorem can be used to construct ω -categorical structures, because homogeneous structures with finite relational signature are ω -categorical. More generally, we have the following.

Lemma 3.2.10. Let \mathfrak{C} be a countably infinite homogeneous structure such that for each k only a finite number of distinct k-ary relations can be defined by atomic formulas. Then \mathfrak{C} is ω -categorical.

PROOF. By homogeneity of \mathfrak{C} , the atomic formulas that hold on the elements of a tuple t in \mathfrak{C} determine the orbit of t in $\operatorname{Aut}(\mathfrak{C})$. Since there are only finitely many inequivalent such atomic formulas, there are finitely many orbits of k-tuples in $\operatorname{Aut}(\mathfrak{C})$. The claim follows by Theorem 3.1.4.

It is sometimes convenient to define an ω -categorical τ -structure \mathfrak{B} by specifying an amalgamation class \mathcal{C} with a signature that is larger than τ such that \mathfrak{B} is a reduct of the Fraïssé-limit of \mathcal{C} . If the Fraïssé-limit of \mathcal{C} satisfies the condition of Lemma 3.2.10, it will be ω -categorical, and therefore also all its reducts are ω -categorical (Lemma 3.1.10). This method is for instance used in Section 4.1.

This technique has also been used in [122] to give another proof of a theorem due to Cherlin, Shelah, and Shi (Theorem 3.2.11). The result appears in [75] for the special case where τ has a single binary relation denoting the edge relationship of undirected graphs. The statement for general relational signatures τ also follows from a result of [79]. The theorem of Cherlin, Shelah, and Shi will be useful in Section 4.5.

Let \mathcal{N} be a finite set of finite structures with a finite relational signature τ . Recall that a τ -structure \mathfrak{B} is called \mathcal{N} -free if there is no homomorphism from any structure in \mathcal{N} to \mathfrak{B} . A structure \mathfrak{A} in a class of structures \mathcal{C} is called *universal* for \mathcal{C} if it contains all structures in \mathcal{C} as an induced substructure. Recall that a structure is connected if it cannot be given as the disjoint union of non-empty structures.

THEOREM 3.2.11 (of [75]; also see [122]). Let \mathcal{N} be a finite set of finite connected τ -structures. Then there is an ω -categorical \mathcal{N} -free τ -structure \mathfrak{B} that is universal for the class of all countable \mathcal{N} -free structures. The structure \mathfrak{B} can be expanded by finitely many primitive positive definable relations whose complement is existential positive definable so that the expanded structure is homogeneous.

We want to remark that the structure \mathfrak{B} from Theorem 3.2.11 is uniquely (up to isomorphism) given by the fact that it is \mathcal{N} -free, universal for the class of all finite \mathcal{N} -free graphs, and model-complete (Theorem 3.6.8); model-completeness will be treated in Section 3.6.2.

3.3. Oligomorphic Permutation Groups

We have seen in Section 3.1 that a structure is ω -categorical if and only if its automorphism group is *oligomorphic*, i.e., has for each $n \ge 1$ only finitely many orbits of n-tuples. This section describes this connection between logic and permutation groups in more detail.

3.3.1. Closure. Automorphism groups of relational structures \mathfrak{B} have the property that they are *closed*, in the following sense. We write S(B) for the set of all permutations of the set B.

DEFINITION 3.3.1. A set of permutations \mathscr{P} of a set B is called closed (in S(B)) if \mathscr{P} contains all $\alpha \in S(B)$ with the property that for every finite subset A of B there exists $\beta \in \mathscr{P}$ such that $\alpha x = \beta x$ for all $x \in A$.

Proposition 3.3.2. Let \mathscr{P} be a set of permutations of some base set B. Then the following are equivalent.

- (1) \mathscr{P} is the automorphism group of a relational structure;
- (2) \mathscr{P} is a closed permutation group;
- (3) \mathscr{P} is the automorphism group of a homogeneous relational structure.

In the proof of this proposition, the following concept is useful. When \mathscr{F} is a subset of $B \to B$, then $\operatorname{Inv}(\mathscr{F})$ denotes the set of all relations over B that are preserved by all functions from \mathscr{F} . A relational structure over the base set B whose relations are exactly the relations from $\operatorname{Inv}(\mathscr{F})$ is called a *canonical structure*² for \mathscr{F} .

PROOF OF PROPOSITION 3.3.2. For the implication from (1) to (2), let \mathscr{P} be the automorphism group of a relational structure \mathfrak{B} with domain B, and let $\alpha \in \overline{\mathscr{P}}$. Then α must preserve all relations from \mathfrak{B} , because if α violates a relation from \mathfrak{B} ,

²Here, we slightly deviate from the definition given in [70], which only includes a k-ary relation for each orbit of k-tuples, for all k. The difference does not matter here, but becomes important in later sections.

then this can be seen from the restriction of α to a finite subset of the domain. Hence, $\alpha \in \mathscr{P}$.

For the implication from (2) to (3), first note that canonical structures \mathfrak{B} for \mathscr{P} are homogeneous: when i is an isomorphism between finite substructures of \mathfrak{B} , say i has domain $\{a_1,\ldots,a_n\}$, consider the relation $\{(\alpha a_1,\ldots,\alpha a_n)\mid \alpha\in\mathscr{P}\}$. This relation is preserved by all operations in \mathscr{P} and hence belongs to the relations of \mathfrak{B} . Thus, i preserves this relation, and $(i(a_1),\ldots,i(a_n))=(\alpha a_1,\ldots,\alpha a_n)$ for some $\alpha\in\mathscr{P}$. This shows that there is an automorphism of \mathfrak{B} that extends i. In fact, since \mathscr{P} is closed, this also shows that every automorphism of \mathfrak{B} is from \mathscr{P} .

The implication from (3) to (1) is trivial.

3.3.2. The Inv-Aut Galois Connection. When \mathfrak{B} is a relational structure, we denote by $\langle \mathfrak{B} \rangle_{fo}$ the set of all relations that are first-order definable in \mathfrak{B} . We will see in this section that the set

 $\{\langle \mathfrak{B} \rangle_{\text{fo}} \mid \mathfrak{B} \text{ first-order definable in } \mathfrak{C} \}$,

partially ordered by inclusion, is closely connected to the set of all closed permutation groups that contain the automorphisms of \mathfrak{C} , again partially ordered by inclusion; the connection is one-to-one when \mathfrak{C} is ω -categorical.

Recall that the automorphism group of a relational structure \mathfrak{B} , i.e., the set of all automorphisms of \mathfrak{B} , is denoted by $\operatorname{Aut}(\mathfrak{B})$. In the following it will be convenient to define the operator Aut also on sets \mathcal{R} of relations over the same domain B, in which case $\operatorname{Aut}(\mathcal{R})$ denotes the set of all permutations α of B such that α and its inverse α^{-1} preserve all relations form \mathcal{R} .

DEFINITION 3.3.3. An (anti-tone) Galois connection is a pair of functions $F: U \to V$ and $G: V \to U$ between two posets U and V, such that $v \leq F(u)$ if and only if $u \leq G(v)$ for all $u \in U, v \in V$.

It follows immediately from $F(u) \leq F(u)$ and Definition 3.3.3 that $u \leq G(F(u))$ for all $u \in U$, and similarly that $F(G(v)) \geq v$ for all $v \in V$. Moreover, F(u) = F(G(F(u))) and G(v) = G(F(G(v))) for all $u \in U$, $v \in V$.

Proposition 3.3.4. The operators Inv and Aut form a Galois connection between sets of relations over the base set B and permutation groups of the set B, both partially ordered by inclusion.

PROOF. Let \mathcal{R} be a set of relations over the set B, and let \mathcal{G} be a permutation group on the set B. First suppose that $\mathcal{G} \subseteq \operatorname{Aut}(\mathcal{R})$, and let $R \in \mathcal{R}$ and $g \in \mathcal{G}$. Then $g \in \operatorname{Aut}(\mathcal{R})$ and hence g preserves \mathcal{R} . Thus, $\mathcal{R} \subseteq \operatorname{Inv}(\mathcal{G})$.

Conversely, suppose that $\mathcal{R} \subseteq \text{Inv}(\mathcal{G})$, and again let $g \in \mathcal{G}$ and $R \in \mathcal{R}$. Then $R \in \text{Inv}(\mathcal{G})$, and hence g preserves R. Since $g^{-1} \in \mathcal{G}$, and g^{-1} also preserves R, we have that $g \in \text{Aut}(\mathcal{R})$. Thus, $\mathcal{G} \subseteq \text{Aut}(\mathcal{R})$.

We now present descriptions of the closure operators $\mathscr{G} \mapsto \operatorname{Aut}(\operatorname{Inv}(\mathscr{G}))$ and $\mathcal{R} \mapsto \operatorname{Inv}(\operatorname{Aut}(\mathcal{R}))$.

DEFINITION 3.3.5. Let \mathscr{G} is a permutation group over a set B. Then $\overline{\mathscr{G}}$, the closure of \mathscr{G} in S(B), is the smallest subset of S(B) that is closed in S(B) and contains \mathscr{G} .

The following is a special case of Corollary 1.9 in [190] (which will be presented in full generality in Proposition 5.2.1 of Chapter 5).

PROPOSITION 3.3.6. Let \mathscr{G} be a permutation group. Then $\operatorname{Aut}(\operatorname{Inv}(\mathscr{G})) = \overline{\mathscr{G}}$.

PROOF. To show that $\operatorname{Aut}(\operatorname{Inv}(\mathscr{G})) \supseteq \overline{\mathscr{G}}$, let $\alpha \in \overline{\mathscr{G}}$ be arbitrary, and let R be from $\operatorname{Inv}(\mathscr{G})$. We have to show that α and α^{-1} preserve R. Let $t \in R$; since $\alpha \in \overline{\mathscr{G}}$, we have that $\alpha t = \beta t$ for some $\beta \in \mathscr{G}$. Since β preserves R, we have that $\alpha t \in R$. The argument for α^{-1} is analogous.

To show that $\operatorname{Aut}(\operatorname{Inv}(\mathscr{G})) \subseteq \mathscr{G}$, let α be from $\operatorname{Aut}(\operatorname{Inv}(\mathscr{G}))$. It suffices to show that for every finite subset $\{a_1,\ldots,a_n\}$ of B there is a $\beta \in \mathscr{G}$ such that $\alpha a_i = \beta a_i$ for all $i \in \{1,\ldots,n\}$. Consider the relation $R := \{(\beta a_1,\ldots,\beta a_n) \mid \beta \in \mathscr{G}\}$. Note that R is preserved by all permutations in \mathscr{G} . Therefore, α preserves R. Since \mathscr{G} contains the identity, R contains (a_1,\ldots,a_n) , and hence $(\alpha(a_1),\ldots,\alpha(a_n)) \in R$. Thus, $(\alpha(a_1),\ldots,\alpha(a_n)) = (\beta(a_1),\ldots,\beta(a_n))$ for some $\beta \in \mathscr{G}$ as required. \square

We now turn to characterisations of the hull operator $\mathfrak{B} \mapsto \operatorname{Inv}(\operatorname{Aut}(\mathfrak{B}))$. First observe the following, which is straightforward to prove.

PROPOSITION 3.3.7. Let \mathfrak{B} be any structure. Then $Inv(Aut(\mathfrak{B}))$ contains $\langle \mathfrak{B} \rangle_{fo}$, the set of all relations that are first-order definable in \mathfrak{B} .

An exact characterisation of $\mathfrak{B} \mapsto \operatorname{Inv}(\operatorname{Aut}(\mathfrak{B}))$ can be given when \mathfrak{B} is ω -categorical. The analogous statement of Proposition 3.3.8 below has been observed for finite structures \mathfrak{B} by Krasner [139]. The fact that it extends to ω -categorical structures is a direct consequence of the Theorem of Ryll-Nardzewski (Theorem 3.1.4).

PROPOSITION 3.3.8. Let \mathfrak{B} be a countable ω -categorical structure with base set B, and let $R \subseteq B^k$ be a relation. Then R is first-order definable in \mathfrak{B} if and only if R is preserved by the automorphisms of \mathfrak{B} , in symbols,

$$Inv(Aut(\mathfrak{B})) = \langle \mathfrak{B} \rangle_{fo}$$
.

As we have seen in the proof of Proposition 3.3.2, it follows in particular that the expansion of every ω -categorical structure by all *first-order* definable relations is homogeneous. Recall that Theorem 3.1.4 even states that for countable structures the conclusion in Proposition 3.3.8 holds *if and only if* \mathfrak{A} is ω -categorical.

We have the following consequence of Proposition 3.3.6 and Proposition 3.3.8. An *anti-isomorphism* between two posets U and V is a bijection f from the elements of U to the elements of V such that $u \le v$ in U if and only if $f(u) \ge f(v)$ in V.

Corollary 3.3.9. Let \mathfrak{C} be a countable ω -categorical structure.

- The set of sets of the form $\langle \mathfrak{B} \rangle_{fo}$, where \mathfrak{B} is first-order definable in \mathfrak{C} , ordered by inclusion, forms a lattice.
- The set of closed permutation groups that contain $Aut(\mathfrak{C})$, ordered by inclusion, forms a lattice.
- The operator Inv is an anti-isomorphism between those two lattices, and Aut is its inverse.

We explicitly state another consequence. Recall that two structures $\mathfrak{B},\mathfrak{C}$ on the same domain are said to be *first-order interdefinable* iff all relations of \mathfrak{B} have a first-order definition in \mathfrak{C} and vice-versa. Then it follows from the above that two ω -categorical structures are first-order interdefinable if and only if they have the same automorphisms.

- **3.3.3. Transitivity and Primitivity.** We define concepts from permutation group theory that will be needed later. A permutation group \mathcal{G} on a set B is
 - k-transitive if for any two k-tuples s, t of distinct elements from B there is an $\alpha \in \mathscr{G}$ such that $\alpha s = t$, where the action of α on tuples is componentwise, i.e., $\alpha(s_1, \ldots, s_k) = (\alpha s_1, \ldots, \alpha s_k)$. We say that \mathscr{G} is transitive if it is 1-transitive.

• k-set transitive if for any two sets $S, T \subseteq B$ of cardinality k there is an $\alpha \in \mathcal{G}$ such that $\alpha S = \{\alpha s \mid s \in S\} = T$.

It is easy to see that a 2-set transitive permutation group \mathscr{G} on an infinite set is also transitive. We prove the contraposition: assume that \mathscr{G} has more than one orbit. There must be an orbit O with two distinct elements c_1, c_2 . Let c_3 be an element not from O. Then there is no automorphism that maps $\{c_1, c_2\}$ to $\{c_1, c_3\}$, and hence \mathscr{G} is not 2-set transitive. More generally, it holds that the number of orbits of n-subsets is a non-decreasing sequence [70].

A congruence of \mathscr{G} is an equivalence relation on B that is preserved by all permutations in \mathscr{G} . The equivalence classes of a congruence are also called blocks. A congruence is trivial if each block contains only one element (and non-trivial otherwise), and it is called proper if it is distinct from the equivalence relation that has only one block. When \mathfrak{B} is an ω -categorical structure and \mathscr{G} its oligomorphic automorphism group, then the congruences of \mathscr{G} are exactly the first-order definable equivalence relations in \mathfrak{B} (and so we apply the terminology that we have for congruences also to those equivalence relations). A permutation group \mathscr{G} is called primitive if \mathscr{G} is transitive and every proper congruence of \mathscr{G} is trivial, and imprimitive otherwise. Clearly, 2-transitive structures are always primitive.

An *orbital* is an orbit of pairs, that is, a set of the form $\{(\alpha a, \alpha b) \mid \alpha \in \mathcal{G}\}$ for $a, b \in B$. The *trivial* orbital is the orbital $\{(a, a) \mid a \in B\}$. When O is an orbital, the *orbital graph* is the directed graph with vertex set B and edges O. The $rank \ r(\mathcal{G})$ of \mathcal{G} is the number of distinct orbitals of \mathcal{G} .

For a sequence \bar{a} of elements of B, the pointwise stabilizer $\mathscr{G}_{\bar{a}}$ of \mathscr{G} is the set of all elements of \mathscr{G} that fix \bar{a} . For a subset A of B, the setwise stabilizer \mathscr{G}_A of \mathscr{G} is the set of all elements α of \mathscr{G} that fix A set-wise, that is, satisfy $\alpha A = A$.

- **3.3.4.** Products. In this section we review the classical theory how to describe a permutation group in terms of transitive ones. The same idea can be used to construct new oligomorphic permutation groups from known ones.
- 3.3.4.1. Group actions. It will be convenient to take a more general perspective on permutation groups (and this will again be used in Chapter 7). We now consider abstract groups, that is, algebraic structures \mathbf{G} over a set G of group elements, with a function symbol for multiplication of group elements, a function for the inverse of a group element, and the constant for the identity. The link to permutation groups is given by the concept of an action of such a group on a set, which is described below.

Let $\operatorname{Sym}(X)$ be the abstract group whose domain is the set of all permutations of X, and where composition is defined as composition of permutations, the inverse of an element g of $\operatorname{Sym}(X)$ is the inverse of g as a permutation of X, and the identity is the identity permutation.

DEFINITION 3.3.10. A (left) group action of an (abstract) group \mathbf{G} on a set X is a binary function $\cdot : G \times X \to X$ which satisfies that $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in S$, and $e \cdot x = x$ for every $x \in X$. The action is faithful if for any two distinct $g, h \in G$ there exists an $x \in X$ such that $g \cdot x \neq h \cdot x$.

Equivalently, a group action of \mathbf{G} on a set X can be viewed as a homomorphism from \mathbf{G} into $\mathrm{Sym}(X)$, and a faithful group action as an isomorphism between \mathbf{G} and a subgroup of $\mathrm{Sym}(X)$. Clearly, to every action of \mathbf{G} on X we can associate a permutation group as considered before, namely the image of the action in $\mathrm{Sym}(X)$. An action is called *oligomorphic* if the associated permutation group is oligomorphic. Conversely, to every permutation group G on a set G0 whose domain is G1, where composition and inverse are defined in the obvious

way, and which acts on X faithfully by $g \cdot x = g(x)$. When \mathfrak{B} is a structure, we call G the abstract automorphism group of \mathfrak{B} if there is an action of G on B such that the image of G under this action is the automorphism group of \mathfrak{B} .

When $x \in X$, the *orbit* of x with respect to an action of G on X is the set $\{g \cdot x \mid g \in G\}$. Hence, an *orbit* of k-tuples in the corresponding permutation group on X is an orbit of the action of G on X^k that is defined componentwise, that is, g maps (x_1, \ldots, x_k) to (gx_1, \ldots, gx_k) . In this way we can also use other terminology introduced for permutation groups (such a transitivity, congruences, primitivity, etc.) for group actions.

The product of a sequence of groups $(\mathbf{G}_i)_{i\in I}$ is the product of this sequence as defined in general in Chapter 2; note that the product is again a group. Products appear in several ways when studying permutation groups; the first is when we want to describe the relation between a permutation group and its 'transitive constituents', described in the following.

3.3.4.2. The intransitive action of a group product. When **G** acts on a set X and $S \subset X$ is an orbit with respect to this action, then **G** naturally acts transitively on S by restriction; we call the corresponding group **H** the group induced by S, or a transitive constituent.

PROPOSITION 3.3.11 (see [70]). Let G be a group acting on a set X, and let $(G_i)_{i\in I}$ be the groups induced by the orbits of G on X. Then G is isomorphic to a subgroup of $\prod_{i\in I} G_i$, and there are surjective homomorphisms from G to G_i , for each i.

We can use the same idea to construct new oligomorphic permutation groups from known ones.

DEFINITION 3.3.12. Let \mathbf{G}_1 and \mathbf{G}_2 be groups acting on disjoint countable sets X and Y, respectively. Then the action of $\mathbf{G}_1 \times \mathbf{G}_2$ on $X \cup Y$ defined by $(g_1, g_2) \cdot z = g_1 z$ if $z \in X$ and $g_2 z$ if $y \in Y$ is called the natural intransitive action of $\mathbf{G}_1 \times \mathbf{G}_2$ on $X \cup Y$.

Note that when \mathbf{G}_1 and \mathbf{G}_2 act oligomorphically on X and Y, respectively, then the natural intransitive action of $\mathbf{G}_1 \times \mathbf{G}_2$ is also oligomorphic: when $F_1(n)$ is the number of orbits of the componentwise action of \mathbf{G}_1 on X^n , and $F_2(n)$ is the number of orbits of the componentwise action of \mathbf{G}_2 on Y, then the number of orbits of the componentwise of $\mathbf{G}_1 \times \mathbf{G}_2$ on $X \cup Y$ is $\sum_{0 \le i \le n} F_1(i)F_2(n-i)$, and hence finite for all n.

When G_1 and G_2 are the automorphism groups of ω -categorical relational structures $\mathfrak A$ and $\mathfrak B$ with disjoint domains A and B, respectively, then the image of the natural intransitive action on $A \cup B$ (as a homomorphism from $G_1 \times G_2$ to $\operatorname{Sym}(A \cup B)$) can also be described as the automorphism group of a relational structure $\mathfrak C$: we can take for $\mathfrak C$ the disjoint union of $\mathfrak A$ and $\mathfrak B$ (defined in Section 1.1), expanded by a unary predicate that contains exactly the elements of A. Since reducts of ω -categorical structures are again ω -categorical, this shows in particular that the disjoint union of two ω -categorical structures is again ω -categorical.

3.3.4.3. The product action. When G_1 is a group acting on a set X, and G_2 a group acting on a set Y, there is another important natural action of $G := G_1 \times G_2$ besides the intransitive natural action of G, which is called the product action of G. In this action, G acts on $X \times Y$ by $(g_1, g_2) \cdot (x, y) = (g_1 x, g_1 y)$. If the actions of G_1 and G_2 are transitive, then the product action is clearly transitive, too. We claim that when the actions of G_1 and G_2 are oligomorphic, then the product action is also oligomorphic. Let $F_1(n)$ and $F_2(n)$ then the number of orbits of the

componentwise action of G_1 on X^n and Y^n , respectively. Then the number of orbits of the componentwise action of G on $X \times Y$ is $F_1(n)F_2(n)$, and in particular finite, which proves the claim.

When \mathbf{G}_1 and \mathbf{G}_2 are the automorphism groups of ω -categorical relational structures \mathfrak{A} and \mathfrak{B} , then the image of the product action of \mathbf{G} in $\mathrm{Sym}(A \times B)$ is the automorphism group of the following structure, which we call the *full product structure* of \mathfrak{A} and \mathfrak{B} , and denote by $\mathfrak{A} \boxtimes \mathfrak{B}$. Let σ be the signature of \mathfrak{A} , and τ be the signature of \mathfrak{B} ; we assume that σ and τ are disjoint, otherwise we rename the relations so that the assumption is satisfied. For each k-ary $R \in \sigma$, the structure $\mathfrak{A} \boxtimes \mathfrak{B}$ contains the relation $\{((a_1,b_1),\ldots,(a_k,b_k)) \mid (a_1,\ldots,a_k) \in R^{\mathfrak{A}},b_1,\ldots,b_k \in B\}$, and for each k-ary $R \in \tau$, it contains the relation $\{((a_1,b_1),\ldots,(a_k,b_k)) \mid (b_1,\ldots,b_k) \in R^{\mathfrak{B}},a_1,\ldots,a_k \in A\}$. Finally, we also add the relations $P_1 = \{((a_1,b_1),(a_2,b_2)) \mid a_1 = a_2\}$ and $P_2 = \{((a_1,b_1),(a_2,b_2)) \mid b_1 = b_2\}$ to $\mathfrak{A} \boxtimes \mathfrak{B}$.

PROPOSITION 3.3.13. The automorphism group of $\mathfrak{C} := \mathfrak{A} \boxtimes \mathfrak{B}$ is $\operatorname{Aut}(\mathfrak{A}) \times \operatorname{Aut}(\mathfrak{B})$ in its product action on $A \times B$.

PROOF. Let h be the product action of $\mathbf{G} := \operatorname{Aut}(\mathfrak{A}) \times \operatorname{Aut}(\mathfrak{B})$ on $A \times B$, viewed as a homomorphism from \mathbf{G} to $\operatorname{Sym}(A \times B)$. Let (g_1, g_2) be an element of \mathbf{G} . Then $h((g_1, g_2))$ is the permutation $(x, y) \mapsto (g_1 x, g_2 y)$ of $A \times B$, and this map preserves \mathfrak{C} : when $((a_1, b_1), \ldots, (a_k, b_k)) \in R^{\mathfrak{C}}$, for $R \in \sigma$, then $(a_1, \ldots, a_k) \in R^{\mathfrak{A}}$, and so $(g_1 a_1, \ldots, g_1 a_k) \in R^{\mathfrak{A}}$. Therefore, $((g_1 a_1, g_2 b_1), \ldots, (g_1 a_k, g_2 b_k)) \in R^{\mathfrak{C}}$. The proof for the relation symbols $R \in \tau$ is analogous.

We now show that conversely, every automorphism g of \mathfrak{C} is in the image of h. Note that P_1 and P_2 are congruences of the automorphism group of \mathfrak{C} . Fix elements $a_0 \in A, b_0 \in B$. Let g_1 be the permutation of A that maps $a \in A$ to a' such that $g((a, b_0)) = (a', b')$. Similarly, let g_2 be the permutation of B that maps $b \in B$ to b' such that $g((a_0, b)) = (a', b')$. Since g preserves P_1, P_2 , the definition of g_1 and g_2 does not depend on the choice of a_0 and b_0 . Moreover, $g_1 \in \operatorname{Aut}(\mathfrak{A})$, since g preserves the relations for the symbols from σ . Similarly, $g_2 \in \operatorname{Aut}(\mathfrak{B})$. Then $g' := h((g_1, g_2))$ equals g, since $g'((a, b)) = (g_1 a, g_2 b) = g(a, b)$. Hence, g is a permutation of $A \times B$ that lies in the image of h.

Note that Proposition 3.3.13 becomes false in general when we omit the relations P_1 and P_2 in $\mathfrak{A} \boxtimes \mathfrak{B}$. Consider for example the structure without structure \mathfrak{B} (that is, \mathfrak{B} has empty signature). Then the automorphism group of $\mathfrak{B} \boxtimes \mathfrak{B}$ is imprimitive, but without the relations P_1 and P_2 , the structure is isomorphic to \mathfrak{B} and hence primitive. Also note that when \mathfrak{A} and \mathfrak{B} are ordered structures (and this will be a typical assumption in Chapter 8), we could omit P_1 and P_2 in the definition of the full product without sacrificing Proposition 3.3.13, since $P_1(x,y)$ is definable from the order < of \mathfrak{A} by the formula $\neg(x < y) \land \neg(y < x)$, and similarly P_2 is definable from the order of \mathfrak{B} .

Finally we remark that $(\mathfrak{A} \boxtimes \mathfrak{B}) \boxtimes \mathfrak{C}$ and $\mathfrak{A} \boxtimes (\mathfrak{B} \boxtimes \mathfrak{C})$ have the same automorphism group (on the domain $A \times B \times C$). We explicitly define the d-fold full product as follows.

DEFINITION 3.3.14 (Full product of d structures). Let $\mathfrak{B}_1, \ldots, \mathfrak{B}_d$ be structures with disjoint relational signatures τ_1, \ldots, τ_d . We denote by $\mathfrak{B}_1 \boxtimes \cdots \boxtimes \mathfrak{B}_d$ the structure with domain $B := B_1 \times \cdots \times B_d$ that contains for every $i \leq d$, and every m-ary $R \in (\tau_i \cup \{=\})$ an m-ary relation defined by

$$\{((x_1^1,\ldots,x_1^d),\ldots,(x_m^1,\ldots,x_m^d))\in B^m\mid (x_1^i,\ldots,x_m^i)\in R^{\mathfrak{B}_i}\}$$
.

If $\mathfrak{B} := \mathfrak{B}_1 = \cdots = \mathfrak{B}_k$, then we first rename $R \in \tau_i$ into R_i so that the factors have pairwise disjoint signatures, and then write $\mathfrak{B}^{[d]}$ for $\mathfrak{B}_1 \boxtimes \cdots \boxtimes \mathfrak{B}_d$.

When $\mathfrak A$ and $\mathfrak B$ have the same signature τ , then the automorphism group of the τ -structure $\mathfrak A \times \mathfrak B$ (see Definition 3.3.4) contains the automorphism group of $\mathfrak A \times \mathfrak B$, and hence $\mathfrak A \times \mathfrak B$ is ω -categorical, by Theorem 3.1.4. As a consequence, the class of all ω -categorical structures forms a lattice with respect to the homomorphism order (where disjoint union is the join, and product the meet of two ω -categorical structures).

3.4. Preservation Theorems

Model-theoretic preservation theorems typically link definability in (a syntactically restricted fragment of) a given logic with certain 'semantic' closure properties. For the syntactic restrictions on first-order formulas that we have introduced in Chapter 2 we have already made remarks about various types of mappings that automatically preserve the respective formulas. Surprisingly, very often these maps can be used to obtain an exact characterisation of definability in the corresponding fragment of first-order logic.

In this text, preservation theorems become relevant in two contexts. The first is that they can be used to give exact characterizations of existential, existential positive, and quantifier-free definability of relations over an ω -categorical structure, in a similar way as we characterized first-order definability in Section 3.3. These characterizations require that we pass from automorphism groups to endomorphism monoids, and they turn out to be useful for the complexity analysis of CSPs. The various relevant connections are displayed in Figure 3.1.

first-order definitions	automorphisms
existential definitions	self-embeddings
positive definitions	surjective endomorphisms
existential positive definitions	endomorphisms
quantifier-free definitions	partial automorphisms

FIGURE 3.1. Syntactically restricted definabilities and the corresponding preservation properties.

The second context where we encounter model-theoretic preservation theorems is when giving syntactic descriptions of ω -categorical theories themselves (rather than relations in ω -categorical structures). For instance, we will see that for every ω -categorical structure $\mathfrak A$ there exists a homomorphically equivalent ω -categorical structure $\mathfrak B$ whose first-order theory is $\forall \exists^+$.

3.4.1. Model-theoretic preservation theorems. When T is a first-order theory, we say that ϕ and ψ are equivalent modulo T if $T \models (\phi \Leftrightarrow \psi)$ (see Section 1.3). The following theorems are well-known and can be found in most model theory books.

Theorem 3.4.1 (Los-Tarski; see e.g. Corollary in 5.4.5 of [120]). Let T be a first-order theory. A first-order formula ϕ is equivalent to an existential formula modulo T if and only if ϕ is preserved by all embeddings between models of T.

Theorem 3.4.2 (Lyndon; see e.g. Corollary in 8.3.5 of [120]). Let T be a first-order theory. A first-order formula ϕ is equivalent to a positive formula modulo T if and only if ϕ is preserved by all surjective homomorphisms between models of T.

Note that here the assumption that \bot is always part of first-order logic becomes relevant: the first-order formula $\exists x. x \neq x$ is preserved by all homomorphisms between models of T, but without \bot it might not be equivalent to a positive formula modulo T (for instance when T is the empty theory).

Theorem 3.4.3 (Homomorphism Preservation Theorem; see e.g. Exercise 2 in Section 5.5 of [120]). Let T be a first-order theory. A first-order formula ϕ is equivalent to an existential positive formula modulo T if and only if ϕ is preserved by all homomorphisms between models of T.

THEOREM 3.4.4 (Chang-Łoś-Suszko Theorem; Theorem 5.4.9 in [120] and remarks after the proof). Let T be a first-order τ -theory.

- A set of first-order τ -formulas Φ is equivalent to a set of $\forall \exists$ -formulas Ψ modulo T if and only if Φ is preserved in unions of chains of models of T.
- A first-order τ -formula ϕ is equivalent to a $\forall \exists$ -formula ψ modulo T if and only if ϕ is preserved in unions of chains of models of T.

Our next preservation theorem, Theorem 3.4.6, is a positive variant of the Chang-Loś-Suszko preservation theorem, which we could not find in explicit form in the literature. Its proof can be derived from the proof of the Chang-Łoś-Suszko theorem given in [120] by modification of a sequence of lemmata given there; since some of them require some care, we will present those modifications in full detail here. Besides the existential positive amalgamation theorem (Lemma 2.4.4), we need the following lemma.

LEMMA 3.4.5. Let T be a first-order theory, and let $\mathfrak A$ be a model of the $\forall \exists^+$ consequences of T. Then $\mathfrak A$ can be extended to a model $\mathfrak B$ of T such that every
existential positive formula that holds on a tuple $\bar a$ in $\mathfrak B$ also holds on $\bar a$ in $\mathfrak A$.

PROOF. Let \mathfrak{A}' be an expansion of \mathfrak{A} by constants such that all elements of \mathfrak{A}' are denoted by a constant symbol. It suffices to prove that $T \cup \operatorname{diag}(\mathfrak{A}') \cup \operatorname{diag}_{\forall^-}(\mathfrak{A}')$ has a model \mathfrak{B} . Suppose for contradiction that it were inconsistent; then by compactness, there exists a finite subset U of $\operatorname{diag}_{\forall^-}(\mathfrak{A}') \cup \operatorname{diag}(\mathfrak{A}')$ such that $T \cup U$ is inconsistent. Let ϕ be the conjunction over U where all new constant symbols are existentially quantified. Then $T \cup \{\phi\}$ is inconsistent as well. But $\neg \phi$ is equivalent to a $\forall \exists^+$ formula, and a consequence of T. Hence, $\mathfrak{A} \models \neg \phi$, a contradiction.

The following is a positive version of the Chang-Loś-Suszko theorem (Theorem 3.4.4).

Theorem 3.4.6. Let T be a first-order τ -theory, and Φ a set of τ -formulas. Then the following are equivalent.

- (1) Φ is modulo T equivalent to a set of $\forall \exists^+$ -formulas Ψ .
- (2) Φ is preserved in direct limits of sequences of models of T;
- (3) Φ is preserved in direct limits of countable sequences of models of T.

PROOF. The implication from (1) to (2) is Proposition 2.5.5. The implication from (2) to (3) is trivial. For the implication from (3) to (1), assume that ϕ is preserved by direct limits of sequences (\mathfrak{A}_i) as in the statement of the proposition. We can assume that Φ is a set of sentences (by adding constants, Lemma 2.3.2). Let Ψ be the set of all $\forall \exists^+$ -sentences that are consequences of $T \cup \Phi$. We first show that $T \cup \Psi$ implies ϕ . It suffices to show that every model of $T \cup \Psi$ is elementary equivalent to a direct limit of a sequence $(\mathfrak{B}_i)_{i<\omega}$ of models of $T \cup \Phi$ where there are coherent homomorphisms $f_{ij} : \mathfrak{B}_i \to \mathfrak{B}_j$ with $f_{jk} \circ f_{ij} = f_{ik}$ for all $i \leq j \leq k$.

To construct this sequence, we define an elementary chain of models $(\mathfrak{A}_i)_{i<\omega}$ of $T\cup\Psi$ such that there are

• homomorphisms $f_i: \mathfrak{A}_i \to \mathfrak{B}_i$, with $\mathfrak{B}_i \models T \cup \Phi$, such that for every tuple \bar{a}_i of elements from \mathfrak{A}_i and every existential positive formula θ , if $\mathfrak{B}_i \models \theta(f_i(\bar{a}_i))$, then $\mathfrak{A}_i \models \theta(\bar{a}_i)$, and

• homomorphisms $g_i \colon \mathfrak{B}_i \to \mathfrak{A}_{i+1}$, such that $g_i \circ f_i$ is the identity on \mathfrak{A}_i .

Let \mathfrak{A}_0 be a countable model of $T \cup \Psi$. To construct the rest of the sequence, suppose that \mathfrak{A}_i has been chosen. Since \mathfrak{A}_0 is an elementary substructure of \mathfrak{A}_i , in particular all the $\forall \exists^+$ -consequences of $T \cup \Phi$ hold in \mathfrak{A}_i . By Lemma 3.4.5, the structure \mathfrak{A}_i can be extended to a model \mathfrak{B}_i of $T \cup \Phi$ such that every ep-sentence that holds in $(\mathfrak{B}_i, \bar{a}_i)$ also holds in $(\mathfrak{A}_i, \bar{a}_i)$. By Lemma 2.4.4 there are an elementary extension \mathfrak{A}_{i+1} of \mathfrak{A}_i and a homomorphism $g_i \colon \mathfrak{B}_i \to \mathfrak{A}_{i+1}$ such that $g_i \circ f_i$ is the identity on \mathfrak{A}_i . Then $\mathfrak{C} := \bigcup_{i < \omega} \mathfrak{A}_i$ equals $\lim_{i < \omega} \mathfrak{B}_i$, and by the Tarski-Vaught elementary chain theorem (Theorem 2.5.2) \mathfrak{A}_0 is an elementary substructure of \mathfrak{C} . So \mathfrak{C} is a model of T, and the direct limit of models \mathfrak{B}_i of $T \cup \Phi$, and hence $\mathfrak{C} \models \phi$. This shows that $T \cup \Psi$ implies Φ .

By compactness one can show that when Φ is finite, then the formula Ψ from item (1) in Theorem 3.4.6 above can be chosen to be finite as well.

3.4.2. Endomorphisms and self-embeddings. We apply the model-theoretic preservation theorems from the previous section to characterize existential, positive, and existential positive definability of relations in ω -categorical structures.

For *finite* structures and existential positive definability the corresponding preservation theorem has already been noted by Krasner [139] (for finite structures, self-embeddings are necessarily automorphisms, and existential definability is the same as first-order definability).

THEOREM 3.4.7 (from [24] and [39,51]). Let \mathfrak{B} be an ω -categorical structure with base set B, and $R \subseteq B^k$ be a relation.

- (1) R has an existential positive definition in \mathfrak{B} if and only if R is preserved by all endomorphisms of \mathfrak{B} .
- (2) R has an existential definition in \mathfrak{B} if and only if R is preserved by all self-embeddings of \mathfrak{B} .
- (3) R has a positive definition in \mathfrak{B} if and only if R is preserved by all surjective endomorphisms of \mathfrak{B} .

PROOF. We have already remarked in Chapter 2 that existential positive formulas are preserved by endomorphisms, and existential formulas are preserved by self-embeddings of \mathfrak{B} .

For the other direction, note that the endomorphisms and self-embeddings of \mathfrak{B} contain the automorphisms of \mathfrak{B} , and hence Theorem 3.1.4 shows that R has a first-order definition ϕ in \mathfrak{B} . Suppose for contradiction that R were preserved by all endomorphisms of \mathfrak{B} but has no existential positive definition in \mathfrak{B} . We use the homomorphism preservation theorem (Theorem 3.4.3). Since by assumption ϕ is not equivalent to an existential positive formula in \mathfrak{B} , there are models \mathfrak{B}_1 and \mathfrak{B}_2 of the first-order theory of \mathfrak{B} and a homomorphism h from \mathfrak{B}_1 to \mathfrak{B}_2 that violates ϕ . By the theorem of Löwenheim-Skolem (Theorem 2.3.3) the first-order theory of the two-sorted structure $(\mathfrak{B}_1,\mathfrak{B}_2;h)$ has a countable model $(\mathfrak{B}'_1,\mathfrak{B}'_2;h')$. Since both \mathfrak{B}'_1 and \mathfrak{B}'_2 must be countably infinite, and because \mathfrak{B} is ω -categorical, we have that \mathfrak{B}'_1 and \mathfrak{B}'_2 are isomorphic to \mathfrak{B} , and h' can be seen as an endomorphism of \mathfrak{B} that violates ϕ ; a contradiction.

The argument for existential definitions and positive definitions is similar, but instead of the homomorphism preservation theorem we use the theorem of Los-Tarski (Theorem 3.4.1) and Lyndon's theorem (Theorem 3.4.2).

We now present a Galois connection for existential positive definability and transformation monoids, similar to the Galois connection for first-order definability and

permutation groups. For a structure \mathfrak{B} , we denote the set of relations with an existential positive definition in \mathfrak{B} by $\langle \mathfrak{B} \rangle_{\text{ep}}$. Similarly as in Section 3.3, we say that a set of operations $\mathscr{F} \subseteq (B \to B)$ is (locally) closed if it contains every operation $f \colon B \to B$ such that for every finite subset A of B there exists a $g \in \mathscr{F}$ such that f(a) = g(a) for all $a \in A$. The closure of \mathscr{F} is the smallest locally closed set of operations that contains \mathscr{F} .

When \mathscr{F} is a transformation monoid, then $\langle \mathscr{F} \rangle$ denotes the smallest locally closed transformation monoid that contains \mathscr{F} . The set of endomorphisms of a relational structure \mathfrak{B} (or the set of operations from $B \to B$ that preserve a set of relations \mathscr{R} over the domain B) is denoted by $\operatorname{End}(\mathfrak{B})$ (or by $\operatorname{End}(\mathscr{R})$, respectively). The following can be shown in a similarly straightforward way as Proposition 3.3.2.

PROPOSITION 3.4.8. For every $\mathscr{F} \subseteq (B \to B)$, the following are equivalent.

- (1) \mathscr{F} is the transformation monoid of a relational structure;
- (2) \mathscr{F} is a locally closed monoid.

The proof of the following statement is similar to the proof of Proposition 3.3.6.

PROPOSITION 3.4.9. Let $\mathscr{F} \subseteq (B \to B)$ be a transformation monoid. Then $g \colon B \to B$ is in the closure of \mathscr{F} if and only if g preserves all relations in $\operatorname{Inv}(\mathscr{F})$. In symbols,

$$\operatorname{End}(\operatorname{Inv}(\mathscr{F})) = \langle \mathscr{F} \rangle$$
.

Theorem 3.4.7 now implies the following analog to Corollary 3.3.9.

COROLLARY 3.4.10. Let $\mathfrak C$ be an ω -categorical structure. Then the lattice of locally closed transformation monoids that contain $\operatorname{Aut}(\mathfrak C)$ is anti-isomorphic to the lattice of sets of the form $\langle \mathfrak B \rangle_{\operatorname{ep}}$ where $\mathfrak B$ is first-order definable in $\mathfrak C$.

To illustrate the use of this Galois connection, we present a simple and typical application.

LEMMA 3.4.11. Let \mathfrak{B} be such that $\operatorname{Aut}(\mathfrak{B})$ is 2-set transitive. If \mathfrak{B} has a non-injective endomorphism f, then \mathfrak{B} also has a constant endomorphism.

PROOF. Let f be an endomorphism of \mathfrak{B} such that f(b) = f(b') for two distinct values $b, b' \in B$. Let b_1, b_2, \ldots be an enumeration of B. We construct an infinite sequence of endomorphisms e_1, e_2, \ldots , where e_i is an endomorphism that maps all of the values b_1, \ldots, b_i to b_1 . This suffices, since then by local closure the mapping defined by $e(x) := b_1$ for all x is an endomorphism of \mathfrak{B} .

For e_1 , we take the identity map, which clearly is an endomorphism with the desired properties. To define e_i for $i \geq 2$, let α be an automorphism of \mathfrak{B} that maps $\{b_1, e_{i-1}(b_i)\}$ to $\{b, b'\}$; such an automorphism exists because $\operatorname{Aut}(\mathfrak{B})$ is 2-set transitive. Then the endomorphism $f(\alpha e_{i-1}(x))$ is constant on b_1, \ldots, b_i ; recall that $b_1 = e_{i-1}(b_1) = \cdots = e_{i-1}(b_{i-1})$. Since \mathfrak{B} is 2-transitive, it is in particular transitive, and there is an automorphism β that maps f(b) to b_1 . Then $e_i \colon x \mapsto \beta f(\alpha e_{i-1}(x))$ is an endomorphism of \mathfrak{B} with the desired properties.

- **3.4.3.** Locally invertible self-embeddings. Let $\mathfrak A$ and $\mathfrak B$ be τ -structures, let e be an embedding of $\mathfrak A$ into $\mathfrak B$, and let f be an embedding of $\mathfrak B$ into $\mathfrak A$. We say that e and f locally invert each other if
 - for every tuple \bar{a} of elements of \mathfrak{A} there are $\beta \in \operatorname{Aut}(\mathfrak{B})$ and $\alpha \in \operatorname{Aut}(\mathfrak{A})$ such that $\alpha f(\beta e(\bar{a})) = \bar{a}$, and
 - for every tuple \bar{b} of elements of \mathfrak{B} there are $\alpha \in \operatorname{Aut}(\mathfrak{A})$ and $\beta \in \operatorname{Aut}(\mathfrak{B})$ such that $\beta e(\alpha f(\bar{b})) = \bar{b}$.

We say that e is *locally invertible* if there exists a self-embedding f such that e and f locally invert each other.

We will show that locally invertible self-embeddings preserve first-order formulas. To do so, we need the following concept. Let $\mathfrak A$ and $\mathfrak B$ be τ -structures. A back-and-forth system from $\mathfrak A$ to $\mathfrak B$ (our definition is taken from [120]) is a non-empty set I of pairs $(\bar a, \bar b)$ of tuples, with $\bar a$ from $\mathfrak A$ and $\bar b$ from $\mathfrak B$, such that the following hold.

- (1) If $(\bar{a}, \bar{b}) \in I$ then \bar{a} and \bar{b} have the same length and (\mathfrak{A}, \bar{a}) satisfies the same atomic formulas as (\mathfrak{B}, \bar{b}) .
- (2) (Going Forth.) For every pair $(\bar{a}, \bar{b}) \in I$ and every element c of \mathfrak{A} there is an element d of \mathfrak{B} such that the pair $(\bar{a}c, \bar{b}d) \in I$.
- (3) (Going Back.) For every pair $(\bar{a}, \bar{b}) \in I$ and every element d of \mathfrak{B} there is an element c of \mathfrak{A} such that the pair $(\bar{a}c, \bar{b}d) \in I$.

There is a back-and-forth system from $\mathfrak A$ to $\mathfrak B$ if and only if $\mathfrak A$ and $\mathfrak B$ are isomorphic (combination of Lemma 3.2.2 and Theorem 3.2.3 (b) in [120]).

THEOREM 3.4.12. A relation R has a first-order definition in an ω -categorical structure \mathfrak{B} if and only if R is preserved by all locally invertible self-embeddings of \mathfrak{B} .

PROOF. We are in the remarkable situation (in comparison to the other preservation theorems discussed here) that the "if" direction of the statement is easy (it follows directly from the theorem of Ryll-Nardzewski, since automorphisms are locally inverted by their inverse), and that we only have to show the 'only if' direction.

Let e and f be self-embeddings of $\mathfrak B$ that locally invert each other, and suppose that $\bar a$ is a tuple from $\mathfrak B$ that satisfies a first-order formula ϕ . We claim that $e(\bar a)$ satisfies ϕ as well. It clearly suffices to show that the structures $(\mathfrak B, \bar a)$ and $(\mathfrak B, e(\bar a))$ are isomorphic. We claim that the set

$$\begin{split} I := \{ (\bar{u}, \bar{v}) \mid \text{ there are } \gamma, \delta \in \operatorname{Aut}(\mathfrak{B}) \\ \text{ so that } \delta e \gamma(\bar{u}) = \bar{v} \} \end{split}$$

is a back-and-forth system from (\mathfrak{B}, \bar{a}) to $(\mathfrak{B}, e(\bar{a}))$.

The set I is non-empty, since $(\bar{a}, e(\bar{a})) \in I$ (we have $\gamma = \delta = id$ in the definition of I). It is obvious that I satisfies item (1) in the definition of back-and-forth systems since all involved operations are embeddings. Now, let (\bar{u}, \bar{v}) be from I. By definition of I, there are $\gamma \in \operatorname{Aut}(\mathfrak{B})$ and $\delta \in \operatorname{Aut}(\mathfrak{B})$ so that $\delta e(\gamma \bar{u}) = \bar{v}$. For going forth, let c be an arbitrary element of \mathfrak{B} . Let d be $\delta e(\gamma c)$. The clearly $(\bar{u}c, \bar{v}d) \in I$.

For going back, let d be an arbitrary element of \mathfrak{B} . Since e is locally inverted by f, there exist $\alpha, \beta \in \operatorname{Aut}(\mathfrak{B})$ such that $\alpha f(\beta e(\gamma \bar{u})) = \gamma \bar{u}$. Since $e(\gamma \bar{u}) = \delta^{-1} \bar{v}$, this is the same as saying that $\alpha f(\beta \delta^{-1} \bar{v}) = \gamma \bar{u}$, and by multiplication with α^{-1} we note

$$f(\beta \delta^{-1} \bar{v}) = \alpha^{-1} \gamma \bar{u} . \tag{7}$$

We now set c to $\gamma^{-1}\alpha f(\beta\delta^{-1}d)$, claiming that $(\bar{u}c,\bar{v}d) \in I$, which completes the proof. To show the claim, we have to find $\gamma', \delta' \in \operatorname{Aut}(\mathfrak{B})$ such that $\delta' e(\gamma'(\bar{u}c)) = \bar{v}d$. Let \bar{p} be the tuple $\beta\delta^{-1}(\bar{v}d)$. By the second item in the definition of local inversion, there are $\alpha', \beta' \in \operatorname{Aut}(\mathfrak{B})$ such that $\beta' e(\alpha' f(\bar{p})) = \bar{p}$.

Choose
$$\gamma' = \alpha' \alpha^{-1} \gamma$$
 and $\delta' = \delta \beta^{-1} \beta'$. Then
$$\delta' e(\gamma'(\bar{u}c)) = \delta \beta^{-1} \beta' e(\alpha' \alpha^{-1} \gamma(\bar{u}c))$$

$$= \delta \beta^{-1} \beta' e(\alpha' (\alpha^{-1} \gamma \bar{u}, \alpha^{-1} \gamma \bar{c}))$$

$$= \delta \beta^{-1} \beta' e(\alpha' (f(\beta \delta^{-1} \bar{v}), f(\beta \delta^{-1} d)))$$

$$= \delta \beta^{-1} \beta' e(\alpha' f(\beta \delta^{-1} (\bar{v}d)))$$

$$= \delta \beta^{-1} \beta \delta^{-1} (\bar{v}d)$$

$$= \bar{v}d,$$
(see (7))

and so $(\bar{u}c, \bar{v}d) \in I$.

Theorem 3.4.12 will be used in Section 3.6, and later also in Chapter 10, as a tool for proving that all automorphisms of certain structures \mathfrak{B} are locally generated by the self-embeddings of \mathfrak{B} . We note the following consequence of Theorem 3.4.12.

COROLLARY 3.4.13. An endomorphism e of an ω -categorical structure is locally invertible if and only if e is locally generated by the automorphisms of Γ .

PROOF. If e is locally generated by the automorphisms, then e is clearly locally invertible. The converse follows from Theorem 3.4.12 in combination with Proposition 3.4.9.

3.4.4. Partial Automorphisms. Recall that a formula is called *quantifier-free* if it can be constructed from atomic formulas by usage of Boolean connectives only. Also quantifier-free definability can be characterized by a model-theoretic preservation theorem; in this case, this is very easy to prove.

PROPOSITION 3.4.14. Let T be a first-order theory over a relational signature. A first-order formula ϕ is equivalent to a quantifier-free formula modulo T if and only if ϕ is preserved by partial isomorphisms between models of T.

PROOF. It is clear that quantifier-free formulas are preserved by partial isomorphisms between models of T. For the converse, let ϕ be preserved by all partial isomorphisms between models of T. Let Ψ be the set of all quantifier-free formulas ψ such that $T \models \forall \bar{x}(\phi(\bar{x}) \Rightarrow \psi(\bar{x}))$. It suffices to prove that Ψ implies ϕ . Let \mathfrak{A} be a model of T and \bar{a} a tuple from \mathfrak{A} such that \bar{a} satisfies Ψ in \mathfrak{A} . Let \mathfrak{B} be a model of T and \bar{b} a tuple from \mathfrak{B} such that \bar{b} satisfies ϕ in \mathfrak{B} (if no such \mathfrak{B} exists, the statement of the proposition is trivial). Since \bar{a} and \bar{b} satisfy the same atomic formulas, the mapping that sends \bar{b} to \bar{a} is a partial isomorphism. Since ϕ is by assumption preserved by partial isomorphisms, \bar{a} satisfies ϕ in \mathfrak{A} . The set Ψ modulo T equivalent to a single quantifier-free formula by compactness of first-order logic and this concludes the proof.

The following can be derived from the previous proposition.

Proposition 3.4.15. Let \mathfrak{B} be an ω -categorical structure. Then a relation R has a quantifier-free definition in \mathfrak{B} if and only if R is preserved by all partial automorphisms of \mathfrak{B} , i.e., preserved by isomorphisms between induced substructures of \mathfrak{B} .

PROOF. If a relation R is preserved by all partial endomorphisms of \mathfrak{B} , then it is in particular preserved by all endomorphisms of \mathfrak{B} . By Theorem 3.4.7, R has an existential positive definition in \mathfrak{B} . We can therefore use Proposition 3.4.14 in the same way as we used model-theoretic preservation theorems to prove Theorem 3.4.7 to conclude the argument.

3.5. Existential Positive Completion

It might be that the same CSP can be formulated with different templates. Consider for example the random graph $(\mathbb{V}; E)$, which has exactly the same CSP as the template $(\mathbb{N}; \{(x,y) \mid x \neq y\})$. Recall that two structures \mathfrak{B} , \mathfrak{C} have the same CSP if and only if they have the same existential positive theory T, or equivalently, if they have the same universal negative theory S. Moreover, any model of $T \cup S$ has the same CSP as \mathfrak{B} and \mathfrak{C} . The topic of this section is how to produce a model \mathfrak{B} of $T \cup S$ such that \mathfrak{B} has many good properties for studying CSP(\mathfrak{B}). Good candidates for such models \mathfrak{B} are existential-positively closed models, which will be introduced here. Much of the material presented here is analogous to the classical facts about existential completion, which we briefly review in Section 3.5.1. Existential positive completion is discussed in Section 3.5.2. The results in this section have been published in [35].

3.5.1. Existential Completion. Let T be a first-order theory. A model \mathfrak{A} of T is existentially closed for T (we also say that \mathfrak{A} is an existentially closed model of T) if $\mathfrak{A} \models \phi(\bar{a})$ for any embedding e from \mathfrak{A} into another model \mathfrak{B} of T, any tuple \bar{a} from A, and any primitive formula ϕ with $\mathfrak{B} \models \phi(e(\bar{a}))$.

To construct existentially closed models of T, we use unions of elementary chains (see Section 2.1.3). A first-order theory T is *inductive* if the union of every chain of models of T is also a model of T. Note that by Proposition 2.5.3, when T is a $\forall \exists$ -theory, then T is inductive. The following lemma implies that if T is inductive, then it has an existentially closed model. For a proof, see [120], or the proof of Lemma 3.5.3 below, which is very similar.

LEMMA 3.5.1 (Corollary 7.2.2 in [120]). Let T be an inductive theory and let κ be an infinite cardinal. Then any model $\mathfrak A$ of T of cardinality at most κ embeds into an existentially closed model $\mathfrak B$ of T of cardinality at most κ .

3.5.2. Existential positive completion. Again, let T be a first-order theory.

DEFINITION 3.5.2. A model \mathfrak{A} of T is existential-positively closed for T (or short an epc model of T) if $\mathfrak{A} \models \phi(\bar{a})$ for any homomorphism h from \mathfrak{A} into another model \mathfrak{B} of T, any tuple \bar{a} from A, and any existential positive formula ϕ with $\mathfrak{B} \models \phi(h(\bar{a}))$.

Note that we can equivalently replace 'existential positive' by 'primitive positive' in the previous definition.

To show the existence of epc models we apply the *direct limit* construction from Section 2.1.3.

LEMMA 3.5.3 (from [35]; also see [20]). Let T be a $\forall \exists^+ \tau$ -theory and let $\kappa = \max(\omega, |\tau|)$. Then any model $\mathfrak A$ of T of cardinality at most κ admits a homomorphism to an epc model $\mathfrak B$ of T of cardinality at most κ .

PROOF. Set $\mathfrak{B}_0 := \mathfrak{A}$. Having constructed \mathfrak{B}_i of cardinality at most κ , for $i < \omega$, let $\{(\phi_{\alpha}, \bar{a}_{\alpha}) \mid \alpha < \kappa\}$ be an enumeration of all pairs (ϕ, \bar{a}) where ϕ is existential positive with free variables x_1, \ldots, x_n , and \bar{a} is an n-tuple from B_i . We construct a sequence $(\mathfrak{B}_i^{\alpha})_{0 \leq \alpha < \kappa}$ of models of T of cardinality at most κ and a coherent sequence $(f_i^{\mu,\alpha})_{0 \leq \mu < \alpha < \kappa}$ where $f_i^{\mu,\alpha}$ is a homomorphism from \mathfrak{B}_i^{μ} to \mathfrak{B}_i^{α} , as follows.

 $(f_i^{\mu,\alpha})_{0 \leq \mu < \alpha < \kappa}$ where $f_i^{\mu,\alpha}$ is a homomorphism from \mathfrak{B}_i^{μ} to \mathfrak{B}_i^{α} , as follows. Set $\mathfrak{B}_i^0 = \mathfrak{B}_{i-1}$. Now let $\alpha = \beta + 1 < \kappa$ be a successor ordinal. Let \bar{b}_{β} be the image of \bar{a}_{β} in \mathfrak{B}_i^{β} under $f_i^{0,\beta}$. If there is a model \mathfrak{C} of T and a homomorphism $h \colon \mathfrak{B}_i^{\beta} \to \mathfrak{C}$ such that $\mathfrak{C} \models \phi_{\beta}(h(\bar{b}_{\beta}))$, then by the theorem of Löwenheim-Skolem there is also a model \mathfrak{C}' of cardinality at most κ of T and a homomorphism $h' \colon \mathfrak{B}_i^{\beta} \to \mathfrak{C}'$ such that $\mathfrak{C}' \models \phi_{\beta}(h'(\bar{b}_{\beta}))$. Set $\mathfrak{B}_i^{\alpha} \coloneqq \mathfrak{C}'$ and $f_i^{\mu,\alpha} \coloneqq h' \circ f_i^{\mu,\beta}$ for all $\mu < \alpha$. Otherwise, if there is no such model \mathfrak{C} , we set $\mathfrak{B}_i^{\alpha} := \mathfrak{B}_i^{\beta}$ and $f_i^{\beta,\alpha} := \mathrm{id}$ (the identity mapping) and $f_i^{\mu,\alpha} := f_i^{\mu,\beta}$. Finally, for limit ordinals $\alpha < \kappa$, set $\mathfrak{B}_i^{\alpha} := \lim_{\mu < \alpha} \mathfrak{B}_i^{\mu}$ and let $f_i^{\mu,\alpha}$ be the corresponding limit homomorphism from \mathfrak{B}_i^{μ} to \mathfrak{B}_i^{α} .

Let \mathfrak{B}_i be $\lim_{\alpha<\kappa}\mathfrak{B}_i^{\alpha}$ and let $g_i\colon \mathfrak{B}_{i-1}\to \mathfrak{B}_i$ be the limit homomorphism mapping each element of $\mathfrak{B}_{i-1}=\mathfrak{B}_i^0$ to its equivalence class in \mathfrak{B}_i . In the natural way, the g_i give raise to a coherent sequence of homomorphisms, and by Proposition 2.5.5, $\mathfrak{B}:=\lim_{i<\omega}\mathfrak{B}_i$ is a model of T; let $h_i\colon \mathfrak{B}_i\to \mathfrak{B}$ for $i<\omega$ be the corresponding limit homomorphisms.

The structure \mathfrak{B} is epc in T. To verify this, let g be a homomorphism from \mathfrak{B} to a model \mathfrak{C} of T, and suppose that there is a tuple \bar{b} over B and an existential positive formula ϕ such that $\mathfrak{C} \models \phi(g(\bar{b}))$. Then there is an $i < \omega$ and an $\bar{a} \in B_i$ such that $h_i(\bar{a}) = \bar{b}$. Then $g \circ h_i$ is a homomorphism from \mathfrak{B}_i to \mathfrak{C} , and by construction we have that $\mathfrak{B}_{i+1} \models \phi(g_{i+1}(\bar{a}))$. Note that $h_{i+1} \circ g_{i+1} = h_i$. Thus, since h_{i+1} preserves existential positive formulas, we also have that $\mathfrak{B} \models \phi(\bar{b})$, which is what we had to show.

For an equivalent characterization of existentially closed models in terms of maximal pp-types (Proposition 3.5.5), we need the following lemma, a close relative of Theorem 10.3.1 in [119].

LEMMA 3.5.4. Let $\mathfrak A$ and $\mathfrak B$ be τ -structures, where $\mathfrak B$ is pp-|A|-saturated. Suppose that $\mu < |A|$ and that f is a mapping from $\{a_{\alpha} \mid \alpha < \mu\} \subseteq A$ to B such that all $pp-(\tau \cup \{c_{\alpha} \mid \alpha < \mu\})$ -sentences true on $(\mathfrak A, (a_{\alpha})_{\alpha < \mu})$ are true on $(\mathfrak B, (f(a_{\alpha}))_{\alpha < \mu})$. Then f can be extended to a homomorphism from $\mathfrak A$ to $\mathfrak B$.

PROOF. Let $(a'_{\alpha})_{\alpha<|A|}$ well-order A such that $\{a'_{\alpha} \mid \alpha < \mu\} = \{a_{\alpha} \mid \alpha < \mu\}$ (there is the implicit and harmless assumption that $(a_{\alpha})_{\alpha<\mu}$ contains no repetitions). Set $(b_{\alpha})_{\alpha<\mu} := (f(a_{\alpha}))_{\alpha<\mu}$.

We will construct by transfinite induction on β (up to |A|) a sequence $(b_{\alpha})_{\alpha<\beta}$ such that we maintain the inductive hypothesis

- (*) all pp- $(\tau \cup \{c_{\alpha} \mid \alpha < \beta\})$ -sentences true on $(\mathfrak{A}; (a'_{\alpha})_{\alpha < \beta})$ are true on $(\mathfrak{B}; (b_{\alpha})_{\alpha < \beta})$.
 - (Base Case.) $\beta = \mu$. Follows from the hypothesis of the lemma.
 - (Inductive Step. Limit ordinals.) $\beta = \lambda$. Property (*) holds, since a sentence can only mention a finite collection of constants, whose indices must all be less than some $\gamma < \lambda$.
 - (Inductive Step. Successor ordinals.) $\beta = \gamma^+ < |A|$. Set

$$\Sigma := \left\{ \phi(x) \mid \phi \text{ is a pp-}(\tau \cup \{c_{\alpha} \mid \alpha < \gamma\}) \text{-formula such that} \right.$$
$$\left(\mathfrak{A}; (a'_{\alpha})_{\alpha < \gamma}\right) \models \phi(a'_{\gamma}) \right\}.$$

By (*), for every $\phi \in \Sigma$, $(\mathfrak{B}; (b_{\alpha})_{\alpha < \gamma}) \models \exists x.\phi(x)$. By compactness, since Σ is closed under conjunction, we have that Σ is a pp-1-type of $(\mathfrak{B}; (b_{\alpha})_{\alpha < \gamma})$. Then Σ is realized by some element $b_{\gamma} \in B$ because \mathfrak{B} is pp-|A|-saturated. By construction we maintain that all pp- $(\tau \cup \{c_{\alpha} \mid \alpha < \gamma^{+}\})$ -sentences true on $(\mathfrak{A}; (a'_{\alpha})_{\alpha < \gamma^{+}})$ are true on $(\mathfrak{B}; (b_{\alpha})_{\alpha < \gamma^{+}})$.

The result follows by reading f as the function that maps a'_{α} to b_{α} for all $\alpha < |A|$. \square

PROPOSITION 3.5.5. Let T be a theory, and let \mathfrak{A} be a model of T. Then \mathfrak{A} is epc for T if and only if every complete pp-n-type of \mathfrak{A} is a maximal pp-type of T.

PROOF. (Forwards.) Suppose $p(x_1, \ldots, x_n)$ is an pp-n-type, realized in $\mathfrak A$ by the tuple (a_1, \ldots, a_n) . Let c_1, \ldots, c_n be new constant symbols that denote a_1, \ldots, a_n in $\mathfrak A$. Let $\phi(x_1, \ldots, x_n)$ be a primitive positive formula such that $T \cup p(c_1, \ldots, c_n) \cup$

 $\{\phi(c_1,\ldots,c_n)\}\$ has a model $(\mathfrak{C};c_1,\ldots,c_n)$. Now, let $(\mathfrak{C}_{\mathrm{sat}};c_1,\ldots,c_n)$ be an |A|-saturated model of $\mathrm{Th}(\mathfrak{C};c_1,\ldots,c_n)$; such a model always exists by Theorem 2.6.1. Clearly $(\mathfrak{C}_{\mathrm{sat}};c_1,\ldots,c_n)$ is pp-|A|-saturated, and all primitive positive formulas true on $(\mathfrak{A},c_1,\ldots,c_n)$ are true on $(\mathfrak{C}_{\mathrm{sat}};c_1,\ldots,c_n)$. By Lemma 3.5.4, there is a homomorphism h from $(\mathfrak{A},c_1,\ldots,c_n)$ to $(\mathfrak{C}_{\mathrm{sat}})$. Now, since $\phi(c_1,\ldots,c_n)$ holds on $\mathfrak{C}_{\mathrm{sat}}$ and \mathfrak{A} is epc for T, we find that $(\mathfrak{A},c_1,\ldots,c_n)\models\phi(c_1,\ldots,c_n)$, and conclude that p is a maximal pp-type of T.

(Backwards.) Take $\mathfrak{B} \models T$, $h \colon \mathfrak{A} \to \mathfrak{B}$ a homomorphism, \bar{a} a tuple of elements of \mathfrak{A} , and $\phi(x_1, \ldots, x_n)$ a primitive positive formula such that $\mathfrak{B} \models \phi(h(\bar{a}))$. Let p be the pp-type of \bar{a} in \mathfrak{A} . Since \mathfrak{B} is a model of T and h preserves all primitive positive formulas, it follows that $T \cup p \cup \{\phi\}$ is satisfiable. By maximality of p, we have that $\phi \in p$, and therefore $\mathfrak{A} \models \phi(\bar{a})$.

We close this section with an observation that will be needed later on.

Lemma 3.5.6. The class of all epc models of a theory T is closed under direct limits.

PROOF. Suppose that $\mathfrak{A} = \lim_{\lambda < \kappa} \mathfrak{A}_{\lambda}$ for a sequence $(\mathfrak{A}_{\lambda})_{\lambda < \kappa}$ of epc models of T, \bar{a} is a tuple from \mathfrak{A} , ϕ an existential positive formula, and h is a homomorphism from \mathfrak{A} into another model of T such that $\mathfrak{B} \models \phi(h(\bar{a}))$. Then there exists a $\lambda < \kappa$ such that $\bar{a} = g_{\lambda}(\bar{a}')$ for \bar{a}' from \mathfrak{A}_{λ} (where g_{λ} is as in the definition of direct limits). Note that $h \circ g_{\lambda}$ is a homomorphism from \mathfrak{A}_{λ} to \mathfrak{B} , and since \mathfrak{A}_{λ} is an epc model of T, $\mathfrak{A}_{\lambda} \models \phi(\bar{a}')$. Since g_{λ} preserves existential positive formulas, we thus also have that $\mathfrak{A} \models \phi(\bar{a})$.

3.6. Quantifier-elimination, Model-completeness, Cores

This section is concerned with structures \mathfrak{B} where various forms of syntactic restrictions of first-order logic have equal expressive power. In particular, we consider the situation that in \mathfrak{B} every first-order formula is equivalent to

- a quantifier-free formula (Section 3.6.1),
- an existential formula (Section 3.6.2),
- an existential positive formula (Sections 3.6.3 and 3.6.4).

For ω -categorical structures, such a definability collaps translates nicely into a property of the operations that preserve \mathfrak{B} , using the preservation theorems from Section 3.4. A survey picture is given in Figure 3.2. All these collapse results will be useful when studying the complexity of CSPs. For example, these results clarify when the so-called *constraint entailment problem for* \mathfrak{A} can be reduced to the constraint satisfaction problem for \mathfrak{A} (see Section 3.6.4).

We also develop a theory that can be viewed as a positive variant of the classical theory of model-completeness and model companions (Section 3.6.4 and 3.6.5). This allows us to clarify the question which CSPs can be formulated with an ω -categorical template (Section 3.6.6).

3.6.1. Quantifier-elimination. We say that a τ -structure $\mathfrak A$ admits quantifier elimination if for every first-order τ -formula there exists an equivalent quantifier-free τ -formula.

In this context, our assumption that we allow \bot as a first-order formula (denoting the empty 0-ary relation) becomes relevant; Hodges [119] does not make this assumption, and therefore has to distinguish between *quantifier-elimination* and what he calls *quantifier-elimination for non-sentences*. We will later often make use of the following fact.

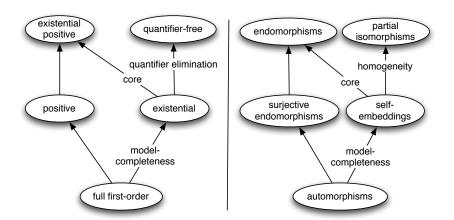


FIGURE 3.2. Various forms of definability (left side), ordered by relative strength, and the corresponding class of operations (right side). The labels on the arrows indicate the condition on the structure when the corresponding two forms of definability coincide (left side), and correspondingly when one set of operations locally generates the other (right side).

LEMMA 3.6.1 (Statement 2.22 in [70]). An ω -categorical structure \mathfrak{B} admits quantifier elimination if and only if it is homogeneous.

PROOF. By Theorem 3.1.4, the orbit of a k-tuple of elements of \mathfrak{B} is first-order definable. Suppose that \mathfrak{B} has quantifier-elimination. Then two k-tuples $\bar{a} = (a_1, \ldots, a_k)$ and $\bar{b} = (b_1, \ldots, b_k)$ are in the same orbit if and only if the mapping that sends a_i to b_i , for $1 \leq i \leq n$, is an isomorphism between the structures induced by $\{a_1, \ldots, a_k\}$ and by $\{b_1, \ldots, b_k\}$. This proves homogeneity.

Now suppose that \mathfrak{B} is homogeneous, and let $\phi(x_1,\ldots,x_k)$ be a first-order formula. By the theorem of Ryll-Nardzewski (Theorem 3.1.4), there are finitely many orbits O_1,\ldots,O_m of orbits of k-typles that satisfy ϕ . Clearly, it suffices to show that each of those orbits can be defined by a quantifier-free formula. Let $a \in B^k$ be such that $\mathfrak{B} \models \phi(a)$. We claim that the set of quantifier-free formulas that hold on (a_1,\ldots,a_k) defines the orbit of a over \mathfrak{B} . To see this, let (b_1,\ldots,b_k) be another k-tuple that satisfies the same quantifier-free formulas as (a_1,\ldots,a_k) . Then the mapping that sends a_i to b_i is a partial isomorphism, and by homogeneity can be extended to an automorphism of \mathfrak{B} . Since automorphisms preserve first-order formulas, (b_1,\ldots,b_k) also satisfies ϕ , which proves the claim.

3.6.2. Model-Completeness. The purpose of this section is to recall classical results about model-completeness; they inspired the new results of the next section about model-complete cores. A theory T is model-complete if every embedding between models of T is elementary, i.e., preserves all first-order formulas. There are several equivalent characterizations of model-completeness, stated in Theorem 3.6.2 below.

Theorem 3.6.2 (Theorem 7.3.1 in [120]). Let T be a theory. Then the following are equivalent.

- (1) T is model-complete.
- (2) Every model of T is an existentially closed model of T.

(3) Every first-order formula is equivalent to an existential formula modulo T.

For the proof, we refer to [120]; but note that the theorem has a positive variant (Theorem 3.6.11 below) with an analogous proof that will be presented in full length.

EXAMPLE 3.6.3. The structure $(\mathbb{Q}_0^+;<)$, where \mathbb{Q}_0^+ denotes the non-negative rational numbers, is not model-complete, because the self-embedding $x \mapsto x+1$ of $(\mathbb{Q}_0^+;<)$ does not preserve the formula $\phi(x) = \forall y \, (y>x \Rightarrow \exists z \, (x < z \land z < y))$ (which is satisfied only by 0).

When $\mathfrak A$ is not model-complete, we can sometimes find a model-complete structure $\mathfrak B$ that satisfies the same universal first-order sentences as $\mathfrak A$.

Definition 3.6.4. A theory U is a model companion of a theory T if

- *U* is model-complete;
- Every model of U embeds into a model of T; and
- every model of T embeds into a model of U.

Note that the last two conditions in this definition are equivalent to saying that U and T imply exactly the same existential sentences (equivalently, the same universal sentences); the proof is analogous to the one of Proposition 2.4.5.

If T has a model-companion U, then U is unique up to equivalence of theories.

THEOREM 3.6.5 (Theorem 7.3.6. in [120]). For any two model-companions U_1, U_2 of a theory T we have that $U_1 \vdash U_2$ and $U_2 \vdash U_1$.

The following theorem by Simmons [188] will not be used in this thesis; however, it has an existential positive version, Theorem 3.6.23 below, which has important consequences for the study of the CSP. Recall the *joint embedding property*, which has been defined for classes of structures in Section 3.2; a theory T has the joint embedding property (JEP) if for any two models $\mathfrak{B}_1, \mathfrak{B}_2$ of T there exists a model \mathfrak{C} of T that embeds both \mathfrak{B}_1 and \mathfrak{B}_2 .

Theorem 3.6.6 (from [188]). Let T be a theory with the JEP. Then the following are equivalent.

- T has an ω -categorical model companion.
- For every n, T has finitely many maximal existential n-types.

In particular, every ω -categorical theory has an ω -categorical model companion [181].

The consequence stated for ω -categorical theories T at the end of Theorem 3.6.6 is an earlier result by Saracino [181], and clearly follows from the first part.

We say that a structure \mathfrak{A} is model-complete if and only if the first-order theory $\operatorname{Th}(\mathfrak{A})$ of \mathfrak{A} is model-complete. As we see below, for ω -categorical structures \mathfrak{A} model-completeness of \mathfrak{A} can be translated into a property of the self-embedding monoid of \mathfrak{A} , and into a property concerning the axiomatization of $\operatorname{Th}(\mathfrak{A})$. In the following theorem, the equivalence of (1) and (4) can be found in [51]. The implication from (5) to (1) has been observed in [41].

Theorem 3.6.7. Let \mathfrak{B} be ω -categorical. Then the following are equivalent.

- (1) The structure \mathfrak{B} is model-complete.
- (2) $Th(\mathfrak{B})$ is equivalent to a $\forall \exists$ -theory.
- (3) \mathfrak{B} has a homogeneous expansion by relations R_1, R_2, \ldots such that both the R_i and their complements have existential definitions in \mathfrak{B} .
- (4) Every self-embedding of \mathfrak{B} is locally generated by the automorphisms of \mathfrak{B} .
- (5) Every self-embedding of \mathfrak{B} is locally invertible (see Section 3.4.3).

PROOF. The implication from (1) to (2) holds for all structures \mathfrak{B} (we do not need ω -categoricity; see e.g. Theorem 7.3.3 in [120]). The reverse direction is a direct consequence of a result known as Lindstr"om's test (Theorem 7.3.4. in [120]).

We now prove $(1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$.

Suppose that (1) holds. By Theorem 3.6.2, every first-order definable relation has an existential definition in \mathfrak{B} . Hence, when we expand \mathfrak{B} by all existentially definable relations, every first-order formula has a quantifier-free definition. So Lemma 3.6.1 shows that the expansion is homogeneous.

Now suppose that (3) holds. We claim that every self-embedding e of \mathfrak{B} is in the closure of the automorphisms of \mathfrak{B} . The restriction e' of e to a finite subset S of the domain of \mathfrak{B} is an isomorphism between finite induced substructures of \mathfrak{B} , and also an isomorphism between the expansion of \mathfrak{B} by all existentially definable relations. Homogeneity of this expansion implies that e' can be extended to an automorphism of \mathfrak{B} , which proves the claim.

The implication from (4) to (5) is trivial. Finally, the implication from (5) to (1) it is a direct consequence of Theorem 3.4.12.

Note that all finite structures are model-complete: self-embeddings of $\mathfrak B$ are automorphisms, and hence they are elementary. Every relation that is first-order definable in a finite structure also has an existential definition.

Using the concept of model completeness, we can restate Theorem 3.2.11, clarifying in which sense the structure \mathfrak{B} constructed in Theorem 3.2.11 is unique. In the following, τ is a finite relational signature.

THEOREM 3.6.8 (a variant of Theorem 3.2.11). Let \mathcal{N} be a finite set of finite connected τ -structures. Then there is a model-complete τ -structure \mathfrak{B} whose age is the class of all finite \mathcal{N} -free structures. The structure \mathfrak{B} is unique up to isomorphism.

PROOF. Theorem 3.2.11 states the existence of an ω -categorical \mathcal{N} -free structure which is universal for the class of all countable \mathcal{N} -free structures which has a homogeneous expansion by primitive positive definable relations. By $(3) \Rightarrow (1)$ in Theorem 3.6.7, the structure \mathfrak{B} is indeed model-complete. We have to show that every model-complete structure \mathfrak{C} with the same age as \mathfrak{B} is isomorphic to \mathfrak{B} . Let T be the first-order theory of \mathfrak{C} . Since \mathfrak{B} and \mathfrak{C} have the same age, S and T imply the same existential sentences. By Theorem 3.6.5, S and T are equivalent theories. By ω -categoricity of T, \mathfrak{B} and \mathfrak{C} are isomorphic. \square

- **3.6.3.** Cores. We have already encountered the concept of a *core* of a finite structure in Section 1.1. To recall, a *core* is a structure \mathfrak{B} such that all endomorphisms of \mathfrak{B} are embeddings, and a structure \mathfrak{B} is a *core* of \mathfrak{A} if \mathfrak{B} is a core and homomorphically equivalent to \mathfrak{A} . The concept of the core of a finite relational structure plays an important role in the classification program for finite-domain CSPs. Three crucial properties of finite cores are:
 - every finite structure \mathfrak{A} has a core \mathfrak{B} (Proposition 1.1.10);
 - all core structures \mathfrak{B} of \mathfrak{A} are isomorphic (Proposition 1.1.10);
 - orbits of k-tuples in finite cores \mathfrak{B} are primitive positive definable (Proposition 1.2.9).

Also for every *infinite* structure \mathfrak{A} there is a core \mathfrak{B} such that $CSP(\mathfrak{A}) = CSP(\mathfrak{B})$. This follows from Lemma 3.5.3 and the following proposition.

Proposition 3.6.9. If \mathfrak{B} is an epc model for its universal negative theory, then \mathfrak{B} is a core.

PROOF. Suppose \mathfrak{B} is epc for its universal negative theory T, and let h be an endomorphism of \mathfrak{B} . By epc, for b_1, \ldots, b_k in B, if $\mathfrak{B} \models R(h(b_1), \ldots, h(b_k))$ or $\mathfrak{B} \models (h(b_1) = h(b_2))$, then $\mathfrak{B} \models R(b_1, \ldots, b_k)$ or $\mathfrak{B} \models (b_1 = b_2)$, respectively. It follows that h is an embedding.

There are many equivalent definitions of when a *finite* structure is a core: for example, a finite structure is a core if and only if all endomorphisms are surjective, or injective, or bijective, or all endomorphisms are automorphisms. For infinite structures (even when they are ω -categorical), these definitions are in general not equivalent, see [17, 18, 25]. As we will see in this section, our definition of cores is the most appropriate definition in many contexts, in particular in the context of constraint satisfaction for ω -categorical templates.

EXAMPLE 3.6.10. The structure $(\mathbb{Q};<)$ is easily seen to be a core: every endomorphism of $(\mathbb{Q};<)$ must be injective, and must be strong. In contrast, the random graph $(\mathbb{V};E)$ is not a core. By the defining property of the random graph, $(\mathbb{V};E)$ contains arbitrarily large finite cliques. By Lemma 3.1.5, it even has an infinite clique as a subgraph. Therefore, $(\mathbb{V};E)$ has endomorphisms with the property that they map pairs of non-adjacent vertices to pairs of adjacent vertices, and thus is not a core.

Before we prove general results about existence and uniqueness of cores, we state important properties of cores and model-complete cores for ω -categorical structures that follow in a straightforward way from previous facts³.

Theorem 3.6.11. Let \mathfrak{B} be ω -categorical. Then \mathfrak{B} is a core if and only if every existential formula is equivalent to an existential positive formula over \mathfrak{B} . Moreover, the following are equivalent.

- (1) **B** is a model-complete core;
- (2) \mathfrak{B} has a homogeneous expansion by relations R_1, R_2, \ldots such that the relations R_i and their complements have existential positive definitions;
- (3) Every first-order formula is equivalent to an existential positive one over \mathfrak{B} ;
- (4) The orbits of n-tuples in \mathfrak{B} are primitive positive definable in \mathfrak{B} ;
- (5) The automorphisms locally generate the endomorphisms of \mathfrak{B} .

PROOF. The first statement is straightforward from Theorem 3.4.7. To prove the equivalence of (1)-(5), we show implications in cyclic order.

For the implication from (1) to (2), consider the expansion of \mathfrak{B} by all relations with an existential positive definition in \mathfrak{B} . Since all endomorphisms of \mathfrak{B} also preserve the complements of those relations, the complements also have an existential positive definition by Theorem 3.4.7, and hence the expansion is of the desired type. The orbits of n-tuples of \mathfrak{B} (and its expansion) are by assumption preserved by all endomorphisms of \mathfrak{B} , and therefore have an existential positive definition in \mathfrak{B} , and thus a quantifier-free definition in the expansion. It follows that the expansion is homogeneous.

For the implication from (2) to (3), let ϕ be a first-order formula. Then ϕ has in the homogeneous expansion of \mathfrak{B} a quantifier-free definition ψ ; assume without loss of generality that ψ is written in conjunctive normal form. If we replace all positive literals that involve relations R_i by their existential positive definition in \mathfrak{B} , and all negative literals that involve relations R_i by the existential positive definition of the complement of R_i in \mathfrak{B} , we arrrive at an equivalent formula which is existential positive in the signature of \mathfrak{B} .

³Yet another characterization of when an ω -categorical structure is a model-complete core can be found in Proposition 3.6.20.

For the implication from (3) to (4), let O be an orbit of n-tuples in \mathfrak{B} . By Theorem 3.3.8, O has a first-order definition. Assuming (2), O even has an existential positive definition. Note that every existential positive formula can be written as a disjunction of primitive positive formulas, so let ϕ be such a definition of O. We can also assume without loss of generality that none of the disjuncts in ϕ implies another (otherwise, we simply omit it). Since O is a minimal first-order definable relation, ϕ can only contain a single disjunct, and therefore is primitive positive.

- (4) implies (5). Assume (4), and let e be an endomorphism of \mathfrak{B} . To show that e is locally generated by the automorphism of \mathfrak{B} , let t be a finite tuple of elements of \mathfrak{B} . We have to show that there is an automorphism α of \mathfrak{B} such that $e(t) = \alpha(t)$. The orbit of t is primitive positive definable, and hence preserved by e. So e(t) is in the same orbit as t, and we are done.
- (5) implies (1). Suppose (5), that is, suppose that all endomorphisms are generated by the automorphisms of \mathfrak{B} . Since the automorphisms preserve all first-order formulas in \mathfrak{B} , the same is true for the endomorphisms of \mathfrak{B} , by Proposition 3.4.8. \square

The fact that in ω -categorical model-complete cores the orbits of n-tuples are primitive positive definable is one of the three key facts for finite cores \mathfrak{B} that we have mentioned above. The other two facts, existence and uniqueness of model-complete cores for ω -categorical structures, follow from more general theorems that apply not only to ω -categorical structures, as we will see in Section 3.6.6.

3.6.4. Core Theories. A theory T is called a *core theory* if every homomorphism between models of T is an embedding. Note that a finite or ω -categorical structure \mathfrak{B} is a core if and only if it has a core theory (for ω -categorical \mathfrak{B} , this is an easy consequence of the Löwenheim-Skolem theorem – Theorem 2.3.3).

PROPOSITION 3.6.12. Let T be a first-order τ -theory. Then T is a core theory if and only if every existential formula is equivalent to an existential positive formula.

PROOF. First assume that T is a core theory, and let ϕ be an existential formula. Then ϕ is preserved by all embeddings between models of T. Since all homomorphism between models of T are embeddings, ϕ is also preserved by all homomorphisms between models of T. Hence, Theorem 3.4.3 implies that ϕ is equivalent modulo T to an existential positive formula. The converse implication is trivial.

We would like to point out an interesting corollary for CSPs. Let \mathfrak{B} be a structure with finite relational signature τ . The constraint entailment problem for \mathfrak{B} is the following computational problem. The input consists of a primitive positive τ -formula ϕ , and a single atomic τ -formula ψ , both ϕ and ψ with free variables x_1, \ldots, x_n . The question is whether ϕ implies (entails) ψ in \mathfrak{B} , i.e., whether

$$\mathfrak{B} \models \forall x_1, \dots, x_n \ (\phi \Rightarrow \psi) \ .$$

COROLLARY 3.6.13. Let τ be a finite relational signature, and let \mathfrak{B} be a τ -structure whose first-order theory T is a core theory. Then there is a polynomial-time Turing reduction from the constraint entailment problem for \mathfrak{B} to $\mathrm{CSP}(\mathfrak{B})$.

PROOF. Let ϕ, ψ be an input to the constraint entailment problem for \mathfrak{B} . Since T is a core theory, $\neg \psi$ is by Proposition 3.6.12 equivalent to an existential positive τ -formula, and hence equivalent to a disjunction $\psi_1 \lor \cdots \lor \psi_m$ of primitive positive formulas. Since the signature τ is finite, we can consider the size of this disjunction is bounded by a constant, for all possible inputs. Then ϕ entails ψ if and only if for all $1 \le i \le m$, we have that $\exists x_1, \ldots, x_k \ (\phi \land \psi_i)$ is false in \mathfrak{B} . Like this we have reduced the entailment problem to solving a constant number of constraint satisfaction problems for the structure \mathfrak{B} .

If we combine the assumption that T is a core theory with the assumption that it is model-complete, we arrive at Theorem 3.6.14. Its proof closely follows the proof of Theorem 7.3.1 in [120].

Theorem 3.6.14. Let T be a first-order theory over signature τ . Then the following are equivalent.

- (1) T is a model-complete core theory.
- (2) Every model of T is an existential positive complete model of T.
- (3) If $\mathfrak{A}, \mathfrak{B}$ are models of T and h is a homomorphism from \mathfrak{A} to \mathfrak{B} then there are an elementary extension \mathfrak{C} of \mathfrak{A} and an embedding g of \mathfrak{B} into \mathfrak{C} such that gh is the identity on \mathfrak{A} .
- (4) Every first-order formula is equivalent to an existential positive formula modulo T.

PROOF. (1) implies (2) is immediate from the definition of epc models: if $\mathfrak A$ and $\mathfrak B$ are models of T and $h\colon A\to B$ is a homomorphism from $\mathfrak A$ to $\mathfrak B$, then h must be an embedding since T is a core theory, and in fact must be elementary since T is model-complete. Hence, for every tuple $\bar a$ from A and any existential positive formula ϕ such that $h(\bar a)$ satisfies ϕ we have that $\bar a$ also satisfies ϕ .

- (2) implies (3). Assume (2). Let \mathfrak{A} and \mathfrak{B} be models of T, and let h be a homomorphism from \mathfrak{A} to \mathfrak{B} . Choose \bar{a} to be a vector that enumerates the elements of \mathfrak{A} . Since \mathfrak{A} is an epc model of T, h is an embedding. Hence, every existential sentence that holds in $(\mathfrak{B}, h(\bar{a}))$ also holds in (\mathfrak{A}, \bar{a}) . Proposition 2.4.3 now directly implies (3).
- (3) implies (4). We first claim that if (3) holds, then every homomorphism between models of T preserves all universal τ -formulas. For if h is a homomorphism of $\mathfrak A$ into $\mathfrak B$, $\bar a$ a tuple from A and $\phi(\bar x)$ a universal τ -formula such that $\mathfrak A \models \phi(\bar a)$, then taking $\mathfrak C$ and g as in (3) we have $\mathfrak C \models \phi(g(h(\bar a)))$ and so $\mathfrak B \models \phi(h(\bar a))$ since ϕ is a universal formula. This proves the claim. It follows from Theorem 3.4.3 that all universal τ -formulas are equivalent to existential positive τ -formulas.

To finally prove (4), let $\phi(\bar{x})$ be any first-order τ -formula, wlog. in prenex normal form. By a simple induction on the number of quantifier-blocks we can transform ϕ to an existential formula, using the fact that the innermost quantifier block is either existential or universal, and can therefore be transformed into an existential formula (see Theorem 7.3.1 in [120]). Finally, existential τ -formulas are preserved by homomorphisms between models of T, since such homomorphisms must be embeddings. Hence, the entire formula is even equivalent to an existential positive formula by Theorem 3.4.3.

(4) implies (1). Existential positive formulas are preserved by homomorphisms between models of T.

From this we obtain a positive version of a fact known as Lindstr"om's test (Theorem 7.3.4 in [120]).

PROPOSITION 3.6.15. Let T be a λ -categorical τ -theory, for $\lambda \geq |\tau|$, which has no finite models, and whose unique model of cardinality λ is epc for T. Then T is a model-complete core theory.

PROOF. We prove that every model of T is an epc model of T and use Theorem 3.6.14. So let \mathfrak{A} and \mathfrak{B} be two models of T and let h be a homomorphism from \mathfrak{A} to \mathfrak{B} . Let \bar{a} be a tuple such that $\mathfrak{B} \models \phi(h(\bar{a}))$ and suppose for contradiction that $\mathfrak{A} \not\models \phi(\bar{a})$. Then we can put those two structures into a new 2-sorted structure (comprising \mathfrak{A} , \mathfrak{B} , and the homomorphism h between them) with first-order theory T', and apply the Löwenheim-Skolem theorem (Theorem 2.3.3; here we use the assumption

that $\lambda \geq |\tau|$) to produce a countable model of T' (where both sorts have the same cardinality since T has no finite models). By applying Löwenheim-Skolem again, this time to T' augmented by sentences expressing a bijection between the two sorts over a signature expanded by a new function symbol, we obtain a two-sorted model of T' where each sort has cardinality λ , inducing structures $\mathfrak C$ and $\mathfrak D$, respectively. By assumption there exists an epc model of cardinality λ , and by λ -categoricity $\mathfrak C$ is an epc model of T. This contradicts the fact that we can express in T' that $\mathfrak A$ is not an epc model of T.

PROPOSITION 3.6.16. Let T be a model-complete core theory. Then T is equivalent to a $\forall \exists^+$ -theory.

PROOF. This is an immediate consequence of Theorem 3.4.6 (where the theory denoted by T in Theorem 3.4.6 is empty and Φ from Theorem 3.4.6 equals the theory T from the statement here) because for any sequence $(\mathfrak{B}_i)_{i<\kappa}$ of models of T with homomorphisms $g_{ij}: \mathfrak{B}_i \to \mathfrak{B}_j$, the g_{ij} are elementary. By the Tarski-Vaught theorem (Theorem 2.5.2), we have that $(\lim_{i<\kappa}\mathfrak{B}_i) \models T$.

3.6.5. Core Companions. In this section we study when we can pass from a theory T to a model-complete core theory T' that has the same CSP.

Definition 3.6.17. Let T be a first-order τ -theory. Then a τ -theory U is called a core companion of T if

- *U* is a model-complete core theory;
- every model of U homomorphically maps to a model of T;
- ullet every model of T homomorphically maps to a model of U.

Recall from Proposition 2.4.5 that the last two items in Definition 3.6.17 are equivalent to requiring that T and U imply the same universal negative sentences.

PROPOSITION 3.6.18. Let T be a $\forall \exists^+$ -theory with signature τ . If T has a core companion U, then U is up to equivalence of theories unique, and is the theory of the class of all epc models of T.

PROOF. It suffices to show that the epc models of T are precisely the models of U. We first show that every model \mathfrak{B} of U is an epc model for T; that is, we have to show that \mathfrak{B} is a model of T, and that \mathfrak{B} is epc for T.

Since U is a core companion of T, there is a homomorphism e from \mathfrak{B} to a model \mathfrak{A} of T. The assumption that U is a core companion of T also implies that there exists a homomorphism f from \mathfrak{A} into a model \mathfrak{C} of U. Then $f \circ e$ is a homomorphism between two models of U, and since U is a model-complete core theory it must be an elementary embedding. This shows in particular that e is an embedding.

We claim that \mathfrak{B} is a model of the $\forall \exists^+$ -theory T. Let $\phi = \forall \bar{y}.\psi$ be a sentence from T where ψ is a disjunction of existential positive and negated atomic τ -formulas, and let \bar{b} be a tuple from \mathfrak{B} . Since \mathfrak{A} is a model of T and therefore satisfies $\forall y.\psi$, in particular the tuple $e(\bar{b})$ satisfies ψ . If $e(\bar{b})$ satisfies a negated atom in the disjunction ψ then \bar{b} also satisfies ψ as e is an embedding. Otherwise, $e(\bar{b})$ satisfies an existential positive formula in the disjunction ψ , and $f(e(\bar{b}))$ satisfies ϕ in \mathfrak{C} as well since f is a homomorphism. But this shows that \bar{b} satisfies ψ in \mathfrak{B} since $f \circ e$ is elementary. Since this holds for all \bar{b} , we have proven that \mathfrak{B} satisfies ϕ .

The verification that \mathfrak{B} is an epc model for T is similar, and as follows. Let g be a homomorphism from \mathfrak{B} into another model \mathfrak{A} of T, \bar{b} a tuple from \mathfrak{B} , and ϕ an existential positive formula with $\mathfrak{A} \models \phi(g(\bar{b}))$. We have to show that $\mathfrak{B} \models \phi(\bar{b})$. Again, since U is a core companion of T there exists a homomorphism h from \mathfrak{A} into a model \mathfrak{C} of U. Since U is a model-complete core theory, the mapping $h \circ g$

is elementary. Since h preserves existential positive formulas, $\mathfrak{C} \models \phi(h(g(\bar{b})))$. Since $h \circ g$ is elementary, $\mathfrak{B} \models \phi(\bar{b})$.

Conversely, we show that every epc model \mathfrak{B} of T satisfies U. By Proposition 3.6.16, U is equivalent to a $\forall \exists^+$ -theory, and thus it suffices to show that \mathfrak{B} satisfies all $\forall \exists^+$ -consequences $\forall \bar{y}.\psi(\bar{y})$ of U, where ψ is a disjunction of existential positive and negative atomic τ -formulas. Let \bar{b} be a tuple of elements of \mathfrak{B} . We have to show that $\mathfrak{B} \models \psi(\bar{b})$. Since U is a core companion, there is a homomorphism h from \mathfrak{B} to a model \mathfrak{A} of U. Since $\mathfrak{A} \models \forall \bar{y}.\psi(\bar{y})$, at least one disjunct $\theta(h(\bar{b}))$ of ψ is true in \mathfrak{A} . If θ is a negative atomic formula, then $\theta(\bar{b})$ is also true in \mathfrak{B} since h is a homomorphism. Now suppose that θ is an existential positive formula. Since U is a core companion of T, there is a homomorphism g from \mathfrak{A} to a model \mathfrak{C} of T. Since g preserves θ we have that $\mathfrak{C} \models \theta(g(h(\bar{b})))$. Now $\mathfrak{B} \models \theta(\bar{b})$, since \mathfrak{B} is an epc model of T. In both cases we can conclude that $\mathfrak{B} \models \psi(\bar{b})$.

PROPOSITION 3.6.19. Let T be a $\forall \exists^+$ -theory with signature τ . Then T has a core companion if and only if the class of epc models of T is axiomatizable by a τ -theory.

PROOF. If T has a core companion U, then Proposition 3.6.18 above implies that U axiomatizes the epc models of T.

For the converse, suppose that the class of epc models of T is the class of all models of a τ -theory U. Then every model of U is in particular a model of T, and every model of T homomorphically maps to a model of U by Lemma 3.5.3. So we only have to verify that U is a model-complete core theory to show that U is the core companion of T. Every model $\mathfrak A$ of U is an epc model of T, and in fact an epc model of U. It follows by the equivalence of (1) and (2) in Theorem 3.6.14 that U is a model-complete core theory.

3.6.6. ω -categorical model-complete cores. We have already seen in Theorem 3.6.11 that whether an ω -categorical structure is a model-complete core can be characterized in many different ways. The results in Section 3.6.4 provide a further characterization of ω -categorical model-complete cores in terms of their axiomatization, as we see in the following.

PROPOSITION 3.6.20. Let \mathfrak{B} be a countable ω -categorical structure. Then \mathfrak{B} is a model-complete core if and only if its theory is equivalent to a $\forall \exists^+$ -theory.

PROOF. One direction is Proposition 3.6.16. For the other direction, assume that the theory of \mathfrak{B} is equivalent to a $\forall \exists^+$ -theory T. Then T has no finite models, and hence T has a countably infinite model that is epc for T, by Lemma 3.5.3. Hence, by Proposition 3.6.15, T is a model-complete core theory (since T is ω -categorical, it is easy to see that we can assume for this application that the signature of T is countable), and so \mathfrak{B} is a model-complete core.

We now present an existential positive version of Simmons' theorem (Theorem 3.6.6), which answers the question which CSPs can be formulated with an ω -categorical template. For a satisfiable theory T, let \sim_n^T be the equivalence relation defined on existential positive formulas with n free variables x_1, \ldots, x_n (we could have equivalently used primitive positive formulas here) as follows. For two such formulas ϕ_1 and ϕ_2 , let $\phi_1 \sim_n^T \phi_2$ if for all existential positive formulas ψ with free variables x_1, \ldots, x_n we have that $\{\phi_1, \psi\} \cup T$ is satisfiable if and only if $\{\phi_2, \psi\} \cup T$ is satisfiable.

Theorem 3.6.21. Let T be a theory with the joint homomorphism property (JHP; confer Proposition 2.4.6). Then the following are equivalent.

 T has a core companion that is either ω-categorical or the theory of a finite structure.

- (ii) \sim_n^T has finitely many equivalence classes for each n.
- (iii) T has finitely many maximal existential positive n-types for each n.
- (iv) There is a finite or ω -categorical model-complete core \mathfrak{B} that satisfies an existential positive sentence ϕ if and only if $T \cup \{\phi\}$ is satisfiable.

PROOF. We show $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$.

 $(i)\Rightarrow (ii)$. Let U be the core companion of T. Since U and T entail the same universal negative sentences, we can deduce that for every existential positive formula ψ the theory $U\cup\{\psi\}$ is satisfiable if and only if $T\cup\{\psi\}$ is satisfiable; from which it follows that the indices of \sim_n^U and \sim_n^T coincide.

For a proof by contraposition, assume that \sim_n^U has infinite index for some n. Let ψ_1 and ψ_2 be two existential positive formulas from different equivalence classes of \sim_n^U . Hence, there is an existential positive formula ψ_3 such that exactly one of $\{\psi_1,\psi_3\} \cup U$ and $\{\psi_2,\psi_3\} \cup U$ is satisfiable. This shows that ψ_1 and ψ_2 are inequivalent modulo U. Therefore there are infinitely many first-order formulas with n variables that are inequivalent modulo U, and U can neither be ω -categorical by Theorem 3.1.4 nor the theory of a finite structure.

 $(ii)\Rightarrow (iii)$. We show that every maximal ep-n-type p is determined completely by the \sim_n^T equivalence classes of the existential positive formulas contained in p. Since there are finitely many such classes, the result follows. Let p and q be maximal ep-n-types such that for every $\phi_1\in p$ there exists $\phi_1'\in q$ such that $\phi_1\sim_n^T\phi_1'$ and for every $\phi_2\in q$, there exists $\phi_2'\in p$ such that $\phi_2\sim_n^T\phi_2'$. We aim to prove that p=q. If not then there exists, without loss of generality, $\psi\in p$ such that $\psi\notin q$. Since q is maximal, $T\cup q\cup \{\psi\}$ is not satisfiable. By compactness, $T\cup \{\theta,\psi\}$ is not satisfiable for some finite conjunction θ of formulas from q. Now, $\theta\in q$ by maximality and there exists by assumption $\theta'\in p$ such that $\theta\sim_n^T\theta'$. By definition of \sim_n^T we deduce $T\cup \{\theta',\psi\}$ satisfiable iff $T\cup \{\theta,\psi\}$ satisfiable. Since the latter is not satisfiable, we deduce that neither is the former, which yields the contradiction that $T\cup p\cup \{\psi\}$ is not satisfiable.

 $(iii) \Rightarrow (iv)$. An existential positive formula $\phi(\bar{x})$ is said to *isolate* a maximal ep-n-type $p(\bar{x})$ of T, if p is the only maximal ep-n-type of T of which ϕ is a member. If there is only a finite number of maximal ep-n-types of T, then it follows that each has an isolating formula. Assume (iii), and let S be the set of all existential positive sentences ϕ such that $T \cup \{\phi\}$ is satisfiable, together with the set of all universal negative consequences of T. By Proposition 2.4.6, S has a model \mathfrak{C} , and by Theorem 2.3.3 we can assume that \mathfrak{C} is either finite or countable. Lemma 3.5.3 gives a homomorphism from $\mathfrak C$ to a finite or countable epc τ -model $\mathfrak B$ of S. Note that also B satisfies exactly those existential positive sentences that are satisfiable together with T. We consider the signature τ' , which is the expansion of τ by μ_n relations of each arity n, corresponding to the maximal pp-n-types of T. Any model of T has a canonical (unique) expansion to a τ' -structure, by the new relation symbols labeling tuples that attain their type. Consider this canonical τ' -expansion \mathfrak{B}' of \mathfrak{B} . We will shortly prove that \mathfrak{B}' is homogeneous. From this it will follow that \mathfrak{B}' and \mathfrak{B} are finite, or ω -categorical by Lemma 3.2.10 (since variable identifications are primitive positive, there is only a finite number of inequivalent atomic formulas of each arity n), whereupon ω -categoricity is inherited by its τ -reduct \mathfrak{B} .

To prove that \mathfrak{B} is a model-complete core, we use Theorem 3.6.11, and show that every first-order formula ϕ is equivalent to an existential positive formula over \mathfrak{B} . Since \mathfrak{B} has a homogeneous expansion \mathfrak{B}' by primitive positive definable relations, ϕ is equivalent to a Boolean combination of primitive positive formulas. Because we have added a relation symbol for each maximal pp-n-type of T, and there are finitely many of those for each n, the negation of a primitive positive formula is equivalent to

a finite disjunction of maximal pp-n-types. This proves that ϕ is in $\mathfrak B$ equivalent to an existential positive formula.

It remains to be shown that \mathfrak{B}' is homogeneous. Let $f:(a_1,\ldots,a_m)\mapsto (b_1,\ldots,b_m)$ be a partial automorphism of \mathfrak{B}' (in the signature τ'). Let a' be an arbitrary element of B'. Consider the ep-n-types $p(x_1,\ldots,x_m)$ of (a_1,\ldots,a_m) and $q(x_1,\ldots,x_m,y)$ of (a_1,\ldots,a_m,a') in \mathfrak{B} .

By Proposition 3.5.5, each of these types is maximal, and is isolated by the epformulas $\theta_p(x_1,\ldots,x_m)$ and $\theta_q(x_1,\ldots,x_m,y)$, respectively. Furthermore, the type of (b_1,\ldots,b_m) in \mathfrak{B} is p, because the partial automorphism of \mathfrak{B}' respects the signature τ' . But now, since $\exists y.\theta_q(x_1,\ldots,x_m,y)$ is in p (by maximality), we may deduce a p' such that $\mathfrak{B}' \models \theta_q(b_1,\ldots,b_m,b')$ and consequently $\mathfrak{B}' \models q(b_1,\ldots,b_m,b')$. It follows that $p': (a_1,\ldots,a_m,a') \mapsto (b_1,\ldots,b_m,b')$ is a partial automorphism of \mathfrak{B}' (in the signature τ'). A simple back-and-forth argument shows that we may extend to an automorphism of \mathfrak{B}' , and the result follows.

For the implication $(iv) \Rightarrow (i)$, observe that a finite or ω -categorical structure \mathfrak{B} is a model-complete core if and only if it has a model-complete core theory – this is an easy consequence of the Löwenheim-Skolem theorem (Theorem 2.3.3). So it suffices to show that the first-order theory of \mathfrak{B} and T have the same universal negative consequences, by Proposition 2.4.5. A universal negative sentence ϕ is implied by T if and only if $T \cup \{\neg \phi\}$ is unsatisfiable, which is the case if and only if \mathfrak{B} does not satisfy $\neg \phi$ (and hence satisfies ϕ).

Theorem 3.6.21 implies a necessary and sufficient condition when an CSP can be formulated with an ω -categorical template.

Corollary 3.6.22. Let \mathfrak{A} be a structure with a finite relational signature. Then the following are equivalent.

- (1) There is an ω -categorical template \mathfrak{B} such that $CSP(\mathfrak{B}) = CSP(\mathfrak{A})$;
- (2) $\sim_n^{\text{Th}(\mathfrak{A})}$ has finite index for all n;
- (3) There exists a structure \mathfrak{B} with $CSP(\mathfrak{B}) = CSP(\mathfrak{A})$ which has for all $n \geq 1$ finitely many primitive positive definable relations of arity n.

PROOF. The implications from (1) to (3) and from (3) to (2) are easy. The implication from (2) to (1) follows from $(ii) \Rightarrow (iv)$ in Theorem 3.6.21.

We present a simple new proof of the following result from [25].

Theorem 3.6.23 (of [25]). Every ω -categorical structure $\mathfrak A$ is homomorphically equivalent to an ω -categorical model-complete core $\mathfrak B$. All model-complete cores of $\mathfrak A$ are isomorphic to $\mathfrak B$.

PROOF. Let T be the first-order theory of \mathfrak{A} ; clearly, T has the JHP. Since T is ω -categorical, \sim_n^T has finite index for each n (Theorem 3.1.4), and Theorem 3.6.21 implies that T has a core companion S which is either ω -categorical or the theory of a finite structure. By Proposition 3.6.18, the core companion of $\operatorname{Th}(\mathfrak{A})$ is unique up to equivalence of first-order theories. Since $\operatorname{Th}(\mathfrak{B})$ is ω -categorical or the theory of a finite structure, it follows that \mathfrak{B} is unique up to isomorphism.

Since the model-complete core \mathfrak{B} of \mathfrak{A} from the previous theorem is unique up to isomorphism, we call it the model-complete core of \mathfrak{A} . The following gives an indication that the model-complete core of an ω -categorical structure \mathfrak{A} is typically 'simpler' than \mathfrak{A} .

Proposition 3.6.24. Let $\mathfrak A$ be an ω -categorical structure, and let $\mathfrak B$ its model-complete core. Then

- for every n, the number of orbits of n-tuples in B is at most the number of orbits of n-tuples in A;
- if $\mathfrak A$ is homogeneous, then $\mathfrak B$ is homogeneous as well.

PROOF. Let f be a homomorphism from $\mathfrak A$ to $\mathfrak B$, and g be a homomorphism from $\mathfrak B$ to $\mathfrak A$. It suffices to show that when two n-tuples t_1,t_2 from $\mathfrak B$ are mapped by g to tuples s_1,s_2 in the same orbit in $\mathfrak A$, then t_1 and t_2 lie in the same orbit in $\mathfrak B$. Let α be an automorphism of $\mathfrak A$ that maps s_1 to s_2 . Since $\mathfrak B$ is an ω -categorical model-complete core, there are primitive positive definitions ϕ_1 and ϕ_2 of the orbits of t_1 and t_2 . Since g, α , and f preserve primitive positive formulas, the tuple $t_3 := f(\alpha g(t_1))$ satisfies ϕ_1 . As $\alpha g(t_1) = s_2 = g(t_2)$, the tuple t_3 can also be written as $f(g(t_2))$, and hence also satisfies ϕ_2 . Thus, ϕ_1 and ϕ_2 define the same orbit, and t_1 and t_2 are in the same orbit.

For the second part of the statement, suppose that h is an isomorphism between two finite substructures \mathfrak{C} and \mathfrak{C}' of \mathfrak{B} . Then $g(\mathfrak{C})$ induces in \mathfrak{A} a structure that is isomorphic to \mathfrak{C} , since otherwise the endomorphism $e\colon x\mapsto f(g(x))$ of \mathfrak{B} would not preserve all first-order formulas, contradicting the assumption that \mathfrak{B} is a model-complete core. Similarly, $g(\mathfrak{C}')$ induces in \mathfrak{A} a structure that is isomorphic to \mathfrak{C}' and \mathfrak{C} , and by homogeneity of \mathfrak{A} there exists an automorphism α of \mathfrak{A} that maps $g(\mathfrak{C})$ to $g(\mathfrak{C}')$. The mapping $e'\colon x\mapsto f(\alpha g(x))$ is an endomorphism of \mathfrak{B} . By Theorem 3.6.11, this mapping is locally generated by the automorphisms of \mathfrak{B} , and in particular there exists an automorphism β of \mathfrak{B} such that $\beta(x)=e'(x)$ for all elements x of \mathfrak{C} . Since e is locally generated by the automorphisms of \mathfrak{B} , too, there exists $\gamma\in \mathrm{Aut}(\mathfrak{B})$ such that $\gamma(x)=e(x)$ for all elements x of \mathfrak{C}' . Then $\gamma^{-1}\circ\beta\in \mathrm{Aut}(\mathfrak{B})$ maps \mathfrak{C} to \mathfrak{C}' . This proves the homogeneity of \mathfrak{B} .

When the template of a CSP is a model-complete core, then this can be exploited in the study of the CSP in many ways. For instance, we have the following consequence of Theorem 3.6.11 and Lemma 1.2.8, which is essentially from [25].

COROLLARY 3.6.25. Let \mathfrak{B} be an ω -categorical model-complete core, and let \mathfrak{C} be the expansion of \mathfrak{B} by finitely many unary singleton relations, that is, relations of the form $\{c\}$ for some element c of \mathfrak{B} . Then for every finite signature reduct \mathfrak{C}' of \mathfrak{C} there exists a finite signature \mathfrak{B}' of \mathfrak{B} such that $\mathrm{CSP}(\mathfrak{C}')$ has a polynomial-time reduction to $\mathrm{CSP}(\mathfrak{B}')$.

PROOF. Let $\{c_1\}, \ldots, \{c_k\}$ be the relations of \mathfrak{C} that have been added to \mathfrak{B} , and let \mathfrak{C}' be a finite signature reduct of \mathfrak{C} . By Theorem 3.6.11, the orbit of (c_1, \ldots, c_k) in \mathfrak{B} has a primitive positive definition ϕ in \mathfrak{B} . Let \mathfrak{B}' be the reduct of \mathfrak{B} whose signature contains all relation symbols mentioned in ϕ , and the relation symbols of \mathfrak{C}' that are also relation symbols in \mathfrak{B} ; observe that the signature of \mathfrak{B}' is finite. Lemma 1.2.8 implies that there is a polynomial-time reduction from $\mathrm{CSP}(\mathfrak{C}')$ to $\mathrm{CSP}(\mathfrak{B}')$.

CHAPTER 4

Examples



The running examples in the previous chapter were the linear order $(\mathbb{Q};<)$ and the random graph $(\mathbb{V};E)$. Structures definable in those structures provide further examples of ω -categorical structures, and they will be studied in great detail in Chapter 6, 9, and 10. In this chapter, we present other ω -categorical structures $\mathfrak A$ that will not be treated at the same level of detail as it will be done for $(\mathbb{Q};<)$ and for $(\mathbb{V};E)$ in Chapter 9 and Chapter 10. For example, we treat homogeneous C-relations, dense semi-linear orders, and the atomless Boolean algebra. In each case, we give a brief discussion on what is known about CSPs with templates that can be defined in those ω -categorical structures. Thereby, we revisit many problems from Section 1.5. We also discuss ω -categorical structures that serve as templates for problems in connected monotone monadic SNP.

The ω -categorical structures presented in this chapter are chosen so that they illuminate the diversity of the class of all ω -categorical structures, and so that many computational problems and classes of computational problems from the literature can be formulated as CSPs for those structures.

4.1. Phylogeny Constraints and Homogeneous C-relations

The rooted-triple satisfiability problem from Section 1.5.2 can be formulated as $CSP(\mathfrak{B})$ for an ω -categorical template \mathfrak{B} (an observation from [25]). There are various different ways how to define such a structure \mathfrak{B} ; the most convenient for us is via amalgamation.

Let \mathcal{T} be the class of all finite rooted binary trees \mathfrak{T} . The leaf structure \mathfrak{C} of a tree $\mathfrak{T} \in \mathcal{T}$ with leaves L is the relational structure (L; |) where | is a ternary relation symbol, and ab|c holds in \mathfrak{C} iff yca(a,b) lies below yca(b,c) in \mathfrak{T} (recall that yca(a,b) denotes the youngest common ancestor of a and b in a rooted tree \mathfrak{T} ; see Section 1.5.2). We also call \mathfrak{T} the underlying tree of \mathfrak{C} . Let \mathcal{C} be the class of all leaf structures for trees from \mathcal{T} .

Proposition 4.1.1. The class C is an amalgamation class.

PROOF. Closure under isomorphisms and induced substructures is by definition. For the amalgamation property, let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{C}$ be such that $\mathfrak{A} = \mathfrak{B}_1 \cap \mathfrak{B}_2$ is an induced substructure of both \mathfrak{B}_1 and \mathfrak{B}_2 . We want to show that there is an amalgam of \mathfrak{B}_1 and \mathfrak{B}_2 over \mathfrak{A} in \mathcal{C} . We inductively assume that the statement has been shown for all triples $(\mathfrak{A}, \mathfrak{B}'_1, \mathfrak{B}'_2)$ where $B'_1 \cup B'_2$ is a proper subset of $B_1 \cup B_2$.

Let \mathfrak{T}_1 be the rooted binary tree underlying \mathfrak{B}_1 , and \mathfrak{T}_2 the rooted binary tree underlying \mathfrak{B}_2 . Let $\mathfrak{B}_1^1 \in \mathcal{C}$ be the substructure of \mathfrak{B}_1 induced by the vertices below the left child of \mathfrak{T}_1 , and $\mathfrak{B}_1^2 \in \mathcal{C}$ be the substructure of \mathfrak{B}_1 induced by the vertices below the right child of \mathfrak{T}_1 . The structures \mathfrak{B}_2^1 and \mathfrak{B}_2^2 are defined analogously for \mathfrak{B}_2 instead of \mathfrak{B}_1 .

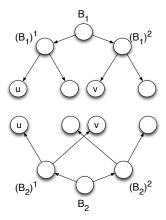


FIGURE 4.1. Illustration for the proof of Proposition 4.1.1.

First consider the case that there is a vertex u that lies in both \mathfrak{B}^1_1 and \mathfrak{B}^1_2 , and a vertex v that lies in both \mathfrak{B}^1_2 and \mathfrak{B}^2_1 (see Figure 4.1 for an illustration). We claim that in this case no vertex w from \mathfrak{B}^2_2 can lie inside \mathfrak{B}_1 : for otherwise, w is either in \mathfrak{B}^1_1 , in which case we have uw|v in \mathfrak{B}_1 , or in \mathfrak{B}^2_1 , in which case we have vw|u in \mathfrak{B}_1 . But since u, v, w are in A, this is in contradiction to the fact that uv|w holds in \mathfrak{B}_2 . Let $\mathfrak{C}' \in \mathcal{C}$ be the amalgam of \mathfrak{B}_1 and \mathfrak{B}^1_2 over \mathfrak{A} , which exists by inductive assumption, and let $\mathfrak{T}' \in \mathcal{T}$ be its underlying tree. Now let \mathfrak{T} be the tree with root v and v as a left subtree, and the underlying tree of v as a right subtree. It is straightforward to verify that the leaf structure of v is in v, and that it is an amalgam of v and v over v (via the identity embeddings).

Up to symmetry, the only remaining essentially different case we have to consider is that $B_1^1 \cup B_2^1$ and $B_1^2 \cup B_2^2$ are disjoint. In this case it is similarly straightforward to first amalgamate \mathfrak{B}_1^1 with \mathfrak{B}_2^1 and \mathfrak{B}_1^2 with \mathfrak{B}_2^2 to obtain the amalgam of \mathfrak{B}_1 and \mathfrak{B}_2 ; the details are left to the reader.

Let $\mathfrak B$ denote the Fraïssé-limit of $\mathcal C$. The structure $\mathfrak B$ is homogeneous, so it is a fortiori model-complete. It is straightforward to verify that $\mathfrak B$ is a core. Since the age of $\mathfrak B$ is the class of all leaf structures for structures from $\mathcal T$, it is obvious that $\mathrm{CSP}(\mathfrak B)$ is the rooted triple consistency problem. The relation | in $\mathfrak B$ is closely related to so-called C-relations, following the terminology of [2]. C-relations became an important concept in model theory, see e.g. [111]. They are given axiomatically; the presentation here follows [2,45].

A ternary relation C is said to be a C-relation on a set L if for all $a, b, c, d \in L$ the following conditions hold:

```
C1 C(a;b,c) \rightarrow C(a;c,b);

C2 C(a;b,c) \rightarrow \neg C(b;a,c);

C3 C(a;b,c) \rightarrow C(a;d,c) \lor C(d;b,c);

C4 a \neq b \rightarrow C(a;b,b).
```

A C-relation is called *dense* if it satisfies

C7
$$C(a; b, c) \rightarrow \exists e (C(e; b, c) \land C(a; b, e)).$$

The structure (L; C) is also called a C-set. A structure Γ is said to be relatively k-transitive if for every partial isomorphism f between induced substructures of Γ of size k there exists an automorphism of Γ that extends f. Note that a relatively 3-transitive C-set is necessarily 2-transitive (i.e., has a 2-transitive automorphism group, as defined in Section 3.3.3).

THEOREM 4.1.2 (Theorem 14.7 in [2]). Let (L; C) be a relatively 3-transitive C-set. Then (L; C) is ω -categorical.

In fact, there is, up to isomorphism, a unique relatively 3-transitive countable C-set which is dense and uniform with branching number 2, that is, satisfies $\forall x, y, z$ ($(x \neq y \lor x \neq z \lor y \neq z) \Rightarrow (C(x; y, z) \lor C(y; x, z) \lor C(z; x, y))$) (see the comments in [2] after the statement of Theorem 14.7).

It is straightforward to verify that the Fraïssé-limit \mathfrak{B} of the amalgamation class from Proposition 4.1.1 has the same automorphism group (equivalently, is first-order interdefinable; see Section 3.3) as the relatively 3-transitive dense countable C-set that is uniform with branching number 2.

A similar approach is possible to define a homogeneous template for the quartet satisfiability problem from Section 1.5.2. Alternatively, an ω -categorical template (B;Q) for the quartet satisfiability problem can be given via a first-order definition in the structure $\mathfrak{B} = (B;|)$ constructed above for the rooted triple consistency problem.

The following definition will be useful now, and later. Since yca is associative as a binary operation, it makes sense to write $yca(v_1, \ldots, v_l)$ for

$$yca(v_1,\ldots,yca(v_{l-1},v_l)\ldots)$$
.

DEFINITION 4.1.3. When u_1, \ldots, u_k and v_1, \ldots, v_l are leaves in a rooted tree \mathfrak{T} , then we write $u_1 \ldots u_k | v_1 \ldots, v_l$ if $u := yca(u_1, \ldots, u_k)$ and $v := yca(v_1, \ldots, v_l)$ are disjoint in \mathfrak{T} , i.e., neither u lies above v nor v lies above u in \mathfrak{T} .

The first-order definition of Q(x, y, u, v) is

$$(xy|uv) \lor (uv|x \land vx|y) \lor (xy|u \land yu|v)$$
.

Indeed, when $u, v, x, y \in B$, and \mathfrak{T} is the tree underlying the substructure of (B; |) induced by $\{u, v, x, y\}$, then the given formula describes the situation that the shortest path from x to y in \mathfrak{T} does not intersect the shortest path from u to v in \mathfrak{T} . Note that whether this is true is in fact independent from the position of the root of \mathfrak{T} . We leave the verification to the reader that $\mathrm{CSP}((B;Q))$ indeed describes the quartet

satisfiability problem studied in comptuational biology. Lemma 3.1.10 implies ω -categoricity of (B;Q). Similarly as for the C-relation given above, an axiomatic treatment of (B;Q) has been given in [2]; there, the relation Q has been called a D-relation, and this became standard terminology in model theory. As we have mentioned above, the structure (B;Q) can also be defined as a Fraïssé-limit of finite D-structures (see Cameron [70]).

4.2. Branching-Time Constraints

The branching-time satisfaction problem from Section 1.5.3 can be formulated as $CSP(\mathfrak{B})$ for an ω -categorical structure $\mathfrak{B} = (D; \leq, \parallel, \neq)$; this has already been observed in [47]. This time, \mathfrak{B} has an explicit description. The domain B consists of the set of all non-empty finite sequences of rational numbers. For $a = (q_1, q_1, \ldots, q_n), b = (q'_1, q'_1, \ldots, q'_m), n \leq m$, we write a < b if one of the following conditions holds:

- a is a proper initial subsequence of b, i.e., n < m and $q_i = q_i'$ for $1 \le i \le n$;
- $q_i = q'_i$ for $1 \le i < n$, and $q_n < q'_n$.

We use $a \leq b$ to denote $(a < b) \vee (a = b)$, and \parallel denotes the binary relation that contains all pairs of elements that are incomparable with respect to <; in particular, we have $a \parallel a$ for all a. A proof that $\mathfrak{B} = (B; \leq, \parallel, \neq)$ is 1-transitive and ω -categorical can be found in [2] (Section 5).

The reduct $(B; \leq)$ of this structure is a semi-linear order, i.e., for all $x \in D$, the set $\{y \mid y \leq x\}$ is linearly ordered by <; such structures have been studied systematically in the context of infinite permutation groups; see [70, 87]. The structure $\mathfrak B$ is not homogeneous, and therefore cannot be described as the Fraïssé-limit of a class of finite structures. However, it has an expansion by primitive positive definable relations which is homogeneous. The following is well-known and straightforward to verify.

Proposition 4.2.1. The expansion of \mathfrak{B} by the ternary relation with the primitive positive definition

$$\exists u ((u \leq x) \land (u \leq y) \land (u \parallel z))$$

and the ternary relation with the primitive positive definition

$$\exists u \, (x | | y \land x \neq y \land (u \leq x) \land (u \leq y) \land (z \leq u) \land (z \neq u))$$

is homogeneous.

Therefore it is possible to describe the structure $(D; \leq, \parallel, \neq)$ as the reduct of the Fraïssé-limit of an amalgamation class; the respective proof is similar to the proof of Proposition 4.1.1. Proposition 4.2.1 in combination with Theorem 3.6.7 also shows that \mathfrak{B} is model-complete. The structure \mathfrak{B} has four orbitals, with the primitive positive definitions $x \leq y \land x \neq y, \ y \leq x \land x \neq y, \ x \parallel y \land x \neq y, \ \text{and} \ x = y$. Since all relations of \mathfrak{B} are binary, this implies that every endomorphism of \mathfrak{B} must be an embedding, and hence \mathfrak{B} is a core.

The expansion \mathfrak{B}' of \mathfrak{B} by the first of the two ternary relations given in Proposition 4.2.1 also has an age that has amalgamation; however, its Fraïssé-limit \mathfrak{B}'' is not isomorphic to \mathfrak{B}' because it *lacks joins*, that is, the second of the ternary relations in Proposition 4.2.1 holds for *all* triples x, y, z with $x||y, x \neq y, z < y$, and z < y.

4.3. Set Constraints

Here we discuss how the set constraint satisfaction problems discussed in Section 1.5.5, and many other set constraint satisfaction problems, can be formulated with ω -categorical templates, following [34].

Here, a set constraint satisfaction problem is a CSP for a template with a first-order definition in the structure \mathfrak{S} with domain $\mathcal{P}(\mathbb{N})$, the set of all subsets of natural numbers, and with signature $\{\cap, \cup, c, \mathbf{0}, \mathbf{1}\}$, where

- \cap is a binary function symbol that denotes intersection, i.e., $\cap^{\mathfrak{S}} = \cap$;
- \cup is a binary function symbol for union, i.e., $\cup^{\mathfrak{S}} = \cup$;
- c is a unary function symbol for complementation, i.e., $c^{\mathfrak{S}}$ is the function that maps $S \subseteq \mathbb{N}$ to $\mathbb{N} \setminus S$;
- 0 and 1 are constants (treated as 0-ary function symbols) denoting the empty set \emptyset and the full set \mathbb{N} , respectively.

A set constraint language is a relational structure with a set of relations with a quantifier-free first-order definition in \mathfrak{S} ; we always allow equality in first-order formulas. For example, the relation $\{(x,y,z)\in\mathcal{P}(\mathbb{N})^3\mid x\cap y\subseteq z\}$ has the quantifier-free first-order definition $z\cap(x\cap y)=x\cap y$ over \mathfrak{S} .

PROPOSITION 4.3.1 (follows from Proposition 5.8 in [160]). Let \mathfrak{B} be a set constraint language with a finite signature. Then $CSP(\mathfrak{B})$ is in NP.

The first-order theory of the structure \mathfrak{S} is certainly not ω -categorical – it is easy to verify that there are infinitely many pairwise inequivalent first-order formulas with one free variable. However, all set constraint satisfaction problems can be formulated with an ω -categorical template. To see this, first note that the structure $(\mathcal{P}(\mathbb{N}); \cup, \cap, c, \mathbf{0}, \mathbf{1})$ is a Boolean algebra, with

- 0 playing the role of false, and 1 playing the role of true;
- c playing the role of \neg ;
- \cap and \cup playing the role of \wedge and \vee , respectively.

To facilitate the notation, we write \bar{x} instead of c(x), and $x \neq y$ instead of $\neg(x = y)$. An atom in a Boolean algebra is an element $x \neq 0$ such that for all y with $x \cap y = y$ and $x \neq y$ we have y = 0. If a Boolean algebra does not contains atoms, it is called atomless. It is well-known that there exist countable atomless Boolean algebras, and that all countable atomless Boolean algebras are isomorphic (Corollary 5.16 in [137]; also see Example 4 on page 100 in [119]). Let \mathfrak{A} denote such a countable atomless Boolean algebra; the domain \mathfrak{A} is denoted by \mathbb{A} . Since the axioms of Boolean algebras and the property of not having atoms can all be written as first-order sentences, it follows that \mathfrak{A} is ω -categorical. We also remark that the structure \mathfrak{A} has quantifier elimination (see Exercise 17 on Page 391 in [119]). The link between the set constraint satisfaction problems over $2^{\mathbb{N}}$ mentioned in Section 1.5 and the atomless Boolean algebra is the following.

PROPOSITION 4.3.2. Let \mathfrak{C} be a set constraint language. Then there exists an ω -categorical structure \mathfrak{B} such that \mathfrak{B} and \mathfrak{C} have the same existential theory. In particular, when \mathfrak{C} has finite signature, then \mathfrak{B} and \mathfrak{C} have the same CSP.

PROOF. Let ϕ_1, ϕ_2, \ldots be quantifier-free first-order formulas that define the relations $R_1^{\mathfrak{C}}, R_2^{\mathfrak{C}}, \ldots$ of \mathfrak{C} over $\mathfrak{S} = (\mathcal{P}(\mathbb{N}); \cup, \cap, c, \mathbf{0}, \mathbf{1})$. Let $R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}}, \ldots$ be the relations defined by ϕ_1, ϕ_2, \ldots over the atomless Boolean algebra \mathfrak{A} . The structure $\mathfrak{B} = (\mathbb{A}; R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}}, \ldots)$ is ω -categorical (see the comment after Lemma 3.1.10). To verify that \mathfrak{B} and \mathfrak{C} have the same existential theory, let Φ be a conjunction of atomic formulas over the signature $\{R_1, R_2, \ldots\}$. Replace each atomic formula of the form $R_i(x_1, \ldots, x_k)$ in Φ by the formula $\phi_i(x_1, \ldots, x_k)$. The resulting formula is a quantifier-free first-order formula in the language of Boolean algebras, $\{\cup, \cap, c, \mathbf{0}, \mathbf{1}\}$. We claim that Φ is satisfiable in \mathfrak{S} if and only if it is satisfiable in \mathfrak{A} . This follows from Corollary 5.7 in [160]: Φ is satisfiable in some infinite Boolean algebra if and only if Φ is satisfiable in all infinite Boolean algebras.

A large class of tractable set constraint languages has been described in [34]; the class given there is *maximal tractable* in the sense that every strictly larger class of set constraint languages contains a finite subset with an NP-hard CSP.

4.4. Spatial Reasoning

The essential reasons why the network satisfaction problem for RCC5 (introduced in Section 1.5.6) is *not* a set constraint satisfaction problem as introduced in the previous section is that in RCC5 we exclude the empty set as a possible value for the variables. To formulate the CSP for the network satisfaction problem of RCC5 and its fragments with ω -categorical templates, we again use structures with a first-order definition in the atomless Boolean algebra, but restrict those structures to non-zero elements (this observation has already been made in [89], Proposition 4.4).

Formally, let $\mathfrak A$ be the atomless Boolean algebra, and let DR, PO, PP, PPI, EQ be the binary relations with the following first-order definition in $\mathfrak A$ (and their intuitive meaning in quotes).

$$\begin{aligned} \operatorname{DR}(x,y) & \text{ iff } & (x \cap y = \mathbf{0}) \land x \neq y \land x, y \notin \{\mathbf{0},\mathbf{1}\} \\ & `x \text{ and } y \text{ are disjoint'} \end{aligned}$$

$$\operatorname{PP}(x,y) & \text{ iff } & (x \cap y = x) \land x \neq y \land x, y \notin \{\mathbf{0},\mathbf{1}\} \\ & `y \text{ properly contains } x' \end{aligned}$$

$$\operatorname{PPI}(x,y) & \text{ iff } & (x \cap y = y) \land x \neq y \land x, y \notin \{\mathbf{0},\mathbf{1}\} \\ & `x \text{ properly contains } y' \end{aligned}$$

$$\operatorname{EQ}(x,y) & \text{ iff } & x = y \land x, y \notin \{\mathbf{0},\mathbf{1}\} \\ & `x \text{ equals } y' \end{aligned}$$

$$\operatorname{PO}(x,y) & \text{ iff } & \neg \operatorname{DR}(x,y) \land \neg \operatorname{PPI}(x,y) \land x \neq y \land x, y \notin \{\mathbf{0},\mathbf{1}\} \\ & `x \text{ and } y \text{ properly overlap'} \end{aligned}$$

When \mathfrak{D} is the structure that contains all binary relations that are first-order definable in $(A \setminus \{0,1\}; \mathrm{DR}, \mathrm{PO}, \mathrm{PP}, \mathrm{PPI}, \mathrm{EQ})$ (so that we can associate a binary relation from \mathfrak{D} to every element of RCC5 in the natural way), then $\mathrm{CSP}(\mathfrak{D})$ and the network satisfaction problem for RCC5 are essentially the same problem (in the sense of Section 1.3.2). The structure \mathfrak{D} has a (1-dimensional) first-order interpretation in \mathfrak{A} , and hence is ω -categorical by Lemma 3.1.10. It can be shown that the model companion of \mathfrak{D} gives a representation of RCC5.

4.5. CSPs and Fragments of SNP

Recall that a constraint satisfaction problem can be viewed as a class of finite structures with finite relational signature (as described in Section 1.1 and Section 1.4), namely the class of all satisfiable instances of the CSP. In this section we study the question when this class can be described by a τ -sentence Φ from a fixed logic \mathcal{L} in the sense that for all finite structures \mathfrak{A} , we have that $\mathfrak{A} \models \Phi$ if and only if $\mathfrak{A} \in \mathcal{C}$. If there is such a sentence then we say that $CSP(\mathfrak{B})$ is in \mathcal{L} .

The first two logics considered here will be first-order logic, and monadic SNP. It turns out that CSPs that can be described by a sentence from those logics can be formulated with ω -categorical templates. Finally, we present a new and more expressive logic, called Amalgamation SNP. Again, every problem in amalgamation SNP describes a problem in NP that can be formulated as CSP(\mathfrak{B}) for an ω -categorical template \mathfrak{B} , and the universal-algebraic techniques presented in later sections can be applied to study the computational complexity of the problems in this logic.

4.5.1. First-order Definable CSPs. We first consider the situation where $CSP(\mathfrak{B})$ is in FO, i.e., can be described by a first-order sentence (in the sense just described). The class of CSPs in FO is quite restricted. It is not hard to see that when $CSP(\mathfrak{B})$ is in FO, then in particular it can be solved in deterministic logarithmic space. We will see that when $CSP(\mathfrak{A})$ is in FO, then there exists an ω -categorical structure \mathfrak{B} that has the same CSP^1 . We use the following famous result.

Theorem 4.5.1 (Homomorphism Preservation in the Finite [180]). Let τ be a finite relational signature, and let Φ be a first-order τ -sentence. Then Φ is equivalent to an existential positive first-order sentence on all finite τ -structures if and only if the class of all finite τ -models of Φ is closed under homomorphisms (Definition 1.1.6).

In the rest of this section, τ is a finite relational signature, and \mathfrak{B} be a τ -structure. Recall that $\mathrm{CSP}(\mathfrak{B})$, viewed as a class of finite τ -structures, is closed under inverse homomorphisms and disjoint unions. In particular, the class of all finite τ -structures that do *not* homomorphically map to \mathfrak{B} is closed under homomorphisms, and by Theorem 4.5.1 describable by an existential positive τ -sentence Ψ . This leads us to the following.

THEOREM 4.5.2. If $CSP(\mathfrak{B})$ is in FO, then there exists an ω -categorical structure \mathfrak{B}' that has the same CSP.

PROOF. From the remarks that precede the statement of the theorem, Theorem 4.5.1 shows that there is an existential positive τ -sentence Ψ such that $\mathfrak A$ homomorphically maps to $\mathfrak B$ if and only if $\mathfrak A$ satisfies $\neg \Psi$. The sentence $\neg \Psi$ can be re-written as a universal negative sentence in conjunctive normal form; let Φ be such a universal negative sentence of minimal size. We claim that the canonical database $\mathfrak C$ for each conjunct in Φ is connected. To see this, suppose that $\mathfrak C$ has several connected components. If one of them does not homomorphically map to $\mathfrak B$, then Φ was not of minimal size, since the corresponding conjunct could have been replaced by the (smaller) conjunctive query for the component. If all components homomorphically map to $\mathfrak B$, then so does $\mathfrak C$, a contradiction to the fact that $\mathfrak C$ is the canonical database of a conjunct of Φ .

Therefore each conjunct in Φ is connected, and we can apply Theorem 3.2.11 to the finite set \mathcal{N} of canonical databases for all the conjuncts in Φ . From Theorem 3.2.11 we obtain an ω -categorical τ -structure which is universal for CSP(\mathfrak{B}), which is what we had to show.

4.5.2. CSPs in Monadic SNP. Also every CSP in monadic SNP can be formulated with an ω -categorical template.

THEOREM 4.5.3 (from [31]). Let \mathfrak{C} be a structure with a finite relational signature. If $\mathrm{CSP}(\mathfrak{C})$ can be described by a monadic SNP sentence Φ , then there is an ω -categorical \mathfrak{B} such that $\mathrm{CSP}(\mathfrak{B}) = \mathrm{CSP}(\mathfrak{C})$.

PROOF. By Corollary 1.4.15, we can assume without loss of generality that Φ is a *connected* and *monotone* monadic SNP sentence.

We assume without loss of generality that Φ is written in negation normal form. Let P_1, \ldots, P_k be the existential monadic predicates in Φ . Let τ' be the signature containing the input relations from τ , the existential monadic relations P_i , and new symbols P'_i for the negative occurrences of the existential relations.

By monotonicity, all such literals with input relations are positive. For each existential monadic relation P_i we introduce an existentially quantified monadic relation

¹An exact characterization of those ω -categorical structures that are in FO can be found in [35], obtained by a slight modification of a proof for finite domain CSPs in [144].

symbol P_i' , and replace negative literals of the form $\neg P_i(x)$ in Φ by $P_i'(x)$. We shall denote the τ' -formula obtained from Φ after this transformation by Φ' . We define \mathcal{N} to be the set of τ' -structures containing for each clause ψ in Φ' its canonical database (as defined in Section 1.4.3). We shall use the fact that a τ' -structure \mathfrak{A} satisfies a clause ψ if and only if the canonical database of ψ is not homomorphic to \mathfrak{A} . Since Φ is connected, all structures in \mathcal{N} are connected.

Then Theorem 3.2.11 asserts the existence of a \mathcal{N} -free ω -categorical τ' -structure \mathfrak{B}' that is universal for all \mathcal{N} -free structures. We use \mathfrak{B}' to define the template \mathfrak{B} with the properties required in the statement of the theorem we are about to prove. The structure \mathfrak{B} is the τ -reduct of the restriction of \mathfrak{B}' to the points with the property that for all existential monadic predicates P_i , $1 \leq i \leq k$, either P_i or P_i' holds (but not both P_i and P_i'). It follows from Theorem 3.1.10 that reducts of ω -categorical structures, and restrictions to first-order definable subsets of ω -categorical structures are again ω -categorical. Hence, the resulting τ -structure \mathfrak{B} is ω -categorical.

We claim that a τ -structure $\mathfrak A$ satisfies Φ if and only if $\mathfrak A$ homomorphically maps to $\mathfrak B$. First, let $\mathfrak A$ be a structure that has a homomorphism h to $\mathfrak B$. Let $\mathfrak A'$ be the τ' -expansion of $\mathfrak A$ such that for all $i \leq k$ and $a \in A$ the relation $P_i(a)$ holds in $\mathfrak A'$ if and only if $P_i(h(a))$ holds in $\mathfrak B'$, and $P_i'(a)$ holds in $\mathfrak A'$ if and only if $P_i(h(a))$ holds in $\mathfrak B'$. Clearly, h defines a homomorphism from $\mathfrak A'$ to $\mathfrak B'$. In consequence, none of the structures from $\mathcal N$ maps to $\mathfrak A'$. Hence, the τ -reduct $\mathfrak A$ of $\mathfrak A'$ satisfies Φ .

Conversely, let \mathfrak{A} be a τ -structure satisfying Φ . Consequently, there exists a τ' -expansion \mathfrak{A}' of \mathfrak{A} that satisfies the first-order part of Φ' , and where for every $a \in A$ exactly one of $P_i(a)$ or $P_i'(a)$ holds. Clearly, no structure in \mathcal{N} is homomorphic to \mathfrak{A}' , and by universality of \mathfrak{B}' the τ' -structure \mathfrak{A}' is an induced substructure of \mathfrak{B}' . Since for every $a \in A$ exactly one of $P_i(a)$ and $P_i'(a)$ holds, \mathfrak{A}' is also an induced substructure of the restriction of \mathfrak{B}' to B. Consequently, \mathfrak{A} is homomorphic to the τ -reduct of this restriction. This completes the proof.

4.5.3. Amalgamation SNP. In this section we introduce a logic that describes only CSPs with ω -categorical templates, and which we call *Amalgamation SNP*, or short ASNP.

DEFINITION 4.5.4. Amalgamation SNP is the logic that consists of all monotone SNP sentences Φ where the class of all finite models of the first-order part of Φ has the amalgamation property.

It can be verified that the examples of SNP sentences given for $CSP((\mathbb{Z};<))$ in Example 1.4.2 and for $CSP((\mathbb{Z};Betw))$ in Example 1.4.3 are in fact Amalgamation SNP sentences. Recall that a structure is called *finitely bounded* if its age can be described by finitely many forbidden induced substructures (Definition 3.2.8).

PROPOSITION 4.5.5. Every sentence in Amalgamation SNP describes a problem of the form $CSP(\mathfrak{B})$ for an ω -categorical structure \mathfrak{B} that can be expanded to a homogeneous finitely bounded structure.

PROOF. To prove the first statement, let Φ be a sentence in ASNP, let τ be the input signature of Φ (i.e., the free relation symbols in Φ), and let ϕ be the first-order part of Φ (which is a σ -formula, for $\sigma \supseteq \tau$). Since the class of all finite models of ϕ has the amalgamation property, Theorem 3.2.2 asserts the existence of a countable homogeneous σ -structure $\mathfrak C$ whose age is exactly the class of all finite σ -structures that satisfy ϕ . The structure $\mathfrak C$ is finitely bounded; this is witnessed by the set $\mathcal N$ of all σ -structures $\mathfrak N$ with a minimal number of vertices that do not satisfy ϕ . None of those structures $\mathfrak N$ can have larger size than the number of variables of ϕ , and hence $\mathcal N$ is finite. To see that $\mathcal N$ indeed bounds the age of $\mathfrak C$, let $\mathfrak A$ be a finite

induced substructure of \mathfrak{C} . Then \mathfrak{A} cannot have a substructure from \mathcal{N} , since any such substructure would falsify ϕ , and hence also \mathfrak{A} would not satisfy ϕ . Now suppose that \mathfrak{C} does not contain \mathfrak{A} as a finite induced substructure. This means that \mathfrak{A} falsifies ϕ . Then there is a minimal number of elements witnessing that the universal formula ϕ is false in \mathfrak{A} , and those elements induce a structure from \mathcal{N} , which proves the claim. By Lemma 3.2.10 the structure \mathfrak{C} is ω -categorical. The τ -reduct \mathfrak{B} of \mathfrak{C} is also ω -categorical (Lemma 3.1.10).

We finally show that every finite τ -structure \mathfrak{A} satisfies Φ if and only if it homomorphically maps (in fact, embeds) into \mathfrak{B} . When \mathfrak{A} is a τ -structure that satisfies Φ , then there exists a σ -expansion \mathfrak{A}' of \mathfrak{A} that satisfies ϕ . By universality of \mathfrak{C} the same map is an embedding of \mathfrak{A}' into \mathfrak{C} , and this gives an embedding of \mathfrak{A} into \mathfrak{B} .

Conversely, suppose that there is a homomorphism f from a τ -structure \mathfrak{A} to \mathfrak{B} . Then we construct the σ -expansion \mathfrak{A}' of \mathfrak{A} by putting for all $S \in \sigma \setminus \tau$ the tuple (t_1, \ldots, t_n) into $S^{\mathfrak{A}'}$ if and only if $(f(t_1), \ldots, f(t_n)) \in S^{\mathfrak{C}}$. Suppose for contradiction that there were a tuple $t = (t_1, \ldots, t_k)$ of elements that violates a clause of ϕ in this expansion \mathfrak{A}' . Then the image of f induces a substructure in \mathfrak{C} that also violates ϕ , since all relation symbols from τ appear negatively in ϕ . This is a contradiction, and hence \mathfrak{A}' satisfies ϕ . We conclude that \mathfrak{A} satisfies Φ .

We want to give an example of an ω -categorical structure \mathfrak{B} such that $\mathrm{CSP}(\mathfrak{B})$ is in NP, but not in Amalgamation SNP. To show that $\mathrm{CSP}(\mathfrak{B})$ is not in ASNP, the sequence $(m_n)_{n\geq 1}$ of numbers of maximal pp-n-types of $\mathrm{Th}(\mathfrak{B})$ turns out to be useful (see Section 3.6.6). Note that when two structures have the same CSP, then they have the same number of maximal pp-n-types. Hence, when Φ is a sentence from ASNP, there is a unique sequence $(m_n)_{n\geq 1}$ such that m_n equals the of maximal pp-n-types of $\mathrm{Th}(\mathfrak{B})$ for any \mathfrak{B} such that Φ describes $\mathrm{CSP}(\mathfrak{B})$. We call $(m_n)_{n\geq 1}$ the characteristic sequence of Φ .

PROPOSITION 4.5.6. Let Φ be a sentence from Amalgamation SNP. Then the characteristic sequence of Φ is in $O(2^{P(n)})$, for some polynomial P.

PROOF. Proposition 4.5.5 shows that there exists an ω -categorical structure \mathfrak{B} such that Φ describes $\mathrm{CSP}(\mathfrak{B})$. The proof of Proposition 4.5.5 shows that there exists an expansion \mathfrak{C} of \mathfrak{B} by finitely many relation symbols which is homogeneous. Hence, an orbit of n-tuples in \mathfrak{C} is uniquely described by the atomic formulas that hold on a (equivalently, all) tuples from this orbit, and since the signature of \mathfrak{C} is finite, this gives a bound of $2^{P(n)}$ on the number of orbits of n-tuples in \mathfrak{C} , for some polynomial whose degree equals the maximal arity of the relations in \mathfrak{C} . The number of orbits of n-tuples is an upper bound on the number of maximal pp-n-tpyes (since two tuples with the same orbit clearly have the same pp-type).

We can now present the example of an ω -categorical structure \mathfrak{B} such that $CSP(\mathfrak{B})$ is in NP, but not in ASNP. We use a CSP for a set constraint languages, as introduced in Section 1.5.5.

EXAMPLE 4.5.7. Let \mathfrak{B} be the structure that contains all relations of arity at most three with a quantifier-free first-order definition in the atomless Boolean algebra \mathfrak{A} . Since \mathfrak{A} is ω -categorical, the signature of \mathfrak{B} is finite. By Proposition 4.3.1, $CSP(\mathfrak{B})$ is in NP. To prove that $CSP(\mathfrak{B})$ is not in ASNP it suffices to show that the characteristic sequence of Φ grows faster than $O(2^{P(n)})$, for any polynomial P. We first show that \mathfrak{B} is a model-complete core. Trivially, \mathfrak{B} is a core, since with each relation also the complement of the relation is a relation of \mathfrak{B} . To see that \mathfrak{B} is model-complete, let ϕ be a first-order formula that defines a first-order relation R over \mathfrak{B} ; we have to show that R also has an existential definition over \mathfrak{B} . By quantifier-elimination of \mathfrak{A}

(recall that \mathfrak{A} has function symbols \cup , \cap , c, $\mathbf{0}$, $\mathbf{1}$), there is a quantifier-free first-order formula ψ that defines R over \mathfrak{A} . By un-nesting terms in ψ with the help of new existentially quantified variables, and replacing occurrences of atomic formulas by the corresponding formulas in the language of \mathfrak{B} (for instance replacing formulas of the form $x \cap y = z$ by S(x, y, z) where S is the relation of \mathfrak{B} defined by $x \cap y = z$), we find the required existential definition of R in \mathfrak{B} . By Theorem 3.6.11, the orbits of n-tuples in \mathfrak{B} are primitive positive definable, and so m_n equals the number of orbits of n-tuples of \mathfrak{B} .

We show that $m_n \geq 2^{2^{\lfloor n/2 \rfloor}}$. To see this, let $l = \lfloor n/2 \rfloor$ and $X := \{x_1, \ldots, x_l\}$ be elements such that for any two distinct subsets S_1, S_2 of X the elements $\cap S_1$ and $\cap S_2$ of \mathfrak{A} are distinct. Hence, there are 2^l many elements a_1, \ldots, a_{2^l} that can by formed from x_1, \ldots, x_l by applying \cap , and there are 2^{2^l} many ways of selecting a tuple (y_1, \ldots, y_l) of elements from $\{a_1, \ldots, a_{2^l}\}$. For any such selection, and since the relations $\{(x, y, z) \mid x \cap y = z\}$ and $\{(x, y, z) \mid x \cup y = z\}$ are in \mathfrak{B} , the tuple $(x_1, \ldots, x_l, y_1, \ldots, y_l)$ will lie in a distinct orbit, which shows the claim.

From this example and the proof of Proposition 4.5.6 we can also see the following, which we note for later use.

COROLLARY 4.5.8. There is no relational structure \mathfrak{B} that can define the atomless Boolean algebra and is homogeneous in a finite relational signature.

4.5.4. CSPs in SNP without an ω -categorical template. We have seen in Section 1.4.3 that every CSP in SNP is also in connected monotone SNP. We have also seen a characterization of those CSPs that can be formulated with an ω -categorical template (Theorem 3.6.21). So it is natural to ask for a concrete example of a connected monotone SNP sentence that cannot be formulated with an ω -categorical template.

Example 4.5.9. Let Φ be the following connected monotone SNP sentence.

```
\exists E, T \ \forall x, y, z \ (`E \ \text{is equivalence relation'} \\ \land `T \ \text{is transitive irreflexive and extends } \textit{succ'} \\ \land ((\textit{succ}(x,y) \land E(x,z)) \Rightarrow \textit{succ}(z,y)) \\ \land ((\textit{succ}(x,y) \land E(y,z)) \Rightarrow E(x,z)) \\ \land ((\textit{succ}(x,y) \land \textit{succ}(x,z)) \Rightarrow E(y,z)) \\ \land ((\textit{succ}(x,y) \land \textit{succ}(z,y)) \Rightarrow E(y,z)) \\ \land (\neg E(x,y) \lor \neg \textit{succ}(x,y)))
```

The sentence Φ describes $\mathrm{CSP}((\mathbb{Z};succ))$ where $succ = \{(x,y) \in \mathbb{Z}^2 \mid y=x+1\}$ (as in Section 1.7.4.1). The idea is that an $\{succ, E, T\}$ -structure satisfies the quantifier-free part of Φ if

- E(x,y) holds if for all homomorphisms from the $\{succ\}$ -reduct of the structure to $(\mathbb{Z}; succ)$ we have h(x) = h(y), and
- T(x, y) holds for all homomorphisms h from the $\{succ\}$ -reduct of the structure to $(\mathbb{Z}; succ)$ we have h(x) < h(y).

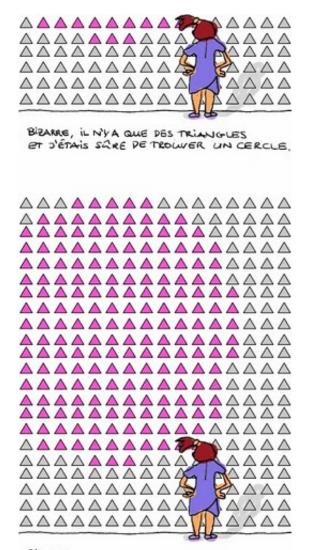
Proposition 4.5.10. $CSP((\mathbb{Z}; succ))$ cannot be formulated with an ω -categorical template.

PROOF. The number of maximal pp-n-types is the same in any structure \mathfrak{B} where $CSP(\mathfrak{B})$ is described by Φ , so by Corollary 3.6.22 it suffices to check that $(\mathbb{Z}; succ)$

has an infinite number of maximal pp-2-types. But this is clear since the formula $\phi_n(x_0, x_n)$ defined by $\exists x_1, \dots, x_{n-1} \ \bigwedge_{i=1}^n succ(x_{i-1}, x_i)$ is for each n in a different pp-2-type.

CHAPTER 5

Universal Algebra



BIZARRE

One of the central concerns of universal algebra, similarly as in model-theory, is *classification* of mathematical structures. Often, model-theory is considered to be an extension of universal algebra, as formulated by Chang and Keisler in

model-theory = universal algebra + logic.

Universal algebra leads to classification results with finer distinctions: while model theory often considers two relational structures to be equal when they are first-order interdefinable, universal algebra provides methods that allow to distinguish relational structures up to primitive positive definability. To do so, we study higher-dimensional generalizations of endomorphisms monoids, called *polymorphism clones*; from the perspective of this text, we therefore have

model-theory = one-dimensional universal algebra.

The strongest universal-algebraic classification results are available on *finite* domains [118]. In recent years, strong links between deep and central questions in universal algebra and the Feder-Vardi conjecture have led to renewed activity. In fact, several important new and purely algebraic results, for example from [13,123, 159,187], were originally motivated by questions about CSPs.

There has been less work on algebras over infinite domains. However, a considerable amount of universal-algebraic techniques also applies when the algebra under consideration contains as operations all the permutations from an oligomorphic permutation group; we will call such algebras *oligomorphic*. The assumption that the algebra be oligomorphic seems to provide the necessary amount of 'finiteness' that we need for applying universal-algebraic methods.

We have decided to give a self-contained presentation of this section, even though that this implies a certain redundancy for the reader who has already followed Chapter 3. The step from algebras with only unary functions to algebras that contain higher-ary functions is the step where universal algebra becomes interesting. At the same time, the step from studying automorphisms and embeddings to studying polymorphisms is the step that is new to model-theorists, so we found it natural to divide the material into a chapter on model theoretic and a chapter on universal algebraic background.

This chapter contains original material from [26, 27, 29, 30, 47].

5.1. Oligomorphic Clones

Fix a countably infinite base set B, also called the *domain* or *base set*. For $n \geq 1$, denote by $\mathscr{O}^{(n)}$ the set $B^{B^n} = (B^n \to B)$ of n-ary operations on B. Then $\mathscr{O} := \bigcup_{n \geq 1} \mathscr{O}^{(n)}$ is the set of all operations on B of finite arity. A *clone* \mathscr{C} is a subset of \mathscr{O} satisfying the following two properties:

- \mathscr{C} contains all projections, i.e., for all $1 \leq k \leq n$ the operation $\pi_k^n \in \mathscr{O}^{(n)}$ defined by $\pi_k^n(x_1, \ldots, x_n) = x_k$, and
- \mathscr{C} is closed under composition, that is, for all $f \in \mathscr{C} \cap \mathscr{O}^{(n)}$ and $g_1, \ldots, g_n \in \mathscr{C} \cap \mathscr{O}^{(m)}$ the operation $f(g_1, \ldots, g_n) \in \mathscr{O}^{(m)}$ defined by

$$(x_1,\ldots,x_m)\mapsto f(g_1(x_1,\ldots,x_m),\ldots,g_n(x_1,\ldots,x_m))$$

is an element of \mathscr{C} .

For our applications of universal algebra in logic, we are interested in clones that satisfy an additional topological closure property¹. A clone \mathscr{C} is called *locally closed*² if and only if it satisfies the following interpolation property:

for all $n \ge 1$ and all $g \in \mathcal{O}^{(n)}$, if for all finite $A \subseteq B^n$ there exists an n-ary $f \in \mathcal{C}$ which agrees with g on A, then $g \in \mathcal{C}$.

¹The corresponding topology is defined as follows. Equip B with the discrete topology, and $\mathscr{O}^{(n)} = B^{B^n}$ with the corresponding product (Tychonoff) topology, for every $n \geq 1$. (For background in topology, see Chapter 7.) A clone \mathscr{C} is *closed* with respect to this topology iff each of its n-ary fragments $\mathscr{C} \cap \mathscr{O}^{(n)}$ is a closed subset of $\mathscr{O}^{(n)}$.

²In universal algebra, locally closed clones are often just called *local clones*. Topologists would call such objects just *closed clones* since the reference to the specific topology under consideration is clear. Our choice to call those objects *locally closed* is a compromise, and standard [190].

The following proposition is folklore in universal algebra, see e.g. [190]. Its proof is very similar to the proof of Proposition 3.3.2, and we leave it to the reader.

Proposition 5.1.1. Let $\mathscr{F} \subseteq \mathscr{O}$ be a set of operations. Then the following are equivalent.

- (1) \mathscr{F} is the polymorphism clone of a relational structure;
- (2) \mathcal{F} is a locally closed clone.

Arbitrary intersections of clones are clones, and arbitrary intersections of locally closed sets are locally closed. In fact, the set of all locally closed clones on B, partially ordered by inclusion, forms a complete lattice. The join of a family $(\mathscr{C}_i)_{i\in I}$ can be obtained as follows. First, take the set of all operations on B which can be obtained by composing operations from $\bigcup_{i\in I}\mathscr{C}_i$; this set is a clone, but might not be locally closed. For this reason, we have to additionally form the topological closure of this set in order to arrive at the join in this lattice. For a set of operations $\mathscr{F}\subseteq\mathscr{O}$, we write $\langle\mathscr{F}\rangle$ for the clone locally generated by \mathscr{F} , i.e., for the smallest locally closed clone containing \mathscr{F} .

DEFINITION 5.1.2. Let $\mathscr{C} \subseteq \mathscr{O}$ be a clone. A unary operation $e \in \mathscr{C}$ is called invertible in \mathscr{C} if there exists a unary $i \in \mathscr{C}$ such that i(e(x)) = e(i(x)) = x for all $x \in B$.

Suppose that $\mathscr C$ is the polymorphism clone of a structure $\mathfrak B$. Then obviously any invertible operation of $\mathscr C$ is an automorphism of $\mathfrak B$.

DEFINITION 5.1.3. A clone $\mathscr{C} \subseteq \mathscr{O}$ is oligomorphic if the set of operations in \mathscr{C} that are invertible in \mathscr{C} forms an oligomorphic permutation group.

It is immediate from Theorem 3.1.4 that a locally closed clone is oligomorphic if and only if it is the polymorphism clone of an ω -categorical structure.

5.2. The Inv-Pol Galois Connection

The exposition in this section parallels that of Section 3.3. Let f be from $\mathcal{O}^{(n)}$, and let $R \subseteq B^m$ be a relation. Then we say that f preserves R (and that R is invariant under f) iff $f(r_1, \ldots, r_n) \in \rho$ whenever $r_1, \ldots, r_n \in R$, where $f(r_1, \ldots, r_n)$ is calculated componentwise. For a relational structure \mathfrak{B} (or for a set of relations \mathcal{R}) with domain B, we write $\text{Pol}(\mathfrak{B})$ (or $\text{Pol}(\mathcal{R})$, respectively) for the set of those operations in \mathcal{O} which preserve all relations from \mathfrak{B} (all relations in \mathcal{R}). The operations in $\text{Pol}(\mathfrak{B})$ are called polymorphisms of \mathfrak{B} . Note that the polymorphisms of \mathfrak{B} are exactly the homomorphism from finite powers of \mathfrak{B} to \mathfrak{B} .

We have seen how to assign sets of operations to sets of relations; likewise, we can go the other way. Given a set of operations $\mathscr{F} \subseteq \mathscr{O}$, we write $\operatorname{Inv}(\mathscr{F})$ for the set of all relations which are invariant under all $f \in \mathscr{F}$. Using the Galois connection defined by the operators Pol and Inv, we obtain the following well-known description of the hull operator $\mathscr{F} \mapsto \operatorname{Pol}(\operatorname{Inv}(\mathscr{F}))$ (confer [190], in particular Corollary 1.9).

Proposition 5.2.1. Let $\mathscr{F} \subseteq \mathscr{O}$, and $g \in \mathscr{O}$. Then the following are equivalent.

- (1) $q \in \langle \mathscr{F} \rangle$;
- (2) g is in the local closure of the operations of the clone generated by \mathscr{F} ;
- (3) For all $n \ge 1$ and all $a_1, \ldots, a_k \in B^n$ there is a f in the clone generated by \mathscr{F} such that $g(a_1, \ldots, a_k) = f(a_1, \ldots, a_k)$.
- (4) $g \in \text{Pol}(\text{Inv}(\mathscr{F}));$

In particular, $\langle \mathscr{F} \rangle = \text{Pol}(\text{Inv}(\mathscr{F})).$

PROOF. Note that the set \mathscr{F}' of all operations that are in the local closure of the clone generated by \mathscr{F} is a clone and locally closed. Therefore, the clone $\langle \mathscr{F} \rangle$ is contained in \mathscr{F}' , and (1) implies (2).

For the implication from (2) to (3), let g be a k-ary operation that is in the local closure of the clone \mathscr{F}' generated by \mathscr{F} . Let a_1, \ldots, a_k be from B^n for some $n \geq 1$. Suppose $a_i = (a_i^1, \ldots, a_i^n)$ for $i \leq k$, and let $a^j = (a_1^j, \ldots, a_k^j)$ for $j \leq n$. Since g is in the closure of \mathscr{F}' , there exists an $f \in \mathscr{F}'$ that agrees with g on $\{a^1, \ldots, a^n\} \subseteq B^k$. In particular, $g(a_1, \ldots, a_k) = f(a_1, \ldots, a_k)$.

For the implication from (3) to (4), assume (3), and let R be from $\operatorname{Inv}(\mathscr{F})$. We have to show that g preserves R. Let t_1, \ldots, t_k be from R. By assumption $f(t_1, \ldots, t_k) = g(t_1, \ldots, t_k)$ for some operation g generated from operations in \mathscr{F} and projections. Since all those operations preserve R, we have that $f(t_1, \ldots, t_k) \in R$.

To show that (4) implies (1), let f be a k-ary operation from $\operatorname{Pol}(\operatorname{Inv}(\mathscr{F}))$. Let \mathscr{C} be the clone generated by \mathscr{F} . It suffices to show that for every finite subset A of B there is an operation $g \in \mathscr{C}$ such that $f(\bar{a}) = g(\bar{a})$ for all $\bar{a} \in A^k$. List all elements of A^k by a^1, \ldots, a^n , and consider the relation

$$R := \{ (g(a^1), \dots, g(a^n)) \mid g \in \mathscr{C} \}.$$

Note that R is preserved by all operations in \mathscr{F} and so by assumption f preserves R. Also note that $(a_i^1,\ldots,a_i^n)\in R$ since \mathscr{C} contains the projections. Therefore, $(f(a^1),\ldots,f(a^n))\in R$, and hence $(f(a^1),\ldots,f(a^n))=(g(a^1),\ldots,g(a^n))$ for a $g\in\mathscr{C}$, as required.

The following is straightforward.

PROPOSITION 5.2.2. Let \mathfrak{B} be any structure. Then $Inv(Pol(\mathfrak{B}))$ contains $\langle \mathfrak{B} \rangle_{pp}$, the set of all relations that are primitive positive definable in \mathfrak{B} .

PROOF. Suppose that R is k-ary, has a primitive positive definition ψ , and let f be an l-ary polymorphism of \mathfrak{B} . To show that f preserves R, let t_1,\ldots,t_l be tuples from R. Then there must be witnesses for the existentially quantified variables x_{l+1},\ldots,x_n of ψ that show that $\psi(t_i)$ holds in \mathfrak{B} , for all $1 \leq i \leq n$. Write s_i for the extension of t_i such that s_i satisfies the quantifier-free part $\psi'(x_1,\ldots,x_l,x_{l+1},\ldots,x_n)$ of ψ (we assume that ψ is written in prenex normal form). Then the tuple

$$(f(s_1[1],\ldots,s_l[1]),\ldots,f(s_1[n],\ldots,s_l[n]))$$

satisfies ψ' as well. This shows that $(f(s_1[1], \ldots, s_l[1]), \ldots, f(s_1[k], \ldots, s_l[k]))$ satisfies ψ in \mathfrak{B} , which is what we had to show.

A relation R has a primitive positive definition in a *finite* structure if and only if R is preserved by all polymorphisms of this structure. This was discovered by Geiger [102] and independently by Bodnarcuk et al. [56], and is of central importance in universal algebra. We have the following generalization of this theorem to ω -categorical structures [46].

THEOREM 5.2.3 (from [46]). Let \mathfrak{B} be an ω -categorical or a finite structure. A relation R has a primitive positive definition in \mathfrak{B} if and only if R is preserved by all polymorphisms of \mathfrak{B} ; in symbols, $\operatorname{Inv}(\operatorname{Pol}(\mathfrak{B})) = \langle \mathfrak{B} \rangle_{pp}$.

PROOF. One direction has been shown in Proposition 5.2.2. For the other direction, let R be a k-ary relation that is preserved by all polymorphisms of \mathfrak{B} . In particular, R is preserved by all automorphisms of \mathfrak{B} , and hence R is a union of orbits of k-tuples in the automorphism group of \mathfrak{B} . By item (2) of Theorem 3.1.4, there is a finite number of such orbits, O_1, \ldots, O_w . If R is empty, there is nothing to show (but we again use the assumption that \bot is allowed as a primitive positive formula),

so let us assume that $w \geq 1$. Fix for each $1 \leq j \leq w$ a k-tuple a_j from O_j . Let B be the domain of \mathfrak{B} . Let b_1, b_2, \ldots be an enumeration of all w-tuples in B^w with the additional property that for $1 \leq i \leq k$ we have $b_i = (a_1[i], \ldots, a_w[i])$.

By Lemma 3.1.8, if for every finite substructure \mathfrak{A} of \mathfrak{B}^w that contains b_1, \ldots, b_k there is a homomorphism from \mathfrak{A} to \mathfrak{B} that maps (b_1, \ldots, b_k) to a tuple that is not in R, then there is also a homomorphism from \mathfrak{B}^k to \mathfrak{B} that maps (b_1, \ldots, b_k) to a tuple that is not in R, and this would be a polymorphism of \mathfrak{B} violating R (the properties of a mapping to be a polymorphism and to violate R have universal axiomatizations). So there must be a finite substructure \mathfrak{A} containing the vertices b_1, \ldots, b_k of B^w such that all homomorphisms from \mathfrak{A} to \mathfrak{B} map b_1, \ldots, b_k to a tuple in R.

Let q_1, \ldots, q_l be the vertices of $\mathfrak A$ without b_1, \ldots, b_k . Write ψ for the quantifier-free part of the canonical query of $\mathfrak A$ (see Section 1.2.1). We claim that the formula $\exists q_1, \ldots, q_l. \psi$ is a primitive positive definition of R. The above argument shows that $\exists q_1, \ldots, q_l. \psi$ implies $R(b_1, \ldots, b_k)$. To show that every tuple in R satisfies $\exists q_1, \ldots, q_l. \psi$, let $f: \mathfrak A \to \mathfrak B$ be a homomorphism such that the tuple $(f(b_1), \ldots, f(b_k))$ is from R. Then $(f(b_1), \ldots, f(b_k)) \in O_j$ for some $1 \leq j \leq w$. There is an automorphism α of $\mathfrak B$ sending a_j to $(f(b_1), \ldots, f(b_k))$. So we can extend f to a homomorphism from $\mathfrak B^w$ to $\mathfrak B$ by setting $f(x_1, \ldots, x_n) := \alpha(x_j)$. This shows in particular that $f(b_1), \ldots, f(b_k)$) satisfies $\exists q_1, \ldots, q_l. \psi$.

Analogously to Corollary 3.3.9, we obtain a Galois connection between structures with a first-order definition in an ω -categorical structure \mathfrak{C} , considered up to primitive positive interdefinability, and locally closed clones containing $\operatorname{Aut}(\mathfrak{C})$.

Theorem 5.2.4. Let \mathfrak{C} be a countable ω -categorical structure. Then we have the following.

- (1) The set of sets of the form $\langle \mathfrak{B} \rangle_{pp}$, where \mathfrak{B} is first-order definable in \mathfrak{C} , ordered by inclusion, forms a lattice.
- (2) The set of locally closed clones that contain $Aut(\mathfrak{C})$, ordered by inclusion, forms a lattice.
- (3) The operator Inv is an anti-isomorphism between those two lattices, and Pol is its inverse.

The above theorem tells us that classifying the reducts of an ω -categorical structure \mathfrak{C} up to primitive positive interdefinability really amounts to understanding the lattice of locally closed clones containing the automorphisms of \mathfrak{C} .

5.3. Essential Arity

This section investigates the *bottom* of the lattice of locally closed clones \mathscr{C} that contain a fixed oligomorphic permutation group.

5.3.1. Essentially unary operations. We say that $f \in \mathcal{O}^{(k)}$ depends on the argument $i \in \{1, \ldots, k\}$ if there is no (k-1)-ary operation f' such that $f(x_1, \ldots, x_k) = f'(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$ for all $x_1, \ldots, x_k \in B$. We can equivalently characterize k-ary operations that depend on the i-th argument by requiring that there are $x_1, \ldots, x_k \in B$ and $x_i' \in B$ such that

$$f(x_1,\ldots,x_k) \neq f(x_1,\ldots,x_{i-1},x'_i,x_{i+1},\ldots,x_k)$$
.

We say that an operation f is essentially unary iff there is an $i \in \{1, ..., k\}$ and a unary operation f_0 such that $f(x_1, ..., x_k) = f_0(x_i)$. Operations that are not essentially unary are called essential.³

³This is standard in clone theory, and it makes sense also for us, since the essential operations are those that are essential for complexity classification.

Definition 5.3.1. For any set B, the relations P_3^B and P_4^B over B are defined as follows.

$$P_3^B = \{(a, b, c) \in B^3 \mid a = b \text{ or } b = c\}$$

$$P_4^B = \{(a, b, c, d) \in B^4 \mid a = b \text{ or } c = d\}$$

Lemma 5.3.2. Let $f \in \mathcal{O}$ be an operation on a set B. Then the following are equivalent.

- (1) f is essentially unary.
- (2) f preserves P_3^B . (3) f preserves P_4^B .
- (4) f depends on at most one argument.

PROOF. Let k be the arity of f. The implication from (1) to (2) is obvious, since unary operations clearly preserve P_3^B .

To show the implication from (2) to (3), we show the contrapositive, and assume that f violates P_4^B . By permuting arguments of f, we can assume that there are an $l \leq k$ and 4-tuples $a^1, \ldots, a^k \in P_4^B$ with $f(a^1, \ldots, a^k) \notin P_4^B$ such that in a^1, \ldots, a^l the first two coordinates are equal, and in a^{l+1}, \ldots, a^k the last two coordinates are equal. Let c be the tuple $(a_1^1, \ldots, a_1^l, a_4^{l+1}, \ldots, a_4^k)$. Since $f(a^1, \ldots, a^k) \notin P_4^B$ we have $f(a_1^1, \ldots, a_1^k) \neq f(a_2^1, \ldots, f_2^k)$, and therefore $f(c) \neq f(a_1^1, \ldots, a_1^k)$ or $f(c) \neq f(a_2^1, \ldots, a_2^k)$. Let $d = (a_1^1, \ldots, a_1^k)$ in the first case, and $d = (a_2^1, \ldots, a_2^k)$ in the second case. Likewise, we have $f(c) \neq f(a_3^1, \ldots, a_3^k)$ or $f(c) \neq f(a_4^1, \ldots, a_4^k)$, and let $e = (a_1^3, \ldots, a_3^k)$ in the first, and $e = (a_4^1, \ldots, a_4^k)$ in the second case. Then for each $i \leq k$ the tuple (d_1, c, e_1) is from P_2^B but $(f(d), f(c), f(e)) \notin P_2^B$ $i \leq k$, the tuple (d_i, c_i, e_i) is from P_3^B , but $(f(d), f(c), f(e)) \notin P_3^B$.

The proof of the implication from (3) to (4) is again by contraposition. Suppose f depends on the i-th and j-th argument, $1 \le i \ne j \le k$. Hence there exist tuples $a_1, b_1, a_2, b_2 \in B^k$ such that a_1, b_1 and a_2, b_2 only differ at the entries i and j, respectively, and such that $f(a_1) \neq f(b_1)$ and $f(a_2) \neq f(b_2)$. Then $(a_1(l), b_1(l), a_2(l), b_2(l)) \in P_4^B$ for all $l \leq k$, but $(f(a_1), f(b_1), f(a_2), f(b_2)) \notin P_4^B$, which shows that f violates P_4^B .

For the implication from (4) to (1), suppose that f depends only on the first argument. Let $i \leq k$ be maximal such that there is an operation g with $f(x_1, \ldots, x_k) =$ $g(x_1,\ldots,x_i)$. If i=1 then f is essentially unary and we are done. Otherwise, observe that since f does not depend on the i-th argument, neither does g, and so there is an (i-1)-ary operation g' such that for all $x_1, \ldots, x_n \in B$ we have $f(x_1,\ldots,x_n)=g(x_1,\ldots,x_i)=g'(x_1,\ldots,x_{i-1}),$ contradicting the choice of i.

Combined with Theorem 3.4.7, we obtain for ω -categorical structures a characterization of the situation where disjunction can be eliminated from existential positive formulas.

Proposition 5.3.3. Let \mathfrak{B} be an ω -categorical structure, and let \mathscr{C} be its polymorphism clone. Then the following are equivalent.

- (1) All relations with an existential positive definition in B also have a primitive positive definition in \mathfrak{B} .
- (2) The relation P₃^B is contained in Inv(\$\mathcal{C}\$).
 (3) All operations in \$\mathcal{C}\$ are essentially unary.

PROOF. (1) implies (2). The formula $(x = y) \lor (y = z)$ is existential positive, and thus has a primitive positive definition in \mathfrak{B} ; such formulas are preserved by \mathscr{C} .

(2) implies (3). Follows from Lemma 5.3.2.

(3) implies (1). Unary operations preserve all existentially positive formulas. Hence, when ϕ is an existential positive formula, then by assumption all polymorphisms of \mathfrak{B} preserve ϕ , and ϕ is equivalent to a primitive positive formula by Theorem 3.4.7.

If all operations of a clone & are essentially unary, we say that & is essentially unary.

5.3.2. Elementary Clones. If every polymorphism of an ω -categorical structure \mathfrak{B} is locally generated by the automorphisms of \mathfrak{B} , then \mathfrak{B} has the remarkable property that every first-order formula is equivalent to a primitive positive formula over \mathfrak{B} . In this case, the polymorphism clone of \mathfrak{B} is the smallest element of the lattice of locally closed clones described in Section 5.2. The facts in this section are straightforward combinations of previous results. We state them for future use.

COROLLARY 5.3.4. Let \mathfrak{B} be an ω -categorical structure, and let \mathscr{C} be its polymorphism clone. Then the following are equivalent.

- (1) Every relation with a first-order definition also has a primitive positive definition in \mathfrak{B} .
- (2) \mathfrak{B} is a model-complete core, and P_3^B is primitive positive definable in \mathfrak{B} . (3) \mathscr{C} is locally generated by the invertible operations in \mathscr{C} .
- (4) All operations in $\mathscr C$ are elementary, i.e., preserve all first-order definable relations in B.
- PROOF. (1) implies (2). We assume that every first-order definable relation has a primitive positive definition, and hence is preserved by all polymorphisms of \mathfrak{B} . In particular, the endomorphisms of $\mathfrak B$ preserve all first-order definable relations, and hence \mathfrak{B} is a model-complete core. Moreover, the relation P_3^B is clearly first-order definable, and therefore also primitive positive definable in \mathfrak{B} .
- (2) implies (3). Assume (2). Then Proposition 5.3.3 implies that all polymorphisms of \mathfrak{B} are essentially unary. Thus, for every n-ary polymorphism f of \mathfrak{B} there is an endomorphism g of \mathfrak{B} and an $j \leq n$ such that $f(x_1,\ldots,x_n)=g(x_i)$. Since $\mathfrak B$ is a model-complete core, and by Theorem 3.6.11, g is locally generated by the automorphisms of \mathfrak{B} , and in particular by the invertible operations in \mathscr{C} . Hence, f is locally generated by the locally invertible operations in \mathscr{C} , which proves (3).
- (3) implies (4). Invertible operations of $\mathscr C$ preserve all first-order definable relations in \mathfrak{B} . Hence, the implication follows from Proposition 5.2.1.
 - (4) implies (1). By Theorem 5.2.3.

We will see in Corollary 5.5.10 that when the equivalent conditions from Corollary 5.3.4 apply, then \mathfrak{B} has a finite signature reduct \mathfrak{B}' whose CSP is NP-hard.

- **5.3.3.** Arity Reduction. For many combinatorial arguments with oligomorphic clones it is crucial to have bounds on the arity of operations with certain properties. A basic, yet extremely useful observation to obtain such bounds is the following (which holds for arbitrary structures \mathfrak{B}).
- LEMMA 5.3.5. Let \mathfrak{B} be a relational structure and let R be a k-ary relation contained in m orbits of k-tuples of $Aut(\mathfrak{B})$. If \mathfrak{B} has a polymorphism f that violates R, then \mathfrak{B} also has an at most m-ary polymorphism that violates R.
- PROOF. Let f' be an polymorphism of \mathfrak{B} of smallest arity l that violates R. Then there are k-tuples $t_1, \ldots, t_l \in R$ such that $f'(t_1, \ldots, t_l) \notin R$. For l > m there are two tuples t_i and t_j that lie in the same orbit of k-tuples, and therefore \mathfrak{B} has

an automorphism α such that $\alpha t_j = t_i$. By permuting the arguments of f', we can assume that i = 1 and j = 2. Then the (l-1)-ary operation g defined as

$$g(x_2,\ldots,x_l):=f'(\alpha x_2,x_2,\ldots,x_l)$$

is also a polymorphism of \mathfrak{B} , and also violates R, a contradiction. Hence, $l \leq m$. \square

We present applications of Lemma 5.3.5. Recall that $r(\mathcal{G})$ denotes the rank of \mathcal{G} , i.e., the number of orbitals of \mathcal{G} (see Section 3.3).

COROLLARY 5.3.6. Let \mathfrak{B} be a structure with an automorphism group \mathscr{G} . If \mathfrak{B} has an essential polymorphism, then it must also have an essential polymorphism of arity at most $2r(\mathscr{G}) - 1$.

PROOF. The structure \mathfrak{B} has an essential polymorphism if and only if it has a polymorphism that violates the relation P_3^B , where B is the domain of \mathfrak{B} , by Proposition 5.3.3. The relation P_3^B consists of at most $2r(\mathscr{G}) - 1$ orbits of triples: there are at most $r(\mathscr{G})$ orbits of triples (t_1, t_2, t_3) where $t_1 = t_2 \neq t_3$, and at most that many where $t_1 \neq t_2 = t_3$. Only the orbit of the tuple where $t_1 = t_2 = t_3$ is counted twice. The statement follows from Lemma 5.3.5.

COROLLARY 5.3.7. Let $\mathfrak B$ be first-order definable in $(\mathbb Q;<)$, and suppose there is a polymorphism of $\mathfrak B$ that violates <. Then there is also an endomorphism of $\mathfrak B$ that violates <.

PROOF. Observe that < consists of a single orbit of pairs in $\operatorname{Aut}((\mathbb{Q};<))$, and therefore also in $\operatorname{Aut}(\mathfrak{B})$.

COROLLARY 5.3.8. Let $\mathscr{F} \subseteq \mathscr{O}$ be a local clone that contains a 2-transitive permutation group \mathscr{G} . If there is an $f \in \mathscr{F}$ that violates \neq , then \mathscr{F} contains a constant operation.

PROOF. The relation \neq consists of a single orbit of pairs in \mathscr{G} . Hence, there is a unary operation in \mathscr{F} that violates \neq by Lemma 5.3.5. The rest follows by Lemma 3.4.11.

Another application of Lemma 5.3.5 can be found in Section 5.4, and many applications can be found in Chapters 9 and 10.

5.3.4. Kára's method. We present another method for showing that an oligomorphic clone with essential operations must contain a *binary* essential operation. The method applies in many cases where the arity bounds from Corollary 5.3.6 are too weak. The idea is taken from [40], where it has been stated only for structures that are preserved by all permutations of the domain, and it has been generalized slightly in [51]. To state the method in full generality, we introduce the following, apparently new, concept.

DEFINITION 5.3.9. A permutation group \mathscr{G} on a set B has the orbital extension property if there is an orbital O such that for all $b_1, b_2 \in B$ there is an element $c \in B$ such that $(c, b_1) \in O$ and $(c, b_2) \in O$.

Note that every permutation group with the orbital extension property is transitive, and that every 2-transitive infinite permutation group has the orbital extension property. More examples of oligomorphic permutation groups with the orbital extension property are the automorphism group of the Random graph, (\mathbb{Q} ; <), the universal homogeneous poset, the universal homogeneous C-relation, and many more. An example of a structure without the orbital extension property is $K_{\omega,\omega}$, the complete bipartite graph where both parts are countably infinite. An example of an imprimitive

oligomorphic permutation group with the orbital extension property is the automorphism group of an equivalence relation on an infinite set with infinitely many infinite classes.

Lemma 5.3.10. Let $\mathscr C$ be a clone that contains a permutation group $\mathscr G$ with the orbital extension property. Then, if $\mathscr C$ contains an essential operation, it must also contain a binary essential operation.

PROOF. Let f be an essential operation in \mathscr{C} , and let k be the arity of f. Assume without loss of generality that f depends all its arguments and is at least ternary. In particular, there are a_1, \ldots, a_k and a'_1 such that $f(a_1, \ldots, a_k) \neq f(a'_1, a_2, \ldots, a_k)$. Let O be the orbital of \mathscr{G} which exist due to the orbital extension property of \mathscr{G} . We distinguish two cases.

Case 1. There are elements b_1, \ldots, b_k such that $(b_i, a_i) \in O$ for $2 \le i \le k$ and $f(b_1, a_2, \ldots, a_k) \ne f(b_1, \ldots, b_k)$. Then there are $\alpha_3, \ldots, \alpha_k \in \mathscr{G}$ such that $\alpha_i(a_2) = a_i$ and $\alpha_i(b_2) = b_i$. We define

$$g(x,y) := f(x,y,\alpha_3(y),\ldots,\alpha_k(y)) ,$$

which clearly depends on both arguments.

Case 2. For all b_1, \ldots, b_k , if $(a_i, b_i) \in O$ for $2 \le i \le k$, then $f(b_1, a_2, \ldots, a_k) = f(b_1, b_2, \ldots, b_k)$. Since f depends on its second coordinate, there are c_1, \ldots, c_k and c'_2 such that $f(c_1, c_2, c_3, \ldots, c_k) \ne f(c_1, c'_2, c_3, \ldots, c_k)$. Then $f(c_1, a_2, \ldots, a_k)$ can be equal to either $f(c_1, c_2, c_3, \ldots, c_k)$, or to $f(c_1, c'_2, c_3, \ldots, c_k)$, but not to both. We assume without loss of generality that $f(c_1, a_2, \ldots, a_k) \ne f(c_1, c_2, c_3, \ldots, c_k)$. Since $\mathscr G$ has the orbital extension property, we can choose d_2, \ldots, d_k such that for each $1 \le i \le k$, the pairs $1 \le i \le k$, the pairs $1 \le i \le k$ such that $1 \le i \le k$

$$g(x,y) := f(x,y,\alpha_3(y),\ldots,\alpha_k(y))$$

depends on both arguments. Indeed, we know that $g(a_1,d_2)=f(a_1,d_2,\ldots,d_k)=f(a_1,\ldots,a_k)$, and that $f(a_1',d_2)=f(a_1',d_2,\ldots,d_k)=f(a_1',a_2,\ldots,a_k)$. By the choice of the values a_1,\ldots,a_k and a_1' these two values are distinct, and we have that g depends on the first argument. For the second argument, note that $g(c_1,d_2)=f(c_1,d_2,\ldots,d_k)=f(c_1,a_2,\ldots,a_k)$ and that $g(c_1,c_2)=f(c_1,c_2,\ldots,c_k)$. Because $f(c_1,a_2,\ldots,a_k)$ and $f(c_1,c_2,\ldots,c_k)$ are distinct, the function g also depends on the second argument.

Corollary 5.3.11. Let \mathfrak{B} be 2-transitive with an essential polymorphism. Then \mathfrak{B} also has a binary essential polymorphism.

5.3.5. Minimal Operations. In some classification results, e.g. in Chapter 9 and Chapter 10 it turns out that a *bottom-up* approach works best: for example in Chapter 9 we first classify all the minimal (with respect to set inclusion) locally closed clones that strictly contain the automorphism group of the random graph, and then the classification argument is organised according to those *minimal* clones.

Definition 5.3.12. Let $\mathscr{C} \subseteq \mathscr{O}$ be a locally closed clone. We say that

- a local clone $\mathscr{D} \subseteq \mathscr{O}$ is minimal above \mathscr{C} if $\mathscr{C} \subsetneq \mathscr{D}$, and $\mathscr{C} \subsetneq \mathscr{E} \subseteq \mathscr{D}$ implies $\mathscr{E} = \mathscr{D}$ for all locally closed clones \mathscr{E} .
- a function $f \in \mathcal{O}$ is minimal above \mathcal{C} if f is not from \mathcal{C} , and of minimal arity such that for every $g \notin \mathcal{C}$ that is locally generated by $\mathcal{C} \cup \{f\}$ we have that $\mathcal{C} \cup \{g\}$ locally generates f.

The following is obvious from the definitions.

PROPOSITION 5.3.13. Let \mathscr{C} be a locally closed clone. Then every minimal operation above \mathscr{C} locally generates a clone that is minimal above \mathscr{C} , and every clone that is minimal above \mathscr{C} is locally generated by a minimal operation above \mathscr{C} .

For oligomorphic clones \mathscr{C} , minimality translates into maximality of $\operatorname{Inv}(\mathscr{C})$, and we obtain the following.

PROPOSITION 5.3.14. Let \mathfrak{B} be an ω -categorical structure, and let \mathfrak{C} be primitive positive definable in \mathfrak{B} . Then $\operatorname{Pol}(\mathfrak{C})$ is minimal above $\operatorname{Pol}(\mathfrak{B})$ if and only if for every $R \in \langle \mathfrak{B} \rangle_{pp} \setminus \langle \mathfrak{C} \rangle_{pp}$ the structure \mathfrak{B} has a primitive positive definition in (\mathfrak{C}, R) .

PROOF. The equivalence follows from Proposition 5.2.4.

It is well-known that every clone over a finite domain contains a clone \mathscr{D} which is minimal above the trivial clone that just contains the projections [81]. The same is not true for infinite domains: the clone with domain \mathbb{N} generated by the operation $x \mapsto x+1$ does not contain a clone that is minimal above the set of all projections over \mathbb{N} . The situation is again better when \mathscr{C} is oligomorphic.

Theorem 5.3.15 (from [27]). Let \mathfrak{B} be a finite or ω -categorical structure with a finite relational signature, and let \mathscr{B} be its polymorphism clone. Then any clone \mathscr{C} that contains \mathscr{B} contains a locally closed clone \mathscr{D} that is minimal above \mathscr{B} .

PROOF. By Proposition 5.3.14, it suffices to show that there is a structure \mathfrak{D} whose relations are a subset of $\operatorname{Inv}(\mathscr{B})$ such that for every $R \in \operatorname{Inv}(\mathscr{B}) \setminus \operatorname{Inv}(\mathscr{C})$ there is a primitive positive definition of \mathfrak{B} in (\mathfrak{D}, R) .

Consider the partially ordered set of all locally closed clones that contain \mathscr{B} and that are contained in \mathscr{C} , ordered by inclusion. From this poset we remove \mathscr{B} , which is the unique minimal element; the resulting poset will be denoted by P. We claim that in P, all descending chains $S_1 \supseteq S_2 \supseteq \cdots$ are bounded, i.e., for all such chains there exists an $S \in P$ such that $S_i \supseteq S$ for all $i \ge 1$. To see this, observe that $\bigcup_{i \ge 1} \operatorname{Inv}(S_i)$ is closed under primitive positive definability in the sense that it can be written as $\langle \mathfrak{S} \rangle_{\operatorname{pp}}$ for some relational structure \mathfrak{S} (since only a finite number of relations can be mentioned in a formula, and since each of the S_i is closed under primitive positive definability).

Moreover, there must be a relation $R \in \mathfrak{B}$ that is not contained in $\bigcup_{i\geq 1} \operatorname{Inv}(S_i)$; otherwise, since \mathfrak{B} has finitely many relations, there is a $j < \omega$ such that $\operatorname{Inv}(S_j)$ contains all relations from \mathfrak{B} , and hence equals $\operatorname{Inv}(\mathscr{B})$, which is impossible since \mathfrak{B} is not an element of P. Hence, by Theorem 5.2.4, the structure \mathfrak{S} has a polymorphism f that is not from \mathfrak{B} . Then $\mathfrak{B} \cup \{f\}$ is contained in $\bigcap_{i\geq 1} S_i$ and is a lower bound of the descending chain $(S_i)_{i\geq 0}$. We can thus apply Zorn's lemma and conclude that P contains a minimal element S. Any structure \mathfrak{D} whose relations are exactly the relations from S satisfies the initial requirements.

For essentially unary oligomorpic clones \mathcal{B} , we can bound the arity of minimal functions above \mathcal{B} ; this follows from the following.

PROPOSITION 5.3.16 (from [54]). Let \mathfrak{B} be any relational structure with a finite number p of orbitals. Then every minimal clone above $\operatorname{End}(\mathfrak{B})$ is generated by a function of arity at most $2 \cdot p - 1$ together with $\operatorname{End}(\mathfrak{B})$.

PROOF. Let $\mathscr C$ be a minimal clone above $\operatorname{End}(\mathfrak B)$. If all the functions in $\mathscr C$ are essentially unary, then $\mathscr C$ is generated by a unary operation together with $\operatorname{End}(\mathfrak B)$ and we are done. Otherwise, let f be an essential operation in $\mathscr C$. By Lemma 5.3.2 the operation f violates P_3^B over the domain B of $\mathfrak B$; recall that P_3^B is defined by the formula $(x=y)\vee (y=z)$. The subset of P_3^B that contains all tuples of the form

(a,a,b), for $a,b \in B$, clearly consists of p orbits in \mathfrak{B} . Similarly, the subset of P_3^B that contains all tuples of the form (a,b,c), for $a,b \in B$, consists of the same number of orbits. The intersection of these two relations consists of exactly one orbit (namely, the triples with three equal entries), and therefore P_3^3 is the union of $2 \cdot p - 1$ different orbits. The assertion now follows from Lemma 5.3.5.

In Section 8.3 we will see that under further Ramsey-theoretic assumptions on the structure \mathfrak{B} , there are only *finitely many* minimal closed clones above the polymorphism clone of \mathfrak{B} .

5.4. Schaefer's Theorem

Schaefer's theorem states that every CSP for a 2-element template is either in P or NP-hard. Via the Inv-Pol Galois connection (Section 5.2), most of the classification arguments in Schaefer's article follow from earlier work of Post [176], who classified all clones on a two-element domain. We present a short proof of Schaefer's theorem here, using the results and ideas from Section 5.3.

The following operations are important in this context. A ternary function $f ext{:} B^3 \to B$ is called a *minority function* if it satisfies f(x,y,y) = f(y,x,y) = f(y,y,x) = x for all $x,y \in B$. It is called a *majority function* if f(x,x,y) = f(x,y,x) = f(y,x,x) = x for all $x,y \in B$. Note that on Boolean domains, the given identities determine f uniquely. A k-ary operation f is called *idempotent* iff $f(x,\ldots,x) = x$. Also recall the definition of NAE and 1IN3.

$$\begin{aligned} \text{NAE} = & \{ (0,0,1), (0,1,0), (1,0,0), (1,1,0), (1,0,1), (1,1,0) \} \\ & \text{1IN3} = & \{ (0,0,1), (0,1,0), (1,0,0) \} \end{aligned}$$

LEMMA 5.4.1. Let f be an idempotent function on the domain $\{0,1\}$ that violates 1IN3.

- If f is binary, then f must be $(x,y) \mapsto min(x,y)$ or $(x,y) \mapsto max(x,y)$.
- If f is ternary, then f generates min, max, the majority, or the minority operation.

PROOF. There are only four binary idempotent operations on $\{0,1\}$, two of which are projections and therefore preserve 1IN3. The other two operations are min and max. If f is ternary, then the operations defined by f(x,x,y), f(x,y,x), and f(y,x,x) must be projections, or otherwise they generate min or max and we are done. So we consider the following eight cases.

Cases	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
f(x, x, y)	x	x	x	x	y	y	y	y
f(x, y, x)		x	y	y	\boldsymbol{x}	x	y	y
f(y, x, x)	x	y	\boldsymbol{x}	y	\boldsymbol{x}	y	\boldsymbol{x}	y

In case (1) the operation f is a majority, and in case (8) a minority. The cases (2), (3), and (5) are impossible since in this case f would preserve 1IN3. In the remaining three cases, which are symmetric with respect to permuting arguments of f, the function defined by f(x, f(x, y, z), z) is a majority.

The following is well-known; the short proof is taken from [72].

Theorem 5.4.2. If R is a Boolean relation preserved by the minority operation, then R has a definition by a conjunction of linear equalities modulo 2.

PROOF. The proof is by induction on the arity k of R. The statement is clear when R is unary. Otherwise, let R_0 be the Boolean relation of arity k-1 defined by $R_0(x_2,\ldots,x_k)\equiv R(0,x_2,\ldots,x_k)$, and let $R_1\subseteq\{0,1\}^{k-1}$ be defined by $R_1(x_2,\ldots,x_k)\equiv R(1,x_2,\ldots,x_k)$. By the inductive assumption, there are conjunctions of linear equalities ψ_0 and ψ_1 defining R_0 and R_1 , respectively. If R_0 is empty, we may express $R(x_1,\ldots,x_k)$ by $x_1=1\wedge\psi_1$. The case that R_1 is empty can be treated analogously. When both R_0 and R_1 are non-empty, fix a tuple $(c_2^0,\ldots,c_k^0)\in R_0$ and a tuple $(c_2^1,\ldots,c_k^1)\in R_1$. Define c^0 to be $(0,c_2^0,\ldots,c_k^0)$ and c^1 to be $(1,c_2^0,\ldots,c_k^0)$. Let b be an arbitrary tuple from $\{0,1\}^k$. Observe that if $b\in R$, then $minority(b,c^0,c^1)\in R$, since $c^0\in R$ and $c^1\in R$. Moreover, if $minority(b,c^0,c^1)\in R$, then $minority(minority(b,c^0,c^1),c^0,c^1)=b\in R$. Thus, $b\in R$ if and only if $minority(b,c^0,c^1)\in R$. Specializing this to $b_1=1$, we obtain

$$(b_2,\ldots,b_k)\in R_1 \Leftrightarrow (minority(b_2,c_2^0,c_2^1),\ldots,minority(b_k,c_k^0,c_k^1))\in R_0$$
.

This implies

$$(b_1,\ldots,b_k)\in R \Leftrightarrow (minority(b_2,c_2^0b_1,c_2^1b_1),\ldots,minority(b_k,c_k^0b_1,c_k^1b_1))\in R_0.$$

Thus, $R(x_1, \ldots, x_k)$ is defined by the formula

$$\exists x_2', \dots, x_k' (\phi_0(x_2', \dots, x_k') \land \bigwedge_{i \in \{2, \dots, k\}} (x_i + c_i^0 x_1 + c_i^1 x_1 = x_i')) .$$

A binary relation is called *bijunctive* if it can be defined by a propositional formula that is a conjunction of clauses of size two (aka 2CNF formulas).

THEOREM 5.4.3 (of Post [176] and Schaefer [182]). Let \mathfrak{B} be a structure over a two-element universe. Then either ($\{0,1\}$; NAE) has a primitive positive definition in \mathfrak{B} , and CSP(\mathfrak{B}) is NP-complete, or

- (1) B is preserved by a constant operation.
- (2) \mathfrak{B} is preserved by min. In this case, every relation of \mathfrak{B} has a definition by a propositional Horn formula.
- (3) \mathfrak{B} is preserved by max. In this case, every relation of \mathfrak{B} has a definition by a dual-Horn formula, that is, by a propositional formula in CNF where every clause contains at most one negative literal.
- (4) \mathfrak{B} is preserved by the majority operation. In this case, every relation of \mathfrak{B} is bijunctive.
- (5) \mathfrak{B} is preserved by the minority operation. In this case, every relation of \mathfrak{B} can be defined by a conjunction of linear equations modulo 2.

In case (1) to case (5), $CSP(\mathfrak{B})$ can be solved in polynomial time.

PROOF. If the relation $1IN3 = \{(0,0,1), (0,1,0), (1,0,0)\}$ has a primitive positive definition in \mathfrak{B} , then Lemma 1.2.6 shows that the NP-hard problem positive 1-in-3-3SAT [101] (see Example 1.2.2) can be reduced to $CSP(\mathfrak{B})$. In this case, also the relation NAE is primitive positive definable in \mathfrak{B} , as we have seen in Example 1.2.7.

If 1IN3 is not primitive positive definable in \mathfrak{B} , then by Theorem 5.2.3 there is a polymorphism f of \mathfrak{B} that violates 1IN3; let f be such an operation of minimal arity. Because 1IN3 consists of three tuples only, Lemma 5.3.5 asserts that f is at most ternary.

If f is not unary, then $\hat{f} : \{0,1\} \to \{0,1\}$ defined by $x \mapsto f(x,\ldots,x)$ must preserve 1IN3 by the choice of f. Hence, \hat{f} is the identity and f is idempotent. If f is binary, then by Lemma 5.4.1 it is either min or max. In case that f is min, we show that all relations in \mathfrak{B} can be defined by propositional Horn formulas. It is well-known that

positive unit-resolution is a polynomial-time decision procedure for the satisfiability problem of propositional Horn-clauses [183]. The case that f is max is dual to this case.

So let R be a Boolean relation preserved by min. Let ϕ be a propositional formula in CNF that defines ϕ . We can assume without loss of generality that for all literals in clauses of ϕ , when we remove this literal from the clause, the resulting formula is not equivalent to ϕ . (Otherwise, we keep on removing literals until the formula has the required property.) Now suppose for contradiction that ϕ contains a clause C with two positive literals x and y. Since ϕ is reduced, there is an assignment s_1 that satisfies ϕ such that $s_1(x) = 1$, and such that all other literals of C evaluate to 0. Similarly, there is a satisfying assignment s_2 for ϕ such that $s_2(y) = 1$ and all other literal s of C evaluate to 0. Then $s_0: x \mapsto min(s_1(x), s_2(y))$ does not satisfy C, and does not satisfy ϕ , in contradiction to the assumption that min preserves R.

If f is ternary, then f either generates the minority or the majority operation, by Lemma 5.4.1 and the choice of f. If f generates the majority operation, we show that every relation of \mathfrak{B} is bijunctive. Hence, in this case $\mathrm{CSP}(\mathfrak{B})$ is equivalent to the 2SAT problem, and can be solved in linear time [7]. Let again ϕ be a reduced definition of a relation from R, and suppose that ϕ contains a clause C with three literals l_1, l_2, l_3 . Since ϕ is reduced, there must be satisfying assignments s_1, s_2, s_3 to ϕ such that under s_i all literals of C evaluate to 0 except for l_i . Then the mapping $s_0: x \mapsto majority(s_1(x), s_2(x), s_3(x))$ does not satisfy C and therefore does not satisfy ϕ , in contradiction to the assumption that majority preserves R.

If f generates the minority operation, and R is a relation of \mathfrak{B} , then by Theorem 5.4.2 the relation R has a definition by a conjunction of linear equalities modulo 2. Then $CSP(\mathfrak{B})$ can be solved in polynomial time by Gaussian elimination.

Finally, if f is unary, then f is either constant, and we are done, or f is the operation \neg defined by $x \mapsto 1-x$. If f is \neg , then NAE consists of three orbits of triples. If NAE is primitive positive definable in Φ , then CSP(\mathfrak{B}) is NP-hard by reduction from positive not-all-equal-3SAT [101] (see again Example 1.2.2). Otherwise, by Lemma 5.3.5, there is an at most ternary operation g that violates NAE. Since all non-constant unary operations preserve NAE, we can assume that g is at least binary. If g is binary and violates NAE, then there are $t_1, t_2 \in \text{NAE}$ such that $t_0 = g(t_1, t_2) \notin \text{NAE}$. For $i \in \{1, 2\}$, if $t_i \in \text{IIN3}$, set α_i to be id, otherwise set α_i to be \neg , and note that $\alpha_i t_i \in \text{IIN3}$. Then either $h: (x, y) \mapsto g(\alpha_1 x, \alpha_2 y)$ or $h: (x, y) \mapsto \neg g(\alpha_1 x, \alpha_2 y)$ is idempotent and violates 1IN3, and the statement follows from the above proof when we take $h \in \text{Pol}(\mathfrak{B})$ in place of f. The argument for ternary g follows the same lines.

Hard Boolean constraint languages can be characterized in many equivalent ways via Corollary 5.3.4. To see this, we need the following proposition.

Proposition 5.4.4. Let \mathfrak{B} be a structure over a two-element universe. Then the following are equivalent.

- (1) $(\{0,1\}; NAE)$ has a primitive positive definition in \mathfrak{B} .
- (2) **B** is neither preserved by min, max, minority, majority, nor the constant operations.
- (3) The polymorphism clone of \mathfrak{B} either contains only projections, or is generated by the unary operation $x \mapsto -x$.
- (4) In B every first-order formula is equivalent to a primitive positive formula.

PROOF. The implication from (1) to (2) follows from the fact that NAE is not preserved by *min*, *max*, *minority*, *majority*, and constant operations, which is straightforward to verify. A proof that (2) implies (3) can for instance be found in [72]

(Theorem 5.1). The implication from (3) to (4) follows from Corollary 5.3.4. For the implication from (4) to (1), note that NAE is preserved by $x \mapsto -x$, and hence preserved by all automorphisms of \mathfrak{B} . In particular, NAE is first-order definable in \mathfrak{B} . So (4) implies that NAE also has a primitive positive definition in \mathfrak{B} .

5.5. Pseudo-varieties and Primitive Positive Interpretations

Primitive positive definability is a strong tool to prove that certain CSPs are hard, but in some cases this tool is not strong enough. In this section we discuss the concept of *primitive positive interpretations*, and the matching universal-algebraic concept, which is the concept of *pseudo-varieties*.

5.5.1. Algebras. Algebras have been defined in Chapter 2: they are simply structures with a purely functional signature. When **A** is an algebra with signature τ and domain A, we denote by $\text{Clo}(\mathbf{A})$ the set of all functions with domain A of the form $(x_1, \ldots, x_n) \mapsto t(x_1, \ldots, x_n)$ where t is any term over the signature τ whose set of variables is contained in $\{x_1, \ldots, x_n\}$; clearly, $\text{Clo}(\mathbf{A})$ is closed under compositions, and contains the projections, and therefore forms a clone. In this section we recall some basic universal-algebraic facts that will be used in the following subsections.

When K is a class of algebras of the same signature, then

- P(K) denotes the class of all products of algebras from K.
- $P^{fin}(\mathcal{K})$ denotes the class of all finite products of algebras from \mathcal{K} .
- S(K) denotes the class of all subalgebras of algebras from K.
- H(K) denotes the class of all homomorphic images of algebras from K.

(Products, subalgebras, and homomorphic images have been defined in Chapter 2.) Note that closure under homomorphic images implies in particular closure under isomorphism. For the operators P, P^{fin} , S and H we often omit the brackets when applying them to single algebras, i.e., we write $H(\mathbf{A})$ instead of $H(\{\mathbf{A}\})$. The elements of $HS(\mathbf{A})$ are also called the *factors* of \mathbf{A} .

A class \mathcal{V} of algebras with the same signature τ is called a *pseudo-variety* if \mathcal{V} contains all homomorphic images, subalgebras, and direct products of algebras in \mathcal{V} , i.e., $H(\mathcal{V}) = S(\mathcal{V}) = P^{fin}(\mathcal{V})$. The class \mathcal{V} is called a *variety* if \mathcal{V} also contains all (finite and infinite) products of algebras in \mathcal{V} . So the only difference between pseudo-varieties and varieties is that pseudo-varieties need not be closed under direct products of infinite cardinality. The smallest pseudo-variety (variety) that contains an algebra \mathbf{A} is called the pseudo-variety (variety) *generated* by \mathbf{A} .

DEFINITION 5.5.1. Let \mathfrak{B} be a relational structure with domain B. An algebra with domain B whose operations are exactly the polymorphisms of \mathfrak{B} is called a polymorphism algebra of \mathfrak{B} .

Note that a relational structure can have many different polymorphism algebras, since Definition 5.5.1 does not prescribe how to assign function symbols to the polymorphisms of \mathfrak{B} . In our applications, the precise choice of the signature never plays a role, and therefore we sometimes refer to the polymorphism algebra of \mathfrak{B} , and denote it by $Alg(\mathfrak{B})$. So statements about a polymorphism algebra of \mathfrak{B} (or about the polymorphism algebra $Alg(\mathfrak{B})$) can typically be translated to statements that hold for all polymorphism algebras of \mathfrak{B} (e.g. in Theorem 5.5.6 below).

Also note that when \mathfrak{B} is ω -categorical, then the signature of the polymorphism algebra has cardinality 2^{ω} . This follows directly from Theorem 7.3.2 and the remark after Lemma 3.1.10.

Congruences and Quotients. When $\mu \colon C \to D$ is a map, then the *kernel* of μ is the equivalence relation E on C where $(c,c') \in E$ if $\mu(c) = \mu(c')$. For $c \in C$, we denote by c/E the equivalence class of c in E, and by C/E the set of all equivalence classes of elements of C. A *congruence* of an algebra \mathbf{A} is an equivalence relation that is preserved by all operations in \mathbf{A} . The results in Section 5.2 show that for ω -categorical structures $\mathfrak B$ with polymorphism algebra $\mathbf B$, the congruences of $\mathbf B$ are exactly the primitive positive definable equivalence relations over $\mathfrak B$.

PROPOSITION 5.5.2 (see [68]). Let A be an algebra. Then E is a congruence of A if and only if E is the kernel of a homomorphism from A to some other algebra B.

When K is a congruence of a τ -algebra **A**, then \mathbf{A}/K denotes τ -algebra with domain A/K where

$$f^{\mathbf{A}/K}(a_1/K,\ldots,a_k/K) = f^{\mathbf{A}}(a_1,\ldots,a_k)/K$$

where $a_1, \ldots, a_k \in A$ and $f \in \tau$ is k-ary. This is well-defined since K is preserved by all operations of **A**. If K is the kernel of μ then we also write \mathbf{A}/μ instead of \mathbf{A}/K . The following is well-known.

LEMMA 5.5.3 (The Homomorphism Lemma). Let **A** be a τ -algebra, let K be a congruence of **A**, and let $\mu_1 \colon A \to B_1$ and $\mu_2 \colon A \to B_2$ be two mappings with kernel K. Then \mathbf{A}/μ_1 is isomorphic to \mathbf{A}/μ_2 .

The following is also well known (see e.g. Theorem 6.3 in [68]).

LEMMA 5.5.4. Let **A** and **B** be algebras with the same signature, and let $\mu: \mathbf{A} \to \mathbf{B}$ be a homomorphism. Then the image of any subalgebra \mathbf{A}' of \mathbf{A} under μ is a subalgebra of \mathbf{B} , and the preimage of any subalgebra \mathbf{B}' of \mathbf{B} under μ is a subalgebra of \mathbf{A} .

PROOF. Let $f \in \tau$ be k-ary. Then for all $a_1, \ldots, a_k \in A'$,

$$f^{\mathbf{B}}(\mu(a_1),\ldots,\mu(a_k)) = \mu(f^{\mathbf{A}}(a_1,\ldots,a_k)) \in h(A')$$
,

so $\mu(A')$ is a subalgebra of **C**. Now suppose that $\mu(a_1), \ldots, \mu(a_k)$ are in B'; then $f^{\mathbf{B}}(\mu(a_1), \ldots, \mu(a_k)) \in B'$ and hence $\mu(f^{\mathbf{A}}(a_1, \ldots, a_k)) \in B'$. So, $f^{\mathbf{A}}(a_1, \ldots, a_k)) \in \mu^{-1}(B')$ which shows that $\mu^{-1}(B')$ induces a subalgebra of **A**.

5.5.2. Primitive Positive Interpretations. In Chapter 3, we have seen that first-order interpretations are a convenient tool to construct ω -categorical structures from other ω -categorical structures. Primitive positive interpretations are interpretations I where the domain formula δ_I and all the defining formulas ϕ_I are primitive positive. As we will see, such interpretations can be used to study the computational complexity of constraint satisfaction problems.

DEFINITION 5.5.5. Let I be an interpretation. If the domain formula δ_I and the interpreting formulas ϕ_I are primitive positive, then we say that I is a primitive positive interpretation.

We first start with a result that is known for finite domain constraint satisfaction, albeit not using the terminology of primitive positive interpretations [66]. In the present form, it appears first in [26].

THEOREM 5.5.6. Let \mathfrak{B} and \mathfrak{C} be structures with finite relational signatures. If there is a primitive positive interpretation of \mathfrak{B} in \mathfrak{C} , then there is a polynomial-time reduction from $CSP(\mathfrak{B})$ to $CSP(\mathfrak{C})$.

PROOF. Let d be the dimension of the primitive positive interpretation I of the τ -structure \mathfrak{B} in the σ -structure \mathfrak{C} , let $\delta_I(x_1,\ldots,x_d)$ be the domain formula, let $h:\delta_I(\mathfrak{C}^d)\to D(\mathfrak{B})$ be the coordinate map, and let $\phi_I(x_1,\ldots,x_{dk})$ be the formula for the k-ary relation R from \mathfrak{B} .

Let ϕ be an instance of CSP(\mathfrak{B}) with variable set $U = \{x_1, \ldots, x_n\}$. We construct an instance ψ of CSP(\mathfrak{C}) as follows. For distinct variables $V := \{y_1^1, \ldots, y_n^d\}$, we set ψ_1 to be the formula

$$\bigwedge_{1 \le i \le n} \delta(y_i^1, \dots, y_i^d) .$$

Let ψ_2 be the conjunction of the formulas $\theta_I(y_{i_1}^1,\ldots,y_{i_1}^d,\ldots,y_{i_k}^1,\ldots,y_{i_k}^d)$ over all conjuncts $\theta=R(x_{i_1},\ldots,x_{i_k})$ of ϕ . By moving existential quantifiers to the front, the sentence

$$\exists y_1^1, \ldots, y_n^d \ (\psi_1 \wedge \psi_2)$$

can be re-written to a primitive positive σ -formula ψ , and clearly ψ can be constructed in polynomial time in the size of \mathfrak{A} .

We claim that ϕ is true in \mathfrak{B} if and only ψ is true in \mathfrak{C} . Let C be the domain of \mathfrak{C} , B the domain of \mathfrak{B} , and suppose that $f: U \to C$ satisfies all conjuncts of ψ in \mathfrak{C} . Hence, by construction of ψ , if ϕ has a conjunct $\theta = R(x_{i_1}, \ldots, x_{i_k})$, then

$$\mathfrak{C} \models \theta_I((f(y_{i_1}^1), \dots, f(y_{i_l}^d)), \dots, (f(y_{i_k}^1), \dots, f(y_{i_k}^d)))$$
.

By the definition of interpretations, this implies that

$$\mathfrak{B} \models R(h(f(y_{i_1}^1), \dots, f(y_{i_1}^d)), \dots, h(f(y_{i_k}^1), \dots, f(y_{i_k}^d)))$$
.

Hence, the mapping $g: U \to B$ that sends x_i to $h(f(y_i^1), \dots, f(y_i^d))$ satisfies all conjuncts of ϕ in \mathfrak{B} .

Now, suppose that $f: U \to B$ satisfies all conjuncts of ϕ over \mathfrak{B} . Since h is a surjective mapping from $\delta(\mathfrak{C}^d)$ to B, there are elements c_i^1, \ldots, c_i^d in \mathfrak{C} such that $h(c_i^1, \ldots, c_i^d) = f(x_i)$, for all $i \in \{1, \ldots, n\}$. We claim that the mapping $g: V \to C$ that maps y_i^j to c_i^j is a homomorphism from ψ to \mathfrak{C} . By construction, any constraint in ψ either comes from ψ_1 or from ψ_2 . If it comes from ψ_1 then it must be of the form $\delta_I(y_i^1, \ldots, y_i^d)$, and is satisfied since the pre-image of h is $\delta_I(\mathfrak{C}^d)$. If the constraint comes from ψ_2 , then it must be a conjunct of a formula $\theta_I(y_{i_1}^1, \ldots, y_{i_1}^d, \ldots, y_{i_k}^1, \ldots, y_{i_k}^d)$ that was introduced for a constraint $\theta = R(x_{i_1}, \ldots, x_{i_k})$ in \mathfrak{A} . It therefore suffices to show that

$$\mathfrak{C} \models \theta_I(g(y_{i_1}^1), \dots, g(y_{i_1}^d), \dots, g(y_{i_k}^1), \dots, g(y_{i_k}^d)) .$$

By assumption, $R(f(x_{i_1}), \ldots, f(x_{i_k}))$ holds in \mathfrak{B} . By the choice of c_1^1, \ldots, c_n^d , this shows that $R(h(c_{i_1}^1, \ldots, c_{i_1}^d), \ldots, h(c_{i_k}^1, \ldots, c_{i_k}^d))$ holds in \mathfrak{C} . By the definition of interpretations, this is the case if and only if $\theta_I(c_{i_1}^1, \ldots, c_1^d, \ldots, c_{i_k}^d)$ holds in \mathfrak{C} , which is what we had to show.

We describe how to compose interpretations, and observe that compositions of primitive positive interpretations are again primitive positive. Note that if \mathfrak{C}_2 has a d_1 -dimensional interpretation I_1 in \mathfrak{C}_1 , and \mathfrak{C}_3 has an d_2 -dimensional interpretation I_2 in \mathfrak{C}_2 , then \mathfrak{C}_3 has a natural (d_1d_2) -dimensional interpretation in \mathfrak{C}_1 , which we denote by $I_2 \circ I_1$. To formally describe $I_2 \circ I_1$, suppose that the signature of \mathfrak{C}_i is τ_i for i = 1, 2, 3, and that $I_1 = (d_1, S_1, h_1)$ and $I_2 = (d_2, S_2, h_2)$. When ϕ is a τ_2 -formula, let ϕ_{I_1} denote the τ_1 -formula obtained from ϕ by replacing each atomic τ_2 formula ψ in ϕ by the τ_1 -formula ψ_{I_1} . Note that when ϕ is primitive positive (existential positive),

and the interpreting formulas of I_1 are primitive positive (existential positive), then ϕ_{I_2} is again primitive positive (existential positive)⁴.

Now the interpretation $I_2 \circ I_1$ is given by (d_1d_2, S, h) where $S := (\delta_{I_2})_{I_1}((\mathfrak{C}_1)^{d_1d_2})$, and where the coordinate map $h : S \to \mathfrak{C}_3$ is defined by

$$(a_1^1,\ldots,a_1^{d_1},\ldots,a_{d_2}^1,\ldots,a_{d_2}^{d_1}) \; \mapsto \; h_2(h_1(a_1^1,\ldots,a_1^{d_1}),\ldots,h_1(a_{d_2}^1,\ldots,a_{d_2}^{d_1})) \; .$$

Observe that when I_1 and I_2 are primitive positive interpretations, then $I_2 \circ I_1$ is also primitive positive.

In many hardness proofs we use Theorem 5.5.6 in the following way.

COROLLARY 5.5.7. Let \mathfrak{B} be an ω -categorical relational structure. If there is a primitive positive interpretation of ($\{0,1\}$; IIN3) or ($\{0,1\}$; NAE) in \mathfrak{B} , then \mathfrak{B} has a reduct with finite signature whose CSP is NP-hard.

PROOF. The primitive positive formulas involved in the primitive positive interpretation can mention only finitely many relations from \mathfrak{B} . Let \mathfrak{B}' be the reduct of \mathfrak{B} that contains exactly those relations. Then NP-hardness of CSP(\mathfrak{B}') follows from the NP-hardness of CSP(($\{0,1\};IIN3$)) and CSP(($\{0,1\};NAE$)) (see Section 1.2 in Chapter 1, Example 1.2.2) via Theorem 5.5.6.

We present an application of Theorem 5.5.6 and prove a hardness result that becomes useful at several occasions in later sections.

Definition 5.5.8. For any set B, we write I_6^B for the relation

$$\{(x_1, x_2, y_1, y_2, z_1, z_2) \in B^6 \mid (x_1 = x_2 \land y_1 \neq y_2 \land z_1 \neq z_2)$$

$$\lor (x_1 \neq x_2 \land y_1 = y_2 \land z_1 \neq z_2)$$

$$\lor (x_1 \neq x_2 \land y_1 \neq y_2 \land z_1 = z_2) \}.$$

PROPOSITION 5.5.9. For any set B with $|B| \ge 2$, the structure ($\{0,1\}$; 1IN3) has a primitive positive interpretation in $(B; I_6^B)$, and $CSP((B; I_6^B))$ is NP-hard.

PROOF. Recall that IIN3 denotes the boolean relation $\{(1,0,0),(0,1,0),(0,0,1)\}$. We give a primitive positive interpretation I of the structure $\mathfrak{B} := (\{0,1\}; 1IN3)$ in $(B; I_6^B)$. The dimension of I is 2, and the domain formula is $\delta_I := \top$ (for true). The formula $1IN3(x_1, x_2, y_1, y_2, z_1, z_2)_I$ is $I_6^B(x_1, x_2, y_1, y_2, z_1, z_2)$, and the formula $=_I(x_1, x_2, y_1, y_2)$ is

$$\exists a_1, a_2, u_1, u_2, u_3, u_4, z_1, z_2 \ \left(a_1 = a_2 \wedge I_6^B(a_1, a_2, u_1, u_2, u_3, u_4) \right. \\ \left. \qquad \qquad \wedge I_6^B(u_1, u_2, x_1, x_2, z_1, z_2) \wedge I_6^B(u_3, u_4, z_1, z_2, y_1, y_2)\right).$$

Note that the primitive positive formula $=_I (x_1, x_2, y_1, y_2)$ is equivalent to $x_1 = x_2 \Leftrightarrow y_1 = y_2$. The map h maps $(b_1, b_2) \in B^2$ to 1 if $b_1 = b_2$, and to 0 otherwise. NP-hardness of $CSP((B; I_6^B))$ then follows from Corollary 5.5.7.

COROLLARY 5.5.10. Let \mathfrak{B} be an ω -categorical structure where all first-order formulas are equivalent to primitive positive formulas (or that satisfies some of the other equivalent conditions from Corollary 5.3.4). Then there is a primitive positive interpretation of ({0,1}; IIN3) in \mathfrak{B} , and \mathfrak{B} has a finite signature reduct \mathfrak{B}' such that $CSP(\mathfrak{B}')$ is NP-hard.

PROOF. Since I_6^B is first-order definable, it also has a primitive positive definition in \mathfrak{B} by assumption. Proposition 5.5.9 implies that the structure ($\{0,1\}$; 1IN3) has a primitive positive interpretation in \mathfrak{B} . The last part of the statement follows from Corollary 5.5.7.

⁴Note that this is in general false for *existential* formulas: there are existential formulas ϕ and existential interpretations I_1 such that ϕ_{I_1} is no longer existential.

In fact, we could have weakened the assumptions in Corollary 5.5.10 by only requiring that all polymorphisms of \mathfrak{B} are essentially unary, and that all endomorphisms of \mathfrak{B} are injective, because it is then easy to see that the relation I_6^B is preserved by all polymorphisms of \mathfrak{B} , and hence primitive positive definable in \mathfrak{B} , by Theorem 5.2.3.

There are many situations where Theorem 5.5.6 can be combined with Lemma 1.2.8 to prove hardness of CSPs, as described in the following.

PROPOSITION 5.5.11. Let \mathfrak{A} be a structure with finite relational signature, and let \mathfrak{B} be a structure with elements c_1, \ldots, c_k such that

- the orbit of (c_1, \ldots, c_k) in \mathfrak{B} is primitive positive definable, and
- \mathfrak{A} has a primitive positive interpretation in $(\mathfrak{B}, c_1, \ldots, c_k)$.

Then there is a finite signature reduct \mathfrak{B}' of \mathfrak{B} and a polynomial-time reduction from $CSP(\mathfrak{A})$ to $CSP(\mathfrak{B}')$.

PROOF. Let \mathfrak{C} denote the expansion of \mathfrak{B} by the unary relations $\{c_1\}, \ldots, \{c_k\}$. Then the interpretation of \mathfrak{A} in $(\mathfrak{B}, c_1, \ldots, c_k)$ shows that there is also a primitive positive interpretation of \mathfrak{A} in \mathfrak{C} , and this interpretation mentions only finitely many relations of \mathfrak{C} . Let \mathfrak{C}' be the finite signature reduct of \mathfrak{C} that contains exactly those relations and the relations that are mentioned in the primitive positive definition of the orbit of (c_1, \ldots, c_k) . Since \mathfrak{C}' still interprets \mathfrak{A} , there is a polynomial-time reduction from $\mathrm{CSP}(\mathfrak{A})$ to $\mathrm{CSP}(\mathfrak{C}')$ by Theorem 5.5.6. Since there is still a primitive positive definition of the orbit of (c_1, \ldots, c_k) in \mathfrak{C}' , we can apply Corollary 3.6.25 and get a polynomial-time reduction from $\mathrm{CSP}(\mathfrak{C}')$ to $\mathrm{CSP}(\mathfrak{B}')$, where \mathfrak{B}' is the reduct of \mathfrak{B} that only contains the relations that are also in \mathfrak{C}' ; note that \mathfrak{B}' has finite signature. Composing reductions, we conclude that there is a polynomial-time reduction from $\mathrm{CSP}(\mathfrak{A})$ to $\mathrm{CSP}(\mathfrak{B}')$.

Together with Corollary 3.6.25 we have the following consequence.

COROLLARY 5.5.12. Let \mathfrak{B} be an ω -categorical model-complete core, and let \mathfrak{A} be a structure with a finite signature and a hard CSP. If \mathfrak{A} has a primitive positive interpretation with parameters in \mathfrak{B} , then \mathfrak{B} has a reduct with finite signature whose CSP is NP-hard.

We give an application of this technique in Proposition 5.5.13 below. Many more applications can be found in Section 9.2 and Section 10.2.3. We have defined in Example 1.1.3 the relation Betw on \mathbb{Z} ; we use the analogous definition for Betw over \mathbb{Q} , that is,

Betw :=
$$\{(x, y, z) \in \mathbb{Q}^3 \mid x < y < z \lor z < y < x\}$$
.

PROPOSITION 5.5.13. The structure ($\{0,1\}$; NAE) has a primitive positive interpretation in (\mathbb{Q} ; Betw, 0), and CSP((\mathbb{Q} ; Betw)) is NP-hard.

PROOF. Recall that the relation NAE is $\{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$. The dimension of our interpretation I is one, and the domain formula is $\exists z$. Betw(x,0,z), which is equivalent to $x \neq 0$. The formula $=_I (x_1, y_1)$ is

$$\exists z \ (\operatorname{Betw}(x_1, 0, z) \land \operatorname{Betw}(z, 0, y_1)) \ .$$

Note that $=_I$ is over $(\mathbb{Q}; \text{Betw}, 0)$ equivalent to $(x_1 > 0 \Leftrightarrow y_1 > 0)$. Finally, the formula $\text{NAE}(x_1, y_1, z_1)_I$ is

$$\exists u \; (\operatorname{Betw}(x_1, u, y_1) \wedge \operatorname{Betw}(u, 0, z_1)) \; .$$

The map h maps positive points to 1, and all other points from \mathbb{Q} to 0.

Since the orbit of 0 is the entire set \mathbb{Q} it is in particular primitive positive definable, and we can show NP-hardness of CSP ((\mathbb{Q} ; Betw)) using Proposition 5.5.11 and the fact that CSP (($\{0,1\}; NAE$)) is NP-hard.

5.5.3. Pseudo-varieties. We present the mentioned connection between primitive positive interpretations and pseudo-varieties.

THEOREM 5.5.14 (from [26]). Let \mathfrak{C} be a finite or ω -categorical relational structure, and let \mathbf{C} be a polymorphism algebra of \mathfrak{C} . Then a structure \mathfrak{B} has a primitive positive interpretation in \mathfrak{C} if and only if there is an algebra \mathbf{B} in the pseudo-variety generated by \mathbf{C} such that all operations of \mathbf{B} are polymorphisms of \mathfrak{B} .

PROOF. Let τ be the signature of \mathbf{C} , and let \mathcal{V} be the pseudo-variety generated by \mathbf{C} . Similarly to the famous HSP theorem for varieties (see e.g. [68]), every algebra in \mathcal{V} is the homomorphic image of a subalgebra of a finite direct product of \mathbf{C} . To see this, we have to verify that $\mathrm{HSP^{fin}}(\mathbf{C})$ is closed under H , S , and $\mathrm{P^{fin}}$. It is clear that $\mathrm{H}(\mathrm{HSP^{fin}}(\mathbf{C})) = \mathrm{HSP^{fin}}(\mathbf{C})$. Lemma 5.5.4 implies that $\mathrm{S}(\mathrm{HSP^{fin}}(\mathbf{C})) \subseteq \mathrm{HSP^{fin}}(\mathbf{C})$. Finally,

$$P^{\mathrm{fin}}(\mathrm{HSP^{\mathrm{fin}}}(\mathbf{C})) \subseteq \mathrm{HP^{\mathrm{fin}}}\,\mathrm{S}\,\mathrm{P^{\mathrm{fin}}}(\mathbf{C}) \subseteq \mathrm{HSP^{\mathrm{fin}}}\,\mathrm{P^{\mathrm{fin}}}(\mathbf{C}) = \mathrm{HSP^{\mathrm{fin}}}(\mathbf{C}) \;.$$

First assume that there is an algebra \mathbf{B} in \mathcal{V} all of whose operations are polymorphisms of \mathfrak{B} . Then there exists a finite number $d \geq 1$, a subalgebra \mathbf{D} of \mathbf{C}^d , and a surjective homomorphism h from \mathbf{D} to \mathbf{B} . We claim that \mathfrak{B} has a first-order interpretation I of dimension d in \mathfrak{C} . All operations of \mathbf{C} preserve D (viewed as a d-ary relation over \mathfrak{C}), since \mathbf{D} is a subalgebra of \mathbf{C}^d . By Theorem 5.2.3, this implies that D has a primitive positive definition $\delta(x_1,\ldots,x_d)$ in \mathfrak{C} , which becomes the domain formula δ_I of I. As coordinate map we choose the mapping h.

If R is a k-ary relation in \mathfrak{B} , let $R' \subseteq C^{dk}$ be defined by

$$(a_1^1, \dots, a_1^d, \dots, a_k^1, \dots, a_k^d) \in R' \Leftrightarrow (h(a_1^1, \dots, a_1^d), \dots, h(a_k^1, \dots, a_k^d)) \in R$$
.

Let $f \in \tau$ be arbitrary. By assumption, $f^{\mathbf{B}}$ preserves R. It is easy to verify that then $f^{\mathbf{C}}$ preserves R'. Hence, all polymorphisms of \mathfrak{C} preserve R', and because \mathfrak{C} is ω -categorical, the relation R' has a primitive positive definition in \mathfrak{C} (Theorem 5.2.3), which becomes the defining formula for $R(x_1, \ldots, x_k)$ in I. Finally, since h is an algebra homomorphism, the kernel K of h is a congruence of \mathbf{D} . It follows that K, viewed as a 2d-ary relation over C, is preserved by all operations from \mathbf{C} . Theorem 5.2.3 implies that K has a primitive positive definition in \mathfrak{C} . This definition becomes the formula $=_I$. It is straightforward to verify that I is a primitive positive interpretation of \mathfrak{B} in \mathfrak{C} .

To prove the opposite direction, suppose that \mathfrak{B} has a primitive positive interpretation I in \mathfrak{C} . We have to show that \mathcal{V} contains a τ -algebra \mathbf{B} such that all operations in \mathbf{B} are polymorphisms of \mathfrak{B} . Let d be the dimension and δ be the primitive positive domain formula of I. Clearly, the set $\delta(\mathfrak{C}^d)$ is preserved by all operations in \mathbf{C} , and therefore induces a subalgebra \mathbf{D} of \mathbf{C}^d .

We first show that the kernel K of the coordinate map h of the interpretation is a congruence of \mathbf{D} . For all d-tuples $\overline{a}, \overline{b} \in D$, the 2d-tuple $(\overline{a}, \overline{b})$ satisfies $=_I$ in $\mathfrak C$ if and only if $h(\overline{a}) = h(\overline{b})$. Let S be the 2d-ary relation defined by $=_I$ over $\mathfrak C$. Then S can be viewed as a binary relation over C^d , and we have $S \cap D^2 = K$. Since $=_I$ is primitive positive definable in $\mathfrak C$, S is preserved by all polymorphisms of $\mathfrak C$. To show that K is a congruence of $\mathbf D$, let $f \in \tau$ be k-ary, and let $(a^1, b^1), \ldots, (a^k, b^k)$ be pairs from K. Let $a = f^{\mathbf D}(a^1, \ldots, a^k)$ and $b = f^{\mathbf D}(b^1, \ldots, b^k)$. We have to show that $(a, b) \in K$. Since $\mathbf D$ is a subalgebra of $\mathbf C^d$, $a, b \in D$, and hence it suffices to show that $(a, b) \in S$. Recall that $f^{\mathbf D}$ is defined by applying $f^{\mathbf C}$ component-wise. Since $(a^i, b^i) \in S$ for all $i \leq k$ and $f^{\mathbf C}$ preserves S, we thus have that $(a, b) \in S$. Hence, K is a congruence of $\mathbf D$ and K is a surjective homomorphism from $\mathbf D$ to $\mathbf B := \mathbf D/h$.

We finally verify that every operation in **B** is a polymorphism of \mathfrak{B} , i.e., for every $f \in \tau$, every relation R of \mathfrak{B} is preserved by $f^{\mathbf{B}}$. The operation $f^{\mathbf{B}}$ preserves

 $\phi := R(x_1, \dots, x_k)$ if and only if $f^{\mathbf{C}}$ preserves ϕ_I . Since $f^{\mathbf{C}}$ is a polymorphism of \mathfrak{C} , and since ϕ_I is primitive positive over \mathfrak{C} , the operation $f^{\mathbf{C}}$ indeed preserves ϕ_I . \square

The proof of Theorem 5.5.14 above gives more information about the link between the algebras in $HSP^{fin}(Alg(\mathfrak{B}))$ and the primitive positive interpretations in \mathfrak{B} , and we state it explicitly.

THEOREM 5.5.15. Let \mathfrak{C} be a finite or ω -categorical structure, and let \mathfrak{B} be an arbitrary structure. Then the following are equivalent.

- (1) there is a polymorphism algebra \mathbf{C} of \mathfrak{C} , an algebra $\mathbf{S} \in \mathrm{S}(\mathbf{C}^d)$ with domain S, and a surjective homomorphism h from \mathbf{S} to an algebra \mathbf{B} such that $\mathrm{Clo}(\mathbf{B}) \subseteq \mathrm{Pol}(\mathfrak{B})$;
- (2) \mathfrak{B} has the primitive positive interpretation (d, S, h) in \mathfrak{C} .

We return to applications of these concepts to CSPs.

COROLLARY 5.5.16. Let \mathfrak{B} be ω -categorical. If there is an expansion \mathfrak{C} of the model-complete core of \mathfrak{B} by finitely many constants such that the pseudo-variety \mathcal{V} generated by $Alg(\mathfrak{C})$ contains a 2-element algebra where all operations are projections, then \mathfrak{B} has a finite signature reduct with an NP-hard CSP.

PROOF. Let **D** be the 2-element algebra in \mathcal{V} where all operations are projections. All operations of **D** preserve the relation 1IN3. By Theorem 5.5.14, the structure ($\{0,1\}; 1IN3$) has a primitive positive interpretation in \mathfrak{C} . Then Corollary 5.5.7 shows that \mathfrak{B} has a finite signature reduct \mathfrak{B}' with an NP-hard CSP.

All templates \mathfrak{B} with a first-order definition in a homogeneous structure with finite relational signature known to the author that have an NP-complete CSP satisfy the condition from Corollary 5.5.16. For finite templates \mathfrak{B} there is the conjecture (and strong evidence) that $CSP(\mathfrak{B})$ is NP-hard if and only if \mathfrak{B} satisfies this condition (see Section 5.6.1).

THEOREM 5.5.17. Let \mathfrak{B} be any structure. Then the following are equivalent.

- (1) there is a primitive positive interpretation of $(\{0,1\}; 1IN3)$ in \mathfrak{B} .
- (2) there is a primitive positive interpretation of $(\{0,1\}; NAE)$ in \mathfrak{B} ;
- (3) B interprets a structure with at least two elements where all first-order formulas are equivalent to primitive positive formulas;
- (4) all finite structures have a primitive positive interpretation in \mathfrak{B} .

If \mathfrak{B} is ω -categorical, the following two conditions are equivalent to the conditions above.

- (5) the pseudo-variety V generated by $Alg(\mathfrak{B})$ contains for all n an algebra on n elements all of whose operations are projections;
- (6) the pseudo-variety V generated by $Alg(\mathfrak{B})$ contains a 2-element algebra all of whose operations are projections.

PROOF. The first statement can be shown by proving implications in cyclic order, $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$. Obviously, (4) implies (1). We have given a primitive positive definition of NAE in ($\{0,1\}$; IIN3) in the proof of Theorem 5.4.3, which implies that (1) implies (2). The implication from (2) to (3) is by Proposition 5.4.4.

For the implication $(3) \Rightarrow (4)$, let \mathfrak{B}' be the structure that has a primitive positive interpretation in \mathfrak{B} , has at least two elements, and where all first-order formulas are equivalent to primitive positive formulas. Let \mathfrak{A} be a τ -structure with domain $\{1,\ldots,n\}$. We prove that \mathfrak{A} has a first-order interpretation in \mathfrak{B}' . This yields in fact a primitive positive interpretation since every first-order formula is equivalent to a

primitive positive formula in \mathfrak{B}' . The claim then follows by composing the primitive positive interpretation of \mathfrak{B}' in \mathfrak{B} with that of \mathfrak{A} in \mathfrak{B}' .

Our first-order interpretation I of \mathfrak{A} in \mathfrak{B}' is 2n-dimensional. The domain formula $\delta_I(x_1,\ldots,x_n,x_1',\ldots,x_n')$ expresses that for exactly one $i \leq n$ we have $x_i = x_i'$; clearly this is first-order. Equality is interpreted by the formula

$$=_{I} (x_{1}, \dots, x_{n}, x'_{1}, \dots, x'_{n}, y_{1}, \dots, y_{n}, y'_{1}, \dots, y'_{n}) := \bigwedge_{i=1}^{n} ((x_{i} = x'_{i}) \Leftrightarrow (y_{i} = y'_{i})).$$

Note that the equivalence relation defined by $=_I$ on $\delta((\mathfrak{B}')^{2n})$ has exactly n equivalence classes, and the coordinate map sends $(x_1,\ldots,x_n,x_1',\ldots,x_n')$ to i if and only if $x_i=x_i'$. It is now straightforward to write down first-order formulas ϕ_I that interpret atomic τ -formulas ϕ . When $R \in \tau$ is k-ary, then the formula $R(x_1,\ldots,x_k)_I$ is a disjunction of conjunctions with the 2nk variables $x_{1,1},\ldots,x_{k,n},x_{1,1}',\ldots,x_{k,n}'$. For each tuple (t_1,\ldots,t_k) from $R^{\mathfrak{A}}$ the disjunction contains the conjunct $\bigwedge_{i\leq k} x_{i,t_i} = x_{i,t_i}'$.

Now suppose that \mathfrak{B} is ω -categorical. We prove that (4) implies (5), that (5) implies (6), and that (6) implies (1). For (4) \Rightarrow (5), let \mathfrak{A} be the structure with domain $A = \{1, \ldots, n\}$, the relations P_3^A , and for each $i \in \{1, \ldots, n\}$ the unary relation $\{i\}$. By (4) there is a primitive positive interpretation of \mathfrak{A} in \mathfrak{B} . Hence, Theorem 5.5.14 implies that there is an algebra $\mathbf{A}' \in \mathcal{V}$ such that all operations of \mathbf{A}' are polymorphisms of \mathfrak{A} . But all polymorphisms of \mathfrak{A} are projections (Corollary 5.3.4). The implication (5) \Rightarrow (6) is trivial. The implication (6) \Rightarrow (1) follows from Theorem 5.5.14 and the fact that the projections preserve 1IN3.

5.5.4. Bi-interpretations and Classification Transfer. Let $\mathfrak C$ be a structure with finite relational signature. By the classification project for $\mathfrak C$ we mean a complexity classification for $\mathrm{CSP}(\mathfrak B)$ for all first-order expansions $\mathfrak B$ of $\mathfrak C$ that have finite relational signature. For instance, the classification project for the random graph $(\mathbb V;E)$ is treated in Chapter 9, and the classification project for $(\mathbb Q;<)$ is treated in Chapter 10.

Sometimes, it is possible to derive the complexity classification project for \mathfrak{C} from the complexity classification project for \mathfrak{D} , for another ω -categorical structure \mathfrak{D} . For instance, we will show below how to derive the classification project for the directed graph

$$\mathfrak{C} := (\mathbb{N}^2; \{(x, y), (u, v) \mid y = u\})$$

from the classification project for $\mathfrak{D} := (\mathbb{N}; =)$ (which will be given in Chapter 6); a more advanced application of such a classification transfer can be found in Theorem 5.5.23 below.

Primitive positive interpretability is a crucial concept for the transfer of complexity classifications. In particular, this section studies primitive positive bi-interpretations in this context. Two interpretations of \mathfrak{C} in \mathfrak{D} with coordinate maps h_1 and h_2 are called $homotopic^5$ if the relation $\{(\bar{x},\bar{y}) \mid h_1(\bar{x}) = h_2(\bar{y})\}$ is first-order definable in \mathfrak{D} . If this relation is even primitive positive definable in \mathfrak{D} , we say that the two interpretations are pp-homotopic. The identity interpretation of a τ -structure \mathfrak{C} is the interpretation I = (1, true, h) of \mathfrak{C} in \mathfrak{C} whose coordinate map h is the identity (note that the identity interpretation is primitive positive). Recall that we write $I_1 \circ I_2$ for the natural composition of two interpretations I_1 and I_2 , defined in Section 5.5.2.

DEFINITION 5.5.18. Two structures \mathfrak{C} and \mathfrak{D} with an interpretation I of \mathfrak{C} in \mathfrak{D} and an interpretation J of \mathfrak{C} in \mathfrak{D} are called mutually interpretable. If both $I \circ J$ and

⁵We are following the terminology from [3].

 $J \circ I$ are homotopic to the identity interpretation (of \mathfrak{D} and of \mathfrak{C} , respectively), then we say that \mathfrak{C} and \mathfrak{D} are bi-interpretable.

When both interpretations I and J are primitive positive, then $\mathfrak C$ and $\mathfrak D$ are called mutually pp-interpretable. If moreover $I \circ J$ and $J \circ I$ are pp-homotopic to the identity interpretation, then $\mathfrak C$ and $\mathfrak D$ are called primitive positive bi-interpretable.

EXAMPLE 5.5.19. The directed graph $\mathfrak{C} := (\mathbb{N}^2; M)$ where

$$M := \{((u_1, u_2), (v_1, v_2)) \mid u_2 = v_1\}$$

and the structure $\mathfrak{D}:=(\mathbb{N};=)$ are primitive positive bi-interpretable. The interpretation I of \mathfrak{C} in \mathfrak{D} is 2-dimensional, the domain formula is true, and the coordinate map h is the identity. The interpretation J of \mathfrak{D} in \mathfrak{C} is 1-dimensional, the domain formula is true, and the coordinate map g sends (x,y) to x. Both interpretations are clearly primitive positive.

Then g(h(x,y)) = z is definable by the formula x = z, and hence $I \circ J$ is pphomotopic to the identity interpretation of \mathfrak{D} . Moreover, h(g(u), g(v)) = w is primitive positive definable by

$$M(w,v) \wedge \exists p (M(p,u) \wedge M(p,w))$$
,

so $J \circ I$ is also pp-homotopic to the identity interpretation of \mathfrak{C} .

EXAMPLE 5.5.20. The structures $\mathfrak{C} := (\mathbb{N}^2; \{(x,y), (u,v) \mid x=u\})$ and $\mathfrak{D} := (\mathbb{N}; =)$ are mutually primitive positive interpretable, but *not* primitive positive biinterpretable. There is a primitive positive interpretation I_1 of \mathfrak{D} in \mathfrak{C} , and a primitive positive interpretation of \mathfrak{C} in \mathfrak{D} such that $I_2 \circ I_1$ is pp-homotopic to the identity interpretation. However, the two structures are not even *first-order* bi-interpretable, as we will see in Example 7.4.2 in Section 7.4.

Here comes the central lemma for complexity classification transfer.

LEMMA 5.5.21. Suppose $\mathfrak D$ has a primitive positive interpretation I in $\mathfrak C$, and $\mathfrak C$ has a primitive positive interpretation J in $\mathfrak D$ such that $J \circ I$ is pp-homotopic to the identity interpretation of $\mathfrak C$. Then for every first-order expansion $\mathfrak C'$ of $\mathfrak C$ there is a first-order expansion $\mathfrak D'$ of $\mathfrak D$ such that $\mathfrak C'$ and $\mathfrak D'$ are mutually pp-interpretable.

PROOF. Let I=(c,U,g) and J=(d,V,h) be the primitive positive interpretations from the statement, and let \mathfrak{C}' be a first-order expansion of \mathfrak{C} . Then we set \mathfrak{D}' to be the expansion of \mathfrak{D} that contains for every k-ary R in the signature of \mathfrak{C}' the (dk)-ary relation S defined as follows. When ϕ is the first-order definition of R in \mathfrak{C} , then S is the relation defined by ϕ_J in \mathfrak{D} (see Section 3.1 and Section 5.5.2).

We claim that \mathfrak{C}' has the primitive positive interpretation (d,V,h) in \mathfrak{D}' . First note that V is primitive positive definable in \mathfrak{D}' since \mathfrak{D}' is an expansion of \mathfrak{D} . An atomic formula ψ with free variables x_1,\ldots,x_k in the signature of \mathfrak{C}' can be interpreted in \mathfrak{D}' as follows. We replace the relation symbol in ψ by its definition in \mathfrak{C} , and obtain a formula ϕ in the language of \mathfrak{C} . Let S be the symbol in the language of \mathfrak{D}' for the relation defined by $\phi_J(x_1^1,\ldots,x_1^d,\ldots,x_k^1,\ldots,x_k^d)$ over \mathfrak{D}' . Then indeed $S(x_1^1,\ldots,x_1^d,\ldots,x_k^1,\ldots,x_k^d)$ is a defining formula for ψ , because

$$\mathfrak{C}' \models \psi(h(a_1^1, \dots, a_1^d), \dots, h(a_k^1, \dots, a_k^d)) \Leftrightarrow \mathfrak{D}' \models S(a_1^1, \dots, a_1^d, \dots, a_k^1, \dots, a_k^d)$$
 for all $a_1, \dots, a_k \in V$.

Conversely, we claim that \mathfrak{D}' has the primitive positive interpretation (c, U, g) in \mathfrak{C}' . Again, U is primitive positive definable in \mathfrak{C}' since \mathfrak{C}' is an expansion of \mathfrak{C} . Let ϕ be an atomic formula in the (relational) signature of \mathfrak{D}' . If the relation symbol in ϕ is already in the signature of \mathfrak{D} , then there is a primitive positive interpreting formula in \mathfrak{C} and therefore also in \mathfrak{C}' . Otherwise, by definition of

 \mathfrak{D}' , the relation symbol in ϕ has arity dk, and has been introduced for a k-ary relation R from \mathfrak{C}' . We have to find a defining formula having kcd variables. Let $\theta(x_0, x_{1,1}, \ldots, x_{1,c}, \ldots, x_{d,1}, \ldots, x_{c,d})$ be the primitive positive formula of arity cd+1 that shows that $h(g(x_{1,1}, \ldots, x_{c,1}), \ldots, g(x_{1,d}, \ldots, x_{c,d})) = x_0$ is primitive positive definable in \mathfrak{C} . Then the defining formula for the atomic formula $\phi(x_1^1, \ldots, x_d^k)$ has free variables $x_{1,1}^1, \ldots, x_{c,d}^k$ and equals

$$\exists x^1, \dots, x^k \ (R(x^1, \dots, x^k) \ \land \bigwedge_{i=1}^k \theta(x^i, x_{1,1}^i, \dots, x_{c,d}^i)).$$

In particular, when $\mathfrak{C}, \mathfrak{D}, \mathfrak{C}'$ and \mathfrak{D}' are as in Lemma 5.5.21, and \mathfrak{C}' and \mathfrak{D}' have a finite relational signature, then $\mathrm{CSP}(\mathfrak{C}')$ and $\mathrm{CSP}(\mathfrak{D}')$ have the same computational complexity, by Theorem 5.5.6. Hence, Lemma 5.5.21 shows that the classification project for \mathfrak{C} can be reduced to the classification project for \mathfrak{D} . With a slightly stronger assumption we can get the following consequence.

COROLLARY 5.5.22. Let $\mathfrak C$ and $\mathfrak D$ be primitive positive bi-interpretable ω -categorical structures. Then every first-order expansion of $\mathfrak C$ is primitive positive bi-interpretable with a first-order expansion of $\mathfrak D$.

More about *first-order* bi-interpretability can be found in Section 7.4. Let us conclude with a concrete application of Corollary 5.5.22.

THEOREM 5.5.23. Let \mathfrak{B} be a reduct of Allen's interval algebra (Example 3.1.11) that contains the relation $m = \{((u_1, u_2), (v_1, v_2)) | u_2 = v_1\}$. Then CSP(\mathfrak{B}) is either in P or NP-complete.

PROOF. We show that the structure $(\mathbb{I}; m)$ is primitive positive bi-interpretable with $(\mathbb{Q}; <)$. It follows that \mathfrak{B} is primitive positive bi-interpretable with a temporal constraint language \mathfrak{B}' , and the result follows by the main result of Chapter 10 and Corollary 5.5.22.

Let I be the 2-dimensional interpretation of $(\mathbb{I};m)$ in $(\mathbb{Q};<)$ with domain formula x < y, the formula $(y_1 = y_2)_I$ is true, and the formula $(m(y_1,y_2))_I$ has variables $x_1^1, x_2^1, x_1^2, x_2^2$ and is given by $x_2^1 = x_1^2$. The coordinate map g sends $(x,y) \in \mathbb{Q}^2$ with x < y to the interval $[x,y] \in \mathbb{I}$.

Let J be the 1-dimensional interpretation with domain formula true, and where the coordinate map h is $[x,y] \mapsto x$. The formula $(x < y)_I$ is the primitive positive formula

$$\exists u, v (m(u, x_1) \land m(u, v) \land m(v, x_2))$$
.

We show that $J \circ I$ and $J \circ I$ are pp-homotopic to the identity interpretation. The relation $\{(x_1, x_2, y) \mid h(g(x_1, x_2)) = y\}$ has the primitive positive definition $x_1 = y$. To see that the relation $R := \{(u, v, w) \mid g(h(u), h(v)) = w\}$ has a primitive positive definition in $(\mathbb{I}; m)$, first note that the relation

$$\{(u,v) \mid u = [u_1, u_2], v = [v_1, v_2], u_1 = v_1\}$$

has the primitive positive definition $\phi_1(u,v) = \exists w \ (m(w,u_1) \land m(w,u_2))$ in $(\mathbb{I};m)$. Similarly, $\{(u,v) \mid u = [u_1,u_2], v = [v_1,v_2], u_2 = v_2\}$ has a primitive positive definition $\phi_2(u,v)$. Then the formula $\phi_1(u,w) \land \phi_2(v,w)$ is equivalent to a primitive positive formula over $(\mathbb{I};m)$, and defines R.

5.6. Varieties

Varieties (which we have introduced briefly in Section 5.5) are a fascinatingly powerful concept to study classes of algebras. For a finite structure $\mathfrak B$ with finite signature, the complexity of $\mathrm{CSP}(\mathfrak B)$ only depends on the variety generated by the polymorphism algebra of $\mathfrak B$. This is in particular related to the fact that a finite algebra is in the variety generated by a finite algebra $\mathbf B$ if and only if it is in the pseudo-variety generated by $\mathbf B$; the link between the pseudo-variety generated by $\mathbf B$ and the CSP has already been explained in Section 5.5.

The section has two parts. In Section 5.6.1 we explain the role of varieties for the study of CSPs with finite templates. In particular, we present various equivalent forms of the *tractability conjecture* for finite domain constraint satisfaction.

The second part studies the situation for ω -categorical templates. It is open whether the complexity of an ω -categorical model-complete core only depends on the variety generated by the polymorphism algebra of \mathfrak{B} . But it will turn out that the tractability frontier in the classification results in Chapter 9 and Chapter 10 can be described elegantly using varieties, and the description is very similar to the tractability conjecture for finite domain constraint satisfaction. In Section 5.6.2 we provide some partial explanation for this phenomenon.

5.6.1. The Tractability Conjecture. In this section, we present some classical results that specifically hold for *finite* algebras and are relevant to constraint satisfaction. We also discuss more recent universal-algebraic results about finite algebras. We cannot cover all recent developments here, but sketch in this section some of the highlights.

We have already mentioned in the introduction the dichotomy conjecture of Feder and Vardi [95], which we state here since it is one of the central stimulating conjectures for finite domain constraint satisfaction.

Conjecture 5.1 (Dichotomy Conjecture [95]). Let \mathfrak{B} be a structure with finite relational signature and finite domain. Then $CSP(\mathfrak{B})$ is in P or NP-complete.

We will now see a stronger conjecture, due to [65, 66], that exactly describes which finite-domain CSPs are NP-hard, and which can be solved in polynomial time. This conjecture is called the *tractability conjecture*, and it has been confirmed in many important cases, for example for

- finite structures B that contain a unary relation symbol for each subset of the domain of B, due to [61] (see also [11]),
- structures over a 3-element domain [63], and
- digraphs without sources and sinks [15], and which includes the case of undirected graphs.

The tractability conjecture can be formulated in terms of primitive positive interpretability (Section 5.5) as follows.

Conjecture 5.2 (Tractability Conjecture). Let $\mathfrak B$ be a finite structure with finite relational signature, and let $\mathfrak C$ be the core of $\mathfrak B$. Then $\mathrm{CSP}(\mathfrak B)$ is NP-hard if there is a primitive positive interpretation of $(\{0,1\};\mathrm{IIN3})$ with parameters in $\mathfrak C$, and can be solved in polynomial-time otherwise.

We remark that the first part of this conjecture follows directly from Corollary 5.5.12. Moreover, we have also seen that $CSP(\mathfrak{B})$ is polynomial-time equivalent to $CSP(\mathfrak{C})$, so all that remains to be shown is to prove polynomial-time tractability of $CSP(\mathfrak{C})$ when \mathfrak{C} does not admit a primitive positive interpretation of a hard Boolean

CSP. By the results in Section 5.5, this condition can be translated into a condition of the pseudo-variety generated by the polymorphism algebra \mathbf{C} of \mathfrak{C} .

When \mathbf{B} is a finite algebra, it turns out that a finitely generated algebra (and in particular a finite algebra) \mathbf{A} is in the pseudo-variety generated by \mathbf{B} if and only if \mathbf{A} is in the variety generated by \mathbf{B} (see [68]; the claim follows from Exercise 11.5 in combination with the proof of Lemma 11.8 there). Varieties have the advantage that they can be described by the equations satisfied by its members.

THEOREM 5.6.1 (Birkhoff; see e.g. [119] or [68]). Let τ be a functional signature, **A** a τ -algebra, and C be a class of τ -algebras. Then the following are equivalent.

- All universal conjunctive sentences that hold in all members of C also hold in A.
- A is in the variety generated by C.
- $\mathbf{A} \in \mathrm{HSP}(\mathcal{C})$.

Theorem 5.6.1 is important for constraint satisfaction since it can be used to transform the 'negative' statement of not interpreting certain hard boolean CSPs into a 'positive' statement of having polymorphisms satisfying non-trivial identities. The following theorem is an application of this philosophy, and goes back to Walter Taylor (Corollary 5.3 in [192]; see also Lemma 9.4 in [118]).

Theorem 5.6.2. Let \mathfrak{B} be a finite structure, and suppose that the polymorphism algebra \mathbf{B} of \mathfrak{B} is idempotent. Then the following are equivalent.

- $(\{0,1\}; IIN3)$ does not have a primitive positive interpretation in \mathfrak{B} .
- every 2-element algebra in the pseudo-variety generated by **B** contains an essential operation.
- every 2-element algebra in HSP(B) contains an essential operation.
- **B** has a Taylor term, that is, an n-ary operation, for $n \geq 2$, such that for every $1 \leq i \leq n$ there are $x_1, \ldots, x_n, y_1, \ldots, y_n \in \{x, y\}$ such that (B; f) satisfies

$$\forall x, y. \ f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)$$

= $f(y_1, \dots, y_{i-1}, y, y_{i+1}, \dots, y_n)$.

The equations satisfied by Taylor terms are a special form of a *linear equation*, that is, an equation of the form

$$\forall x_1, \dots, x_n. f(u_1, \dots, u_n) = f(v_1, \dots, v_n)$$

where $u_1, ..., u_n, v_1, ..., v_n \in \{x_1, ..., x_n\}.$

Even though Theorem 5.6.2 is of central importance in universal algebra, it was discovered only recently and under the influence of work in the context of constraint satisfaction that the existence of Taylor terms is equivalent to the existence of various other terms satisfying stronger conditions, which are likely to be of greater use in the quest for polynomial-time algorithms for CSPs.

Theorem 5.6.3 (of [14, 187]). Let **B** be a finite idempotent algebra. Then the following are equivalent.

- **B** has a Taylor term.
- B has a weak near unanimity, that is, an n-ary idempotent operation f, for $n \geq 2$, that satisfies

$$\forall x, y. \ f(x, \dots, x, y) = f(x, \dots, x, y, x) = \dots = f(y, x, \dots, x).$$

- **B** has a Siggers term⁶, that is, a four-ary operation f that satisfies $\forall x, y. \ f(y, y, x, x) = f(x, x, x, y) = f(y, x, y, x)$.
- **B** has a cyclic term, i.e., an n-ary operation f, for $n \ge 2$, that satisfies $\forall x_1, \ldots, x_n \cdot f(x_1, \ldots, x_n) = f(x_2, \ldots, x_n, x_1)$.

Note that weak near unanimities, Siggers terms and cyclic terms are (special) Taylor terms, and that cyclic terms are (special) weak near unanimities. Also note that the existence of a Siggers term can be decided (for an explicitly given finite idempotent algebra \mathbf{B}), and hence the condition of the tractability conjecture for finite domain constraint satisfaction is decidable. We would also like to remark that binary commutative operations, that is, operations f satisfying f(x,y) = f(y,x), are Taylor terms, and that Conjecture 5.2 is already open in this special case.

Another improvement of Theorem 5.6.2 is the following result from [65] (Proposition 4.14).

THEOREM 5.6.4 (of [65]). Let **B** be a finite idempotent algebra. Then HSP(**B**) contains an algebra without essential operations if and only if HS(**B**) does.

Since all algebras in $HS(\mathbf{B})$ are smaller than \mathbf{B} (or isomorphic to \mathbf{B}), this leads to another algorithm that decides whether a given structure \mathfrak{B} satisfies the equivalent conditions in Theorem 5.6.2 (besides the approach via searching for Siggers terms mentioned above).

5.6.2. Canonical Clones. We cannot offer an full analog of Theorem 5.6.2 for ω -categorical structures \mathfrak{B} . This section treats the special case where the polymorphism algebra of an ω -categorical structure \mathfrak{B} resembles a finite algebra in a certain formal sense; in this case, an analog of Theorem 5.6.2 can be transferred from the finite.

Let \mathfrak{B} be a structure. Then $f: B^k \to B$ is called m-canonical (with respect to \mathfrak{B}) if for all m-tuples t_1, \ldots, t_k , the m-type of $f(t_1, \ldots, t_k)$ in \mathfrak{B} only depends on the m-types of t_1, \ldots, t_k in \mathfrak{B} . It is called canonical if it is m-canonical for all finite m. A clone (or an algebra) is called canonical if all its operations are canonical (this is still with respect to some base structure \mathfrak{B}).

LEMMA 5.6.5. Let \mathfrak{B} be a structure with a finite number q of m-types, and let \mathbf{B} be an algebra with signature τ such that all operations of \mathbf{B} are m-canonical with respect to \mathfrak{B} . Then there exists a τ -algebra \mathbf{A} of size q and a surjective homomorphism μ from \mathbf{B}^m to \mathbf{A} .

PROOF. Let p_1, \ldots, p_q be the m-types of \mathfrak{B} . Define $\mu \colon B^m \to \{1, \ldots, q\}$ by $g(b_1, \ldots, b_m) = i$ if (b_1, \ldots, b_m) has type p_i . Since the operations of \mathbf{B} are m-canonical, the kernel of μ is a congruence K of \mathbf{B}^m . Then $\mathbf{A} := \mathbf{B}^m/K$ satisfies the requirements of the statement.

The algebra **A** constructed from **B** in the proof of the previous lemma will be called the *type algebra* of **B**, denoted by $T_m(\mathbf{B})$.

LEMMA 5.6.6. Let \mathfrak{B} be an ω -categorical model-complete core, and suppose that all operations of the polymorphism algebra \mathbf{B} of \mathfrak{B} are m-canonical with respect to \mathfrak{B} . Then $T_m(\mathbf{B})$ is idempotent.

⁶Originally, Siggers gave equations for a six-ary operation, using the universal-algebraic formulation from [62] of the dichotomy theorem for the CSPs for undirected graphs \mathfrak{H} from [113]. This was later improved by an anonymous referee of [187] to the given identities for a 4-ary operation, using the main result from [15]; see concluding comments in [187].

PROOF. When \mathfrak{B} is an ω -categorical model-complete core, then all orbits of m-tuples in \mathfrak{B} are preserved by the endomorphisms of \mathfrak{B} . It follows that every operation f of $T_m(\mathbf{B})$ satisfies $\forall x. f(x, \ldots, x) = x$.

Note that if \mathfrak{B} is homogeneous in a relational signature with maximal arity m (or first-order interdefinable with such a structure), then being m-canonical implies being n-canonical for all $n \geq m$. In this case, we simply write $T(\mathbf{B})$ instead of $T_m(\mathbf{B})$.

We say that an operation $f: B^n \to B$ is cyclic modulo $e_1, e_2: B \to B$ if

$$\forall x_1, \dots, x_n. \ e_1(f(x_1, \dots, x_n)) = e_2(f(x_2, \dots, x_n, x_1)) \ .$$

Similarly, we say that $f: B^n \to B$ is a weak near unanimity modulo $e_1, \ldots, e_n: B \to B$ if the following is satisfied.

$$\forall \bar{x}. \ e_1(f(x,...,x,y)) = e_2(f(x,...,y,x)) = \cdots = e_n(f(y,x,...,x))$$

The same definition can be made for any type of equation, and we therefore also define Taylor operations modulo unary operations and Siggers polymorphisms modulo unary operations analogously.

The idea of the following lemma comes from the proof of Proposition 6.6 in Bodirsky, Pinsker, and Pongracz [53], and has been used in [36].

LEMMA 5.6.7. Let \mathfrak{B} be ω -categorical, and $f \in \operatorname{Pol}(\mathfrak{B})$. Suppose that for every finite $A \subset B$ there exists an $\alpha \in \operatorname{Aut}(\mathfrak{B})$ such that $f(x_1, \ldots, x_n) = \alpha f(x_2, \ldots, x_n, x_1)$ for all $x_1, \ldots, x_n \in A$. Then \mathfrak{B} has a cyclic polymorphism modulo endomorphisms. The analogous statement holds linear equations modulo unary operations in general.

PROOF. We show that there are $e_1, e_2 \in \overline{\operatorname{Aut}(\mathfrak{B})}$ such that $e_1(f(x_1, \ldots, x_n)) = e_2(f(x_2, \ldots, x_n, x_1))$ for all x_1, \ldots, x_n from the domain B of \mathfrak{B} . Construct a rooted tree as follows. Each vertex of the tree lies on some level $n \in \mathbb{N}$. Let d_1, d_2, \ldots be an enumeration of B. Let F_n be the set of partial isomorphisms of \mathfrak{B} with domain $D_n := \{d_1, \ldots, d_n\}$, and define the equivalence relation \sim on F_n^2 as follows: $(\alpha_1, \alpha_2) \sim (\beta_1, \beta_2)$ if there exists a $\delta \in \operatorname{Aut}(\mathfrak{B})$ such that $\alpha_i = \delta \circ \beta_i$ for $i \in \{1, 2\}$. Note that for each n, the relation \sim has finitely many equivalence classes on F_n^2 , by the ω -categoricity of \mathfrak{B} and Theorem 3.1.4. The vertices of the tree on level n are precisely the equivalence classes E of \sim on F_n^2 such that for all $(\alpha_1, \alpha_2) \in E$ and $x_1, \ldots, x_n \in B$ satisfying $\{f(x_1, \ldots, x_n), f(x_2, \ldots, x_n, x_1)\} \subseteq D_n := \{d_1, \ldots, d_n\}$ we have $\alpha_1(f(x_1, \ldots, x_n)) = \alpha_2(f(x_2, \ldots, x_n, x_1))$.

The equivalence class of the partial map with the empty domain D_0 becomes the root of the tree, on level n=0. We define adjacency in the tree by restriction as follows: when E is a vertex on level n, and E' a vertex on level n+1, and E contains (α_1, α_2) and E' contains (α'_1, α'_2) such that $\alpha_1 = \alpha'_1 \upharpoonright_{D_n}$ and $\alpha_2 = \alpha'_2 \upharpoonright_{D_n}$, then we make E and E' adjacent in the tree. Note that the resulting rooted tree is finitely branching. By assumption, the tree has vertices on all levels. Hence, by König's tree lemma, there exists an infinite path E_0, E_1, E_2, \ldots in the tree, where E_i is from level $i \in \mathbb{N}$

We define $e_1, e_2 \in \overline{\operatorname{Aut}(\mathfrak{B})}$ as follows. Suppose e_1, e_2 are already defined on D_n such that $\alpha_1 := e_1 \upharpoonright_{D_n}$, $\alpha_2 := e_2 \upharpoonright_{D_n}$, and $(\alpha_1, \alpha_2) \in E_n$. We want to define e_1 and e_2 on d_{n+1} , and we will do it in such a way that $(e_1 \upharpoonright_{D_{n+1}}, e_2 \upharpoonright_{D_{n+1}}) \in E_{n+1}$. Since E_n and E_{n+1} are adjacent, there exist $(\beta_1, \beta_2) \in E_n$ and $(\beta'_1, \beta'_2) \in E_{n+1}$ such that $\beta_1 = \beta'_1 \upharpoonright_{D_n}$ and $\beta_2 = \beta'_2 \upharpoonright_{D_n}$. By the definition of \sim there exists a $\delta \in \operatorname{Aut}(\mathfrak{B})$ such that $\alpha_1 = \delta \circ \beta_1$ and $\alpha_2 = \delta \circ \beta_2$. For $j \in \{1, 2\}$, define $\alpha'_j := \delta \circ \beta'_j$ so that $(\alpha'_1, \alpha'_2) \in E_{n+1}$ and observe that

$$\alpha'_j \upharpoonright_{D_n} := (\delta \circ \beta'_j) \upharpoonright_{D_n} = \delta \circ \beta_j = \alpha_j$$
,

and hence that α'_j extends α_j . Define $e_j(d_{n+1}) := \alpha'_j(d_{n+1})$. The proof for general linear equations modulo unary operations is analogous.

COROLLARY 5.6.8. Let $\mathfrak C$ be a structure which is homogeneous in a finite relational language, and let $\mathfrak B$ be a model-complete core which is first-order interdefinable with $\mathfrak C$ such that all the operations of the polymorphism algebra $\mathbf B$ of $\mathfrak B$ are canonical with respect to $\mathfrak B$. Then the following are equivalent:

- (1) The type algebra $T(\mathbf{B})$ contains a cyclic operation;
- (2) B has a cyclic polymorphism modulo endomorphisms;
- (3) B has a weak near unanimity polymorphism modulo endomorphisms;
- (4) B has a Siggers polymorphism modulo endomorphisms.

PROOF. Let m be the maximal arity of τ . By Lemma 5.6.6 the finite algebra $T(\mathbf{B})$ is idempotent. Hence, by Theorem 5.6.3 it has a cyclic operation if and only if it has a weak near unanimity operation if and only if it has a Siggers operation.

Suppose now that **B** has a cyclic polymorphism $g^{\mathbf{B}}$ modulo endomorphisms $e_1^{\mathbf{B}}$ and $e_2^{\mathbf{B}}$. Let μ be the homomorphism from **B** to $T(\mathbf{B})$., $e_1^{T(\mathbf{B})} = e_2^{T(\mathbf{B})}$ must be the identity, and $g^{T(\mathbf{B})}$ is a cyclic operation. Hence, 2 implies 1. Analogously one can show that 3 implies 1 and that 4 implies 1.

Conversely, suppose that $T(\mathbf{B})$ has an n-ary cyclic operation $f^{T(\mathbf{B})}$. We claim that $f^{\mathbf{B}}$ satisfies the assumptions of Lemma 5.6.7. To show this, it suffices to prove that for all finite m and all $b^1, \ldots, b^m \in B^n$, the m-tuples $(f^{\mathbf{B}}(b^1), \ldots, f^{\mathbf{B}}(b^m))$ and $(f^{\mathbf{B}}(b^1_2, \ldots, b^1_n, b^1_1), \ldots, f^{\mathbf{B}}(b^m_2, \ldots, b^m_n, b^m_1))$ lie in the same orbit of \mathfrak{C} . Indeed, we have

$$\begin{aligned} \operatorname{tp}^{\mathfrak{C}}(f^{\mathbf{B}}(b^{1}), \dots, f^{\mathbf{B}}(b^{m})) &= f^{T(\mathbf{B})}(\operatorname{tp}(b_{1}^{1}, \dots, b_{1}^{m}), \dots, \operatorname{tp}(b_{n}^{1}, \dots, b_{n}^{m})) \\ &= f^{T(\mathbf{B})}(\operatorname{tp}(b_{2}^{1}, \dots, b_{2}^{m}), \dots, \operatorname{tp}(b_{n}^{1}, \dots, b_{n}^{m}), \operatorname{tp}(b_{1}^{1}, \dots, b_{1}^{m})) \\ &= \operatorname{tp}(f^{\mathbf{B}}(b_{2}^{1}, \dots, b_{n}^{1}, b_{1}^{1}), \dots, f^{\mathbf{B}}(b_{2}^{m}, \dots, b_{n}^{m}, b_{1}^{m})) \end{aligned}$$

Lemma 5.6.7 shows the existence of a cyclic polymorphism modulo endomorphisms of \mathfrak{B} . Analogously one can show the existence of weak near unanimity polymorphisms and Siggers polymorphisms modulo endomorphisms.

Another important consequence of Lemma 5.6.7 is that the existence of cyclic polymorphisms modulo endomorphisms of an ω -categorical model-complete core is inherited by expansions by constants. I am thankful to Trung Van Pham for communicating the proof of the following proposition to me which strengthens a weaker statement in earlier versions of this text.

Proposition 5.6.9. Let \mathfrak{B} be an ω -categorical model-complete core. If \mathfrak{B} has an n-ary cyclic polymorphism modulo endomorphisms, then the expansion of \mathfrak{B} by finitely many constants also has an n-ary cyclic polymorphism modulo endomorphisms. Analogous statements hold for weak near unanimity operations modulo unary operations, and other operations satisfying linear equations modulo unary operations.

PROOF. By the assumption there exist $f \in \text{Pol}(\mathfrak{B})$ of arity k and $e_1, e_2 \in \text{End}(\mathfrak{B})$ such that $e_1(f(x_1, x_2, \ldots, x_k)) = e_2(f(x_2, x_3, \ldots, x_k, x_1))$ for all x_1, \ldots, x_n from the domain B of \mathfrak{B} . Let $\hat{f} \colon B \to B$ be given by $\hat{f}(x) := f(x, x, \ldots, x)$ for all $x \in B$. Clearly, \hat{f} is an endomorphism of \mathfrak{B} . Let $a = (a_1, \ldots, a_n) \in B^n$ be an arbitrary tuple of n constants. Then $a, \hat{f}(a)$, and $e_1(\hat{f}(a)) = e_2(\hat{f}(a))$ lie in the same orbit of $\text{Aut}(\mathfrak{B})$ because \mathfrak{B} is a model-complete core. Let $\alpha, \beta \in \text{Aut}(\mathfrak{B})$ be such that $\alpha e_1(\hat{f}(a)) = a$ and $\beta(\hat{f}(a)) = a$. Let $h_1 := \alpha e_1 \beta^{-1}$ and $h_2 := \alpha e_2 \beta^{-1}$, and $g := \beta f$. Clearly, we

have $g(a, ..., a) = \beta \hat{f}(a) = a$ by the choice of β . We will show that $h_1(a) = a$ and $h_2(a) = a$. We have

$$h_1(a) = h_1(g(a, ..., a)) = \alpha e_1 \beta^{-1}(\beta f(a, ..., a)) = \alpha e_1(\hat{f}(a)) = a.$$

Similarly one can show that $h_2(a) = a$. It follows that $h_1, h_2 \in \text{End}(\mathfrak{B}, a_1, a_2, \dots, a_n)$ and that $g \in \text{Pol}(\mathfrak{B}, a_1, a_2, \dots, a_n)$. Moreover, for all $x_1, x_2, \dots, x_k \in B$ we have that

$$h_1(g(x_1, x_2, \dots, x_k)) = \alpha e_1 \beta^{-1}(\beta f(x_1, x_2, \dots, x_k))$$

$$= \alpha e_1(f(x_1, \dots, x_k))$$

$$= \alpha e_2(f(x_2, \dots, x_k, x_1))$$

$$= \alpha e_2 \beta^{-1}(\beta f(x_2, \dots, x_k, x_1))$$

$$= h_2(g(x_2, x_3, \dots, x_k, x_1)).$$

This shows that $(\mathfrak{B}, a_1, \ldots, a_n)$ has a cyclic polymorphism modulo endomorphisms. The proof for other linear equations modulo endomorphisms is analogous.

How strong is the assumption that every operation of an oligomorphic clone is canonical with respect to some homogeneous structure $\mathfrak C$ over a finite relational signature? In Chapter 8, we see that under fairly general Ramsey-theoretic assumptions on $\mathfrak C$ we can find canonical operations in a natural way. Indeed, it turns out that when $\mathfrak B$ is first-order definable over the random graph $(\mathbb V;E)$ and does not primitively positively interpret ($\{0,1\};1IN3$), then $\mathfrak B$ is the reduct of a structure where all polymorphisms are canonical, and which still does not primitively positively interpret ($\{0,1\};1IN3$) (Theorem 9.8.3). The same statement is not true when we replace the random graph by ($\mathbb Q$; <). However, we make the following general conjecture.

Conjecture 5.3. Let \mathfrak{B} be a countable ω -categorical model-complete core. Then either \mathfrak{B} interprets all finite structures primitively positively with parameters, or \mathfrak{B} has a k-ary weak near unanimity operation modulo endomorphisms.

This conjecture has been confirmed for all \mathfrak{B} that have a first-order definition over $(\mathbb{Q}; <)$, and for all \mathfrak{B} that are definable over the random graph.

Let us remark that in order to 'prevent' primitive positive interpretations of all finite structures, it suffices to have a Taylor term f modulo unary operations applied to the arguments of f and to the function value of f, in the following sense.

PROPOSITION 5.6.10. Let \mathfrak{B} be an ω -categorical structure whose polymorphism algebra \mathbf{B} contains an n-ary f and unary $a_0, \ldots, a_n, b_0, \ldots, b_n$ such that for all $i \leq n$ there are $x_1, \ldots, x_n, y_1, \ldots, y_n \in \{x, y\}$ such that \mathbf{B} satisfies

$$\forall \bar{x}. \, a_0(f(a_1(x_1), \dots, a_{i-1}(x_{i-1}), a_i(x), a_{i+1}(x_{i+1}), \dots, a_n(x_n)))$$

$$= b_0(f(b_1(y_1), \dots, b_{i-1}(y_{i-1}), b_i(y), b_{i+1}(y_{i+1}), \dots, b_n(y_n))).$$
(8)

Then there is no primitive positive interpretation of $(\{0,1\}; 1IN3)$ in \mathfrak{B} .

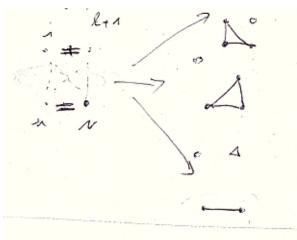
PROOF. Suppose that $(\{0,1\}; 1IN3)$ had a primitive positive interpretation in \mathfrak{B} . Then by Theorem 5.5.17 there is a 2-element algebra \mathbf{A} in the pseudo-variety generated by \mathbf{B} all of whose operations are projections. The algebra \mathbf{A} is in particular in the variety generated by \mathbf{B} , and satisfies (8). Since the unary function symbols of \mathbf{B} must denote the identity in \mathbf{A} , and \mathbf{A} has more than one element, the *i*-th equation prevents that f is the *i*-th projection. Because we have such an equation for each argument of f, the operation f cannot be a projection.

In particular, if \mathfrak{B} is a structure with a weak near unanimity polymorphism modulo endomorphisms, then ($\{0,1\}$; 1IN3) is not primitively positively interpretable in \mathfrak{B} . Using Proposition 5.6.9, we even get the following.

COROLLARY 5.6.11. The two cases in Conjecture 5.3 are indeed disjoint: every ω -categorical model-complete core with a weak near unanimity polymorphism modulo endomorphisms does not interpret ($\{0,1\}$; IIN3) primitively positively with parameters

CHAPTER 6

Equality Constraint Satisfaction Problems



Jan Kára, 2005

This section is about structures with a first-order definition in $(\mathbb{N}; =)$; such structures will be called *equality constraint languages*. From a model-theoretic perspective, equality constraint languages appear to be trivial altogether. However, the set of all such structures, taken up to primitive positive interdefinability and ordered by inclusion, is a quite complicated object (there are actually 2^{ω} many equality constraint languages that pairwise do not define each other primitively positively [30]).

By the results of Section 3.3, a structure \mathfrak{B} is isomorphic to an equality constraint language if and only if \mathfrak{B} is preserved by all permutations of its domain. Therefore, this chapter is about locally closed clones that contain all permutations of the domain. On a *finite* domain, such clones have been completely described in [110]; it turns out that the number of clones that contain all permutations of a fixed finite domain is finite. Clones on infinite sets that contain all permutations are of independent interest in universal algebra [112,152,173,174]. Local closure is a strong additional assumption, which allows a good understanding of the lattice of all locally closed clones that contain all permutations [30].

The CSP for a finite equality constraint language is called an *equality constraint* satisfaction problem. Equality CSPs are of fundamental importance in infinite domain constraint satisfaction; we mention some reasons.

• NP-hard equality relations are very good candidates for establishing hardness results of other infinite domain CSPs. For instance, it follows from the results presented in this section that every structure which admits a primitive positive definition of the relation $\{(x,y,z) \mid (x=y\neq z) \lor (x\neq y=z)\}$ has an NP-hard CSP.

- When analyzing an ω -categorical structure \mathfrak{B} with the universal-algebraic approach, the question which equality constraint languages can be defined in \mathfrak{B} is of crucial importance, as we will see for instance in Chapter 9 and 10. For example, if the relation $\{(x,y,u,v) \mid x=y \Leftrightarrow u=v\}$ is primitive positive definable in \mathfrak{B} , then every polymorphism of \mathfrak{B} that depends on all its arguments must be injective (Proposition 6.1.4).
- Suppose we want to classify the computational complexity of CSP(B) when B has a first-order definition in a fixed infinite structure C; examples of such classifications will be given in Chapter 9 and 10. Then such a classification includes the classification of equality constraint satisfaction problems.

The complexity of equality CSPs has been completely classified [40]; those problems are in P or NP-hard. In this chapter we present a new proof of this result, and show that either the template \mathfrak{B} has a binary injective polymorphism, in which case \mathfrak{B} has a quantifier-free Horn definition in $(\mathbb{N}; =)$, and $CSP(\mathfrak{B})$ is in P, or $(\{0, 1\}; NAE)$ has a primitive positive interpretation in \mathfrak{B} , and $CSP(\mathfrak{B})$ is NP-complete. The new proof has the advantage that it divides the argument into several steps that each holds for a much larger class of structures. Indeed, several results of this chapter turn out to be useful in later classification arguments, in particular in Chapter 9.

We would also like to mention that the fact that satisfiability of quantifier-free Horn clauses over $(\mathbb{N}; =)$ can be decided in polynomial time has already been observed in [129]. Here, we derive the algorithm from more general principles that will also be important for our algorithmic results in Chapter 9 and Chapter 10.

6.1. Independence of Disequality

The importance of *independence* in constraint satisfaction has been recognized several times; the first appearance of this concept in the literature seems to be in [148], and, subsequently, in [149]. In this thesis, we focus on *independence of disequality*, which found most applications; the general definition of independence has been worked out in [76]. Applications of this concept have been studied in metric temporal reasoning [126,138] and qualitative reasoning cacluli [59,60]; also see [37].

DEFINITION 6.1.1 (Independence of Disequality). Let $\mathfrak B$ be a structure with relational signature τ . Then we say that \neq is independent from $\mathfrak B$ if for all primitive positive τ -formula ϕ , if both $\phi \wedge x \neq y$ and $\phi \wedge u \neq v$ are satisfiable over $\mathfrak B$, then $\phi \wedge x \neq y \wedge u \neq v$ is satisfiable over $\mathfrak B$ as well.

In this section we prove that for many ω -categorical structures independence of disequality is equivalent to the existence of a binary injective polymorphism. The following definition comes from [40].

DEFINITION 6.1.2. A relation $R \subseteq B^k$ is called intersection-closed if for all k-tuples $(u_1, \ldots, u_k), (v_1, \ldots, v_k) \in R$ there is a tuple $(w_1, \ldots, w_k) \in R$ such that for all $1 \le i, j \le k$ we have $w_i \ne w_j$ whenever $u_i \ne u_j$ or $v_i \ne v_j$.

LEMMA 6.1.3. Let \mathfrak{B} be an ω -categorical structure where \neq has an primitive positive definition. Then the following are equivalent.

- (1) Disequality is independent from \mathfrak{B} .
- (2) Every finite induced substructure of \mathfrak{B}^2 admits an injective homomorphism into \mathfrak{B} .
- (3) B has a binary injective polymorphism.
- (4) All primitive positive definable relations in $\mathfrak B$ are intersection-closed.

PROOF. Throughout the proof, let b_1, b_2, \ldots be an enumeration of the domain B of \mathfrak{B} . If f is a binary injective polymorphism of \mathfrak{B} , then clearly every relation

in \mathfrak{B} is intersection-closed, so (3) implies (4). The implication from (4) to (1) is straightforward as well.

We now show the implication from (1) to (2). Let \mathfrak{A} be a finite induced substructure of \mathfrak{B}^2 . Then the domain of \mathfrak{A} is contained in $\{b_1, \ldots, b_n\}^2$, for sufficiently large n. It clearly suffices to show that the structure induced by $\{b_1, \ldots, b_n\}^2$ in \mathfrak{B}^2 homomorphically and injectively maps to \mathfrak{B} , so let us assume without loss of generality that the domain of \mathfrak{A} is $\{b_1, \ldots, b_n\}^2$.

Consider the formula ϕ whose variables x_1, \ldots, x_{n^2} are the elements of \mathfrak{A} ,

$$x_1 := (b_1, b_1), \dots, x_n := (b_1, b_n), \dots, x_{n^2 - n + 1} := (b_n, b_1), \dots, x_{n^2} := (b_n, b_n),$$

and which is the conjunction over all literals $R((b_{i_1}, b_{j_1}), \ldots, (b_{i_k}, b_{j_k}))$ such that $R(b_{i_1}, \ldots, b_{i_k})$ and $R(b_{j_1}, \ldots, b_{j_k})$ hold in \mathfrak{A} . So ϕ states precisely which relations hold in \mathfrak{A} .

Using induction over the number m of disequalities, we will now show that for any conjunction $\sigma := \bigwedge_{1 \leq k \leq m} x_{i_k} \neq x_{j_k}$ with the property that $i_k \neq j_k$ for all $1 \leq k \leq m$, the formula $\phi \wedge \sigma$ is satisfiable over \mathfrak{B} . This implies that there exists an n^2 -tuple t in \mathfrak{B} with pairwise distinct entries which satisfies ϕ ; the assignment that sends every x_i to t_i is an injective homomorphism from \mathfrak{A} into \mathfrak{B} .

For the induction beginning, let $x_i \neq x_j$ be any disequality. Let r, s be the n^2 -tuples defined as follows.

$$r := (b_1, \dots, b_1, b_2, \dots, b_2, \dots, b_n, \dots, b_n)$$

$$s := (b_1, b_2, \dots, b_n, b_1, b_2, \dots, b_n, \dots, b_1, b_2, \dots, b_n).$$

These two tuples satisfy ϕ , because the projections to the first and second coordinate, respectively, are homomorphisms from $\mathfrak A$ to $\mathfrak B$. Now either r or s satisfies $x_i \neq x_j$, proving that $\phi \wedge x_i \neq x_j$ is satisfiable in $\mathfrak B$.

In the induction step, let a conjunction $\sigma := \bigwedge_{k \in \{1, ..., m\}} x_{i_k} \neq x_{j_k}$ be given, where $m \geq 2$. Set $\sigma' := \bigwedge_{3 \leq k \leq m} x_{i_k} \neq x_{j_k}$, and $\phi' := \phi \wedge \sigma'$. Observe that ϕ' has a primitive positive definition in \mathfrak{B} , as ϕ and \neq have such definitions. By induction hypothesis, both $\phi' \wedge x_{i_1} \neq x_{j_1}$ and $\phi' \wedge x_{i_2} \neq x_{j_2}$ are satisfiable in \mathfrak{B} . But then $\phi' \wedge x_{i_1} \neq x_{j_1} \wedge x_{i_2} \neq x_{j_2}$, which is equivalent to $\phi \wedge \sigma$, is satisfiable over \mathfrak{B} as well by (1), concluding the proof.

The implication from (2) to (3) follows from Lemma 3.1.8, because the property that a function is injective can be described by the universal first-order sentence $\forall \bar{x}, \bar{y} (\bar{x} \neq \bar{y} \Rightarrow f(\bar{x}) \neq f(\bar{y})).$

Also the situation that the polymorphisms f of an ω -categorical structure are 'essentially injective' can be characterized using equality relations.

PROPOSITION 6.1.4. Let f be an operation from B^k to B that depends on all arguments. Then the following is equivalent.

- (1) f is injective.
- (2) f preserves the relation defined by $x = y \Leftrightarrow u = v$.
- (3) f preserves the relation defined by $x = y \Rightarrow u = v$.

PROOF. For the implication from (1) to (2), suppose that f is injective. We check that f preserves $x = y \Leftrightarrow u = v$. Let a, b, c, d be elements of B^k such that $a_i = b_i \Leftrightarrow c_i = d_i$ for all $i \leq k$, and let t be the tuple (f(a), f(b), f(c), f(d)). If a = b, we thus have that $c_i = d_i$ for all $i \leq k$, and so c = d. In this case, t satisfies $t_1 = t_2$ and $t_3 = t_4$, and we are done. Similarly, if c = d then a = b and we are done. Otherwise, $a \neq b$ and $c \neq d$, and by injectivity of f we have $t_1 \neq t_2$ and $t_3 \neq t_4$. So we have in all cases that $t_1 = t_2$ if and only if $t_3 = t_4$.

For the implication from (2) to (3), note that $x = y \Rightarrow u = v$ is equivalent to a primitive positive formula over (B; R) where $R = \{(a, b, c, d) \mid a = b \Leftrightarrow c = d\}$). The primitive positive formula is

$$\exists w (R(x, y, u, w) \land R(x, y, w, v))$$
.

Finally, for the implication from (3) to (1), suppose that there are distinct $a, b \in B^k$ such that f(a) = f(b). We want to prove that f violates $x = y \Rightarrow u = v$. Let I be the set of all $i \in \{1, \ldots, k\}$ such that $a_i \neq b_i$. Since a and b are distinct, I is non-empty; let $i \in I$ be arbitrary. Since f depends on the i-th argument, there are $c, d \in D^k$ with $c_j = d_j$ for all $j \neq i$, and $c_i \neq d_i$. We claim that (a, b, c, d) shows that f violates $x = y \Rightarrow u = v$. First, note that for all $j \in \{1, \ldots, k\} \setminus I$, we have that $a_j = b_j$ and $c_j = d_j$. Next, note that for all $j \in I$ we have that $a_j \neq b_j$. We conclude that for all $j \in \{1, \ldots, k\}$ we have that $a_i = b_i$ implies $c_i = d_i$. However, f(a) = f(b) and $f(c) \neq f(d)$.

We close with an application to CSPs. When $\mathfrak A$ is an instance of CSP($\mathfrak B$), then an injective homomorphism from $\mathfrak A$ to $\mathfrak B$ is also called an *injective solution* for $\mathfrak A$.

PROPOSITION 6.1.5. Suppose that \mathfrak{B} has a binary injective polymorphism h. Then every satisfiable instance \mathfrak{A} of $\mathrm{CSP}(\mathfrak{B})$ either has an injective solution, or \mathfrak{A} has two distinct elements a, a' such that s(a) = s(a') in all solutions s for \mathfrak{A} .

PROOF. Suppose that $\mathfrak A$ has a solution, but no injective solution. Let f be a solution such that the cardinality of f is maximal. Since there is no injective solution, there are two elements a, a' of $\mathfrak A$ such that f(a)=f(a'). We claim that s(a)=s(a') in all solutions s of $\mathfrak A$. Otherwise, if $s(a)\neq s(a')$ for some solution s, then by the choice of f there must be another pair b,b' such that $s(b)\neq s(b')$ but $f(b)\neq f(b')$. Then the mapping $s\mapsto h(f(s),s(s))$ is also a solution to $\mathfrak A$, but has a strictly larger image than s(s)0 and s(s)1 solution.

6.2. Two-transitive Templates

We show that two-transitive structures (in particular, equality constraint languages) with essential but without constant polymorphisms also have binary injective polymorphisms. Here we use Corollary 5.3.11 about existence of binary essential polymorphisms, and Lemma 6.1.3 about the existence of binary injective polymorphisms.

Theorem 6.2.1. Let $\mathfrak B$ be a two-transitive structure without a constant polymorphism, but with an essential polymorphism. Then $\mathfrak B$ has a binary injective polymorphism.

PROOF. Corollary 5.3.11 implies that $\mathfrak B$ is preserved by a binary essential operation f. Since $\mathfrak B$ has no constant polymorphism and is 2-set transitive, Corollary 5.3.8 implies that all polymorphisms of $\mathfrak B$ preserve \neq , and hence \neq is primitive positive definable in $\mathfrak B$. So we can apply Lemma 6.1.3, and have to show that for every primitive positive formula ϕ the formula $\phi \wedge x \neq y \wedge u \neq v$ is satisfiable over $\mathfrak B$ whenever $\phi \wedge x \neq y$ and $\phi \wedge u \neq v$ are satisfiable over $\mathfrak B$.

Let V be the variables of ϕ , and let $s \colon V \to B$ be a satisfying assignment for $\phi \land x \neq y$, and $t \colon V \to B$ be a satisfying assignment for $\phi \land u \neq v$. We can assume that s(u) = s(v) and t(x) = t(y), otherwise we are done. Let k be the cardinality of the set $\{s(x), s(y), s(u), t(u), t(v), t(x)\}$; note that $4 \leq k \leq 6$. Suppose that k = 6, the other cases are simpler. Since f is essential, it violates the relation P_3^B (Lemma 5.3.2).

Since f preserves \neq , we can therefore assume that there are tuples (a, a, b) for $a \neq b$ and (c, d, d) for $c \neq d$ such that $f(a, c) \neq f(a, d)$ and $f(a, d) \neq f(b, d)$. By 2-transitivity of Aut(\mathfrak{B}), there are $\alpha, \beta \in \text{Aut}(\mathfrak{B})$ such that $\alpha(s(u), s(x)) = (a, b)$, and

 $\beta(t(u), t(x)) = (c, d)$. Since $s(x) \neq s(y)$ and $t(v) \neq t(u)$, and f preserves \neq , we have $f(\alpha(s(x)), \beta(t(v))) \neq f(\alpha(s(y)), \beta(t(u)))$. This implies that $f(\alpha(s(x)), \beta(t(v))) = f(\alpha(s(y)), \beta(t(v)))$ and $f(\alpha(s(y)), \beta(t(v))) = f(\alpha(s(y)), \beta(t(u)))$ cannot both be true. By 2-transitivity of Aut(\mathfrak{B}), there exist α', β' such that $\alpha'(a, \alpha(u)) = (\alpha(u), a)$, and $\beta'(d, \beta(v)) = (\beta(v), d)$.

If $f(\alpha(s(x)), \beta(t(v))) \neq f(\alpha(s(y)), \beta(t(v)))$, then $z \mapsto f(\alpha s(z), \beta' t(z))$ is a satisfying assignment for $\phi \land x \neq y \land u \neq v$. If $f(\alpha(s(y)), \beta(t(v))) \neq f(\alpha(s(y)), \beta(t(u)))$, then $z \mapsto f(\alpha' s(z), \beta t(z))$ is a satisfying assignment for $\phi \land x \neq y \land u \neq v$.

6.3. Horn Formulas

In this section we show that \mathfrak{B} has certain binary injective polymorphisms if and only if all relations in \mathfrak{B} have a quantifier-free Horn definition over a 'base' structure \mathfrak{C} ; this will often be useful to design algorithms for $CSP(\mathfrak{B})$.

The structure $(\mathbb{N};=)$ has quantifier-elimination: this follows from Lemma 3.6.1 by the observation that every bijection between finite subsets of \mathbb{N} can be extended to a permutation of \mathbb{N} . So we could have defined equality constraint languages as those relational structures where all relations have a *quantifier-free* definition in $(\mathbb{N};=)$. Note that not all equality constraint languages have quantifier-elimination; however, all equality constraint languages are model-complete.

Proposition 6.3.1. Every equality constraint language \mathfrak{B} is model-complete.

PROOF. The permutations of \mathbb{N} locally generate all injective self-maps on \mathbb{N} . Hence, the statement follows from Theorem 3.6.7 by the observation that the embeddings of \mathfrak{B} are locally generated by automorphisms of \mathfrak{B} .

The following is a simple, but very useful definition to prove syntactic results.

DEFINITION 6.3.2 (as in [30]). A quantifier-free first-order formula ϕ in conjunctive normal form is called reduced (over a structure \mathfrak{B}) if every formula obtained from ϕ by removing a literal or a clause is not equivalent to ϕ (over \mathfrak{B}).

Clearly, every quantifier-free formula is equivalent to a reduced formula over \mathfrak{B} , because we can find one by successively removing literals and clauses from ϕ . The following theorem is from [29] and [35] (stated there for quantifier-free Horn formulas only).

THEOREM 6.3.3. Let \mathfrak{B} be a structure with an embedding e from \mathfrak{B}^2 into \mathfrak{B} . Then a relation R with a quantifier-free definition in \mathfrak{B} has a quantifier-free Horn definition in \mathfrak{B} if and only if R is preserved by e.

PROOF. (Backwards.) Let δ be a quantifier-free Horn definition of R over \mathfrak{B} , written in prenex conjunctive normal form. It suffices to demonstrate that e preserves each clause in δ . Note that a Horn clause ψ of δ can always be written in the form $(\phi_1 \wedge \cdots \wedge \phi_l) \to \phi_0$, for atomic τ -formulas ϕ_0, \ldots, ϕ_l . Let V be the variables of ψ , and let $s_1, s_2 \colon V \to \mathbb{N}$ be two assignments that satisfy the clause. We claim that $s_3 \colon V \to \mathbb{N}$ defined by $s_3(x,y) = e(s_1(x), s_2(y))$ satisfies ψ . There are two cases cases to consider. Either there is an $i \leq l$ such that s_1 or s_2 does not satisfy ϕ_i . In this case, since e is an embedding from \mathfrak{B}^2 to \mathfrak{B} , s_3 does not satisfy ϕ_i , and therefore satisfies ψ . Or, if for all $i \leq l$ both s_1 and s_2 satisfies ϕ_i , then they also satisfy ϕ_0 . Since e is a polymorphism of \mathfrak{B} , it follows that s_3 satisfies ϕ_0 , and therefore also ψ .

(Forwards.) Consider a quantifier-free definition δ of R in \mathfrak{B} such that δ is in prenex normal form, and that the quantifier-free part η of δ is a reduced CNF formula over \mathfrak{B} . Assume for contradiction that δ is not Horn, that is, η has a clause $\psi = \phi_1 \vee \phi_2 \vee \phi_3 \vee \cdots \vee \phi_l$ where ϕ_1, ϕ_2 are positive literals, and ϕ_3, \ldots, ϕ_l are

positive or negative literals. Let V be the variables of η . Since η is reduced, it has a satisfying assignment $s_1 \colon V \to \mathbb{N}$ such that ϕ_i is false for all $i \leq l$ except for i=1; otherwise, we could remove ϕ_i from ψ and would obtain a formula that is equivalent to η over \mathfrak{B} , contradicting the assumption that η is reduced. Similarly, η has a satisfying assignment $s_2 \colon V \to \mathbb{N}$ such that ϕ_i is false for all $i \leq l$ except for i=2. Then $s_3 \colon V \to \mathbb{N}$ defined by $s_3(x,y) = e(s_1(x),s_2(y))$ does not satisfy ψ , a contradiction.

The structure $\mathfrak{B} := (\mathbb{N}; =)$ is an obvious example with an embedding from \mathfrak{B}^2 into \mathfrak{B} . When \mathfrak{B} is a relational structure, then \mathfrak{B}^{\neg} denotes the expansion of \mathfrak{B} by all relations that are the complement of a relation from \mathfrak{B} . The following is from [29] (note that we do not assume ω -categoricity of \mathfrak{B} and \mathfrak{C}).

THEOREM 6.3.4. Let \mathfrak{C} be a structure with an embedding e from \mathfrak{C}^2 into \mathfrak{C} . Let \mathfrak{B} be a relational structure with finite signature σ that is preserved by e and has a quantifier-free definition in \mathfrak{C} . Then there is a polynomial-time Turing reduction from $CSP(\mathfrak{B})$ to $CSP(\mathfrak{C}^{\neg})$.

```
// Input: An instance \phi of CSP(\mathfrak{B})
// Assumption: \mathfrak{B} has a quantifier-free Horn definition in a \tau-structure \mathfrak{C}.
Replace each constraint R(x_1, \ldots, x_n) from \phi by \delta(x_1, \ldots, x_n),
   where \delta is a quantifier-free Horn definition of R in \mathfrak{C}.
Let \psi be the resulting \tau-sentence, written in prenex conjunctive normal form.
Repeat := true
While Repeat = true do
   Repeat := false
   Let \Psi be the set of all singleton clauses in \psi.
   If \Psi is unsatisfiable over \mathfrak{C} then reject.
   For each negative literal \eta of \psi do
      If \Psi \cup \eta, considered as an instance of CSP(\mathfrak{C}^{\neg}), is unsatisfiable
        Remove \eta from its clause in \psi
        Repeat := true
   End for
Loop
Accept
```

FIGURE 6.1. A polynomial-time Turing reduction from $CSP(\mathfrak{B})$ to $CSP(\mathfrak{C}^{\neg})$ when \mathfrak{B} has a polymorphism that is an embedding of \mathfrak{C}^2 into \mathfrak{C} .

PROOF. We use the algorithm shown in Figure 6.1, which is due to [76]. By Theorem 6.3.3, every relation of \mathfrak{B} has a quantifier-free Horn definition in \mathfrak{C} . Let ϕ be an input instance of $\mathrm{CSP}(\mathfrak{B})$, and let ψ be the sentence in the language of \mathfrak{C} obtained from ϕ as described in the algorithm. Since σ is finite and fixed, and does not depend on the input, there is only a linear number of literals that can be deleted from ψ in the course of the algorithm. It is thus clear that the algorithm works in polynomial time.

To show that the algorithm is correct, observe that ϕ is false in $\mathfrak B$ if and only if ψ is false in $\mathfrak C$. We first show that if the algorithm rejects, then ψ is false in $\mathfrak B$. The reason is that whenever a negative literal η is removed from a clause of ψ , then in fact $\neg \eta$ is implied by the other clauses in ψ , and therefore removing η from ψ leads to an equivalent formula.

Finally, we show that if the algorithm accepts, then ψ is true in \mathfrak{C} . Let B be the domain of \mathfrak{B} and \mathfrak{C} , and let V be the set of variables of ψ . Consider the negative literals η_1, \ldots, η_m that are in clauses of ψ at the final stage of the algorithm. For all $i \leq m$, let $t_i \colon V \to B$ be an assignment that satisfies all clauses of ψ without negative literals, and which also satisfies η_i . Such an assignment must exist, since otherwise η_i would have been false in all solutions, and our algorithm would have removed η_i in the inner loop of the algorithm. We claim that $s \colon V \to B$ given by

$$s(x) = e(t_1(x), e(t_2(x), \dots e(t_{m-1}(x), t_m(x)) \dots))$$

satisfies all clauses of ψ . Negative literals η_k are satisfied because t_k satisfies η_k , and e is an embedding of \mathfrak{C}^2 into \mathfrak{C} . Positive literals from ψ are satisfied by s because they are satisfied by all the t_i , and since e is a polymorphism of \mathfrak{C} .

6.4. Classification

We now finish the complexity classification for $CSP(\mathfrak{B})$ where \mathfrak{B} is an equality constraint language, combining the results from the previous sections of this chapter.

Corollary 6.4.1. Let $\mathfrak B$ be an equality constraint language. Then one of the following cases applies.

- (1) B has a constant polymorphism.
- (2) B has a binary injective polymorphism.
- (3) In \mathfrak{B} every first-order formula is equivalent to a primitive positive formula.

PROOF. Suppose that \mathfrak{B} does not have a constant polymorphism. Since equality constraint languages have 2-transitive automorphism groups, we can use the contrapositive of Corollary 5.3.8 to derive that all polymorphisms of \mathfrak{B} must preserve \neq . The endomorphisms of \mathfrak{B} are therefore injective, and locally generated by the automorphisms of \mathfrak{B} . If \mathfrak{B} does not have essential polymorphisms, then Corollary 5.3.4 shows that all relations that are first-order definable in \mathfrak{B} are also primitive positive definable in \mathfrak{B} , and we are in case (3). If \mathfrak{B} has an essential polymorphism, then \mathfrak{B} has a binary injective polymorphism by Theorem 6.2.1.

We can now give the complexity classification for equality constraint languages, which confirms Conjecture 5.3 in a special case.

Theorem 6.4.2. Let \mathfrak{B} be an equality constraint language. Then exactly one of the following cases applies.

• \mathfrak{B} has a polymorphism f and an automorphism α such that

$$f(x,y) = \alpha f(y,x)$$

for all elements x and y of \mathfrak{B} . In this case, for every finite reduct \mathfrak{B}' of \mathfrak{B} the problem $\mathrm{CSP}(\mathfrak{B}')$ can be solved in polynomial time.

• There is a primitive positive interpretation of $(\{0,1\}; \text{IIN3})$ in \mathfrak{B} . In this case, there is a finite reduct \mathfrak{B}' of \mathfrak{B} such that $\text{CSP}(\mathfrak{B}')$ is NP-complete.

PROOF. By Proposition 5.6.10, the two cases are disjoint. If \mathfrak{B} has a constant polymorphism, then clearly there are f and α such that $f(x,y) = \alpha f(y,x)$ for all $x,y \in B$. The claim for finite reducts of \mathfrak{B} follows from Proposition 1.1.11.

Now suppose that \mathfrak{B} has a binary injective polymorphism f. Such an operation is an isomorphism between $(\mathbb{N};=)^2$ and $(\mathbb{N};=)$, and we can find a permutation α of B such that $f(x,y)=\alpha f(y,x)$ for all $x,y\in\mathbb{N}$. Since every relation of \mathfrak{B} has a quantifier-free definition over $(\mathbb{N};=)$, Theorem 6.3.3 shows that every relation of \mathfrak{B} even has a quantifier-free Horn definition over \mathfrak{B} . By Theorem 6.3.4, the CSP for every finite signature reduct of \mathfrak{B} can be reduced to $CSP((\mathbb{N};=,\neq))$ in polynomial time.

Tractability of $\mathrm{CSP}((\mathbb{N};=,\neq))$ has been shown in Section 1.1. By Corollary 6.4.1, the only remaining case is that over $\mathfrak B$ all first-order formulas are equivalent to primitive positive formulas. In this case the claim follows from Corollary 5.5.10.

CHAPTER 7

Topology



Several important properties of ω -categorical structures only depend on their automorphism group considered as a topological group, that is, on their automorphism group viewed as an abstract group, with the topology of pointwise convergence on the group elements. This is in particular the case for certain Ramsey properties that become important in the next chapter. We therefore give a self-contained introduction to basic topological background, with a focus on the topics that become relevant for our applications to automorphism groups of ω -categorical structures.

7.1. Topological Spaces

A topological space is a set S together with a collection of subsets of S, called the open sets of S, such that

- (1) the empty set and S are open;
- (2) arbitrary unions of open sets are open;
- (3) the intersection of two open sets is open.

Complements of open sets are called *closed*. For $E \subseteq S$, the *closure* of E is the set of all points x such that every open set in S that contains x also contains a point from

E. Clearly, the closure of E is a closed set. A subset E of S is called dense (in S) if its closure is the full space S. The subspace of S induced on E is the topological space on E where the open sets are exactly the intersections of E with the open sets of S.

DEFINITION 7.1.1. A mapping between two topological spaces is called continuous if the pre-images of open sets are open, and open if images of open sets are open. A bijective open and continuous map is called a homeomorphism.

A basis of S is a collection of open subsets of S such that every open set in S is the union of sets from the collection. For $s \in S$, a collection of open subsets of S is called a basis at s if each set from the collection contains s, and every open set containing s also contains an open set from the collection. For a sequence $(s_n)_{n\geq 1}$ of elements of S, we write $\lim_{n\to\infty} s_n = s$, and say that s_n converges (against the limit s) if for every open set U of S that contains s there exists an n_0 such that $s_n \in U$ for all $n > n_0$. A topological space S is called

- ullet discrete if every subset of S is open (and hence also closed);
- compact if for an arbitrary collection $\{U_i\}_{i\in A}$ of open subsets of S with $S = \bigcup_{i\in A} U_i$ there is a finite subset B of A such that $S = \bigcup_{i\in B} U_i$;
- Hausdorff if for any two distinct points u, v of S there are disjoint open sets U and V that contain u and v, respectively;
- first-countable if for all $s \in S$ there exists a countable basis at s.
- separable if there is a countable dense set;

The following equivalent characterization of continuity of maps from a first-countable space S to a topological space T is often easier to work with. For $x \in S$, we say that f is continuous at x if for every open $V \subseteq T$ containing f(x) there is an open $U \subseteq S$ containing x whose image f(U) is contained in V.

Proposition 7.1.2. Let S be a first-countable and T an arbitrary topological space. Then for every $f: S \to T$ the following are equivalent.

- (1) f is continuous.
- (2) For all s_n , if $\lim_{n\to\infty} s_n = s$ then $\lim_{n\to\infty} f(s_n) = f(s)$.
- (3) f is continuous at every $x \in S$.

PROOF. The implication from (1) to (2) is true even without the assumption that S is first-countable. Let $(s_n)_{n\geq 1}$ be such that $\lim_{n\to\infty} s_n=s$, and let V be open so that $f(s)\in V$. Then $U:=f^{-1}(V)$ is open, and $s\in U$. So there exists an n with $s_n\in U$. For this $n,\,f(s_n)\in V$. So $\lim_{n\to\infty} f(s_n)=f(s)$.

For the implication from (2) to (3), we show the contraposition. Suppose that f is not continuous at some $s \in S$. That is, there exists an open set V containing f(s) such that all open sets U that contain x have an image that is not contained in V. Since S is first-countable, there exists a countable collection U_n of open sets containing x so that any open V that contains x also contains some U_n . Replacing U_n by $\bigcap_{k=1}^n U_k$ where necessary, we may assume that $U_1 \supset U_2 \supset \cdots$. If $U_n \subseteq f^{-1}(V)$, then $f(U_n) \subseteq V$, in contradiction to our assumption; so we can pick an $x_n \in U_n \setminus f^{-1}(V)$ for all n, and obtain a sequence that converges to x. But $s_n \notin f^{-1}(V)$ for all n, and so $f(s_n)$ does not converge to $f(s) \in V$.

Finally, the implication from (3) to (1) again holds in arbitrary topological spaces. Let $V \subseteq T$ be open. We want to show that $U := f^{-1}(V)$ is empty. When s is a point from U, then because f is continuous at s, and V contains f(s) and is open, there is an open set $U_s \subseteq S$ containing s whose image $f(U_s)$ is contained in V. Then $\bigcup_{s \in U} U_s = U$ is open as a union of open sets.

Important examples of topologies come from metric spaces. A sequence $(s_n)_{n\in\mathbb{N}}$ of elements of a metric space (S;d) is called a *Cauchy sequence* if for every $\epsilon>0$ there is an $n_0\in\mathbb{N}$ such that for all $n,m>n_0$ we have that $d(s_n,s_m)<\epsilon$. A topological space S is called

- metrizable if there exists a metric d on S which is compatible, i.e., the open sets are unions of sets of the form $\{y \in S \mid d(c,y) \le r\}$, for $x \in S$, $0 \le r \in \mathbb{R}$;
- completely metrizable if it has a compatible complete metric d, i.e., a metric d on S where every Cauchy sequence converges against an element of S;
- Polish if S is separable and completely metrizable.

The $product \prod_{i \in I} S_i$ of a family of topological spaces $(S_i)_{i \in I}$ is the topological space on the cartesian product $\times_{i \in I} S_i$ where the open sets are unions of sets of the form $\times_{i \in I} U_i$ where U_i is open in S_i for all $i \in I$, and $U_i = S_i$ for all but finitely many $i \in I$. When I has just two elements, say 1 and 2, we also write $S_1 \times S_2$ for the product (this operation is clearly associative and commutative). We denote by S^k for the k-th power $S \times \cdots \times S$ of S, equipped with the product topology as described above.

We also write S^I to a |I|-th power of S, where the factors are indexed by the elements of I. In this case, we can view each element of $T := S^I$ as a function from I to S in the obvious way. The product topology on T is also called the *topology of pointwise convergence*, due to the following.

PROPOSITION 7.1.3. Let S be a topological space, and I be a set. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of elements of the product space $T:=S^I$. Then $\lim_{n\to\infty} f_n=f$ if and only if $\lim_{n\to\infty} f_n(j)=f(j)$ in S for all $j\in I$.

PROOF. Suppose first that $\lim_{n\to\infty} f_n = f$ in T. Let $j \in I$ be arbitrary and let V be an open set that contains f(j). Then the set $U := \prod_{i\in I} T_i$ where $T_i = V$ if i = j, and $T_i = S$ otherwise, is open in T and contains f. So there is an n_0 such that $f_n \in U$ for all $n \geq n_0$. But then $f_n(j) \in V$ for all $n \geq n_0$, and so $\lim_{n\to\infty} f_n(j) = f(j)$.

Now suppose that $\lim_{n\to\infty} f_n(j) = f(j)$ in S for all $j \in I$, and let V be an open set of T that contains f. Then there exists a finite $J \subseteq I$ and open subsets $(V_j)_{j\in J}$ of S such that $f \in \prod_{i\in I} T_i$ where $T_i = V_i$ if $i \in J$ and $T_i = S$ otherwise. For each $j \leq J$ there exists an n_j so that $f_n(j) \in V_j$ for all $n \geq n_j$. Then $f_n \in V$ for all $n \geq \max_{i \in J} n_i$, and hence $\lim_{n\to\infty} f_n = f$.

Example 7.1.4. When we equip the natural numbers \mathbb{N} with the discrete topology, then $\mathbb{N}^{\mathbb{N}}$ with the topology of pointwise convergence is called the *Baire space*. The open sets are exactly the unions of sets of the form $\{g \in \mathbb{N} \to \mathbb{N} \mid g(\bar{a}) = \bar{b}\}$ for some $\bar{a}, \bar{b} \in \mathbb{N}^k, k \in \mathbb{N}$.

Theorem 7.1.5 (Tychonoff; see e.g. [124]). Products of compact spaces are compact.

7.2. Topological Groups

A topological group is an (abstract) group \mathbf{G} together with a topology on the elements G of \mathbf{G} such that $(x,y)\mapsto xy^{-1}$ is continuous from G^2 to G. In other words, we require that the binary group operation and the inverse function are continuous. Two topological groups are said to be *isomorphic* if the groups are isomorphic, and the isomorphism is a homeomorphism between the respective topologies.

EXAMPLE 7.2.1. The elements of the group $\operatorname{Sym}(\mathbb{N})$ form a (non-closed) subset of the Baire space $\mathbb{N}^{\mathbb{N}}$ (Example 7.1.4), and the topology induced by the Baire space on $\operatorname{Sym}(\mathbb{N})$ is also called the *topology of pointwise convergence*. The open sets are the

unions of sets of the form $\{g \in \operatorname{Sym}(\mathbb{N}) \mid g(\bar{a}) = \bar{b}\}$ for some finite tuples \bar{a}, \bar{b} over \mathbb{N} .

An action of a topological group G on a topological space S is continuous if it is continuous as a function from $G \times S$ into S. A continuous action of G on S gives rise to a homomorphism from G into the group of all homeomorphisms of S. An action is faithful if this homomorphism is injective. If G is a subgroup of $Sym(\mathbb{N})$ and the space S is countable and equipped with the discrete topology, it makes sense to call a such a homomorphism topologically faithful if additionally the homomorphism is a homeomorphism whose image is closed in Sym(S) (equipped with the product topology).

An important example of a continuous action of topological groups is the following. A left coset of a subgroup V of G is a set of the form $\{hg \mid g \in V\}$ for $h \in G$, also written hV. Clearly, the set of all left cosets of G partitions G, and is denoted by G/V. The cardinality of G/V is the index of V in G. The set G/V can be viewed a topological space where a set of left-cosets is open if their union is open in G. We can define a continuous action of G on G/V by setting $g \cdot hV = ghV$. This action is also called the action of G on G/V by left translation. Analogously we define the space G/V of all right-cosets Vh, and the action of G on G/V by right translation.

Every open subgroup **H** of **G** is closed, since the complement of H in G is the open set given by the union of open sets gH for $g \in G \setminus H$. A topological group **G** is

- Hausdorff (metrizable, Polish) if the topology of **G** is Hausdorff (metrizable, Polish, respectively);
- first-countable if it has a countable basis at the identity.
- non-archimedian if it has a basis at the identity consisting of open subgroups. We also recall the following.

PROPOSITION 7.2.2 (Proposition 13 and Proposition 14 in [58]). Let G be a topological group, and let H be a subgroup of G. Then

- G/H is discrete if and only if H is open in G;
- G/H is Hausdorff if and only if H is closed in G.

The following is sometimes useful to verify that an action is topologically faithful.

PROPOSITION 7.2.3 (Proposition 2.2.1 in [100]). Let G be a Polish group and H a subgroup of G with the subspace topology. Then H is Polish if and only if H is closed in G.

The topological automorphism group of a structure \mathfrak{B} with domain B is a topological group obtained from the abstract automorphism group G of \mathfrak{B} (see Section 3.3.4) by equiping the elements G of G with the topology of pointwise convergence, that is, the topology induced on G by the one on Sym(B) as given in Example 7.2.1.

PROPOSITION 7.2.4. A set \mathcal{B} of permutations of a set B is a closed subset of Sym(B) if and only if it is locally closed as defined in Definition 3.3.1.

PROOF. The set of operations \mathscr{B} is *not* closed in the topology of pointwise convergence if and only if there exists a permutation $g \in \operatorname{Sym}(B) \setminus \mathscr{B}$ such that every open set containing g also contains an element of \mathscr{B} . This is the case if and only that for every tuple \bar{a} , \mathscr{B} contains an operation h such that the restriction of g to the elements of \bar{a} equals the restriction of h to those elements. According to Definition 3.3.1, this is exactly the case when g is in the local closure by \mathscr{B} .

In the following, let G be a topological group that is the automorphism group of a relational structure \mathfrak{B} , and let G be its domain (equipped with the topology

of pointwise convergence). Note that if G is compact then all orbits of G must be finite. Hence, when $\mathfrak B$ is ω -categorical, G cannot be compact. It is clear that G is non-archimedian. The topology on G has the following compatible metric d. When b_1, b_2, \ldots is an enumeration of the domain B of $\mathfrak B$, then for elements $f, g \in G$ we define $d(f,g) = 1/2^{n+1}$ where n is the least natural number such that $f(b_n) \neq g(b_n)$. In fact, d is an ultrametric, that is, it satisfies $d(x,z) \leq max(d(x,y),d(y,z))$ for all x,y,z. Moreover, d is left-invariant, i.e., d(gx,gy) = d(x,y) for all $g,x,y \in G$. This metric is not complete: to see this, let f be an arbitrary injective non-surjective mapping from $B \to B$. For each n, there exists a permutation h_n of B such that $h_n(b_i) = f(b_i)$ for all $i \leq n$. Hence, the sequence $(h_n)_{n \geq 1}$ is Cauchy, but it does not converge to a permutation.

The topology on G is also completely metrizable. To see this, we define a compatible complete metric d' by setting $d'(f,g) = 1/2^{n+1}$ for elements $f,g \in G$ where n is the least natural number such that $f(b_n) \neq g(b_n)$ or $f^{-1}(b_n) \neq g^{-1}(b_n)$. Alternatively, and more generally, when d is a compatible left-invariant metric, then $d(x,y) + d(x^{-1},y^{-1})$ defines a compatible complete metric (see [19]).

Finally, **G** is separable: for all finite tuples \bar{a} , \bar{b} that lie in the same orbit we fix an element of G that maps \bar{a} to \bar{b} ; the (countable) set of all the selected elements of G is clearly dense in G.

In this thesis, we will be exclusively interested in topological groups that arise as automorphism groups of countable structures. Those groups can be characterised in topological terms, as demonstrated in Proposition 7.2.5 below.

PROPOSITION 7.2.5 (Section 1.5 in [19]; also see Theorem 2.4.1 and Theorem 2.4.4 in [100]). Let **G** be a topological group. Then the following are equivalent.

- (1) **G** is isomorphic to the topological automorphism group of a countable relational structure.
- (2) **G** is isomorphic to a closed subgroup of $Sym(\mathbb{N})$.
- (3) **G** is Polish and admits a compatible left-invariant ultrametric.
- (4) **G** is Polish and non-archimedian.
- (5) **G** is Polish and has an at most countable basis closed under left multiplication, that is, an at most countable basis \mathcal{B} of **G** so that for any $U \in \mathcal{B}$ and $g \in G$ we have $gU \in \mathcal{B}$.

PROOF. The equivalence of (1) and (2) has been shown in Proposition 3.3.2. The implication from (1) to (3) has been explained in the paragraphs preceding the statement of the proposition. So it suffices to show $(3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (2)$.

For the implication from (3) to (4), let d be a left-invariant ultrametric on G. Let $U_n = \{x \in G \mid d(x,1) < 2^{-n}\}$, for $n \in \mathbb{N}$. We claim that the set of all those U_n forms a basis at the identity consisting of open subgroups. Since d is a left-invariant ultra-metric, for $x, y \in G$ we have

$$d(x^{-1}y, 1) = d(y, x) \le max(d(y, 1), d(1, x))$$

and thus U_n is a indeed a subgroup.

For the implication from (4) to (5), assume (4). Let $\{U_1, U_2, \dots\}$ be an at most countable basis at the identity (which exists since G is metrizable). Each U_i has an open subset V_i which is a subgroup, since G has a basis at the identity consisting of open subgroups. Then $\{V_1, V_2, \dots\}$ is a countable basis of the identity consisting of open subgroups. Each V_i has at most countably many cosets since G is separable. So the set of all cosets of those groups gives an at most countable basis that is closed under left multiplication.

Finally, we show that (5) implies (2). Let $\mathcal{B} = \{U_1, U_2, \dots\}$ be an at most countable basis closed under left multiplication. If \mathcal{B} is infinite, we define the map

 $\xi \colon G \to \operatorname{Sym}(\mathbb{N})$ by setting

$$\xi(g)(n) = m \Leftrightarrow gU_n = U_m$$
.

If $|\mathcal{B}| = n_0$ is finite, we define the map $\xi \colon G \to \operatorname{Sym}(\mathbb{N})$ similarly, but set $\xi(g)(n) = n$ for all $n > n_0$. It is straightforward to verify that $\xi(fg) = \xi(f)\xi(g)$. The mapping ξ is injective: when f and g are distinct, then there are disjoint open subsets U and V with $f \in U$ and $g \in V$, because the topology is Hausdorff; since \mathcal{B} is a basis, we can assume that $U = U_{n_1}$ and $V = U_{n_2}$, for some $n_1, n_2 \geq 1$. If $fU_{n_1} = gU_{n_1}$, then $g \in U_{n_1} = U$ since $f \in U_{n_1}$, contradicting the assumption that U and V are disjoint. Hence, $\xi(f)(n_1) \neq \xi(g)(n_1)$, and so $\xi(f) \neq \xi(g)$. Since bijective algebra homomorphisms are isomorphisms, ξ is an isomorphism between \mathbf{G} and a subgroup of $\operatorname{Sym}(\mathbb{N})$. To verify that ξ is continuous, let $g \in G$ be arbitrary, and let $V \subseteq \operatorname{Sym}(\mathbb{N})$ be an open set containing $\xi(g)$. Then V is a union of basic open sets of the form $V_{\bar{a},\bar{b}} := \{f \in \operatorname{Sym}(\mathbb{N}) \mid f(\bar{a}) = \bar{b}\}$ for some $\bar{a},\bar{b} \in \mathbb{N}^n$. The preimage of $V_{\bar{a},\bar{b}}$ under ξ is $\{g \in G \mid gU_{a_1} = U_{b_1} \wedge \cdots \wedge gU_{a_n} = U_{b_n}\}$. Since multiplication in G is continuous, this set is open. Hence the preimage of V is a union of open sets and therefore open as well, which concludes the proof that ξ is continuous.

It can also be verified that ξ is open; for the details of this last step, we refer to [100] (Theorem 2.4.4). Therefore, ξ is a homeomorphism between \mathbf{G} and its image $\xi(G)$, which is therefore also Polish, and a subgroup of the Polish group Sym(\mathbb{N}). By Proposition 7.2.3, $\xi(G)$ is a *closed* subgroup of Sym(\mathbb{N}).

A subgroup **N** of **G** with domain N is called *normal* if gN = Ng for all elements g of **G**. Recall the following equivalent characterizations of normality of subgroups, which can be seen as a refinement of Proposition 5.5.2 for the case of groups.

Proposition 7.2.6. Let G be a group, and N be a subgroup of G. Then the following are equivalent.

- (1) N is normal.
- (2) **G** has the congruence $E = \{(a,b) \mid ab^{-1} \in N\}$.
- (3) There is a homomorphism h from **G** to some group such that $N = h^{-1}(0)$.
- (4) For every $g \in G$ and every $v \in N$ we have $gvg^{-1} \in N$.

PROOF. (1) \Rightarrow (2): to verify that E is a congruence, we have to show that for all $(a_1,b_1),(a_2,b_2)\in E, (a_1a_2,b_1b_2)\in E$. Indeed, $(a_1a_2)(b_1b_2)^{-1}=a_1(a_2b_2^{-1})b_1^{-1}\in a_1Nb_1^{-1}=Na_1b_1^{-1}\subseteq NN=N$.

- (2) \Rightarrow (3): follows from Proposition 5.5.2: $g \mapsto gN$ is a group homomorphism from \mathbf{G} to $\mathbf{G/N}$.
- (3) \Rightarrow (4): For $g \in G$ and $v \in h^{-1}(0)$, we must show that $gvg^{-1} \in h^{-1}(0)$. Indeed, $h(gvg^{-1}) = h(g)h(v)h(g)^{-1} = h(g)0h(g)^{-1} = 0$.
- $(4) \Rightarrow (1)$: assume that $gNg^{-1} \subseteq N$ for all $g \in G$. Let $a \in G$ be arbitrary. Applying the assumption for g = a we find that $aN \subseteq Na$. Applying the assumption for $g = a^{-1}$ we find that $a^{-1}N(a^{-1})^{-1} = a^{-1}Na \subseteq N$, and hence $Na \subseteq aN$. We conclude that aN = Na.

When G is an automorphism group, then *closed normal subgroups* of G typically arise as the subgroups consisting of those elements of G that fix the equivalence classes of a congruence relation on the elements of G. This can be made precise as follows¹.

PROPOSITION 7.2.7. Let G be the automorphism group of a relational structure \mathfrak{B} with domain B. If E is a G-invariant equivalence relation on B^n , for some n, then the subgroup of G that preserves each equivalence class of E is closed and normal.

¹I thank Todor Tsankov for pointing this out to me.

Conversely, every closed normal subgroup of G is the intersection of closed normal subgroups that arise in this way.

PROOF. Let $\mathfrak C$ be the expansion of $\mathfrak B$ by a unary relation for each equivalence class of E. Then $\operatorname{Aut}(\mathfrak C)$ is closed by Proposition 3.3.2, and it is a normal subgroup of $\operatorname{Aut}(\mathfrak B)$: when $g \in \operatorname{Aut}(\mathfrak B)$ and $h \in \operatorname{Aut}(\mathfrak C)$, then $g \circ h \circ g^{-1}$ preserves each equivalence class of E, and thus is an automorphism of $\mathfrak C$. Normality of $\operatorname{Aut}(\mathfrak C)$ follows from Proposition 7.2.6.

For the second part, suppose that ${\bf G}$ has a closed normal subgroup ${\bf N}.$ Consider the relation

$$R_n := \{(x,y) \mid x,y \in B^n \text{ and there is } h \in N \text{ such that } h(x) = y\}$$
.

This relation is obviously an equivalence relation, and it is preserved by all the automorphisms of \mathfrak{B} . For this, we have to show that for all $g \in G$ and all $(x,y) \in R_n$ we have that $(g(x), g(y)) \in R_n$. So suppose that $x, y \in B^n$ such that h(x) = y for some $h \in N$. Then $g(y) = g(h(x)) \in (gN)(x) = (Ng)(x) = N(g(x))$ by normality of \mathbb{N} . Hence there exists an $h' \in N$ such that h'(g(x)) = g(y), which shows that $(g(x), g(y)) \in R_n$.

Let \mathfrak{C} be the structure that contains for all n the n-ary relations given by the equivalence classes of the relations R_n for all $n \geq 0$. We claim that \mathbf{N} is precisely the automorphism group of \mathfrak{C} . As in the first part we can verify that every $h \in N$ is an automorphism of \mathfrak{C} . The converse follows by local closure as follows. Let g be an automorphism of \mathfrak{C} , and let x, y be from B^n so that g(x) = y. Since g preserves the equivalence classes of R_n , there exists an $h \in N$ such that h(x) = y. Hence, g lies in the closure of \mathbf{N} , which implies that g is from \mathbf{N} since \mathbf{N} is closed.

EXAMPLE 7.2.8. The automorphism group \mathbf{G} of the structure $\mathfrak{B} = (\mathbb{Q}; \text{Betw})$, where $\text{Betw} = \{(x, y, z) \mid (x < y < z) \lor (z < y < x)\}$, is 2-transitive and therefore primitive. However, the relation $\{((x_1, x_2), (y_1, y_2)) \mid (x_1 < x_2 \land y_1 < y_2) \lor (x_1 > x_2 \land y_1 > y_2) \lor (x_1 = x_2 \land y_1 = y_2)\}$ is a \mathbf{G} -invariant equivalence relation on \mathbb{Q}^2 . And indeed, \mathbf{G} has a closed normal subgroup \mathbf{N} that is isomorphic to the automorphism group of $(\mathbb{Q}; <)$, and \mathbf{G}/\mathbf{N} has two elements, corresponding to the automorphisms that reverse the order <, and the automorphisms that preserve the order.

7.3. Oligomorphic Groups

In the last section we have seen conditions that describe when a topological group is the automorphism group of a countable structure. In this section, we see conditions that describe when a topological group is the automorphism group of a countable ω -categorical structure. As a permutation group, we have seen that these groups are precisely the closed oligomorphic permutation groups (as we have seen in Section 3.3); we therefore call a topological group oligomorphic if it is isomorphic to an oligomorphic subgroup of Sym(N). In fact, Theorem 7.3.1 below shows that the information whether a topological group \mathbf{G} is oligomorphic can be expressed quite naturally in terms of the open subgroups of \mathbf{G} without referring to any particular oligomorphic action of \mathbf{G} .

A topological group G is called *Roelcke precompact* if for every open set $U \subseteq G$ that contains the identity there exists a finite set $F \subseteq G$ such that G = UFU. The following theorem is essentially from Tsankov [195]; there, the focus has been a characterization of Roelcke precompact groups in terms of oligomorphic groups. Here, on the other hand, the focus will be the characterization of oligomorphic groups in

terms of Roelcke precompact ones, and this motivates the following formulation of Tsankov's theorem².

THEOREM 7.3.1 (of Tsankov [195]). Let G be isomorphic to a closed subgroup of $Sym(\mathbb{N})$. Then the following are equivalent.

- (1) **G** is the automorphism group of a countably infinite ω -categorical structure.
- (2) **G** is Roelcke precompact, and **G** has an open subgroup V of countably infinite index such that for all open subgroups U of **G** there are $g_1, \ldots, g_n \in G$ such that $\bigcap_{i \le n} g_i V g_i^{-1} \subseteq U$.
- (3) For every open subgroup U of \mathbf{G} the set $\{UfU \mid f \in G\}$ is finite, and \mathbf{G} has an open subgroup V of countably infinite index such that for all open subgroups U of \mathbf{G} there are $g_1, \ldots, g_n \in G$ such that $\bigcap_{i < n} g_i V g_i^{-1} \subseteq U$.
- (4) **G** has a topologically faithful transitive action on a countably infinite set with the discrete topology, and every such action of **G** is oligomorphic.
- (5) **G** is the automorphism group of a countably infinite ω -categorical structure with only one orbit.

PROOF. The implication from (5) to (1) is trivial, and we prove (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5). For the implication from (1) to (2), suppose that \mathbf{G} is the automorphism group of an ω -categorical structure \mathfrak{B} , and let G be the domain of \mathbf{G} , which is a set of permutations of the domain B of \mathfrak{B} . Since \mathfrak{B} is ω -categorical, it has a finite number k of orbits by Theorem 3.1.4; choose orbit representatives $b_1, \ldots, b_k \in B$, and write \bar{b} for (b_1, \ldots, b_k) . Then the stabilizer $V := G_{\bar{b}}$ is an open subgroup of \mathbf{G} of countably infinite index. Let U be an arbitrary open subgroup of \mathbf{G} . Then U contains $G_{\bar{a}}$ for some $\bar{a} \in B^n$. For $j \leq n$, let $g_j \in G$ be such that $g_j(a_j) = b$ where $b \in \{b_1, \ldots, b_k\}$ is from the same orbit as a_j . We claim that $K := \bigcap_{j \leq n} g_j^{-1} V g_j \subseteq U$. To see this, let $h \in K$ be arbitrary. Since $h \in g_j^{-1} V g_j$ we find that $h(a_j) = a_j$. Hence, $h \in G_{\bar{a}} \subseteq U$.

To show that \mathbf{G} is Roelcke precompact, let $U \subseteq G$ be open with $1 \in U$. Then there exists an n such that U contains the stabilizer $G_{\bar{a}}$ for an n-tuple \bar{a} of elements of B. It suffices to show the existence of a finite number of elements g_1, \ldots, g_k of G such that $G = \bigcup_{i \leq k} G_{\bar{a}} g_i G_{\bar{a}}$. By Theorem 3.1.4, G has finitely many orbits of 2n-tuples; so let $(\bar{a}, g_1 \cdot \bar{a}), \ldots, (\bar{a}, g_k \cdot \bar{a})$ be a complete list of representatives for those orbits of 2n-tuples that are contained in $G \cdot \bar{a} \times G \cdot \bar{a}$. We claim that $G_{\bar{a}} g_1 G_{\bar{a}} \cup \cdots \cup G_{\bar{a}} g_k G_{\bar{a}} = G$. Let $f \in G$ be arbitrary. Let $i \leq k$ be such that $(\bar{a}, f \cdot \bar{a})$ and $(\bar{a}, g_i \cdot \bar{a})$ lie in the same orbit of n-tuples in $G_{\bar{a}}$. So there exists an $h \in G_{\bar{a}}$ such that $f \cdot \bar{a} = hg_i \cdot \bar{a}$. Then $f^{-1} \circ h \circ g_i$ lies in $G_{\bar{a}}$, so $f \in G_{\bar{a}} g_i G_{\bar{a}}$ as required.

For the implication (2) implies (3), let U be an open subgroup of G. Since G is Roelcke precompact there exists a finite set $F \subseteq G$ such that G = UFU. Then |F| bounds the sets of the form $\{UfU \mid f \in G\}$ because those sets partition G.

(3) implies (4). Since V is open, \mathbf{G}/\mathbf{V} has the discrete topology, and the action of \mathbf{G} on the countably infinite set \mathbf{G}/\mathbf{V} by left translation is continuous and transitive. We show that this action, as a map ξ from \mathbf{G} to $\mathrm{Sym}(\mathbf{G}/\mathbf{V})$, is open. Let $U \subset G$ be open. By assumption, there are $g_1, \ldots, g_n \in G$ such that $K := \bigcap_{i \leq n} g_i V g_i^{-1} \subseteq U$. Note that every $h \in K$ fixes $g_i V$ for all $i \leq n$. Hence, $\xi(U)$ contains the stabilizer of finitely many elements, and hence is open. It also follows that the action is faithful: to see this, let $f, g \in G$ be distinct. We have to show that $\phi(fg^{-1}) \neq 1$. Since $fg^{-1} \neq 1$ there is an open subgroup U that contains 1 but not fg^{-1} . Since $\xi(U)$ is open, there are finitely many h_1, \ldots, h_m such that

$$U' := \{ h \in U \mid hh_1V = h_1V \land \cdots \land hh_mV = h_mV \} \subseteq U.$$

²I am grateful to Todor Tsankov for his help with the presented reformulation of his result.

Then U' still contains 1 and not fg^{-1} , so assume in the following that U' = U. Since the sets of the form gVg^{-1} are precisely the point stabilizers of \mathbf{G}/\mathbf{V} , we have that the kernel $\phi^{-1}(1)$ of ξ can be expressed as $\phi^{-1} = \bigcap_{g \in G} gVg^{-1}$. Since $fg^{-1} \notin U = \bigcap_{g \in \{h_1, \dots, h_m\}} hVh^{-1}$, it follows in particular that $fg^{-1} \notin \phi^{-1}(1)$, which is what we wanted to show.

Now ξ is a homeomorphism between the Polish group **G** and its image $\mathbf{H} := \xi(U)$ and hence **H** is Polish as well. We can apply Proposition 7.2.3 to the subgroup **H** of the Polish group $\operatorname{Sym}(\mathbf{G}/\mathbf{V})$, and conclude that H is closed in $\operatorname{Sym}(\mathbf{G}/\mathbf{V})$. Hence, the action is topologically faithful.

We show by induction on n that this action has only finitely many orbits of n-tuples, for all n. Since the action is transitive, this is true for n=1. For the induction step, fix $\bar{a}=(a_1,\ldots,a_n)\in D^n$, and let c be an arbitrary element from $D\setminus\{a_1,\ldots,a_n\}$. By Roelcke precompactness of \mathbf{G} , there exists a finite set $\{f_1,\ldots,f_k\}\subseteq G$ such that $G=G_{\bar{a}c}f_1G_{\bar{a}c}\cup\cdots\cup G_{\bar{a}c}f_kG_{\bar{a}c}$. Let $B(\bar{a})$ be $\{f_1\cdot c,\ldots,f_k\cdot c\}$.

Claim 1. For every $d \in D \setminus \{a_1, \ldots, a_n\}$ there is an $h \in G_{\bar{a}}$ and $b \in B(\bar{a})$ such that $d = h \cdot b$. By transitivity of G, there is a $g \in G$ so that $d = g \cdot c$, for arbitrary $d \in D \setminus \{a_1, \ldots, a_n\}$. Let i, h_1, h_2 be such that $h_1, h_2 \in G_{\bar{a}c}$ and $g = h_1 f_i h_2$. Then $d = gc = h_1 f_i h_2 \cdot c = h_1 f_i \cdot c$, proving Claim 1.

Claim 2. When $\{\bar{a}_1, \dots, \bar{a}_s\}$ is a complete set of representatives for the orbits of n-tuples of the permutation group G, then

$$\{(\bar{a}_i, b) \mid i \le s, b \in B(\bar{a}_i)\}$$

is a complete set of representatives for the orbits of (n+1)-tuples. Let $(\bar{c},d) \in D^{n+1}$. By assumption there exists $g \in G$ such that $g \cdot \bar{a}_i = \bar{c}$. Find $h \in G_{\bar{a}_i}$ and $b \in B(\bar{a}_i)$ such that $g^{-1} \cdot d = h \cdot b$. Then one has

$$gh \cdot (\bar{a}_i, b) = g \cdot (\bar{a}_i, h \cdot b) = (\bar{c}, d)$$
.

This shows that G has finitely many orbits of (n + 1)-tuples, and concludes the induction step.

The implication from (4) to (5) follows from Corollary 3.3.9: let D be the countably infinite set on which \mathbf{G} acts continuously, transitively, and topologically faithfully. Then the set of permutations of D induced by this action is a closed oligomorphic permutation group, and hence the automorphism group of an ω -categorical relational structure with domain D. Since the action is transitive, the structure has only one orbit.

Note that the groups from Theorem 7.3.1 must always have continuum cardinality; this follows from the following and the remarks after Lemma 3.1.10.

Theorem 7.3.2 (Corollary 4.1.5 in [120]). Let \mathfrak{B} be a countable structure. Then the following are equivalent.

- (1) $|\operatorname{Aut}(\mathfrak{B})| \leq \omega$
- (2) $|\operatorname{Aut}(\mathfrak{B})| < 2^{\omega}$
- (3) There is a finite tuple \bar{a} in \mathfrak{B} such that $|\operatorname{Aut}((\mathfrak{B},\bar{a}))| = 1$

7.4. Bi-interpretations

When two ω -categorical structures share the same topological automorphism group, then the relationship between the two structures can be described model-theoretically.

THEOREM 7.4.1 (Ahlbrandt and Ziegler [3]). Two ω -categorical structures \mathfrak{B} and \mathfrak{C} are bi-interpretable if and only if $\operatorname{Aut}(\mathfrak{B})$ and $\operatorname{Aut}(\mathfrak{C})$ are isomorphic as topological groups.

The subgroup of G consisting of the identity element only is called *trivial*, and subgroups of G that are distinct from G are called *proper*.

EXAMPLE 7.4.2. The structures $\mathfrak{C} := (\mathbb{N}^2; \{(x,y), (u,v) \mid x=u\})$ and $\mathfrak{D} := (\mathbb{N}; =)$ are mutually primitive positive interpretable, but *not* bi-interpretable. To see this, observe that $\operatorname{Aut}(\mathfrak{C})$ has a proper non-trivial closed normal subgroup \mathbf{N} such that $\operatorname{Aut}(\mathfrak{C})/\mathbf{N}$ is isomorphic to $\operatorname{Aut}(\mathfrak{D})$ (see Proposition 7.2.7), whereas $\operatorname{Aut}(\mathfrak{D})$, the symmetric permutation group of a countably infinite set, has no proper non-trivial closed normal subgroups (it has exactly three proper non-trivial normal subgroups [184], none of which is closed).

Theorem 7.4.1 has many consequences. For instance, it shows in combination with Theorem 7.3.1 that every ω -categorical structure is bi-interpretable with an ω -categorical structure that has only one orbit. Ahlbrandt and Ziegler also showed the following.

THEOREM 7.4.3 (From [3]; also see Theorem 5.3.5 and 7.3.7 in [119]). Let \mathfrak{C} be an ω -categorical structure with at least two elements. Then a structure \mathfrak{B} has a first-order interpretation in \mathfrak{C} if and only if \mathfrak{B} is the reduct of a structure \mathfrak{B}' such that there is a surjective continuous group homomorphism $f \colon \operatorname{Aut}(\mathfrak{C}) \to \operatorname{Aut}(\mathfrak{B}')$.

Several fundamental properties of ω -categorical structures \mathfrak{B} are preserved by bi-interpretability, and therefore, by Theorem 7.4.1, only depend on the topological automorphism group of \mathfrak{B} . As we will see in Chapter 8, this is for instance the case for the property whether an ordered homogeneous structure has the Ramsey property. We give another property of this type.

DEFINITION 7.4.4. Let \mathfrak{B} be an ω -categorical structure. Then \mathfrak{B} has essentially infinite signature if every relational structure \mathfrak{C} that is interdefinable with \mathfrak{B} (equivalently, has the same set of automorphisms as \mathfrak{B} , by Proposition 3.3.8) has an infinite signature.

We show that the property to have essentially infinite language is preserved by bi-interpretability.

PROPOSITION 7.4.5. Let \mathfrak{B} and \mathfrak{C} be ω -categorical structures that are first-order bi-interpretable. Then \mathfrak{B} has essentially infinite signature if and only if \mathfrak{C} has.

PROOF. Let τ be the signature of \mathfrak{B} . We have to show that if \mathfrak{C} has finite signature, then \mathfrak{B} is interdefinable with a structure \mathfrak{B}' with a finite signature. Let $\sigma \subseteq \tau$ be the set of all relation symbols that appear in all the formulas of the interpretation of \mathfrak{C} in \mathfrak{B} . Since the signature of \mathfrak{C} is finite, the cardinality of σ is finite as well.

We will show that there is a first-order definition of \mathfrak{B} in the σ -reduct \mathfrak{B}' of \mathfrak{B} . Suppose that the interpretation I_1 of \mathfrak{C} in \mathfrak{B} is d_1 -dimensional, and that the interpretation I_2 of \mathfrak{B} in \mathfrak{C} is d_2 -dimensional. Let $\theta(x, y_{1,1}, \ldots, y_{d_1, d_2})$ be the formula that shows that $I_2 \circ I_1$ is homotopic to the identity interpretation. That is, θ defines in \mathfrak{B} the (d_1d_2+1) -ary relation that contains a tuple $(a, b_{1,1}, \ldots, b_{d_1, d_2})$ iff

$$a = h_2(h_1(b_{1,1}, \ldots, b_{1,d_2}), \ldots, h_1(b_{d_1,1}, \ldots, b_{d_1,d_2}))$$
.

Let ϕ be an atomic τ -formula with k free variables x_1, \ldots, x_k . We will specify a σ -formula that is equivalent to ϕ over \mathfrak{B}' .

$$\exists y_{1,1}^1, \dots, y_{d_1,d_2}^k \left(\bigwedge_{i \le k} \theta(x_i, y_{1,1}^i, \dots, y_{d_1,d_2}^i) \right.$$

$$\land \phi_{I_1 I_2}(y_{1,1}^1, \dots, y_{1,d_2}^k, y_{2,d_2}^1, \dots, y_{2,d_2}^k, \dots, y_{d_1,d_2}^k) \right)$$

is equivalent to $\phi(x_1,\ldots,x_k)$ over \mathfrak{B} . Indeed, by surjectivity of h_2 , for every element a_i of \mathfrak{B} there are elements $c_1^i,\ldots,c_{d_2}^i$ of \mathfrak{C} such that $h_2(c_1^i,\ldots,c_{d_2}^i)=a_i$, and by surjectivity of h_1 , for every element c_j^i of \mathfrak{C} there are elements $b_{1,j}^i,\ldots,b_{d_1,j}^i$ of \mathfrak{B} such that $h_1(b_{1,j}^i,\ldots,b_{d_1,j}^i)=c_j^i$. Then

$$\mathfrak{B} \models R(a_1, \dots, a_k) \iff \mathfrak{C} \models \phi_{I_2}(c_1^1, \dots, c_{d_2}^1, \dots, c_1^k, \dots, c_{d_2}^k)$$

$$\Leftrightarrow \mathfrak{B}' \models \phi_{I_1 I_2}(b_{1,1}^1, \dots, b_{1,d_2}^k, b_{2,d_2}^1, \dots, b_{2,d_2}^k, \dots, b_{d_1,d_2}^k)$$

Note that if $\mathfrak B$ and $\mathfrak C$ are ω -categorical structures that are even primitive positive bi-interpretable, then the above proof even shows that $\mathfrak B$ is primitively positive interdefinable with a structure with a finite domain if and only if $\mathfrak C$ is.

Proposition 7.4.5 shows via Theorem 7.4.1 that having essentially infinite signature only depends on the topological automorphism group of \mathfrak{B} . But unlike the property of ω -categoricity and Theorem 7.3.1, we are not aware of any elegant characterization of those properties that is directly stated in terms of the topological group.

CHAPTER 8

Ramsey Theory



À LA CITÉ DE L'ARCHITECTURE ET DU PATRIHOINE IL YA UNE EXPO SUR UN ARCHI-TECTE QUI A DESK'NÉ DES PLANS QUI N'ONT DAMAIS ÉTÉ CONSTRUIT.

ON PENSE AUX ALGORITHMES QUI SONT DÉCRITS DANS DES ARTICLES, MAIS JAHAIS IMPLÉMENTÉS.

The application of Ramsey theory to study the expressive power of constraint languages via polymorphisms is one of the central contributions of this thesis. The idea is that polymorphisms must behave in a regular way on large parts of their domain. This also leads us to decidability results for several meta-questions about the expressive power of constraint languages. The same idea can be used to show statements of the type 'every polymorphism that violates R must locally generate g', for certain fixed operations g with good properties. Such statements will be crucial in the classification projects in Chapters 9 and 10.

In this chapter we first revisit classical concepts and results from structural Ramsey theory in Section 8.1. In order to apply Ramsey theory to analyze polymorphism clones, we need the product Ramsey theorem, but also other fundamental facts from Ramsey theory, some of which appear to be new (such as Corollary 8.2.13). Those facts will be derived from a recently discovered fundamental connection between Ramsey theory and topological dynamics due to Kechris, Pestov, and Todorcevic [132]. This connection allows a more systematic understanding of Ramsey-theoretic principles, and we present it in Section 8.2. The way in which we apply Ramsey theory to polymorphisms is described in Section 8.3. We close with an application of our technique in Section 8.4, and prove the decidability of various meta-problems concerning constraint satisfaction problems, that is, problems where the input is a description of a template \mathfrak{B} , and the question is whether the corresponding CSP has certain properties (for instance, whether certain relations are primitive positive definable).

Some of the results presented here have been published in [54]; there is also a survey article [50] that additionally covers the applications of our technique for the classification of 'the first-order reducts' of a given homogeneous structure \mathfrak{C} , that is, the structures that are first-order definable in \mathfrak{C} .

8.1. Ramsey Classes

This section is about classes \mathcal{C} of finite structures that satisfy the following Ramsey-type property: for all $\mathfrak{A},\mathfrak{B}\in\mathcal{C}$ there exists a $\mathfrak{C}\in\mathcal{C}$ such that \mathfrak{B} embeds into \mathfrak{C} , and when we assign finitely many 'colors' to the substructures of \mathfrak{C} that are isomorphic to \mathfrak{A} , then we can find a 'monochromatic' copy of \mathfrak{B} in \mathfrak{C} , i.e., an induced substructure of \mathfrak{C} that is isomorphic to \mathfrak{B} and in which all copies of \mathfrak{A} in this substructure have the same color. Before we formalize this in detail, we give the classical result of Ramsey, which provides a prototype of a class with the Ramsey property.

From now on, we denote the set $\{1, \ldots, n\}$ also by [n]. Subsets of a set of cardinality m will be called m-subsets in the following. Let $\binom{S}{m}$ denote the set of all m-subsets of S. We also refer to mappings $f:\binom{S}{m} \to [r]$ as a coloring of S (with the colors [r]).

Theorem 8.1.1 (Ramsey's theorem). Let B be a countably infinite set, and let m,r be finite integers. For every $\chi\colon\binom{B}{m}\to[r]$ there exists an infinite $P\subseteq B$ such that χ is constant on all m-element subsets of P.

A proof of Theorem 8.1.1 can be found in [120] (Theorem 5.6.1); for a broader introduction to Ramsey theory see [105]. It is easy to derive the following finite version of Ramsey's theorem from Theorem 8.1.1 via a compactness argument.

THEOREM 8.1.2 (Finite version of Ramsey's theorem). For all positive integers r, m, k there is a positive integer l such that for every $\chi: \binom{[l]}{m} \to [r]$ there exists a k-subset S of [l] such that χ is constant on $\binom{S}{m}$.

PROOF. A proof by contradiction: suppose that there are positive integers r, m, k such that for all positive integers l there is a $\chi \colon \binom{[l]}{m} \to [r]$ such that for all k-subsets S of [l] the mapping χ is not constant on $\binom{S}{m}$. Since the property that for all k-subsets S of [l] the mapping χ is not constant on $\binom{S}{m}$ is universal first-order, and since the image of χ is finite, we can apply Lemma 3.1.8 and get the existence of a mapping χ with the same property but defined on all integers. This contradicts Theorem 8.1.1. \square

We write $\mathbf{R}(r, m, k)$ for the *smallest* number l whose existence is asserted by Theorem 8.1.2.

More generally, when $\mathfrak A$ and $\mathfrak B$ are τ -structures, we write $\mathfrak A$ for the set of all induced substructures of $\mathfrak B$ that are isomorphic to $\mathfrak A$. When $\mathfrak A, \mathfrak B, \mathfrak C$ are τ -structures, and $r \geq 1$ is finite, we write

$$\mathfrak{C} o (\mathfrak{B})^{\mathfrak{A}}_r$$

if for all $\chi: \binom{\mathfrak{C}}{\mathfrak{A}} \to [r]$ there exists $\mathfrak{B}' \in \binom{\mathfrak{C}}{\mathfrak{B}}$ such that χ is constant on $\binom{\mathfrak{B}'}{\mathfrak{A}}$.

DEFINITION 8.1.3. A class of relational structures that is closed under isomorphisms and induced substructures is called Ramsey if for every $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ and for every finite $k \geq 1$ there exists a $\mathfrak{C} \in \mathcal{C}$ such that \mathfrak{B} embeds into \mathfrak{C} , and $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_k$.

Our first example of a Ramsey class is the class of all finite structures over the empty signature; this is an immediate consequence of Theorem 8.1.2. We also observe the following. Recall that the age of a τ -structure \mathfrak{B} is the class of all finite τ -structures that embed into \mathfrak{B} .

COROLLARY 8.1.4. The age of $(\mathbb{Q}; <)$ is Ramsey.

PROOF. This is again a direct consequence of Theorem 8.1.2, since whether or not an m-element substructure is isomorphic to an n-element substructure of $(\mathbb{Q}; <)$ only depends on n and m.

We will now present further examples of Ramsey classes; the proofs are non-trivial and fall out of the scope of this thesis, but we provide references.

EXAMPLE 8.1.5. The class of all finite Boolean algebras $\mathfrak{B} = (B; \sqcup, \sqcap, c, \mathbf{0}, \mathbf{1})$ has the Ramsey property. This is explicitly stated in [132], page 147, line 3ff (see also page 112, line 9ff), where it is observed that this follows from a result by Graham and Rothschild [104].

It might be instructive to also see an example of a class of structures that is *not* Ramsey. Typical examples come from classes that contain structures with non-trivial automorphism groups, as in the following.

EXAMPLE 8.1.6. The class of all finite graphs is not a Ramsey class. To see this, let \mathfrak{A} be the (undirected) graph ($\{0,1,2\};\{\{0,1\},\{1,2\}\}$) (since the edge relation is symmetric we write edges as 2-element subsets of the vertices) with three vertices and two edges, and let \mathfrak{B} be C_4 , that is, the undirected four-cycle

$$(\{0,1,2,3\};\{\{0,1\},\{1,2\},\{2,3\},\{3,0\}\})$$
.

Let $\mathfrak C$ be an arbitrary graph. We want to show that there is a way to color the copies of $\mathfrak A$ in $\mathfrak C$ without producing a monochromatic copy of $\mathfrak B$. For that, fix an arbitrary linear order < on the vertices of $\mathfrak C$. We define a coloring $\chi \colon \binom{\mathfrak C}{\mathfrak A} \to \{0,1\}$ as follows. If there is an embedding h of $\mathfrak A$ into $\mathfrak C$ such that h(0) < h(1) < h(2), then we color the corresponding copy of $\mathfrak A$ in $\mathfrak C$ with 0; all other copies of $\mathfrak A$ in $\mathfrak C$ are colored by 1. We claim that any copy of $\mathfrak B$ in $\mathfrak C$ contains a copy of $\mathfrak A$ that is colored by 1, and one that is colored by 0. The reason is that for any ordering of the vertices of $\mathfrak B$ there is an embedding h' of $\mathfrak A$ into $\mathfrak B$ such that h'(0) < h'(1) < h'(2), and an embedding h'' of $\mathfrak A$ into $\mathfrak B$ such that not h''(0) < h''(1) < h''(2). Hence, $\mathfrak C \not \to (\mathfrak B)^{\mathfrak A}_{\mathfrak A}$.

Frequently, a class without the Ramsey property can be made Ramsey by expanding its members appropriately with a linear ordering. We will see several examples.

EXAMPLE 8.1.7. Nešetřil and Rödl [168] and independently Abramson and Harrington [1] showed that for any relational signature τ , the class \mathcal{C} of all finite ordered τ -structures is a Ramsey class. That is, the members of \mathcal{C} are finite structures $\mathfrak{A} = (A; <, R_1, R_2, \ldots)$ for some fixed signature $\tau = \{<, R_1, R_2, \ldots\}$, where < is a total linear order of A.

A shorter and simpler proof of this substantial result can be found in [169] and [166]; the proof there uses the *partite method*, which uses amalgamation to reduce the statement to proving the so-called *partite lemma*; the proof of the partite lemma relies on the Hales-Jewett theorem from Ramsey theory (see [105]).

Example 8.1.8. Recall the homogeneous structure $\mathfrak{B}=(B;|)$ carrying a C-relation, introduced in Section 4.1. We consider the expansion of \mathfrak{B} by a linear order, defined as follows. It is easy to see that for every finite tree \mathfrak{T} there is an ordering < on the leaves L of \mathfrak{T} such that for all $u, v, w \in L$ with u < v < w we have either u|vw or uv|w (recall Definition 4.1.3). This can be seen from the obvious existence of an embedding of \mathfrak{T} on the plane so that all leaves lie on a line and none of the edges cross, and take the linear order induced by the line. We call such an ordering of L compatible with \mathfrak{T} . The class of all finite substructures \mathfrak{C} of \mathfrak{B} expanded by a compatible ordering of the underlying tree of \mathfrak{C} is a Ramsey class; this follows from more general results by Milliken (Theorem 4.3 in [163], building on work in [83]), and has been shown explicitly in [49].

EXAMPLE 8.1.9. The Ramsey classes from Example 8.1.7 have been further generalized by Nešetřil and Rödl as follows [168]. Suppose that \mathcal{N} is a (not necessarily

finite) class of structures with finite relational signature τ whose Gaifman graph (Definition 2.1.2) is a clique – such structures have been called *irreducible* in the Ramsey theory literature. It can be readily verified that $\mathcal{C} := \operatorname{Forb}(\mathcal{N})$ is an amalgamation class. Then the class of all expansions of the structures in \mathcal{C} by a linear order is a Ramsey class; again, this can been shown by the partite method [169]. This example is indeed a generalization of Example 8.1.7 since we obtain the previous result by taking $\mathcal{N} = \emptyset$.

The fact that all the above Ramsey classes could be described as the age of a homogeneous structure is not a coincidence. We have the following (the proof is from [167], and presented here for the convenience of the reader).

THEOREM 8.1.10 (of [167]). Let τ be a relational signature, and let \mathcal{C} be a class of ordered finite τ -structures that is closed under induced substructures, isomorphism, and has the joint embedding property (see Section 3.2). If \mathcal{C} is Ramsey, then it has the amalgamation property.

PROOF. Let $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2$ be members of \mathcal{C} such that \mathfrak{A} is an induced substructure of both \mathfrak{B}_1 and \mathfrak{B}_2 . Since \mathcal{C} has the joint embedding property, there exists a structure $\mathfrak{C} \in \mathcal{C}$ with embeddings e_1, e_2 of \mathfrak{B}_1 and \mathfrak{B}_2 into \mathfrak{C} . If e_1, e_2 have the same restriction to \mathfrak{A} , then we are done, so assume otherwise.

Let $\mathfrak{D} \in \mathcal{C}$ be such that $\mathfrak{D} \to (\mathfrak{C})_2^{\mathfrak{A}}$. Define a coloring $\chi : \binom{\mathfrak{D}}{\mathfrak{A}} \to \{1,2\}$ as follows. When $\mathfrak{A}' \in \binom{\mathfrak{D}}{\mathfrak{A}}$, and $f : \mathfrak{A} \to \mathfrak{A}'$ is an isomorphism, then $\chi(\mathfrak{A}') = 1$ if and only if there is an embedding h of \mathfrak{C} into \mathfrak{D} such that $f = h \circ e_1$.

Since $\mathfrak{D} \to (\mathfrak{C})_2^{\mathfrak{A}}$, there exists $\mathfrak{C}' \in {\mathfrak{D} \choose \mathfrak{C}}$, witnessed by an embedding h' of \mathfrak{C} into \mathfrak{D} such that χ is constant on ${\mathfrak{C} \choose \mathfrak{A}}$. Now any copy of \mathfrak{C} in \mathfrak{D} contains a copy \mathfrak{A}' of \mathfrak{A} with $\chi(\mathfrak{A}') = 1$. Thus χ is constant 1 on ${\mathfrak{C} \choose \mathfrak{A}}$.

Consider the embedding $h' \circ e_2$ of $\mathfrak A$ into $\mathfrak D$; as we have seen above, the corresponding copy of $\mathfrak A$ in $\mathfrak D$ is colored 1. Thus there exists an embedding h'' of $\mathfrak C$ into $\mathfrak D$ such that $f = h'' \circ e_1 = h' \circ e_2$ (here we use the assumption that the structure $\mathfrak A$ is ordered). This shows that $\mathfrak D$ together with the embeddings $h'' \circ e_1 \colon B_1 \to D$ and $h' \circ e_2 \colon B_2 \to D$ is the amalgam of $\mathfrak B_1$ and $\mathfrak B_2$ over $\mathfrak A$.

It is often convenient to work with the Fraïssé-limit of a Ramsey class rather than the class itself. Indeed, we have the following.

PROPOSITION 8.1.11. Let C be an amalgamation class, and let \mathfrak{C} be the Fraïssélimit of C. Then C has the Ramsey property if and only if $\mathfrak{C} \to (\mathfrak{B})_k^{\mathfrak{A}}$ for all $\mathfrak{A}, \mathfrak{B} \in C$, and $k \geq 2$.

PROOF. Let $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$, and $k \geq 2$ an integer. When k is the cardinality of $\binom{\mathfrak{B}}{\mathfrak{A}}$, then for any structure \mathfrak{D} the fact that $\mathfrak{D} \to (\mathfrak{B})_r^{\mathfrak{A}}$ can equivalently be expressed in terms of r-colorability of a certain k-uniform hypergraph, defined as follows. Let $\mathfrak{G} = (V; E)$ be the structure whose vertex set V is $\binom{\mathfrak{D}}{\mathfrak{A}}$, and where $(\mathfrak{A}_1, \ldots, \mathfrak{A}_k) \in E$ if there exists a $\mathfrak{B}' \in \binom{\mathfrak{D}}{\mathfrak{B}}$ such that $\binom{\mathfrak{B}'}{\mathfrak{A}} = \{\mathfrak{A}_1, \ldots, \mathfrak{A}_k\}$. Let $\mathfrak{H} = ([r]; E)$ be the structure where E contains all tuples except the tuples $(1, \ldots, 1), \ldots, (r, \ldots, r)$. Then $\mathfrak{D} \to (\mathfrak{B})_r^{\mathfrak{A}}$ if and only if \mathfrak{G} homomorphically maps to \mathfrak{H} . By Lemma 3.1.5, this is the case if and only if all finite substructures of \mathfrak{G} homomorphically map to \mathfrak{H} . Thus, $\mathfrak{C} \to (\mathfrak{B})_r^{\mathfrak{A}}$ if and only if $\mathfrak{C}' \to (\mathfrak{B})_r^{\mathfrak{A}}$ for all finite substructure \mathfrak{C}' of \mathfrak{C} .

When \mathfrak{B} is a homogeneous structure with a finite relational signature whose age is a Ramsey class, then this fact is useful for studying which relations are primitive positive definable in \mathfrak{B} , as we will see for instance in Section 8.4. In fact, for those applications of Ramsey theory it suffices that \mathfrak{B} can be expanded to a structure \mathfrak{C}

that is homogeneous, Ramsey, and has a finite relational signature. The following question has been asked.

QUESTION 8.1. Let \mathfrak{B} be a homogeneous structure with a finite relational signature. Then there is a homogeneous expansion \mathfrak{C} of \mathfrak{B} with finite relational signature whose age has the Ramsey property.

8.2. Extremely Amenable Groups

This section presents a link between Ramsey classes and Polish groups that are extremely amenable. The link rests on the theorem of Kechris, Pestov, and Todorcevic that characterizes those ordered homogeneous structures that are Ramsey in terms of their topological automorphism group, and will be presented in Section 8.2.1. In Section 8.2.2, 3.3.4, and 8.2.4, we use this result to obtain a more systematic understanding of Ramsey classes.

8.2.1. Extreme Amenability. The Ramsey property for ordered homogeneous structures \mathfrak{B} has an elegant characterization in terms of the topological automorphism group of \mathfrak{B} : the age of \mathfrak{B} is Ramsey if and only if the topological automorphism group of \mathfrak{B} is extremely amenable. Extreme amenability is a concept from group theory studied since the 60s [106].

Definition 8.2.1. A topological group is extremely amenable iff every continuous action of the group on a compact Hausdorff space has a fixed point.

The following is the combination of Proposition 4.2, Proposition 4.3, Theorem 4.5, and Theorem 4.7 from [132].

THEOREM 8.2.2 (Kechris, Pestov, Todorcevic [132]). Let \mathfrak{B} be a homogeneous relational structure, and let G be the topological automorphism group of \mathfrak{B} . Then the following are equivalent.

- (1) The age of \mathfrak{B} has the Ramsey property, and G preserves a linear order < on B.
- (2) The age of \mathfrak{B} only contains rigid structures, and has the Ramsey property.
- (3) (a) For any finite subset F of B, the substructure induced by F in $\mathfrak B$ is rigid, and (b) for all orbits O_1, O_2 of finite subsets in $\mathfrak B$, and for every $\chi \colon O_1 \to [r]$ there is an $i \leq r$ and an $F \in O_2$ such that $\chi(F') = i$ for all $F' \subseteq F$ where $F' \in O_1$.
- (4) For any open subgroup V of G, every $\chi \colon G/V \to [r]$, and every finite $A \subseteq G/V$ there is $g \in G$ and $1 \le i \le r$ such that $\chi(ga) = i$ for all $a \in A$.
- (5) \mathbf{G} is extremely amenable.

For structures \mathfrak{B} that are not homogeneous but ω -categorical, the equivalence between (3), (4), and (5) remains valid, since every ω -categorical structure has a homogeneous expansion by first-order definable relations – and such an expansion has the same automorphism group as \mathfrak{B} ; we can then apply Theorem 8.2.2 to the expansion. It therefore makes sense to call an ω -categorical structure \mathfrak{B} Ramsey if the age of the expansion of \mathfrak{B} by all first-order definable relations is Ramsey. There are also interesting applications of Theorem 8.2.2 when \mathfrak{B} is not ω -categorical, see [132].

We point out a remarkable consequence of Theorem 8.2.2 in combination with Proposition 3.3.8.

COROLLARY 8.2.3. Suppose that \mathfrak{B} is an ω -categorical Ramsey structure where all finite induced substructures are rigid. Then a linear order is first-order definable in \mathfrak{B} .

When a Ramsey structure \mathfrak{B} has substructures with non-trivial automorphisms, Theorem 8.2.2 is therefore not directly applicable. For the applications of Ramsey theory we have in mind, though, it is sufficient to know that \mathfrak{B} has a first-order definition in an ordered Ramsey structure that is homogeneous in a finite relational signature (see Section 8.3 and 8.4).

The choice of the order is not arbitrary, but plays an important role when we want to preserve the Ramsey property. To give another example, consider again the countable atomless Boolean algebra, which is an example of an ω -categorical structure that is *not* homogeneous in a finite relational signature (Corollary 4.5.8). In this case an order expansion with an extremely amenable automorphism group has been specified in [132], and can be found below.

EXAMPLE 8.2.4. Let $\mathfrak{B} = (B; \sqcup, \sqcap, c, \mathbf{0}, \mathbf{1})$ be a finite Boolean algebra and A its set of atoms (see Example 8.1.5 in Section 4.3). Then every ordering $a_1 < \cdots < a_n$ of A gives an ordering of B as follows (we follow [132]). For $x, y \in B$, we set x < y if there exists an $i_0 \in \{1, \ldots, n\}$ such that

- for all $i \in \{1, \dots, i_0 1\}$ we have that $a_i \sqcap x = a_i \sqcap y$, and
- $x \sqcap a_{i_0} = \mathbf{0}$ and $y \cap a_{i_0} \neq \mathbf{0}$.

Such an ordering of the elements of \mathfrak{B} is called a *natural ordering*. It can be shown that the class \mathcal{C} of all naturally ordered finite atomless Boolean algebras has the Ramsey property (see the comments preceding Theorem 6.14 in [132], and Proposition 5.6 in [132]). By Theorem 8.1.10, \mathcal{C} is an amalgamation class. The reduct of the Fraïssé-limit of \mathcal{C} with signature $\{\sqcup, \sqcap, c, \mathbf{0}, \mathbf{1}\}$ is the atomless Boolean algebra (Propositions 5.2 and 6.13 in [132]), so we indeed found an extremely amenable order expansion of the atomless Boolean algebra.

The main focus of the article by Kechris, Pestov, and Todorcevic [132] is the application of Theorem 8.2.2 to prove that certain groups are extremely amenable, using known and deep Ramsey results. Here we are rather interested in the opposite direction: we are applying Theorem 8.2.2 in the following sections to obtain a more systematic understanding of which classes of structures have the Ramsey property.

8.2.2. Continuous Homomorphisms. Interestingly, whether an ω -categorical structure \mathfrak{B} is Ramsey only depends on the automorphism group of \mathfrak{B} viewed as a *topological* group. For this observation we do not need the full power of Theorem 8.2.2: the equivalence of (1) and (3) suffices. More generally, we have the following.

PROPOSITION 8.2.5. Let G be an extremely amenable group, and let H be a Polish group. If there is a continuous homomorphism $\xi \colon G \to H$ such that $\xi(G)$ is dense in H, then H is also extremely amenable.

PROOF. Let $a: H \times S \to S$ be a continuous action of \mathbf{H} on a compact Hausdorff space S. Then $b: G \times S \to S$ given by $(g,s) \mapsto a(\xi(g),s)$ is a continuous action of \mathbf{G} on S. Since \mathbf{G} is extremely amenable, b has a fixed point s_0 . Now let $h \in H$ be arbitrary. Since $\xi(G)$ is dense in H, there exists a sequence $(g_i)_{i\geq 1}$ of elements of G such that $\lim_i \xi(g_i)$ converges against h in H. Therefore there exists an i_0 such that for all $j \geq i_0$ we have that

$$a(h, s_0) = a(\xi(g_i), s_0) = b(g_i, s_0) = s_0$$

and hence s_0 is also a fixed point under a.

Example 8.2.6. Recall Example 3.1.11, where we introduced an ω -categorical structure \mathfrak{A} with binary relations and a two-dimensional first-order interpretation

over $(\mathbb{Q}; <)$ – the corresponding relation algebra is also called *Allen's Interval Algebra* (see Section 1.5.1).

It follows from the proof of Theorem 5.5.23 that $\mathfrak A$ is first-order bi-interpretable with $(\mathbb Q;<)$. Since the automorphism group of $(\mathbb Q;<)$ is extremely amenable, Theorem 8.2.2 therefore shows that the automorphism group of $\mathfrak A$ is extremely amenable, and that $\mathfrak A$ is Ramsey. Since $\mathfrak A$ is homogeneous, we also have the Ramsey property for the age of $\mathfrak A$.

Unfortunately, Theorem 8.2.2 leaves Question 8.1 unresolved. An important variant of Question 8.1 with a reformulation in terms of topological groups turned out to be false.

Theorem 8.2.7 (Evans [93]). There exists a closed oligomorphic permutation group without a closed oligomorphic extremely amenable subgroup. Equivalently, there exists an ω -categorical structure without an ordered ω -categorical Ramsey expansion.

8.2.3. Products. In this section we present an important tool to build new extremely amenable groups, Theorem 8.2.8 below. Recall the definition of direct products of two groups from Section 3.3.4. The direct product of two topological groups $\mathbf{G_1}$ and $\mathbf{G_2}$ is the direct product of the respective abstract groups, together with the product topology on the group elements.

Theorem 8.2.8 (Proposition 6.7 in [132]). Let G be a topological group.

- (1) Let \mathbf{N} a normal subgroup of \mathbf{G} . If both \mathbf{N} and \mathbf{G}/\mathbf{N} are extremely amenable, then so is \mathbf{G} .
- (2) When **G** is a finite direct product of extremely amenable groups, then **G** is extremely amenable.

Item (1) has been stated in [132] under the additional assumption that **N** is *closed*; however, as we see in the proof, this assumption is not necessary. Item (2) can be generalized to infinite products of groups, but we do not need this here, and refer to [132] instead.

PROOF. We first show (1). Suppose **G** acts continuously on a compact Hausdorff space S. Let $S_{\mathbf{N}}$ be the subspace of S induced on $\{x \in S \mid h \cdot x = x \text{ for all } h \in N\}$, which is clearly also Hausdorff. Moreover, it is closed. To see this, let $x \in S$ and $f \in N$ be such that $f \cdot x \neq x$. Then there exists an open set $V(x, f) \subseteq S$ that contains x such that $f \cdot y \neq y$ for all $y \in V(x, f)$. Otherwise, there exists a sequence $(x_n)_{n \geq 1}$ with $x_n \to x$ such that $f(x_n) = x_n$ for all $n \geq 1$. But $f \cdot x = f \cdot \lim x_n = \lim f \cdot x_n = \lim x_n = x$, a contradiction. Then $\bigcup_{f \in N, x \in S} V(f, x)$ defines the complement of $S_{\mathbf{N}}$ and is open. So $S_{\mathbf{N}}$ is closed, and also compact since S is compact and closed subsets of compact spaces are compact.

As **N** is extremely amenable, $S_{\mathbf{N}}$ is non-empty. The set $S_{\mathbf{N}}$ is preserved by the action of **G** on S: when $x \in S_{\mathbf{N}}$ and $g \in G$, then for any $h \in N$

$$h \cdot (g \cdot x) = hg \cdot x = g \cdot (g^{-1}hg) \cdot x = g \cdot x$$

(where the last equality is by normality of N and Proposition 7.2.6), and so $g \cdot x \in S_{\mathbf{N}}$. Now, consider the action of \mathbf{G}/\mathbf{N} on $S_{\mathbf{N}}$ defined by $(gN) \cdot x = g \cdot x$, which is clearly well-defined. To verify continuity of this action with Proposition 7.1.2, let $(g_nN, s_n) \to (gN, s)$. Then $(g_nN) \cdot s_n = g_n \cdot s_n \to g \cdot s = (gN)s$ since the action of \mathbf{G} on S is continuous

By extreme amenability of \mathbf{G}/\mathbf{N} there is a point $p \in S_{\mathbf{N}}$ such that $f \cdot p = p$ for all $f \in \mathbf{G}/\mathbf{N}$. But then, p is also a fixed point for the action of \mathbf{G} on S, since $g \cdot p = (gN) \cdot p = p$ for any $g \in G$.

(2) follows from (1): suppose that $\mathbf{G} = \mathbf{H}_1 \times \mathbf{H}_2$, and that \mathbf{H}_1 and \mathbf{H}_2 are extremely amenable. Then \mathbf{H}_1 is a normal subgroup of \mathbf{G} , and \mathbf{G}/\mathbf{H}_1 is isomorphic to \mathbf{H}_2 , so \mathbf{G} is extremely amenable by (1). The statement for $\mathbf{H}_1 \times \cdots \times \mathbf{H}_n$ follows by induction on n.

When we later apply Ramsey theory to analyze polymorphism clones in Section 8.3 of this chapter, and when the polymorphisms are higher-ary, it will be crucial to apply the so-called *product Ramsey theorem*. For every ordered Ramsey class there is a corresponding product Ramsey theorem (which usually has various slightly different formulations). This can be shown either directly, or by applying the general results from topological dynamics.

For illustration, we give a direct proof for the class of all finite linear orders (Theorem 8.2.9, which will be used extensively for d=m=2 in Chapter 10). For concreteness, we give specialized terminology in this case. If S_1, \ldots, S_d are sets, we call a set of the form $S_1 \times \cdots \times S_d$ a grid, and also write S^d for a product of the form $S \times \cdots \times S_d$ with d factors. A $[k]^d$ -subgrid of a grid $S_1 \times \cdots \times S_d$ is a subset of $S_1 \times \cdots \times S_d$ of the form $S'_1 \times \cdots \times S'_d$, where S'_i is a k-element subset of S_i .

THEOREM 8.2.9 (Product Ramsey Theorem). For all positive integers d, r, m, and $k \ge m$, there is a positive integer $L = \mathbf{R}(d, r, m, k)$ such that for every coloring of the $[m]^d$ subgrids of $[L]^d$ with r colors there exists a monochromatic $[k]^d$ subgrid G of $[L]^d$, i.e., G is such that all its $[m]^d$ subgrids have the same color.

PROOF. Let d, r, m, and $k \ge m$ be positive integers. We claim that we can choose $L = \mathbf{R}(d, r, m, k)$ to be $\mathbf{R}(r, dm, dk)$. To verify this, let χ be a coloring of the $[m]^d$ subgrids of $[L]^d$ with r colors. We have to find a monochromatic subgrid of $[L]^d$.

We use χ to define an r-coloring ξ of the dm-subsets of [L] as follows. Let $S = \{s_1, s_2, \ldots, s_{dm}\}$ be a dm-subset of [L], with $s_1 < s_2 < \cdots < s_{dm}$. Then define

$$\xi(S) = \chi(\{s_1, \dots, s_m\} \times \dots \times \{s_{m(d-1)+1}, \dots, s_{dm}\})$$
.

By Theorem 8.1.2, there is a dk-subset $\{t_1, t_2, \ldots, t_{dk}\}$ of [L] such that ξ is constant on the dm-element subsets of $\{t_1, \ldots, t_{dk}\}$. Suppose that $t_1 < t_2 < \cdots < t_{dk}$. Then $G = \{t_1, \ldots, t_k\} \times \cdots \times \{t_{k(d-1)+1}, \ldots, t_{dk}\}$ is a subgrid of $[L]^d$ that is monochromatic with respect to χ .

We next present a formulation of the product Ramsey theorem for arbitrary ordered Ramsey structures. The proof uses Theorem 8.2.2 and Theorem 8.2.8.

THEOREM 8.2.10. Let $\mathfrak{B}_1, \ldots, \mathfrak{B}_d$ be ω -categorical ordered Ramsey structures. Then $\mathfrak{P} := \mathfrak{B}_1 \boxtimes \cdots \boxtimes \mathfrak{B}_d$ is Ramsey.

PROOF. By Theorem 8.2.2, the automorphism groups $\mathbf{G}_1, \ldots, \mathbf{G}_d$ of $\mathfrak{B}_1, \ldots, \mathfrak{B}_d$ are extremely amenable, and it suffices to show that the automorphism group \mathbf{G} of \mathfrak{P} is extremely amenable. The group \mathbf{G} is given by the product action of $\mathbf{G}_1 \times \cdots \times \mathbf{G}_d$ on B_1, \ldots, B_d (see Section 3.3.4.3). Hence, extreme amenability of \mathbf{G} follows from Theorem 8.2.8.

Theorem 8.2.10 indeed generalizes Theorem 8.2.9, which can be seen as follows. Let r, d, m, k be positive integers. We consider the ω -categorical ordered Ramsey structure (\mathbb{Q} ; <), and apply Theorem 8.2.10 where d in Theorem 8.2.10 equals the d given above. Let \mathfrak{A} be the structure induced in $\mathfrak{P} := (\mathbb{Q}; <)^{[d]}$ by some (equivalently, every) $[m]^d$ subgrid of \mathbb{Q}^d , and let \mathfrak{B} be the structure induced in \mathfrak{P} by some (equivalently, every) $[k]^d$ subgrid of \mathbb{Q}^d . Since \mathfrak{C} is Ramsey, there exists an induced substructure \mathfrak{C} of \mathfrak{P} such that $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_r$. If \mathfrak{C} is not induced by an $[L]^d$ subgrid of \mathbb{Q}^d , for some large enough L, we can clearly choose a larger substructure \mathfrak{C} with this

property, such that still $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_r$. The occurrences of \mathfrak{A} in \mathfrak{C} correspond precisely to the $[m]^d$ -subgrids of $[L]^d$, which proves the claim.

8.2.4. Open Subgroups. In this section we show that open subgoups of extremely amenable groups are again extremely amenable. This fact will be important in Section 8.3.4 and 8.4 when it comes to the applications for analyzing polymorphism clones. We first show the following basic fact.

Proposition 8.2.11. Let X be a topological space, and Y be any set. Let G be a topological group that acts on X^Y . Then the action is continuous if for every $y \in Y$, the map $f_y : \mathbf{G} \times X^Y \to X$ given by $f_y(g,\xi) := (g \cdot \xi)(y)$ is continuous.

PROOF. Suppose that $\lim_{n\to\infty}(g_n,\xi_n)=(g,\xi)$. Then by Proposition 7.1.3 we have $\lim_{n\to\infty} g_n = g$ and $\lim_{n\to\infty} \xi_n(y) = \xi(y)$ for all $y \in Y$. Since f_y is continuous and by Proposition 7.1.2

$$\lim_{n \to \infty} (g_n \cdot \xi_n)(y) = \lim_{n \to \infty} f_y(g_n, \xi_n) = f_y(g, \xi) = (g \cdot \xi)(y)$$

for all $y \in Y$. We again apply Proposition 7.1.3 and obtain that $\lim_{n\to\infty} (g_n \cdot x_n) = g \cdot \xi$, which implies continuity of the action of G, again using Proposition 7.1.2.

Proposition 8.2.12 (from [54]). Let G be an extremely amenable group, and let **H** be an open subgroup of **G**. Then **H** is also extremely amenable.

In the proof, it is not essential but technically more convenient to use right cosets instead of left cosets.

PROOF. Let **H** act continuously on a compact space X; we will show that this action has a fixed point. Denote by $\pi \colon \mathbf{G} \to \mathbf{H} \backslash \mathbf{G}$ the quotient map and let $s \colon \mathbf{H} \backslash \mathbf{G} \to \mathbf{H} \backslash \mathbf{G}$ **G** be a section for π (i.e., a mapping satisfying $\pi \circ s = \mathrm{id}$) such that s(H) = 1. Let α be the map from $\mathbf{H}\backslash\mathbf{G}\times\mathbf{G}\to\mathbf{H}$ defined by

$$\alpha(w, q) = s(w)qs(wq)^{-1}.$$

For $w \in \mathbf{H} \setminus \mathbf{G}$ and $g \in \mathbf{G}$, note that s(w)g and s(wg) lie in the same right coset of **H**, namely wg, and hence the image of α is H. The map α satisfies

$$\alpha(w, g_1 g_2) = s(w) g_1 g_2 (s(w g_1 g_2))^{-1}$$

$$= s(w) g_1 s(w g_1) s(w g_1)^{-1} g_2 (s(w g_1 g_2))^{-1}$$

$$= \alpha(w, g_1) \alpha(w g_1, g_2) .$$

As H is open, $\mathbf{H}\backslash\mathbf{G}$ is discrete. Hence, s is continuous, and therefore α is contin-

uous as a composition of continuous maps. Now consider the product space $X^{\mathbf{H}\backslash\mathbf{G}}$ which is Hausdorff and compact by Theorem 7.1.5. The *co-induced action* of \mathbf{G} on $X^{\mathbf{H}\backslash\mathbf{G}}$ is defined by

$$(g \cdot \xi)(w) = \alpha(w, g) \cdot \xi(wg).$$

We claim that this action is continuous. By Proposition 8.2.11, it suffices to verify that the map $(q,\xi) \mapsto (q \cdot \xi)(w)$ is a continuous map from $G \times X^{H \setminus G} \to X$ for every fixed $w \in H \backslash G$. We already know that α is continuous and that the action of **H** on X is continuous. To see that $(g,\xi) \mapsto \xi(wg)$ is continuous, suppose that $\lim_{n\to\infty}(g_n,\xi_n)=(g,\xi)$. Let w=Hk. As $\lim_{n\to\infty}g_n=g$ and $k^{-1}Hk$ is open, we will have that eventually $g_ng^{-1}\in k^{-1}Hk$, giving that $kg_n(kg)^{-1}\in H$, or, which is the same, $Hkg_n = Hkg$. We obtain that for sufficiently large n, $wg_n = wg$. Therefore $\lim_{n\to\infty} \xi_n(wg_n) = \xi(wg).$

By the extreme amenability of \mathbf{G} , the co-induced action has a fixed point ξ_0 . Now we check that $\xi_0(H) \in X$ is a fixed point of the action $H \cap X$. Indeed, for any $h \in H$, $h \cdot \xi_0 = \xi_0$ and we have

$$\xi_0(H) = (h \cdot \xi_0)(H)$$

$$= \alpha(H, h) \cdot \xi_0(Hh)$$

$$= s(H)hs(Hh)^{-1}\xi_0(H)$$

$$= h \cdot \xi_0(H),$$

finishing the proof.

Proposition 8.2.12 can be applied to provide a short and elegant proof of the following.

COROLLARY 8.2.13 (from [54]). Let \mathfrak{B} be ordered homogeneous Ramsey, and let c_1, \ldots, c_n be elements of \mathfrak{B} . Then $(\mathfrak{B}, c_1, \ldots, c_n)$ is ordered homogeneous Ramsey as well.

PROOF. It is easy to see that the expansion of any homogeneous structure \mathfrak{B} by finitely many constants is again homogeneous. When \mathfrak{B} is additionally ordered Ramsey, then $\operatorname{Aut}(\mathfrak{B})$ is extremely amenable. The automorphism group of $(\mathfrak{B}, c_1, \ldots, c_n)$ is an open subgroup of $\operatorname{Aut}(\mathfrak{B})$. The statement thus follows directly from Proposition 8.2.12 and Theorem 8.2.2.

Note that in order to preserve homogeneity, we have to add constants c as unary function symbols, and not as unary singleton relations $\{c\}$. Consider for example the homogeneous structure $(\mathbb{V}; E)$, and let $u \in \mathbb{V}$ (see Example 3.2.6). Then $(\mathbb{V}; E, \{u\})$ is *not* homogeneous, since there are no automorphisms that map neighbours of u to non-neighbours.

8.3. Canonization

In this section we apply Ramsey theory to analyse endomorphism monoids and polymorphism clones of Ramsey structures \mathfrak{B} . The central idea is that for arbitrary finite substructures \mathfrak{C} of \mathfrak{B} , any mapping from $B^k \to B$ must 'behave canonically' on a copy of \mathfrak{C} in \mathfrak{B} . We first consider the more general case of functions between two possibly distinct structures, and introduce a refinement of the notion of canonicity from Section 5.6.2.

DEFINITION 8.3.1. Let \mathfrak{C} be a structure with domain C, and S a subset of C. Let \mathfrak{B} be a structure with domain B, and let $f: C \to B$ be a function. We say that f is canonical on S as a map from \mathfrak{C} to \mathfrak{B} if for all n and every n-tuple t over S the n-type of f(t) in \mathfrak{B} only depends on the n-type of t in \mathfrak{C} .

The basic lemma to apply Ramsey theory in the analysis of functions is the following.

LEMMA 8.3.2. Let \mathfrak{C} be an ω -categorical ordered Ramsey structure with domain C and finite relational signature τ , let \mathfrak{B} be an ω -categorical structure with domain B, and let $f: C \to B$ be an operation. Then for all finite subsets S of C there is an automorphism α of \mathfrak{C} so that the operation $x \mapsto f(\alpha x)$ is canonical on S as a map from \mathfrak{C} to \mathfrak{B} .

PROOF. Let m be the maximal arity of the relations in τ , and < be the linear order in the signature of \mathfrak{C} . Let \mathfrak{C}' be the homogeneous ω -categorical expansion of \mathfrak{C} by all relations that are first-order definable in \mathfrak{C} . Then the age of \mathfrak{C}' is a Ramsey class. Let \mathfrak{S} be the substructure induced by S in \mathfrak{C}' , and n := |S|.

When \mathfrak{A} is a substructure of \mathfrak{C}' of size m, then $a^{\mathfrak{A}}$ is the tuple (a_1, \ldots, a_m) such that $\{a_1, \ldots, a_m\}$ are the elements of \mathfrak{A} and $t_1 < \cdots < t_m$.

Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ be a list all non-isomorphic substructures of \mathfrak{S} of cardinality m. Since \mathfrak{C}' is Ramsey, there is a substructure \mathfrak{C}_1 of \mathfrak{C}' such that $\mathfrak{C}_1 \to (\mathfrak{S})_r^{\mathfrak{A}_1}$. Further, there is a substructure \mathfrak{C}_2 of \mathfrak{C}' such that $\mathfrak{C}_2 \to (\mathfrak{C}_1)_r^{\mathfrak{A}_2}$. We iterate this k times, arriving at a structure \mathfrak{C}_k . For each $i \leq k$, the operation f defines a coloring χ_i of $\binom{\mathfrak{C}_k}{\mathfrak{A}_i}$ with finitely many colors as follows: the color of a copy \mathfrak{A} of \mathfrak{A}_i is just the type of $f(a^{\mathfrak{A}})$ in \mathfrak{B} ; since \mathfrak{B} is ω -categorical, the number r of m-types in \mathfrak{B} is finite.

Now going back the argument, we find that \mathfrak{C}_k contains a copy of \mathfrak{S} on which all colorings χ_1, \ldots, χ_k are constant. Since \mathfrak{C}' is homogeneous, there exists an automorphism α of \mathfrak{C}' that sends S to this copy. Then $x \mapsto f(\alpha x)$ is canonical on S as a map from \mathfrak{C} to \mathfrak{B} .

Note that the assumption that \mathfrak{B} is ordered is necessary in Lemma 8.3.2: for instance, if f is an injective function from $X \to (\mathbb{Q}; <)$, then f is not canonical as a map from (X; =) to $(\mathbb{Q}; <)$ on any two-element subset of X.

8.3.1. Multivariate functions. The appropriate generalisation of canonicity for multivariate functions is the following.

DEFINITION 8.3.3. Let \mathfrak{B} be a structure with domain B, and \mathfrak{C} a structure with domain C. When $f: C^d \to B$ is a function, and S is a subset of C^d we say that f is canonical on S if for all n and all n-tuples t^1, \ldots, t^d where $(t_i^1, \ldots, t_i^d) \in S$ for all $i \leq n$ the n-type of $f(t^1, \ldots, t^d)$ in \mathfrak{B} only depends on the n-types of t^1, \ldots, t^d in \mathfrak{C} . We say that f is canonical (as a d-ary map from \mathfrak{C} to \mathfrak{B}) if f is canonical on B^d .

When $\mathfrak{B} = \mathfrak{C}$ in the definition above, we say that f is canonical on \mathfrak{B} when it is canonical as a d-ary map from \mathfrak{B} to \mathfrak{B} .

EXAMPLE 8.3.4. Let lex be a binary operation on \mathbb{Q} such that lex(a, b) < lex(a', b') if either a < a', or a = a' and b < b'. Clearly, such an operation exists. Note that lex is injective, that it preserves <, and that it is canonical as a binary polymorphism of $(\mathbb{Q}; <)$.

In the proof of the following we use the product Ramsey theorem, Theorem 8.2.10.

THEOREM 8.3.5. Let \mathfrak{B} be an ω -categorical ordered Ramsey structure with finite relational signature and domain B, and let $f: B^d \to B$ be any operation. Then for all finite subsets S_1, \ldots, S_d of B there are automorphisms $\alpha_1, \ldots, \alpha_d$ of \mathfrak{B} so that the operation $(x_1, \ldots, x_d) \mapsto f(\alpha_1 x_1, \ldots, \alpha_d x_d)$ is canonical on $S_1 \times \cdots \times S_d$.

PROOF. By Theorem 8.2.10, the structure $\mathfrak{B}^{[d]}$ is Ramsey. Hence, Lemma 8.3.2 shows the existence of an automorphism α of $\mathfrak{B}^{[d]}$ such that $x \mapsto f(\alpha x)$ is canonical on $S_1 \times \cdots \times S_d$ as a map from $\mathfrak{B}^{[d]}$ to \mathfrak{B} .

Let **G** be the topological automorphism group of \mathfrak{B} . Since the automorphism group of $\mathfrak{B}^{[d]}$ is induced by the product action of \mathbf{G}^k on B^k , there are group elements a_1, \ldots, a_d of \mathfrak{G} so that $\alpha(x_1, \ldots, x_d) = (\alpha_1 x_1, \ldots, \alpha_d x_d)$. Now clearly the function $(x_1, \ldots, x_d) \mapsto f(\alpha_1 x_1, \ldots, \alpha_d x_d)$ is canonical on $S_1 \times \cdots \times S_d$ as a multivariate function on \mathfrak{B} .

8.3.2. Interpolation modulo automorphisms. One of the central questions when analysing a polymorphism of a structure \mathfrak{B} is to find out what functions it generates (since those functions will also be polymorphisms of \mathfrak{B} , see Section 5.2). Theorem 8.3.5 can be used for this purpose; to illustrate this, we present the following.

COROLLARY 8.3.6. Let \mathfrak{B} be an ω -categorical ordered Ramsey structure with finite relational signature. Then every injective operation $f \colon B^k \to B$ together with the automorphisms of \mathfrak{B} locally generates a canonical injective operation g.

This corollary follows in a straightforward way from Theorem 8.3.5 and a compactness argument, which we do not present here since we will present a proper generalisation of it in full detail, Theorem 8.3.8.

Note that when we drop the injectivity assumption for f in the statement of Corollary 8.3.6 then the statement of the corollary becomes trivially true, since every operation locally generates the projections, which are canonical on the entire domain. We therefore need a concept that is weaker than local closure, but stronger than interpolation, to turn the idea of Corollary 8.3.6 into a meaningful statement for all functions from B^k to B.

DEFINITION 8.3.7. Let \mathfrak{B} be an ω -categorical structure with domain B, and $f,g \colon B^d \to B$ be functions. Then f interpolates g modulo automorphisms of \mathfrak{B} if for every finite $S \subseteq B$ there are automorphisms $\alpha_0, \alpha_1, \ldots, \alpha_d$ of \mathfrak{B} such that $g(x_1, \ldots, x_d) = \alpha_0(f(\alpha_1 x_1, \ldots, \alpha_d x_d))$.

Theorem 8.3.8. Let \mathfrak{B} be an ordered ω -categorical Ramsey structure with finite relational signature and domain B, and $f \colon B^d \to B$ any operation. Then there is a canonical operation $g \colon B^d \to B$ that is interpolated by f modulo automorphisms.

Theorem 8.3.8 is still not in its most general and most useful form. For this, we need a further generalisation of the notion of interpolation modulo automorphisms to the situation where f is a function from a structure \mathfrak{C} to a different structure \mathfrak{B} .

DEFINITION 8.3.9. Let \mathfrak{B} , \mathfrak{C} be structures with domains B and C, and let $f,g: C \to B$ be functions. Then f interpolates g modulo automorphisms of \mathfrak{C} if for every finite $S \subseteq C$ there is an $\alpha \in \operatorname{Aut}(\mathfrak{C})$ and a $\beta \in \operatorname{Aut}(\mathfrak{B})$ such that $g(x) = \beta(f(\alpha x))$.

The following is the central statement about Ramsey structures and interpolation modulo automorphisms.

THEOREM 8.3.10. Let \mathfrak{C} be ω -categorical ordered Ramsey with finite relational signature and domain C, and let \mathfrak{B} be ω -categorical with domain B. Then every $f: C \to B$ interpolates a canonical operation $g: C \to B$ modulo automorphisms of \mathfrak{C} .

PROOF. By Lemma 3.1.8, it suffices to show that for every finite subset C' of C there is a function from $C \to B$ that is canonical on C' and interpolated by f modulo automorphisms of \mathfrak{C} , since the property to be canonical on C' is a universal first-order statement about f. This follows from Lemma 8.3.2.

PROOF OF THEOREM 8.3.8. We apply Theorem 8.3.10 to the structure $\mathfrak{C} := \mathfrak{B}^{[d]}$, expanded by the first-order definable lexicographic ordering on $\mathfrak{B}^{[d]}$, which is Ramsey when \mathfrak{B} is Ramsey, by Theorem 8.2.10. As in the proof of Theorem 8.3.5, canonicity of a function g from $\mathfrak{B}^{[d]}$ to \mathfrak{B} translate into canonicity of g as a d-ary function on \mathfrak{B} , and interpolation operations modulo automorphisms of $\mathfrak{B}^{[d]}$ and \mathfrak{B} corresponds to interpolation of d-ary functions modulo automorphisms of \mathfrak{B} .

Note that Theorem 8.3.8 is indeed a generalization of Corollary 8.3.6, since clearly operations that are interpolated by injective operations modulo automorphisms are again injective.

'Canonization' of operations as exhibited in Theorem 8.3.10 becomes particularly powerful when we combine it with expansions by constants. The following theorem has numerous applications.

THEOREM 8.3.11 (fom [54]). Let \mathfrak{B} be an ω -categorical ordered Ramsey structure with domain B and finite relational signature. Let $c_1, \ldots, c_m \in B$, and let $f : B^k \to B$ be any function. Then $\{f\} \cup \operatorname{Aut}(\mathfrak{B})$ locally generates a function which is canonical as a function from $(\mathfrak{B}, c_1, \ldots, c_m)^k$ to \mathfrak{B} , and which equals f on all tuples containing only values c_i .

PROOF. By Corollary 8.2.13, also the structure $(\mathfrak{B},c_1,\ldots,c_m)$ is ordered Ramsey. By Theorem 8.2.10, the structure $\mathfrak{C}:=(\mathfrak{B},c_1,\ldots,c_m)^{[k]}$ is ordered Ramsey (and still ω -categorical). Let d_1,\ldots,d_n be an enumeration of the image of the restriction of f to $\{c_1,\ldots,c_m\}$. The structure $(\mathfrak{B},d_1,\ldots,d_n)$ is still ω -categorical. Then Theorem 8.3.10 shows that f interpolates modulo automorphisms of \mathfrak{C} and $(\mathfrak{B},d_1,\ldots,d_n)$ an operation which is canonical as a function to $(\mathfrak{B},d_1,\ldots,d_n)$ and therefore also as a function to \mathfrak{B} . In particular, g is locally generated by $\{f\} \cup \operatorname{Aut}(\mathfrak{B})$, and the restrictions of f and g to $\{c_1,\ldots,c_m\}$ are equal.

8.3.3. Behavior of Operations. It is sometimes important to work with operations that exhibit a 'behavior' that is only partially canonical. The following definition from [50] gives us some flexibility in specifying such functions.

DEFINITION 8.3.12. Let $\mathfrak C$ and $\mathfrak B$ be structures with domains C and B, and let $k \geq 1$. An (n-)type condition between $\mathfrak C$ and $\mathfrak B$ is a k+1-tuple (t^1,\ldots,t^d,s) , where each t_i is an n-type in $\mathfrak C$, and s is an n-type in $\mathfrak B$. A d-ary function $f\colon C^d\to B$ satisfies an n-type condition (t^1,\ldots,t^d,s) on $S\subseteq C^d$ if for all n-tuples a^i of type t^i in $\mathfrak C$ with $(a_1^i,\ldots,a_d^i)\in S$ for all $i\leq d$, the n-tuple $(f(a_1^1,\ldots,a_1^d),\ldots,f(a_n^1,\ldots,a_n^d))$ is of type s in $\mathfrak B$.

A behavior between two structures \mathfrak{C} and \mathfrak{B} is a set Λ of type conditions. A function $f \colon C^d \to B$ has behavior Λ on $S \subseteq C^d$ if it satisfies all the type conditions of Λ on S. We say that f has behavior Λ if it has behavior Λ on all of C^d .

Note that a d-ary operation $f: \mathfrak{C}^d \to \mathfrak{B}$ is canonical if for all $n \geq 1$ and all d-tuples (t^1, \ldots, t^d) of types of n-tuples in \mathfrak{C} there exists a type s of an n-tuple in \mathfrak{B} such that f satisfies the type condition (t^1, \ldots, t^d, s) . When \mathfrak{B} is homogeneous in a relational signature with maximal arity n, then already the n-type conditions determine the behavior of functions over \mathfrak{B} .

When $\mathfrak B$ is ω -categorical then the clone generated by $\operatorname{Aut}(\mathfrak B)$ and a canonical d-ary function f over $\mathfrak B$ is completely described by the behavior of f. In fact, when $f,g\colon B^d\to B$ are functions with the same behavior, then $\{f\}\cup\operatorname{Aut}(\mathfrak B)$ generates g, and $\{g\}\cup\operatorname{Aut}(\mathfrak B)$ generates f, by local closure.

LEMMA 8.3.13. Let \mathfrak{B} be ordered ω -categorical with domain B. Let Λ be a behaviour for functions from \mathfrak{B}^k to \mathfrak{B} , and let $g \colon B^k \to B$ be arbitrary. If for every finite substructure \mathfrak{A} of \mathfrak{B} there are copies $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ of \mathfrak{A} in \mathfrak{B} such that g has behavior Λ on $\mathfrak{A}_1 \times \cdots \times \mathfrak{A}_k$ then $\{g\} \cup \operatorname{Aut}(\mathfrak{B})$ locally generates a function f of behaviour Λ .

Proof. A direct consequence of Lemma 3.1.8.

An orbit of $(\mathfrak{B}, c_1, \ldots, c_m)$ is called *full* if it contains copies of all finite substructures of \mathfrak{B} . The following follows from Lemma 8.3.13. As we will see in Chapter 9 and Chapter 10, it becomes important in the context of canonization after expansions by constants (Theorem 8.3.11).

LEMMA 8.3.14. Let \mathfrak{B} be ω -categorical, and let c_1, \ldots, c_m be elements from \mathfrak{B} . When \mathfrak{A} is the substructure induced in \mathfrak{B} by a full orbit O of $(\mathfrak{B}, c_1, \ldots, c_m)$, and f is a function from $(\mathfrak{B}, c_1, \ldots, c_m)^k$ to \mathfrak{B} with behaviour Λ on O, then $\{f\} \cup \operatorname{Aut}(\mathfrak{B})$ locally generates a function from \mathfrak{B}^k to \mathfrak{B} with behaviour Λ .

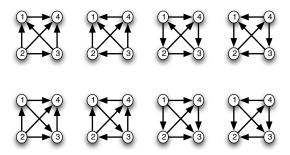


FIGURE 8.1. Canonical behavior on $[2]^2$ grids.

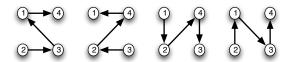


FIGURE 8.2. When f is a canonical binary injective polymorphism of $(\mathbb{Q}; <)$, then there is one linear order of a $[2]^2$ grid as depicted here such that all $[2]^2$ subgrids of \mathbb{Q}^2 are linearly ordered in this way.

Not every behavior Λ between $\mathfrak C$ and $\mathfrak B$ is realized by a function in the sense that there exists a function from $C \to B$ that has behavior Λ . We give an example. There are eight distinct candidates for canonical behavior of injective maps from $(\mathbb Q;<)^2$ to $(\mathbb Q;<)$; they are illustrated in Figure 8.1. However, only four of those canonical behaviors are realized by binary injective polymorphisms of $(\mathbb Q;<)$; those are illustrated in Figure 8.2. The others would imply the existence of three points x,y,z in the image such that x < y and y < z and z < x, which is impossible over $(\mathbb Q;<)$. The not necessarily injective case can be analyzed similarly, and we get the following.

LEMMA 8.3.15. Let f be a canonical binary polymorphism of $(\mathbb{Q}; <)$. Then f has the same behavior as one out of the following seven operations.

- lex(x,y) or lex(y,x);
- lex(x, -y) or lex(y, -x).
- $(x,y) \mapsto x \text{ or } (x,y) \mapsto y.$
- a constant operation

Together with Corollary 8.3.6, we find that every binary injective polymorphism of $(\mathbb{Q}; <)$ locally generates lex(x, y), lex(y, x), lex(x, -y), or lex(y, -x).

8.3.4. Canonical Violation. Let \mathfrak{C} be an ω -categorical ordered Ramsey structure, and let \mathfrak{B} be a structure with a first-order definition in \mathfrak{C} . Suppose that a relation R does not have a primitive positive definition in \mathfrak{B} . We wish to show that then there exists a polymorphism of \mathfrak{B} that violates R and is canonical as a function over \mathfrak{C} . Boldly stated like this, this cannot hold true. However, the results from Section 8.2.4 show us how to fix the statement. To illustrate the basic idea, we first discuss the unary case, with existential positive definability instead of primitive positive definability.

THEOREM 8.3.16. Let $\mathfrak C$ be an ω -categorical ordered Ramsey structure, and let $\mathfrak B$ be a structure with a first-order definition in $\mathfrak C$. Suppose that the k-ary relation R does not have an existential positive definition in $\mathfrak B$. Then there exists an endomorphism e of $\mathfrak B$ and a k-tuple $t=(t_1,\ldots,t_k)\in R$ such that

- $e(t) \notin R$
- e is canonical as a map from $(\mathfrak{C}, t_1, \ldots, t_k)$ to \mathfrak{C} .

PROOF. The structure \mathfrak{B} is ω -categorical. If R does not have an existential definition, then by Theorem 3.4.7 there is an endomorphism e' of \mathfrak{B} which violates R, that is, there is a k-tuple $t = (t_1, \ldots, t_k) \in R$ such that $e'(t) \notin R$. By Theorem 8.3.11, $\{e'\} \cup \operatorname{Aut}(\mathfrak{B})$ locally generates an operation e that is canonical as a function from $(\mathfrak{B}, t_1, \ldots, t_k)$ to \mathfrak{B} that has the same restriction to $\{t_1, \ldots, t_k\}$ as e', and e has the required properties from the statement of the theorem.

Here comes the multivariate analog of Theorem 8.3.16, whose proof is analogous to the proof of the previous theorem.

THEOREM 8.3.17. Let $\mathfrak C$ be an ω -categorical ordered Ramsey structure, let $\mathfrak B$ be a structure with a first-order definition in $\mathfrak C$, and suppose that the k-ary relation R does not have a primitive positive definition in $\mathfrak B$. Then there exists a finite d, a d-ary polymorphism f of $\mathfrak B$, and k-tuples $t^1, \ldots, t^d \in R$ such that

- $f(t^1, \dots, t^d) \notin R$
- f is canonical as a map from $(\mathfrak{C}, t^1) \boxtimes \cdots \boxtimes (\mathfrak{C}, t^d)$ to \mathfrak{C} .

PROOF. If R does not have a primitive positive definition in \mathfrak{B} , then since \mathfrak{B} is ω -categorical, by Theorem 5.2.3 there is a polymorphism f' of \mathfrak{B} which violates R. By Lemma 5.3.5, we can assume that the arity of f' equals the number d of orbits of k-tuples contained in R, which is bounded by $o^{\mathfrak{B}}(k)$. So there are k-tuples $t^1, \ldots, t^d \in R$ such that $f'(t^1, \ldots, t^m) \notin R$. By Corollary 8.2.13, for all $i \leq d$ the structure (\mathfrak{C}, t^i) is Ramsey, and by Theorem 8.2.10 the structure $(\mathfrak{C}, t^1) \boxtimes \cdots \boxtimes (\mathfrak{C}, t^d)$ is Ramsey. Then Theorem 8.3.10 shows that f' interpolates modulo automorphisms of $(\mathfrak{C}, t^1) \boxtimes \cdots \boxtimes (\mathfrak{C}, t^d)$ a canonical operation f, and $f(t^1, \ldots, t^d) = f'(t^1, \ldots, t^d) \notin R$. Since f is in particular locally generated by polymorphisms of \mathfrak{B} , it is itself an polymorphism of \mathfrak{B} .

We are now in the situation to prove the following, which has been announced already in Section 5.3.5.

Theorem 8.3.18 (from [54]). Let \mathfrak{B} be a structure with finite relational signature, and with a first-order definition in an ordered homogeneous Ramsey structure \mathfrak{C} with a relational signature of maximal arity k. Then there are finitely many minimal closed clones above $\operatorname{Pol}(\mathfrak{B})$.

PROOF. Every minimal closed clone above $\operatorname{Pol}(\mathfrak{B})$ is locally generated by a minimal operation f (Proposition 5.3.14), and by Theorem 5.2.4 there must be a relation R in \mathfrak{B} that is violated by f, that is, there are $t^1, \ldots, t^d \in R$ such that $f(t^1, \ldots, t^d) \notin R$. Since f is a minimal operation, Theorem 8.3.17 implies that f must be canonical as a map from $(\mathfrak{C}, t^1) \boxtimes \cdots \boxtimes (\mathfrak{C}, t^d) \to \mathfrak{C}$. But since \mathfrak{C} is homogeneous is a finite relational signature, and \mathfrak{B} has finite relational signature, there are only finitely many canonical behaviors of such operations; since two minimal operations with the same behavior locally generate the same closed clone above $\operatorname{Pol}(\mathfrak{B})$, we are done.

8.4. Decidability Results for Meta-Problems

We turn to another application of the ideas of the previous sections. For a fixed structure \mathfrak{C} with a finite relational signature τ and domain C, consider the following computational problem.

Expr-fo(\mathfrak{C})

INSTANCE: Quantifier-free first-order τ -formulas ϕ_0, \ldots, ϕ_n defining the relations R_0, \ldots, R_n over \mathfrak{C} .

QUESTION: Is there a first-order definition of R_0 in $(C; R_1, \ldots, R_n)$?

We are also interested in the variants of this problem where we replace first-order definability by other syntactically restricted versions of definability, in particular by primitive positive definability. The corresponding computational problem for primitive positive definability is denoted by $\operatorname{Expr-pp}(\mathfrak{C})$ (and the problem for existential and existential positive definability by $\operatorname{Expr-ex}(\mathfrak{C})$ and $\operatorname{Expr-ep}(\mathfrak{C})$, respectively).

For finite structures \mathfrak{C} the problem Expr-pp(\mathfrak{C}) is in co-NEXPTIME (and in particular decidable). For the variant where the finite structure Γ is part of the input, the problem has recently shown to be also co-NEXPTIME-hard [196]. An algorithm for Expr-pp(\mathfrak{C}) has theoretical and practical consequences in the study of the computational complexity of CSPs for structures that are first-order definable in \mathfrak{C} , as illustrated in the following examples.

EXAMPLE 8.4.1. We can use an algorithm for Expr-pp(\mathfrak{C}) to decide whether all polymorphisms of a structure $(C; R_1, \ldots, R_n)$, given by τ -formulas ϕ_1, \ldots, ϕ_n that define R_1, \ldots, R_n over \mathfrak{C} , are essentially unary. For that, we simply apply the algorithm to $x \neq y \lor y \neq z, \phi_1, \ldots, \phi_n$, for each i; here we use Proposition 5.3.3.

Example 8.4.2. To decide whether a structure $(C; R_1, \ldots, R_n)$, again given by τ -formulas ϕ_1, \ldots, ϕ_n that define R_1, \ldots, R_n over \mathfrak{C} , is a core, we apply the algorithm for Expr-pp(\mathfrak{C}) to $\neg \phi_i, \phi_1, \ldots, \phi_n$, for each i. Additionally, we apply the algorithm to $x \neq y, \phi_1, \ldots, \phi_n$. The structure $(C; R_1, \ldots, R_n)$ is a core if and only if none of those calls reports false, that is, all the relations defined by $\neg \phi_i$ or by $x \neq y$ are primitive positive definable in $(C; R_1, \ldots, R_n)$.

EXAMPLE 8.4.3. We can use an algorithm for Expr-pp(\mathfrak{C}) to effectively test the hardness condition for $CSP(\mathfrak{B})$ given in Proposition 5.5.9 for structures \mathfrak{B} with a first-order definition in \mathfrak{C} and a finite relational signature.

The main result of this section is the decidability of Expr-pp(\mathfrak{C}) for a certain class of structures \mathfrak{C} . Even for the simplest of countable structures, namely the structure (X;=) having no relations but equality, the decidability of Expr-pp(Γ) is not obvious (see [30]). Recall the concept of *finitely bounded* structures \mathfrak{C} (Definition 3.2.8): we require that the age of \mathfrak{C} is given by a finite set of finite forbidden induced substructures.

Theorem 8.4.4 (from [54]). Let $\mathfrak C$ be of finite relational signature, and first-order definable over a structure $\mathfrak D$ which is homogeneous, ordered, Ramsey, finitely bounded, and with finite relational signature. Then $\operatorname{Expr-pp}(\mathfrak C)$ is decidable.

PROOF. Let D be the domain of \mathfrak{D} and \mathfrak{C} . The input consists of formulas $\phi_0, \phi_1, \ldots, \phi_k$ in the signature of \mathfrak{C} . Those formulas define the relations R_0, R_1, \ldots, R_k over \mathfrak{C} . Set \mathfrak{B} to be the structure $(D; R_1, \ldots, R_k)$. We will decide whether there is a primitive positive definition of R_0 in \mathfrak{B} . We can without loss of generality assume that in each formula ϕ_0, \ldots, ϕ_k , the variables are called x_1, \ldots, x_p , for some p.

By Theorem 8.3.17, if R_0 is m-ary and does not have a primitive positive definition in \mathfrak{B} , then there exists a finite d, a d-ary polymorphism f of \mathfrak{B} , and m-tuples $t^1, \ldots, t^d \in R_0$ such that $f(t^1, \ldots, t^d) \notin R_0$, and f is canonical as a map from $(\mathfrak{D}, t^1) \boxtimes \cdots \boxtimes (\mathfrak{D}, t^d)$ to \mathfrak{D} . Such a polymorphism of \mathfrak{B} will be called a witness at t^1, \ldots, t^d (for the fact that R_0 is not primitive positive definable in \mathfrak{B}). The question whether such a witness exists for a specific choice of tuples t^1, \ldots, t^d does of course only depend on the orbits of t^1, \ldots, t^d in \mathfrak{D} , and by ω -categoricity of \mathfrak{D} there are only finitely many such orbits. Moreover, by homogeneity of \mathfrak{D} , the orbits of n-tuples are in one-to-one correspondence to the n-element induced substructures of \mathfrak{D} , which can be effectively stored and enumerated on a computer. So it suffices in the following to consider the case where t^1, \ldots, t^d are fixed, and to show how to decide whether a witness exists at this choice of t^1, \ldots, t^d .

Since expansions of homogeneous structures by finitely many constants are homogeneous, the one-to-one correspondence between orbits of l-tuples, maximal l-types, and induced l-element substructures of $\mathfrak D$ extends to the structures $(\mathfrak D, t^i)$. In the following, let n be max(3, n'+1) where n' is the maximal arity of the relations in $\mathfrak D$. Then by homogeneity of $\mathfrak D$ the behavior of f is determined by the n-type conditions that are satisfied by f (for this property we only need that $n \geq n'$; the requirement $n \geq 3$ is motivated by the way how we treat equality in our approach, as we will see later). When f is canonical, then the set Λ of n-type conditions can be viewed as a function from $S_n^{(\mathfrak D,t^1)} \times \cdots \times S_n^{(\mathfrak D,t^d)}$ to $S_n^{\mathfrak D}$. By ω -categoricity of $\mathfrak D$ and of $(\mathfrak D,t^i)$ there are only finitely many such functions Λ .

We decide the existence of a witness by reduction to a finite-domain constraint satisfaction problem. The domain of the CSP is the set of all n-types of \mathfrak{D} . The instance of the CSP has a variable for every d-tuple (S_1, \ldots, S_d) where S_i is an n-type of (\mathfrak{D}, t^i) ; in fact, we identify the variables of the instance and those d-tuples of n-types. The constraints are described below. The idea is that the solutions to this CSP are exactly the functions Λ for witnesses as described above.

To implement this in detail, it will be convenient to make the assumption that \mathcal{N} is *minimal* in the sense that it does not contain structures $\mathfrak{A}_1, \mathfrak{A}_2$ such that \mathfrak{A}_1 is an induced substructure of \mathfrak{A}_2 ; this assumption is without loss of generality since otherwise we remove \mathfrak{A}_2 from \mathcal{N} , and find that the resulting set of structures still bounds \mathfrak{D} .

- (Compatibility.) Note that every behavior of a witness must have an extension to a function from $S_l^{(\mathfrak{D},t^1)} \times \cdots \times S_l^{(\mathfrak{D},t^d)}$ to $S_l^{\mathfrak{D}}$, for all $1 \leq l \leq n$. Hence, when $(S_1,\ldots,S_d), (T_1,\ldots,T_d) \in S_n^{(\mathfrak{D},t^1)} \times \cdots \times S_n^{(\mathfrak{D},t^d)}$, and $I \subset [n]$, and if for all $i \leq d$ the subtype of S_i induced by I and the subtype of T_i induced by I coincide, then we impose the binary constraint that I induces the same subtype in $\Lambda(S_1,\ldots,S_d) \in S_n^{\mathfrak{D}}$ and in $\Lambda(T_1,\ldots,T_d) \in S_n^{\mathfrak{D}}$.
- (Realizability.) We also want to make sure that the behavior Λ can be realized by an operation (recall the example given in Section 8.3.3). The idea is that when $\mathfrak D$ is finitely bounded, then Λ should not force the existence of one of the forbidden substructures in the image, since in this case it would be impossible to find an operation with image in $\mathfrak D$ whose behavior is Λ . As we will see, it suffices here to consider structures $\mathfrak A \in \mathcal N$ whose number of elements s exceeds s.

For each structure $\mathfrak{A} \in \mathcal{N}$ with s > n elements and each sequence S_1, \ldots, S_d with $S_i \in S_s^{(\mathfrak{D};t^i)}$ for all $i \leq d$ we have a constraint of arity $r := \binom{s}{n}$. Let a_1, \ldots, a_s be the elements of \mathfrak{A} . Observe that for every subset $I \subseteq [s]$ with |I| = n the structure induced by $\{a_i \mid i \in I\}$ in \mathfrak{A} is an

induced substructure of \mathfrak{D} , by the minimality assumption on \mathcal{N} . Let $\phi^{\mathfrak{A}}[I]$ be the formula with variables x_1, \ldots, x_n that contains for $i_1, \ldots, i_m \in I$ the conjunct $R(x_{i_1}, \ldots, x_{i_m})$ if and only if $(a_{i_1}, \ldots, a_{i_m}) \in R^{\mathfrak{A}}$. By the observation we just made, $\phi^{\mathfrak{A}}[I]$ is contained in a unique n-type of \mathfrak{D} .

The constraint of arity r requires that for some $I \subseteq [s]$ with |I| = n the subtype of $\Lambda(S_1, \ldots, S_d)$ induced by I does not contain $\phi^{\mathfrak{A}}[I]$.

- (Violation.) We want that Λ is the behavior of an operation that violates the m-ary relation R_0 . For simplicity of presentation, we assume that $m \geq n$; this is without loss of generality, since we can otherwise add dummy variables to ϕ_0 .
 - For $t = (t_1, \ldots, t_m)$ and $i_1, \ldots, i_n \in [m]$ with $i_1 < \cdots < i_n$, denote by $t[\{i_1, \ldots, i_n\}]$ the tuple $(t_{i_1}, \ldots, t_{i_n})$. When $I \subseteq [m]$ we denote by $\phi_0[I]$ the subtype of $\{\phi_0\}$ induced by I; here, $\{\phi_0\}$ is viewed as a type over the empty theory. We add the $\binom{m}{n}$ -ary constraint that for some $I \subset [m]$ of cardinality n the type $\Lambda(\operatorname{tp}^{(\mathfrak{D},t^1)}(t^i[I]), \ldots, \operatorname{tp}^{(\mathfrak{D},t^d)}(t^i[I]))$ does not contain $\phi_0[I]$.
- (Preservation.) We also want that Λ is the behavior of an operation that preserves \mathfrak{B} . Let $j \leq k$, and suppose that the relation R_j of \mathfrak{B} defined by ϕ_j is p-ary. For simplicity of presentation, we assume that $p \geq n$, otherwise we add dummy arguments to R_j . For every list S_1, \ldots, S_d such that $S_i \in S_p^{(\mathfrak{D}, t^i)}$ contains ϕ_j for all $i \leq d$, we impose the following constraint of arity $q = \binom{p}{n}$. For all $I \subseteq [p]$ with |I| = n, let S_i^I be the subtype of S_i induced by I, and let S_0^I be the subtype of $\{\phi_j\}$ (of the empty theory) induced by I. We add the constraint that $\Lambda(S_1^I, \ldots, S_d^I)$ contains S_0^I for all $I \subseteq [p]$.

We now prove that there is a witness f at t^1, \ldots, t^d for the fact that R_0 is not primitive positive definable in \mathfrak{B} if and only if the described CSP instance has a satisfying assignment, which concludes the proof. For the easy direction, suppose that there exists such a witness, and let Λ be its behavior. Then Λ clearly satisfies compatibility, realizability, violation, and preservation constraints.

For the opposite direction, suppose that α is a solution to the described CSP, i.e., a mapping from $S_n^{(\mathfrak{D},t^1)} \times \cdots \times S_n^{(\mathfrak{D},t^d)}$ to $S_n^{\mathfrak{D}}$ that satisfies compatibility, realizability, violation and preservation constraints. We show the existence of a witness f at t^1, \ldots, t^d in three steps.

We first construct an infinite structure \mathfrak{E} with domain D^d of the same signature τ as \mathfrak{D} as follows. When $a^1, \ldots, a^d \in D^n$ are such that $a^i \in S_n^{(\mathfrak{D}, t^i)}$, then for $R \in \sigma$ the relation $R((a_1^1, \ldots, a_1^d), \ldots, (a_n^1, \ldots, a_n^d))$ holds in \mathfrak{E} if and only if $R(x_1, \ldots, x_n)$ is contained in $\Lambda(\operatorname{tp}^{(\mathfrak{D}, t^1)}(a^1) \times \cdots \times \operatorname{tp}^{(\mathfrak{D}, t^d)}(a^d))$. This is well-defined by the compatibility constraints.

Next, we consider the relation \sim on the domain of $\mathfrak E$ such that $a_1 \sim a_2$ for $a_1, a_2 \in D^d$ if and only if there exist $a_3, \ldots, a_n \in D^d$ such that the subtype of $\Lambda((\operatorname{tp}^{(\mathfrak D,t^1)}((a_1^1,\ldots,a_n^1))\times \cdots \times \operatorname{tp}^{(\mathfrak D,t^d)}(a_1^d,\ldots,a_n^d))$ induced by $\{1,2\}$ contains $x_1=x_2$. Note that since $n\geq 3$, the relation \sim must be an equivalence relation (since the properties of an equivalence relation can be formulated with a universal formula with three variables). Then the quotient structure $\mathfrak E/_{\sim}$ is defined to be the τ -structure whose elements are the equivalence classes $E/_{\sim}$ of \sim , and where $R(E_1,\ldots,E_p)$ holds for a p-ary $R\in \tau$ and $E_1,\ldots,E_p\in E/_{\sim}$ if and only if there are $b_1\in E_1,\ldots,b_p\in E_p$ such that $R(b_1,\ldots,b_p)$ holds in $\mathfrak E$.

The final step is to show that there exists an embedding f of $\mathfrak{E}/_{\sim}$ into \mathfrak{D} . By ω -categoricity of \mathfrak{D} and Lemma 3.1.5, it suffices to show every finite substructure \mathfrak{A} of $\mathfrak{E}/_{\sim}$ embeds into \mathfrak{D} . Since \mathfrak{D} is finitely bounded by \mathcal{N} , we thus have to show that no structure in \mathcal{N} embeds into \mathfrak{A} . Suppose to the contrary that there is an embedding

e of $\mathfrak{F} \in \mathcal{N}$ into \mathfrak{A} . Let u_1, \ldots, u_s be the elements of \mathfrak{F} . Pick any elements v_1, \ldots, v_s from the equivalence classes $e(u_1), \ldots, e(u_s)$, respectively. The rest of the paragraph is devoted to the argument that the mapping e' that maps u_i to v_i is an embedding of \mathfrak{F} into \mathfrak{E} , which contradicts the realizability constraints. It is obvious that e' is injective. To see that it is a strong homomorphism, let R be an (n-1)-ary symbol from τ ; the case that R has a smaller arity can be dealt with by adding dummy variables. Note that in the following we use the assumption that n is strictly larger than the maximal arity of \mathfrak{D} ; intuitively, we implement Leibniz' law for equality. We have $R(e(u_{i_1}), \ldots, e(u_{i_p}))$ if and only if there are $v'_{i_1} \in e(u_{i_1}), \ldots, v'_{i_p} \in e(u_{i_p})$ so that $R(v'_{i_1}, \ldots, v'_{i_p})$ holds in \mathfrak{E} . Since the formula $R(x_1, \ldots, x_p) \wedge x_p = x_{p+1}$ is contained in

$$\Lambda\left(\operatorname{tp}^{(\mathfrak{D},t^1)}(v'_{i_1}[1],\ldots,v'_{i_p}[1],v_n[1]),\ldots,\operatorname{tp}^{(\mathfrak{D},t^d)}(v'_{i_1}[d]),\ldots,v'_{i_p}[d],v_{i_n}[d])\right)\,,$$

this type of \mathfrak{D} must also contain $R(x_1, \ldots, x_{p-1}, x_{p+1})$. In this way we argue successively for all arguments of R, and finally obtain that $R(v_{i_1}, \ldots, v_{i_p})$ holds in \mathfrak{E} . The argument can be reverted, and we have that e' is a strong homomorphism. We conclude that e' is an embedding.

Observe that the mapping g from D^d to D that maps \bar{u} to $f(\bar{u}/_{\sim})$ is a polymorphism of \mathfrak{B} , by the preservation constraints, and it is canonical by construction. By the violation constraints, g violates R_0 , and hence is the desired witness.

Analogously to the proof of this theorem, one can show the following.

Theorem 8.4.5 (from [54]). Let $\mathfrak C$ be with finite relational signature, and first-order definable over a structure $\mathfrak D$ which is ordered, homogeneous, Ramsey, finitely bounded, and with finite relational signature. Then $\operatorname{Expr-ex}(\mathfrak C)$ and $\operatorname{Expr-ep}(\mathfrak C)$ are decidable.

An important open problem is whether the method can be extended to show decidability of $\operatorname{Expr-fo}(\mathfrak{C})$, under the same assumptions on \mathfrak{C} as in Theorems 8.4.4 and 8.4.5. By the theorem of Ryll-Nardzewski, first-order definability is characterized by preservation under automorphisms, i.e., surjective self-embeddings. But the requirement of surjectivity is difficult to deal with in our approach.

QUESTION 8.2. Let \mathfrak{B} be with finite relational signature, and definable in a structure \mathfrak{C} which is ordered, homogeneous, Ramsey, finitely bounded, and with finite relational signature. Is Expr-fo(\mathfrak{B}) decidable?

While the conditions of Theorem 8.4.4 might appear rather restrictive at first sight, they actually are quite general: we want to point out that we only require that $\mathfrak C$ is first-order definable over an ordered, homogeneous, Ramsey, and finitely bounded structure, rather than requiring that $\mathfrak C$ itself to have these properties. We do not know of a single homogeneous structure $\mathfrak C$ with finite relational signature which does not satisfy the conditions of Theorems 8.4.4 and 8.4.5. Examples of structures $\mathfrak C$ that do satisfy the assumptions, and the corresponding references, see Section 8.1.

We finally show that the assumption in Theorem 8.4.4 of ${\mathfrak C}$ being finitely bounded is necessary.

PROPOSITION 8.4.6. There exists a homogeneous ordered Ramsey structure \mathfrak{C} with finite relational signature such that $\operatorname{Expr-fo}(\mathfrak{C})$, $\operatorname{Expr-pp}(\mathfrak{C})$, $\operatorname{Expr-ep}(\mathfrak{C})$, and $\operatorname{Expr-ex}(\mathfrak{C})$ is undecidable.

PROOF. Recall the definition of *Henson digraphs* from Example 3.2.7. When \mathcal{C}' is the age of a Henson digraph \mathfrak{C}' , then the class \mathcal{C} consisting of all structures obtained from the digraphs in \mathcal{C}' by adding an arbitrary linear order on the vertices, is again

an amalgamation class. In fact, it is even a Ramsey class by the results described in Example 8.1.9. Let $\mathfrak C$ denote its Fraïssé-limit.

We first show that non-isomorphic Henson digraphs \mathfrak{C}_1 and \mathfrak{C}_2 have distinct Expr-pp problems. In fact, we show the existence of a first-order formula ϕ_1 over digraphs such that the input with $\phi_0 := E(x,y)$ and ϕ_1 is a yes-instance of Expr-pp(\mathfrak{C}_1) and a no-instance of Expr-pp(\mathfrak{C}_2), or vice versa. Since there are uncountably many Henson digraphs, but only countably many algorithms, this clearly shows the existence of Henson digraphs \mathfrak{C}' such that Expr-pp(\mathfrak{C}') is undecidable. It follows that for the ordered Ramsey structure \mathfrak{C} described above the problem Expr-pp(\mathfrak{C}) is undecidable as well.

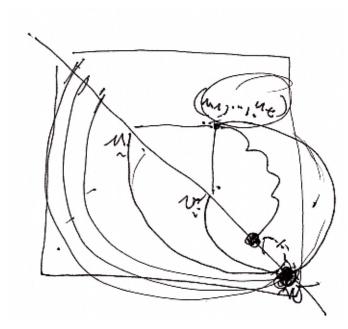
Since \mathfrak{C}_1 and \mathfrak{C}_2 are non-isomorphic, there must be a structure \mathfrak{A} that embeds to \mathfrak{C}_1 but not to \mathfrak{C}_2 , or that embeds to \mathfrak{C}_2 but not to \mathfrak{C}_1 . Assume the former is the case; in the latter, simply exchange \mathfrak{C}_1 and \mathfrak{C}_2 . Let s be the number of elements of \mathfrak{A} , and denote the elements by a_1, \ldots, a_s . Let ψ be the formula with variables x_1, \ldots, x_s that has for distinct $i, j \leq s$ a conjunct $E(x_i, x_j)$ if $E(a_i, a_j)$ holds in \mathfrak{A} , and a conjunct $\neg E(x_i, x_j) \wedge x_i \neq x_j$ otherwise. Let ϕ be the formula $\psi \Rightarrow E(x_{s+1}, x_{s+2})$.

Let C_1 be the domain of \mathfrak{C}_1 , and consider the relation $R_1 \subseteq (C_1)^{s+2}$ defined by ϕ in \mathfrak{C}_1 . Let R be a relation symbol of arity s+2, and \mathfrak{B} be the structures with signature $\{R\}$, domain C_1 , and where R denotes the relation R_1 . It is clear that $\exists x_1, \ldots, x_s. R(x_1, \ldots, x_s, x, y)$ is a primitive positive definition of E(x, y) in \mathfrak{B} .

Now consider the relation R_2 defined by ϕ in \mathfrak{C}_2 over the domain C_2 . Since \mathfrak{A} does not embed into \mathfrak{C}_2 , the precondition of ϕ is never satisfied, and the relation R_2 is empty. Hence, the structure $(C_2; R_2)$ is preserved by all permutations. But the relation E(x, y) is certainly not first-order definable over a structure that is preserved by all permutations.

CHAPTER 9

Schaefer's Theorem for Graphs



Jaroslav Nešetřil, 2004

This chapter is based on results from [51, 52, 193].

9.1. Motivation and the Result

In an influential paper in 1978, Schaefer [182] classified the computational complexity of $CSP(\mathfrak{B})$ for all structures \mathfrak{B} over a two-element universe; the result and a simple proof have been presented in Section 5.4. Schaefer's theorem can be viewed as a complexity classification for systematic syntactic restrictions of the Boolean satisfiability problem, as reflected in the following formulation of the result.

Let $\Psi = \{\psi_1, \dots, \psi_n\}$ be a finite set of propositional (Boolean) formulas.

Boolean-SAT (Ψ)

INSTANCE: Given a finite set of variables W and a propositional formula of the form $\Phi = \phi_1 \wedge \cdots \wedge \phi_l$ where each ϕ_i for $1 \leq i \leq l$ is obtained from one of the formulas ψ in Ψ by substituting the variables of ψ by variables from W.

QUESTION: Is there a satisfying Boolean assignment to the variables of W (equivalently, those of Φ)?

Schaefer's theorem (Theorem 5.4.3) states that Boolean-SAT(Ψ) can be solved in polynomial time if and only if Ψ is a subset of one of six sets of Boolean formulas

(called θ -valid, 1-valid, Horn, dual-Horn, affine, and bijunctive), and is NP-complete otherwise.

We prove a similar classification result, but for the first-order logic of graphs instead for propositional logic. More precisely, let E be a relation symbol which denotes an antireflexive and symmetric binary relation and hence stands for the edge relation of a (simple, undirected) graph. We consider formulas that are constructed from atomic formulas of the form E(x,y) and x=y by the usual boolean connectives (negation, conjunction, disjunction), and call formulas of this form graph formulas. A graph formula $\Phi(x_1, \ldots, x_m)$ is satisfiable if there exists a graph H and an m-tuple a of elements in H such that $\Phi(a)$ holds in H.

The problem to decide whether a given graph formula is satisfiable can be very difficult. For example the question whether the Ramsey number R(5,5) is larger than 43 (which is an open question, see e.g. [94]) can be easily formulated in terms of satisfiability of a single graph formula. Recall that R(5,5) is the least number k such that every graph with at least k vertices either contains a clique of size 5 or an independent set of size 5. So the question whether R(5,5) is greater than 43 can be formulated using 43 variables x_1, \ldots, x_{43} by imposing the constraints that all variables denote a different vertex in the graph, and by imposing for every subset of the variables of cardinality five a constraint that forbids that the corresponding five variables form a clique or an independent set. If there is a solution to this instance, then this implies that R(5,5) > 43, and otherwise $R(5,5) \le 43$.

Similarly as in Schaefer's theorem, we systematically investigate restrictions of the satisfiability problem for graph formulas that can be solved in polynomial time. Let $\Psi = \{\psi_1, \dots, \psi_n\}$ be a finite set of graph formulas. Then Ψ gives rise to the following computational problem.

Graph-SAT(Ψ)

INSTANCE: Given a set of variables W and a graph formula of the form $\Phi = \phi_1 \wedge \cdots \wedge \phi_l$ where each ϕ_i for $1 \leq i \leq l$ is obtained from one of the formulas ψ in Ψ by substituting the variables from ψ by variables from W.

QUESTION: Is Φ satisfiable?

Example 9.1.1. Let Ψ be the set that just contains the formula

$$(E(x,y) \land \neg E(y,z) \land \neg E(x,z))$$

$$\lor (\neg E(x,y) \land E(y,z) \land \neg E(x,z))$$

$$\lor (\neg E(x,y) \land \neg E(y,z) \land E(x,z)).$$

Then Graph-SAT(Ψ) is the problem of deciding whether there exists a graph such that certain prescribed subsets of its vertex set of cardinality at most three induce subgraphs with exactly one edge. The problem Graph-SAT(Ψ) is NP-complete.

Example 9.1.2. There are also many interesting tractable Graph-SAT problems, for instance when Ψ consists of the formula

$$(E(x,y) \land \neg E(y,z) \land \neg E(x,z))$$

$$\lor (\neg E(x,y) \land E(y,z) \land \neg E(x,z))$$

$$\lor (\neg E(x,y) \land \neg E(y,z) \land E(x,z))$$

$$\lor (E(x,y) \land E(y,z) \land E(x,z)).$$

It is obvious that the problem Graph-SAT(Ψ) is for all Ψ contained in NP. The goal of this chapter is to prove the following dichotomy result.

THEOREM 9.1.3. For all Ψ , the problem Graph-SAT(Ψ) is NP-complete or in P. Moreover, the problem to decide for given Ψ whether Graph-SAT(Ψ) is NP-complete or in P is decidable.

We establish our result by translating Graph-SAT problems into CSPs. More specifically, for every set of formulas Ψ we present a relational structure \mathfrak{B}_{Ψ} such that Graph-SAT(Ψ) is equivalent to CSP(\mathfrak{B}_{Ψ}). The relational structure \mathfrak{B}_{Ψ} has a first-order definition in the random graph (\mathbb{V} ; E) introduced in Chapter 3. This perspective allows us to use polymorphisms to classify the computational complexity of Graph-SAT problems as outlined in Chapter 5. Our proof also relies crucially on strong results from structural Ramsey theory. Following the technique from Chapter 8, we use such results to find regular patterns in the behavior of polymorphisms of structures on (\mathbb{V} ; E), which in turn allows us to find analogies with polymorphisms of structures on Boolean domains. Our dichotomy result can be stated as follows.

THEOREM 9.1.4. Let \mathfrak{B} be a relational structure with a first-order definition in $(\mathbb{V}; E)$. Then exactly one of the following two statements applies.

- (1) there is a primitive positive interpretation of all finite structures in the model-complete core of \mathfrak{B} . In this case, \mathfrak{B} has a finite-signature reduct with an NP-hard CSP, by Corollary 5.5.7.
- (2) \mathfrak{B} has a cyclic polymorphism f modulo an endomorphism, i.e., there are $f \in \operatorname{Pol}(\mathfrak{B})$ and $e \in \operatorname{End}(\mathfrak{B})$ satisfying

$$f(x_1,\ldots,x_n)=e(f(x_2,\ldots,x_n,x_1))$$

for all $x, y \in \mathbb{V}$. In this case, every finite-signature reduct of \mathfrak{B} has a polynomial-time tractable CSP.

The proof of this theorem can be found at the end of Section 9.8. The algorithmic part of Theorem 9.1.4 is obtained by combinations of ideas from Section 6.3 and reductions to the tractable cases of Schaefer's theorem (Theorem 5.4.3).

In the remainder of this chapter, \mathfrak{B} denotes a relational structure with a first-order definition in the random graph $(\mathbb{V}; E)$. Since all the polymorphism clones of this chapter contain the automorphism group $\operatorname{Aut}(\mathbb{V}; E)$ of the random graph, we also make the following convention, which exclusively holds for this chapter. For a set of functions F and a function g on the domain \mathbb{V} , we say that F generates g when $F \cup \operatorname{Aut}(\mathbb{V}; E)$ locally generates g; also, we say that a function f generates g if f generates g. That is, in this paper we consider the automorphisms of $(\mathbb{V}; E)$ be present in all sets of functions when speaking about the local generating process.

9.2. Endomorphisms

The goal of this section is the proof of Proposition 9.2.2, which allows us to reduce the classification task for \mathfrak{B} to the classification of those structures \mathfrak{B} where the $\operatorname{Aut}(\mathfrak{B})$ is dense in $\operatorname{End}(\mathfrak{B})$; moreover, we give a description of the five possible automorphism groups that can appear, due to Thomas [193].

We write N for the relation $\{(x,y) \in \mathbb{V}^2 \mid x \neq y \land \neg E(x,y)\}$. Note that $(\mathbb{V};N)$ is the complement of the graph $(\mathbb{V};E)$, and that $(\mathbb{V};N)$ is isomorphic to $(\mathbb{V};E)$ (it is straightforward to verify the extension property). Let - be such an isomorphism. For any finite subset S of \mathbb{V} , if we flip edges and non-edges between S and $\mathbb{V} \setminus S$ in $(\mathbb{V};E)$, then the resulting graph is isomorphic to $(\mathbb{V};E)$ (it is straightforward to verify the extension property). For any non-empty set S, we write S for such an isomorphism. Note that when S and S are two finite non-empty subsets of S, then S and S are two finite non-empty subsets of S, where S is any fixed element of S.

DEFINITION 9.2.1 ($R^{(k)}$ and $S^{(k)}$). Let $R^{(k)}$ ($S^{(k)}$) be the k-ary relation that holds on $x_1, \ldots, x_k \in \mathbb{V}$ if x_1, \ldots, x_k are pairwise distinct, and the number of edges between these k vertices is odd (even).

Observe that $R^{(3)}$ and $R^{(4)}$ are preserved by sw, that $R^{(4)}$ and $R^{(4)}$ are preserved by -, and that $R^{(5)}$ and $S^{(5)}$ are preserved by - and by sw, but not by all permutations of \mathbb{V} .

PROPOSITION 9.2.2. Let \mathfrak{B} be first-order definable in $(\mathbb{V}; E)$. Then at least one of the following holds.

- (a) The endomorphisms of \mathfrak{B} are generated by $\operatorname{Aut}(\mathbb{V}; E)$.
- (b) \mathfrak{B} has a constant endomorphism. In this case $CSP(\mathfrak{B})$ is trivial.
- (c) \mathfrak{B} is homomorphically equivalent to a countably infinite structure that is preserved by all permutations of its domain. Such structures have been classified in Chapter 6.
- (d) The endomorphisms of \mathfrak{B} are precisely the functions generated by sw; equivalently, $\operatorname{End}(\mathfrak{B}) = \operatorname{End}(\mathbb{V}; R^{(3)}, S^{(3)})$.
- (e) The endomorphisms of \mathfrak{B} are precisely the functions generated by -; equivalently, $\operatorname{End}(\mathfrak{B}) = \operatorname{End}(\mathbb{V}; R^{(4)}, S^{(4)})$.
- (f) The endomorphisms of \mathfrak{B} are precisely the functions generated by $\{-, sw\}$; equivalently, $\operatorname{End}(\mathfrak{B}) = \operatorname{End}(\mathbb{V}; R^{(5)}, S^{(5)})$.

To prove the proposition, we first cite a result about structures definable over the random graph due to Thomas [194]; the formulation of the result presented here first appeared in [51]. The graph ($\mathbb{V}; E$) contains all countable graphs as induced subgraphs. In particular, it contains an infinite complete subgraph, denoted by K_{ω} . It is clear that any two injective operations from $\mathbb{V} \to \mathbb{V}$ whose images induce K_{ω} in ($\mathbb{V}; E$) generate one another. Let e_E be one such operation. Similarly, ($\mathbb{V}; E$) contains an infinite independent set, denoted by I_{ω} . Let e_N be an injective operation from $\mathbb{V} \to \mathbb{V}$ whose image induces I_{ω} in ($\mathbb{V}; E$).

THEOREM 9.2.3 (of [51,194]). Let \mathfrak{B} be first-order definable in $(\mathbb{V}; E)$. Then one of the following cases applies.

- (1) B has a constant endomorphism.
- (2) \mathfrak{B} has the endomorphism e_E .
- (3) \mathfrak{B} has the endomorphism e_N .
- (4) The endomorphisms of \mathfrak{B} are locally generated by the automorphisms of \mathfrak{B} .

COROLLARY 9.2.4 (from [51]). All relational structures \mathfrak{B} with a first-order definition in $(\mathbb{V}; E)$ are model-complete.

PROOF. By Theorem 3.6.7, an ω -categorical structure \mathfrak{B} is model-complete if and only if $\operatorname{Aut}(\mathfrak{B})$ is dense in the monoid \mathscr{M} of self-embeddings of \mathfrak{B} . We apply Theorem 9.2.3 to \mathscr{M} , which, as a closed monoid containing $\operatorname{Aut}((\mathbb{V};E))$, is also an endomorphism monoid of a structure \mathfrak{B}' with a first-order definition in $(\mathbb{V};E)$. Clearly, \mathfrak{B}' and \mathfrak{B} have the same automorphisms, namely those permutations in \mathscr{M} whose inverse is also in \mathscr{M} . Therefore we are done if the last case of Theorem 9.2.3 holds. Note that \mathscr{M} cannot contain a constant operation as all its operations are injective. So suppose that \mathscr{M} contains e_N – the argument for e_E is analogous. Let R be any relation of \mathfrak{B} , and ϕ_R be its defining quantifier-free formula; ϕ_R exists since $(\mathbb{V};E)$ has quantifier-elimination. Let ψ_R be the formula obtained by replacing all occurrences of E by false; so ψ_R is a formula over the empty language. Then a tuple a satisfies ϕ_R in $(\mathbb{V};E)$ iff $e_N(a)$ satisfies ϕ_R in $(\mathbb{V};E)$ (because e_N is an embedding) if and only if $e_N(a)$ satisfies ψ_R in $(\mathbb{V};E)$ (as there are no edges on $e_N(a)$) if and only if

 $e_N(a)$ satisfies ψ_R in the substructure induced by $e_N[\mathbb{V}]$ (since ψ_R does not contain any quantifiers). Thus, \mathfrak{B} is isomorphic to the structure on $e_N[\mathbb{V}]$ which has the relations defined by the formulas ψ_R . Therefore, \mathfrak{B} is isomorphic to a structure with a first-order definition over the empty signature. This structure has, of course, all injections as self-embeddings, and all permutations as automorphisms, and hence is model-complete; the same is true for \mathfrak{B} .

The last case in Theorem 9.2.3 splits into five sub-cases, corresponding to the five locally closed permutation groups that contain $\operatorname{Aut}((\mathbb{V}; E))$ exhibited by Thomas [193]

DEFINITION 9.2.5. For all $k \geq 3$, let $P^{(k)}$ denote the k-ary relation that holds on $x_1, \ldots, x_k \in \mathbb{V}$ if x_1, \ldots, x_k are pairwise distinct, and the graph induced by $\{x_1, \ldots, x_k\}$ in $(\mathbb{V}; E)$ is neither an independent set nor a clique.

THEOREM 9.2.6 (of [193]). Let \mathfrak{B} be a relational structure with a first-order definition in $(\mathbb{V}; E)$. Then exactly one of the following is true.

- (1) \mathfrak{B} is first-order interdefinable with $(\mathbb{V}; E)$; equivalently, $\operatorname{Aut}(\mathfrak{B}) = \operatorname{Aut}((\mathbb{V}; E))$.
- (2) \mathfrak{B} is first-order interdefinable with $(\mathbb{V}; R^{(4)})$; equivalently, \mathfrak{B} is preserved by -, but not by sw.
- (3) \mathfrak{B} is first-order interdefinable with $(\mathbb{V}; R^{(3)})$; equivalently, \mathfrak{B} is preserved by sw, but not by –.
- (4) \mathfrak{B} is first-order interdefinable with $(\mathbb{V}; R^{(5)})$; equivalently, \mathfrak{B} is preserved by and by sw, but not by all permutations of \mathbb{V} .
- (5) \mathfrak{B} is first-order interdefinable with $(\mathbb{V};=)$; equivalently, \mathfrak{B} is preserved by all permutations of \mathbb{V} .

We are now ready to prove Proposition 9.2.2.

PROOF OF PROPOSITION 9.2.2. Theorem 9.2.3 states that \mathfrak{B} has a constant endomorphism, or the endomorphism e_E , or the endomorphism e_N , or all endomorphisms of \mathfrak{B} are generated by its automorphisms. If \mathfrak{B} has a constant endomorphism we are in Case (b) and done. If \mathfrak{B} has the endomorphisms e_E or e_N , then we are in Case (c) since $e_E[\mathbb{V}]$ and $e_N[\mathbb{V}]$ induce structures in $(\mathbb{V}; E)$ which are invariant under all permutations of their domain. So assume in the following that \mathfrak{B} has neither e_E , nor e_N , nor a constant as an endomorphism, and that all endomorphisms of \mathfrak{B} are generated by $\operatorname{Aut}(\mathfrak{B})$. The statement now follow from Theorem 9.2.6.

Proposition 9.2.2 allows us to focus in the following on the situation where $Aut(\mathfrak{B})$ is dense in $End(\mathfrak{B})$. Moreover, there are only five possibilities for $Aut(\mathfrak{B})$, and the case $Aut(\mathfrak{B}) = Aut(\mathbb{V}; =)$ has already been solved.

9.3. First-order Expansions of $(\mathbb{V}; E, N)$

We remark that since $(\mathbb{V}; E)$ has only binary relations, a function $f \colon \mathbb{V}^k \to \mathbb{V}$ is canonical if and only if it is 2-canonical (Section 5.6.2). The polymorphisms that imply tractability of $\mathrm{CSP}(\mathfrak{B})$ will be canonical with respect to $(\mathbb{V}; E)$. We now define different behaviors that some of these canonical functions might have. For $Q_1, \ldots, Q_k \in \{E, N, =, \neq\}$, we will in the following write $Q_1 \cdots Q_k$ for the binary relation on \mathbb{V}^k that holds between two k-tuples $x, y \in \mathbb{V}^k$ iff $Q_i(x_i, y_i)$ holds for all $1 \leq i \leq k$.

Definition 9.3.1. We say that a binary injective operation $f: \mathbb{V}^2 \to \mathbb{V}$ is

• balanced in the first argument if for all $u, v \in \mathbb{V}^2$ we have that E=(u, v) implies E(f(u), f(v)) and N=(u, v) implies N(f(u), f(v));

- balanced in the second argument if $(x,y) \mapsto f(y,x)$ is balanced in the first argument;
- balanced if f is balanced in both arguments, and unbalanced otherwise;
- E-dominated (N-dominated) in the first argument if for all $u, v \in \mathbb{V}^2$ with $\neq = (u, v)$ we have that E(f(u), f(v)) (N(f(u), f(v)));
- E-dominated (N-dominated) in the second argument if $(x, y) \mapsto f(y, x)$ is E-dominated (N-dominated) in the first argument;
- E-dominated (N-dominated) if it is E-dominated (N-dominated) in both arguments;
- of type min if for all $u, v \in \mathbb{V}^2$ with $\neq \neq (u, v)$ we have E(f(u), f(v)) if and only if EE(u, v);
- of type max if for all $u, v \in \mathbb{V}^2$ with $\neq \neq (u, v)$ we have N(f(u), f(v)) if and only if NN(u, v);
- of type p_1 if for all $u, v \in \mathbb{V}^2$ with $\neq \neq (u, v)$ we have E(f(u), f(v)) if and only if $E(u_1, v_1)$;
- of type p_2 if $(x,y) \mapsto f(y,x)$ is of type p_1 ;
- of type projection if it is of type p_1 or p_2 .

Note that, for example, being of type max is a behavior of binary functions that does not force a function to be canonical, since the condition only talks about certain types of pairs in \mathbb{V}^2 , but not all such types; however, being of type max and balanced does mean that a function is canonical. The next definition contains some important behaviors of ternary functions.

Definition 9.3.2. An injective ternary function $f: \mathbb{V}^3 \to \mathbb{V}$ is of type

- majority if for all $u, v \in \mathbb{V}^3$ we have that E(f(u), f(v)) if and only if EEE(u, v), EEN(u, v), ENE(u, v), or NEE(u, v);
- minority if for all $x, y \in \mathbb{V}^3$ we have E(f(x), f(y)) if and only if EEE(u, v), NNE(u, v), NEN(u, v), or ENN(u, v).

While the polynomial-time tractability results of this section will be shown by means of a number of different canonical functions, all hardness cases will be established by the following single relation.

Definition 9.3.3. We define the 6-ary relation $H_1(x_1, y_1, x_2, y_2, x_3, y_3)$ on \mathbb{V} by

$$\bigwedge_{i,j \in \{1,2,3\}, i \neq j, u \in \{x_i, y_i\}, v \in \{x_j, y_j\}} N(u, v)$$

$$\wedge \left((E(x_1, y_1) \wedge N(x_2, y_2) \wedge N(x_3, y_3)) \right)$$

$$\vee \left(N(x_1, y_1) \wedge E(x_2, y_2) \wedge N(x_3, y_3) \right)$$

$$\vee \left(N(x_1, y_1) \wedge N(x_2, y_2) \wedge E(x_3, y_3) \right).$$
(9)

The goal of this section is to prove the following proposition.

PROPOSITION 9.3.4. Let $\mathfrak{B} = (\mathbb{V}; E, N, \neq, ...)$ be first-order definable in $(\mathbb{V}; E)$. Then at least one of the following holds:

- (a) There is a primitive positive definition of H_1 in \mathfrak{B} . In this case, $CSP(\mathfrak{B})$ is NP-complete.
- (b) \mathfrak{B} has a canonical polymorphism of type minority, as well as a canonical binary injection which of type p_1 and E-dominated or N-dominated in the second argument. In this case, $CSP(\mathfrak{B})$ is polynomial-time tractable.
- (c) \mathfrak{B} has a canonical polymorphism of type majority, as well as a canonical binary injection which of type p_1 and E-dominated or N-dominated in the second argument. In this case, $CSP(\mathfrak{B})$ is polynomial-time tractable.

- (d) \mathfrak{B} has a canonical polymorphism of type minority, as well as a canonical binary injection which is balanced and of type projection. In this case, $CSP(\mathfrak{B})$ is polynomial-time tractable.
- (e) \mathfrak{B} has a canonical polymorphism of type majority, as well as a canonical binary injection which is balanced and of type projection. In this case, $CSP(\mathfrak{B})$ is polynomial-time tractable.
- (f) \mathfrak{B} has a canonical polymorphism of type max or min. In this case, $CSP(\mathfrak{B})$ is polynomial-time tractable.

The remainder of this section contains the proof of Proposition 9.3.4, except for the polynomial-time tractability proofs, which will be given in Section 9.7. For the other statements of Proposition 9.3.4, we proceed as follows. In Section 9.3.1, we show that the relation H_1 is hard. In particular, a structure $\mathfrak{B} = (\mathbb{V}; E, N, \neq, \ldots)$ with a first-order definition in $(\mathbb{V}; E)$ must have an essential polymorphism, or has a finite reduct with an NP-hard CSP. In Section 9.3.2 we show when \mathfrak{B} has an essential polymorphism, it must also have a binary injective polymorphism. We finally prove in Section 9.3.4 that \mathfrak{B} has one of the polymorphisms listed in cases (b) to (f) of the proposition.

9.3.1. Hardness of H_1 . This section is devoted to case (a) of Proposition 9.3.4.

PROPOSITION 9.3.5. There is a primitive positive interpretation of ($\{0,1\}$; IIN3) in (\mathbb{V} ; H_1), and CSP(\mathbb{V} ; H_1) is NP-hard.

PROOF. We give a 2-dimensional interpretation I of $(\{0,1\}; 1IN3)$ in \mathfrak{B} . The domain formula is true. The formula $=_I (x_1, x_2, y_1, y_2)$ is

$$\exists z_1, z_2, u_1, u_2, v_1, v_2 \left(H_1(x_1, x_2, u_1, u_2, z_1, z_2) \land N(u_1, u_2) \right. \\ \left. \land H_1(z_1, z_2, v_1, v_2, y_1, y_2) \land N(v_1, v_2) \right)$$

This formula is equivalent to a primitive positive formula over \mathfrak{B} since N(x,y) is primitive positive definable by H_1 . The formula $1\mathrm{IN}3_I(x_1,x_2,y_1,y_2,z_1,z_2)$ is

$$\exists x'_1, x'_2, y'_1, y'_2, z'_1, z'_2 \left(H_1(x'_1, x'_2, y'_1, y'_2, z'_1, z'_2) \right.$$

$$\land =_I \left(x_1, x_2, x'_1, x'_2 \right) \land =_I \left(y_1, y_2, y'_1, y'_2 \right) \land =_I \left(z_1, z_2, z'_1, z'_2 \right) \right)$$

The coordinate map sends a tuple (x_1, x_2) to 1 if $E(x_1, x_2)$ and to 0 otherwise. The second part of the statement follows from Corollary 5.5.7.

9.3.2. Producing binary injections. We now show that if a structure $\mathfrak{B} = (\mathbb{V}; E, N, \neq, \ldots)$ with a first-order definition in $(\mathbb{V}; E)$ has an essential polymorphism, then \mathfrak{B} must also have a binary injective polymorphism. This is in particular the case when the relation H_1 from the previous section is not primitive positive definable in \mathfrak{B} : since E and N are among the relations of \mathfrak{B} , and since any essentially unary polymorphism preserving both E and N preserves all relations with a first-order definition in $(\mathbb{V}; E)$, we have that the polymorphism violating H_1 must be essential.

THEOREM 9.3.6 (from [51]). Let $\mathfrak{B} = (\mathbb{V}; E, N, \neq, ...)$ be first-order definable in $(\mathbb{V}; E)$, and suppose that \mathfrak{B} has an essential polymorphism. Then \mathfrak{B} also has a binary injective polymorphism.

PROOF. Let $f: \mathbb{V}^k \to \mathbb{V}$ be an essential polymorphism of \mathfrak{B} of minimal arity. By Lemma 5.3.10, f must be binary. Hence, we may apply Lemma 6.1.3 to \mathfrak{B} , and in order to show that \mathfrak{B} is preserved by a binary injection, it suffices to show that if ϕ is a primitive positive formula over \mathfrak{B} such that both $\phi \land x \neq y$ and $\phi \land s \neq t$ are satisfiable over \mathfrak{B} , then $\phi \land x \neq y \land s \neq t$ is satisfiable over \mathfrak{B} as well. The proof follows the idea of the proof of Theorem 6.2.1.

Let ϕ be a primitive positive formula over the signature of \mathfrak{B} such that

- there is a tuple t_1 that satisfies $\phi \land x \neq y$
- there is a tuple t_2 that satisfies $\phi \land s \neq t$.

Let a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 be the values for x, y, s, t in t_1 and t_2 , respectively. We have $a_1 \neq a_2$ and $b_3 \neq b_4$. We want to show that $\phi \wedge x \neq y \wedge s \neq t$ is satisfiable over \mathfrak{B} . Thus, if $a_3 \neq a_4$ or $b_1 \neq b_2$, there is nothing to show, and so we assume that $a_3 = a_4$ and $b_1 = b_2$.

We claim that there are automorphisms α, β of $(\mathbb{V}; E)$ such that in the tuple $t_3 := f(\alpha(t_1), \beta(t_2))$ the value of x is different from the value of y, and the value of s is different from the value of t. Then, since f preserves \mathfrak{B} , the tuple t_3 shows that $\phi \wedge x \neq y \wedge s \neq t$ is satisfiable over \mathfrak{B} , and concludes the proof.

To prove the claim, we will find tuples $c := (c_1, c_2, c_3, c_4)$ and $d := (d_1, d_2, d_3, d_4)$ of the same type as (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) , respectively, such that the tuple e := f(c, d) satisfies $e_1 \neq e_2$ and $e_3 \neq e_4$. Then, by the homogeneity of $(\mathbb{V}; E)$, we can find automorphisms α and β of $(\mathbb{V}; E)$ sending a to c and b to d, which suffices for the proof of our claim.

In the sequel, we will assume that $X(a_1, a_2)$ and $Y(b_3, b_4)$, where $X, Y \in \{E, N\}$.

Case 1. Suppose first that $a_3 = a_4 \in \{a_1, a_2\}$ and $b_1 = b_2 \in \{b_3, b_4\}$; without loss of generality, $a_3 = a_2$ and $b_1 = b_3$.

Case 1.1 There exists $u \in \mathbb{V}$ such that for all $p, v \in \mathbb{V}$ with $(u, v) \in Y$ we have f(p, u) = f(p, v). Then, because f preserves \neq , we have $f(p, u) \neq f(q, u)$ for all $p \neq q$. If for all $p, v \in \mathbb{V}$ we have that f(p, u) = f(p, v), then this implies that for all $p, v, v' \in \mathbb{V}$ we have that f(p, v') = f(p, v), contradicting our assumption that f is essential. So there are $p, v \in \mathbb{V}$ such that $f(p, u) \neq f(p, v)$. Pick $w \in \mathbb{V}$ such that $(w, u), (w, v) \in Y$. Pick moreover $q \in \mathbb{V}$ such that $(p, q) \in X$. We have $f(p, v) \neq f(p, u) = f(p, w)$. Moreover, $f(p, w) = f(p, u) \neq f(q, u) = f(q, w)$. Hence, the tuples c := (q, p, p, p) and d := (w, w, w, v) prove the claim.

Case 1.2 For all $u \in \mathbb{V}$ there exist $p, v \in \mathbb{V}$ with $(u, v) \in Y$ such that $f(p, u) \neq f(p, v)$. Pick $m, n, u \in \mathbb{V}$ with $(m, n) \in X$ and $f(m, u) \neq f(n, u)$. Pick $p, v \in \mathbb{V}$ such that $(u, v) \in Y$ and $f(p, u) \neq f(p, v)$. If we can pick p in such a way that $(p, m), (p, n) \in X$, then since either $f(m, u) \neq f(p, u)$ or $f(n, u) \neq f(p, u)$ we have that either (m, p, p, p) or (n, p, p, p) proves the claim together with the tuple (u, u, u, v). So suppose that this is impossible. Then for any $q \in \mathbb{V}$ with $(q, m), (q, n) \in X$ we have $f(q, u) = f(q, v) \neq f(p, u)$, so we have that (q, p, p, p) and (u, u, u, v) satisfy the claim.

Case 2. Now suppose that $a_3 = a_4 \in \{a_1, a_2\}$ and $b_1 = b_2 \notin \{b_3, b_4\}$; wlog $a_3 = a_2$. Write $(b_1, b_3) \in Q_3$ and $(b_1, b_4) \in Q_4$, where $Q_3, Q_4 \in \{E, N\}$.

Case 2.1 There exists $u \in \mathbb{V}$ such that for all p, v, r with $(v, r) \in Y$, $(u, v) \in Q_3$ and $(u, r) \in Q_4$ we have f(p, v) = f(p, r). Then one easily concludes that for all $p \in \mathbb{V}$ and all $v, v' \in \mathbb{V}$ with $v, v' \neq u$ we have f(p, v) = f(p, v'). This implies that $f(p, v) \neq f(q, v)$ whenever $p \neq q$ and $v \neq u$. Since f is essential, there exist $p, v \in \mathbb{V}$ with $(u, v) \in Y$ such that $f(p, u) \neq f(p, v)$. Now pick $w, q \in \mathbb{V}$ such that $(w, u) \in Q_3$, $(w, v) \in Q_4$, and $(q, p) \in X$. Then $f(p, w) \neq f(q, w)$, and so the tuples (q, p, p, p) and (w, w, u, v) prove the claim.

Case 2.2 For all u there exist p, v, r with $(v, r) \in Y$, $(u, v) \in Q_3$, $(u, r) \in Q_4$ and $f(p, v) \neq f(p, r)$. Pick m, n, u with $(m, n) \in X$ and $f(m, u) \neq f(n, u)$. Pick $p, v, r \in \mathbb{V}$ such that $(v, r) \in Y$, $(u, v) \in Q_3$, $(u, r) \in Q_4$ and $f(p, v) \neq f(p, r)$. If we can pick p in such a way that $(p, m), (p, n) \in X$, then either (m, p, p, p) and (u, u, v, r) or (n, p, p, p) and (u, u, v, r) prove the claim. So suppose that this is impossible. Then for any q with $(q, m), (q, n) \in X$ and all $v, r \in \mathbb{V}$ with $(v, r) \in Y$, $(u, v) \in Q_3$, $(u, r) \in Q_4$

we have f(q,v) = f(q,r). This implies that for all such q and all $v,v' \neq u$ we have f(q,v) = f(q,v'). Pick w such that $(w,v) \in Q_3$, $(w,r) \in Q_4$. Pick q such that $(q,p) \in X$. We have $f(q,w) \neq f(p,w)$, and so (q,p,p,p) and (w,w,v,r) prove the claim.

Case 3. To finish the proof, suppose that $a_3 = a_4 \notin \{a_1, a_2\}$ and $b_1 = b_2 \notin \{b_3, b_4\}$. Write $(a_3, a_1) \in P_1$, $(a_3, a_2) \in P_2$, $(b_1, b_3) \in Q_3$ and $(b_1, b_4) \in Q_4$, where $P_i, Q_i \in \{E, N\}$.

Case 3.1 There exists u such that for all p, v, r with $(v, r) \in Y$, $(u, v) \in Q_3$ and $(u, r) \in Q_4$ we have f(p, v) = f(p, r). Then one easily concludes that for all $p \in \mathbb{V}$ and all $v, v' \in \mathbb{V}$ with $v, v' \neq u$ we have f(p, v) = f(p, v'). This implies that $f(p, v) \neq f(q, v)$ whenever $p \neq q$ and $v \neq u$. We claim that there exist p, v with $(u, v) \in Y$ such that $f(p, u) \neq f(p, v)$. Otherwise, if f(p, u) = f(p, v) for all p, \mathbb{V} , then f(p, v) = f(p, v') for all p, v, v', and f depends only on its first variable, contradicting the assumption that f is essential. Now pick w, m, n such that $(w, u) \in Q_3$, $(w, v) \in Q_4$, $(m, n) \in X$, $(m, p) \in P_1$, and $(n, p) \in P_2$. Then the tuples (m, n, p, p) and (w, w, u, v) prove the claim.

Case 3.2 For all u there exist p, v, r with $(v, r) \in Y$, $(u, v) \in Q_3$, $(u, r) \in Q_4$ and $f(p, v) \neq f(p, r)$. Pick m, n, u with $(m, n) \in X$ and $f(m, u) \neq f(n, u)$. Pick p, v, r such that $(v, r) \in Y$, $(u, v) \in Q_3$, $(u, r) \in Q_4$ and $f(p, v) \neq f(p, r)$. If we can pick p in such a way that $(p, m) \in P_1$ and $(p, n) \in P_2$, then (m, n, p, p) and (u, u, v, r) prove the claim, so suppose that this is impossible. Then for any q with $(q, m) \in P_1$ and $(q, n) \in P_2$ and all v, r with $(v, r) \in Y$, $(u, v) \in Q_3$, $(u, r) \in Q_4$ we have f(q, v) = f(q, r). This is easily seen to imply that for all such q and all $v, v' \neq u$ we have f(q, v) = f(q, v'). Pick w such that $(w, v) \in Q_3$, $(w, r) \in Q_4$, and $w \neq u$. Pick q, q' such that $(q, q') \in X$, $(q, p) \in P_1$ and $(q', p) \in P_2$. We have $f(q, w) \neq f(q', w)$, and thus (q, q', p, p) and (w, w, v, r) prove the claim.

9.3.3. Minimal Binary Functions. Let \mathscr{C} be the clone generated by $\operatorname{Aut}(\mathbb{V}; E)$. We know from Theorem 9.2.3 and Theorem 9.3.6 that all essential functions that are minimal above \mathscr{C} are binary, injective, and preserve both E and N. It is the goal of this section to determine these binary minimal functions. To state the main result, we define the *dual* of an operation f on $(\mathbb{V}; E)$, which can be imagined as the function obtained from f by exchanging the roles of E and N.

DEFINITION 9.3.7. The dual of a function $f(x_1, ..., x_n)$ on $(\mathbb{V}; E)$ is the function $-f(-x_1, ..., -x_n)$.

THEOREM 9.3.8 (from [51]). If $\mathfrak{B} = (\mathbb{V}; E, N, \neq, ...)$ is first-order definable in $(\mathbb{V}; E)$ and has an essential polymorphism, it must also have at least one of the following binary injective canonical polymorphisms.

- a balanced operation of type p_1 ;
- a balanced operation of type max;
- an E-dominated operation of type max;
- an E-dominated operation of type p_1 ;
- a binary operation of type p_1 that is balanced in the first and E-dominated in the second argument;

or one of the duals of the last four operations (the first operation is self-dual).

Our proof of Theorem 9.3.8 makes essential use of the Ramsey techniques from Chapter 8. As we have seen in Example 8.1.6, the class of all finite graphs is not a Ramsey class. However, the class of all finite ordered graphs is a Ramsey class (see Example 8.1.7 for a more general result). This class is clearly an amalgamation

class, and we denote its Fraïssé-limit by $(\mathbb{V}; E, <)$. Note that the reduct of this structure without the order has the extension property, and hence is isomorphic to the random graph. It therefore makes sense to use the same symbol \mathbb{V} for the elements of $(\mathbb{V}; E, <)$ and the elements of the random graph. Also note that the reduct $(\mathbb{V}; <)$ is isomorphic to $(\mathbb{Q}; <)$. By the argument above, the following is a direct consequence of Corollary 8.3.6.

COROLLARY 9.3.9. Every essential function that is minimal above the clone generated by $\operatorname{Aut}(\mathbb{V}; E)$ is a binary injection that is canonical as a function from $(\mathbb{V}; E, <)^2$ to $(\mathbb{V}; E, <)$.

In the rest of this section, canonical means canonical as a function from $(\mathbb{V}; E, <)^2$ to $(\mathbb{V}; E, <)$, and minimal means minimal as an operation above \mathscr{C} . The following behavior of functions from $(\mathbb{V}; E, <)^2 \to (\mathbb{V}; E, <)$ is useful to describe canonical functions.

DEFINITION 9.3.10. Let $f: \mathbb{V}^2 \to \mathbb{V}$, and let $\{R_1, R_2\} = \{E, N\}$. If for all $(x_1, x_2), (y_1, y_2) \in \mathbb{V}^2$ with $x_1 < y_1, x_2 < y_2, R_1(x_1, y_1), and R_2(x_2, y_2)$ we have

- $N(f(x_1, x_2), f(y_1, y_2))$, then we say that f behaves like min on input (<, <).
- $E(f(x_1, x_2), f(y_1, y_2))$, then we say that f behaves like max on input (<, <).
- $R_1(f(x_1,x_2),f(y_1,y_2))$, then we say that f behaves like p_1 on input (<,<).
- $R_2(f(x_1, x_2), f(y_1, y_2))$, then we say that f behaves like p_2 on input (<, <).

Analogously, we define behavior on input (<,>) using pairs $(x_1,x_2),(y_1,y_2) \in \mathbb{V}^2$ with $x_1 < y_1$ and $x_2 > y_2$.

Of course, we could also have defined "behavior on input (>, >)" and "behavior on input (>, <)"; however, behavior on input (>, >) equals behavior on input (<, <), and behavior on input (>, <) equals behavior on input (<, >) since graphs are symmetric. Thus, there are only two kinds of inputs to be considered, namely "straight input" (<, <) and "twisted input" (<, >).

PROPOSITION 9.3.11. Let $f: \mathbb{V}^2 \to \mathbb{V}$ be injective and canonical, and suppose it preserves E and N. Then it behaves like min, max, p_1 or p_2 on input (<,<). Moreover, it behaves like on min, max, p_1 or p_2 on input (<,>).

PROOF. By canonicity it suffices to check the statement for all possible types of pairs $x, y \in \mathbb{V}^2$.

We remark that the four possibilities correspond to the four binary operations g on the two-element domain $\{E, N\}$ that are *idempotent*, i.e., that satisfy that g(E, E) = E and g(N, N) = N.

DEFINITION 9.3.12. If $f: \mathbb{V}^2 \to \mathbb{V}$ behaves like X on input (<,<) and like Y on input (<,>), where $X,Y \in \{max, min, p_1, p_2\}$, then we say that f is of type X/Y.

Fix an automorphism \leftrightarrow of the graph $(\mathbb{V};E)$ that reverses the order on \mathbb{V} ; such an automorphism clearly exists since $(\mathbb{V};E,<)$ and $(\mathbb{V};E,>)$ are isomorphic. Lemma 8.3.15 shows that any canonical binary injective polymorphism of $(\mathbb{V};<)$ has the same behavior as lex(x,y), $lex(x,\leftrightarrow y)$, lex(y,x), or $lex(y,\leftrightarrow x)$. If f is any function that is canonical as a map from $(\mathbb{V};<;E)^2$ to $(\mathbb{V};<)$, and does not preserve <, then $(x,y)\mapsto \leftrightarrow f(x,y)$ preserves <. Moreover, by passing from f to $(x,y)\mapsto f(\leftrightarrow x,y)$ or $(x,y)\mapsto f(\leftrightarrow x,y)$ we can assume that f behaves either like lex(x,y) or lex(y,x).

We will now prove that minimal binary canonical injections are never of mixed type, i.e., they have to behave the same way on straight and twisted inputs.

LEMMA 9.3.13. Suppose that $f: \mathbb{V}^2 \to \mathbb{V}$ is injective and canonical, and suppose that it is of type max/p_i or of type p_i/max , where $i \in \{1, 2\}$. Then f is not minimal.

PROOF. We prove that f generates a binary injective canonical function g which is of type max/max. Clearly, all binary injective canonical functions generated by g then are also of type max/max, so g cannot generate f, proving the lemma.

Assume without loss of generality that f is of type max/p_i , and note that we assume that f behaves like lex(x,y). Set $h(u,v) := f(u, \leftrightarrow v)$. Then h behaves like p_i on input (<, <) and like max on input (<, >); moreover, $f(x_1, x_2) < f(y_1, y_2)$ iff $h(x_1, x_2) < h(y_1, y_2)$, for all $x_1 \neq y_1$ and $x_2 \neq y_2$. We then have that g(u, v) := f(f(u, v), h(u, v)) is of type max/max, finishing the proof.

LEMMA 9.3.14. Suppose that $f: \mathbb{V}^2 \to \mathbb{V}$ is injective and canonical, and suppose that it is of type min/p_i or of type p_i/min , where $i \in \{1, 2\}$. Then f is not minimal.

PROOF. The dual proof works. \Box

LEMMA 9.3.15. Suppose that $f: \mathbb{V}^2 \to \mathbb{V}$ is injective and canonical, and suppose that it is of type max/min or of type min/max. Then f is not minimal.

PROOF. Assume without loss of generality that f is of type max/min, and recall that we assume that f behaves like lex(x,y). Consider $h(u,v) := f(f(u,v), \leftrightarrow v)$. Then h is of type p_2/p_2 , so it cannot reproduce f.

LEMMA 9.3.16. Suppose that $f: \mathbb{V}^2 \to \mathbb{V}$ is injective and canonical, and suppose that it is of type p_1/p_2 or of type p_2/p_1 . Then f is not minimal.

PROOF. If f is of type p_1/p_2 , then $h(u,v) := f(f(u,v), \leftrightarrow v)$ is of type p_2/p_2 and cannot reproduce f. If f is of type p_2/p_1 , then $g(u,v) := f(u, \leftrightarrow v)$ is of type p_1/p_2 and still behaves like lex(x,y); hence, we are back in the first case.

We have seen that actually no "mixed" types appear for minimal functions. In other words, minimal functions that are canonical as functions from $(\mathbb{V}; E, <)^2 \to (\mathbb{V}; E, <)$ are also canonical as functions from $(\mathbb{V}; E)^2 \to (\mathbb{V}; E)$. This motivates the following definition.

DEFINITION 9.3.17. Let $f: \mathbb{V}^2 \to \mathbb{V}$. We say that f behaves like $min\ (max,\ p_1,\ p_2)$ on input (\neq,\neq) iff it behaves like $min\ (max,\ p_1,\ p_2)$ both on input (<,<) and on input (<,>). We also say that f is of type $min\ (max,\ p_1,\ p_2)$. If f is of type p_1 or p_2 then we also say that f is of type projection.

Our observations so far can be summarized as follows.

PROPOSITION 9.3.18. Let $f: \mathbb{V}^2 \to \mathbb{V}$ be essential and minimal. Then it is injective, canonical as a function from $(\mathbb{V}; E)^2 \to (\mathbb{V}; E)$, and behaves like min, max, p_1 or p_2 on input (\neq, \neq) .

In the following, we consider further types of tuples $x, y \in \mathbb{V}^2$. So far, we did not consider the case where $x_1 = y_1$ or $x_2 = y_2$.

DEFINITION 9.3.19. Let $f: \mathbb{V}^2 \to \mathbb{V}$. We say that f behaves like e_E $(e_N, id, -)$ on input $(\neq, =)$ iff for every fixed $c \in \mathbb{V}$, the function g(x) := f(x, c) behaves like e_E $(e_N, id, -)$. Similarly we define behavior on input $(=, \neq)$.

If f is canonical and injective, then it behaves like one of the mentioned functions on input $(\neq, =)$ and $(=, \neq)$, respectively. This motivates the following.

DEFINITION 9.3.20. We say that $f: \mathbb{V}^2 \to \mathbb{V}$ is of type E/N iff f behaves like e_E on input $(\neq, =)$ and like e_N on input $(=, \neq)$. Similarly we define the types E/E, N/E, E/id , E/-, etc. Moreover, we say that f is balanced iff it is of type $\operatorname{id}/\operatorname{id}$, we say it is E-dominated iff it is of type E/E, and we say it is N-dominated iff it is of type N/N.

In the following theorem, we finally characterize those canonical behaviors that yield minimal functions.

Theorem 9.3.21. The minimal polymorphisms of (V; E, N) are precisely the binary injective canonical operations of the following types:

- (1) Projection and balanced.
- (2) max and balanced.
- (3) min and balanced.
- (4) max and E-dominated.
- (5) min and N-dominated.
- (6) Projection and E-dominated.
- (7) Projection and N-dominated.
- (8) p_2 and E/id, or p_1 and id/E.
- (9) p_2 and N/id, or p_1 and id/N.

Moreover, these 9 different kinds of minimal functions do not generate one another, and any two functions in the same group do generate one another.

If e is essential, then it must be binary and injective by Theorem 9.3.6. The rest of the theorem follows from Proposition 9.3.18 and the following lemmas. By the homogeneity of $(\mathbb{V}; E)$ and local closure, it is easy to see that a binary canonical injection in one of the classes of Theorem 9.3.21 generates all other functions in the same class. The verification of Lemmas 9.3.22 to 9.3.25 is left to the reader; the proof always uses induction over terms.

Lemma 9.3.22. Any binary essential function generated by a binary canonical injection of type min, max, or projection, respectively, is of the same type.

Lemma 9.3.23. Any binary essential function generated by a binary canonical injection that is balanced and preserves E and N is balanced.

We thus have that the first three classes of functions of Proposition 9.3.21 are indeed minimal. The following lemma proves minimality for items (4) and (5).

Lemma 9.3.24. Any binary essential function generated by an E-dominated binary canonical injection of type max is E-dominated. Dually, any binary essential function generated by an N-dominated binary canonical injection of type min is N-dominated.

The following lemma proves minimality for items (6) and (7).

Lemma 9.3.25. Any binary essential function generated by an E-dominated binary canonical injection of type projection is E-dominated. Dually, any binary essential function generated by an N-dominated binary canonical injection of type projection is N-dominated.

It remains to prove minimality for items (8) and (9), which is achieved in the following lemma.

Lemma 9.3.26. Any binary essential function generated by a binary canonical injection of type E/id and p_2 is either of the same type or of type id/E and p_1 . Dually, any binary essential function generated by a binary canonical injection of type N/id and p_2 generates is either of the same type or of type id/N and p_1 .

PROOF. Let f(u, v) be of type E/id and p_2 . f(v, u) is of type id/E and p_1 . Both f(u, f(u, v)) and f(v, f(u, v)) are of type E/id and p_2 . So is f(f(u, v), v). The function f(f(u, v), u) is of type id/E and p_1 . Finally, f(f(u, v), f(v, u)) also is of type id/E and p_1 , so f cannot generate any new behaviors.

Next we claim that no other functions except for those listed in Theorem 9.3.21 are minimal. This will be achieved in the following lemmas.

Lemma 9.3.27. Let f be a binary canonical injection of type max. If f is not balanced or E-dominated, then f is not minimal.

PROOF. If f is of type E/id , then g(x,y) := f(f(x,y),x) is E-dominated. By Lemma 9.3.24, g cannot reproduce f. If f is of type E/N, then g is E-dominated as well. So it is if f is of type E/-.

If f is of type N/id, then g(x,y) := f(x, f(x,y)) is balanced, so f is not minimal by Lemma 9.3.23. If f is of type N/-, then g is balanced as well.

If f is of type id /- or of type -/-, then g(x,y) := f(x,f(x,y)) is of type E/id , which we have already shown not to be minimal.

By symmetry, if we switch the arguments in a type of f, e.g., if f is of type id /E, then f is not minimal either. We have thus covered all possible types.

Analogously, we find that every minimal binary injection of type \min is balanced or N-dominated.

LEMMA 9.3.28. Let f be a binary canonical injection of type p_1 . If f is not balanced, E-dominated, N-dominated, of type id/E, or of type id/N, then f is not minimal.

PROOF. If f is of type E/id , E/-, $-/\operatorname{id}$, or -/-, then g(x,y) := f(x,f(x,y)) is balanced and cannot reproduce f. If it is of type E/N or $\operatorname{id}/-$, then g is of type E/id , and we are back in the preceding case. Dually, if f is of type N/id or N/-, then g is balanced. If it is of type N/E, then g is of type N/id , bringing us back to the preceding case. If it is of type -/E, then g is of type O0 id O1, and hence cannot reproduce O1 by Lemma 9.3.26. The dual argument works if O2 is of type O3.

Analogously, we find that every minimal binary injection of type p_2 is balanced, E-dominated, N-dominated, or of type E/ id.

9.3.4. Producing functions that are not of type projection. Theorem 9.3.8 and the following proposition together imply that indeed, if case (a) of Proposition 9.3.4 does not apply, then one of the other cases does.

PROPOSITION 9.3.29. Suppose that f is an operation on \mathbb{V} that preserves the relations E and N and violates the relation H_1 . Then f generates a binary injective canonical operation of type min or max, or a ternary injective canonical operation of type minority or majority.

We first prove the following.

LEMMA 9.3.30. Let f be an operation on $(\mathbb{V}; E)$ which preserves E and N and violates H_1 . Then f generates a binary or ternary injection which shares the same properties.

PROOF. Since the relation H_1 consists of three orbits of 6-tuples, by Lemma 5.3.5 f generates an at most ternary function that violates H_1 , and hence we can assume without loss of generality that f itself is at most ternary. The operation f must certainly be essential, since essentially unary operations that preserve E and N also preserve H_1 . Applying Theorem 9.3.8, we get that f generates a binary injective canonical function of type min, max, or p_1 . In the first two cases we are done, since binary injections of type min and max violate H_1 . So consider the last case and

denote the function of type p_1 by g. By adding a dummy variable, we may assume that f is ternary. Now consider

$$h(x, y, z) := g(g(g(f(x, y, z), x), y), z)$$
.

Then h is clearly injective, and still violates H_1 – the latter can easily be verified combining the facts that f violates H_1 , g is of type p_1 , and all tuples in H_1 have pairwise distinct entries.

It will turn out that just as in the proof of Lemma 9.3.30, there are two cases for f in the proof of Proposition 9.3.29: either all binary canonical injections generated by f are of type projection, and f generates an edge majority or an edge minority, or f generates a binary canonical injection of type min or max. We start by considering the first case, which is combinatorially less involved.

PROPOSITION 9.3.31. Let f be an operation on (V; E) which preserves E and N and violates H. Suppose moreover that all binary injections generated by f are of type p_1 or p_2 . Then f generates a canonical ternary injection of type majority or minority.

PROOF. By Lemma 9.3.30, we can assume that f is a binary or ternary injection; if it was binary, it would be of type projection and thus preserve H_1 , so it must be ternary. Because f violates H_1 , there are $x^1, x^2, x^3 \in H$ such that $(f(x_1), \ldots, f(x_6)) \notin H$, where $x_i := (x_i^1, x_i^2, x_i^3)$ for $1 \le i \le 6$.

If there was an automorphism α such that $\alpha x^i = x^j$ for $i \neq j \leq 3$, then f generates a binary injection that still violates H_1 , which contradicts the assumption that all binary injections generated by f are of type projection. By permuting arguments of f if necessary, we can therefore assume without loss of generality that

$$\begin{split} E(x_1^1,x_2^1), N(x_3^1,x_4^1), N(x_5^1,x_6^1), \\ N(x_1^2,x_2^2), E(x_3^1,x_4^1), N(x_5^1,x_6^1), \\ N(x_1^3,x_2^3), N(x_3^1,x_4^1), E(x_5^1,x_6^1). \end{split}$$

We set

$$S := \{ y \in \mathbb{V}^3 \mid NNN(x_i, y) \text{ for all } i \le 6 \} .$$

Consider the binary relations $Q_1Q_2Q_3$ on \mathbb{V}^3 , where $Q_i \in \{E,N\}$ for $1 \leq i \leq 3$; each of these relations defines a 2-type in $(\mathbb{V};E)^{[3]}$. We claim that for every 2-type s defined by one of those relations there is a 2-type s' of $(\mathbb{V};E)$ such that f satisfies the type condition (s,s') on S. To prove the claim, fix a relation $Q_1Q_2Q_3$ and let $u,v \in S$ be such that $Q_1Q_2Q_3(u,v)$ holds; we must show that whether E(f(u),f(v)) or N(f(u),f(v)) depends only on $Q_1Q_2Q_3$ (and not on u,v). We go through all possibilities of $Q_1Q_2Q_3$.

- (1) $Q_1Q_2Q_3 = ENN$. Let $\alpha \in \text{Aut}(\mathbb{V}; E)$ be such that $\alpha(x_1^2, x_2^2, u_2, v_2) = (x_1^3, x_2^3, u_3, v_3)$; such an automorphism exists since $NNN(x_1, u), NNN(x_1, u), NNN(x_2, u), NNN(x_2, v)$ and since (x_1^2, x_2^2) has the same type as $(x_1^3, x_2^3),$ and (u_2, v_2) has the same type as (u_3, v_3) . By assumption, the operation g defined by $g(x, y) := f(x, y, \alpha y)$ must be of type projection. Hence, $E(g(u_1, u_2), g(v_1, v_2))$ iff $E(g(x_1^1, x_1^2), g(x_2^1, x_2^2))$. Combining this with the equations $(f(u), f(v)) = (g(u_1, u_2), g(v_1, v_2))$ and $(g(x_1^1, x_1^2), g(x_2^1, x_2^2)) = (f(x_1), f(x_2))$, we get that E(f(u), f(v)) iff $E(f(x_1), f(x_2))$, and so we are done
- (2) $Q_1Q_2Q_3 = NEN$ or $Q_1Q_2Q_3 = NNE$. These cases are analogous to the previous case.

- (3) $Q_1Q_2Q_3 = NEE$. Let α be defined as in the first case. By assumption, the operation defined by $f(x, y, \alpha y)$ must be of type projection. Reasoning as above, one gets that E(f(u), f(v)) iff $N(f(x_1), f(x_2))$.
- (4) $Q_1Q_2Q_3 = ENE$ or $Q_1Q_2Q_3 = EEN$. These cases are analogous to the previous case.
- (5) $Q_1Q_2Q_3 = EEE$ or $Q_1Q_2Q_3 = NNN$. These cases are trivial since f preserves E and N.

To show that f generates an operation of type majority or minority, it suffices to prove that f generates a function of type majority or minority on S (that is, has on S the same behavior as a function of type majority), since S contains copies of arbitrary finite products of substructures of $(\mathbb{V}; E)$, and by Lemma 8.3.13. We prove this by another case distinction, based on the fact that $(f(x_1), \ldots, f(x_6)) \notin H$.

- (1) Suppose that $E(f(x_1), f(x_2)), E(f(x_3), f(x_4)), E(f(x_5), f(x_6))$. Then f itself is of type minority on S.
- (2) Suppose that $N(f(x_1), f(x_2)), N(f(x_3), f(x_4)), N(f(x_5), f(x_6))$. Then f itself is of type majority on S.
- (3) Suppose that $E(f(x_1), f(x_2)), E(f(x_3), f(x_4)), N(f(x_5), f(x_6))$. Let e be a self-embedding of $(\mathbb{V}; E)$ such that for all $w \in V$ and all $i \leq 6$ we have that $N(x_i, e(w))$. Then $(u_1, u_2, e(f(u_1, u_2, u_3))) \in S$ for all $(u_1, u_2, u_3) \in S$. Hence, by the above, the ternary operation defined by f(x, y, e(f(x, y, z))) is of type majority on S.
- (4) Suppose that $E(f(x_1), f(x_2))$, $N(f(x_3), f(x_4))$, and $E(f(x_5), f(x_6))$, or that $N(f(x_1), f(x_2))$, $E(f(x_3), f(x_4))$, and $E(f(x_5), f(x_6))$. These cases are analogous to the previous case.

Let h(x, y, z) be a ternary injection of type majority or minority generated by f; it remains to make h canonical. By Theorem 9.3.8, f generates a binary canonical injection g(x, y), which is of type projection by our assumption on f. Assume without loss of generality that it is of type p_1 and set t(x, y, z) := g(x, g(y, z)). Then the function h(t(x, y, z), t(y, z, x), t(z, x, y)) is still of type majority or minority and canonical; we leave the straightforward verification to the reader.

In order to obtain a full proof of Proposition 9.3.29, it remains to show the following proposition.

PROPOSITION 9.3.32. Let $f: \mathbb{V}^2 \to \mathbb{V}$ be a binary injection preserving E and N that is neither of type p_1 nor of type p_2 . Then f generates a binary injection of type min or of type max.

In the remainder of this section we will show this by a Ramsey theoretic analysis of f. The global strategy behind what follows now is to take a binary injection f and fix finitely many constants \bar{c} from \mathbb{V}^2 on which it can be seen that f is not of type projection. Then, using Theorem 8.3.11, we generate a binary canonical function which is identical with f on all tuples with elements from \bar{c} ; this canonical function then still is not of type projection, and can be handled more easily because it is canonical. To reduce the number of cases that we have to consider, we rule out some behaviors of canonical functions already before introducing the constants.

LEMMA 9.3.33. Suppose that $f: \mathbb{V}^2 \to \mathbb{V}$ is injective and canonical, and suppose that it is of type max/g or of type g/max, where $g \in \{min, p_1, p_2\}$. Then f generates a binary injection of type max.

PROOF. Assume without loss of generality that f is of type max/g (when f is of type g/max, replace f by $f(x, \leftrightarrow y)$, which is of type g/max). We also assume that f obeys p_1 for the order (otherwise, continue with f(y, x) instead of f(y, x)).

Set $h(u,v) := f(u, \leftrightarrow v)$. Then h behaves like g on input (<,<) and like max on input (<,>); moreover, $f(x_1,x_2) < f(y_1,y_2)$ iff $h(x_1,x_2) < h(y_1,y_2)$, for all $x_1 \neq y_1$ and $x_2 \neq y_2$. We then have that f(f(u,v),h(u,v)) is of type max/max, which means that it is of type max.

LEMMA 9.3.34. Suppose that $f: \mathbb{V}^2 \to \mathbb{V}$ is injective and canonical, and suppose that it is of type \min/p_i or of type p_i/\min , where $i \in \{1, 2\}$. Then f generates a binary injection of type \min .

Proof. The dual proof works.

Lemma 9.3.35. Suppose that $f: V^2 \to V$ is injective and canonical as a function from $(V; E, \prec)^2$ to $(V; E, \prec)$, and suppose that it is of type max/min or of type min/max. Then f generates a binary injection of type max (and by duality, a binary injection of type min).

PROOF. Assume without loss of generality that f is of type max/min, and remember that we may assume that f obeys p_1 for the order. Then g(x,y) := f(x, f(x,y)) is of type max/ p_1 and generates a binary injection of type max by Lemma 9.3.33.

We next consider the last remaining mixed behavior, p_1/p_2 , by combining operational with relational arguments.

LEMMA 9.3.36. Let \mathfrak{B} be a structure that is first-order definable in $(\mathbb{V}; E)$, contains the relations E, N, \neq , and is preserved by a binary injection of type p_1 . Then the following are equivalent.

- (1) B has a binary injective polymorphism of behavior min.
- (2) For every primitive positive formula ϕ over \mathfrak{B} , if $\phi \wedge N(x_1, x_2) \wedge \bigwedge_{1 \leq i < j \leq 4} x_i \neq x_j$ and $\phi \wedge N(x_3, x_4) \wedge \bigwedge_{1 \leq i < j \leq 4} x_i \neq x_j$ are satisfiable over \mathfrak{B} , then $\phi \wedge N(x_1, x_2) \wedge N(x_3, x_4)$ is satisfiable over \mathfrak{B} as well.
- (3) For every finite $F \subseteq \mathbb{V}^2$ there exists a binary injective polymorphism of \mathfrak{B} which behaves like min on F.

PROOF. The implication from (1) to (2) follows directly by applying a binary injective polymorphism of behavior min to tuples r,s satisfying $\phi \wedge N(x_1,x_2) \wedge \bigwedge_{1 \leq i < j \leq 4} x_i \neq x_j$ and $\phi \wedge N(x_3,x_4) \wedge \bigwedge_{1 \leq i < j \leq 4} x_i \neq x_j$, respectively.

To prove that (2) implies (3), assume (2) and let $F \subset \mathbb{V}^2$ be finite. Without loss of generality we can assume that F is of the form $\{e_1,\ldots,e_n\}^2$, for sufficiently large n. Let \mathfrak{A} be the structure induced by F in \mathfrak{B}^2 . We construct an injective homomorphism h from Δ to \mathfrak{B} ; every homomorphism can clearly be extended to a binary polymorphism of \mathfrak{B} , for example inductively by using universality of $(\mathbb{V}; E)$. We construct h in such a way that the extension behaves as min on F.

To construct h, consider the formula ϕ_0 with variables $x_{i,j}$ for $1 \le i, j \le n$ which is the conjunction over all literals $R(x_{i_1,j_1},\ldots,x_{i_k,j_k})$ such that R is a relation in $\mathfrak B$ and $R(e_{i_1},\ldots,e_{i_k})$ and $R(e_{j_1},\ldots,e_{j_k})$ hold in $\mathfrak B$. So ϕ_0 states precisely which relations hold in $\mathfrak B^2$ on elements from F. Since $\mathfrak B$ is preserved by a binary injection, we have that $\phi_1 := \phi_0 \wedge \bigwedge_{1 \le i,j,k,l \le n, (i,j) \ne (k,l)} x_{i,j} \ne x_{k,l}$ is satisfiable.

Let P be the set of pairs of the form $((i_1, i_2), (j_1, j_2))$ with $i_1, i_2, j_1, j_2 \in \{1, \ldots, n\}$, $i_1 \neq j_1, i_2 \neq j_2$, and where $N(e_{i_1}, e_{j_1})$ or $N(e_{i_2}, e_{j_2})$. We show by induction on the size of $I \subseteq P$ that the formula $\phi_1 \wedge \bigwedge_{((i_1, i_2), (j_1, j_2)) \in I} N(x_{i_1, i_2}, x_{j_1, j_2})$ is satisfiable over \mathfrak{B} . Note that this statement applied to the set I = P gives us the a homomorphism h from \mathfrak{A} to \mathfrak{B} such that for all $a, b \in F$ we have N(h(a), h(b)) whenever EN(a, b)

or NE(a,b) by setting $h(e_i,e_j) := s(x_{i,j})$, where s is the satisfying assignment for $\phi_1 \wedge \bigwedge_{((i_1,i_2),(j_1,j_2))\in P} N(x_{i_1,i_2},x_{j_1,j_2})$.

For the induction beginning, let $p = ((i_1, i_2), (j_1, j_2))$ be any element of P. Let r, s be the n^2 -tuples defined as follows.

$$r := (e_1, \dots, e_1, e_2, \dots, e_2, \dots, e_n, \dots, e_n)$$

$$s := (e_1, e_2, \dots, e_n, e_1, e_2, \dots, e_n, \dots, e_1, e_2, \dots, e_n)$$

In the following we use double indices for the entries of n^2 -tuples; for example, $r = (r_{1,1}, \ldots, r_{1,n}, r_{2,1}, \ldots, r_{n,n})$. The two tuples r and s satisfy ϕ_0 . To see this, observe that by definition of ϕ_0 the tuple

$$((e_1, e_1), \dots, (e_1, e_n), (e_2, e_1), \dots, (e_n, e_n))$$

satisfies ϕ_0 in \mathfrak{B}^2 ; since r and s are projections of that tuple onto the first and second coordinate, respectively, and projections are homomorphisms, r and s satisfy ϕ_0 as well. Let g be a binary injective polymorphism of \mathfrak{B} which is of type p_1 , and set r':=g(r,s) and s':=g(s,r). Then r' and s' satisfy ϕ_1 since g is injective. Since $p \in P$, we have that $N(e_{i_1},e_{j_1})$ or $N(e_{i_2},e_{j_2})$. Assume that $N(e_{i_1},e_{j_1})$; the other case is analogous. Since $r_{i_1,i_2}=e_{i_1},\,r_{j_1,j_2}=e_{j_1},\,r':=g(r,s)$, and g is of type p_1 , we have that $N(r'_{i_1,i_2},r'_{j_1,j_2})$, proving that $\phi_1 \wedge N(x_{i_1,i_2},x_{j_1,j_2})$ is satisfiable in \mathfrak{B} .

In the induction step, let $I \subseteq P$ be a set of cardinality $n \ge 2$, and assume that the statement has been shown for subsets of P of cardinality n-1. Pick any distinct $q_1, q_2 \in I$. We set

$$\psi := \phi_1 \land \bigwedge_{((i_1, i_2), (j_1, j_2)) \in I \setminus \{q_1, q_2\}} N(x_{i_1, i_2}, x_{j_1, j_2})$$

and observe that ψ is a primitive positive formula over $\mathfrak B$ (here we use the assumption that $\mathfrak B$ contains the relations N and \neq). Write $q_1=((u_1,u_2),(v_1,v_2))$ and $q_2=((u_1',u_2'),(v_1',v_2'))$. Then the inductive assumption shows that each of $\psi \wedge N(x_{u_1,u_2},x_{v_1,v_2})$ and $\psi \wedge N(x_{u_1',u_2'},x_{v_1',v_2'})$ is satisfiable in $\mathfrak B$. Note that ψ contains in particular conjuncts that state that the four variables $x_{u_1,u_2},\,x_{v_1,v_2},\,x_{u_1',u_2'},\,x_{v_1',v_2'}$ denote distinct elements. Hence, by (2), the formula $\psi \wedge N(x_{u_1,u_2},x_{v_1,v_2}) \wedge N(x_{u_1',u_2'},x_{v_1',v_2'})$ is satisfiable over $\mathfrak B$ as well, which is what we had to show.

The implication from (3) to (1) follows from Lemma 3.1.8.

LEMMA 9.3.37. Let $f: \mathbb{V}^2 \to \mathbb{V}$ be a binary injection of behavior p_1/p_2 which preserves E and N. Then f generates a binary injection of type min and a binary injection of type max.

PROOF. By Theorem 9.3.8, f generates a binary injection of type max, min, or p_1 . Suppose first that it does not generate a binary injection of type max or min; we will lead this to a contradiction. Let \mathfrak{B} be the structure with domain \mathbb{V} that contains all relations that are first-order definable in $(\mathbb{V}; E)$ and that are preserved by f. Since f generates a binary injection of type p_1 , we may apply implication (2) \rightarrow (1) from Lemma 9.3.36. Let ϕ be a primitive positive formula with variable set S, $\{x_1, \ldots, x_4\} \subseteq S$, such that the formulas $\phi \wedge N(x_1, x_2) \wedge \bigwedge_{i < j \leq 4} x_i \neq x_j$ and $\phi \wedge N(x_3, x_4) \wedge \bigwedge_{i < j \leq 4} x_i \neq x_j$ have in \mathfrak{B} the satisfying assignments r and s from $S \rightarrow \mathbb{V}$, respectively.

We can assume without loss of generality that $r(x_1) < r(x_2)$ and $r(x_3) < r(x_4)$; otherwise, since $r(x_1), \ldots, r(x_4)$ must be pairwise distinct, we can apply an automorphism of $(\mathbb{V}; E)$ to r such that the resulting map has the required property. Similarly,

by applying an automorphism of $(\mathbb{V}; E)$ to s, we can assume without loss of generality that $s(x_1) < s(x_2)$ and $s(x_3) > s(x_4)$. Then the mapping $t: S \to \mathbb{V}$ defined by t(x) = f(r(x), s(x)) shows that $\phi \wedge N(x_1, x_2) \wedge N(x_3, x_4)$ is satisfiable in \mathfrak{B} :

- The assignment t satisfies ϕ since f is a polymorphism of \mathfrak{B} .
- We have that $N(t(x_1), t(x_2))$ since $r(x_1) < r(x_2)$, $s(x_1) < s(x_2)$, f is of type p_1 on input (<,<), and $N(r(x_1), r(x_2))$.
- We have that $N(t(x_3), t(x_4))$ since $r(x_3) < r(x_4)$, $s(x_3) > s(x_4)$, f is of type p_2 on input (<,>), and $N(s(x_3), s(x_4))$.

By Lemma 9.3.36, we conclude that \mathfrak{B} is preserved by a binary injection of type min, and consequently f generates a binary injection of type min – a contradiction.

Therefore, f generates a binary injection of type max or min. Since the assumptions of the lemma are symmetric in E and N, we infer a posteriori that f generates both a binary injection of type max and a binary injection of type min.

Having ruled out some behaviors without constants, we now examine behaviors when we add constants to the language. In the sequel, we will also say that a function $f: \mathbb{V}^2 \to \mathbb{V}$ has behavior B between two points $x, y \in \mathbb{V}^2$ if it has behavior B on $\{x, y\}$.

LEMMA 9.3.38. Let $u \in \mathbb{V}^2$, and set $U := (\mathbb{V} \setminus \{u_1\}) \times (\mathbb{V} \setminus \{u_2\})$. Let $f : \mathbb{V}^2 \to \mathbb{V}$ be a binary injection which preserves E and N, behaves like p_1 between all points $v, w \in U$, and which behaves like p_2 between u and all points in U. Then f generates a binary injection of type min as well as a binary injection of type max.

PROOF. Let \mathfrak{B} be the structure with domain \mathbb{V} that contains all relations that are first-order definable in $(\mathbb{V}; E)$ and that are preserved by f. Since U contains copies of products of arbitrary finite graphs, f behaves like p_1 on arbitrarily large finite substructures of $(\mathbb{V}; E)^2$, and hence generates a binary injection of type p_1 by Lemma 8.3.13. Hence \mathfrak{B} is also preserved by such a function, and we may apply the implication from (2) to (1) in Lemma 9.3.36 to \mathfrak{B} .

Let ϕ be a primitive positive formula with variable set S, $\{x_1, \ldots, x_4\} \subseteq S$, such that $\phi \wedge N(x_1, x_2) \wedge \bigwedge_{1 \leq i < j \leq 4} x_i \neq x_j$ and $\phi \wedge N(x_3, x_4) \wedge \bigwedge_{1 \leq i < j \leq 4} x_i \neq x_j$ are satisfiable over \mathfrak{B} , witnessed by satisfying assignments $r, s \colon S \to \overline{\mathbb{V}}$, respectively.

Let α be an automorphism of $(\mathbb{V}; E)$ that maps $r(x_3)$ to u_1 , and let β be an automorphism of $(\mathbb{V}; E)$ that maps $s(x_3)$ to u_2 . Then $(\alpha r(x_3), \beta s(x_3)) = u$, and $v := (\alpha r(x_4), \beta s(x_4)) \in U$ since $\alpha r(x_4) \neq \alpha r(x_3) = u_1$ and $\beta s(x_4) \neq \beta s(x_3) = u_2$. Thus, f behaves like p_2 between u and v, and since s satisfies $N(x_3, x_4)$, we have that $t : S \to \mathbb{V}$ defined by

$$t(x) = f(\alpha x, \beta x)$$

satisfies $N(x_3, x_4)$, too. Since α, β, f are polymorphisms of \mathfrak{B} , the assignment t also satisfies ϕ . To see that $N(t(x_1), t(x_2))$, observe that $\alpha r(x_1) \neq \alpha r(x_3)$ and $\beta s(x_1) \neq \beta s(x_3)$, and hence $p := (\alpha r(x_1), \beta s(x_1)) \notin U$. Similarly, $q := (\alpha r(x_2), \beta s(x_2)) \notin U$. Hence, f behaves as p_1 between p and q, and since $N(r(x_1), r(x_2))$, so does t.

By Lemma 9.3.36 we conclude that \mathfrak{B} is preserved by a binary injection of type min, and consequently f generates a binary injection of type min.

Since our assumptions on f were symmetric in E and N, it follows that f also generates a binary injection of type max.

LEMMA 9.3.39. Let $u \in \mathbb{V}^2$, and let $f : \mathbb{V}^2 \to \mathbb{V}$ be a binary injection that behaves like p_1 between all points $v, w \in U := (\mathbb{V} \setminus \{u_1\}) \times (\mathbb{V} \setminus \{u_2\})$, and which behaves like min between u and all points in U. Then f generates a binary injection of type min.

PROOF. The proof is identical with the proof in the preceding lemma; note that our assumptions on f here imply more deletions of edges as the assumptions in that lemma, so it can only be easier to generate a binary injection of type min.

LEMMA 9.3.40. Let $u, v \in \mathbb{V}^2$ such that $\neq \neq (u, v)$ and set $W := (\mathbb{V} \setminus \{u_1, v_1\}) \times (\mathbb{V} \setminus \{u_2, v_2\})$. Let $f : \mathbb{V}^2 \to \mathbb{V}$ be a binary injection that

- ullet preserves E and N
- behaves like p_1 between all points $w, r \in W$
- behaves like p_1 between u and all points $w \in W$
- behaves like p_1 between v and all points $w \in W$
- does not behave like p_1 between u and v.

Then f generates a binary injection of type min as wella s a binary injection of type max.

PROOF. We have to consider the case that EN(u, v) and N(f(u), f(v)), and the case that NE(u, v) and E(f(u), f(v)). In the first case we prove that f generates a binary injection of type min; it then follows by duality that in the second case, f generates a binary injection of type max.

As in Lemma 9.3.38, we apply the implication $(2) \to (1)$ from Lemma 9.3.36. Let \mathfrak{B} , ϕ , x_1, \ldots, x_4 , and S be as in the proof of Lemma 9.3.38; by the same argument as before, \mathfrak{B} is preserved by a binary injection of type p_1 . If $N(r(x_3), r(x_4))$, then the assignment r shows that $\phi \wedge N(x_1, x_2) \wedge N(x_3, x_4)$ is satisfiable and we are done. Otherwise, since $r(x_3) \neq r(x_4)$, we have $E(r(x_3), r(x_4))$. Therefore, there is an $\alpha \in \operatorname{Aut}(\mathbb{V}; E)$ such that $(\alpha r(x_3), \alpha r(x_4)) = (u_1, v_1)$. Similarly, since $N(s(x_3), s(x_4))$ and $N(u_2, v_2)$, there is a $\beta \in \operatorname{Aut}(\mathbb{V}; E)$ such that $(\beta s(x_3), \beta s(x_4)) = (u_2, v_2)$. We claim that the map $t \colon S \to \mathbb{V}$ defined by

$$t(x) = f(\alpha x, \beta x)$$

is a satisfying assignment for $\phi \wedge N(x_1, x_2) \wedge N(x_3, x_4)$. The assignment t satisfies ϕ since α, β and f are polymorphisms of \mathfrak{B} . Then $N(t(x_3), t(x_4))$ holds because $(\alpha r(x_3), \beta s(x_3)) = u$ and $(\alpha r(x_4), \beta s(x_4)) = v$, and N(f(u), f(v)). To prove that $N(t(x_1), t(x_2))$ holds, observe that $r(x_1) \neq r(x_3)$ and $r(x_1) \neq r(x_4)$, and hence

$$\alpha r(x_1) \notin \{\alpha r(x_3), \alpha r(x_4)\} = \{u_1, v_1\}$$
.

Similarly, $\beta s(x_1) \notin \{\beta s(x_3), \beta s(x_4)\} = \{u_2, v_2\}$. Hence, $(\alpha r(x_1), \beta s(x_1) \in W$. A similar argument for x_2 in place of x_1 shows that $(\alpha r(x_2), \beta s(x_2) \in W$. Since f behaves like p_1 between all points of W, and since r satisfies $N(x_1, x_2)$, we have proved the claim. This shows that \mathfrak{B} is preserved by a binary injection of type min, and hence f generates such a function.

By symmetry of our assumptions on f in E and N, it follows that f generates a binary injection of type min if and only if it generates a binary injection of type max.

We are now set up to prove Proposition 9.3.32, and hence complete the proof of Proposition 9.3.29.

PROOF OF PROPOSITION 9.3.32. Let f be given. By Theorem 9.3.8, f generates a binary canonical injection g of type projection, min, or max. In the last two cases we are done, so consider the first case. We claim that f also generates a (not necessarily canonical) binary injection h of type min or max. Then h(g(x,y),g(y,x)) is still of type min or max and in addition canonical, and the proposition follows.

To prove our claim, fix a finite set $C := \{c_1, \ldots, c_m\} \subseteq \mathbb{V}$ such that the fact that f does not behave like a projection is witnessed on C. Invoking Theorem 8.3.11, we may henceforth assume that f is canonical as a function from $(\mathbb{V}; E, <, c_1, \ldots, c_m)^2$ to $(\mathbb{V}; E, <)$ (and hence also to $(\mathbb{V}; E)$ since tuples of equal type in $(\mathbb{V}; E, <)$ have equal type in $(\mathbb{V}; E)$). It is clear that this new f must be injective.

In the following we consider orbits of elements in the structure $(V; E, <, c_1, \ldots, c_m)$. The infinite orbits are precisely the sets of the form

$$\{v \in \mathbb{V} \mid Q_i(v, c_i) \text{ and } R_i(v, c_i) \text{ for all } 1 \leq i \leq m\},$$

for $Q_1, \ldots, Q_m \in \{E, N\}$, and $R_1, \ldots, R_m \in \{<, >\}$. The finite orbits are of the form $\{c_i\}$ for some $1 \leq i \leq m$. Each infinite orbit of $(\mathbb{V}; E, <, c_1, \ldots, c_m)$ contains copies of arbitrary linearly ordered finite graphs, and in particular, forgetting about the order, of all finite graphs.

Therefore, if f behaves like min or max on an infinite orbit of $(V; E, <, c_1, \ldots, c_m)$, then by Lemma 8.3.13 it generates a function which behaves like min or max everywhere, and we are done. Moreover, if f is of mixed type on an infinite orbit, then, again by Lemma 8.3.13, f generates a canonical function which has the same mixed behavior everywhere. But then we are done by Lemmas 9.3.33, 9.3.34, and 9.3.37. Hence, we may henceforth assume that f behaves like a projection on every infinite orbit. Fix in the following an infinite orbit O and assume without loss of generality that f behaves like p_1 on O.

Now suppose that there exists an infinite orbit W such that f behaves like p_2 between all points $u \in O^2$ and $v \in W^2$ for which $u_1 < v_1$ and $u_2 < v_2$. Then fix any $v \in W^2$, and set $O_1 := \{o \in O \mid o < v_1\}$ and $O_2 := \{o \in O \mid o < v_2\}$. Set $O'_1 := O_1 \cup \{v_1\}$ and $O'_2 := O_2 \cup \{v_2\}$. We then have that f behaves like p_2 between v and any point u of $(O'_1 \setminus \{v_1\}) \times (O'_2 \setminus \{v_2\})$, and like p_1 between any two points of $(O'_1 \setminus \{v_1\}) \times (O'_2 \setminus \{v_2\})$. Since $(O'_i; E, v_i)$ contains copies of all finite substructures of $(V; E, v_i)$, for $i \in \{1, 2\}$, by Lemma 8.3.13 we get that f generates a function which behaves like p_2 between v and any point v of $(V \setminus \{v_1\}) \times (V \setminus \{v_2\})$, and which behaves like v between any two points of $(V \setminus \{v_1\}) \times (V \setminus \{v_2\})$. Then Lemma 9.3.38 implies that v generates a binary injection of type v and we are done.

This argument is easily adapted to any situation where there exists an infinite orbit W such that f behaves like p_2 between all points $u \in O^2$ and $v \in W^2$ with $R_1(u_1, v_1)$ and $R_2(u_2, v_2)$, for $R_1, R_2 \in \{<,>\}$.

When there exists an infinite orbit W such that f behaves like min between all points $u \in O^2$ and $v \in W^2$ with $R_1(u_1, v_1)$ and $R_2(u_2, v_2)$, then we can argue similarly, invoking Lemma 9.3.39 at the end. Replacing min by max we can use the dual argument, with the notable difference that f generates a binary injection of type max rather than min.

Since f is canonical, one of the situations described so far must occur. Putting this together, we conclude that for every infinite orbit W and all points $u \in O^2$ and $v \in W^2$, f behaves like p_1 between u and v. Having that, suppose that for an infinite orbit W, f behaves like p_2 on W. Then exchanging the roles of O and W and of p_1 and p_2 above, we can again conclude that f generates a binary injection of type min. We may thus henceforth assume that f behaves like p_1 between all points $u, v \in (V \setminus C)^2$.

Pick any $u \in C^2$. Suppose that there exists $v \in (\mathbb{V} \setminus C)^2$ such that f does not behave like p_1 between u and v; say without loss of generality that EN(u,v) and N(f(u), f(v)). Let O_i be the (infinite) orbit of v_i , for $i \in \{1, 2\}$. Then for all $v \in O_1 \times O_2$ we have EN(u,v) and N(f(u), f(v)) since f is canonical. Now let $w \in O_2 \times O_1$. We distinguish the two cases E(f(u), f(w)) and N(f(u), f(w)). In the first case, f behaves like p_2 between u and all $v \in (O_1 \cup O_2)^2$. We can then argue as above and are done. In the second case, f behaves like min between u and all $v \in (O_1 \cup O_2)^2$, and we are again done by the corresponding argument above. We conclude that we may assume that for all $u \in C^2$ and all $v \in (\mathbb{V} \setminus C)^2$, f behaves like p_1 between u and v as well.

Now pick $u, v \in C^2$ such that f does not behave like p_1 between u and v, say without loss of generality EN(u,v) and N(f(u),f(v)); this is possible since the fact that f does not behave like p_1 everywhere is witnessed on C. Pick any 16 infinite orbits O_1,\ldots,O_{16} such that for all $Q_1,Q_2,R_1,R_2\in\{E,N\}$ there exists $w\in(O_1\cup\ldots\cup O_{16})^2$ with $Q_1Q_2(u,w)$ and $R_1R_2(v,w)$. Set $S_1:=\{u_1,v_1\}\cup O_1\cup\ldots\cup O_{16}$ and $S_2:=\{u_2,v_2\}\cup O_1\cup\ldots\cup O_{16}$. Then S_i contains copies of all finite substructures of $(\mathbb{V};E,u_i,v_i)$, for $i\in\{1,2\}$, and hence applying Lemma 8.3.13 to functions from $(\mathbb{V};E,u_1,v_1)\times(\mathbb{V};E,u_2,v_2)$ to $(\mathbb{V};E)$ we see that f generates a function which behaves like p_2 between u and v, like p_1 between v and all points v0. Then v1 is v2 in v3, and like v3 between any two points v4 and all points v5. But then we are done by Lemma 9.3.40.

9.4. First-order Expansions of $(\mathbb{V}; R^{(3)}, S^{(3)})$

The structure of this section will be similar to the one of Section 9.3, but $R^{(3)}$ will take the role of E, and $S^{(3)}$ will take the role of N. The relation H_1 will be replaced by the following relation.

DEFINITION 9.4.1. Let H_2 be the smallest 9-ary relation that is preserved by $\{sw\}$ and contains all tuples $(x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3) \in \mathbb{V}^9$ such that

$$\bigwedge_{i,j \in \{1,2,3\}, i \neq j, u \in \{x_i, y_i, z_i\}, v \in \{x_j, y_j, z_j\}} N(u, v)$$

$$\wedge \left((R^{(3)}(x_1, y_1, z_1) \wedge S^{(3)}(x_2, y_2, z_2) \wedge S^{(3)}(x_3, y_3, z_3) \right)$$

$$\vee \left(S^{(3)}(x_1, y_1, z_1) \wedge R^{(3)}(x_2, y_2, z_2) \wedge S^{(3)}(x_3, y_3, z_3) \right)$$

$$\vee \left(S^{(3)}(x_1, y_1, z_1) \wedge S^{(3)}(x_2, y_2, z_2) \wedge R^{(3)}(x_3, y_3, z_3) \right) \right).$$

PROPOSITION 9.4.2. Let \mathfrak{B} be a reduct of $(\mathbb{V}; E)$ whose endomorphisms are precisely the unary functions generated by $\{sw\}$. Then either H_2 is primitive positive definable in \mathfrak{B} , or \mathfrak{B} satisfies item (b) or (d) of Proposition 9.3.4.

PROPOSITION 9.4.3. There is a primitive positive interpretation of $(\{0,1\}; 1IN3)$ in $(\mathbb{V}; H_2)$, and $CSP(\mathbb{V}; H_2)$ is NP-hard.

PROOF. This can be shown analogously to Proposition 9.3.5, but this time we represent 1 by triples from $R^{(3)}$ instead of pairs that satisfy E, and 0 by triples from $S^{(3)}$, and then use H_2 analogously as we have used H_1 in the proof of Proposition 9.3.5.

9.4.1. Producing canonical functions of type projection. As in Section 9.3.2, we show that if \mathfrak{B} has an essential polymorphism f, then it must also contain a binary injective polymorphism. Every binary injective function generates a binary injective canonical function, and those can be classified similarly as in Section 9.3.3. Luckily, even though we do not work in this section under the assumption that E and N are preserved by f, we are able to reduce to this case in our argument.

Proposition 9.4.4. Suppose that \mathfrak{B} has an essential polymorphism. Then \mathfrak{B} is preserved by a constant function, e_E , e_N , or by a canonical binary injection of type min, max, or p_1 .

PROOF. If there is a primitive positive definition of E and N, then the statement follows from Theorem 9.3.8. So suppose that this is not that case; also suppose that \mathfrak{B} is not preserved by e_E , e_N , or a constant function. Then $\operatorname{Aut}(\mathfrak{B})$ is dense in $\operatorname{End}(\mathfrak{B})$ by Proposition 9.2.2, and so they must violate E and N as otherwise these relations

would have a primitive positive definition. By Theorem 9.2.6, we then see that $\operatorname{Aut}(\mathfrak{B})$ is 2-transitive. By Theorem 6.2.1, \mathfrak{B} has a binary injective polymorphism g. Since $(\mathbb{V}; E, \prec)$ is Ramsey (Example 8.1.7), we can apply Corollary 8.3.6 and obtain that g generates a binary injective function h which is canonical as a function from $(\mathbb{V}; E, \prec)^2$ to $(\mathbb{V}; E, \prec)$. The function $x \mapsto h(x, x)$ either preserves E and N, or behaves like -, e_E or e_N . We can assume that it does not behave like e_E or e_N , and if it behaves like -, we can replace h by -h and assume that $x \mapsto h(x, x)$ preserves E and N. Now consider the function $x \mapsto h(x, \alpha(x))$, where $\alpha \in \operatorname{Aut}(\mathbb{V}; E)$ reverses \prec . Again, we may exclude the possibility that it behaves like e_E or e_N . But then the function $(x, y) \mapsto h(h(x, y), h(y, x))$ preserves E and N and we can apply Theorem 9.3.8 to conclude that it generates a binary injection which is canonical as a function from $(\mathbb{V}; E)^2$ to $(\mathbb{V}; E)$ and of type min, max, or p_1 .

COROLLARY 9.4.5. Let $\mathfrak{B} = (\mathbb{V}; R^{(3)}, S^{(3)}, \ldots)$ be first-order definable over $(\mathbb{V}; E)$ with an essential polymorphism. Then \mathfrak{B} is preserved by a binary canonical injection of type p_1 .

PROOF. Since e_N and functions of type min do not preserve $R^{(3)}$ and e_E and functions of type max do not preserve $S^{(3)}$, Proposition 9.4.4 implies that \mathfrak{B} is preserved by a binary canonical injection of type p_1 .

9.4.2. Eliminating mixed behavior.

LEMMA 9.4.6. Let $f: \mathbb{V}^2 \to \mathbb{V}$ be a binary injection that preserves $R^{(3)}$ and $S^{(3)}$. Then f is not of type p_1/p_2 .

PROOF. Suppose for contradiction that f does have the behavior p_1/p_2 . Let $u_1, u_2, u_3 \in \mathbb{V}$ with $u_1 \prec u_2 \prec u_3$, $E(u_1, u_2)$, $N(u_2, u_3)$, and $N(u_1, u_3)$. Let $v_1, v_2, v_3 \in \mathbb{V}$ with $v_1 \prec v_2 \prec v_3$ and $N(v_1, v_2)$, $E(v_2, v_3)$, $N(v_1, v_3)$. Then $E(f(u_1, v_1), f(u_2, v_3))$ and $N(f(u_1, v_1), f(u_3, v_2))$ since f behaves like p_1 on input (\prec, \prec) . Moreover, we have $E(f(u_2, v_3), f(u_3, v_2))$ since f behaves like p_2 on input (\prec, \succ) . Then $(u_1, u_2, u_3) \in R^{(3)}$ and $(u_1, u_2, u_3) \in R^{(3)}$, but $(f(u_1, v_2), f(u_2, v_3), f(u_3, v_2)) \notin R^{(3)}$, in contradiction to our assumptions.

9.4.3. Behaviors relative to vertices.

LEMMA 9.4.7. Let $u \in \mathbb{V}^2$, and set $U := (\mathbb{V} \setminus \{u_1\}) \times (\mathbb{V} \setminus \{u_2\})$. Let $f : \mathbb{V}^2 \to \mathbb{V}$ be a binary injection which behaves like p_1 on U, and which behaves like p_2 or max between u and all points in U. Then f does not preserve $R^{(3)}$.

PROOF. Let $v, w \in U$ be such that NE(u, v), EN(v, w), and NN(u, w). Then we have E(f(u), f(v)), E(f(v), f(w)), and N(f(u), f(w)). Hence, $R^{(3)}(u_i, v_i, w)$ for $i \in \{1, 2\}$, but $S^{(3)}(f(u), f(v), f(w))$.

Definition 9.4.8. We say that a binary injective function $f: \mathbb{V}^2 \to \mathbb{V}$ is

- of type $R^{(3)}$ - p_i , for $i \in \{1, 2\}$, iff for all $u, v, w \in \mathbb{V}^2$ with $\neq \neq (u, v), \neq \neq (v, w)$, and $\neq \neq (u, w)$ we have $R^{(3)}(f(u), f(v), f(w))$ if and only if $R^{(3)}(u_i, v_i, w_i)$.
- of type $R^{(3)}$ -projection iff it is of type $R^{(3)}$ - p_1 or of type $R^{(3)}$ - p_2 .

PROPOSITION 9.4.9. Suppose that $f: \mathbb{V}^2 \to \mathbb{V}$ preserves $R^{(3)}$ and $S^{(3)}$). Then f is of type $R^{(3)}$ -projection.

PROOF. The proof is similar to the proof of Proposition 9.3.32. Fix a finite set $C := \{c_1, \ldots, c_m\} \subseteq \mathbb{V}$ such that the fact that f is not of type $R^{(3)}$ -projection is witnessed on C. Invoking Theorem 8.3.11, we may henceforth assume that f is canonical as a function from $(\mathbb{V}; E, \prec, c_1, \ldots, c_m)^2$ to $(\mathbb{V}; E, \prec)$.

In the following we consider orbits in the structure $(\mathbb{V}; E, \prec, c_1, \ldots, c_m)$. The infinite orbits are precisely the sets of the form

$$\{v \in \mathbb{V} \mid Q_i(v, c_i) \text{ and } R_i(v, c_i) \text{ for all } 1 \leq i \leq m\},$$

for $Q_1, \ldots, Q_m \in \{E, N\}$, and $R_1, \ldots, R_m \in \{\prec, \succ\}$. The finite orbits are of the form $\{c_i\}$ for some $1 \leq i \leq m$. Each infinite orbit of $(\mathbb{V}; E, \prec, c_1, \ldots, c_m)$ induces in $(\mathbb{V}; E, \prec)$ a structure isomorphic to $(\mathbb{V}; E, \prec)$. Lemma 8.3.14 implies that if f has a certain behaviour on such an infinite orbit, then it generates a canonical function which has the same behaviour everywhere. Therefore we have for all infinite orbits O that f

- cannot be of type min or max on O since it preserves $R^{(3)}$ and $S^{(3)}$;
- cannot have behaviour max $/p_i$ or p_i / max for $i \in \{1, 2\}$ on O, by Lemma 9.3.33;
- cannot have behaviour min p_i or p_i min for $i \in \{1, 2\}$ on O, by 9.3.34;
- it cannot have behaviour max / min or min / max on O, by Lemma 9.3.35;
- it cannot have behavior p_1/p_2 or p_2/p_1 on O, by Lemma 9.4.6.

Hence, we may assume that f behaves like a projection on every infinite orbit. Fix in the following an infinite orbit O and assume without loss of generality that f behaves like p_1 on O.

Let W be any infinite orbit. Then since f is canonical, it behaves like p_1 , p_2 , min, or max between all u, v with $u \in O^2$, $v \in W^2$ and $u_1 \prec v_1$ and $u_2 \prec v_2$. Consider the case where there exists an infinite orbit W such that f behaves like p_2 or max between all points $u \in O^2$ and $v \in W^2$ for which $u_1 \prec v_1$ and $u_2 \prec v_2$. Then fix any $v \in W^2$, and set $O_1 := \{o \in O \mid o \prec v_1\}$ and $O_2 := \{o \in O \mid o \prec v_2\}$. Set $O_1' := O_1 \cup \{v_1\}$ and $O_2' := O_2 \cup \{v_2\}$. We then have that f behaves like p_2 or max between v and any point u of $(O_1' \setminus \{v_1\}) \times (O_2' \setminus \{v_2\})$, and like p_1 between any two points of $(O_1' \setminus \{v_1\}) \times (O_2' \setminus \{v_2\})$. Since $(O_i'; E, v_i)$ is isomorphic to $(\mathbb{V}; E, v_i)$, for $i \in \{1, 2\}$, by Lemma 8.3.14 we get that f generates a function which behaves like p_2 or max between v and any point v of $(\mathbb{V} \setminus \{v_1\}) \times (\mathbb{V} \setminus \{v_2\})$, and which behaves like v between any two points of $(\mathbb{V} \setminus \{v_1\}) \times (\mathbb{V} \setminus \{v_2\})$. This is impossible by Lemma 9.4.7. This argument is easily adapted to any situation where there exists an infinite orbit v such that v behaves like v between all points v of v and v of v with v or v with v or v and v and v or v with v or v and v or v with v or v with v or v and v or v with v or v with v or v or v or v or v with v or v

Since f is canonical, one of the situations described so far must occur. Putting this together, we conclude that for every infinite orbit W and all points $u \in O^2$ and $v \in W^2$, f behaves like p_1 between u and v. Having that, suppose that for an infinite orbit W, f behaves like p_2 on W. Then exchanging the roles of O and W and of p_1 and p_2 above, we again arrive at a contradiction. We may thus henceforth assume that f behaves like p_1 on $(\mathbb{V} \setminus C)^2$.

Pick any $u \in C^2$. Suppose that there exists $v \in (\mathbb{V} \setminus C)^2$ such that f does not behave like p_1 between u and v. Assume first that EN(u,v) and N(f(u),f(v)). Let O_i be the (infinite) orbit of v_i , for $i \in \{1,2\}$. Then for all $v \in O_1 \times O_2$ we have EN(u,v) and N(f(u),f(v)) since f is canonical. Now let $w \in O_2 \times O_1$. We distinguish the two cases E(f(u),f(w)) and N(f(u),f(w)). In the first case, f behaves like p_2 between u and all $v \in (O_1 \cup O_2)^2$. We can then argue as above and are done. In the second case, f behaves like min between u and all $v \in (O_1 \cup O_2)^2$, and we are again done by the corresponding argument above. The dual argument works when NE(u,v) and E(f(u),f(v)). Now assume that EE(u,v) and N(f(u),f(v)). We claim that EE(u,v') implies N(f(u),f(v')) and NN(u,v') implies E(f(u),f(v')) for all $v' \in (\mathbb{V} \setminus C)^2$. Suppose that $v' \in (\mathbb{V} \setminus C)^2$ is a counterexample. We can find

 $v'' \in (\mathbb{V} \setminus C)^2$ such that v'_1, v''_1 and v'_2, v''_2 belong to the same orbit and such that $R^{(3)}(u_i, v_i, v''_i)$ for $i \in \{1, 2\}$. But then $S^{(3)}(f(u), f(v), f(v''))$, a contradiction. By applying a version of sw which switches edges and non-edges with respect to $f[C^2]$ to f from the left, we may assume that f behaves like p_1 between all $u \in C^2$ and all $v \in (\mathbb{V} \setminus C)^2$

Since f does not behave like $R^{(3)}$ - p_1 on C^2 , in particular it does not behave like p_1 on C^2 . Pick $u, v \in C^2$ witnessing this. Then f behaves like p_1 between any point in $\{u, v\}$ and any point in $(\mathbb{V} \setminus C)^2$. Since $(\mathbb{V} \setminus C) \cup \{u_i, v_i\}$ induces an isomorphic copy of the random graph for $i \in \{1, 2\}$, we can refer to Lemma 9.3.40 to arrive at a contradiction: f generates e_E , e_N , or a binary injection of type min or max, all of which violate either $R^{(3)}$ or $S^{(3)}$.

Definition 9.4.10. We say that a ternary injective function $f: \mathbb{V}^3 \to \mathbb{V}$ is

- of type $R^{(3)}$ -majority iff for all $u, v, w \in \mathbb{V}^3$ with $\neq \neq \neq (u, v), \neq \neq \neq (u, w), \neq \neq \neq (v, w)$ we have $R^{(3)}(f(u), f(v), f(w))$ if and only if $R^{(3)}R^{(3)}R^{(3)}(u, v, w), R^{(3)}R^{(3)}S^{(3)}(u, v, w), R^{(3)}S^{(3)}(u, v, w), \text{ or } S^{(3)}R^{(3)}R^{(3)}(u, v, w).$
- of type $R^{(3)}$ -minority iff for all $u, v, w \in \mathbb{V}^3$ with $\neq \neq \neq (u, v), \neq \neq \neq (u, w), \neq \neq \neq (v, w)$ we have $R^{(3)}(f(u), f(v), f(w))$ if and only if $R^{(3)}R^{(3)}R^{(3)}(u, v, w), R^{(3)}S^{(3)}S^{(3)}(u, v, w), S^{(3)}R^{(3)}S^{(3)}(u, v, w), \text{ or } S^{(3)}S^{(3)}R^{(3)}(u, v, w).$

LEMMA 9.4.11. Functions $f: \mathbb{V}^3 \to \mathbb{V}$ of type $R^{(3)}$ -majority do not preserve $R^{(3)}$.

PROOF. Let $u^1, u^2, u^3 \in \mathbb{V}^4$ be such that

- $E(u_1^1, u_2^1)$ and $N(u_i^1, u_j^1)$ for all pairs (i, j) of distinct elements from $\{1, \ldots, 4\}$ that are distinct from (1, 2).
- $E(u_2^2, u_3^2)$ and $N(u_i^1, u_j^1)$ for all pairs (i, j) of distinct elements from $\{1, \ldots, 4\}$ that are distinct from (2, 3).
- $E(u_1^3, u_3^3)$ and $N(u_i^3, u_j^3)$ for all pairs (i, j) of distinct elements from $\{1, \ldots, 4\}$ that are distinct from (1, 3).

Since f is of type $R^{(3)}$ -majority $S^{(3)}(f(u_1), f(u_2), f(u_4))$, $S^{(3)}(f(u_1), f(u_3), f(u_4))$, and $S^{(3)}(f(u_2), f(u_3), f(u_4))$. Since for all four-element subsets of \mathbb{V} there must always be an even number of three-element subsets in $R^{(3)}$, we have $S^{(3)}(f(x_1), f(x_2), f(x_3))$, and hence f does not preserve $R^{(3)}$.

LEMMA 9.4.12. Let $f: \mathbb{V}^3 \to \mathbb{V}$ be of type $R^{(3)}$ -minority. Then $\{f, sw\}$ generates a function of type minority.

PROOF. Let g be any ternary injection of type minority, and let $u, v, w \in \mathbb{V}^3$ with $\neq \neq \neq (u, v), \neq \neq \neq (u, w), \neq \neq \neq (v, w)$ be given. We will show that $R^{(3)}(g(u), g(v), g(w))$ if and only if $R^{(3)}(f(u), f(v), f(w))$. Recall that $R^{(3)}(f(u), f(v), f(w))$ if and only if

$$\begin{split} R^{(3)}S^{(3)}S^{(3)}(u,v,w), \\ S^{(3)}R^{(3)}S^{(3)}(u,v,w), \\ S^{(3)}S^{(3)}R^{(3)}(u,v,w), \\ \text{or } R^{(3)}R^{(3)}R^{(3)}(u,v,w) \,. \end{split}$$

This is in turn the case if and only if the cardinality of the set

$$E \cap \bigcup_{i \in \{1,2,3\}} \{(u_i, v_i), (u_i, w_i), (v_i, w_i)\}$$

is odd, which is the case if and only if $E \cap \{(g(u), g(v)), (g(u), g(w)), (g(v), g(w))\}$ is odd, which is the case if and only if $R^{(3)}(g(u), g(v), g(w))$ holds.

By Corollary 9.4.5, f generates a binary canonical injection s(x,y) of type p_1 . Set t(x,y,z) := s(x,s(y,z)). As in the proof of Proposition 9.3.31 the function p(x,y,z) := f(t(x,y,z),t(y,z,x),t(z,x,y)) is still of type $R^{(3)}$ -minority, and the function q(x,y,z) := g(t(x,y,z),t(y,z,x),t(z,x,y)) is still of type minority. Moreover, by the above we have $R^{(3)}(p(u),p(v),p(w))$ if and only if $R^{(3)}(q(u),q(v),q(w))$ for all $u,v,w \in \mathbb{V}^3$, since t is injective. Therefore, the homogeneity of $(\mathbb{V};R^{(3)})$ implies that for all finite $S \subseteq \mathbb{V}^3$ there exists a unary operation a generated by $\{sw\}$ such that the ternary function a(p(x,y,z)) agrees with q(x,y,z) on S. By local closure, q is thus generated by $\{f,sw\}$.

LEMMA 9.4.13. Let $\mathfrak{B} = (\mathbb{V}; R^{(3)}, S^{(3)}, \ldots)$ be a reduct of $(\mathbb{V}; E)$ such that H_2 is not primitive positive definable. Then \mathfrak{B} has a ternary injective polymorphism which violates H_2 .

PROOF. Since the relation H_2 consists of three orbits of 9-tuples in $\operatorname{Aut}(\mathbb{V}; R^{(3)})$, Lemma 5.3.5 implies that f generates an at most ternary function that violates H_2 , and hence we can assume that f itself is at most ternary; by adding a dummy variable if necessary, we may assume that f is actually ternary. Moreover, f must certainly be essential, since essentially unary operations that preserve $R^{(3)}$ and $S^{(3)}$ are generated by $\{sw\}$ and hence also preserve H_2 . Corollary 9.4.5 implies that \mathfrak{B} is preserved by a binary canonical injection g of type p_1 . Consider

$$h(x, y, z) := g(g(g(f(x, y, z), x), y), z)$$
.

Then h is clearly injective, and still violates H_2 – the latter can easily be verified combining the facts that f violates H_2 , g is of type p_1 , and all tuples in H_2 have pairwise distinct entries.

PROPOSITION 9.4.14. Let f be an operation on $(\mathbb{V}; E)$ that preserves $R^{(3)}$ and $S^{(3)}$ and violates H_2 . Then $\{f, sw\}$ generates a ternary canonical injection of type minority.

PROOF. The proof is similar to the proof of Proposition 9.3.31. By Lemma 9.4.13, we can assume that f is a ternary injection. Because f violates H_2 , there are $x^1, x^2, x^3 \in H_2$ such that $f(x^1, x^2, x^3) \notin H_2$. In the following, we will write $x_i := (x_i^1, x_i^2, x_i^3)$ for $1 \le i \le 9$. So $(f(x_1), \ldots, f(x_9)) \notin H_2$. If there were a map a generated by sw such that $a(x^i) = x^j$ for $1 \le i \ne j \le 3$, then $\{f, sw\}$ would generate a binary injection that still violates H_2 . Proposition 9.4.9 asserts that all binary injections generated by $\{f, sw\}$ are of type $R^{(3)}$ -projection, so we have reached a contradiction since operations of type $R^{(3)}$ -projection preserve H_2 . By permuting arguments of f if necessary, we can therefore assume without loss of generality that

$$R^{(3)}S^{(3)}S^{(3)}(x_1, x_2, x_3), S^{(3)}R^{(3)}S^{(3)}(x_4, x_5, x_6), \text{ and } S^{(3)}S^{(3)}R^{(3)}(x_7, x_8, x_9).$$

We set

$$S:=\{y\in\mathbb{V}^3\mid NNN(x_i,y) \text{ for all } 1\leq i\leq 9\}$$
 .

Consider the ternary relations $Q_1Q_2Q_3$ on \mathbb{V}^3 , where $Q_i \in \{R^{(3)}, S^{(3)}\}$ for $1 \leq i \leq 3$; each of these relations defines a 3-type in $(\mathbb{V}; R^{(3)})$. We claim that for fixed $Q_1Q_2Q_3$, whether or not $R^{(3)}(f(u), f(v), f(w))$ holds for $u, v, w \in S$ with $Q_1Q_2Q_3(u, v, w)$ does not depend on u, v, w. We go through all possibilities of $Q_1Q_2Q_3$.

(1) $Q_1Q_2Q_3 = R^{(3)}S^{(3)}S^{(3)}$. Let $\alpha \in \text{Aut}(\mathbb{V}; R^{(3)})$ be such that the tuple $(x_1^2, x_2^2, x_3^2, u_2, v_2, w_2)$ is mapped to $(x_1^3, x_2^3, x_3^3, u_3, v_3, w_3)$; such an automorphism exists since $NNN(x_1, u), NNN(x_1, v), NNN(x_1, w), NNN(x_2, u), NNN(x_2, v), NNN(x_2, w)$ and since the tuple (x_1^2, x_2^2, x_3^2) has the same type

as (x_1^3, x_2^3, x_3^3) , and (u_2, v_2, w_2) has the same type as (u_3, v_3, w_3) in $(\mathbb{V}; \mathbb{R}^{(3)})$. By Proposition 9.4.9, the operation g defined by $g(x,y) := f(x,y,\alpha(y))$ must be of type $R^{(3)}$ -projection. Hence, $R^{(3)}(g(u_1, u_2), g(v_1, v_2), g(w_1, w_2))$ iff $R^{(3)}(g(x_1^1, x_1^2), g(x_2^1, x_2^2), g(x_3^1, x_3^2))$. Combining this with the equations

$$(f(u), f(v), f(w)) = (g(u_1, u_2), g(v_1, v_2), g(w_1, w_2)) \text{ and } (g(x_1^1, x_1^2), g(x_2^1, x_2^2), g(x_3^1, x_3^2)) = (f(x_1), f(x_2), f(x_3))$$

we get that $R^{(3)}(f(u), f(v), f(w))$ iff $R^{(3)}(f(x_1), f(x_2), f(x_3))$, and so we are

- (2) $Q_1Q_2Q_3 = S^{(3)}R^{(3)}S^{(3)}$ or $Q_1Q_2Q_3 = S^{(3)}S^{(3)}R^{(3)}$. These cases are analogous to the previous case.
- (3) $Q_1Q_2Q_3 = S^{(3)}R^{(3)}R^{(3)}$. Let α be defined as in the first case. By Proposition 9.4.9, the operation defined by $f(x, y, \alpha(y))$ must be of type projection. Reasoning as above, one obtains that $R^{(3)}(f(u), f(v), f(w))$ iff $S^{(3)}(f(x_1), f(x_2), f(x_3)).$
- (4) $Q_1Q_2Q_3 = R^{(3)}S^{(3)}R^{(3)}$ or $Q_1Q_2Q_3 = R^{(3)}R^{(3)}S^{(3)}$. These cases are analo-
- gous to the previous case. (5) $Q_1Q_2Q_3 = R^{(3)}R^{(3)}R^{(3)}$ or $Q_1Q_2Q_3 = S^{(3)}S^{(3)}S^{(3)}$. These cases are trivial since f preserves $R^{(3)}$ and $S^{(3)}$.

To show that f generates an operation of type minority, by Lemma 8.3.14 it suffices to prove that f generates a function of type minority on S, since S is the product of isomorphic copies of $(\mathbb{V}; E)$. We show this by another case distinction, based on the fact that $(f(x_1), \ldots, f(x_9)) \notin H_2$.

- (1) Suppose that $R^{(3)}(f(x_1), f(x_2), f(x_3)), R^{(3)}(f(x_4), f(x_5), f(x_6))$ and that $R^{(3)}(f(x_7), f(x_8), f(x_9))$. By the above, note that $R^{(3)}(f(u), f(v), f(w))$ for $u, v, w \in S$ if and only if $R^{(3)}S^{(3)}S^{(3)}(u, v, w)$, $S^{(3)}R^{(3)}S^{(3)}(u, v, w)$, $S^{(3)}S^{(3)}R^{(3)}(u,v,w)$, or $R^{(3)}R^{(3)}R^{(3)}(u,v,w)$. Hence, f behaves like an $R^{(3)}$ -minority on S, and we are done by Lemma 9.4.12.
- (2) Suppose that $S^{(3)}(f(x_1), f(x_2), f(x_3)), S^{(3)}(f(x_4), f(x_5), f(x_6)),$ and that $S^{(3)}(f(x_7), f(x_8), f(x_9))$. Then f behaves like an $R^{(3)}$ -majority on S, which is impossible by Lemma 9.4.11.
- (3) Suppose that $R^{(3)}(f(x_1), f(x_2), f(x_3)), R^{(3)}(f(x_4), f(x_5), f(x_6)),$ and that $S^{(3)}(f(x_7), f(x_8), f(x_9))$. Let e be a self-embedding of $(\mathbb{V}; E)$ such that for all $w \in \mathbb{V}$, all $1 \leq j \leq 3$, and all $1 \leq i \leq 9$ we have that $N(x_i^j, e(w))$. Then $(u_1, u_2, e(f(u_1, u_2, u_3))) \in S$ for all $(u_1, u_2, u_3) \in S$. Hence, by the above, the ternary operation defined by f(x, y, e(f(x, y, z))) is of type $R^{(3)}$ -majority on S; but this is impossible by Lemma 9.4.11.
- (4) Suppose that $R^{(3)}(f(x_1), f(x_2), f(x_3)), S^{(3)}(f(x_4), f(x_5), f(x_6)),$ and that $R^{(3)}(f(x_7), f(x_8), f(x_9))$. Analogous to the previous case.
- (5) Suppose that $S^{(3)}(f(x_1), f(x_2), f(x_3)), R^{(3)}(f(x_4), f(x_5), f(x_6)),$ and that $R^{(3)}(f(x_7), f(x_8), f(x_9))$. Analogous to the previous case.

Let h(x, y, z) be a ternary injection of type minority generated by f; it remains to make h canonical. By Corollary 9.4.5, f generates a binary canonical injection g(x,y)of type p_1 . Set t(x,y,z) := g(x,g(y,z)). As in the proof of Proposition 9.3.31 the function h(t(x,y,z),t(y,z,x),t(z,x,y)) is still of type minority and canonical.

PROOF OF PROPOSITION 9.4.2. Assume that H_2 is not primitive positive definable; by Theorem 5.2.3 there exists a polymorphism f of \mathfrak{B} that violates H_2 . Since $Aut(\mathfrak{B})$ contains sw, the relations $R^{(3)}$ and $S^{(3)}$ consist of only one orbit of triples in B. Therefore, since they are preserved by all endomorphisms of B, it follows by Theorem 5.2.3 and Lemma 5.3.5 that these relations are primitive positive definable in \mathfrak{B} .

We can now apply Proposition 9.4.14 and obtain that $\{f, sw\}$ generates a ternary injection of type minority which is canonical as a function from $(\mathbb{V}; E)$ to $(\mathbb{V}; E)$. Corollary 9.4.5 implies that \mathfrak{B} is preserved by a binary injection of type p_1 which is canonical as a function from $(\mathbb{V}; E)$ to $(\mathbb{V}; E)$, and the statement follows from Theorem 9.3.8.

9.5. First-order Expansions of $(\mathbb{V}; R^{(4)}, S^{(4)})$

We assume that the endomorphisms of \mathfrak{B} are exactly the functions generated by $\{-\}$. In particular, $\operatorname{Aut}(\mathfrak{B})$ contains – but not sw, and the automorphisms of \mathfrak{B} generate its endomorphisms.

DEFINITION 9.5.1. Let H_1' be the smallest 6-ary relation that is preserved by $\{-\}$ and contains H_1 .

PROPOSITION 9.5.2. There is a primitive positive interpretation of $(\{0,1\}; NAE)$ in $(\mathbb{V}; H'_1)$, and $CSP(\mathbb{V}; H'_1)$ is NP-hard.

PROOF. Similar to the proof of Proposition 9.3.5.

The following is an analog of Proposition 9.3.4 for the situation of this section.

PROPOSITION 9.5.3. Let \mathfrak{B} be a reduct of $(\mathbb{V}; E)$ whose endomorphisms are precisely the unary functions generated by $\{-\}$. Then either H'_1 is primitive positive definable in \mathfrak{B} , or one of the cases (b)-(e) of Proposition 9.3.4 applies.

PROOF. Note that H_1' consists of three orbits of 6-tuples in $\operatorname{Aut}(\mathfrak{B})$, and hence, if H_1' is not primitive positive definable in \mathfrak{B} , then there exists by Theorem 5.2.3 and Lemma 5.3.5 a ternary polymorphism f of \mathfrak{B} that violates H_1' . That is, there are $t^1, t^2, t^3 \in H_1'$ such that $f(t^1, t^2, t^3) \notin H_1'$. Note that for each t^j , either t^j or $-t^j \in H_1$. In the first case we set g_j to be the identity function on \mathbb{V} , in the second case we let g_j be the operation -. Now consider the function f' defined by $f'(x_1, x_2, x_3) := f(g_1(x_1), g_2(x_2), g_3(x_3))$. We have that $s^j := g_j^{-1}(t^j) \in H_1$, but $f'(s^1, s^2, s^3) = f(t^1, t^2, t^3)$ is not in H_1' . Consider the function h(x) := f'(x, x, x); since the endomorphisms of \mathfrak{B} are generated by $\{-\}$, h either preserves E and N, or it flips them. By replacing f' by -(f') in the latter case we may assume that h preserves E and N. Note that we still have that $f'(s^1, s^2, s^3)$ is not in H_1' , and therefore not in H_1 either. Hence, f' violates H_1 .

Now suppose that f' violates E or N; we will derive a contradiction. Say without loss of generality that there are $u,v\in\mathbb{V}^3$ with EEE(u,v) such that E(f'(u),f'(v)) does not hold. Pick distinct $a,b,c,d\in\mathbb{V}$ such that $\{a,b,c,d\}$ induces a clique in $(\mathbb{V};E)$, and such that each element is connected to all entries on u,v by an edge. Pick then $\alpha_1,\alpha_2,\alpha_3\in\mathrm{Aut}(\mathbb{V};E)$ such that $\alpha_i(a)=u_i$ and $\alpha_i(b)=v_i$ for all $i\in\{1,2,3\}$, and such that $\alpha_1(c)=\alpha_2(c)=\alpha_3(c)=c$ and $\alpha_1(d)=\alpha_2(d)=\alpha_3(d)=d$. We then have that the function $x\mapsto f'(\alpha_1(x),\alpha_2(x),\alpha_3(x))$ maps (c,d) to an edge since h(x) preserves E, but it does not map (a,b) to an edge, by our assumption on u and v. This is, however, impossible, since the function must be generated by $\{-\}$.

Therefore, f' preserves E and N. Then Proposition 9.3.4 implies that f' generates functions with the desired properties, or a binary canonical injection of type max or min. A binary canonical injection of type max together with $\{-\}$ generates a binary canonical injection of type min, and vice versa. Then

 $\max(\min(x, y), \min(y, z), \min(x, z))$

is a ternary canonical injection of type majority with the desired properties, and we are also done in this case, since identifying two of its variables yields a binary canonical injection of type projection. \Box

9.6. First-order Expansions of $(\mathbb{V}; R^{(5)}, S^{(5)})$

We assume that the endomorphisms of \mathfrak{B} are precisely the unary functions generated by $\{-, sw\}$. In particular, $\operatorname{Aut}(\mathfrak{B})$ contains -, sw, and the automorphisms of \mathfrak{B} generate its endomorphisms.

DEFINITION 9.6.1. Let H'_2 be the smallest 9-ary relation that is preserved by – and contains H_2 .

PROPOSITION 9.6.2. There is a primitive positive interpretation of $(\{0,1\}; NAE)$ in $(\mathbb{V}; H_2')$, and $CSP(\mathbb{V}; H_2')$ is NP-hard.

Proof. Similar to Proposition 9.4.3.

The following is an analog of Proposition 9.3.4 for the situation of this section.

PROPOSITION 9.6.3. Let \mathfrak{B} be a reduct of $(\mathbb{V}; E)$ whose endomorphisms are precisely the unary functions generated by $\{-, sw\}$. Then H'_2 is primitive positive definable in \mathfrak{B} , or (b) or (d) from Proposition 9.3.4 applies.

PROOF. Note that H'_2 consists of three orbits of 9-tuples in $\operatorname{Aut}(\mathfrak{B})$, and hence, if H'_2 is not primitive positive definable in \mathfrak{B} , then there exists by Theorem 5.2.3 and Lemma 5.3.5 a ternary polymorphism f of \mathfrak{B} that violates H'_2 . That is, there are $t^1, t^2, t^3 \in H'_2$ such that $f(t^1, t^2, t^3) \notin H'_2$. Note that for each t^j , either t^j or $-t^j \in H_2$. In the first case we set g_j to be the identity function on \mathbb{V} , in the second case we let g_j be the operation -. Now consider the function f' defined by $f'(x_1, x_2, x_3) := f(g_1(x_1), g_2(x_2), g_3(x_3))$. We have that $s^j := g_j^{-1}(t^j) \in H_2$, but $f'(s^1, s^2, s^3) = f(t^1, t^2, t^3)$ is not in H'_2 , and therefore not in H_2 either. Hence, f' violates H_2 . The function h(x) := f'(x, x, x) is generated by $\{-, sw\}$, and hence h either preserves $R^{(3)}$ and $S^{(3)}$, or it flips them. Since $f'(s^1, s^2, s^3)$ is not in H'_2 , neither is $-f'(s^1, s^2, s^3)$, and in particular not in H_2 , so also -f' violates H_2 . Hence, by replacing f' with -f' if necessary, we may assume that h preserves $R^{(3)}$ and $S^{(3)}$.

We claim that f' preserves $R^{(3)}$ and $S^{(3)}$. Suppose for contradiction that there are $u, v, w \in \mathbb{V}^3$ with $R^{(3)}(u_i, v_i, w_i)$ for all $i \in \{1, 2, 3\}$ such that $R^{(3)}(f'(u), f'(v), f'(w))$ does not hold; the case where f' violates $S^{(3)}$ can be treated similarly. If (u_1, v_1, w_1) , (u_2, v_2, w_2) , and (u_3, v_3, w_3) all lie in the same orbit of triples in $(\mathbb{V}; E)$, then we choose $a, b, c \in \mathbb{V}$ with $R^{(3)}(a, b, c)$ such that N(x, y) for $x \in \{a, b, c\}$ and $y \in \{u_1, v_1, w_1, u_2, v_2, w_2, u_3, v_3, w_3\}$. Then by the homogeneity of $(\mathbb{V}; E)$ there is for each $i \in \{2, 3\}$ a unary operation $\alpha_i \in \operatorname{Aut}(\mathbb{V}; E)$ such that $\alpha_i(u_1, v_1, w_1, a, b, c) = (u_i, v_i, w_i, a, b, c)$. We then have that the unary function $g(x) := f'(x, \alpha_2(x), \alpha_3(x))$ maps $(u_1, v_1, w_1) \in R^{(3)}$ to $(f'(u), f'(v), f'(w)) \notin R^{(3)}$. But g and the function h above agree on $\{a, b, c\}$, and hence g preserves $R^{(3)}$ on $\{a, b, c\}$, but violates it on $\{u_1, v_1, w_1\}$. This contradicts the assumption that g is generated by $\{-, sw\}$.

So suppose in the following that $R^{(3)}(f'(u), f'(v), f'(w))$ for all $u, v, w \in \mathbb{V}^3$ with $R^{(3)}(u_i, v_i, w_i)$ for all $i \in \{1, 2, 3\}$ such that u, v, w belong to the same orbit of triples in $(\mathbb{V}; E)$. We now show that $R^{(3)}(f'(u), f'(v), f'(w))$ for all $u, v, w \in \mathbb{V}^3$ with $R^{(3)}(u_i, v_i, w_i)$ for all $i \in \{1, 2, 3\}$. To this end, note that for each $i \in \{2, 3\}$ there is a subset S_i of $\{u_i, v_i, w_i\}$ such that $(sw_{S_i}(u_i), sw_{S_i}(v_i), sw_{S_i}(w_i))$ and (u_1, v_1, w_1) belong to the same orbit in $(\mathbb{V}; E)$. Hence, there is $\beta_i \in \operatorname{Aut}(\mathbb{V}; E)$ such that $\beta_i(sw_{S_i}(u_1)) = u_i$, $\beta_i(sw_{S_i}(v_1)) = v_i$, and $\beta_i(sw_{S_i}(w_1)) = w_i$. Pick $a, b, c \in V \setminus \bigcup_{i \in \{1, 2, 3\}} \{u_i, v_i, w_i\}$. Note that for both $i \in \{2, 3\}$ we have that the

triples (a, b, c) and $(sw_{S_i}(a), sw_{S_i}(b), sw_{S_i}(c))$ lie in the same orbit. We then have that the function $x \mapsto f'(x, \beta_2(sw_{S_2}(x)), \beta_3(sw_{S_3}(x)))$ maps $(u_1, v_1, w_1) \in R^{(3)}$ to $(f'(u), f'(v), f'(w)) \notin R^{(3)}$. But the same unary function also maps $(a, b, c) \in R^{(3)}$ to a tuple in $R^{(3)}$ since f' by assumption preserves $R^{(3)}$ on tuples $R^{(3)}$ that lie in the same orbit, and indeed we have that for $i \in \{2,3\}$ the triples (a,b,c) and $(\beta_i(sw_{S_i}(a)), \beta_i(sw_{S_i}(b)), \beta_i(sw_{S_i}(c)))$ lie in the same orbit. This again contradicts the assumption that the unary function is generated by $\{-, sw\}$.

We therefore have that f' preserves $R^{(3)}$ and $S^{(3)}$. Since it violates H_2 , Proposition 9.4.3 implies that $\{f', sw\}$ generates a ternary canonical injection of type minority, and we are done.

9.7. Algorithms for Graph-SAT problems

Throughout this section we assume that \mathfrak{B} is a structure with finite relational signature τ and a first-order definition in $(\mathbb{V}; E)$.

- **9.7.1. The unbalanced case.** We now prove tractability of the CSP for templates \mathfrak{B} as in cases (b) and (c) of Proposition 9.3.4, that is, for structures with a first-order definition in $(\mathbb{V}; E)$ that have
 - a ternary polymorphism of type majority or minority, and
 - a binary polymorphism of type p_1 which is either E-dominated or N-dominated in the second argument.

By duality, we may assume that the polymorphism of type p_1 is E-dominated in the second argument.

It turns out that for such templates \mathfrak{B} we can reduce $CSP(\mathfrak{B})$ to the CSP of a structure that we call the *injectivization* of \mathfrak{B} . This implies in turn that the CSP can be reduced to a CSP over a Boolean domain.

Definition 9.7.1. A tuple is called injective if all its entries have pairwise distinct entries. A relation is called injective if all its tuples are injective. A structure is called injective if all its relations are injective.

Definition 9.7.2. We define injectivizations for relations, atomic formulas, and structures.

- Let R be any relation. Then the injectivization of R, denoted by inj(R), is the (injective) relation consisting of all injective tuples of R.
- Let $\phi(x_1, \ldots, x_n)$ be an atomic formula in the language of \mathfrak{B} , where x_1, \ldots, x_n is a list of the variables that appear in ϕ . Then the injectivization of $\phi(x_1, \ldots, x_n)$ is the formula $R_{\phi}^{\text{inj}}(x_1, \ldots, x_n)$, where R_{ϕ}^{inj} is a relation symbol which stands for the injectivization of the relation defined by ϕ .
- The injectivization of a relational structure \mathfrak{B} , denoted by $\operatorname{inj}(\mathfrak{B})$, is the relational structure \mathfrak{C} with the same domain as \mathfrak{B} whose relations are the injectivizations of the atomic formulas over \mathfrak{B} , i.e., the relations $R_{\phi}^{\operatorname{inj}}$.

Note that $\operatorname{inj}(\mathfrak{B})$ also contains the injectivizations of relations that are defined by atomic formulas in which one variable might appear several times. In particular, the injectivization of an atomic formula ϕ might have smaller arity than the relation symbol that appears in ϕ .

To state the reduction to the CSP of an injectivization, we also need the following operations on instances of CSP(\mathfrak{B}). Here, it will be convenient to view instances of CSP(\mathfrak{B}) as primitive positive τ -sentences (see Section 1.2).

```
// Input: An instance \phi of CSP(\mathfrak{B}) with variables W
While \phi contains a constraint that implies x=y for x,y\in W do Replace each occurrence of x by y in \phi.

If \phi contains a false constraint then reject Loop
Accept if and only if \operatorname{inj}(\phi) is satisfiable in \operatorname{inj}(\mathfrak{B}).
```

FIGURE 9.1. Algorithm for $CSP(\mathfrak{B})$ when \mathfrak{B} is preserved by an unbalanced binary injection, using an algorithm for $inj(\mathfrak{B})$.

DEFINITION 9.7.3. Let ϕ be an instance of $CSP(\mathfrak{B})$. Then the injectivization of ϕ , denoted by $inj(\phi)$, is the instance ψ of $CSP(inj(\mathfrak{B}))$ obtained from ϕ by replacing each conjunct $\phi(x_1,\ldots,x_n)$ of ϕ by $R_{\phi}^{inj}(x_1,\ldots,x_n)$.

We say that a constraint in an instance of $CSP(\mathfrak{B})$ is false if it defines an empty relation in \mathfrak{B} . Note that a constraint $R(x_1, \ldots, x_k)$ might be false even if the relation R is non-empty (simply because some of the variables from x_1, \ldots, x_k might be equal).

PROPOSITION 9.7.4. Let \mathfrak{B} be preserved by a binary injection f of type p_1 that is E-dominated in the second argument. Then the algorithm shown in Figure 9.1 is a polynomial-time reduction of $CSP(\mathfrak{B})$ to $CSP(inj(\mathfrak{B}))$.

PROOF. In the main loop, when the algorithm detects a constraint that is false and therefore rejects, then ϕ cannot hold in \mathfrak{B} , because the algorithm only contracts variables x and y when x=y in all solutions to ϕ – and contractions are the only modifications performed on the input formula ϕ . So suppose that the algorithm does not reject, and let ψ be the instance of CSP(\mathfrak{B}) computed by the algorithm when it reaches the final line of the algorithm.

By the observation we just made it suffices to show that ψ holds in $\mathfrak B$ if and only if $\operatorname{inj}(\psi)$ holds in $\operatorname{inj}(\mathfrak B)$. It is clear that when $\operatorname{inj}(\psi)$ holds in $\operatorname{inj}(\mathfrak B)$ then ψ holds in $\mathfrak B$ (since the constraints in $\operatorname{inj}(\psi)$ have been made stronger). We now prove that if ψ has a solution s in $\mathfrak B$, then there is also a solution for $\operatorname{inj}(\psi)$ in $\operatorname{inj}(\mathfrak B)$.

Let s' be any mapping from the variable set W of ψ to $\mathbb V$ such that for all distinct $x,y\in W$ we have that

```
• if E(s(x), s(y)) then E(s'(x), s'(y));
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- if N(s(x), s(y)) then N(s'(x), s'(y));
- if s(x) = s(y) then E(s'(x), s'(y)).

By universality of (V; E), such a mapping exists. We claim that s' is a solution to ψ in \mathfrak{B} . Since s' is injective, it is then clearly also a solution to $\operatorname{inj}(\psi)$. To prove the claim, let γ be a constraint of ψ on the variables $x_1,\ldots,x_k\in W$. Since we are at the final stage of the algorithm, we can conclude that $\gamma(x_1,\ldots,x_k)$ does not imply equality of any of the variables x_1,\ldots,x_k , and so there is for all $1\leq i< j\leq k$ a tuple $t^{(i,j)}$ such that $R(t^{(i,j)})$ and $t_i\neq t_j$ hold. Since $\gamma(x_1,\ldots,x_k)$ is preserved by a binary injection, it is also preserved by injections of arbitrary arity (it is straightforward to build such terms from a binary injection). Application of an injection of arity $\binom{k}{2}$ to the tuples $t^{(i,j)}$ shows that $\gamma(x_1,\ldots,x_k)$ is satisfied by an injective tuple (t_1,\ldots,t_k) .

Consider the mapping $r: \{x_1, \ldots, x_k\} \to \mathbb{V}$ given by $r(x_l) := f(s(x_l), t_l)$. This assignment has the property that for all $i, j \in S$ if $E(s(x_i), s(x_j))$, then E(r(x), r(y)), and if $N(s(x_i), s(x_j))$ then $N(r(x_i), r(x_j))$, because f is of type p_1 . Moreover, if $s(x_i) = s(x_j)$ then $E(r(x_i), r(x_j))$ because f is E-dominated in the second argument. Therefore, $(s'(x_1), \ldots, s'(x_n))$ and $(r(x_1), \ldots, r(x_n))$ have the same type in $(\mathbb{V}; E)$. Since f is a polymorphism of \mathfrak{B} , we have that $(r(x_1), \ldots, r(x_n))$ satisfies the constraint

 $\gamma(x_1,\ldots,x_n)$. Hence, s' satisfies $\gamma(x_1,\ldots,x_n)$ as well. We conclude that s' satisfies all the constraints of ψ , proving our claim.

To reduce the CSP for injective structures to Boolean CSPs, we make the following definition.

DEFINITION 9.7.5. Let t be a k-tuple of distinct vertices of $(\mathbb{V}; E)$, and let q be $\binom{k}{2}$. Then Boole(t) is the q-tuple $(a_{1,2}, a_{1,3}, \ldots, a_{1,k}, a_{2,3}, \ldots, a_{k-1,k}) \in \{0,1\}^q$ such that $a_{i,j} = 0$ if $N(t_i, t_j)$ and $a_{i,j} = 1$ if $E(t_i, t_j)$. If R is a k-ary injective relation, then Boole(R) is the q-ary Boolean relation $\{\text{Boole}(t) \mid t \in R\}$. If ϕ is a formula that defines an injective relation R over $(\mathbb{V}; E)$, then we also write Boole(ϕ) instead of Boole(inj(R)). Finally, for injective \mathfrak{B} , we write Boole(\mathfrak{B}) for the structure over a Boolean domain with the relation Boole(R) for each relation R of \mathfrak{B} .

PROPOSITION 9.7.6. Let \mathfrak{B} be injective. Then there is a polynomial-time reduction from $CSP(\mathfrak{B})$ to $CSP(Boole(\mathfrak{B}))$.

PROOF. Let ϕ be an instance of $\mathrm{CSP}(\mathfrak{B})$ with variable set W. We create an instance ψ of $\mathrm{CSP}(\mathrm{Boole}(\mathfrak{B}))$ as follows. The variable set of ψ is the set of unordered pairs of variables from ϕ . When $\gamma = R(x_1, \ldots, x_k)$ is a constraint in ϕ , then ψ contains the constraint $\mathrm{Boole}(R)(x_{1,2}, x_{1,3}, \ldots, x_{1,k}, x_{2,3}, \ldots, x_{k-1,k})$. It is straightforward to verify that ψ can be computed from ϕ in polynomial time, and that ϕ is a satisfiable instance of $\mathrm{CSP}(\mathfrak{B})$ if and only if ψ is a satisfiable instance of $\mathrm{CSP}(\mathfrak{B})$. \square

The following proposition, together with Propositions 9.7.4 and 9.7.6, solves the case where $Pol(\mathfrak{B})$ contains a ternary injection of type minority or majority as well as one of the functions of Theorem 9.3.8 which are unbalanced and of type projection. It thus shows tractability of cases (b) and (c) of Proposition 9.3.4 given that none of the other cases applies.

PROPOSITION 9.7.7. Let \mathfrak{B} be injective, and suppose it has an polymorphism of type minority (majority). Then Boole(\mathfrak{B}) has a minority (majority) polymorphism, and CSP(Boole(\mathfrak{B})) can be solved in polynomial time.

PROOF. It is straightforward to show that $Boole(\mathfrak{B})$ has a minority (majority) polymorphism. We have seen in Theorem 5.4.3 that $CSP(Boole(\mathfrak{B}))$ can then be solved in polynomial time.

9.7.2. Tractability for type minority. We show tractability of $CSP(\mathfrak{B})$ when \mathfrak{B} has a polymorphism of type minority as well as a binary canonical injection of type p_1 which is balanced. We start by proving that in this case the relations of \mathfrak{B} can be defined in (V; E) by first-order formulas of a certain restricted syntactic form; this normal form will later be essential for our algorithm.

Recall that a Boolean relation R is affine if it can be defined by a conjunction of linear equations modulo 2, which is the case if and only if R is preserved by the Boolean minority operation (see Theorem 5.4.3). In the following, we denote the Boolean exclusive-or connective (xor) by \oplus .

DEFINITION 9.7.8. A graph formula is called edge affine if it is a conjunction of formulas of the form

$$x_1 \neq y_1 \vee \ldots \vee x_k \neq y_k$$

$$\vee (u_1 \neq v_1 \wedge \cdots \wedge u_l \neq v_l$$

$$\wedge E(u_1, v_1) \oplus \cdots \oplus E(u_l, v_l) = p)$$

$$\vee (u_1 = v_1 \wedge \cdots \wedge u_l = v_l),$$

where $p \in \{0, 1\}$, variables need not be distinct, and each of k and l can be 0.

DEFINITION 9.7.9. A ternary operation $f: \mathbb{V}^3 \to \mathbb{V}$ is called balanced if for every $c \in \mathbb{V}$, the binary operations $(x,y) \mapsto f(x,y,c)$, $(x,z) \mapsto f(x,c,z)$, and $(y,z) \mapsto f(c,y,z)$ are balanced injections of type p_1 .

Observe that the existence of balanced operations and even balanced minority operations follows from the fact that $(\mathbb{V}; E)$ contains all countable graphs as induced subgraphs.

PROPOSITION 9.7.10. Let R be a relation with a first-order definition in $(\mathbb{V}; E)$. Then the following are equivalent:

- (1) R can be defined by an edge affine formula;
- (2) R is preserved by every injection of type minority which is balanced;
- (3) R is preserved by an injection of type minority, and a balanced binary injection of type p_1 .

PROOF. We first show the implication from 1 to 2, that every n-ary relation R defined by an edge affine formula $\psi(x_1,\ldots,x_n)$ is preserved by balanced functions f of type minority. We verify that each clause ϕ from ψ is preserved by f. By injectivity of f, it is easy to see that we only have to show this for the case that ϕ does not contain disequality disjuncts (i.e., for the case k = 0). In this case ϕ is of the following form, for $p \in \{0,1\}$ and $u_1,\ldots,u_l,v_1,\ldots,v_l \in \{x_1,\ldots,x_n\}$.

$$\phi = (u_1 \neq v_1 \land \dots \land u_l \neq v_l \land (E(u_1, v_1) \oplus \dots \oplus E(u_l, v_l) = p))$$

$$\lor (u_1 = v_1 \land \dots \land u_l = v_l)$$

In the following, it will sometimes be notationally convenient to consider tuples in $(\mathbb{V}; E)$ satisfying a formula as mappings from the variable set of the formula to \mathbb{V} . Let $t_1, t_2, t_3 : \{x_1, \ldots, x_n\} \to \mathbb{V}$ be three mappings that satisfy ϕ . We have to show that the mapping $t_0 : \{x_1, \ldots, x_n\} \to \mathbb{V}$ defined by $t_0(x) = f(t_1(x), t_2(x), t_3(x))$ satisfies ϕ .

Suppose first that each of t_1, t_2, t_3 satisfies $u_1 \neq v_1 \land \cdots \land u_l \neq v_l$. In this case, $t_0(u_1) \neq t_0(v_1) \land \cdots \land t_0(u_l) \neq t_0(v_l)$, since f preserves \neq . Note that $E(t_0(u_i), t_0(v_i))$, for $1 \leq i \leq l$, if and only if $E(t_1(u_i), t_1(v_i)) \oplus E(t_2(u_i), t_2(v_i)) \oplus E(t_3(u_i), t_3(v_i)) = 1$. Therefore, since each t_1, t_2, t_3 satisfies $E(u_1, v_1) \oplus \cdots \oplus E(u_l, v_l) = p$, we find that t_0 also satisfies $E(u_1, v_1) \oplus \cdots \oplus E(u_l, v_l) = p \oplus p \oplus p = p$.

Next, suppose that one of t_1, t_2, t_3 satisfies $u_i = v_i$ for some (and therefore for all) $1 \le i \le l$. By permuting arguments of f, we can assume that $t_1(u_i) = t_1(v_i)$ for all $i \in \{1, \ldots, l\}$. Since the function f is balanced, the operation $g \colon (y, z) \mapsto f(t_1(u_i), y, z)$ is a balanced injection of type p_1 . Suppose that $t_2(u_i) = t_2(v_i)$. Then $E(t_0(u_i), t_0(v_i))$ if and only if $E(t_3(u_i), t_3(v_i))$, since g is balanced. Hence, t_0 satisfies ϕ . Now suppose that $t_2(u_i) \ne t_2(v_i)$. Then $E(t_0(u_i), t_0(v_i))$ if and only if $E(t_2(u_i), t_2(v_i))$, since g is of type p_1 . Again, t_0 satisfies ϕ . This shows that f preserves ϕ .

The implication from 2 to 3 is trivial, since every balanced function of type minority generates a balanced binary injection of type p_1 by identification of two of its variables. It is also here that we have to check the existence of balanced injections of type minority; as mentioned above, this follows easily from the universality of $(\mathbb{V}; E)$.

We show the implication from 3 to 1 by induction on the arity n of the relation R. Let g be the balanced binary injection of type p_1 , and let h be the operation of type minority. For n=2 the statement of the theorem holds, because all binary relations with a first-order definition in $(\mathbb{V}; E)$ can be defined over $(\mathbb{V}; E)$ by expressions as in Definition 9.7.8:

- For $x \neq y$ we set k = 1 and l = 0.
- For $\neg E(x, y)$ we can set k = 0, l = 1, p = 0.
- For $\neg N(x, y)$ we can set k = 0, l = 1, p = 1.
- Then, E(x,y) can be expressed as $(x \neq y) \land \neg N(x,y)$.
- N(x,y) can be expressed as $(x \neq y) \land \neg E(x,y)$.
- x = y can be expressed as $\neg E(x, y) \land \neg N(x, y)$.
- The empty relation can be expressed as $E(x,y) \wedge N(x,y)$.
- Finally, \mathbb{V}^2 can be defined by the empty conjunction.

For n > 2, we construct a formula ϕ that defines the relation $R(x_1, \ldots, x_n)$ as follows. If there are distinct $i, j \in \{1, \ldots, n\}$ such that for all tuples t in R we have $t_i = t_j$, consider the relation defined by $\exists x_i.R(x_1, \ldots, x_n)$. This relation is also preserved by g and h, and by inductive assumption has a definition ψ as required. Then the formula $\phi := (x_i = x_j) \land \psi$ proves the claim. So let us assume that for all distinct i, j there is a tuple $t \in R$ where $t_i \neq t_j$. Note that since R is preserved by the binary injective operation g, this implies that R also contains an injective tuple.

Since R is preserved by an operation of type minority, the relation Boole(inj(R)) is preserved by the Boolean minority operation, and hence has a definition by a conjunction of linear equations modulo 2 (Theorem 5.4.3). From this definition it is straightforward to obtain a definition $\psi(x_1,\ldots,x_n)$ of inj(R) which is the conjunction of $\bigwedge_{i< j< n} x_i \neq x_j$ and of formulas of the form

$$E(u_1, v_1) \oplus \cdots \oplus E(u_l, v_l) = p$$
,

for $u_1, \ldots, u_l, v_1, \ldots, v_l \in \{x_1, \ldots, x_n\}$. It is clear that we can assume that none of the formulas of the form $E(u_1, v_1) \oplus \cdots \oplus E(u_l, v_l) = p$ in ψ can be equivalently replaced by a conjunction of shorter formulas of this form.

For all $i, j \in \{1, ..., n\}$ with i < j, let $R_{i,j}$ be the relation that holds for the tuple $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ iff $R(x_1, ..., x_{i-1}, x_j, x_{i+1}, ..., x_n)$ holds. Because $R_{i,j}$ is preserved by g and h, but has arity n-1, it has a definition $\psi_{i,j}$ as in the statement by inductive assumption. We call the conjuncts of $\psi_{i,j}$ also the *clauses* of $\psi_{i,j}$.

Let ϕ be the conjunction composed of conjuncts from the following two groups:

- (1) $\gamma \vee (x_i \neq x_j)$ for all $i < j \le n$ and each clause γ of $\psi_{i,j}$;
- (2) when $\eta = (E(u_1, v_1) \oplus \cdots \oplus E(u_l, v_l) = p)$ is a conjunct of ψ , then ϕ contains the formula

$$(u_1 \neq v_1 \wedge \cdots \wedge u_l \neq v_l \wedge \eta)$$

\(\forall (u_1 = v_1 \lambda \cdots \lambda u_l = v_l)\).

Obviously, ϕ is a formula of the required form. We have to verify that ϕ defines R.

Let t be an n-tuple such that $t \notin R$. If t is injective, then t violates a formula of the form

$$E(u_1, v_1) \oplus \cdots \oplus E(u_l, v_l) = p$$

from the formula ψ defining inj(R), and hence it violates a conjunct of ϕ of the second group. If there are i,j such that $t_i=t_j$ then the tuple $t^i:=(t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n)\notin R_{i,j}$. Therefore some conjunct γ of $\psi_{i,j}$ is not satisfied by t^i , and $\gamma\vee(x_i\neq x_j)$ is not satisfied by t. Thus, in this case t does not satisfy ϕ either.

It remains to verify that all $t \in R$ satisfy ϕ . Let $\gamma \vee (x_i \neq x_j)$ be a conjunct of ϕ created from some clause in $\psi_{i,j}$. If $t_i \neq t_j$, then t satisfies $x_i \neq x_j$. If $t_i = t_j$, then $(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n) \in R_{i,j}$ and thus this tuple satisfies $\psi_{i,j}$. This also implies that t satisfies ϕ . Now, let η be a conjunct of ϕ from the second group. We distinguish three cases.

(1) For all $1 \le i \le l$ we have that t satisfies $u_i = v_i$. In this case we are clearly done since t satisfies the second disjunct of η .

- (2) For all $1 \leq i \leq l$ we have that t satisfies $u_i \neq v_i$. Suppose for contradiction that t does not satisfy $E(u_1, v_1) \oplus \cdots \oplus E(u_l, v_l) = p$. Let $r \in R$ be injective, and consider the tuple s := g(t, r). Then $s \in R$, and s is injective since the tuple r and the function g are injective. However, since g is of type p_1 , we have $E(s(u_i), s(v_i))$ if and only if $E(t(u_i), t(v_i))$, for all $1 \leq i \leq l$. Hence, s violates the conjunct $E(u_1, v_1) \oplus \cdots \oplus E(u_l, v_l) = p$ from ψ , a contradiction since $s \in \text{inj}(R)$.
- (3) The remaining case is that there is a proper non-empty subset S of $\{1,\ldots,l\}$ such that t satisfies $u_i=v_i$ for all $i\in S$ and t satisfies $u_i\neq v_i$ for all $i\in \{1,\ldots,n\}\setminus S$. We claim that this case cannot occur. Suppose that all tuples t' from $\operatorname{inj}(R)$ satisfy that $\bigoplus_{i\in S} E(u_i,v_i)=1$. In this case we could have replaced $E(u_1,v_1)\oplus\cdots\oplus E(u_l,v_l)=p$ by the two shorter formulas $\bigoplus_{i\in S} E(u_i,v_i)=1$ and $\bigoplus_{i\in [n]\setminus S} E(u_i,v_i)=p\oplus 1$, in contradiction to our assumption on ψ . Hence, there is a tuple $s\in \operatorname{inj}(R)$ where $\bigoplus_{i\in S} E(u_i,v_i)=1$. Now, for the tuple g(t,s) we have

$$\bigoplus_{i \in [n]} E(u_i, v_i) = \bigoplus_{i \in S} E(u_i, v_i) \oplus \bigoplus_{i \in [n] \setminus S} E(u_i, v_i)$$

$$= 1 \oplus p$$

$$\neq p$$

which is a contradiction since $g(t,s) \in \operatorname{inj}(R)$.

Hence, all $t \in R$ satisfy all conjuncts of ϕ . We conclude that ϕ defines R.

We now present a polynomial-time algorithm for $CSP(\mathfrak{B})$ for the case that \mathfrak{B} has finitely many relations that are all edge affine.

DEFINITION 9.7.11. Suppose all relations of \mathfrak{B} are edge affine, and let ϕ be an instance of CSP(\mathfrak{B}). Then the graph of ϕ is the (undirected) graph whose vertices are unordered pairs of distinct variables of ϕ , and which has an edge between distinct sets $\{a,b\}$ and $\{c,d\}$ if ϕ contains a constraint whose definition as in Definition 9.7.8 has a conjunct of the form

$$(u_1 \neq v_1 \wedge \dots \wedge u_l \neq v_l \wedge (E(u_1, v_1) \oplus \dots \oplus E(u_l, v_l) = p))$$

$$\vee (u_1 = v_1 \wedge \dots \wedge u_l = v_l)$$

such that
$$\{a,b\} = \{u_i, v_i\}$$
 and $\{c,d\} = \{u_i, v_i\}$ for some $i, j \in \{1, ..., l\}$.

It is clear that for \mathfrak{B} with finite signature, the graph of an instance ϕ of CSP(\mathfrak{B}) can be computed in linear time from ϕ .

DEFINITION 9.7.12. Let \mathfrak{B} only have edge affine relations, and let ϕ be an instance of $\mathrm{CSP}(\mathfrak{B})$. For a set C of 2-element subsets of variables of ϕ , we define $\mathrm{inj}(\Phi,C)$ to be the following affine Boolean formula. The set of variables of $\mathrm{inj}(\phi,C)$ is C. The constraints of $\mathrm{inj}(\phi,C)$ are obtained from the constraints γ of ϕ as follows. If γ has a definition as in Definition 9.7.8 with a clause of the form

$$(u_1 \neq v_1 \wedge \dots \wedge u_l \neq v_l \wedge (E(u_1, v_1) \oplus \dots \oplus E(u_l, v_l) = p))$$

$$\vee (u_1 = v_1 \wedge \dots \wedge u_l = v_l)$$

where all pairs $\{u_i, v_i\}$ are in C, then $\operatorname{inj}(\phi, C)$ contains the conjunct $\{u_1, v_1\} \oplus \cdots \oplus \{u_l, v_l\} = p$.

Tractability of case (d) of Proposition 9.3.4 now follows from the following proposition and Proposition 9.7.10.

```
// Input: An instance \phi of CSP(\mathfrak B)
Repeat
For each connected component C of the graph of \phi do
Let \psi be the affine Boolean formula \operatorname{inj}(\phi,C).
If \psi is unsatisfiable then
For each \{x,y\} \in C do
Replace each occurrence of x by y in \phi.
If \phi contains a false constraint then reject
Loop
Until \operatorname{inj}(\phi,C) is satisfiable for all components C
Accept
```

FIGURE 9.2. A polynomial-time algorithm for $CSP(\mathfrak{B})$ when \mathfrak{B} is preserved by a balanced operation of type minority.

PROPOSITION 9.7.13. Let \mathfrak{B} be a structure with a first-order definition in $(\mathbb{V}; E)$ and a finite signature, and suppose that \mathfrak{B} is preserved by a balanced injection of type minority. Then the algorithm shown in Figure 9.2 solves $CSP(\mathfrak{B})$ in polynomial time.

PROOF. We first show that when the algorithm detects a constraint that is false and therefore rejects in the innermost loop, then ϕ must be unsatisfiable. Since variable contractions are the only modifications performed on the input formula ϕ , it suffices to show that the algorithm only equates variables x and y when x=y in all solutions to ϕ . To see that this is true, assume that $\psi:=\operatorname{inj}(\phi,C)$ is an unsatisfiable Boolean formula for some connected component C. Hence, in any solution s to ϕ there must be a pair $\{x,y\}$ in C such that s(x)=s(y). It follows immediately from the definition of the graph of ϕ that then s(u)=s(v) for all $\{u,v\}$ adjacent to $\{x,y\}$ in the graph of ϕ . By connectivity of C, we have that s(u)=s(v) for all $\{u,v\}\in C$. Since this holds for any solution to ϕ , the contractions in the innermost loop of the algorithm preserve satisfiability.

So we only have to show that when the algorithm accepts, there is indeed a solution to ϕ . When the algorithm accepts, we must have that $\operatorname{inj}(\phi, C)$ has a solution s_C for all components C of the graph of ϕ . Let s be a mapping from the variables of ϕ to the \mathbb{V} such that $E(x_i, x_j)$ if $\{x_i, x_j\}$ is in component C of the graph of ϕ and $s_C(\{x_i, x_j\}) = 1$, and $N(x_i, x_j)$ otherwise. It is straightforward to verify that this assignment satisfies all of the constraints.

9.7.3. Tractability for type majority. We turn to case (e) of Proposition 9.3.4, i.e., the case where \mathfrak{B} has ternary injection of type majority and a binary canonical injection of type p_1 which is balanced.

Recall that a Boolean relation is called *bijunctive* if it can be defined by a conjunction of clauses of size at most two. It is well-known that a Boolean relation is bijunctive if and only if it is preserved by the Boolean majority operation (see Section 5.4).

DEFINITION 9.7.14. A formula is called graph bijunctive iff it is a conjunction of graph bijunctive clauses, i.e., formulas of the form

$$x_1 \neq y_1 \vee \cdots \vee x_k \neq y_k \vee \phi$$

where ϕ is of one of the following forms

- (i) $u_1 = v_1$;
- (ii) $L_1(u_1, v_1)$;
- (iii) $L_1(u_1, v_1) \vee L_2(u_2, v_2)$;

```
(iv) L_1(u_1, v_1) \vee u_1 = v_1;

(v) (L_1(u_1, v_1) \vee u_1 = v_1 \vee L_2(u_2, v_2)) \wedge (u_1 \neq v_1 \vee L_2(u_2, v_2) \vee u_2 = v_2);

for L_1, L_2 \in \{E, N\}, \text{ and } k \geq 0.
```

Note that when M_1, M_2 are such that $\{L_i, M_i\} = \{E, N\}$ for $i \in \{1, 2\}$, then the graph bijunctive clause in Item (v) can be equivalently written in the form

$$(M_1(u_1, v_1) \Rightarrow L_2(u_2, v_2)) \land (u_1 = v_1 \Rightarrow \neg M_2(u_2, v_2))$$
.

Theorem 9.7.15. Let R be a relation with a first-order definition over (V; E). Then the following are equivalent.

- (1) R can be defined by a graph bijunctive formula;
- (2) R is preserved by every ternary injection which is of type majority and balanced:
- (3) R is preserved by some ternary injection of type majority and some binary balanced injection of type p_1 .

PROOF. We show the equivalence of (1) and (2); the equivalence between (2) and (3) is easy and is to be added later.

For the implication $(1) \Rightarrow (2)$, let ψ be a graph bijunctive clause. It suffices to show that ψ is preserved by every balanced injection f of type majority. Let t_1, t_2, t_3 be three tuples that satisfy ψ . If ψ contains an inequality disjunct $x_i \neq y_i$, and one of t_1, t_2, t_3 satisfies $x_i \neq y_i$, then by injectivity of f we have that $t_0 = f(t_1, t_2, t_3)$ satisfies $x_i \neq y_i$ and therefore also ψ . So we can focus on the case k = 0, i.e., ψ does not contain any inequality disjunct. If ψ is of the form $u_1 = v_1$, ψ is clearly preserved. If ψ is of the form $L_1(u_1, v_1)$ or of the form $\neg L_1(u_1, v_1)$, then f preserves ψ since it is of type majority and balanced. Suppose now that ψ is of the form $L_1(u_1, v_1) \vee L_2(u_2, v_2)$ for $L_1, L_2 \in \{E, N\}$. Then at least two of t_1, t_2, t_3 satisfy $L_2(u_2, v_2)$. In the former case, t_0 satisfies $L_1(u_1, v_1)$, in the latter case t_0 satisfies $L_2(u_2, v_2)$, since f is of type majority and balanced.

Finally, suppose that ψ is as in item (v) of the definition of graph bijunctive formulas. If $t_0 = f(t_1, t_2, t_3)$ satisfies $\neg M_1(u_1, v_1) \land u_1 \neq v_1$, then t_0 satisfies both conjuncts of ψ and we are done. We thus may assume that t_0 satisfies either $u_1 = v_1$ or $M_1(u_1, v_1)$. If t_0 satisfies $u_1 = v_1$, then t_0 satisfies the first conjunct of ψ . By injectivity of f we must have that all of t_1, t_2, t_3 satisfy $u_1 = v_1$, and therefore all three tuples satisfy $L_2(u_2, v_2) \lor u_2 = v_2$. Since f is of type majority and balanced, also t_0 satisfies $L_2(u_2, v_2) \lor u_2 = v_2$, which is the second conjunct of ψ , and we are done also in this case.

Suppose now that t_0 satisfies $M_1(u_1, v_1)$. Since f is of type majority and balanced, either

- (a) at least two out of t_1, t_2, t_3 satisfy $M_1(u_1, v_1)$, or
- (b) t_1 satisfies $M_1(u_1, v_1)$ and exactly one out of t_2, t_3 satisfy $u_1 = v_1$, or
- (c) t_1 satisfies $u_1 = v_1$ and t_2 satisfies $M_1(u_1, v_1)$.

If at least two tuples out of t_1, t_2, t_3 satisfy $M_1(u_1, v_1)$, then they also satisfy $L_2(u_2, v_2)$, and so does t_0 since f is of type majority and balanced. We conclude that t_0 satisfies ψ . Now assume (b). Then t_1 satisfies $M_1(u_1, v_1)$, and therefore also satisfies $L_2(u_2, v_2)$. Moreover, one of t_2, t_3 satisfies $u_1 = v_1$, and therefore also $L_2(u_2, v_2) \vee u_2 = v_2$. Since f is balanced and of type majority we have that t_0 satisfies $L_2(u_2, v_2)$, and therefore also ψ . Suppose finally that (c) holds, i.e., t_1 satisfies $u_1 = v_1$ and t_2 satisfies $M_1(u_1, v_1)$. In this case t_1 satisfies $L_2(u_2, v_2) \vee u_2 = v_2$ and t_2 satisfies $L_2(u_2, v_2)$. Again, since f is balanced and of type majority, we have that t_0 satisfies $L_2(u_2, v_2)$, and therefore also ψ .

We next show the implication $(2) \Rightarrow (1)$. Let R be a relation preserved by a ternary injection f which is of type majority and balanced. Let Φ be a formula in CNF that defines R over $(\mathbb{V}; E, N)$ such that all literals of Φ are of the form E(x,y), $N(x,y), x \neq y$, or x=y. This can be achieved by replacing literals of the form $\neg L(x,y)$ by $M(x,y) \lor x=y$, for M such that $\{L,M\}=\{E,N\}$. Also suppose that Φ is minimal in the sense that no clause ϕ of Φ can be replaced by a set of clauses such that

- (1) each replacing clause has fewer literals of the form L(x,y) for $L \in \{E,N\}$ than ϕ , or
- (2) each replacing clause has the same number of literals of the form L(x, y), but fewer literals of the form x = y than ϕ , or
- (3) each replacing clause has the same number of literals of the form L(x, y) and of the form x = y, but fewer literals of the form $x \neq y$ than ϕ .

Let Ψ be the set of all graph bijunctive clauses that are implied by Φ . To prove $(2) \Rightarrow (1)$, it suffices to show that Ψ implies all clauses ϕ of Φ . Let ϕ such a clause. In the entire proof we make the convention that L_1, \ldots, L_n denote elements of $\{E, N\}$, and M_1, \ldots, M_n are such that $\{L_i, M_i\} = \{E, N\}$, for all $i \leq n$.

Observation 1: The clause ϕ cannot contain two different literals of the form $x_1 = y_1$ and $x_2 = y_2$. Otherwise, since Φ is minimal, the formula obtained by removing $x_1 = y_1$ from ϕ is inequivalent to Φ , and hence there exists a tuple t_1 that satisfies Φ , and none of the literals in ϕ except for $x_1 = y_1$. Similarly, there exists a tuple t_2 that satisfies Φ , and none of the literals in ϕ except for $x_2 = y_2$. By the injectivity of f, the tuple $t_0 = f(t_1, t_2, t_2)$ satisfies $x_1 \neq y_1$ and $x_2 \neq y_2$. Moreover, t_0 does not satisfy any other literal of ϕ because the fact that it is of type majority and balanced implies that f preserves the negations of all literals of the form x = y, E(x,y), N(x,y), and $x \neq y$. Therefore, t_0 satisfies none of the literals in ϕ , contradicting the assumption that f preserves Φ .

Observation 2: The clause ϕ contains at most two literals of the form L(x, y), where $L \in \{E, N\}$. Suppose to the contrary that ϕ contains three different literals of the form $L_1(x_1, y_1)$, $L_2(x_2, y_2)$, and $L_3(x_3, y_3)$. Let θ be the clause obtained from ϕ by removing those three literals from ϕ . Note that it is impossible that Φ has satisfying assignments t_1, t_2, t_3 with

$$t_1 \models M_2(x_2, y_2) \land M_3(x_3, y_3) \land \neg \theta$$

$$t_2 \models M_1(x_1, y_1) \land M_3(x_3, y_3) \land \neg \theta$$

$$t_3 \models M_1(x_1, y_1) \land M_2(x_2, y_2) \land \neg \theta$$
.

Otherwise, $t_0 = f(t_1, t_2, t_3)$ satisfies $M_1(x_1, y_1) \wedge M_2(x_2, y_2) \wedge M_3(x_3, y_3)$ since f is of type majority and balanced. Moreover, t_0 satisfies $\neg \theta$, since f preserves the negations of literals of the form x = y, E(x, y), N(x, y), and $x \neq y$. Therefore, t_0 does not satisfy ϕ , in contradiction to the assumption that f preserves Φ .

Suppose without loss of generality that there is no satisfying assignment t_1 as above. In other words, Φ implies the clause

$$\theta \lor L_2(x_2, y_2) \lor (x_2 = y_2) \lor L_3(x_3, y_3) \lor (x_3 = y_3)$$
. (10)

Note that Φ also implies the clauses

$$\theta \lor L_1(x_1, y_1) \lor L_2(x_2, y_2) \lor (x_3 \neq y_3)$$
 (11)

$$\theta \lor L_1(x_1, y_1) \lor (x_2 \neq y_2) \lor L_3(x_3, y_3)$$
 (12)

since they are obvious weakenings of ϕ . We claim that the clauses in (10), (11), and (12) together imply ϕ . To see this, suppose they hold for a tuple t which does not satisfy ϕ . Then t satisfies neither θ nor any of the L_i , and hence it satisfies both $(x_2 \neq y_2)$ and $(x_3 \neq y_3)$, by (11) and (12). On the other hand, in this situation (10) implies $x_2 = y_2 \vee x_3 = y_3$, a contradiction. Hence ϕ is equivalent to the conjunction of these three clauses. Now replacing ϕ by this conjunction in Φ , we arrive at a contradiction to the minimality of Φ .

Taking the two observations together, we conclude that ϕ contains at most one literal of the form x=y, and at most two literals of the form L(x,y). If it has no literal of the form x=y or no literal of the form L(x,y) then it is itself graph bijunctive and hence an element of Ψ , and we are done. So assume henceforth that ϕ contains a literal $x_1=y_1$ and a literal of the form $L_2(x_2,y_2)$. It may or may not contain at most one more literal $L_3(x_3,y_3)$; all other literals of ϕ are of the form $x \neq y$.

Let us first consider the case where ϕ does not contain the literal $L_3(x_3, y_3)$. Let θ be the clause obtained from ϕ by removing $x_1 = y_1$ and $L_2(x_2, y_2)$; all literals in θ are of the form $x \neq y$. We claim that Φ implies the following formula.

$$\theta \lor (x_1 \neq y_1) \lor L_2(x_2, y_2) \lor (x_2 = y_2)$$
 (13)

To show the claim, suppose for contradiction that there is a tuple t_1 that satisfies $\Phi \land \neg \theta \land (x_1 = y_1) \land M_2(x_2, y_2)$. By minimality of Φ , there is also a tuple t_2 that satisfies $\Phi \land \neg \theta \land (x_1 \neq y_1) \land L_2(x_2, y_2)$. Then $f(t_1, t_1, t_2)$ satisfies $\Phi \land \neg \theta \land x_1 \neq y_1 \land M_2(x_2, y_2)$ since f is of type majority and balanced; but this is a contradiction since such a tuple does not satisfy ϕ . We next show that Φ implies the graph bijunctive formulas

$$\theta \lor (E(x_1, y_1) \lor x_1 = y_1 \lor L_2(x_2, y_2)) \land (x_1 \neq y_1 \lor L_2(x_2, y_2) \lor x_2 = y_2)$$
 (14)

$$\theta \lor (N(x_1, y_1) \lor x_1 = y_1 \lor L_2(x_2, y_2)) \land (x_1 \neq y_1 \lor L_2(x_2, y_2) \lor x_2 = y_2). \tag{15}$$

Since Φ implies (13), it suffices to show that Φ implies $\theta \vee E(x_1, y_1) \vee (x_1 = y_1) \vee L_2(x_2, y_2)$ and $\theta \vee N(x_1, y_1) \vee (x_1 = y_1) \vee L_2(x_2, y_2)$. But this is clear since those formulas are weakenings of ϕ . Hence, the formulas (14) and (15) are in Ψ . As $E(x_1, y_1) \vee (x_1 = y_1) \vee L_2(x_2, y_2)$ and $N(x_1, y_1) \vee (x_1 = y_1) \vee L_2(x_2, y_2)$ implies $(x_1 = y_1) \vee L_2(x_2, y_2)$, the formulas (14) and (15) imply ϕ , and therefore Ψ implies ϕ .

Finally, we consider the case where ϕ also contains a literal $L_3(x_3,y_3)$. Let θ be the clause obtained from ϕ by removing $x_1 = y_1$, $L_2(x_2,y_2)$, and $L_3(x_3,y_3)$; all literals of θ are of the form $x \neq y$. If Φ implies $\theta \vee \neg M_2(x_2,y_2)$, then we could have replaced ϕ by the two clauses $\theta \vee L_2(x_2,y_2) \vee (x_2 = y_2)$ and $\theta \vee (x_1 = y_1) \vee (x_2 \neq y_2) \vee L_3(x_3,y_3)$ which together imply ϕ , in contradiction to the minimality of Φ . The same argument shows that Φ does not imply $\theta \vee \neg M_3(x_3,y_3)$. Now observe that Φ implies the following.

$$\theta \lor x_1 = y_1 \lor x_2 \neq y_2 \lor x_3 \neq y_3 \tag{16}$$

$$\theta \vee \neg M_2(x_2, y_2) \vee \neg M_3(x_3, y_3) \tag{17}$$

$$\theta \lor x_2 \neq y_2 \lor L_3(x_3, y_3) \lor x_3 = y_3 \tag{18}$$

$$\theta \lor x_3 \neq y_3 \lor L_2(x_2, y_2) \lor x_2 = y_2 \ . \tag{19}$$

This is obvious for (16). For (17), assume otherwise that there is an assignment t satisfying $\Phi \wedge \neg \theta \wedge M_2(x_2, y_2) \wedge M_3(x_3, y_3)$. By minimality of Φ there is also an assignment t' satisfying $\Phi \wedge \neg \theta \wedge (x_1 \neq x_2)$. Then f(t, t, t') satisfies none of the literals of ϕ , a contradiction. We now show that (18) is implied; the proof for (19) is symmetric. Assume otherwise that t satisfies $\Phi \wedge \neg \theta \wedge (x_2 = y_2) \wedge M_3(x_3, y_3)$. There also exists a tuple t' that satisfies $\Phi \wedge \neg \theta \wedge M_2(x_2, y_2)$ since Φ does not imply

 $\theta \vee \neg M_2(x_2, y_2)$ as we have observed above. Then f(t, t, t') satisfies $\neg \theta \wedge M_2(x_2, y_2) \wedge M_3(x_3, y_3)$, which contradicts (17).

We now claim that Φ also implies at least one of the following two formulas.

$$\theta \lor L_2(x_2, y_2) \lor x_2 = y_2 \lor L_3(x_3, y_3) \tag{20}$$

$$\theta \lor L_3(x_3, y_3) \lor x_3 = y_3 \lor L_2(x_2, y_2)$$
 (21)

Otherwise, there would be a tuple t satisfying $\Phi \wedge \neg \theta \wedge M_2(x_2, y_2) \wedge \neg L_3(x_3, y_3)$ and a tuple t' satisfying $\Phi \wedge \neg \theta \wedge \neg L_2(x_2, y_2) \wedge M_3(x_3, y_3)$. Then f(t, t', t') would satisfy $\neg \theta \wedge M_2(x_2, y_2) \wedge M_3(x_3, y_3)$, which is impossible by (17). Suppose without loss of generality that Φ implies $\theta \vee L_2(x_2, y_2) \vee (x_2 = y_2) \vee L_3(x_3, y_3)$. Since Φ also implies (18), we have that Ψ contains the graph bijunctive formula

$$\theta \lor ((L_2(x_2, y_2) \lor x_2 = y_2 \lor L_3(x_3, y_3)) \land (x_2 \neq y_2 \lor L_3(x_3, y_3) \lor x_3 = y_3)) . \tag{22}$$

We finally show that Ψ implies ϕ . Let t be a tuple that satisfies Ψ . If t satisfies $\theta \vee (x_1 = y_1)$ there is nothing to show, so suppose otherwise. Then (16), which is graph bijunctive and thereofore in Ψ , implies that either $x_2 \neq y_2$ or $x_3 \neq y_3$. If $x_2 \neq y_2$, then by the first conjunct in (22) we have that $L_2(x_2, y_2)$ or $L_3(x_3, y_3)$, in which case t satisfies ϕ and we are done. Otherwise, suppose that $x_2 = y_2$. Then $x_3 \neq y_3$ as we have seen above. But then the second conjunct in (22) implies that $L_3(x_3, y_3)$, and we are again done.

PROPOSITION 9.7.16. Let \mathfrak{B} be a reduct of $(\mathbb{V}; E)$ with finite relational signature, and suppose that \mathfrak{B} has a balanced ternary polymorphism of type majority. Then $CSP(\mathfrak{B})$ can be solved in polynomial time.

PROOF. Let Φ be an instance of $\mathrm{CSP}(\mathfrak{B})$ with variables S, and let Ψ be the set of clauses obtained from Φ by replacing each constraint by its graph bijunctive definition over $(\mathbb{V}; E, N)$ which exists by Theorem 9.7.15. Clearly, Φ is satisfiable in \mathfrak{B} if and only if Ψ is satisfiable in $(\mathbb{V}; E, N)$.

We associate to Ψ a 2SAT instance $\psi = \psi(\Psi)$ as follows. For each unordered pair $\{u, v\}$ of distinct variables u, v of Ψ we have a variable $x_{\{u, v\}}$ in $\psi(\Psi)$. Then

- if Ψ contains the clause E(u,v) or the clause $E(u,v) \vee u = v$ then $\psi(\Psi)$ contains the clause $\{x_{\{u,v\}}\}$;
- if Ψ contains the clause N(u,v) or the clause $N(u,v) \vee u = v$ then $\psi(\Psi)$ contains the clause $\{\neg x_{\{u,v\}}\};$
- if Ψ contains the clause $N(a,b) \vee E(c,d)$ then $\psi(\Psi)$ contains the clause $\{\neg x_{\{a,b\}}, x_{\{c,d\}}\}$. Clauses of the form $L_1(a,b) \vee L_2(c,d)$ are translated correspondingly for all $L_1, L_2 \in \{E, N\}$;
- if Ψ contains the clause $(N(a,b) \lor a = b \lor E(c,d)) \land (a \neq b \lor E(c,d) \lor c = d)$ then $\psi(\Psi)$ contains the clause $\{\neg x_{\{a,b\}}, x_{\{c,d\}}\}$. Clauses of the form $(L_1(u_1,v_1) \lor u_1 = v_1 \lor L_2(u_2,v_2)) \land (u_1 \neq v_1 \lor L_2(u_2,v_2) \lor u_2 = v_2)$ are translated correspondingly for all $L_1, L_2 \in \{E, N\}$.

All other clauses of Ψ are ignored for the definition of $\psi(\Psi)$.

We recall an important and well-known concept to decide satisfiability of 2SAT instances ψ . If ψ contains clauses of size one, we can reduce to the case where all clauses have size two by replacing the clause $\{x\}$ by $\{x,x\}$. The *implication graph* G_{ψ} of a conjunction ψ of propositional clauses of size two is the directed graph whose vertices T are the variables x, y, z, \ldots of ψ , and the negations $\neg x, \neg y, \neg z$ of the variables. The edge set of G_{ψ} contains $(x, x') \in V^2$ if ψ contains the clause $\{\neg x, x'\}$ (here we identify $\neg(\neg x)$ with x). It is well-known that ψ is unsatisfiable if and only if there exists $x \in T$ such that x and $\neg x$ belong to the same strongly connected component (SCC) of G_{ψ} .

```
// Input: A set of graph bijunctive clauses \Psi Do

While \Psi contains a clause of the form u=v do

Replace each occurrence of v by u in \Psi.

Remove literals of the form E(u,u), N(u,u), and u\neq u from \Psi.

If \Psi contains an empty clause then reject.

Loop.

Compute the 2SAT instance \psi=\psi(\Psi), and the graph G_{\psi}.

If G_{\psi} contains x_{\{u,v\}} such that x_{\{u,v\}} and \neg x_{\{u,v\}} are in the same SCC then Replace each occurrence of v by u in \Psi.

Remove literals of the form E(u,u), N(u,u), and u\neq u from \Psi.

If \Psi contains an empty clause then reject.

Loop until \Psi does not change any more.

Accept.
```

FIGURE 9.3. Polynomial-time algorithm to test satisfiability of a given set of graph bijunctive clauses.

Now consider the algorithm displayed in Figure 9.3. We make the following claims.

- (1) Whenever the algorithm replaces all occurrences of a variable v in Ψ by a variable u, then u and v must have the same value in all solutions of Ψ .
- (2) When the algorithm rejects an instance, then Ψ is unsatisfiable.
- (3) When the algorithm accepts, then the input formula indeed is indeed satisfiable.

The first claim can be shown inductively over the execution of the algorithm as follows. When the algorithm replaces all occurrences of v by u in line 4 of the algorithm, the first claim is trivially true. The only other variable contraction can be found in line 10 of the algorithm.

So let Ψ be the set of graph bijunctive clauses when we reach line 10, and suppose that $x_{\{u,v\}}$ and $\neg x_{\{u,v\}}$ lie in the same SCC of G_{ψ} . Since $x_{\{u,v\}}$ and $\neg x_{\{u,v\}}$ belong to the same SCC, there is a path $x_{\{u,v\}} = x_0, x_1, \ldots, x_n = \neg x_{\{u,v\}}$ from $x_{\{u,v\}}$ to $\neg x_{\{u,v\}}$, and a path $\neg x_{\{u,v\}} = y_0, y_1, \ldots, y_m = x$ from $\neg x_{\{u,v\}}$ to $x_{\{u,v\}}$.

Suppose that Ψ has a solution $s \colon S \to V$. We have to show that s(u) = s(v). Suppose otherwise that $s(u) \neq s(v)$; without loss of generality, E(s(u), s(v)) holds. Let $\{u_i, v_i\}$ be the pair of variables of Φ that corresponds to x_i . We show by induction on i that if x_i is positive, then $E(s(u_i), s(v_i))$, and if x_i is negative then $N(s(u_i), s(v_i))$. Suppose without loss of generality that x_i is positive, and suppose inductively that $E(s(u_i), s(v_i))$. There is a clause in Ψ that contributed the edge (x_i, x_{i+1}) to G_{ψ} . If x_{i+1} is a positive literal, then this clause is either of the form $N(u_i, v_i) \vee E(u_{i+1}, v_{i+1})$, or of the form

```
(N(u_i, v_i) \lor u_i = v_i \lor E(u_{i+1}, v_{i+1})) \land (u_i \neq v_i \lor E(u_{i+1}, v_{i+1}) \lor u_{i+1} = v_{i+1}).
```

In both cases, the clause together with $E(s(u_i), s(v_i))$ implies that $E(s(u_{i+1}), s(v_{i+1}))$. The argument in the case that x_{i+1} is a negative literals is similar. For i+1=n we obtain that N(s(u), s(v)), in contradiction to our assumption. Therefore, we conclude that s(u) = s(v), which concludes the proof of the first claim.

Since the only modifications to Ψ are variable contractions, the first claim implies that when at some stage during the execution of the algorithm the formula Ψ contains an empty clause, then there is indeed no solution to the original input formula; this proves the second claim.

To prove the third claim, suppose that the algorithm accepts. Let $\psi = \psi(\Psi)$ be the 2SAT instance in the final round of the main loop of the algorithm, and let T be the set of variables of ψ . The 2SAT formula ψ must have a solution, since otherwise the algorithm would have changed Φ , in contradiction to our assumptions. From a solution $t: T \to \{0,1\}$ for ψ we obtain a solution $s: S \to V$ for the clause set Ψ at the end of the execution of the algorithm by assigning distinct vertices of V to every variable of Ψ such that $(s(u), s(v)) \in E$ if and only if $s(x_{\{u,v\}}) = 0$. We also get a solution to the originally given set of clauses (before contractions of variables) by setting contracted variables to the same value.

The three claims show the correctness of the algorithm. It is easy to see that the algorithm can be implemented in polynomial (in fact, in quadratic) time in the input size. \Box

9.7.4. Tractability of types max and min. We are left with the case where \mathfrak{B} has a canonical binary injective polymorphism of type max or min, which corresponds to case (f) of Proposition 9.3.4.

We claim that we can assume that this polymorphism is either balanced, or of type max and E-dominated, or of type min and N-dominated.

PROPOSITION 9.7.17. If $\mathfrak{B} = (\mathbb{V}; E, N, \neq, ...)$ is first-order definable in $(\mathbb{V}; E)$ and has a canonical binary injective polymorphism of type max (min), then it also has a canonical binary injective polymorphism of type max which is balanced or Edominated (N-dominated).

PROOF. We prove the statement for type max (the situation for min is dual). Let p be the polymorphism of type max. Then h(x,y) := p(x,p(x,y)) is not N-dominated in the first argument; this is easy to see. But then p(h(x,y),h(y,x)) is either balanced or E-dominated, and still of type max.

We apply Theorem 6.3.4 to our setting as follows.

PROPOSITION 9.7.18. Let \mathfrak{B} be a structure with a first-order definition in $(\mathbb{V}; E)$ and a finite relational signature, and suppose \mathbb{B} is preserved by a binary canonical injection which is of type max and balanced or E-dominated, or of type min and balanced or N-dominated. Then $\mathrm{CSP}(\mathfrak{B})$ can be solved in polynomial time.

PROOF. First note that

$$CSP(V; E, \neg E, N, \neg N, =, \neq)$$

can be solved in polynomial time. One way to see this is to verify that all relations are preserved by a balanced polymorphism of type majority, and to use the algorithm presented in Section 9.7.3. We observe the following.

- A canonical binary injection which is of type min and N-dominated is an embedding of $(\mathbb{V}; E, =)^2$ into $(\mathbb{V}; E, =)$.
- A canonical binary injection which is of type max and E-dominated is a an embedding of $(\mathbb{V}; N, =)^2$ into $(\mathbb{V}; N, =)$.
- A canonical binary injection which is of type max and balanced is an embedding of $(\mathbb{V}; \neg E, =)^2$ into $(\mathbb{V}; \neg E, =)$.
- A canonical binary injection which is of type min and balanced is an embedding of $(\mathbb{V}; \neg N, =)^2$ into $(\mathbb{V}; \neg N, =)$.

In each case, polynomial-time tractability of $CSP(\mathfrak{B})$ follows from Theorem 6.3.4. \square

This completes the proof of Proposition 9.3.4.

9.8. Classification

In this section we present a refined description of the polymorphisms of structures with a first-order definition in $(\mathbb{V}; E)$ that imply tractability. This leads to a dichotomy result that has already been stated in Theorem 9.1.4, and which holds without any complexity-theoretic assumptions: either

- (1) there is a primitive positive interpretation of all finite structures in the model-complete core of \mathfrak{B} , or
- (2) \mathfrak{B} has a cyclic polymorphism modulo an endomorphism.

It follows from Proposition 5.6.10 that (1) and (2) are indeed disjoint cases. In order to prove that every \mathfrak{B} satisfies (1) or (2) above, we first determine a list of 17 operations with the following properties:

- (a) every structure \mathfrak{B} with a first-order definition in $(\mathbb{V}; E)$ either interprets $(\{0,1\}; 1IN3)$ or $(\{0,1\}; NAE)$ or is preserved by one of those 17 operations; and
- (b) the list is minimal, that is, if any operation is removed from the list, then the list looses property (a).

Our next step (Proposition 9.8.4) will be the verification that each of the 17 operations generates an operation that is cyclic modulo an endomorphism of $(\mathbb{V}; E)$. It will also turn out that $\mathrm{CSP}(\mathfrak{B})$ can be solved in polynomial time if \mathfrak{B} has one of those operations as a polymorphism (Proposition 9.8.5).

DEFINITION 9.8.1. Let B be a behaviour for binary functions on $(\mathbb{V}; E)$. A ternary injection $f: \mathbb{V}^3 \to \mathbb{V}$ is hyperplanely of type B if the binary functions $(x, y) \mapsto f(x, y, c), (x, z) \mapsto f(x, c, z),$ and $(y, z) \mapsto f(c, y, z)$ have behaviour B for all $c \in \mathbb{V}$.

We have already met a special case of this concept in Definition 9.7.9 of Section 9.7.2: a ternary function is balanced if and only if it is hyperplanely balanced of type p_1 . The following behaviors of binary functions appear hyperplanely in ternary functions of our classification result.

Definition 9.8.2. A binary injection $f: \mathbb{V}^2 \to \mathbb{V}$ is of type

- E-constant if the image of f is a clique;
- N-constant if the image of f is an independent set;
- xnor if for all $u, v \in \mathbb{V}^2$ with $\neq \neq (u, v)$ the relation E(f(u), f(v)) holds if and only if EE(u, v) or NN(u, v) holds;
- xor if for all $u, v \in \mathbb{V}^2$ with $\neq \neq (u, v)$ the relation E(f(u), f(v)) holds if and only if neither EE(u, v) nor NN(u, v) hold.

Recall from Definition 5.5.8 that $I_6 := I_6^{\mathbb{V}}$ denotes the 6-ary relation defined by

$$\{(x_1, x_2, y_1, y_2, z_1, z_2) \in \mathbb{V}^6 \mid (x_1 = x_2 \land y_1 \neq y_2 \land z_1 \neq z_2)$$

$$\lor (x_1 \neq x_2 \land y_1 = y_2 \land z_1 \neq z_2)$$

$$\lor (x_1 \neq x_2 \land y_1 \neq y_2 \land z_1 = z_2) \}.$$

Similarly, we define relations E_6 and N_6 by altering the above definition and replacing all occurrences of \neq by E and N, respectively.

THEOREM 9.8.3. Let \mathfrak{B} be a structure with a first-order definition in $(\mathbb{V}; E)$. Then either one of the following relations is primitive positive definable in \mathfrak{B} : I_6 , E_6 , N_6 , H_1 , H'_1 , H_2 , H'_2 , and $(\{0,1\}; IIN3)$ or $(\{0,1\}; NAE)$ are primitive positive interpretable in \mathfrak{B} , or \mathfrak{B} has a polymorphism of one of the following types.

- (1) A constant operation,
- (2) a balanced binary injection of type max,

- (3) an E-dominated binary injection of type max,
- (4) a function of type majority which is hyperplanely of type projection and balanced.
- (5) a function of type majority which is hyperplanely E-constant,
- (6) a function of type majority which is hyperplanely of type max and E-dominated.
- (7) a function of type minority which is hyperplanely of type projection and balanced,
- (8) a function of type minority which is hyperplanely of type projection and Edominated.
- (9) a function of type minority which is hyperplanely of type xnor and balanced,
- (10) a binary injection which is E-constant,

or the dual of one of the last seven operations.

PROOF. If \mathfrak{B} has a constant endomorphism, then we are in case (1), so we may assume that this is not the case.

First consider the case where all polymorphisms of \mathfrak{B} are essentially unary. Then either I_6 , N_6 , or E_6 is preserved by all polymorphisms of \mathfrak{B} , and hence primitive positive definable in \mathfrak{B} , and we are done. So we assume that \mathfrak{B} has an essential operation. By Lemma 5.3.10, we even have a binary essential polymorphism f.

Consider now the case that $e_E \in \text{Pol}(\mathfrak{B})$. Then consider the structure \mathfrak{D} induced in \mathfrak{B} on the image $D:=e_E[\mathbb{V}]$. This structure \mathfrak{D} is preserved by all permutations of its domain, and hence is first-order definable in (D; =). It follows from Corollary 6.4.1 that \mathfrak{D} either has a constant polymorphism, or a binary injection, or all polymorphisms of \mathfrak{D} are essentially unary. The structure \mathfrak{D} cannot have a constant endomorphism as otherwise also 3 has a constant polymorphism by composing the constant with e_E . Suppose that f(a,a) = f(a,b) for all $a,b \in \mathbb{V}$ with E(a,b). We claim that f(u,u)=f(u,v) for every $u,v\in\mathbb{V}$. To see this, let $w\in\mathbb{V}$ be such that E(u, w) and E(v, w). Then f(u, u) = f(u, w) = f(u, v), as required. It follows that f does not depend on its first variable, a contradiction. Hence, there exist $a, b \in \mathbb{V}$ such that E(a,b) and $f(a,a) \neq f(a,b)$. Similarly, there exist $c,d \in \mathbb{V}$ such that E(c,d)and $f(c,c) \neq f(d,c)$. Let T be an infinite clique adjacent to a,b,c,d. Then f is either essential on $T \cup \{a, b\}$ or on $T \cup \{c, d\}$, both cliques. Suppose without loss of generality that f is essential on $C = T \cup \{a, b\}$. Since all operations with the same behaviour as e_E generate each other, we can also assume that the image of e_E is C. Then the restriction f' of $(x_1, x_2) \mapsto e_E(f(x_1, x_2))$ to $e_E[\mathbb{V}]$ is an essential polymorphism of \mathfrak{D} . Hence, Corollary 6.4.1 implies that \mathfrak{D} has a binary injective polymorphism h'. Then $h(x,y) := h'(e_E(x), e_E(y))$ is a polymorphism of \mathfrak{B} . But h is a binary canonical injection which is E-constant, and so \mathfrak{B} has a polymorphism from Item (10) of our list. When \mathfrak{B} is preserved by e_N the dual argument works.

Hence, by Theorem 9.2.3 it remains to consider the case where the endomorphisms of \mathfrak{B} are generated by the automorphisms of \mathfrak{B} , that is, \mathfrak{B} is a model-complete core (Theorem 3.6.11). By Theorem 9.2.6 there are five possibilities for End(\mathfrak{B}). Suppose first that \mathfrak{B} is preserved by all permutations on \mathbb{V} . Then by Corollary 6.4.1, \mathfrak{B} is preserved by all binary injections, and in particular \mathfrak{B} is preserved by, say, a balanced binary canonical injection operation of type max.

We can therefore assume that $\operatorname{End}(\mathfrak{B})$ does not contain all permutations. We consider the case where the automorphism of $(\mathbb{V}; E)$ are dense in $\operatorname{End}(\mathfrak{B})$. If H_1 is primitive positive definable in \mathfrak{B} , then we have found a primitive positive interpretation of $(\{0,1\}; 1IN3)$ in \mathfrak{B} by Proposition 9.4.3. Otherwise, Proposition 9.3.4 applies, and $\operatorname{Pol}(\mathfrak{B})$ contains a binary canonical injection of type max or min, or a function of type minority or majority. If it contains a canonical injection of type max or min, then \mathfrak{B} is preserved by some of the operations from case (2), (3), or their duals, by

Proposition 9.7.17. Otherwise, $\operatorname{Pol}(\mathfrak{B})$ contains a ternary injection t of type minority or majority, and one of the binary canonical injections of type projection listed in Theorem 9.3.8, which we denote by p. Set s(x,y,z) := t(p(x,y),p(y,z),p(z,x)) and w(x,y,z) := s(p(x,y),p(y,z),p(z,x)). Then w has the same behavior as one of the operations from case (4) to case (9) — see Figure 9.4; we leave the verification to the reader.

Now suppose that $\operatorname{End}(\mathfrak{B})$ is the monoid generated by sw. If H_2 is primitive positive definable in \mathfrak{B} , then we have a primitive positive interpretation of ($\{0,1\}$; 1IN3) in \mathfrak{B} by Proposition 9.4.3. Otherwise we are done by Proposition 9.4.2, since the functions that appear in Proposition 9.4.2 are a subset of the function that appear in Proposition 9.3.4 and that we have treated above.

Next consider the case where $\operatorname{End}(\mathfrak{B})$ is the monoid generated by -. If the relation H'_1 is primitively positively definable in \mathfrak{B} , then ($\{0,1\}$; NAE) has a primitive positive interpretation in \mathfrak{B} by Proposition 9.5.2. Otherwise we are done by Proposition 9.5.3.

If $\operatorname{End}(\mathfrak{B})$ is the monoid generated by $\{sw, -\}$ and H'_2 is primitive positive definable in \mathfrak{B} , then we have a primitive positive interpretation of $(\{0,1\}; \operatorname{NAE})$ in \mathfrak{B} by Proposition 9.4.3. Otherwise we are done by Proposition 9.6.3.

Proposition 9.8.4. The 17 operations listed in Theorem 9.8.3 generate 17 distinct clones, each containing an operation which is cyclic modulo a unary operation.

PROOF. Observe that all of the 17 operations listed in Theorem 9.8.3 are canonical as functions over $(\mathbb{V}; E)$. Let g be one of them. By Lemma 5.6.5 there is a homomorphism μ from $(\mathbb{V}; g)^2$ to an algebra $\mathbf{A} = (\{=, E, N\}, f)$ (where =, E, and N are the image of μ for the pairs $(x, y) \in \mathbb{V}^2$ such that x = y, E(x, y), or N(x, y), respectively).

In case (1), case (10), and its dual, the algebra **A** is not idempotent. In case (1), when g is constant, then g is in particular a cyclic polymorphism. In case (10), the image of g induces an infinite clique in $(\mathbb{V}; E)$. As in the proof of Corollary 3.6.2 we see that \mathfrak{B} is preserved by all permutations of its domain. Also note that **A** has a congruence with the congruence classes $\{=\}$ and $\{E, N\}$ (see Proposition 6.1.4), and in the corresponding quotient algebra g denotes max with respect to the order $\{=\} < \{E, N\}$. It is then easy to see that $f: (x, y, u, v) \mapsto g(g(g(x, y), u), v)$ is a cyclic operation modulo an endomorphism of $(\mathbb{V}; E)$. The dual proof works for the dual of case (10).

By Theorem 5.6.4 in combination with Theorem 5.6.2 and Theorem 5.6.3, every finite idempotent algebra either

- has a 2-element factor all of whose operations are projections, or
- has a cyclic term.

If the second case applies, then Corollary 5.6.8 shows that \mathfrak{B} has a cyclic polymorphism modulo an endomorphism. Therefore, it suffices to show in the remaining 14 cases that all factors of \mathbf{A} contain operations that are not projections. Since in those 14 cases both E and N are preserved by g, the relation \neq is also preserved, and $\{E, N\}$ induce a subalgebra of \mathbf{A} . In this subalgebra and if the operation g is ternary, it either acts as a majority (that is, g(x, x, y) = g(x, y, x) = g(y, x, x) = x), or as a minority (that is, g(x, x, y) = g(x, y, x) = g(y, x, x) = y). If f is binary, it satisfies g(x, y) = g(y, x) in this subalgebra. In all cases, f does not act as a projection. Four out of the 14 remaining operations are balanced, which is equivalent to saying that both $\{E, =\}$ and $\{N, =\}$ induce a subalgebra \mathbf{B} in \mathbf{A} . In this case it is easy to check

Binary injection type p_1	Type majority	Type minority
Balanced	Hp. balanced, type p_1	Hp. balanced, type p_1
E-dominated	Hp. E-constant	Hp. type p_1 , E -dominated
N-dominated	Hp. N-constant	Hp. type p_1 , N-dominated
Balanced in 1st, E-dom. in 2nd arg.	Hp. type \max , E -dom.	Hp. type xnor, balanced.
Balanced in 1st, N -dom. in 2nd arg.	Hp. type min, N -dom.	Hp. type xor, balanced.

FIGURE 9.4. Minimal tractable canonical functions of type majority and minority, and their corresponding canonical binary injections of type projection.

from the description of the balanced operations in Theorem 9.8.3 that

$$g(x,y)$$
 satisfies $g(x,y) = g(y,x)$ if f is binary, and (23)

$$h(x,y) := g(x,x,y)$$
 satisfies $h(x,y) = h(y,x)$ if g is ternary. (24)

So g is not a projection in those factors as well. For five of the remaining non-balanced operations we have that $\{E,=\}$ induces a subalgebra of \mathbf{A} . Again, g satisfies the condition in (23). For the other five remaining operations, the set $\{N,=\}$ induces a subalgebra, and the argument that the operation f is not a projection in those algebras is analogous.

Finally, we have to argue that the operation g is in none of the 2-element homomorphic images of \mathbf{A} a projection. Since all of the 14 remaining operations are injective, they have a congruence with the classes $\{E, N\}$ and $\{=\}$ (Proposition 6.1.4). Then the operation g satisfies (23) in the corresponding factor. It can be verified that from all 14 operations, only

- the balanced operation of type max,
- the N-dominated operation of type min,
- and the edge majority that is hyperplanely of type min and N-dominated

preserve the relation $E(x,y)\Leftrightarrow E(u,v)$. In those cases, the algebra **A** has a congruence with the classes $\{E\}$ and $\{N,=\}$. For the balanced operation of type max, and the N-dominated operation of type min, in the corresponding quotient the operation g satisfies the condition in (23). For the edge majority that is hyperplanely of type min and N-dominated, the condition in (24) applies. Congruences of **A** with the classes $\{N\}$ and $\{E,=\}$ can be checked analogously.

LEMMA 9.8.5. Suppose that $Pol(\mathfrak{B})$ contains one of the 17 operations from Theorem 9.8.3. Then every finite signature reduct \mathfrak{B}' of \mathfrak{B} has a polynomial-time tractable CSP.

PROOF. For the constant operation this is Proposition 1.1.11. Cases (2) and (3) are tractable by case (f) of Proposition 9.3.4. In all cases, the duals can be solved analogously. The functions of type majority or minority are tractable by cases (b) to (e) of Proposition 9.3.4: in those cases, certain binary canonical injections of type projection are required – these are obtained by identifying any two variables of the function of type majority / minority; Figure 9.4 shows which function of type majority / minority yields which type of binary injection. We leave the verification to the reader. Finally, let f(x,y) be an E-constant binary injection (case (10)), and denote the reduct corresponding to this clone by \mathfrak{B} . Then g(x) := f(x,x) is a homomorphism from \mathfrak{B} to the structure \mathfrak{C} induced by the image $g[\mathbb{V}]$ in \mathfrak{B} . This structure \mathfrak{C} is invariant under all permutations of its domain, and hence is definable in $(g[\mathbb{V}];=)$; such structures have been treated in Chapter 6. The structure \mathfrak{C} has a

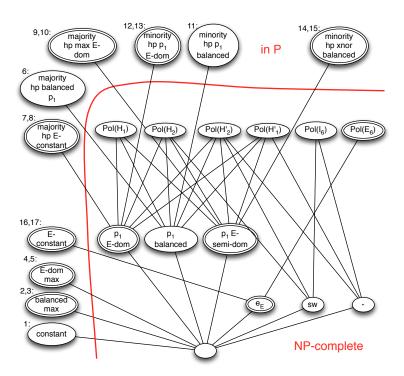


FIGURE 9.5. The border: Minimal tractable and maximal hard clones containing $\operatorname{Aut}((\mathbb{V};E))$.

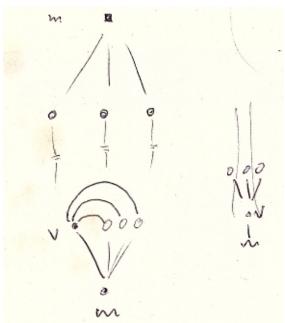
binary injection among its polymorphisms (namely, the restriction of f to \mathfrak{C}). It then follows from Theorem 6.4.2 that $CSP(\mathfrak{C})$ is tractable. Hence, $CSP(\mathfrak{B})$ is tractable as well, since \mathfrak{B} and \mathfrak{C} are homomorphically equivalent.

Figure 9.5 shows the border between the hard and the tractable clones. The picture contains all minimal tractable clones as well as all maximal hard clones, plus some other clones that are of interest in this context. Lines represent containment of clones, but edges that are implied by transitivity of containment are not drawn. Note that lines do not mean to imply that there are no other clones between them which are not shown in the picture. Clones are symbolized with a double border when they have a dual clone (generated by the dual function in the sense of Definition 9.3.7, whose behavior is obtained by exchanging E with N, max with min, and xnor with xor). Of two dual clones, only one representative (the one which has E and max in its definition) is included in the picture. The numbers of the minimal tractable clones refer to the numbers in Theorem 9.8.3. "E-semidom" refers to "balanced in the first and E-dominated in the second argument".

PROOF OF THEOREM 9.1.4. By Theorem 9.8.3, either there is a primitive positive interpretation of $(\{0,1\}; 1IN3)$ in \mathfrak{B} , and the statement follows from Corollary 5.5.7, or \mathfrak{B} is preserved by one out of the 17 canonical operations listed in Theorem 9.8.3. By Proposition 9.8.4, these operations are cyclic modulo an endomorphism of \mathfrak{B} , and polynomial-time tractability for the CSP of finite-signature reducts of \mathfrak{B} follows from Proposition 9.8.5.

CHAPTER 10

Temporal Constraint Satisfaction Problems



Martin Kutz, 2001

This chapter contains results from [41, 42, 50].

10.1. Introduction

A temporal relation is a relation $R \subseteq \mathbb{Q}^k$, for some finite k, with a first-order definition in $(\mathbb{Q};<)$, the ordered rational numbers (which can be thought of time points). A temporal constraint language is a set of temporal relations, and will be treated here as a relational structure with a first-order definition in $(\mathbb{Q};<)$. Constraint satisfaction problems for temporal constraint languages will be called temporal CSPs in the following. We have already discussed some temporal constraint languages in Section 1.5: for instance and/or precedence constraints from scheduling, and Ord-Horn constraints on time points.

There are also several famous NP-complete temporal CSPs. For example the Betweenness Problem [101], which has been introduced in Example 1.1.3 as a CSP with domain \mathbb{Z} , can also be formulated as $\mathrm{CSP}((\mathbb{Q};\mathrm{Betw}))$ where Betw is the ternary relation $\mathrm{Betw} = \{(x,y,z) \in \mathbb{Q}^3 \mid (x < y < z) \lor (z < y < x)\}$. We have seen in Proposition 5.5.13 that this CSP is NP-hard. Similarly, the Cyclic Ordering Problem [101] can be formulated as the CSP for $(\mathbb{Q};\{(x,y,z)\mid (x < y < z) \lor (y < z < x) \lor (z < x < y)\})$, and is also NP-complete [99] (a hardness proof using primitive positive interpretations can be found in Section 10.2.3).

A subclass of temporal CSPs called *ordering CSPs* has been introduced in [109]. An ordering CSP is a temporal CSP where the constraint language only contains relations where the arguments are pairwise distinct (thus, $CSP((\mathbb{Q}; \leq, \neq))$) is not an ordering CSP). Satisfiability thresholds for random instances of ordering CSPs have been studied in [103]. Approximability of ordering CSPs has been studied in [121].

The class of temporal constraint languages is of fundamental importance for infinite domain constraint satisfaction, since CSPs for such languages appear as important special cases in several other classes of CSPs that have been studied, e.g., constraint languages about branching time, partially ordered time, spatial reasoning, and set constraints [59,85,127]. Moreover, several polynomial-time solvable classes of constraint languages on *time intervals* [86,140,165] can be solved by translation into polynomial-time solvable temporal constraint languages.

In this chapter we prove a complete classification of the computational complexity of $CSP(\mathfrak{B})$ when \mathfrak{B} is a temporal constraint language.

THEOREM 10.1.1. Let \mathfrak{B} be a structure with a first-order definition in $(\mathbb{Q};<)$, and let \mathfrak{C} be the model-complete core of \mathfrak{B} . Then exactly one of the following two cases is true.

- \mathfrak{C} has an (at most ternary) weak near unanimity polymorphisms modulo endomorphisms. In this case, $CSP(\mathfrak{B}')$ is in P for every finite reduct \mathfrak{B}' of \mathfrak{B} .
- All finite structures have a primitive positive interpretation with parameters in \mathfrak{C} . In this case, $CSP(\mathfrak{B})$ is NP-hard by Corollary 5.5.7.

Our classification proof is based on the universal-algebraic approach and Ramsey theory as described in Chapter 5 and Chapter 8.

10.2. Preliminaries

10.2.1. Cameron's theorem. In this subsection we recall the classical result of Cameron [69] that describes temporal constraint languages up to first-order interdefinability. For $x_1, \ldots, x_n \in \mathbb{Q}$ write $\overrightarrow{x_1 \cdots x_n}$ when $x_1 < \cdots < x_n$.

Theorem 10.2.1 (Relational version of Cameron's theorem; see e.g. [128]). Let \mathfrak{B} be a temporal constraint language. Then \mathfrak{B} is first-order interdefinable with exactly one out of the following five homogeneous structures.

- The dense linear order $(\mathbb{Q}; <)$ itself,
- The structure (Q; Betw), where Betw is the ternary relation

$$\{(x, y, z) \in \mathbb{Q}^3 \mid \overrightarrow{xyz} \lor \overrightarrow{zyx}\},$$

 \bullet The structure ($\mathbb{Q}; \operatorname{Cycl}),$ where Cycl is the ternary relation

$$\{(x,y,z) \mid \overrightarrow{xyz} \lor \overrightarrow{yzx} \lor \overrightarrow{zxy}\},$$

• The structure (Q; Sep), where Sep is the 4-ary relation

$$\{(x_1, y_1, x_2, y_2) \mid \overrightarrow{x_1 x_2 y_1 y_2} \lor \overrightarrow{x_1 y_2 y_1 x_2} \lor \overrightarrow{y_1 x_2 x_1 y_2} \lor \overrightarrow{y_1 y_2 x_1 x_2} \lor \overrightarrow{y_2 x_1 x_2 y_1} \lor \overrightarrow{y_2 y_1 x_2 x_1} \},$$

• The structure $(\mathbb{Q}; =)$.

The relation Sep is the so-called *separation relation*; note that $\text{Sep}(x_1, y_1, x_2, y_2)$ holds for elements $x_1, y_1, x_2, y_2 \in \mathbb{Q}$ iff all four points x_1, y_1, x_2, y_2 are distinct and the smallest interval over \mathbb{Q} containing x_1, y_1 properly overlaps with the smallest interval containing x_2, y_2 (where properly overlaps means that the two intervals have a non-empty intersection, but none of the intervals contains the other).

The next theorem is also due to Cameron [69], and was his original motivation for the investigation of structures with a first-order definition in $(\mathbb{Q}; <)$. It is not used for our results; however, we would like to state it here because it provides a fundamentally different characterization of the class of temporal constraint languages.

Theorem 10.2.2. A relational structure \mathfrak{B} is highly set-transitive if and only if it is a temporal constraint language.

10.2.2. Polymorphisms of Temporal Constraint Languages. For this chapter only, we make the following convention. We say that a set of operations \mathcal{F} generates an operation g if \mathcal{F} together with all automorphisms of $(\mathbb{Q}; <)$ locally generates g. In case that \mathcal{F} contains just one operation f, we also say that f generates g.

A k-ary operation f on \mathbb{Q} defines a weak linear order \leq on \mathbb{Q}^k , as follows: for $x, y \in \mathbb{Q}^k$, let $x \leq y$ iff $f(x) \leq f(y)$. The following observation follows straightforwardly from Proposition 5.2.1.

Observation 10.2.3. Let f and g be two k-ary operations that define the same weak linear order on \mathbb{Q}^k . Then f generates g and g generates f.

We now define fundamental operations on \mathbb{Q} . The unary operation \leftrightarrow is defined as $\leftrightarrow(x) := -x$ in the usual sense. Let c be any irrational number, and let e be any order-preserving bijection between $(-\infty, c)$ and (c, ∞) . Then the operation \circlearrowleft is defined by e(x) for x < c and by $e^{-1}(x)$ for x > c. With these operations and the notion of generation, Cameron's theorem can be rephrased as follows.

THEOREM 10.2.4 (Operational version of Camerons theorem; see e.g. [128]). Let \mathfrak{B} be a temporal constraint language. Then exactly one of the following holds.

- $\operatorname{Aut}(\mathfrak{B})$ equals $\operatorname{Aut}((\mathbb{Q};<))$;
- The automorphisms of \mathfrak{B} are the permutations generated by \leftrightarrow ;
- The automorphisms of \mathfrak{B} are the permutations generated by \circlearrowleft ;
- The automorphisms of $\mathfrak B$ are the permutations generated by \leftrightarrow and \circlearrowright ;
- Aut(\mathfrak{B}) equals $Sym(\mathbb{Q})$.

If f is a k-ary operation on \mathbb{Q} , then the operation $\leftrightarrow f(\leftrightarrow x_1,\ldots,\leftrightarrow x_k)$ is called the dual of f. Note that if f preserves an m-ary relation R, then the dual of f preserves the relation $\leftrightarrow R$, which is defined to be the relation $\{(\leftrightarrow a_1,\ldots,\leftrightarrow a_m)\mid (a_1,\ldots,a_m)\in R\}$. Clearly, $\mathrm{CSP}((\mathbb{Q};R_1,\ldots,R_k))$ and $\mathrm{CSP}((\mathbb{Q};\leftrightarrow R_1,\ldots,\leftrightarrow R_k))$ are exactly the same computational problem.

10.2.3. Hard temporal CSPs. In this subsection we discuss various important NP-complete temporal constraint satisfaction problems. We have already mentioned in the introduction that the Betweenness and the Cyclic Ordering Problem in [101] can be formulated as temporal CSPs, and that these problems are NP-complete. The corresponding relations Betw and Cycl re-appeared in Cameron's theorem (Theorem 10.2.1). Another important relation for our classification is the relation T_3 , defined as follows.

Definition 10.2.5. Let T_3 be the ternary relation

$$\{(x, y, z) \in \mathbb{Q}^3 \mid (x = y < z) \lor (x = z < y)\}$$

PROPOSITION 10.2.6. The structure ($\{0,1\}$; IIN3) has a primitive positive interpretation in (\mathbb{Q} ; T_3 ,0). The problem $\mathrm{CSP}((\mathbb{Q};T_3))$ is NP-hard.

PROOF. We give a 2-dimensional primitive positive interpretation I of the structure ($\{0,1\}$; 1IN3) in (\mathbb{Q} ; $T_3,0$). The domain formula $\delta_I(x_1,x_2)$ is $T_3(0,x_1,x_2)$; the formula $1IN3_I(x_1,x_2,y_1,y_2,z_1,z_2)$ is

$$\exists u (T_3(u, x_1, y_1) \land T_3(0, u, z_1));$$

the formula $=_I (x_1, x_2, y_1, y_2)$ is $T_3(0, x_1, y_2)$. The coordinate map $h: \delta_I(\mathfrak{B}^2) \to \{0, 1\}$ is defined as follows. Let (b_1, b_2) be a pair of elements of \mathfrak{B} that satisfies δ_I . Then exactly one of b_1, b_2 must have value 0, and the other element is strictly greater than 0. We define $h(b_1, b_2)$ to be 1 if $b_1 = 0$, and to be 0 otherwise.

To see that this is the intended interpretation, let $(x_1,x_2), (y_1,y_2), (z_1,z_2) \in \delta_I(\mathfrak{B}^2)$, and suppose that $t:=(h(x_1,x_2),h(y_1,y_2),h(z_1,z_2))=(1,0,0)\in IIN3$. We have to verify that $(x_1,x_2,y_1,y_2,z_1,z_2)$ satisfies $IIN3_I$ in \mathfrak{B} . Since $h(x_1,x_2)=1$, we have $x_1=0$, and similarly we get that $y_1,z_1>0$. We can then set u to 0 and have $T_3(u,x_1,y_1)$ since $0=u=x_1< y_1$, and we also have $T_3(0,u,z_1)$ since $0=u< z_1$. The case that t=(0,1,0) is analogous. Suppose now that $t=(0,0,1)\in IIN3$. Then $x_1,y_1>0$, and $z_1=0$. We can then set u to $min(x_1,y_1)$, and therefore have $T_3(u,x_1,y_1)$, and $T_3(0,u,z_1)$ since $0=z_1< u$. Conversely, suppose that $(x_1,x_2,y_1,y_2,z_1,z_2)$ satisfies $IIN3_I$ in \mathfrak{B} . Since $T_3(0,u,z_1)$, exactly one out of u,z_1 equals 0. When u=0, then because of $T_3(u,x_1,y_1)$ exactly one out of x_1,y_1 equals 0, and we get that $(h(x_1,x_2),h(y_1,y_2),h(z_1,z_2))\in\{(0,1,0),(1,0,0)\}\subseteq IIN3$. When u>0, then $x_1>0$ and $y_1>0$, and so $(h(x_1,x_2),h(y_1,y_2),h(z_1,z_2))=(0,0,1)\in IIN3$.

Since the orbit of 0 is primitive positive definable, NP-hardness of $CSP((\mathbb{Q}; T_3))$ follows from the NP-hardness of $CSP((\{0,1\}; 1IN3))$ via Proposition 5.5.11.

We will see in Theorem 10.6.3 that if no relation among Betw, Cycl, Sep, T_3 , $\leftrightarrow T_3$, or E_6 is primitive positive definable in a temporal constraint language \mathfrak{B} , then $\mathrm{CSP}(\mathfrak{B})$ is tractable. In fact, when \mathfrak{B} is $(\mathbb{Q};R)$ for one of the relations R above, then we give primitive positive interpretations of $(\{0,1\};1\mathrm{IN}3)$ with finitely many constants in \mathfrak{B} . Thus, hardness of temporal CSPs can always be shown with Proposition 5.5.11. We have already seen this for Betw, T_3 (and thus $\leftrightarrow T_3$), and E_6 , and close by showing it for Cycl and Sep. We thank Trung Van Pham for pointing out a simpler proof for Cycl than our original proof which was inspired by the NP-hardness proof of [99] for the 'Cyclic ordering problem' (see [101]).

THEOREM 10.2.7. The structure ($\{0,1\}$; <, Betw) has a primitive positive interpretation in (\mathbb{Q} ; Cycl, 0, 1). The structure ($\{0,1\}$; 1IN3) has a primitive positive interpretation with parameters in (\mathbb{Q} ; Cycl), and CSP((\mathbb{Q} ; Cycl)) is NP-hard.

PROOF. Our interpretation of $(\mathbb{Q}; <, \text{Betw})$ in $(\mathbb{Q}; \text{Cycl}, 0, 1)$ is 1-dimensional. The domain formula $\delta(x)$ is Cycl(0, x, 1), and defines the open interval $(-1, 1) \subseteq \mathbb{Q}$. The coordinate map c is any isomorphism between $(\mathbb{Q}; <)$ and the substructure induced by these numbers. The interpreting relation for x < y is Cycl(0, x, y). It is easy to verify that the relation Cycl is not preserved by any of the relations listed in Lemma 10.6.2. Hence, Betw has a primitive positive definition in $(\mathbb{Q}; \text{Cycl}, <)$, which is the interpreting formula for Betw in the interpretation.

Since $(\mathbb{Q}; Betw)$ can in turn interpret primitively positively $(\{0, 1\}; 1IN3)$ with parameters by Proposition 5.5.13, the desired interpretation can be obtained by composing interpretations (see Section 5.5.4). We can then apply Proposition 5.5.11, and the NP-hardness of CSP($(\mathbb{Q}; Cycl)$) follows from the NP-hardness of $(\{0, 1\}; 1IN3)$.

Another relation that appeared in Theorem 10.2.1 is the separation relation Sep. The corresponding CSP is again NP-complete.

PROPOSITION 10.2.8. There is a primitive positive interpretation of (\mathbb{Q} ; Betw) in (\mathbb{Q} ; Sep, -1, 1, 2), and therefore also a primitive positive interpretation of ($\{0, 1\}$; IIN3) in (\mathbb{Q} ; Sep) with parameters. The problem $CSP((\mathbb{Q}; Sep))$ is NP-hard.

PROOF. Our interpretation of (\mathbb{Q} ; Betw) in (\mathbb{Q} ; Sep, -1, 1, 2) is 1-dimensional. The domain formula $\delta(x)$ is Sep(-1, 1, x, 2), and defines the open interval (-1, 1) \subseteq

 \mathbb{Q} . The coordinate map c is any isomorphism between $(\mathbb{Q}; <)$ and the substructure induced by these numbers. Then the formula $\operatorname{Sep}(x, z, y, 1)$ interprets Betw: x and y must satisfy δ , and so -1 < x, y < 1. Therefore,

$$Betw(c(x), c(y), c(z)) \Leftrightarrow -1 < x < y < z < 1 \text{ or } -1 < z < y < x < 1$$
$$\Leftrightarrow Sep(x, z, y, 1)$$

A primitive positive interpretation of $(\{0,1\}; 1IN3)$ can be obtained as follows. The argument above shows that the structure $(\mathbb{Q}; Sep)$ can interpret primitively positively $(\mathbb{Q}; Betw, 0)$ with parameters, which in turn can interpret primitively positively $(\{0,1\}; 1IN3)$ by Proposition 5.5.13. Then the desired interpretation can be obtained by composing interpretations (see Section 5.5.4). We can then apply Proposition 5.5.11, and the NP-hardness of $(\{0,1\}; 1IN3)$.

10.3. Endomorphisms

In this section we study the endomorphisms of temporal constraint languages. As an application, we obtain a reduction of the complexity classification for temporal constraint satisfaction problems to the classification for those languages that admit a primitive positive definition of the binary relation <.

Theorem 10.3.1. Let \mathfrak{B} be a temporal constraint language. Then exactly one of the following cases applies.

- (1) B has a constant endomorphism;
- (2) All endomorphisms of \mathfrak{B} preserve <;
- (3) End(\mathfrak{B}) equals the set of unary operations generated by \leftrightarrow ;
- (4) End(\mathfrak{B}) equals the set of unary operations generated by \circlearrowleft ;
- (5) End(\mathfrak{B}) equals the set of unary operations generated by \leftrightarrow and \circlearrowleft ;
- (6) End(3) equals the set of all injective unary operations.

PROOF. First note that all the cases are indeed disjoint: a constant endomorphism violates <, and cannot be generated by a set of injective unary operations; this shows that the first case is distinct from all others. Disjointness of the remaining cases follows from Theorem 10.2.4.

If \mathfrak{B} has a non-injective endomorphism, then Corollary 5.3.7 shows that there is also a constant endomorphism. Otherwise all endomorphisms of \mathfrak{B} are injective. We show that then all endomorphisms e of \mathfrak{B} are locally invertible: for any $a_1, \ldots, a_l \in \mathbb{Q}$ there exists a self-embedding f of \mathfrak{B} into \mathfrak{B} such that $f(e(a_i)) = a_i$ for all $i \in \{1, \ldots, l\}$. Because e is injective, there is an $\alpha \in \operatorname{Aut}((\mathbb{Q}; <))$ such that $\alpha e(\{a_1, \ldots, a_l\}) = \{a_1, \ldots, a_l\}$. Then $(\alpha e)^{l!}$, i.e., the composition of $(\alpha e) \ldots (\alpha e)$ with l-factorial many terms of the form (αe) , maps a_i to itself for all $1 \leq i \leq l$. Then $(\alpha e)^{l!-1}\alpha$ is also an endomorphism of \mathfrak{B} , and we have $((\alpha e)^{l!-1}\alpha)(e(a_1), \ldots, e(a_l)) = (\alpha e)^{l!}(a_1, \ldots, a_l) = (a_1, \ldots, a_l)$. This proves that e is locally invertible.

Theorem 3.6.7 shows that the endomorphisms of \mathfrak{B} are generated by the automorphisms of \mathfrak{B} . The claim of the statement follows directly from Theorem 10.2.4. \square

The following theorem shows that we can focus on constraint languages where < is primitive positive definable.

Theorem 10.3.2. Let \mathfrak{B} be a temporal constraint language. Then it satisfies at least one of the following:

- (a) There is a primitive positive definition of Cycl, Betw, or Sep in \mathfrak{B} .
- (b) Pol(**B**) contains a constant operation.
- (c) $Aut(\mathfrak{B})$ contains all permutations of \mathbb{Q} .

(d) There is a primitive positive definition of < in \mathfrak{B} .

PROOF. If there is a pp definition of Betw in \mathfrak{B} we are in case (a). Otherwise, since Betw consists of two orbits of triples of the automorphism group of $(\mathbb{Q}; <)$, Lemma 5.3.5 shows that \mathfrak{B} has a binary polymorphism that violates Betw. If there is a pp definition of < in \mathfrak{B} , we are in case (d). Otherwise, again by Lemma 5.3.5, there is a unary polymorphism of \mathfrak{B} that violates <. Proposition 10.3.1 shows that \mathfrak{B} is preserved by a constant, -, or \circlearrowright . For each of these three operations we show the claim of the statement separately in the following three paragraphs.

If $\mathfrak B$ is preserved by a constant we are in case (b), so we assume in the following that $\mathfrak B$ is not preserved by a constant.

If $\mathfrak B$ is preserved by -, the relation Betw consists of only one orbit of triples, and Lemma 5.3.5 shows that there is an endomorphism that violates Betw. Proposition 10.3.1 then implies that $\mathfrak B$ is also preserved by \circlearrowright . Thus, the relation Sep consists of only one orbit of 4-tuples. Again, either Sep has a pp definition, and we are in case (a), or there is an endomorphism that violates Sep. Proposition 10.3.1 now shows that $\mathfrak B$ is preserved by all injective unary operations and we are in case (c).

If \mathfrak{B} is preserved by \circlearrowleft , then the relation Cycl consists of only one orbit of triples. If Cycl has a pp definition in \mathfrak{B} , we are in case (a). Otherwise, Lemma 5.3.5 shows that there is an endomorphism that violates Cycl. Proposition 10.3.1 then shows that \mathfrak{B} is also preserved by -. But the statement of the lemma has already been shown in the case that \mathfrak{B} is preserved by both - and \circlearrowleft in the previous paragraph, so we are done.

In case (a), there is a finite signature reduct \mathfrak{B}' of \mathfrak{B} such that $CSP(\mathfrak{B}')$ is NP-hard, as we have seen in Section 10.2.3. In case (b), for all finite signature reducts \mathfrak{B}' of \mathfrak{B} the problem $CSP(\mathfrak{B})$ is trivially in P (see Proposition 1.1.11). In case (c) the complexity of $CSP(\mathfrak{B})$ has been classified in Chapter 6. In the following, we therefore study only those temporal constraint languages where < is pp definable.

10.4. Lex-closed Constraints

First-order expansions of $(\mathbb{Q}; <)$ can be divided into four (non-disjoint) groups: those where the betweenness relation is primitive positive definable, those that are preserved by an operation called pp, an operation called dual-pp, or by the binary injective operation called lex that we have already encountered in Section 8.3.3. None of the three polymorphisms pp, dual-pp, and lex alone guarantees tractability of the CSP. An illustration of the complexity classification result for first-order expansions of $(\mathbb{Q}; <)$ can be found in Figure 10.1.

- 10.4.1. The operations lex and ll. An important class of temporal constraint languages are the languages preserved by the operation lex, introduced in Section 8.3.3. Recall that lex is a binary injective operation on $\mathbb Q$ such that lex(a,b) < lex(a',b') if either a < a', or a = a' and b < b'. By Observation 10.2.3, all such operations generate the same clone. We also write
 - $lex_{y,x}$ for the operation $(x,y) \mapsto lex(y,x)$,
 - $lex_{y,-x}$ for the operation $(x,y) \mapsto lex(y,-x)$,
 - $lex_{x,-y}$ for the operation $(x,y) \mapsto lex(x,-y)$,
 - $lex_{x,y}$ for the operation $(x,y) \mapsto lex(x,y)$,
 - p_x for the operation $(x,y) \mapsto x$, and
 - p_y for the operation $(x, y) \mapsto y$.

A k-ary operation $f: \mathbb{Q}^k \to \mathbb{Q}$ is dominated by the i-th argument when for all $\bar{a}, \bar{b} \in \mathbb{Q}^k$ it holds that $f(a_1, \ldots, a_k) \leq f(b_1, \ldots, b_k)$ if and only if $a_i \leq b_i$. Examples of

in P pp lex NP-complete

FIGURE 10.1. An illustration of the classification result for temporal constraint languages that contain <. Double-circles mean that the corresponding operation has a dual generating a distinct clone which is not drawn in the figure.

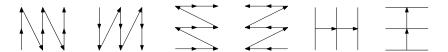


FIGURE 10.2. Illustrations of the six basic operations $lex_{x,y}$, $lex_{x,-y}$, $lex_{y,x}, lex_{y,-x}, p_x, p_y.$

operations dominated by the first argument are p_x , $lex_{x,y}$, and $lex_{x,-y}$, and examples of operations dominated by the second argument are p_y , $lex_{y,x}$, $lex_{y,-x}$.

It is easy to see that the relation Betw is preserved by lex, and more generally by all operations that are dominated by one argument. Therefore, we are interested in further restrictions of languages preserved by lex that imply tractability of the corresponding CSP.

A large tractable temporal constraint language has been introduced in [42]. The language is defined in terms of a binary polymorphism, denoted by ll, and it has a dual version, which is tractable as well. We will see in Proposition 10.4.2 that this language contains the class of Ord-Horn constraints (Section 1.5.9).

Definition 10.4.1. Let $ll: \mathbb{Q}^2 \to \mathbb{Q}$ be such that ll(a,b) < ll(a',b') if

- a ≤ 0 and a < a', or
 a ≤ 0 and a = a' and b < b', or
 a, a' > 0 and b < b', or
 a > 0 and b = b' and a < a'.

All operations satisfying these conditions are by definition injective, and they all generate the same clone. For an illustration of ll and its dual, see Figure 10.3. It is easy to see that ll generates lex.

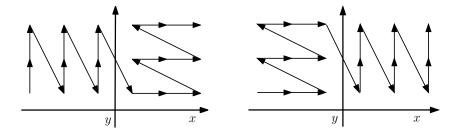


FIGURE 10.3. A visualization of ll (left) and dual-ll (right).

All Ord-Horn relations (Section 1.5.9) are preserved by ll.

Proposition 10.4.2. All relations in Ord-Horn are preserved by ll and dual ll.

PROOF. We give the argument for ll only; the argument for dual ll is analogous. It suffices to show that every relation that can be defined by a formula ϕ of the form $(x_1 = y_1 \land \cdots \land x_{k-1} = y_{k-1}) \rightarrow x_k O y_k$ is preserved by ll, where $O \in \{=, <, \le, \ne\}$. Let t_1 and t_2 be two 2k-tuples that satisfy ϕ . Consider a 2k-tuple t_3 obtained by applying Il componentwise to t_1 and t_2 . Suppose first that there is an $i \leq k-1$ such that one of the tuples does not satisfy $x_i = y_i$. Then $x_i = y_i$ is not satisfied in t_3 as well, by injectivity of ll, and therefore the tuple t_3 satisfies ϕ . Now consider the case that $x_i = y_i$ holds for all $i \leq k-1$ in both tuples t_1 and t_2 . Since t_1 and t_2 satisfy ϕ , the literal $x_k O y_k$ holds in both t_1 and t_2 . Because II preserves all relations in $\{=,<,\leq,\neq\}$, the literal x_kOy_k holds in t_3 , and therefore t_3 satisfies ϕ as well. \square

Since the relation R^{min} defined by $(x > y) \lor (x > z)$ (see Section 1.5.8) is preserved by ll but not by dual ll, the class of ll-closed constraints is strictly larger than Ord-Horn.

10.4.2. Operations generating ll, dual-ll, or lex. In this section we present operations that generate ll, dual-ll, or lex. We again use the concept of a behavior of operations over a relational structure; note that a k-ary operation f behaves like a k-ary operation g on $S = S_1 \times \cdots \times S_k$ if for all $t, t' \in S$ we have $f(t) \leq f(t')$ iff $g(t) \leq g(t')$. That is, the weak linear order induced by f on the tuples from G (in the sense of Observation 10.2.3) is the same as the weak linear order induced on these tuples by g. Let \mathbb{Q}^+ denote the set of all positive rational numbers, and let $\mathbb{Q}_0^$ denote $\mathbb{Q} \setminus \mathbb{Q}^+$.

Definition 10.4.3. Let f, g be from $\mathbb{Q}^2 \to \mathbb{Q}$. Then [f|g] denotes an arbitrary operation from $\mathbb{Q}^2 \to \mathbb{Q}$ with the following properties. For all $x, x', y, y' \in \mathbb{Q}$,

- if $x \le 0$ and x' > 0 then [f|g](x,y) < [f|g](x',y');
- [f|g] behaves like f on Q₀⁻ × Q;
 [f|g] behaves like g on Q⁺ × Q;

For example, if $f = lex_{x,y}$ and $g = lex_{y,x}$, then [f|g] behaves like ll.

LEMMA 10.4.4. Let $f, g \in \{lex_{x,y}, lex_{x,-y}, lex_{y,x}, lex_{y,-x}, p_x, p_y\}$, and let f'(g')be $lex_{x,y}$ if f(g) is dominated by the first argument, and $lex_{y,x}$ otherwise. Then $\{lex, [f|g]\}\ generates\ [f'|g'](x,y).$

PROOF. By Proposition 5.2.1 it suffices to show that every relation R preserved by lex and [f|g] is preserved by [f'|g']. So let R be an arbitrary relation preserved by lex and [f|g], let k denote its arity, and let t_1, t_2 be k-tuples from R. We have to show that $t_3 := [f'|g'](t_1, t_2)$ is in R.

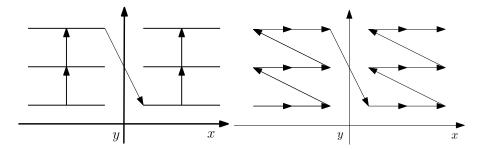


FIGURE 10.4. An illustration of the operation $[p_y|p_y]$ (on the left) and the operation $[lex_{y,x}|lex_{y,x}]$ (on the right).

Let $\alpha \in \operatorname{Aut}((\mathbb{Q}; <))$ be such that for each entry x of t_1 and for each entry y of t_2 , the value of $\alpha lex(x,y)$ is negative when $x \leq 0$, and positive otherwise. We will show that there is an automorphism of $(\mathbb{Q}; <)$ that maps the tuple

$$s := [f|g](\alpha lex(t_1, t_2), lex(t_2, t_1))$$

to t_3 , which proves that t_3 is in R. It suffices to show for $j_1, j_2 \in [k]$ that

$$s[j_1] \le s[j_2]$$
 if and only if $t_3[j_1] \le t_3[j_2]$. (25)

We can assume that $t_1[j_1] \leq t_1[j_2]$ by exchanging the name of j_1 and j_2 if necessary, and distinguish three cases:

- $t_1[j_1] \leq 0$, $t_1[j_2] > 0$. Then $t_3[j_1] < t_3[j_2]$ by definition of [f'|g']. Since for $j \in [k]$, the value of $\alpha lex(t_1[j], t_2[j])$ is positive if and only if the value of $t_1[j]$ is positive, we have $s[j_1] < s[j_2]$ by definition of [f|g]. Thus we have verified (25) in this case.
- $t_1[j_2] \leq 0$. Note that f(lex(x,y), lex(y,x)) behaves like f'(x,y). Thus, writing a[j] for $lex(t_1[j], t_2[j])$ and b[j] for $lex(t_2[j], t_1[j])$, we have the following equivalences.

$$t_{3}[j_{1}] \leq t_{3}[j_{2}] \quad \text{iff} \quad f'(t_{1}[j_{1}], t_{2}[j_{1}]) \leq f'(t_{1}[j_{2}], t_{2}[j_{2}])$$

$$\quad \text{iff} \quad f(a[j_{1}], b[j_{1}]) \leq f(a[j_{2}], b[j_{2}])$$

$$\quad \text{iff} \quad f(\alpha a[j_{1}], b[j_{1}]) \leq f(\alpha a[j_{2}], b[j_{2}])$$

$$\quad \text{iff} \quad s[j_{1}] \leq s[j_{2}]$$

• $t_1[j_1] > 0$. This case is analogous to the previous one and left to the reader.

LEMMA 10.4.5. For $f, g \in \{p_y, lex_{y,x}\}\$ the operation [f|g] generates $[lex_{x,y}|g]$.

In particular, for $f = g = lex_{y,x}$ the lemma shows that [f|g] generates ll. For $f = g = p_y$, the lemma shows that [f|g] generates $[lex_{x,y}|p_y]$ and in particular $lex_{x,y}$. See Figure 10.4 for illustrations of those two cases.

PROOF OF LEMMA 10.4.5. We show that every relation R preserved by [f|g] is preserved by $[lex_{x,y}|g]$, and conclude by Proposition 5.2.1 that [f|g] generates $[lex_{x,y}|g]$. So let R be an arbitrary relation preserved by [f|g], let k denote its arity, and let t_1, t_2 be k-tuples from R. We have to show that $t_3 := [lex_{x,y}|g](t_1, t_2)$ is in R.

Let l denote the number of non-positive values in t_1 . We take $\alpha_1, \ldots, \alpha_l$ from $\operatorname{Aut}((\mathbb{Q}; <))$ such that α_i maps all but the i smallest values in t_1 to positive values.

We define a sequence of tuples s_1, \ldots, s_l as follows: $s_1 = t_2$, and for $i \geq 2$

$$s_i := [f|g](\alpha_i t_1, s_{i-1})$$
.

Clearly, for all $i \in [l]$ the tuple s_i is in R. We will show that there is an automorphism of $(\mathbb{Q}; <)$ that maps s_l to t_3 , which proves that t_3 is also in R. By symmetry it is enough to show for $j_1, j_2 \in [k]$ with $t_1[j_1] \leq t_1[j_2]$ that

$$s_l[j_1] \le s_l[j_2]$$
 if and only if $t_3[j_1] \le t_3[j_2]$. (26)

We distinguish three cases:

- $t_1[j_1] = t_1[j_2] \le 0$. Since $\alpha_i t_1[j_1] = \alpha_i t_1[j_2]$ for all $i \in [l]$, we have $s_l[j_1] \le s_l[j_2]$ if and only if $s_1[j_1] \le s_1[j_2]$. Since $s_1 = t_2$ and $t_1[j_1] \le 0$, and because f is dominated by the second argument, $s_1[j_1] \le s_1[j_2]$ if and only if $t_3[j_1] \le t_3[j_2]$, which proves (26).
- $t_1[j_1] < t_1[j_2], t_1[j_1] \le 0$. Let $i \in [l]$ be such that $\alpha_i t_1[j_1] \le 0$ and $\alpha_i t_1[j_2] > 0$. By definition of [f|g] we see that $s_i[j_1] < s_i[j_2]$. Because $\alpha_i t_1[j_1] < \alpha_i t_1[j_2]$ for all $i \in [l]$, and because [f|g] preserves <, by induction on $i' \ge i$ we have that $s_{i'}[j_1] < s_{i'}[j_2]$. In particular, $s_l[j_1] < s_l[j_2]$. On the other hand, $t_3[j_1] < t_3[j_2]$ by definition of $lex_{x,y}$ and $[lex_{x,y}|g]$, and so (26) also holds in this case.
- $t_1[j_1] > 0$. Observe that by the choice of l we have $\alpha_i t_1[j_1] > 0$ for all $i \in [l]$. Thus (26) holds, because both [f|g] and $[lex_{x,y}|g]$ behave like g on $\mathbb{Q}^+ \times \mathbb{Q}$.

10.4.3. A syntactic description of ll-closed constraints. In this section we present a syntactic characterisation of ll-closed relations. As a consequence, we also obtain a better understanding of the clone generated by *ll*.

Definition 10.4.6. A formula is called ll-Horn if it is a conjunction of formulas of the following form

$$(x_1 = y_1 \wedge \cdots \wedge x_k = y_k) \Rightarrow (z_1 < z_0 \vee \cdots \vee z_l < z_0) , or$$
$$(x_1 = y_1 \wedge \cdots \wedge x_k = y_k) \Rightarrow (z_1 < z_0 \vee \cdots \vee z_l < z_0 \vee (z_0 = z_1 = \cdots = z_l))$$
where $0 \le k, l$.

Note that k or l might be 0: if k=0, we obtain a formula of the form $z_1 < z_0 \lor \cdots \lor z_l < z_0$ or $(z_1 < z_0 \lor \cdots \lor z_l < z_0 \lor (z_0 = z_1 = \cdots = z_l))$, and if l=0 we obtain a disjunction of disequalities. Also note that the variables $x_1, \ldots, x_k, y_1, \ldots, y_k, z_0, \ldots, z_l$ need not be pairwise distinct. On the other hand, the clause $z_1 < z_2 \lor z_3 < z_4$ is an example of a formula that is *not* ll-Horn.

The following result is from [42], but Antoine Mottet found a mistake in the proof presented there; the new proof presented below is also due to him.

Proposition 10.4.7. A temporal relation is preserved by ll if and only if it can be defined by an ll-Horn formula.

PROOF. The proof that every relation defined by an ll-Horn formula is ll-closed is similar to the proof of Proposition 10.4.2. We just need to additionally check that the relation defined by $z_1 < z_0 \lor \cdots \lor z_l < z_0$ and the relation defined by $z_1 < z_0 \lor \cdots \lor z_l < z_0$ are preserved by ll. So let s and t be two assignments that satisfy $\phi := z_1 < z_0 \lor \cdots \lor z_l < z_0$, and let r := ll(s,t). Let $i \in \{1,\ldots,l\}$ be such that $s(z_i) = min(s(z_1),\ldots,s(z_l))$. Note that $s(z_i) < s(z_0)$. Let $j \in \{1,\ldots,l\}$ be such that $t(z_j) < t(z_0)$.

• If $s(z_i) \leq 0$ then $ll(s(z_i), t(z_i)) < ll(s(z_0), s(z_0))$ since $s(z_i) < s(z_0)$, and hence r satisfies ϕ .

• If $s(z_i) > 0$, then $s(z_0) > s(z_i) > 0$ and $s(z_j) > s(z_i) > 0$, and hence $ll(s(z_j), t(z_j)) < ll(s(z_0), s(z_0))$ since $t(z_j) < t(z_0)$, and hence r satisfies ϕ .

When t_1 and t_2 are satisfying assignments of $z_1 < z_0 \lor \cdots \lor z_l < z_0 \lor (z_0 = \cdots = z_l)$ where one of the assignments satisfies the last clause, then the statement follows from the fact that ll is injective and preserves \leq .

Let R be a temporal relation, and let ϕ be a quantifier-free formula in CNF that defines R over $(\mathbb{Q};<)$. In this formula, we replace literals of the form $\neg(y< x)$ by $x< y\vee x=y$, and we use $x\leq y$ as shortcut for those two literals. For reasons that will become clear later, we additionally allow that clauses contain 'clustered equations' which are expressions of the form $x_1=x_2=\cdots=x_n$ and which stand for $x_1=x_2\wedge\cdots\wedge x_1=x_n$; such an expression will be treated as one literal.

We describe four rewriting rules that yield a formula ψ that also defines R over $(\mathbb{Q}; <)$ such that R is preserved by ll if and only if ψ is ll-Horn.

- (1) If ϕ implies x = y for distinct variables x, y of ϕ , replace all occurrences of u in ϕ by x and add the clause x = y.
- (2) Suppose that ϕ contains a clause θ of the form

$$x < y \lor u < v \lor \theta'$$

let ϕ' be the other clauses of ϕ , and suppose that

$$(\phi' \land \neg \theta' \land x < y)$$
 implies $(u \le v \lor x \le v)$

and
$$(\phi' \land \neg \theta' \land u < v)$$
 implies $(x \le y \lor u \le y)$.

Then replace θ by

$$(u \le v \lor x \le v \lor \theta') \land (u \ne v \lor x < y \lor \theta')$$

$$\land (x \le y \lor u \le y \lor \theta') \land (x \ne y \lor u < v \lor \theta').$$

(3) Suppose that ϕ contains a clause θ of the form

$$x < y \lor u < v \lor \theta'$$
.

Let ϕ' be the other clauses of ϕ , and suppose that

$$(\phi' \land \neg \theta' \land x < y)$$
 implies $u < v$.

Then replace θ by

$$(u \le v \lor \theta') \land (x < y \lor u \ne v \lor \theta').$$

(4) Suppose that θ is a clause of ϕ of the form

$$x_1 \neq y_1 \vee \cdots \vee x_k \neq y_k \vee z_1 < z_0 \vee \cdots \vee z_l < z_0 \vee u = v$$

let ϕ' be the other clauses of ϕ , and that

$$\phi' \wedge x_1 = y_1 \wedge \cdots \wedge x_k = y_k \wedge z_0 \leq z_1 \wedge \cdots \wedge z_0 \leq z_l \wedge u = v$$

implies that $z_0 = z_1 = \cdots = z_l$. Then replace θ by

$$(x_1 \neq y_1 \vee \cdots \vee x_k \neq y_k \vee z_0 \neq z_1 \vee \cdots \vee z_0 \neq z_l \vee u = v)$$

$$\wedge (x_1 \neq y_1 \vee \cdots \vee x_k \neq y_k \vee z_1 < z_0 \vee \cdots \vee z_l < z_0 \vee z_0 = z_1 = \cdots = z_l).$$

(5) If ϕ contains a literal such that removing this literal from ϕ results in an equivalent formula, then remove the literal.

We claim that for each of the four rewriting rules, the resulting formula ψ is equivalent to ϕ . This is obvious for rules (1) and (5). To see that ϕ implies the new clauses in rule (2), let s be a satisfying assignment to ϕ . If s satisfies θ' , then s also satisfies the new clauses, so let us assume that θ' is false. Then s satisfies x < y or u < v. The two cases are symmetric, so we only treat the case that s satisfies x < y in the

following. By assumption, s must then satisfy $u \le v \lor y \le v$, and hence the first new clause is satisfied by s. Since x < y, the other new clauses are satisfied, too.

Now suppose conversely that s is a solution to ϕ' and the four new clauses, and suppose for contradiction that θ does not hold. Because of the second and fourth new clause, we then must have $u \neq v$ and $x \neq y$. Then the first new clause implies that $x \leq v$ and the third new clause implies that $u \leq y$. But then $x \leq v \leq u \leq y \leq x$, a contradiction to $x \neq y$.

For rule (3), let s be a solution to ϕ . Then s obviously satisfies the first new clause if u < v or θ' holds; otherwise, s must satisfy x < y because of θ . But then $u \ge v$ by assumption and hence the first new clause also holds in this case. The second new clause is weaker then θ , so it is also satisfied by s. Now suppose conversely that s satisfies ϕ' and the two new clauses, and suppose for contradiction that θ does not hold. Then in particular $v \le u$ holds and the first new clause implies that u = v, and hence x < y because of the second new clause, contradiction to the assumption that θ does not hold.

Finally, for the fourth rule, the first new clause is a weakening of θ , and the second new clause is a consequence of ϕ by assumption. Conversely, suppose that s satisfies all clauses of ϕ except for θ which is not satisfied. Then the first new clause implies that $z_1 < z_1 \lor \cdots \lor z_l < z_0$, and thus the second new clause implies that u = v, and hence θ holds, contradiction. Hence, ψ is indeed equivalent to ϕ .

Note that rules (2) and (3) strictly reduce the number of pairs of literals x < y and u < v in the same clause where y and v are distinct variables. Rule (4) leaves this number invariant, but strictly reduces the number of literals of the form u = v or of the form u < v in the clause (here, we do not count complex equations). Rules (1) and (5) do not increase these numbers, and strictly reduce the number of variables that occur more than once, or strictly reduce the total number of literals. Hence, when we repeatedly apply these rules, the procedure will eventually terminate.

Claim 1. The formula ψ cannot contain a clause θ of the form $x < y \lor u < v \lor \theta'$ where x and u are distinct variables. Since rule (2) is not applicable, there must exist a solution s to $\phi' \land \neg \theta' \land x < y \land v < u \land v < x$ or to $\phi' \land \neg \theta' \land u < v \land y < x \land y < u$. Suppose the former is the case, since the latter case can be treated similarly. Since rule (3) is not applicable, there exists a solution t to $\phi' \land \neg \theta' \land u < v \land y < x$. Let $\alpha \in \operatorname{Aut}(\mathbb{Q};<)$ be such that $\alpha s(v)=0$. We claim that $r=ll(\alpha s,t)$ does not satisfy θ :

- we have r(y) < r(x) since 0 < s(x), s(y) and t(y) < t(x);
- we have r(v) < r(u) since s(v) = 0 and s(u) > 0;
- finally, r does not satisfy θ' since neither s nor t satisfy θ' .

Hence, r does not satisfy ψ , in contradiction to the assumption that ll preserves R.

Claim 2. The formula ψ cannot contain a clause with two distinct literals x=y and u=v. This is because rule (5) and since ϕ is preserved by the injective function ll.

Claim 3. If ψ contains a clause with a literal $z_1 < z_0$ and a literal u = v, then $\{u, v\} = \{x, y\}$. This is because of Claim 1 and Claim 2, any such clause must be of the form $x_1 \neq y_1 \vee \cdots \vee x_k \neq y_k \vee z_1 < z_0 \vee \cdots \vee z_l < z_0 \vee u = v$. Since rule (4) does not apply, there exists a solution s to

$$\phi' \wedge x_1 = y_1 \wedge \dots \wedge x_k = y_k \wedge z_0 \le z_1 \wedge \dots \wedge z_0 \le z_l \wedge u = v$$
$$\wedge z_0 \ne z_1 \vee \dots \vee z_0 \ne z_l.$$

Hence, there exists an $i \in \{1, ..., l\}$ such that $s(z_0) \neq s(z_i)$. Because the literal $z_i < z_0$ cannot be removed from ψ with rule (5), there exists a solution t to ϕ such that $z_i < z_0$ is the only literal in θ satisfied by t. Let $\alpha \in \operatorname{Aut}(\mathbb{Q}; <)$ be such that $\alpha t(z_i) = 0$. Then $r := ll(\alpha t, s)$ does not satisfy θ :

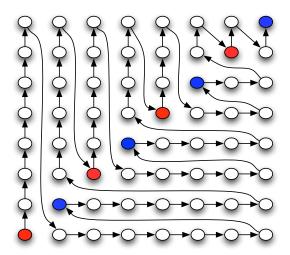


FIGURE 10.5. Illustration of an operation f generated by ll that satisfies $f(x,y) = \alpha f(\beta(x), \beta(y))$.

- r satisfies $\theta' \wedge x_1 = y_1 \wedge \cdots \wedge x_k = y_k$ since both t and s satisfy this formula.
- $r(z_i) < r(z_0)$ since $0 = \alpha t(z_i) < \alpha t(z_0)$.
- $r(z_j) \le r(z_0)$ for all $i \in \{1, \dots, k\} \setminus \{i\}$ since $t(z_0) \le t(z_j)$ and $s(z_0) \le s(z_j)$.
- $r(u) \neq r(v)$ since $t(u) \neq t(v)$ and ll is injective.

The three claims imply that each of the clauses of ψ must be logically equivalent to an implication as in Definition 10.4.6, and this concludes the proof.

- **10.4.4.** Weak Commutativity. In this section we present a different description of ll-closed temporal constraint languages. Let $c_1, \ldots, c_n \in \mathbb{Q}$ be arbitrary. Partition the rationals $\mathbb{Q} = Q_1 \uplus Q_1$ such that each of Q_1 and Q_2 is dense in \mathbb{Q} , and let f be any injective binary operation that preserves < and \le such that
 - $f(c_i, c_i) = c_i$ for all $i \leq n$;
 - for $x \in Q_1$, we have that $f(x, y_1) > f(y_2, x)$ for all $y_1, y_2 > x$;
 - for $x \in Q_2$, we have that $f(x, y_1) < f(y_2, x)$ for all $y_1, y_2 > x$.

We even assume that f is bijective: since the image of any such function will induce a dense linear order without endpoints in $(\mathbb{Q};<)$, the existence of such a function follows from ω -categoricity of $(\mathbb{Q};<)$.

For an illustration of f, see Figure 10.5. The red vertices are the elements of $\{(x,x) \mid x \in Q_1\}$, and the blue blue vertices the elements of $\{(x,x) \mid x \in Q_2\}$. Observe that f(x,y) < f(z,z) for all x,y,z such that x < z or y < z.

PROPOSITION 10.4.8. There are automorphisms α, β of $(\mathbb{Q}; <, c_1, \ldots, c_n)$ such that $f(x,y) = \alpha f(\beta y, \beta x)$ holds for all $x, y \in \mathbb{Q}$.

PROOF. Let β be an automorphism of $(\mathbb{Q}; <, c_1, \ldots, c_n)$ that maps $Q_1 \setminus \{c_1, \ldots, c_n\}$ to $Q_2 \setminus \{c_1, \ldots, c_n\}$ and $Q_2 \setminus \{c_1, \ldots, c_n\}$ to $Q_1 \setminus \{c_1, \ldots, c_n\}$; such an automorphism can easily be constructed by going back-and-forth. To define $\alpha \in \operatorname{Aut}((\mathbb{Q}; <, c_1, \ldots, c_n))$, let $d \in \mathbb{Q}$ be arbitrary. Since f and β are bijective, $f' \colon (x, y) \mapsto f(\beta y, \beta x)$ is bijective as well, so there exists a unique pair $(a, b) \in \mathbb{Q}^2$ such that f'(a, b) = d. Define $\alpha(d) = f(a, b)$. Then by definition $\alpha f(\beta y, \beta x) = f(x, y)$ holds for all $x, y \in \mathbb{Q}$, and it is straightforward to verify that $\alpha \in \operatorname{Aut}((\mathbb{Q}; <, c_1, \ldots, c_n))$.

Proposition 10.4.9. The operation f defined above is generated by ll, and generates ll.

PROOF. It is easy to see that f interpolates ll. For the converse, it suffices to verify that f preserves all ll-Horn formulas, by Proposition 10.4.7. Since f is injective, it suffices to show that f preserves formulas of the form

$$(z_0 > z_1) \lor \cdots \lor (z_0 > z_l)$$

and formulas of the form

$$(z_0 > z_1) \lor \cdots \lor (z_0 > z_l) \lor (z_0 = z_1 = \cdots = z_l)$$
.

Preservation of formulas of the latter type reduces to the former type, since f preserves \leq and is injective binary. Now suppose that $\bar{a} = (a_0, a_1, \ldots, a_l)$ and $\bar{b} = (b_0, b_1, \ldots, b_l)$ are two tuples that satisfy $z_1 < z_0 \lor \cdots \lor z_l < z_0$. Assume that $a_0 \leq b_0$. Let i be such that $a_i = min(a_1, \ldots, a_l)$. Then $a_i < a_0$, and we get that $f(a_0, b_0) \geq f(a_0, a_0) > f(a_i, b_i)$ by the properties of f. Therefore, $min(f(a_1, b_1), \ldots, f(a_l, b_l)) < f(a_0, b_0)$. We can argue analogously in the case that $a_0 \geq b_0$.

10.4.5. Weak near-unanimity modulo endomorphisms. For a uniform presentation of the classification result in Section 10.6, we need yet another description of the clone generated by ll.

We write $lex(x_1, ..., x_n)$ as a shortcut for $lex(x_1, lex(x_2, ... lex(x_{n-1}, x_n) ...))$.

PROPOSITION 10.4.10. There are $a, b, c \in \text{End}(\mathbb{Q}; <)$ such that the ternary function $f: \mathbb{Q}^3 \to \mathbb{Q}$ defined by

$$f(x,y,z) = lex(min(x,y,z), max(min(x,y), min(x,z), min(y,z)), x,y,z)$$
 satisfies for all $x,y \in \mathbb{Q}$

$$a(f(x, x, y)) = b(f(x, y, x)) = c(f(y, x, x)).$$

That is, f is a weak near unanimity modulo endomorphisms of $(\mathbb{Q}; <)$.

PROOF. By Lemma 5.6.7, it suffices to show that for every finite $S \subset \mathbb{Q}$ there are $\alpha, \beta \in \operatorname{Aut}(\mathbb{Q}; <)$ such that for all $x, y \in S$

$$f(y, x, x) = \alpha_1 f(x, y, x) = \alpha_2 f(x, x, y).$$

By the properties of f we have that $f(y, x, x) \leq f(y', x', x')$ if and only if one of the following holds:

- min(x, y) < min(x', y');
- min(x, y) = min(x', y') and x < x';
- min(x, y) = min(x', y'), x = x', and y < y';
- x = x' and y = y'.

Note that this is the case if and only if f(x, y, x) < f(x', y', x'), and if and only if f(x, x, y) < f(x', x', y'). Hence, the existence of α_1 and α_2 follows from the homogeneity of $(\mathbb{Q}; <)$.

Note that the function f defined in Proposition 10.4.10 is injective and preserves \leq .

THEOREM 10.4.11. Let $R \subseteq \mathbb{Q}^n$ be first-order definable over $(\mathbb{Q}; <)$. Then the following are equivalent.

- (1) R is preserved by the operation f as defined in Proposition 10.4.10 (a weak near unanimity modulo endomorphisms).
- (2) R is preserved by ll.
- (3) R has an ll-Horn definition.

PROOF. The implication from (1) to (2) follows from the observation that $(x, y) \mapsto f(x, x, y)$ interpolates ll. The implication from (2) to (3) is Lemma 10.4.7. For the implication from (3) to (1), it suffices to verify that f preserves all ll-Horn formulas. Since f is injective, it suffices to show that f preserves formulas ϕ of the form

$$(z_1 < z_0) \lor \cdots \lor (z_l < z_0)$$

and of the form

$$(z_1 < z_0) \lor \cdots \lor (z_l < z_0) \lor (z_0 = z_1 = \cdots = z_l)$$
.

Suppose that s_1, s_2, s_3 are assignments that satisfy ϕ ; we have to show that the assignment s defined by $s(x) := f(s_1(x), s_2(x), s_3(x))$ satisfies ϕ . Let $j \in \{1, 2, 3\}$ be such that $s_j(z_0) = \min(s_1(z_0), s_2(z_0), s_3(z_0))$.

Suppose first that s_j satisfies $(z_1 < z_0 \lor \cdots \lor z_l < z_0)$. Let i be such that $s_j(z_i) = min(s_j(z_1), \ldots, s_j(z_l))$. Then $s_j(z_i) < s_j(z_0)$ by assumption, and hence

$$min(s_1(z_i), s_2(z_i), s_3(z_i)) < min(s_1(z_0), s_2(z_0), s_3(z_0)).$$

Therefore, $f(s_1(z_i), s_2(z_i), s_3(z_i)) < f(s_1(z_0), s_2(z_0), s_3(z_0))$ by the properties of f, and s satisfies $(z_1 < z_0 \lor \cdots \lor z_l < z_0)$.

Otherwise, s_j must satisfy $z_0 = z_1 = \cdots = z_l$. Let a,b be such that a < b and $\{a,b\} = \{1,2,3\} \setminus \{j\}$. We next consider the case that there exists $c \in \{a,b\}$ and $p \in \{1,\ldots,l\}$ such that $s_j(z_0) > s_c(z_p)$. Let $d \in \{a,b\} \setminus \{c\}$. Note that

$$\min(s_c(z_p), s_d(z_p)) \le s_c(z_p) < s_j(z_p) = s_j(z_0) = \min(s_c(z_0), s_d(z_0))$$

$$\min(s_j(z_p), s_c(z_p)) = s_c(z_p) < s_j(z_p) = s_j(z_0) = \min(s_c(z_0), s_j(z_0))$$

$$\min(s_j(z_p), s_c(z_p)) = s_c(z_p) < s_j(z_p) = s_j(z_0) = \min(s_c(z_0), s_j(z_0))$$

and hence

$$\begin{split} & \max(\min(s_c(z_p), s_d(z_p)), \min(s_j(z_p), s_c(z_p)), \min(s_j(z_p), s_d(z_i))) \\ & < \max(\min(s_c(z_0), s_d(z_0)), \min(s_j(z_0), s_c(z_0)), \min(s_j(z_0), s_d(z_i))) \,. \end{split}$$

Thus, by the definition of f, we have $s(z_p) < s(z_0)$ and s satisfies ϕ .

Otherwise, $s_j(z_0) \leq \min(s_a(z_1), \ldots, s_a(z_l))$ and $s_j(z_0) \leq \min(s_b(z_1), \ldots, s_b(z_l))$. For all $i \in \{0, 1, \ldots, l\}$ we have $s_j(z_i) = s_j(z_0)$ and hence

$$\begin{aligned} & \max(\min(s_a(z_i), s_b(z_i)), \min(s_j(z_i), s_a(z_i)), \min(s_j(z_i), s_b(z_i))) \\ &= \min(s_a(z_i), s_b(z_i)) \geq s_j(z_i) \,. \end{aligned}$$

The definition of f then implies that $f(s_1(z_i), s_2(z_i), s_3(z_i)) < f(s_1(z_0), s_2(z_0), s_3(z_0))$ if and only if $s_a(z_i) < s_a(z_0)$. If there exists an $i \in \{1, \ldots, l\}$ such that $s_k(z_i) < s_k(z_0)$, we therefore have $s(z_i) < s(z_0)$ and s satisfies $(z_1 < z_0) \lor \cdots \lor (z_l < z_0)$. Otherwise, we must have that

$$s_a(z_0) = s_a(z_1) = \dots = s_a(z_l)$$

If also $s_b(z_0) = s_b(z_1) = \cdots = s_b(z_l)$ then s satisfies $z_0 = z_1 = \cdots = z_l$, too. So suppose that there exists a $p \in \{1, \ldots, l\}$ such that $s_b(z_p) < s_b(z_0)$. Since $s_j(z_p) = s_j(z_0)$ and $s_a(z_p) = s_a(z_0)$ we then have $s(z_p) < s(z_0)$ since s is injective and preserves \leq . Hence, s satisfies ϕ also in this case.

10.4.6. An Algorithm for ll-closed Constraints. In this section we present an algorithm for ll-closed constraints. One of the underlying ideas of the algorithm is to use a subroutine that tries to find a solution where every variable has a different value. If this is impossible, the subroutine must return a set of at least two variables that denote the same value in all solutions – since the constraints are preserved by a binary injective operation, such a set must exist (Proposition 6.1.5).

The *i*-th entry in a *k*-tuple *t* is called *minimal* if $t[i] \le t[j]$ for every $j \in [k]$. It is called *strictly minimal* if t[i] < t[j] for every $j \in [k] \setminus \{i\}$.

DEFINITION 10.4.12. Let R be a k-ary temporal relation. A set $S \subseteq [k]$ is called a min-set for the i-th entry in R if there exists a tuple $t \in R$ such that the i-th entry is minimal in t, and for all $j \in [k]$ it holds that $j \in S$ if and only if t[i] = t[j]. We say that t is a witness for this min-set.

Let R be a k-ary relation that is preserved by lex (recall that ll-closed constraints are preserved by lex as well), and suppose that the i-th entry has the min-sets S_1, \ldots, S_l , for $l \geq 1$, with the corresponding witnesses t_1, \ldots, t_l . Consider the tuple $t:=lex(t_1, lex(t_2, \ldots lex(t_{l-1}, t_l)))$. Since the entry i is minimal in every tuple t_1, \ldots, t_l , and since lex preserves both < and \leq , it is also minimal in t. Because lex is injective, we have that t[i]=t[j] if and only if these two entries are equal in each tuple t_1, \ldots, t_l . Hence, the min-set for the i-th entry in R witnessed by the tuple t is a subset of every other min-set S_1, \ldots, S_l . We then call this set the $minimal\ min-set$ for the i-th entry in R.

LEMMA 10.4.13. Let R be a k-ary relation preserved by lex, and let S be the minimal min-set for the i-th entry in R. If $t \in R$ is such that $t[j] \geq t[i]$ for every $j \in S$, then t[i] = t[j] for every $j \in S$.

PROOF. Let $t' \in R$ be the tuple that witnesses the minimal min-set S. Let $t \in R$ be such that not all entries in S are equal (in particular, |S| > 1). Consider the tuple s := lex(t',t). By the properties of lex it holds that s[i] < s[j] for every $j \in [k] \setminus S$. Furthermore, $s[i] \le s[j]$ for $j \in S$ if and only if $t[i] \le t[j]$. Thus, unless s witnesses a smaller min-set for i in R (which would be a contradiction), we have that s[i] > s[j] for some $j \in S$.

To develop our algorithm, we use a specific notion of *constraint graph* of a temporal CSP instance, defined as follows.

DEFINITION 10.4.14. The constraint graph G_{ϕ} of a temporal CSP instance ϕ is a directed graph (V; E) defined on the variables V of ϕ . For each constraint of the form $R(x_1, \ldots, x_k)$ from ϕ we add a directed edge (x_i, x_j) to E if in every tuple from R where the i-th entry is minimal, the j-th entry is minimal as well.

DEFINITION 10.4.15. If an instance of a temporal CSP contains a constraint ϕ imposed on y such that ϕ does not admit a solution where y denotes the minimal value, the we say that y is blocked (by ϕ).

We can easily determine for each constraint which variables are blocked by this constraint: For a constraint represented by weak linear orders we just check all weak linear orders and build a set of variables that are not minimal in any of them. Thus, by inspecting all the constraints it is possible to compute the blocked variables in linear time in the input size. We want to use the constraint graph to identify variables that have to denote the same value in all solutions, and therefore introduce the following concepts.

DEFINITION 10.4.16. A strongly connected component K of the constraint graph G_{ϕ} for a temporal CSP instance ϕ is called a sink component if no edge in G_{ϕ} leaves K, and no variable in K is blocked. A vertex of G that belongs to a sink component of size one is called a sink.

LEMMA 10.4.17. Let \mathfrak{B} be a lex-closed temporal constraint language. Let ϕ be an instance of CSP(\mathfrak{B}) with variables V, and let $K \subseteq V$ be a sink component of the graph G_{ϕ} . Then in every solution of ϕ all variables from K must have equal values.

PROOF. We assume that ϕ has a solution $s\colon V\to \mathbb{Q}$, and that K has at least two vertices; otherwise the statement is trivial. Define $M:=\{x\in K\mid s(x)\leq s(y) \text{ for all }y\in K\}$. We want to show that M=K. Otherwise, because K is a strongly connected component, there is an edge in G_{ϕ} from some vertex $u\in M$ to some vertex $v\in K\setminus M$. By the definition of G_{ϕ} , there is a constraint ψ in ϕ such that whenever u denotes the minimal value of a solution of ψ , then v has to denote the minimal value as well. By permuting arguments, we can assume without loss of generality that ψ is of the form $R(w_1,\ldots,w_k)$ where $w_1=u$. Because K is a sink component, the variable u cannot be blocked, and hence there is a minimal min-set S for the first entry in R.

Note that G_{ϕ} contains an edge from u to w_i for all $i \in S$. Since K is a strongly connected component, all these variables w_i are in K. Because $s(u) \leq s(y)$ for all $y \in K$, there is no variable w_i , $i \in S$, such that $s(w_i) < s(u)$. This contradicts Lemma 10.4.13, because $s(u) \neq s(v)$.

Lemma 10.4.17 immediately implies that we can add constraints of the type x = y for all variables x, y from the same sink component K. Equivalently, we can consider the CSP instance where all the variables in K are contracted, i.e., where all variables from K are replaced by the same variable. When $\phi = \exists x_1, \ldots, x_n \ (\phi_1 \land \cdots \land \phi_m)$ is an instance of a CSP(\mathfrak{B}), and $x_i \in V := \{x_1, \ldots, x_n\}$, then we write $\phi[V \setminus \{x_i\}]$ for the formula $\exists x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \ ((\exists x_i.\phi_1) \land \cdots \land (\exists x_i.\phi_m))$. Note that if \mathfrak{B} contains all primitive positive definable relations whose arity is bounded by the maximal arity of the relations in \mathfrak{B} , then $\phi[V \setminus \{x_i\}]$ can be viewed as an instance of CSP(\mathfrak{B}).

LEMMA 10.4.18. Let \mathfrak{B} be an ll-closed temporal constraint language. Let ϕ be an instance of CSP(\mathfrak{B}) with variables V, and let x be a sink in G_{ϕ} . If $\phi[V \setminus \{x\}]$ has an injective solution, then ϕ has an injective solution as well.

PROOF. Let $s \colon V \to \mathbb{Q}$ be an injective solution to $\phi[V \setminus \{x\}]$. We claim that any extension r of s to x such that r(x) < s(y) for all $y \in V \setminus \{x\}$ is injective and satisfies ϕ . If x appears in no constraint in ϕ , the statement is trivial. Consider a constraint $\psi = R(x_1, \ldots, x_k)$ from ϕ that is imposed on x, and let $S \subseteq [k]$ be such that $i \in S$ if and only if $x = x_i$. By the definition of $\phi[V \setminus \{x\}]$, the mapping s has an extension s' that is also defined on x such that $(s'(x_1), \ldots, s'(x_k)) \in R$. Because x is a sink, there is tuple $t \in R$ such that S is the minimal min-set for the i-th entry of R for each $i \in S$. Let t' be the tuple $(s'(x_1), \ldots, s'(x_k))$, and let $\alpha \in \operatorname{Aut}((\mathbb{Q}; <))$ be such that $\alpha s'(x) = 0$. Then $r := ll(\alpha t', t) \in R$. Note that for $i, j \in [k] \setminus S$, we have that $r[i] \leq r[j]$ if and only if $r(x_i) \leq r(x_j)$. Hence, r satisfies all constraints from ϕ , which is what we had to show.

Our algorithm for ll-closed constraints can be found in Figure 10.7; we are now ready to prove its correctness.

THEOREM 10.4.19. The procedure $Solve(\phi)$ in Algorithm 10.7 decides whether a given set of ll-closed constraints ϕ has a solution. There is an implementation of the algorithm that runs in time O(nm), where n is the number of variables of ϕ and m is the size of the input.

PROOF. The correctness of the procedure Spec immediately implies the correctness of the procedure Solve. In the procedure Spec, after iterated deletion of sinks in G_{ϕ} , we have to distinguish three cases.

First, consider the case V=X. In this case it follows by a straightforward induction from Lemma 10.4.18 that ϕ has an injective solution. Otherwise, consider the case that G_{ϕ} contains a sink component S with $|S| \geq 2$. We claim that for all

```
Spec(\phi)

// Input: An instance \phi of CSP(\mathfrak B) with variables V.

// Output: If algorithm returns false then \phi has no solution.

// If \phi has an injective solution, then algorithm returns true.

// Otherwise return S \subseteq V, |S| \ge 2, such that

// for all x, y \in S we have x = y in all solutions to \phi.

Set X := \emptyset

While G_{\phi} contains a sink s

X := X \cup \{s\}

If X = V then return true

else \phi := \phi[V \setminus X]

If G_{\phi} has sink component S return S

else return false

end if
```

FIGURE 10.6. A polynomial-time algorithm for $CSP(\mathfrak{B})$ when \mathfrak{B} is ll-closed: the sub-procedure Spec.

```
Solve(\Phi)

// Input: An instance \phi.

// Output: accept if \phi is true, reject otherwise.

S := \operatorname{Spec}(\phi)

If S = \operatorname{false} then reject
else if S = \operatorname{true} then accept
else

Let \phi' be contraction of S in \phi.

Return \operatorname{Solve}(\phi').
end if
```

FIGURE 10.7. A polynomial-time algorithm for $CSP(\mathfrak{B})$ when \mathfrak{B} is ll-closed: the main procedure.

variables $x, y \in S$ we have x = y in all solutions to ϕ . Lemma 10.4.17 applied to $\phi[V \setminus X]$ implies that all variables in the same sink component must have the same value in every solution, and hence the output is correct in this case as well.

In the third case we have $X \neq V$ but G_{ϕ} does not contain a sink component. Note that in every solution to ϕ some variable must take the minimal value. However, since each strongly connected component without outgoing edges contains a blocked vertex, there is no variable that can denote the minimal element, and hence ϕ has no solution. Because ϕ is at all times of the execution of the algorithm implied by the original input constraints, the algorithm correctly rejects.

Since in each recursive call of Solve the instance in the argument has at least one variable less, Solve is executed at most n times. It is not difficult to implement the algorithm such that the total running time is cubic in the input size. However, it is possible to implicitly represent the constraint graph and to implement all subprocedures such that the total running time is in O(nm); for the details, we refer to [42].

10.5. Shuffle-closed Constraints

An important subclass of temporal constraint languages are *shuffle-closed* constraint languages. As we will see, there are NP-complete shuffle-closed constraint

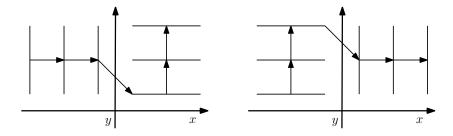


FIGURE 10.8. A visualization of pp (left) and dual-pp (right).

languages. However, in this section we present three additional restrictions for shuffleclosed constraint languages that imply that the corresponding CSPs can be solved in polynomial time.

10.5.1. Shuffle closure. We define shuffle closure, and show how shuffle closure can also be described by a certain binary operation on \mathbb{Q} .

Definition 10.5.1. A k-ary relation R is called shuffle-closed iff for all tuples $t_1, t_2 \in R$ and all indices $l \in [k]$ there is a tuple $t_3 \in R$ such that for all $i, j \in [k]$ we have $t_3[i] \leq t_3[j]$ iff

- $t_1[i] \le t_1[l]$ and $t_1[i] \le t_1[j]$, or $t_1[l] < t_1[i]$, $t_1[l] < t_1[j]$, and $t_2[i] \le t_2[j]$.

Let pp be an arbitrary binary operation on \mathbb{Q} such that $pp(a,b) \leq pp(a',b')$ iff one of the following cases applies:

- $\bullet \ a \leq 0 \text{ and } a \leq a'$ $\bullet \ 0 < a, \ 0 < a', \ \text{and } b \leq b'.$

Clearly, such an operation exists. For an illustration, see the left diagram in Figure 10.8. In diagrams for binary operations f as in Figure 10.8, we draw a directed edge from (a,b) to (a',b') if f(a,b) < f(a',b'). Unoriented lines in rows and columns of the picture for an operation f relate pairs of values that get the same value under f. The right diagram of Figure 10.8 is an illustration of the dual-pp operation. The name of the operation pp is derived from the word 'projection-projection', since the operation behaves as a projection to the first argument for negative first argument, and a projection to the second argument for positive first argument.

Proposition 10.5.2. A temporal relation is shuffle-closed if and only if it is preserved by pp.

PROOF. Let R be a shuffle-closed relation, and let t_1 and t_2 be tuples from R. We want to show that $t_3 = pp(t_1, t_2) \in R$. If t_1 only contains positive values, then there clearly exists an $\alpha \in \text{Aut}((\mathbb{Q};<))$ such that $t_3 = \alpha t_2$, and since R is preserved by the automorphisms of $(\mathbb{Q}; <)$, we are done. Otherwise, let $l \in [k]$ be an index such that $t_1[l]$ is the largest entry in t_1 that is not positive. Because R is shuffle-closed, we know that there exists a tuple $t_3 \in R$ such that $t_3'[i] \leq t_3'[j]$ iff $(t_1[i] \leq t_1[l])$ and $t_1[i] \le t_1[j]$) or $(t_1[l] < t_1[i], t_1[l] < t_1[j], \text{ and } t_2[i] \le t_2[j])$ for all $i, j \in [k]$. By the definition of pp, and the choice of l, the tuple t_3 satisfies the same property, and therefore there exists $\beta \in \operatorname{Aut}((\mathbb{Q};<))$ such that $t_3 = \beta t_3'$, and hence $t_3 \in R$.

For the opposite direction, we assume that R is preserved by pp, and have to show shuffle closure of R. Let t_1, t_2 be tuples in R, and let $l \in [k]$. Choose $\gamma \in \operatorname{Aut}((\mathbb{Q}; <))$ such that γ maps $t_1[l]$ to 0. Then $t_3 = pp(\gamma t_1, t_2)$ is a tuple that satisfies the conditions specified in the definition of shuffle-closure.

Due to Proposition 10.5.2, we use the phrase ' \mathfrak{B} is shuffle-closed' interchangeably with ' \mathfrak{B} is preserved by pp'. The following lemma states an important property of shuffle-closed languages that will be used several times in the next subsections.

LEMMA 10.5.3. Let t_1, \ldots, t_l be tuples from a k-ary shuffle-closed relation R, and let $M_1, \ldots, M_l \subset [k]$ be disjoint sets of indices such that $\bigcup_{i=1}^l M_i = [k]$ and such that for all $i, j \in [l]$ with i < j and for all $i' \in M_i$, $j' \in M_j$ it holds that $t_i[i'] < t_i[j']$. Then there is a tuple $t \in R$ such that

- t[i'] < t[j'] for all $i, j \in [l]$ with i < j and for all $i' \in M_i, j' \in M_j$;
- $t[i'] \le t[i'']$ iff $t_i[i'] \le t_i[i'']$ for all $i \in [l]$ and all $i', i'' \in M_i$.

PROOF. Let $\beta_1, \ldots, \beta_{l-1} \in \operatorname{Aut}((\mathbb{Q}; <))$ be such that β_i maps $\max\{t_i[i']|i' \in M_i\}$ to 0. We set

$$t := pp(\beta_1 t_1, pp(\beta_2 t_2, \dots, pp(\beta_{l-1} t_{l-1}, t_l) \dots)).$$

The tuple t clearly belongs to R.

We prove by induction on l that t satisfies the other conditions of the lemma. Observe that β_1 maps all the entries of t_1 at M_1 to non-positive values. Thus for l=2, it is easy to check from the properties of pp that for each $i \in M_1$ and $i' \in M_2$ we have t[i] < t[i'] as required by the statement of the lemma. Also the second condition is immediate. For l>2 let t' be defined by

$$t' := pp(\beta_2 t_2, pp(\beta_3 t_3, \dots, pp(\beta_{l-1} t_{l-1}, t_l) \dots))$$
.

Then we have $t = pp(\beta_1 t_1, t')$. Now we apply the same argument as for l = 2. Because the order on $[k] \setminus M_1$ is preserved by the application of pp, we know that the conditions are satisfied for the sets M_2, \ldots, M_l . The argument also shows that the entries at M_1 are smaller than the entries at $[k] \setminus M_1$ and that their order is the same as in t_1 . \square

The following lemma is a simple criterion for showing that certain operations generate pp.

LEMMA 10.5.4. Let f be a binary operation preserving < such that for some $\alpha, \beta \in \text{Aut}((\mathbb{Q}; <))$ we have $f(x, y) = \alpha x$ for all $x \le -1$, 0 < y < 1, and $f(x, y) = \beta y$ for all x > 1, 0 < y < 1. Then f generates pp.

PROOF. It suffices to show that every relation preserved by f is also preserved by pp. Let R be preserved by f, and let t_1, t_2 be two tuples from R. Let $\gamma_1 \in \operatorname{Aut}((\mathbb{Q}; <))$ be such that $\gamma_1 = x + 1$ for all positive entries x of t_2 and $\gamma_1 = x + 1$ for all other entries x of t_2 . Let $\gamma_2 \in \operatorname{Aut}((\mathbb{Q}; <))$ be such that all entries of $\gamma_2 t_2$ are larger than 0 and smaller than 1. Then $f(\gamma_1 t_1, \gamma_2 t_2)$ is in the same orbit as $pp(t_1, t_2)$, which is what we wanted to show.

It is easy to verify that the relation T_3 , defined in Section 10.2.3, is shuffle-closed. Proposition 10.2.6 shows that $\mathrm{CSP}((\mathbb{Q};S))$ is NP-complete, and thus the property of shuffle-closure is not strong enough to guarantee tractability.

10.5.2. Min-union closure. This section introduces and studies a stronger property than shuffle-closure, namely preservation under the binary operation min that maps two values x and y to the smaller of the two values; see Figure 10.9 for an illustration of the operation min. We also present a sufficient condition that implies that a temporal constraint language is preserved by min.

For constraint languages over a finite domain, *min*- and *max*-closed relations were studied in [125]. An equivalent clausal description of such constraints is known; however, the equivalence only holds for *finite* domains. The tractability of the CSP where the constraint language has such a clausal description has also been shown for infinite domains [77]. But the algorithm presented in [77] cannot be applied to all

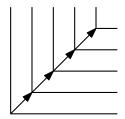


FIGURE 10.9. Illustration of the operation min.

min-closed constraint languages over an infinite domain; it is already not clear how to adapt this approach to deal with the relation $\{(x,y,z) \mid x > y \lor x > z\}$, which is preserved by min. In Section 10.5.7 we describe an algorithm that efficiently solves the CSP for temporal constraint languages that are preserved by min.

Definition 10.5.5. Let t be from \mathbb{Q}^k . The set of indices

$$\{i \in [k] \mid t[i] \le t[j] \text{ for all } j \in [k]\}$$

is called the min-set of t, and denoted by M(t).

DEFINITION 10.5.6. A relation is called min-union closed if for all tuples t_1 , t_2 in R there exists a tuple t_3 in R such that $M(t_3) = M(t_1) \cup M(t_2)$.

We now want to link min-union closure of the relations in the constraint language to the existence of certain polymorphisms.

DEFINITION 10.5.7. Let f be a binary operation preserving <. We say that f provides min-union closure if f(0,0) = f(0,x) = f(x,0) for all integers x > 0.

The operation min is an example of an operation providing min-union closure. The following lemma connects Definition 10.5.6 and Definition 10.5.7.

Lemma 10.5.8. Let R be a temporal relation preserved by an operation f providing min-union closure. Then R is min-union closed.

PROOF. Let t_1 and t_2 be tuples in R, and let a_1 and a_2 be the minimal values among the entries of t_1 and t_2 , respectively. Then there are $\alpha_1, \alpha_2 \in \text{Aut}((\mathbb{Q}; <))$ such that $\alpha_1 a_1 = \alpha_2 a_2 = 0$, and such that α_1 and α_2 map all other entries of t_1 and t_2 to integers. Observe that all entries at $M(t_1) \cup M(t_2)$ in the tuple $t_3 = f(\alpha_1 t_1, \alpha_2 t_2)$ have the same value. Because f preserves <, this value is strictly smaller than the values at all other entries in t_3 . Hence, $M(t_3) = M(t_1) \cup M(t_2)$.

The following proposition implies that $\{f, pp\}$ generates min for every operation f that provides min-union closure.

Proposition 10.5.9. A temporal relation R is preserved by pp and an operation providing min-union closure if and only if R is preserved by min.

PROOF. Clearly, min provides min-union closure. Also observe that min satisfies the conditions of Lemma 10.5.4, and hence generates pp.

For the opposite direction, suppose that R is k-ary and preserved by pp and an operation f providing min-union closure. We show that for any two tuples $t_1, t_2 \in R$ the tuple $t_3 = min(t_1, t_2)$ is in R as well. Let l be the number of distinct values in t_3 and $v_1 < v_2 < \cdots < v_l$ be these values. We define M_i , $i \in [l]$, to be the set of indices of t_3 with the i-th lowest value, i.e., $M_i = \{j \in [k] \mid t_3[j] = v_i\}$.

Now let $\alpha_1, \ldots, \alpha_l \in \operatorname{Aut}((\mathbb{Q}; <))$ be such that $\alpha_i v_i = 0$ and such that the entries of $\alpha_i t_1$ and $\alpha_i t_2$ are integers. Using these automorphisms we define the tuples

 s_1, \ldots, s_l by $s_i = f(\alpha_i t_1, \alpha_i t_2)$. Clearly, these tuples belong to R. It also holds that s_i is constant at M_i because for each $j \in M_i$ at least one of the entries $t_1[j], t_2[j]$ is equal to v_i (the other one can be only greater) which is subsequently mapped to 0 by α_i and f maps all such pairs to the same value. Furthermore, for each $j' \in M_{i'}$ for $i < i' \le l$ we have that $s_i[j']$ is greater than the value of s_i at M_i , because $\min(t_1[j'], t_2[j']) = v_{i'}$ is greater than v_i and f preserves <.

Now we can apply Lemma 10.5.3 to the obtained tuples s_1, \ldots, s_l and the corresponding sets M_1, \ldots, M_l . The lemma gives us some tuple t_3' from R which is constant at each set M_i , $i \leq [l]$, and such that for each $i < j \leq l$ the value of t_3' at M_i is lower than the value of t_3' at M_j . Thus t_3' has the same order of entries as t_3 which shows that t_3 is in R as well.

10.5.3. Min-intersection closure. In this section, we study a different restriction of shuffle-closed constraint languages.

DEFINITION 10.5.10. A relation R is called min-intersection closed if for all tuples t_1 , t_2 in R, if $M(t_1) \cap M(t_2) \neq \emptyset$, then there exists a tuple t_3 in R such that $M(t_3) = M(t_1) \cap M(t_2)$.

DEFINITION 10.5.11. Let f be a binary operation preserving <. We say that f provides min-intersection closure if f(0,0) < f(0,x) and f(0,0) < f(x,0) for all integers x > 0.

Lemma 10.5.12. Let R be a temporal relation that is preserved by an operation f that provides min-intersection closure. Then R is min-intersection closed.

PROOF. Let t_1 and t_2 be two tuples in R such that $M(t_1) \cap M(t_2)$ is non-empty, that is, it contains an index i. Choose $\alpha_1, \alpha_2 \in \operatorname{Aut}((\mathbb{Q}; <))$ such that $\alpha_1 t_1[i] = \alpha_2 t_2[i] = 0$, and such that α_1 and α_2 map all other entries of t_1 and t_2 to integers. Consider the tuple $t_3 = f(\alpha_1 t_1, \alpha_2 t_2)$. Because at the entries from $M(t_1)$ (from $M(t_2)$) the tuple $\alpha_1 t_1$ ($\alpha_2 t_2$) equals 0, and because f(0,0) < f(0,x) and f(0,0) < f(x,0) for all positive integers x, it follows that in t_3 all entries at $M(t_1) \cap M(t_2)$ have a strictly smaller value than all values at the symmetric difference $M(t_1) \triangle M(t_2)$. Because f preserves f0, it also follows that all entries at f1, it also follows that all entries at f2, we conclude that f3, it also follows that f3. We conclude that f4, it also follows that f4.

An example of an operation that provides min-intersection closure is the operation mi, defined by

$$mi(x,y) := \left\{ \begin{array}{ll} a(x) & \text{if } x < y \\ b(x) & \text{if } x = y \\ c(y) & \text{if } x > y \end{array} \right.$$

where a, b, c are unary operations that preserve < such that

$$b(x) < c(x) < a(x) < b(x + \varepsilon)$$

for all $x \in \mathbb{Q}$ and all $0 < \varepsilon \in \mathbb{Q}$ (see Figure 10.10). Operations a,b,c with these properties can be constructed as follows. Let q_1,q_2,\ldots be an enumeration of \mathbb{Q} . Inductively assume that we have already defined a,b,c on $\{q_1,\ldots,q_n\}$ such that $b(q_i) < c(q_i) < a(q_i) < b(q_j)$ whenever $q_i < q_j$, for $i,j \in [n]$. Clearly, this is possible for n=1. If $q_{n+1}>q_i$ for all $i\in [n]$, let q_j be the maximum of $\{q_1,\ldots,q_n\}$, and define $a(q_j) < b(q_{n+1}) < c(q_{n+1}) < a(q_{n+1})$. In the case that $q_{n+1} < q_i$ for all $i\in [n]$ we proceed analogously. Otherwise, let $i,j\in [n]$ such that q_i is the largest possible and q_j is smallest possible such that $q_i < q_{n+1} < q_j$. In this case, define $a(q_i) < b(q_{n+1}) < c(q_{n+1}) < a(q_{n+1}) < b(q_j)$. In this way we define unary operations a,b,c on all of $\mathbb Q$ with the desired properties.

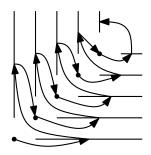


Figure 10.10. Illustration of the operation mi.

In fact, the operation mi will be of special importance, because the following proposition shows that pp together with any operation providing min-intersection closure generates the operation mi.

Proposition 10.5.13. A temporal relation R is preserved by pp and an operation providing min-intersection closure if and only if <math>R is preserved by mi.

PROOF. It is clear that mi provides min-intersection closure, and Lemma 10.5.4 shows that mi generates pp.

For the opposite direction, suppose R is k-ary and preserved by pp and an operation f providing min-intersection closure. We show that for any two tuples $t_1, t_2 \in R$ the tuple $t_3 = mi(t_1, t_2)$ is in R as well. Let a, b, c be the mappings from the definition of the operation mi. Let $v_1 < \cdots < v_l$ be the minimal-length sequence of rational numbers such that for each $i' \in [k]$ it holds that $t_3[i'] \in \bigcup_{j \in [l]} \{a(v_j), b(v_j), c(v_j)\}$. Let M_i be

$$\{i' \in [k] \mid t_3[i'] \in \{a(v_i), b(v_i), c(v_i)\} \}$$
.

Observe that for each $i' \in M_i$ at least one of $t_1[i']$ and $t_2[i']$ is equal to v_i and the other value is greater or equal to v_i . Let M_i^a be the set of those $i' \in M_i$ where $v_i = t_1[i'] < t_2[i']$, M_i^b the set of those $i' \in M_i$ where $v_i = t_1[i'] = t_2[i']$, and M_i^c the set of those $i' \in M_i$ where $v_i = t_2[i'] < t_1[i']$.

Let $\alpha_1, \ldots, \alpha_l \in \operatorname{Aut}((\mathbb{Q}; <))$ be such that α_i maps v_i to 0 and such that the entries of $\alpha_i t_1$ and $\alpha_i t_2$ are integers. Let $\beta \in \operatorname{Aut}((\mathbb{Q}; <))$ be such that $\beta f(0, 0) = 0$. For each $i \in [l]$ we define

$$s_i := pp(\beta f(\alpha_i t_1, \alpha_i t_2)), pp(\alpha_i t_2, t_1). \tag{27}$$

We verify that for all $i \in [l]$ the tuple s_i is constant on each of the sets M_i^a, M_i^b, M_i^c , the value at M_i^b is lower than the value at M_i^c which is lower than the value at M_i^a . Furthermore, for each $j \in [l], j > i$, and each $i' \in M_i, j' \in M_j$, it holds that $s_i[i'] < s_i[j']$. Having this, we can apply Lemma 10.5.3 and obtain a tuple from R with the same ordering of entries as in t_3 , which proves the lemma.

Because α_i maps v_i to 0, the properties of pp imply that the tuple $t_i' = pp(\alpha_i t_2, t_1)$ is constant at $M_i^b \cup M_i^c$ and at M_i^a , and the value at the first set is smaller than the value at the second set. Because the values of t_2 at $M_i^a \cup \bigcup_{j=i+1}^l M_j$ are greater than v_i and the values of t_1 at $\bigcup_{j=i+1}^l M_j$ are also greater than v_i (recall that for each $j \in [l]$, $j' \in M_j$ it holds that $min(t_1[j'], t_2[j']) = v_j$) we conclude that the values of t_i' at $\bigcup_{j=i+1}^l M_j$ are greater than those at M_i .

The application of f in (27) yields a tuple which is constant on M_i^b and its value there (which is consequently mapped to 0 by β) is smaller than the values at

 $M_i^a \cup M_i^c \cup \bigcup_{j=i+1}^l M_j$. Thus it is easy to verify from the properties of pp that the outer application of pp in (27) yields a tuple with the desired properties.

Example 10.5.14. An interesting example of a relation that is preserved by mi but not by min is the 4-ary relation I defined as follows.

$$I(a, b, c, d) \equiv (a = b \land b < c \land c = d)$$

$$\lor (a = b \land b > c \land c = d)$$

$$\lor (a = b \land b < c \land c < d)$$

$$\lor (a > b \land b > c \land c = d)$$

To see that I is preserved by mi, let t_1 and t_2 be two tuples from I. We have to show that $t_3 := mi(t_1, t_2) \in I$. First note that I(a, b, c, d) is equivalent to

$$(a \ge b) \land (b \ne c) \land (c \le d) \land (a = b \lor b > c) \land (b < c \lor c = d)$$

and that mi preserves \leq and \neq .

We distinguish the following cases.

- (1) $t_1[2] < t_1[3]$ and $t_2[2] < t_2[3]$. Then $t_1[1] = t_1[2]$ and $t_2[1] = t_2[2]$, and hence $t_3[1] = t_3[2]$. Since mi preserves <, we have $t_3[2] < t_3[3]$. Since mi preserves \le , we have that $t_3[3] \le t_3[4]$, and hence $t_3[1] = t_3[2] < t_3[3] < t_3[4]$ or $t_3[1] = t_3[2] < t_3[3] = t_3[4]$, which proves the claim.
- (2) $t_1[2] < t_1[3]$ and $t_2[2] > t_2[3]$. Then $t_1[1] = t_1[2]$ and $t_2[3] = t_2[4]$. We verify that t_3 satisfies the equivalent characterization of I given above; since mi preserves \leq and \neq , this amounts to proving that t_3 satisfies the two clauses $(a = b \lor b > c) \land (b < c \land c = d)$.

The first sub-case we consider is $t_3[2] < t_3[3]$. Then by the assumptions on t_1 and t_2 and by definition of mi we have that $t_1[2] < t_2[2]$. Therefore, $t_1[1] = t_1[2] < t_2[2] \le t_2[1]$ and thus $t_3[1] = t_3[2]$ again by the properties of mi; we see that both clauses are satisfied. The second sub-case is that $t_3[2] > t_3[3]$. Then by the assumptions on t_1 and t_2 and by definition of mi we have that $t_1[4] \ge t_1[3] > t_2[3] = t_2[4]$. Thus $t_3[3] = t_3[4]$ and again both clauses are satisfied.

- (3) $t_1[2] > t_1[3]$ and $t_2[2] > t_2[3]$. This is analogous to the first case.
- (4) $t_1[2] > t_1[3]$ and $t_2[2] < t_2[3]$. This is analogous to the second case.

The relation I is not preserved by min since $(0,0,1,2) \in I$ and $(2,1,0,0) \in I$ but $min((0,0,1,2),(2,1,0,0)) = (0,0,0,0) \notin I$.

Example 10.5.15. The following ternary temporal relation U is preserved by min (we omit the easy proof), but not preserved by mi.

$$U(x, y, z) \equiv (x = y \land y < z)$$

$$\lor (x = z \land z < y)$$

$$\lor (x = y \land y = z)$$

To see that U is not preserved by mi, note that the tuple mi((0,0,1),(0,1,0)) has three distinct values and hence is not in U, but $(0,0,1),(0,1,0) \in U$. An algorithm that solves temporal constraint languages preserved by mi can be found in Section 10.5.7.

10.5.4. Weak near-unanimity modulo endomorphisms. For a uniform presentation of the classification result in Section 10.6, we need the following alternative description of the clone generated by mi. When A, B are two subsets of \mathbb{Q}^3 , and $f: \mathbb{Q}^3 \to \mathbb{Q}$, we write $A <_f B$ if for all $(x, y, z) \in A$ and $(x', y', z') \in B$ we have f(x, y, z) < f(x', y', z').

PROPOSITION 10.5.16. There exists a function $f: \mathbb{Q}^3 \to \mathbb{Q}$ whose kernel has the following classes: for each $u \in \mathbb{Q}$

- (1) $x(u) := \{(a, b, c) \mid u = b = c, a > c\};$
- (2) $y(u) := \{(a, b, c) \mid u = a = c, b > a\};$
- (3) $z(u) := \{(a, b, c) \mid u = a = b, c > a\};$
- (4) $X(u) := \{(a, b, c) \mid u = a, b > a, c > a\};$
- (5) $Y(u) := \{(a, b, c) \mid u = b, a > b, c > b\};$
- (6) $Z(u) := \{(a, b, c) \mid u = c, a > c, b > c\};$
- (7) $D(u) := \{(u, u, u)\}.$

Moreover, for u < v, we have

$$D(u) <_f x(u) <_f y(u) <_f z(u) <_f Z(u) <_f Y(u) <_f X(u) <_f D(v)$$
.

PROOF. The specified countable family of subsets of \mathbb{Q}^3 indeed forms a partition of \mathbb{Q}^3 . To see this, note that we distinguish which entries of the tuple are equal to the minimum u of the entries of the tuple. This splits \mathbb{Q}^3 into seven different classes for a given u, all of them pairwise disjoint. See Figure 10.11. Note that $<_f$ defines a linear order on this countable family, and since $(\mathbb{Q};<)$ embeds all countable linear orders, the existence of such a function f follows.

PROPOSITION 10.5.17. Let $f: \mathbb{Q}^3 \to \mathbb{Q}$ be any function with the properties in Proposition 10.5.16. Then there are $a, b, c \in \operatorname{End}(\mathbb{Q}; <)$ such that for all $x, y \in \mathbb{Q}$

$$a(f(y, x, x)) = b(f(x, y, x)) = c(f(x, x, y)).$$

That is, f is a weak near unanimity modulo endomorphisms of $(\mathbb{Q}; <)$.

PROOF. By Lemma 5.6.7, it suffices to show that for all finite $S \subset \mathbb{Q}$ there are $\alpha, \beta \in \operatorname{Aut}((\mathbb{Q}; <))$ such that for all $x, y \in S$

$$f(y, x, x) = \alpha(mi(y, x)) \tag{28}$$

$$f(x, y, x) = \beta(mi(y, x)) \tag{29}$$

$$f(x, x, y) = \gamma(mi(y, x)) \tag{30}$$

Observe that for all $u, v, u', v' \in \mathbb{Q}$ we have $f(v, u, u) \leq f(v', u', u')$ iff one of the following cases applies:

- $(v, u, u) \in D(u), (v', u', u') \in D(u') \cup x(u') \cup X(u'), \text{ and } u \leq u';$
- $(v, u, u) \in x(u), (v', u', u') \in x(u') \cup X(u'), \text{ and } u \leq u';$
- $\bullet \ (v,u,u) \in x(u), \, (v',u',u') \in D(u'), \, \text{and} \, \, u < u';$
- $(v, u, u) \in X(u), (v', u', u') \in X(u'), \text{ and } u \le u';$
- $(v, u, u) \in X(u), (v', u', u') \in D(u') \cup x(u'), \text{ and } u < u'.$

Note that this is the case if and only if

- $u = v \le u' = v'$;
- $u < v, u' \neq v', u \leq u',$
- u < v, u' = v', u < u',
- $v < u, v' < u', u \le u',$
- v < u, u' = v', u < u'.

This in turn is the case if and only if $mi(v,u) \leq mi(v',u')$. Then the statement for (28) follows from homogeneity of $(\mathbb{Q};<)$. The proof for (29) and for (30) is analogous.

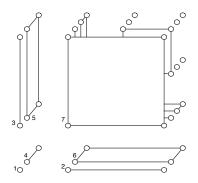


FIGURE 10.11. Illustration of the function f from Proposition 10.5.17.

The addition of the third item in the following proposition is a result of Michał Wrona [198].

THEOREM 10.5.18. Let $R \subseteq \mathbb{Q}^n$ be first-order definable over $(\mathbb{Q}; <)$. Then the following are equivalent.

- (1) R is preserved by the operation f as defined in Proposition 10.5.16 (a weak near unanimity modulo endomorphisms of $(\mathbb{Q}; <)$ by Proposition 10.5.17).
- (2) R is preserved by mi.
- (3) R can be defined by a conjunction of formulas of the form

$$(x_1 \neq z) \vee \cdots \vee (x_k \neq z)$$

$$\vee (y_1 < z) \vee \cdots \vee (y_l < z)$$

$$\vee (y_0 \leq z).$$
(31)

PROOF. The implication from (1) to (2) follows from the observation that $(x, y) \mapsto f(x, x, y)$ induces the same weak linear order on \mathbb{Q}^2 as mi and hence generates mi. The implication from (2) to (3) is an unpublished result from [198]. For the implication from (3) to (1) we verify that f preserves formulas ϕ of the form as in (31). Let t_1, t_2, t_3 be assignments that satisfy ϕ . Suppose for contradiction that $t_0 := f(t_1, t_2, t_3)$ does not satisfy ϕ . In particular, $t_0(z) = t_0(x_1) = \cdots = t_0(x_k)$. By the definition of f, there exists a $u \in \mathbb{Q}$ such that

$$D := \{(t_1(z), t_2(z), t_3(z)), (t_1(x_1), t_2(x_2), t_3(x_3)), \dots, (t_1(x_k), t_2(x_k), t_3(x_k))\}$$

is contained in one of x(u), y(u), z(u), X(u), Y(u), Z(u), or D(u). It follows that there exists an $i \in \{1, 2, 3\}$ such that $t_i(z) = t_i(x_1) = \cdots = t_i(x_k) = u$. Suppose without loss of generality that i = 1. Since t_1 satisfies ϕ , there must be a $j \in \{0, 1, \ldots, l\}$ such that $t_1(y_j) < t_1(z) = u$ or $t_1(y_0) = t_1(z) = u$.

- If $t_1(y_0) = u$ and $t_2(y_0) < t_1(y_0) = u$ or $t_3(y_0) < t_1(y_0) = u$ then $t_0(y_0) < t_0(z)$.
- If $t_2(y_0) > t_1(y_0) = u$ and $t_3(y_0) > t_1(y_0) = u$ then $t_0(y_0) = t_0(z)$.
- If $t_1(y_j) < t_1(z) = u$, then $t_0(y_j) < t_0(z)$.

In each of the three cases, we have reached a contradiction to the assumption that t_0 does not satisfy ϕ .

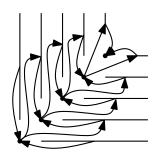


Figure 10.12. Illustration of the operation mx.

10.5.5. Min-xor closure. We now introduce the last of the three mentioned closure conditions.

DEFINITION 10.5.19. A relation is called min-xor closed if for all tuples t_1 , t_2 in R where the symmetric difference $M(t_1) \triangle M(t_2)$ is nonempty there exists a tuple t_3 in R such that $M(t_3) = M(t_1) \triangle M(t_2)$.

DEFINITION 10.5.20. Let f be a binary operation preserving <. We say that f provides min-xor closure if f(0,0) > f(0,x) = f(y,0) for all integers x,y > 0.

For an example of a binary operation that provides min-xor closure, consider the following binary operation, which we denote by mx.

$$mx(x,y) := \left\{ \begin{array}{ll} a(min(x,y)) & \text{if } x \neq y \\ b(x) & \text{if } x = y \end{array} \right.$$

where a and b are unary operations that preserve < such that $a(x) < b(x) < a(x+\varepsilon)$ for all $x \in \mathbb{Q}$ and all $0 < \varepsilon \in \mathbb{Q}$ (see Figure 10.12). Similarly as in the definition of mi, such operations a, b can be easily constructed. It is easy to see that the operation mx neither preserves the relation I nor the relation U introduced in Section 10.5.3.

Lemma 10.5.21. Let R be a temporal relation that is preserved by an operation f providing min-xor closure. Then R is min-xor closed.

PROOF. Let t_1 and t_2 be tuples in R, and suppose that the symmetric difference $M(t_1) \triangle M(t_2)$ of $M(t_1)$ and $M(t_2)$ is non-empty. Let v_1 and v_2 be the minimal values of the entries of t_1 and of t_2 , respectively. Then there are $\alpha_1, \alpha_2 \in \operatorname{Aut}((\mathbb{Q}; <))$ such that $\alpha_1 v_1 = 0$ and $\alpha_2 v_2 = 0$ and such that α_1 and α_2 map all other entries of t_1 and t_2 to integers. Consider the tuple $t_3 = f(\alpha_1 t_1, \alpha_2 t_2)$. Because $\alpha_1 t_1$ is 0 for all entries at $M(t_1), \alpha_2 t_2$ is 0 for all entries at $M(t_2)$, and f(0,0) > f(0,x) = f(y,0) for all x,y>0, it follows that in t_3 all entries at $M(t_1) \cap M(t_2)$ have a strictly larger value than all entries at $M(t_1) \triangle M(t_2)$, which all have the same value. Because f preserves <, all entries of t_3 at $M(t_1) \cap M(t_2)$ have a smaller value than all entries not at $M(t_1) \cup M(t_2)$. We conclude that the tuple $t_3 \in R$ satisfies $M(t_3) = M(t_1) \triangle M(t_2)$.

The following lemma implies that $\{f, pp\}$ generates mx for any operation f that provides min-xor closure.

Proposition 10.5.22. A temporal relation R is preserved by pp and an operation f providing min-xor closure if and only if R is preserved by mx.

PROOF. Clearly, mx provides min-xor closure. Lemma 10.5.4 shows that mx generates pp.

For the opposite direction, suppose that R is k-ary and preserved by pp and an operation f providing min-xor closure. We show that for any two tuples $t_1, t_2 \in R$ the tuple $t_3 = mx(t_1, t_2)$ is in R as well. Let a, b be the mappings as in the definition of the operation mx. Let $v_1 < \cdots < v_l$ be minimal set of rational numbers such that $t_3[i] \in \bigcup_{j \in [l]} \{a(v_j), b(v_j)\}$ for all $i \in [k]$, and let M_i be the set of indices $\{i' \in [k]|t_3[i'] \in \{a(v_i), b(v_i)\}\}$. Observe that for each $i' \in M_i$ at least one of $t_1[i']$ and $t_2[i']$ is equal to v_i and the other value is greater or equal to v_i . Let M_i^a be the set of those $i' \in M_i$ where $t_1[i'] \neq t_2[i']$ and M_i^b the set of those $i' \in M_i$ where $v_i = t_1[i'] = t_2[i']$.

Let $\alpha_1,\ldots,\alpha_l\in \operatorname{Aut}((\mathbb{Q};<))$ be such that α_i maps v_i to 0 and such that the entries of α_it_1 and α_it_2 are integers. For each $i\in [l]$ we define $s_i:=f(\alpha_it_1,\alpha_it_2)$. It is easy to see from the choice of α_i and properties of f that for each $i\in [l]$ the tuple s_i is constant at M_i^a, M_i^b , and that the value at M_i^a is lower than the value at M_i^b . Furthermore, because f preserves <, because the values of t_1 at $\bigcup_{j=i+1}^l M_j$ are greater than v_i , and because the values of t_2 at $\bigcup_{j=i+1}^l M_j$ are greater than v_i , we see that for each $j\in [l], j>i$ and each $i'\in M_i, j'\in M_j$, it holds that $s_i[i']< s_i[j']$. Having this, we can apply Lemma 10.5.3 and obtain a tuple from R with the same ordering of entries as in t_3 , which proves the lemma.

EXAMPLE 10.5.23. An interesting example of a temporal relation that is preserved by mx is the ternary relation X defined as follows.

$$X(x, y, z) \equiv (x = y \land y < z)$$

$$\lor (x = z \land z < y)$$

$$\lor (y = z \land y < x)$$

The relation is not preserved by min and by mi: the tuples $t_1 = (0,0,1)$, $t_2 = (0,1,0)$ are in X, but $min(t_1,t_2) = (0,0,0) \notin R$, and $mi(t_1,t_2)$ has three distinct entries and hence is not in X as well.

An algorithm that solves constraint languages preserved by mx can be found in Section 10.5.7.

10.5.6. Operations generating min, mi, mx. As we have seen in Proposition 10.2.6, if the relation T_3 has a primitive positive definition in \mathfrak{B} , then $CSP(\mathfrak{B})$ is NP-hard. We show that if a temporal constraint language is shuffle-closed and does not admit a primitive positive definition of T_3 , then it is preserved by min, mi, or mx.

If the relation T_3 does not have a primitive positive definition in \mathfrak{B} , then Theorem 5.2.3 implies that \mathfrak{B} has a polymorphism that does not preserve T_3 . By Theorem 10.3.2, it suffices to consider operations that preserve <. We start with a sequence of auxiliary lemmas.

LEMMA 10.5.24. Let f be a binary operation preserving <, and suppose that there is an infinite sequence $x_1 < x_2 < \ldots$ of elements of \mathbb{Q} and $y_1 \in \mathbb{Q}$ such that $f(x_1, y_1) \geq f(x_2, y_1) < f(x_i, y_1)$ for all i > 2. Then f generates an operation providing min-intersection closure.

PROOF. Because f preserves <, we have that for any infinite sequence $y_1 < y_2 < \ldots$ it holds that $f(x_2, y_i) > f(x_1, y_1)$. Hence, the binary operation defined by $f(\alpha(x), \beta(y))$ provides min-intersection closure, where $\alpha \in \operatorname{Aut}((\mathbb{Q}; <))$ maps $0, 1, \ldots$ to x_2, x_3, \ldots and $\beta \in \operatorname{Aut}((\mathbb{Q}; <))$ maps $0, 1, 2, \ldots$ to y, y_1, y_2, \ldots

LEMMA 10.5.25. Suppose f preserves < and generates a sequence of operations f_1, f_2, \ldots such that for each f_k it holds that $f_k(0,0) < f_k(x,0)$ and $f_k(0,0) < f_k(0,x)$

for all integers $x \in [k]$. Then f generates an operation g providing min-intersection closure.

Proof. A direct consequence of Lemma 8.3.13.

LEMMA 10.5.26. Let f be a binary operation preserving < such that there is an infinite sequence $x_1 < x_2 < \ldots$ and $y_1 \in \mathbb{Q}$ satisfying $f(x_i, y_1) > f(x_j, y_1)$ for all $1 \le i < j$. Then $\{f, pp\}$ generates an operation providing min-intersection closure.

PROOF. By Lemma 10.5.25, it suffices to show that there is a sequence of operations f_1, f_2, \ldots , generated by $\{f, pp\}$ such that $f_k(0,0) < f_k(x,0)$ and $f_k(0,0) < f_k(0,x)$ for all $k \ge 1$ and all $x \in [k]$.

So let $k \geq 0$ be a fixed integer, and $y_1 < y_2 < \dots$ be an arbitrary infinite sequence. Let α_k be from $\operatorname{Aut}((\mathbb{Q};<))$ such that

$$\alpha \{ f(x_1, y_i) \mid 1 \le i \le k \} \cup \{ f(x_i, y_1) \mid 1 \le i \le k \} \subseteq \{x_2, \dots, x_{2k}\}$$

and $\beta_1, \beta_2 \in \text{Aut}((\mathbb{Q}; <))$ such that β_1 maps $0, 1, 2, \ldots$ to x_1, x_2, x_3, \ldots and β_2 maps $0, 1, 2, \ldots$ to y_1, y_2, y_3, \ldots We define

$$f_k(x,y) := f(\alpha_k f(\beta_1 x, \beta_2 y), \beta_2 y)$$

and show that f_k has the required properties. It follows from the assumptions on f that for all positive integers x we have $f(\beta_1 0, \beta_2 0) = f(x_1, y) > f(\beta_1 x, y_1) = f(\beta_1 x, \beta_2 0)$, and due to the properties of α_k it holds that $f_k(0,0) < f_k(x,0)$ for all integers $x \in [k]$.

We also have for every $x \in [k]$ that $\beta_2 x > y_1$ and $\alpha_k f(\beta_1 0, \beta_2 x) > x_1$. Because f preserves <, this shows that $f_k(0,x) = f(\alpha_k f(\beta_1 0, \beta_2 x)), \beta_2 x) > f(x_1, y_1)$. Moreover, $f_k(0,0) = f(\alpha_k f(x_1,y_1), y_1) < f(x_1,y_1)$ by the assumptions on f. Hence, $f_k(0,x) > f(x_1,y_1) > f_k(0,0)$ for all $x \in [k]$.

The following lemma applies (a special case of) Ramsey's theorem; more substantial applications of Ramsey theory can be found in Section 10.6.

LEMMA 10.5.27. Let f be a binary operation preserving < such that there is an infinite sequence $x_1 < x_2 < \ldots$ and $y_1 \in \mathbb{Q}$ satisfying $f(x_1, y_1) > f(x_i, y_1) = f(x_j, y_1)$ for all 1 < i < j. Then $\{f, pp\}$ generates an operation providing min-intersection or min-xor closure.

PROOF. By the infinite pigeon-hole principle there must be an infinite sequence $y_2 < y_3 < \dots$ of elements of \mathbb{Q} larger than y_1 such that

- (1) $f(x_2, y_1) = f(x_1, y_i)$ for all $i \ge 2$, or
- (2) $f(x_2, y_1) > f(x_1, y_i)$ for all $i \ge 2$, or
- (3) $f(x_2, y_1) < f(x_1, y_i)$ for all $i \ge 2$.

In case 1, f generates an operation providing min-xor closure and we are done. In case 2, we apply Ramseys theorem (Theorem 8.1.1) in the special case of m=2, r=3 as follows. Let D be $\{y_1,y_2,\dots\}$. For i< j, define $\chi(\{y_i,y_j\})=1$ if $f(x_1,y_i)=f(x_1,y_j),\ \chi(\{y_i,y_j\})=2$ if $f(x_1,y_i)>f(x_1,y_j),\$ and $\chi(\{y_i,y_j\})=3$ if $f(x_1,y_i)< f(x_1,y_j).$ Then Theorem 8.1.1 applied to χ shows that there exists an infinite subsequence $z_1< z_2<\dots$ of $y_1< y_2<\dots$ such that

- 2a. $f(x_1, z_i) = f(x_1, z_j)$ for all $1 \le i < j$, or
- 2b. $f(x_1, z_i) > f(x_1, z_j)$ for all $1 \le i < j$, or
- 2c. $f(x_1, z_i) < f(x_1, z_j)$ for all $1 \le i < j$.

In case 2a, we swap arguments of f and proceed as in case 3. In case 2b, we swap arguments of f, apply Lemma 10.5.26, and conclude that f generates an operation providing min-intersection closure. In case 2c, note that $f(x_1, y_1) > f(x_2, y_1) >$

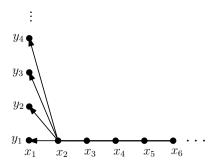


FIGURE 10.13. Illustration for Case 3 of Lemma 10.5.27.

 $f(x_1, y_i)$ for all $i \geq 2$, and thus we can apply Lemma 10.5.24 to conclude that f generates an operation providing min-intersection closure.

In case 3, we show that similarly as in Lemma 10.5.26 there is a sequence of operations f_1, f_2, \ldots generated by $\{f, pp\}$ such that for each f_k it holds that $f_k(0,0) < f_k(x,0)$ and $f_k(0,0) < f_k(0,x)$ for all integers $x \in [k]$, and conclude by application of Lemma 10.5.25. See Figure 10.13 for an illustration.

Let α_k be from $\operatorname{Aut}((\mathbb{Q};<))$ such that $\alpha_k f(x_2,y_1)=x_1$ and $\alpha\{f(x_1,y_i) | 1 \leq i \leq k\} \subseteq \{x_2,\ldots,x_{k+1}\}$. Furthermore let $\beta_1,\beta_2 \in \operatorname{Aut}((\mathbb{Q};<))$ be such that β_1 maps $0,1,2,\ldots$ to x_1,x_2,x_3,\ldots and β_2 maps $0,1,2,\ldots$ to y_1,y_2,y_3,\ldots We define

$$f_k(x,y) := f(\alpha_k f(\beta_1 x, \beta_2 y), \beta_2 y)$$
.

Then for all integers x > 0

$$f_k(0,0) = f(\alpha_k f(x_1, y_1), y_1) = f(x_2, y_1)$$
, and $f_k(x,0) = f(\alpha_k f(\beta_1 x, y_1), y_1) = f(x_1, y_1)$.

Hence $f_k(0,0) < f_k(x,0)$ for all integers x > 0. Finally, since $\beta_2 x > y_1$ and $\alpha_k f(x_1, \beta_2 x) > x_1$ for all integers x > 0, we have that $f_k(0,x) > f(x_1,y_1) > f(x_2,y_1) = f_k(0,0)$.

The previous two lemmas are combined in the following result.

Lemma 10.5.28. Let f be a binary operation that preserves < and violates the relation \leq . Then $\{f, pp\}$ generates an operation providing min-intersection or min-xor closure.

PROOF. As f violates \leq , we can without loss of generality assume that there is $y \in \mathbb{Q}$ and $x_1, x_2 \in \mathbb{Q}$, $x_1 < x_2$, such that $f(x_1, y) > f(x_2, y)$.

We claim that there are only three possibilities:

- a) There is an infinite sequence $x_3 < x_4 < \dots$ such that $x_2 < x_3$ and $f(x_i, y) > f(x_2, y)$ for all i > 2.
- b) There is an infinite sequence $x_3 < x_4 < \dots$ such that $x_2 < x_3$ and $f(x_i, y) > f(x_i, y)$ for all $2 \le i < j$.
- c) There is an infinite sequence $x_3 < x_4 < \dots$ such that $x_2 < x_3$ and $f(x_i, y) = f(x_2, y)$ for all i > 2.

To show this claim, observe that by the infinite pigeon-hole principle there is an infinite sequence $x_3 < x_4 < \ldots$ with $x_2 < x_3$ such that $f(x_i, y) > f(x_2, y)$ for all i > 2, $f(x_i', y) = f(x_2, y)$ for all i > 2, or $f(x_i, y) < f(x_2, y)$ for all i > 2. In the first and the second case the claim holds. In the third case, we repeat the argument with $x_2 < x_3$ instead of $x_1 < x_2$. Again, we distinguish three cases, and as before in two of them we are immediately done. In the third case, we repeat again. If we repeat this

for infinitely many times we obtain a sequence $x_3 = x_3' < x_4' < \dots$ such that $x_2 < x_3$ and $f(x_i', y) > f(x_i', y)$ for all $2 \le i < j$.

In a) the conditions of Lemma 10.5.24 are satisfied and we conclude that $\{f, pp\}$ generates an operation providing min-intersection closure. In b) Lemma 10.5.26 shows that $\{f, pp\}$ generates an operation providing min-intersection closure. In c) we apply Lemma 10.5.27 and conclude that $\{f, pp\}$ generates an operation providing min-intersection or min-xor closure.

The following is the main result of this subsection. Recall that the relation T_3 was defined in Definition 10.2.5 to be

$$\{(x, y, z) \in \mathbb{Q}^3 \mid (x = y < z) \lor (x = z < y) \}$$
.

LEMMA 10.5.29. Let f be a binary operation that preserves < and violates the relation T_3 . Then $\{f, pp\}$ generates min, mi, or mx.

PROOF. By Proposition 10.5.9, 10.5.13, and 10.5.22, it suffices to show that $\{f, pp\}$ generates an operation providing min-intersection, min-union, or min-xor closure. If f violates \leq , then we are immediately done by Lemma 10.5.28. So we further assume that f preserves \leq .

Because f preserves < and violates T_3 , we can assume without loss of generality (possibly after swapping arguments) that there are $x_1, x_2, y_1, y_2 \in \mathbb{Q}$ such that $x_1 < x_2, y_1 < y_2$ and $t := (f(x_1, y_1), f(x_2, y_1), f(x_1, y_2)) \notin T_3$. Because f preserves \leq we have that $f(x_1, y_1) \leq f(x_2, y_1)$ and $f(x_1, y_1) \leq f(x_1, y_2)$. Since $t \notin T_3$, there are only two possibilities:

- (1) t[1] < t[2] and t[1] < t[3]. In this case, choose infinite sequences $x_3 < x_4 < \ldots$ and $y_3 < y_4 < \ldots$ such that $x_2 < x_3$, $y_2 < y_3$. Because f preserves \leq , we have for all i > 1 that $f(x_2, y_1) \leq f(x_i, y_1)$ and $f(x_1, y_2) \leq f(x_1, y_i)$. Since $t[1] = f(x_1, y_1) < t[2] = f(x_2, y_1)$ we have that $f(x_1, y_1) < f(x_i, y_1)$ for all i > 1, and since $t[1] = f(x_1, y_1) < t[3] = f(x_1, y_2)$ we have that $f(x_1, y_1) < f(x_1, y_i)$ for all i > 1. Hence, f provides min-intersection closure.
- (2) t[1] = t[2] = t[3]. In this case we can choose infinite sequences $x'_2 < x'_3 < \dots$ and $y'_2 < y'_3 < \dots$ such that $x_1 < x'_2$, $y_1 < y'_2$, and for all i > 1, $x'_i < x_2$ and $y'_i < y_2$. As f preserves \leq , we see that $f(x'_i, y_1) = f(x_1, y_1) = f(x_1, y'_i)$ for all i > 1 and thus f provides min-union closure.

10.5.7. Algorithms for shuffle-closed languages. In this section we present three algorithms, for the languages preserved by mi, by min, and by mx, respectively. All three algorithms follow a common strategy. They are searching for a subset of the variables that can have the minimal value in a solution. If they have found such a subset, S, the algorithms add equalities and inequalities that are implied by all constraints under the assumption that the variables in S denote the minimal value in all solutions. Next, the algorithms recursively solve the instance consisting of the projections of all constraints to the variables that do not denote the minimal value in all solutions. We later show that for languages preserved by pp it is true that if the instance has a solution, it also has a solution that satisfies all the additional constraints.

Throughout this section we assume that \mathfrak{B} is a structure with a first-order definition in $(\mathbb{Q};<)$ and a finite relational signature. For the formulation of the algorithms and their correctness proofs it will be convenient to work with an expanded constraint language, that contains the binary relation = for the equality relation. We also add to the temporal constraint language \mathfrak{B} several other temporal relations that are primitive positive definable in \mathfrak{B} .

DEFINITION 10.5.30. Let R be an n-ary temporal relation and $L = \{p_1, \ldots, p_k\} \subseteq [n]$ where $p_1 < \cdots < p_k$. Let $\{q_1, \ldots, q_l\}$ be $[n] \setminus L$. Then the ordered projection of R to L is the k-ary relation R' with the primitive positive definition

$$R'(x_{p_1},\ldots,x_{p_k}) \equiv \exists x_{q_1},\ldots,x_{q_l}.R(x_1,\ldots,x_n) \wedge \bigwedge_{i \in [n] \setminus L, \ j \in L} x_i < x_j.$$

Note that if \mathfrak{B} is a finite temporal constraint language, then there are only finitely many projections and ordered projections of relations in \mathfrak{B} . In case that there is a primitive positive definition of < in \mathfrak{B} , ordered projections are primitive positive definable. By Lemma 1.2.6, we can assume in this case that \mathfrak{B} contains all relations that can be defined by ordered projections from relations in \mathfrak{B} .

To formally introduce our algorithms, we also need the concept of an ordered projection of *instances* of the CSP.

DEFINITION 10.5.31. Let \mathfrak{B} be a temporal constraint language that contains all ordered projections of relations from \mathfrak{B} . Let Φ be an instance of $CSP(\mathfrak{B})$ and $X \subseteq V(\Phi)$. Then the ordered projection of Φ to X is the instance of $CSP(\mathfrak{B})$ that contains for each constraint $R(x_1,\ldots,x_n)$ in Φ , with not necessarily distinct variables x_1,\ldots,x_n , the constraint $R'(x_{k_1},\ldots,x_{k_l})$ where $k_1 < \cdots < k_l$ are such that $\{k_1,\ldots,k_l\} = \{k \in [n] \mid x_k \in X\}$, and R' is the ordered projection of R to $\{k_1,\ldots,k_l\}$.

Let Φ be an instance of a temporal CSP.

DEFINITION 10.5.32. If $\psi = R(x_1, \dots, x_k)$ is a constraint from Φ , then a subset X of the variables of ψ is called a min-set (of ψ) if there exists a k-tuple t satisfying ψ such that $x \in X$ iff the value for x in t is the minimum of all entries of t. A set of variables $S \subset V(\Phi)$ is called free iff it is non-empty and for all constraints $R(x_1, \dots, x_k)$ in Φ the set $S \cap \{x_1, \dots, x_k\}$ is either empty or a min-set of R.

We will show how to use the concept of freeness to solve instances of $CSP(\mathfrak{B})$ for shuffle closed temporal constraint languages.

LEMMA 10.5.33. Let Φ be an instance of $CSP(\mathfrak{B})$ for some shuffle closed \mathfrak{B} , and let S be a free set of variables of Φ . Then Φ has a solution if and only if the ordered projection Φ' of Φ to $V(\Phi) \setminus S$ has a solution.

PROOF. First suppose Φ' has a solution s'. Let $\psi = R(x_1, \ldots, x_m)$ be a constraint of Φ such that $V(\psi) \cap S = \{x_{p_1}, \ldots, x_{p_k}\} \neq \emptyset$. Let $\{x_{q_1}, \ldots, x_{q_l}\} = V(\psi) \setminus S$ for $q_1 < \cdots < q_l$. By the definition of an ordered projection, there is a tuple $t_1 \in R$ such that $s'(x_i) = t_1[i]$ for all $i \in \{q_1, \ldots, q_l\}$. Since $V(\psi) \cap S$ is a min-set of R, there is a tuple $t_2 \in R$ such that $M(t_2) = \{p_1, \ldots, p_k\}$. Let $\alpha \in \operatorname{Aut}((\mathbb{Q}; <))$ be such that α maps the minimal value of t_2 to 0. Because R is preserved by p_l , the tuple $t_3 := p_l(\alpha(t_2), t_1)$ is in R. It is easy to verify that $M(t_3) = \{p_1, \ldots, p_k\}$ and that there is $\beta \in \operatorname{Aut}((\mathbb{Q}; <))$ such that $\beta t_3[i] = s'(x_i)$ for $i \in \{q_1, \ldots, q_l\}$. Because we can find such a tuple for all the constraints ψ in Φ where $V(\psi) \cap S \neq \emptyset$, we conclude that a solution s' of Φ' can be extended to a solution s of Φ by setting all the variables in S to some value that is smaller than the smallest value in $\{s'(x) \mid x \in V(\Phi')\}$. Clearly, all the constraints ψ in Φ with $V(\psi) \cap S = \emptyset$ or $V(\psi) \subset S$ are satisfied by s as well.

Now suppose that Φ has a solution s. Let x_1,\ldots,x_n be the variables of Φ , and let $\{x_{r_1},\ldots,x_{r_{|S|}}\}$ be S. Let s' be a mapping from $V(\Phi)$ to $\mathbb Q$ such that $M((s'(x_1),\ldots,s'(x_n)))=\{r_1,\ldots,r_{|S|}\}$, and s'(x)=s(x) for $x\in V(\Phi)\setminus S$. We claim that s' is a solution for Φ' . Let $\psi=R(y_1,\ldots,y_m)$ be a constraint of Φ such that $V(\psi)\cap S\neq\emptyset$. Clearly, $t_1:=(s(y_1),\ldots,s(y_m))$ is in R since s is a solution of Φ . Let $\{y_{p_1},\ldots,y_{p_l}\}$ be $S\cap\{y_1,\ldots,y_m\}$. Since $\{y_{p_1},\ldots,y_{p_l}\}$ is a min-set of R,

```
Solve(\Phi)

// Input: An instance \Phi of CSP(\mathfrak B)

// for a shuffle closed temporal language \mathfrak B

// Output: A solution s to \Phi, or reject if there is no solution.

i:=0

while V(\Phi) \neq \emptyset do begin

S:= \operatorname{FindFreeSet}(\Phi)

if S=false then reject

for each x \in S do s(x):=i

i:=i+1

\Phi:= \operatorname{ordered} projection of \Phi to V(\Phi) \setminus S

end

return s
```

FIGURE 10.14. An algorithm that efficiently solves instances of a shuffle closed constraint language if free sets can be computed efficiently.

there is a tuple $t_2 \in R$ such that $M(t_2) = \{p_1, \ldots, p_l\}$. Let $\alpha \in \operatorname{Aut}((\mathbb{Q}; <))$ be such that α maps the minimal value of t_2 to 0. Because R is preserved by pp, the tuple $t_3 := pp(\alpha t_2, t_1)$ is in R. It is easy to verify that $M(t_3) = \{p_1, \ldots, p_l\}$, and that there is an automorphism β such that $\beta t_3[i] = s(y_i)$ for $i \in [m] \setminus \{p_1, \ldots, p_l\}$. Clearly, the restriction of s' to $V(\Phi) \setminus S$ is a solution to the ordered projection Φ' of Φ to $V(\Phi) \setminus S$ since s' also satisfies all the inequalities imposed by the ordered projection. Therefore Φ' is satisfied by s'.

The above lemma asserts that if we are able to identify a free set for instances of $CSP(\mathfrak{B})$ for a shuffle-closed temporal language \mathfrak{B} in polynomial time, then we also have a polynomial time algorithm that solves $CSP(\mathfrak{B})$. The running time of the algorithm is $O(n \cdot (m + t(n, m)))$, where n = |V|, m is the number of constraints in Φ , and t(n, m) is the running time of the procedure that computes the free set of an instance with n variables and m constraints.

An algorithm for languages preserved by min. Now, we concentrate on the problem to find a free set of Φ if \mathfrak{B} is preserved by the operation min.

Let $\psi = R(x_1, \ldots, x_k)$ be a constraint where R is from \mathfrak{B} and let L be a subset of $\{x_1, \ldots, x_k\}$. Let A_1, \ldots, A_l be all min-sets of ψ that are contained in L. When $l \geq 1$, i.e., when such min-sets exist, there is a unique set A_j , $j \in [l]$, with the property that $A_i \subseteq A_j$ for all $i \in [l]$, because R is preserved by min, and thus min-union closed by Lemma 10.5.8. We call this min-set the maximal min-set of ψ contained in L. Note that for some L it could be that l = 0, i.e., L does not contain min-sets of R.

Figure 10.15 shows our procedure for finding a free set for a min-union closed constraint language. It is straightforward to check that the procedure FindFreeSetUC has a running time O(nm), where n is the number of variables and m is the number of constraints of Φ .

LEMMA 10.5.34. The procedure FindFreeSetUC in Figure 10.15 returns a free set of Φ , or rejects. If it rejects, Φ is unsatisfiable.

PROOF. Suppose that the algorithm returns a (non-empty) set S. Then recheck must be set to false. Therefore, for all constraints $R(x_1, ..., x_k)$ of Φ such that $S \cap \{x_1, ..., x_k\} \neq \emptyset$ the maximal min-set of ψ contained in S equals $S \cap \{x_1, ..., x_k\}$. We conclude that S is a free set of Φ .

```
FindFreeSetUC(\Phi)
// Input: An instance \Phi of CSP(\mathfrak{B}) with variables V
// for a temporal constraint language \mathfrak{B} preserved by min.
// Output: A free set S \subseteq V of \Phi, or reject.
// If the algorithm rejects, \Phi is unsatisfiable
S := V
recheck := \mathit{true}
while recheck do begin
     recheck := false
     for all \psi \in \Phi do begin
         if S \cap V(\psi) \neq \emptyset then begin
             S := (S \setminus V(\psi)) \cup the maximal min-set of \psi contained in S \cap V(\psi)
             if S changed then recheck := true
         end
     end
end
if S \neq \emptyset then return S
else reject
end
```

FIGURE 10.15. A polynomial time algorithm that computes free sets for constraint languages preserved by min.

We now have to argue that in case that Φ is satisfiable, the algorithm does not reject (i.e., it finds a free set). If Φ has a solution, there is some set S' of variables that have the minimal value in this solution. At the beginning of the procedure, S is set to V and therefore $S' \subseteq S$. We show that $S' \subseteq S$ during the entire execution of the procedure. Let $\psi = R(x_1, \ldots, x_k)$ be a constraint from Φ . Because $S' \cap \{x_1, \ldots, x_k\}$ is a min-set of ψ that is contained in S, the maximal min-set of ψ added to $S \setminus \{x_1, \ldots, x_k\}$ certainly contains $S' \cap \{x_1, \ldots, x_k\}$. Therefore, after the modification to S it still holds that $S \supseteq S'$. When the procedure terminates, it returns the set S, because $\emptyset \neq S' \subseteq S$.

Theorem 10.5.35. If $\mathfrak B$ is preserved by min there is an algorithm solving $\mathrm{CSP}(\mathfrak B)$ in time $O(n^2m)$.

PROOF. We use the procedure FindFreeSetUC in Figure 10.15 for the subroutine FindFreeSet in Figure 10.14. Then Lemma 10.5.33 and Lemma 10.5.34 imply the correctness of the resulting algorithm.

An algorithm for languages preserved by mi. In this section we describe how to find free sets in instances of $CSP(\mathfrak{B})$ for languages \mathfrak{B} that are preserved by mi. We define the notion of a minimal min-set: Let $\psi = R(x_1, \ldots, x_k)$ be a constraint from an instance Φ of $CSP(\mathfrak{B})$, and let $L \subseteq \{x_1, \ldots, x_k\}$. Let A_1, \ldots, A_l be all minsets of ψ that contain L. Because R is preserved by mi, and thus is min-intersection closed by Lemma 10.5.12, there is a min-set A_j of ψ that is a subset of every min-set containing L. We call A_j the minimal min-set of R containing L.

The procedure for finding a free set for min-intersection closed constraint languages is given in Figure 10.16. It is straightforward to verify that the above algorithm runs in time $O(n^2m)$ where n is the number of variables and m is the number of constraints in Φ .

LEMMA 10.5.36. The procedure FindFreeSetIC in Figure 10.16 returns a free set S of Φ , or rejects. If it rejects, Φ is unsatisfiable.

```
FindFreeSetIC(\Phi)
// Input: An instance \Phi of CSP(\mathfrak{B}) where \mathfrak{B} is preserved by mi
// Output: A free set S \subseteq V(\Phi) of \Phi, or reject
// If the algorithm rejects, \Phi is unsatisfiable
for all x \in V(\Phi) do begin
     S := \{x\}
     recheck := true; correct := true
     while recheck \wedge correct do begin
         recheck := false
         for all constraints \psi of \Phi such that (V(\psi) \cap S) \neq \emptyset do begin
             if there is no min-set of \psi containing S \cap V(\psi) then correct := false
                  S := S \cup \text{the minimal min-set of } \psi \text{ containing } S \cap V(\psi)
                 if S changed then recheck := true
             end
         end
     end
     if correct then return S
end
reject
```

FIGURE 10.16. A polynomial time algorithm that computes free sets for min-intersection and shuffle closed constraint languages.

PROOF. Suppose that the algorithm returns a set S. The variable *correct* must then be equal to *true*. When the while loop terminates, recheck equals false, and so for all constraints $\psi \in \Phi$ such that $V(\psi) \cap S \neq \emptyset$ the set S did not change. This implies that for all these constraints the minimal min-set of ψ containing $S \cap V(\psi)$ is equal to $S \cap V(\psi)$. We conclude that S is a free set of Φ .

We now have to argue that in case that Φ is satisfiable, the algorithm does not reject. If Φ has a solution, then there is some set S' of variables that have the minimal value in this solution. Consider a run of the while loop in the procedure FindFreeIC for some variable $x \in S'$. In the beginning, it holds that $S = \{x\} \subseteq S'$. For each constraint ψ from Φ we have that $S' \cap V(\psi)$ is a min-set of ψ if $S' \cap V(\psi)$ is non-empty. Therefore, the program variable correct cannot be set to false while $S \subseteq S'$. Because we always add only variables of the minimal min-set of ψ containing $S \cap V(\psi)$ to S, all these variables are always in S'. Therefore, S remains a subset of S' all the time, and the algorithm does not reject.

THEOREM 10.5.37. If \mathfrak{B} is preserved by mi there is an algorithm solving $CSP(\mathfrak{B})$ in time $O(n^3m)$.

PROOF. We use the procedure FindFreeSetIC in Figure 10.16 for the sub-routine FindFreeSet in Figure 10.14. Lemma 10.5.33 and Lemma 10.5.36 imply the correctness of these algorithms.

An algorithm for languages preserved by mx. Finally, we consider languages \mathfrak{B} preserved by mx. Let R be a relation from \mathfrak{B} . For a tuple $t \in R$, we define $\chi_{min}(t)$ to be a vector from $\{0,1\}^k$ such that $\chi_{min}(t)[i]=1$ if and only if t[i] is minimal in t. We define $\chi_{min}(R)$ to be $\{\chi_{min}(t) \mid t \in R\}$. Since R is preserved by mx and hence min-xor closed by Lemma 10.5.21, the set $\chi_{min}(R)$ is closed under addition of distinct vectors over GF(2), and hence in particular closed under the Boolean

minority operation $minority(x, y, z) = x \oplus y \oplus z$. By Theorem 5.4.3, $\chi_{min}(R) \cup \{0^k\}$ is exactly the set of solutions of a system of linear equations.

THEOREM 10.5.38. If \mathfrak{B} is preserved by mx there is an algorithm solving $CSP(\mathfrak{B})$ in time $O(n^4)$.

PROOF. To find a free set of variables of an instance Φ of CSP(\mathfrak{B}) (if it exists), we first construct a system S of linear equations over GF(2) with variable set $\{x_v \mid v \in V\}$ and linear equations as described above for each constraint in Φ . It is well-known that a solution of S that is distinct from 0^n can be computed in cubic time (by Gaussian elimination). If there is such a solution, then the set of variables mapped to 1 is a free set of Φ . If the system has no such solution, then there is no free set of variables, and there is no solution for Φ . Now the claim follows from Lemma 10.5.33 as in Theorem 10.5.35 and Theorem 10.5.37.

10.6. Classification

This section combines the previous results to show that every temporal constraint language has a polynomial-time constraint satisfaction problem, or is NP-complete.

10.6.1. Classification in the presence of <.

LEMMA 10.6.1. Let f be a binary operation violating Betw and preserving <. Then there are $t_1, t_2 \in$ Betw such that $f(t_1, t_2)$ has three distinct entries and $f(t_1, t_2) \notin$ Betw.

PROOF. Since f violates Betw, there are two triples $t_1, t_2 \in$ Betw such that $t := f(t_1, t_2) \notin$ Betw. Because f preserves <, we can assume without loss of generality that $t_1[1] < t_1[2] < t_1[3]$ and $t_2[1] > t_2[2] > t_2[3]$. If t has three distinct entries (in this case, we also say that t is injective), we are done. Otherwise we distinguish two cases:

- (1) t[1] = t[2] = t[3]: In that case, take a triple s_1 such that $s_1[1] < t_1[1]$, $s_1[2] = t_1[2]$, and $s_1[3] = t_1[3]$. We also choose a triple s_2 such that $t_2[2] < s_2[1] < t_2[1]$, $s_2[2] = t_2[2]$, and $s_2[3] = t_2[3]$. It is straightforward to check that $s_1[1] < s_1[2] < s_1[3]$ and $s_2[1] > s_2[2] > s_2[3]$ and thus both triples belong to Betw. Now, consider $s := f(s_1, s_2)$. We have that s[2] = t[2], s[3] = t[3], and s[1] < t[1] = s[2] = s[3] because f preserves s := t[3]. Therefore $s \notin B$ Betw. Take s_1 instead of t_1 , t_2 instead of t_2 and proceed with case 2.
- (2) If exactly two entries in t have the same value, let i, j be their indices and let k be the index of the entry with the unique value. We assume that t[k] > t[i] (the other case is symmetric). It is straightforward to verify that there is an entry in t such that making the value of this entry smaller would make t injective and it would still not be in Betw. We can assume without loss of generality that i is an index of such an entry. We choose s_1 so that $s_1[i] < t_1[i], s_1[j] = t_1[j], s_1[k] = t_1[k], and <math>s_1[1] < s_1[2] < s_1[3]$. We choose s_2 such that $s_2[i] < t_2[i], s_2[j] = t_2[j], s_2[k] = t_2[k], and <math>s_2[1] > s_2[2] > s_2[3]$. Note that $s_1, s_2 \in \text{Betw}$. The tuple $s := f(s_1, s_2)$ satisfies s[i] < t[i], s[j] = t[j], and s[k] = t[k]. By the choice of i we conclude that s is injective, $s \notin \text{Betw}$ and we are done.

We use Ramsey theory via Theorem 8.3.17 to prove the following.

Lemma 10.6.2. Let f be a binary operation that preserves < and violates Betw. Then f generates ll, dual-ll, pp, or dual-pp.

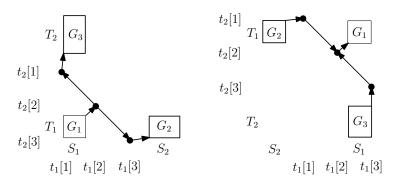


FIGURE 10.17. Grids chosen for the application of the product Ramsey theorem. The depicted ordering on the values of f follows from the choice of t_1, t_2 and because f preserves <.

PROOF. If f violates Betw and preserves <, then Lemma 10.6.1 asserts that there are $t_1, t_2 \in$ Betw such that $t := f(t_1, t_2) \notin$ Betw and t is injective. As f preserves <, we can assume without loss of generality that $t_1[1] < t_1[2] < t_1[3]$ and $t_2[1] > t_2[2] > t_2[3]$ (otherwise, we apply the argument to f(y, x)).

Either the triple t satisfies t[1] > t[2] < t[3] or t[1] < t[2] > t[3]. In the first case, let $S_1 := \{x \in \mathbb{Q} \mid t_1[1] < x < t_1[2]\}$, $S_2 := \{x \in \mathbb{Q} \mid t_1[3] < x\}$, $T_1 := \{y \in \mathbb{Q} \mid t_2[3] < y < t_2[2]\}$, and $T_2 := \{y \in \mathbb{Q} \mid t_2[1] < y\}$. In the second case, let $S_1 := \{x \in \mathbb{Q} \mid t_1[2] < x < t_1[3]\}$, $S_2 := \{x \in \mathbb{Q} \mid x < t_1[1]\}$, $T_1 := \{y \in \mathbb{Q} \mid t_2[2] < y < t_2[1]\}$, and $T_2 := \{y \in \mathbb{Q} \mid y < t_2[3]\}$. See Figure 10.17 for an illustration of these sets.

For each $k \in \mathbb{N}$, we define sets $S_1^{(k)}, T_1^{(k)}, S_2^{(k)}, T_2^{(k)}$ as follows. Apply Lemma 8.2.9 to the grid $S_1 \times T_1$ (both S_1 and T_1 are infinite and in particular larger than $\mathbf{R}(\mathbf{R}(k))$), and obtain subsets $U^{(k)} \subseteq S_1$ and $V^{(k)} \subseteq T_1$ such that $|U^{(k)}| \ge \mathbf{R}(k), |V^{(k)}| \ge \mathbf{R}(k)$, and f is canonical on $U^{(k)} \times V^{(k)}$. Similarly, we apply Theorem 8.2.9 to the grid $U^{(k)} \times T_2$ and obtain subsets $S_1^{(k)} \subseteq U^{(k)}$ and $T_2^{(k)} \subseteq T_2$ of cardinality at least k such that f is homogenous on $S_1^{(k)} \times T_2^{(k)}$. We finally apply Theorem 8.2.9 to the grid $S_2 \times V^{(k)}$ and obtain subsets $S_2^{(k)} \subseteq S_2$ and $T_1^{(k)} \subseteq V^{(k)}$ of cardinality at least k such that f is canonical on $S_2^{(k)} \times T_1^{(k)}$. Note that f is in particular canonical on $S_1^{(k)} \times T_1^{(k)}$.

There are just 6^3 possibilities for how f behaves on those grids for given k. Hence, there is an infinite set $K \subseteq \mathbb{N}$ such that f behaves in the same way on $S_1^{(k)} \times T_1^{(k)}$ for all $k \in K$, in the same way on $S_1^{(k)} \times T_2^{(k)}$ for all $k \in K$, and in the same way on $S_2^{(k)} \times T_1^{(k)}$ for all $k \in K$.

The following observations will be obvious by inspection of Figure 10.17, left side. In case that $S_1^{(k)}$ is before $S_2^{(k)}$ (that is, all elements in $S_1^{(k)}$ are smaller than all elements in $S_2^{(k)}$) and $T_1^{(k)}$ is before $T_2^{(k)}$, then by the choice of $S_1^{(k)}$, $S_2^{(k)}$, $T_1^{(k)}$, and $T_2^{(k)}$, and because f preserves <, we have

$$f(x,y) < f(t_1[2], t_2[2]) < f(t_1[1], t_2[1]) < f(x', y')$$

for all $(x,y) \in S_1^{(k)} \times T_1^{(k)}$ and $(x',y') \in (S_1^{(k)} \times T_2^{(k)})$. Similarly,

$$f(x,y) < f(t_1[2], t_2[2]) < f(t_1[3], t_2[3]) < f(x'', y'')$$

for all $(x,y) \in S_1^{(k)} \times T_1^{(k)}$ and $(x'',y'') \in (S_2^{(k)} \times T_1^{(k)})$. The other case is that $S_2^{(k)}$ is before $S_1^{(k)}$ and $T_2^{(k)}$ is before $T_1^{(k)}$ (see the right side of Figure 10.17 for an

illustration). In this case $f(x,y) > f(t_1[2],t_2[2]) > f(x',y')$ for all $(x,y) \in S_1^{(k)} \times T_1^{(k)}$ and $(x',y') \in (S_1^{(k)} \times T_2^{(k)}) \cup (S_2^{(k)} \times T_1^{(k)})$.

First suppose that f is dominated by the same argument on all the grids $S_1^{(k)} \times T_1^{(k)}$, $S_1^{(k)} \times T_2^{(k)}$, and $S_2^{(k)} \times T_1^{(k)}$ for all $k \in K$. We can assume that f is dominated on these grids by the second argument; otherwise we swap the arguments of f. Let $g, h \in \{lex_{y,x}, lex_{y,-x}, p_y\}$ be such that f behaves like g on $S_1^{(k)} \times T_1^{(k)}$ and like h on $S_2^{(k)} \times T_1^{(k)}$. Then by the above observations and local interpolation f generates [g|h] if S_1 is before S_2 , and [h|g] if S_2 is before S_1 . Moreover, we show that f also generates lex

- If g or h is $lex_{x,y}$ or $lex_{y,x}$, then f clearly generates lex.
- If g or h is $lex_{x,-y}$ or $lex_{y,-x}$, then f generates lex as well, as lex(x, -lex(x, -y)) behaves like lex(x, y).
- If g is p_y and h is p_y , then f generates lex by Lemma 10.4.5.

Now we consider the case that f is dominated by different arguments on the grids $S_1^{(k)} \times T_1^{(k)}$ and $S_1^{(k)} \times T_2^{(k)}$, or by different arguments on the grids $S_1^{(k)} \times T_1^{(k)}$ and $S_2^{(k)} \times T_1^{(k)}$ and $S_2^{(k)} \times T_1^{(k)}$, for all $k \in K$. We only consider the first case; the second case is symmetric under swapping the arguments of f. Let g, h be from $\{lex_{x,y}, lex_{x,-y}, lex_{y,x}, lex_{y,-x}, p_x, p_y\}$ such that f behaves like h on the grids $S_1^{(k)} \times T_1^{(k)}$ and like g on the grids $S_2^{(k)} \times T_1^{(k)}$. Again, by local interpolation f generates [h|g] if S_1 is before S_2 , and [g|h] if S_2 is before S_1 . We assume without loss of generality that f generates [h|g] (in the other case we can exchange the names of h and g and proceed in the same way).

If h is p_y and g is p_x , then [h|g] behaves like pp; hence f generates pp and we are done. Dually, if h is p_x and g is p_y , then f generates dual-pp. In all other cases, either h or g is from $lex_{x,y}$, $lex_{y,x}$, $lex_{x,-y}$, or $lex_{y,-x}$, and thus f generates lex as we have already seen before. But then Lemma 10.4.4 shows that f generates ll or dual-ll. \square

10.6.2. Summary. We summarize our findings in the following classification statement; also see Figure 10.18.

Theorem 10.6.3. Let \mathfrak{B} be a temporal constraint language. Then one of the following applies.

- \mathfrak{B} is preserved by at least one of the following nine operations: ll, min, mi, mx, their duals, or a constant operation.
- Betw, Cycl, Sep, T_3 , $-T_3$, or I_6 is primitive positive definable in \mathfrak{B} .

PROOF. Theorem 10.3.2 asserts that one of the following cases is true:

- (1) There is a primitive positive definition of Cycl, Betw, or Sep in 3.
- (2) Pol(3) contains a constant operation.
- (3) Pol(\mathfrak{B}) contains all permutations of \mathbb{Q} . In this case, Theorem 6.4.2 shows that \mathfrak{B} either has a binary injective polymorphism g, or the relation I_6 has a primitive positive definition in \mathfrak{B} . In the first case, by composing g with a permutation, we see that *all* binary injective operations preserve \mathfrak{B} , and hence in particular the operation ll is a polymorphism of \mathfrak{B} .
- (4) all $f \in \text{Pol}(\mathfrak{B})$ preserve <.

We are done in all cases except the fourth. Also, we can assume that \mathfrak{B} has a polymorphism f that violates Betw. By Lemma 5.3.10, we can assume that f is binary. Then Lemma 10.6.2 implies that the operation f generates pp, dual-pp, ll, or dual-ll. If f generates ll or dual-ll there is nothing to show. If f generates pp then

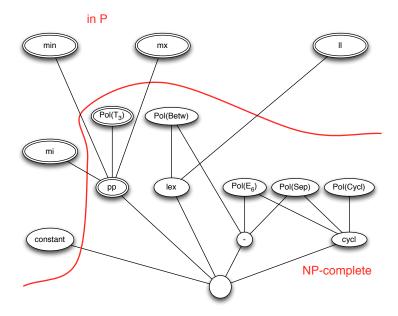


FIGURE 10.18. An illustration of the classification result for temporal constraint languages. Double-circles mean that the corresponding operation has a dual generating a distinct clone which is not drawn in the figure.

Lemma 10.5.29 shows that either T_3 has a primitive positive definition in \mathfrak{B} , or \mathfrak{B} is preserved by min, mi, or mx. Dually, if f generates dual-pp then either $-T_3$ has a primitive positive definition in \mathfrak{B} , or \mathfrak{B} is preserved by one of the duals of min, mi, or mx, which completes the proof.

With the previous theorem it is easy to obtain the full complexity classification for temporal constraint satisfication problems, and finally show Theorem 10.1.1.

PROOF OF THEOREM 10.1.1. When \mathfrak{B} is preserved by ll, min, mi, mx, one of their duals, or the constant operation, then \mathfrak{B} has an at most ternary weak near unanimity polymorphism modulo endomorphisms; this is immediate for the commutative binary functions mx, min, their duals, and for the constant function. For ll, this has been shown in Theorem 10.4.11, and for mi in Theorem 10.5.18. For dual mi and dual ll the dual argument works.

Now let \mathfrak{B}' be a finite signature reduct of \mathfrak{B} . If \mathfrak{B}' is preserved by a constant operation, then tractability of $CSP(\mathfrak{B}')$ follows from Proposition 1.1.11. For the case that \mathfrak{B}' is preserved by ll or dual-ll we have presented a polynomial-time algorithm for $CSP(\mathfrak{B}')$ in Theorem 10.4.19. If \mathfrak{B}' is preserved by min, mi, mx, or one of their duals, tractability of $CSP(\mathfrak{B}')$ is shown in Section 10.5.7.

Now suppose that \mathfrak{B} is not preserved by one of the listed operations. Then by Theorem 10.6.3 we know that one of the relations Betw, Cycl, Sep, T_3 , $-T_3$, or I_6 has a primitive positive definition in \mathfrak{B} . Each of those relations together with finitely many constants primitively positively interprets ($\{0,1\}$; 1IN3):

• For $(\mathbb{Q}; \text{Betw}, 0)$ a primitive positive interpretation of $(\{0, 1\}; \text{NAE})$ has been shown in Proposition 5.5.13, which also gives a primitive positive interpretation of $(\{0, 1\}; \text{IIN3})$ in $(\mathbb{Q}; \text{Betw}, 0)$ via Theorem 5.5.17.

- a primitive positive interpretation of $(\{0,1\}; 1IN3)$ in $(\mathbb{Q}; Cycl)$ with parameters has been given in Theorem 10.2.7.
- The structure (\mathbb{Q} ; Sep, 0, 1) primitively positively interprets ($\{0,1\}$; 1IN3) by Proposition 10.2.8.
- The structure $(\mathbb{Q}; T_3, 0)$ primitively positively interprets $(\{0, 1\}; 1IN3)$ by Proposition 10.2.6; the proof for -T is dual.
- I_6 primitively positively interprets ($\{0,1\}$; 1IN3) by Proposition 5.5.9.

Finally, recall from Theorem 10.3.1 that if \mathfrak{B} does not have a constant endomorphism, then it is a model-complete core, and hence Corollary 5.6.11 shows that the two cases in the statement of Theorem 10.1.1 are distinct.

See Figure 10.19 for an overview over the nine largest tractable temporal constraint languages; the entries also mention *typical relations* for the respective language, i.e., a set of relations that is contained in the language, but not contained in any other of the nine languages – hence, these relations show that all the languages are distinct.

Polymorphism	Typical Relations	Complexity	Reference
min	$\{U,<\}$	$O(n^2m)$	Theorem 10.5.35
mi	$\{I\}$	$O(n^3m)$	Theorem $10.5.37$
mx	$\{X\}$	$O(n^4)$	Theorem $10.5.38$
$\max = \text{dual min}$	$\{-U,<\}$	$O(n^2m)$	
dual mi	$\{-I\}$	$O(n^3m)$	
dual mx	$\{-X\}$	$O(n^4)$	
11	$\{(u \neq v) \lor (x > y) \lor (x > z)\}$	O(nm)	Theorem 10.4.19
dual ll	$\{(u \neq v) \lor (x < y) \lor (x < z)\}$	O(nm)	
constant	$\{(x \le y \le z) \lor (z \le y \le x)\}$	O(m)	

FIGURE 10.19. Summary of the various tractable languages. For the last three operations, the typical relations are given by their first-order definition; in all other cases, see Section 10.5.

10.6.3. Decidability of Tractability. We want to remark that the so-called *meta-problem* for tractability is decidable; this is formally stated in the following corollary.

COROLLARY 10.6.4. There is an algorithm that, given quantifier-free first-order formulas ϕ_1, \ldots, ϕ_n that define over $(\mathbb{Q}; <)$ the relations R_1, \ldots, R_n , decides whether $CSP(\mathbb{Q}; R_1, \ldots, R_n)$ is tractable or NP-complete.

PROOF. Follows from Theorem 8.4.4 in combination with Theorem 10.6.3. \Box

CHAPTER 11

Non-Dichotomies



There are basically two methods for proving that a subclass of NP does not have a complexity dichotomy. The first is to show that for every problem in NP there is a polynomial-time equivalent problem in the subclass. By polynomial-time equivalent we mean that there are polynomial-time Turing reductions between the two problems. The non-dichotomy result then follows from Ladner's theorem [143], which asserts that there are problems in NP that are neither in P nor NP-complete, unless P=NP. This method has been applied to show that, for example, the class of monotone SNP does not exhibit a complexity dichotomy [95]. We will apply this technique in Section 11.1 and Section 11.2 to give two different proofs of the fact that the class of

all constraint satisfaction problems with infinite domains does not have a complexity dichotomy.

The second technique to show a non-dichotomy is to directly use Ladner's proof technique, which is sometimes called *delayed diagonalization*. We will use this method in Section 11.3 to show that there are ω -categorical structures \mathfrak{B} such that $CSP(\mathfrak{B})$ is in coNP, but neither in P nor coNP-complete (unless P=coNP). The question whether there are ω -categorical structures \mathfrak{B} such that $CSP(\mathfrak{B})$ is in NP \ P but not NP-complete is still open.

This chapter contains results from [33] (in Section 11.3) as well as previously unpublished results.

11.1. Arithmetical Templates

In this section we show that for every computational decision problem there exists a polynomial-time equivalent constraint satisfaction problem with an infinite template \mathfrak{B} . This result was first shown in [33]. Here we present a new proof that uses Matiyasevich's theorem. In fact, we prove a stronger result, namely the existence of a single structure \mathfrak{C} such that for every recursively enumerable problem \mathcal{P} there is a structure \mathfrak{B} with a first-order definition in \mathfrak{C} such that $\mathrm{CSP}(\mathfrak{B})$ is polynomial-time equivalent to \mathcal{P} . A second proof, based on the results in Section 1.4.2 of Chapter 1, can be found in the next section.

Previously, Bauslaugh [16] showed that for every recursive function f there exists an infinite structure \mathfrak{B} such that $\mathrm{CSP}(\mathfrak{B})$ is decidable, but has time complexity at least f. More recently, Schwandtner gave upper and lower bounds in the exponential time hierarchy for some infinite domain CSPs [186]; but these bounds leave an exponential gap.

In this section we make essential use of the following theorem, which is due to Davis, Matiyasevich, Putnam, and Robinson.

THEOREM 11.1.1 (See e.g. [162]). A subset of \mathbb{Z} is recursively enumerable if and only if it has a primitive positive definition in $(\mathbb{Z}; *, +, 1)$, the integers with addition and multiplication.

Theorem 11.1.2. For every recursively enumerable problem \mathcal{P} there exists a relational structure \mathfrak{B} with a first-order (in fact, a primitive positive) definition in $(\mathbb{Z};*,+,1)$ such that $\mathrm{CSP}(\mathfrak{B})$ is polynomial-time Turing equivalent to \mathcal{P} .

PROOF. Code \mathcal{P} as a set L of natural numbers, viewing the binary encodings of natural numbers as bit strings. More precisely, $s \in \mathcal{P}$ if and only if the number represented in binary by the string 1s is in L. That is, we append the symbol 1 at the front so that for instance $00 \in \mathcal{P}$ and $01 \in \mathcal{P}$ correspond to different numbers in L. Now consider the structure $\mathfrak{B} := (\mathbb{Z}; S, D, L', N)$ where

 \bullet S is the binary relation defined by

$$S(x,y) \Leftrightarrow ((y=x+1 \land x \ge 0) \lor (x=y=-1))$$

 \bullet *D* is the binary relation defined by

$$D(x,y) \Leftrightarrow ((y=2x \land x \ge 0) \lor (x=y=-1))$$

- $L' := L \cup \{-1\}$
- $N := \{0\}$

Clearly, if \mathcal{P} is recursively enumerable, then L and L' are recursively enumerable, too.

We have to verify that $CSP(\mathfrak{B})$ is polynomial time equivalent to \mathcal{P} . We first show that there is a polynomial-time reduction from \mathcal{P} to $CSP(\mathfrak{B})$. View an instance of \mathcal{P} as a number $n \geq 0$ as above, and let $\eta(x)$ be a primitive positive definition for

x = n in \mathfrak{B} . It is possible to find such a definition in polynomial time by repeatedly doubling (y = x + x) and incrementing (y = x + 1) the value 0 (this also follows from the more general Lemma 1.5.1). It is clear that n codes a yes-instance of \mathcal{P} if and only if $\exists x (\eta(x) \land L'(x))$ is true in \mathfrak{B} .

To reduce $CSP(\mathfrak{B})$ to \mathcal{P} , we present a polynomial-time algorithm for $CSP(\mathfrak{B})$ that uses an oracle for \mathcal{P} (so our reduction will be a polynomial-time Turing reduction). Let ϕ be an instance of $CSP(\mathfrak{B})$, and let H be the undirected graph whose vertices are the variables W of ϕ , and which has an edge between x and y if ϕ contains the constraint S(x,y) or the constraint D(x,y). Compute the connected components of H. If a connected component does not contain x with a constraint N(x) in ϕ , then we can set all variables of that component to -1 and satisfy all constraints involving those variables.

Otherwise, suppose that we have a component C that does contain x_0 with a constraint $N(x_0)$. Observe that by connectivity, if there exists a solution, then all variables in C must take non-negative value. Consider the following linear system: for each constraint of the form S(x,y) for $x,y \in C$ we add y = x+1 and $x \ge 0$ to the system, and for each constraint of the form D(x,y) for $x,y \in D$ we add z = 2x and $x \ge 0$. Subject to $x_0 = 0$ this system has either one or no solution. We can check in polynomial time whether a linear system with 2 variables per constraint has no integer solution [48], and if there is no solution, the algorithm rejects. Otherwise, the algorithm assigns to each variable $x \in C$ its unique integer value, and if ϕ contains a constraint L'(x), we call the oracle for \mathcal{P} with the binary encoding of this value. If any of those oracle calls has a negative result, reject. Otherwise, we have found an assignment that satisfies all constraints, and accept.

The universal-algebraic approach fails badly when it comes to analysing the computational complexity of $CSP(\mathfrak{B})$: the semi-lattice operation $(x,y) \mapsto max(x,y)$ preserves \mathfrak{B} for all structures \mathfrak{B} considered in the previous proof, and from that we cannot draw any consequences for the computational complexity of $CSP(\mathfrak{B})$.

11.2. CSPs in SNP

Another proof that shows that every problem in NP is polynomial-time Turing equivalent to an infinite domain CSP is based on a result by Feder and Vardi, and the results from Section 1.4.3.

Theorem 11.2.1 (Theorem 3 in [95]). Every problem in NP is equivalent to a problem in monotone SNP under polynomial-time reductions.

We show the following.

Proposition 11.2.2. Every problem in monotone SNP is equivalent to a problem in monotone connected SNP under polynomial-time Turing reductions.

PROOF. Let Φ be a monotone SNP sentence of the form $\exists R_1, \ldots, R_k \ \forall x_1, \ldots, x_l. \ \phi$ for ϕ quantifier-free and in conjunctive normal form. The sentence Ψ that we are going to construct from Φ has an additional free relation symbol E, and an existentially quantified relation symbol T, and is defined by

$$\exists R_1, \ldots, R_k, T \ \forall x_1, \ldots, x_l. \ \psi$$

where ψ is the quantifier-free first-order formula with the following clauses.

- (1) $\neg E(x_1, x_2) \lor T(x_1, x_2);$
- (2) $\neg T(x_1, x_2) \lor \neg T(x_2, x_3) \lor T(x_1, x_3);$
- (3) $\neg T(x_1, x_2) \lor T(x_2, x_1);$

(4) for each clause ϕ' of ϕ with variables x_1, \ldots, x_q , the clause

$$\phi' \vee \bigvee_{i < j < q} \neg T(x_i, x_j)$$
.

The sentence Ψ is clearly connected and monotone. We are therefore left with the task to verify that Φ and Ψ are equivalent under polynomial-time Turing reductions.

We start with the reduction from Φ to Ψ . When \mathfrak{A} is a finite τ -structure, we expand \mathfrak{A} to a $(\tau \cup \{E\})$ -structure \mathfrak{A}' by choosing for E the full binary relation. Then also T must denote the full binary relation (so that the clauses from item (1), (2), and (3) above are satisfied), and the clauses introduced in (4) are equivalent to ϕ' . Hence, Φ holds on \mathfrak{A} if and only if Ψ holds on \mathfrak{A}' .

For the reduction from Ψ to Φ , let \mathfrak{A} be an instance of Ψ . We can compute the connected components C_1, \ldots, C_k of the $\{E\}$ -reduct of \mathfrak{A} in polynomial time in the size \mathfrak{A} . For each of those connected components C, we evaluate Φ on the τ -reduct \mathfrak{A}_C of $\mathfrak{A}[C]$. If for one component this evaluation is negative, then $\mathfrak{A}[C]$ and consequently \mathfrak{A} do not satisfy Ψ . Otherwise, for each C there exists an $\tau \cup \{R_1, \ldots, R_k\}$ -expansion of \mathfrak{A}_C that satisfies ϕ . Let \mathfrak{A}' be the expansion of the disjoint union of all those $(\tau \cup \{R_1, \ldots, R_k\})$ -structures by the relation T denotes the equivalence relation with equivalence classes C_1, \ldots, C_k . Clearly, all clauses from items (1), (2), and (3) in the definition of Ψ are satisfied by \mathfrak{A}' . Each q-tuple (a_1, \ldots, a_q) from elements of \mathfrak{A}' either contains entries from different components, and hence satisfies the disjunctions from item (4), or contains only entries from the same component C, but in this case the tuple also satisfies the disjunctions from item (4) since \mathfrak{A}_C satisfies Φ .

COROLLARY 11.2.3. For every problem in NP there is a structure \mathfrak{B} such that the problem is polynomial-time Turing equivalent to $CSP(\mathfrak{B})$.

PROOF. By Theorem 11.2.1, every problem in NP is equivalent to a monotone SNP sentence Φ under polynomial-time reductions. We have shown in Proposition 11.2.2 that Φ is equivalent to a monotone connected SNP sentence Ψ , and by Theorem 1.4.11 there exists an infinite structure \mathfrak{B} such that Ψ describes $CSP(\mathfrak{B})$. \square

In Figure 11.1 the diagram about the fragments of SNP from Section 1.4 has been decorated with information about the complexity classification status.

11.3. coNP-intermediate ω -categorical Templates

In this section we show that there exists an ω -categorical directed graph \mathfrak{B} such that $CSP(\mathfrak{B})$ is in coNP, but neither coNP-complete nor in P (unless coNP=P). All structures in this section will be Fraïssé limits of classes of directed graphs.

Let \mathcal{N} be a class of finite tournaments, and recall that $\operatorname{Forb}(\mathcal{N})$, the class of all finite digraphs that does not embed a tournament from \mathcal{N} , is an amalgamation class (Example 3.2.7). We write $\mathfrak{B}_{\mathcal{N}}$ for the Fraïssé-limit of $\operatorname{Forb}(\mathcal{N})$. Observe that for finite \mathcal{N} the problem $\operatorname{CSP}(\mathfrak{B}_{\mathcal{N}})$ can be solved in deterministic polynomial time, because for a given instance \mathfrak{A} of this problem an algorithm simply has to check whether there is a homomorphism from one of the structures in \mathcal{N} to \mathfrak{A} , which is the case if and only if there is a homomorphism from \mathfrak{A} to $\mathfrak{B}_{\mathcal{N}}$.

When proving that there are uncountably many homogeneous digraphs, Henson specified an infinite set \mathcal{T} of tournaments T_3, T_4, \ldots with the property that T_i does not embed into T_j if $i \neq j$. The tournament T_n , for $n \geq 3$, in Henson's set \mathcal{T} has vertices $0, \ldots, n+1$, and the following edges:

- (i,j) for j=i+1 and $0 \le i \le n$;
- (0, n+1);
- (j,i) for j > i+1 and $(i,j) \neq (0,n+1)$.

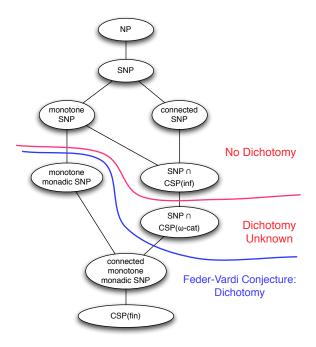


FIGURE 11.1. Dichotomies and non-dichotomies for fragments of SNP. CSP(inf) (CSP(ω -cat), CSP(fin)) refers to the class of all problems CSP(\mathfrak{B}) where \mathfrak{B} is an infinite (ω -categorical, finite, respectively) structure with finite relational signature.

PROPOSITION 11.3.1. The problem $CSP(\mathfrak{B}_{\mathcal{T}})$ is coNP-complete.

PROOF. The problem is contained in coNP, because we can efficiently test whether a sequence v_1, \ldots, v_k of distinct vertices of a given directed graph $\mathfrak A$ induces T_k in $\mathfrak A$, i.e., whether (v_i, v_j) is an arc in $\mathfrak A$ if and only if (i, j) is an arc in T_k , for all $i, j \in \{1, \ldots, k\}$. If for all such sequences of vertices this test is negative, we can be sure that $\mathfrak A$ is from $Forb(\mathcal T)$, and hence homomorphically maps to $\mathfrak B_{\mathcal T}$. Otherwise, $\mathfrak A$ embeds a structure from $\mathcal T$, and hence does not homomorphically map to $\mathfrak B_{\mathcal T}$.

The proof of coNP-hardness goes by reduction from the complement of the NP-complete 3SAT problem (see Example 1.2.2), and is inspired by a classical reduction from 3-SAT to Clique. For a given 3-SAT instance, we create an instance \mathfrak{A} of CSP($\mathfrak{B}_{\mathcal{T}}$) as follows: If

$$\{x_0^1, x_0^2, x_0^3\}, \dots, \{x_{k+1}^1, x_{k+1}^2, x_{k+1}^3\}$$

are the clauses of the 3-SAT formula (we assume without loss of generality that the 3-SAT instance has at least three clauses), then the vertex set of $\mathfrak A$ is

$$\{(0,1),(0,2),(0,3),\ldots,(k+1,1),(k+1,2),(k+1,3)\}$$

and the arc set of \mathfrak{A} consists of all pairs ((i,j),(p,q)) of vertices such that $x_i^j \neq \neg x_p^q$ and such that (i,p) is an arc in T_k .

We claim that a 3-SAT instance is unsatisfiable if and only if the created instance $\mathfrak A$ homomorphically maps to $\mathfrak B_{\mathcal T}$. The 3-SAT instance is satisfiable iff there is a mapping from the variables to true and false such that in each clause at least one literal, say $x_0^{j_0},\ldots,x_{k+1}^{j_{k+1}}$, is true. This is the case if and only if the vertices $(0,j_1),\ldots,(k+1,j_{k+1})$ induce T_k in $\mathfrak A$, i.e., $((i,j_i),(p,j_p))$ is an edge if and only if

(i,p) is an edge in T_k . This is the case if and only if T_k embeds into $\mathfrak A$. To conclude, it suffices to prove that T_k embeds into $\mathfrak A$ if and only if $\mathfrak A$ does not homomorphically map to $\mathfrak B_{\mathcal T}$. It is clear that if T_k embeds into $\mathfrak A$, then $\mathfrak A$ does not homomorphically map to $\mathfrak B_{\mathcal T}$. Conversely, if $\mathfrak A$ does not homomorphically embed to $\mathfrak B_{\mathcal T}$, then there exists a j such that there is an embedding e of T_j into $\mathfrak A$. Then for any (i,j), (p,q) in the image of e we have that (i,p) is an edge of T_k . Therefore, the mapping that sends an element u of T_j to the first component of e(u) is an embedding of T_j into T_k . Since T_j and T_k are homomorphically inequivalent for all distinct $j,k \geq 3$ we obtain that j = k and that T_k embeds into $\mathfrak A$, which finishes the proof.

We now modify the proof of Ladner's Theorem given in [171] (which is basically Ladner's original proof) to create a subset \mathcal{T}_0 of \mathcal{T} such that $CSP(\mathfrak{B}_{\mathcal{T}_0})$ is in coNP, but neither in P nor coNP-complete (unless coNP=P). One of the ideas in Ladner's proof is to 'blow holes into SAT', such that the resulting problem is too sparse to be NP-complete and to dense to be in P. Our modification is that we do not blow holes into a computational problem itself, but that we 'blow holes into the obstruction set \mathcal{T} of $CSP(\mathfrak{B}_{\mathcal{T}})$ '.

In the following, we fix one of the standard encodings of graphs as strings over the alphabet $\{0,1\}$. Let M_1, M_2, \ldots be an enumeration of all polynomial-time bounded Turing machines, and let R_1, R_2, \ldots be an enumeration of all polynomial time bounded reductions. We assume that these enumerations are effective; it is well-known that such enumerations exist.

The definition of \mathcal{T}_0 uses a Turing machine F that computes a function $f : \mathbb{N} \to \mathbb{N}$, which is defined below. The set \mathcal{T}_0 is then defined as follows.

$$\mathcal{T}_0 = \{T_n \mid f(n) \text{ is even } \}$$

The input number n is given to the machine F in unary representation. The computation of F proceeds in two phases. In the first phase, F simulates itself¹ on input 1, then on input 2, 3, and so on, until the number of computation steps of F in this phase exceeds n (we can always maintain a counter during the simulation to recognize when to stop). Let k be the value f(i) for the last input i for which the simulation was completely performed by F.

In the second phase, the machine stops if phase two takes more than n computation steps, and F returns k. We distinguish whether k is even or odd. If k is even, all directed graph $\mathfrak A$ on $s=1,2,3,\ldots$ vertices are enumerated. For each directed graph $\mathfrak A$ in the enumeration the machine F simulates $M_{k/2}$ on the encoding of $\mathfrak A$. Moreover, F computes whether $\mathfrak A$ homomorphically maps to $\mathfrak B_{\mathcal T_0}$. This is the case if for all structures $T_l \in \mathcal T$ that embed into $\mathfrak A$ the value of f(l) is even. So F tests for $l=1,2,\ldots,s$ whether T_l embeds to $\mathfrak A$ (F uses any straightforward exponential time algorithm for this purpose), and if it does, simulates itself on input l to find out whether f(l) is even. If

- (1) $M_{k/2}$ rejects and \mathfrak{A} homomorphically maps to $\mathfrak{B}_{\mathcal{T}_0}$, or
- (2) $M_{k/2}$ accepts and \mathfrak{A} does not homomorphically map to $\mathfrak{B}_{\mathcal{T}_0}$, then F returns k+1 (and f(n)=k+1).

The other case of the second phase is that k is odd. Again F enumerates all directed graphs \mathfrak{A} on $s=1,2,3,\ldots$ vertices, and simulates the computation of $R_{\lfloor k/2 \rfloor}$ on the encoding of \mathfrak{A} . Then F computes whether the output of $R_{\lfloor k/2 \rfloor}$ encodes a directed graph \mathfrak{A}' that homomorphically maps to $\mathfrak{B}_{\mathcal{T}_0}$. The graph \mathfrak{A}' homomorphically maps to $\mathfrak{B}_{\mathcal{T}_0}$ iff for all tournaments T_l that embed into \mathfrak{A}' the value f(l) is

¹Note that by the fixpoint theorem of recursion theory we can assume that F has access to its own description.

even. Whether T_l embeds into \mathfrak{A}' is tested with a straightforward exponential-time algorithm. To test whether f(l) is even, F simulates itself on input l. Finally, F tests with a straightforward exponential-time algorithm whether \mathfrak{A} homomorphically maps to $\mathfrak{B}_{\mathcal{T}}$. If

- (3) \mathfrak{A} homomorphically maps to $\mathfrak{B}_{\mathcal{T}}$ and \mathfrak{A}' does not homomorphically map to $\mathfrak{B}_{\mathcal{T}_0}$, or
- (4) $\mathfrak A$ does not homomorphically map to $\mathfrak B_{\mathcal T}$ and $\mathfrak A'$ homomorphically maps to $\mathfrak B_{\mathcal T_0}$,

then F returns k+1.

LEMMA 11.3.2. The function f is a non-decreasing function, that is, for all n we have $f(n) \leq f(n+1)$.

PROOF. We inductively assume that $f(s-1) \leq f(s)$ for all $s \leq n$, and have to show that $f(n) \leq f(n+1)$. Since F has more time to simulate itself when we run it on n+1 instead of n, the value i computed in the first phase of F cannot become smaller. By inductive assumption, k=f(i) cannot become smaller as well. In the second phase, we either return k or k+1. Hence, if k becomes larger in the first phase, the output of F cannot become smaller. If k does not become larger, then the only difference between the second phase of F for input n+1 compared to input n is that there is more time for the computations. Hence, if the machine F on input n verifies condition (1),(2),(3),(4) for some graph $\mathfrak A$ (and hence returns k+1), then F also verifies this condition for $\mathfrak A$ on input n+1, and returns k+1 as well. Otherwise, f(n)=k, and also here $f(n+1)\geq f(n)$ holds.

LEMMA 11.3.3. For all n_0 there exists an $n > n_0$ such that $f(n) > f(n_0)$ (unless $coNP \neq P$).

PROOF. Assume for contradiction that there exists an n_0 such that f(n) equals a constant k_0 for all $n \ge n_0$. Then there also exists an n_1 such that for all $n \ge n_1$ the value of k computed by the first phase of F on input n is k_0 .

If k_0 is even, then on all inputs $n \geq n_1$ the second phase of F simulates $M_{k_0/2}$ on encodings of an enumeration of graphs. Since the output of F must be k_0 , for all graphs neither (1) nor (2) can apply. Since this holds for all $n \geq n_1$, the polynomial-time bounded machine $M_{k_0/2}$ correctly decides $\mathrm{CSP}(\mathfrak{B}_{\mathcal{T}_0})$, and hence $\mathrm{CSP}(\mathfrak{B}_{\mathcal{T}_0})$ is in P. But then there is the following polynomial-time algorithm that solves $\mathrm{CSP}(\mathfrak{B}_{\mathcal{T}})$, a contradiction to coNP-completeness of $\mathrm{CSP}(\mathfrak{B}_{\mathcal{T}})$ (Proposition 11.3.1) and our assumption that $\mathrm{coNP} \neq \mathrm{P}$.

Input: A directed graph \mathfrak{A} .

If \mathfrak{A} homomorphically maps to $\mathfrak{B}_{\mathcal{T}_0}$ then accept.

Test whether one of the finitely many graphs in $\mathcal{T} \setminus \mathcal{T}_0$ embeds into \mathfrak{A} . Accept if none of them embeds into \mathfrak{A} .

Reject otherwise.

If k_0 is odd, then on all inputs $n \geq n_1$ the second phase of F does not find a graph \mathfrak{A} for which (3) or (4) applies, because the output of F must be k_0 . Hence, $R_{\lfloor k_0/2 \rfloor}$ is a polynomial-time reduction from $\mathrm{CSP}(\mathfrak{B}_{\mathcal{T}})$ to $\mathrm{CSP}(\mathfrak{B}_{\mathcal{T}_0})$, and by Proposition 11.3.1 the problem $\mathrm{CSP}(\mathfrak{B}_{\mathcal{T}_0})$ is coNP-hard. But note that because f(n) equals the odd number k_0 for all but finitely many n, the set \mathcal{T}_0 is finite. Therefore, $\mathrm{CSP}(\mathfrak{B}_{\mathcal{T}_0})$ can be solved in polynomial time, contradicting our assumption that $\mathrm{coNP} \neq \mathrm{P}$.

THEOREM 11.3.4. $CSP(\mathfrak{B}_{\mathcal{T}_0})$ is in coNP, but neither in P nor coNP-complete (unless coNP=P).

PROOF. It is easy to see that $\mathrm{CSP}(\mathfrak{B}_{\mathcal{T}_0})$ is in coNP. On input \mathfrak{A} the algorithm non-deterministically chooses a sequence of l vertices, and checks in polynomial time whether this sequence induces a copy of T_l . If yes, the algorithm computes f(l), which can be done in linear time by executing F on the unary representation of l. If f(l) is even, the algorithm accepts. Recall that \mathfrak{A} does not homomorphically map to $\mathfrak{B}_{\mathcal{T}_0}$ iff a tournament $T_l \in \mathcal{T}_0$ embeds into \mathfrak{A} , which is the case iff there is an accepting computation path for the above non-deterministic algorithm.

Suppose that $CSP(\mathfrak{B}_{\mathcal{T}_0})$ is in P. Then for some i the machine M_i decides $CSP(\mathfrak{B}_{\mathcal{T}_0})$. By Lemma 11.3.2 and Lemma 11.3.3 there exists an n_0 such that $f(n_0) = 2i$. Then there must also be an $n_1 > n_2$ such that the value k computed during the first phase of F on input n_1 equals 2i. Since M_i correctly decides $CSP(\mathfrak{B}_{\mathcal{T}_0})$, the machine F returns 2i on input n_1 . By Lemma 11.3.2, the machine F also returns 2i for all inputs from n_1 to n_2 , and by induction it follows that it F returns 2i for all inputs larger than $n \geq n_0$, in contradiction to Lemma 11.3.3.

Finally, suppose that $CSP(\mathfrak{B}_{\mathcal{T}_0})$ is coNP-complete. Then for some i the machine R_i is a valid reduction from $CSP(\mathfrak{B}_{\mathcal{T}})$ to $CSP(\mathfrak{B}_{\mathcal{T}_0})$. Again, by Lemma 11.3.2 and Lemma 11.3.3 there exists an n_1 such that the value k computed during the first phase of F on input n_1 equals 2i. Since the reduction R_i is correct, the machine F returns 2i on input n_1 , and in fact returns 2i on all inputs greater than n_1 . This contradicts Lemma 11.3.3.

CHAPTER 12

Future Work

We conclude the thesis by mentioning four promising directions of future work.

12.1. Phylogeny Constraints

We have presented a classification of the complexity of $CSP(\mathfrak{B})$ for all structures \mathfrak{B} with a first-order definition over $(\mathbb{Q};<)$, or over the random graph $(\mathbb{V};E)$. One might ask which other structures, besides $(\mathbb{Q};<)$ and the random graph $(\mathbb{V};E)$, are interesting and promising candidates for such a classification. A very interesting candidate is the structure $(\mathbb{L};|)$ (or equivalently, over a relatively 3-transitive C-set), introduced in Section 4.1. The class of CSPs that can be formulated with templates that can be defined over this structure is very large and contains many problems that have been independently studied in the literature, capturing for instance the rooted triple satisfaction problem and the quartet satisfiability problem from phylogenetic analysis. All the tools we needed for complexity classification are available: $(\mathbb{L};|)$ is homogeneous, and an appropriate order expansion of it is Ramsey (see Example 8.1.8).

12.2. Datalog

Feder and Vardi [95] observed that all the known algorithms for solving CSP(B), for a finite structure B, are either based on algebraic algorithms that can be seen as generalizations of Gaussian elimination, or based on simple 'constraint propagation', or combinations of these two paradigms. This is still the case today. An elegant way to formalize algorithms that perform constraint propagation is Datalog. Datalog can be seen as conjunctive queries that have been extended by a recursion mechanisms; alternatively, one can view Datalog as Prolog (see e.g. [183]) without function symbols. In the context of constraint satisfaction Datalog has been introduced in [95] and further studied in [136]. Some of the early contributions were equivalent characterizations of the expressive power of Datalog in terms of bounded treewidth duality and existential pebble games.

Recently, Barto and Kozik [12] presented an exact characterization of those CSPs where CSP(3) can be solved by a Datalog program. The characterization is universal-algebraic (see Chapter 5), and confirming a conjecture of Larose and Zadori [147]. It was later shown to be equivalent to a conjecture made already by Feder and Vardi in [95], see [146].

Datalog programs are very useful to solve infinite-domain constraint satisfaction problems as well. It has been shown in [31] that when $\mathfrak B$ is an ω -categorical structure, then the characterizations of the expressive power of Datalog in terms of bounded treewidth duality and existential pebble games remain valid. This has been applied to show that several fundamental infinite-domain CSPs in the literature cannot be solved by Datalog [42,45]. It would be very interesting to have an algebraic characterization of the expressive power of Datalog for CSPs with ω -categorical templates.

12.3. Topological Clones

We have seen in Section 7.4 that the topological automorphism group of an ω -categorical structure Γ describes Γ up to bi-interpretability; that is, two ω -categorical structures Γ and Δ whose automorphism groups are isomorphic as topological groups are first-order bi-interpretable.

We have also seen that the right tool for the complexity study of CSPs is *primitive* positive interpretability, and not first-order interpretability; see Section 5.5. So it is natural to ask in this context whether primitive positive interpretability can be characterized in terms of the polymorphism clone viewed as a topological clone, that is, viewed as an abstract clone equipped with the topology of point-wise convergence. We did not define abstract clones; but in this context it suffices to know that they relate to clones in the same way as permutation groups relate to abstract groups.

A partial result in this direction has been obtained in joint work with Junker [39]; one of the results proven there is that two ω -categorical structures without constant endomorphism are *existential positive bi-interpretable*¹ if and only if their transformation monoids, viewed as topological monoids, are isomorphic.

The goal to lift this further to primitive positive interpretability amounts to showing that the topological clone of \mathfrak{B} characterizes the pseudo-variety generated by the polymorphism algebra of \mathfrak{B} , via Theorem 5.5.14.

12.4. A Logic for P?

In Section 1.4 we have seen a logical characterization of the complexity class NP: by Fagin's theorem, a problem is in NP if and only if it can be described in existential second-order logic. A similar logic for the complexity class P is not known. The question whether there exists a logic for P has been formalized by Gurevich [108] and became one of the most influential questions in finite model theory [107].

One approach to shed some light on this question is to identify large fragments of existential second-order logic such that the set of sentences in this fragment that describe problems in P has an effective enumeration. For example, consider the logic of connected monotone SNP. Theorem 11.2.1 and Proposition 11.2.2 show that every problem in NP is polynomial-time equivalent to a problem in connected monotone SNP. The proof of Theorem 11.2.1 and Proposition 11.2.2 is constructive in the sense that from a non-deterministic Turing machine we can effectively construct the corresponding connected monotone SNP sentence. So if there were an algorithm that enumerates those connected monotone SNP sentences that describe a problem in P, then the question to Gurevich's question is positive (Gurevich conjectured that the answer is negative). Such an algorithm probably does not exist. But it might exist for fragments of connected monotone SNP. Recall that every sentence in connected monotone SNP describes a CSP. Hence, complexity classification for infinite domain constraint satisfaction can also be motivated by the quest for a logic for P.

¹A first-order interpretation is called *existential positive* if all the involved formulas of the interpretation are existential positive; *existential positive bi-interpretations* are defined by a similar modification of first-order bi-interpretations.

Bibliography

- F. G. Abramson and L. Harrington. Models without indiscernibles. *Journal of Symbolic Logic*, 43(3):572-600, 1978.
- [2] S. A. Adeleke and P. M. Neumann. Relations related to betweenness: their structure and automorphisms, volume 623 of Memoirs of the AMS. American Mathematical Society, 1998.
- [3] G. Ahlbrandt and M. Ziegler. Quasi-finitely axiomatizable totally categorical theories. Annals of Pure and Applied Logic, 30(1):63–82, 1986.
- [4] A. Aho, Y. Sagiv, T. Szymanski, and J. Ullman. Inferring a tree from lowest common ancestors with an application to the optimization of relational expressions. SIAM Journal on Computing, 10(3):405–421, 1981.
- [5] J. F. Allen. Maintaining knowledge about temporal intervals. Communications of the ACM, 26(11):832–843, 1983.
- [6] E. Allender, M. Bauland, N. Immerman, H. Schnoor, and H. Vollmer. The complexity of satisfiability problems: Refining Schaefer's theorem. *Journal of Computer and System Sciences*, 75(4):245–254, 2009.
- [7] B. Aspvall, M. F. Plass, and R. E. Tarjan. A linear-time algorithm for testing the truth of certain quantified boolean formulas. *Information Processing Letters*, 8(3):121–123, 1979.
- [8] A. Atserias. On digraph coloring problems and treewidth duality. In Proceedings of LICS, pages 106–115, 2005.
- [9] A. Atserias, A. A. Bulatov, and A. Dawar. Affine systems of equations and counting infinitary logic. Theoretical Computer Science, 410(18):1666-1683, 2009.
- [10] F. Baader and T. Nipkow. Term rewriting and all that. Cambridge University Press, 1999.
- [11] L. Barto. The dichotomy for conservative constraint satisfaction problems revisited. In Proceedings of the Symposium on Logic in Computer Science (LICS), Toronto, Canada, 2011.
- [12] L. Barto and M. Kozik. Constraint satisfaction problems of bounded width. In Proceedings of the Annual Symposium on Foundations of Computer Science (FOCS), pages 595–603, 2009.
- [13] L. Barto and M. Kozik. New conditions for Taylor varieties and CSP. In *Proceedings of LICS*, pages 100–109, 2010.
- [14] L. Barto and M. Kozik. Absorbing subalgebras, cyclic terms and the constraint satisfaction problem. Logical Methods in Computer Science, 8/1(07):1–26, 2012.
- [15] L. Barto, M. Kozik, and T. Niven. The CSP dichotomy holds for digraphs with no sources and no sinks (a positive answer to a conjecture of Bang-Jensen and Hell). SIAM Journal on Computing, 38(5), 2009.
- [16] B. L. Bauslaugh. The complexity of infinite H-coloring. Journal of Combinatorial Theory, Series B, 61(2):141–154, 1994.
- [17] B. L. Bauslaugh. Core-like properties of infinite graphs and structures. Discrete Mathematics, 138(1):101–111, 1995.
- [18] B. L. Bauslaugh. Cores and compactness of infinite directed graphs. *Journal of Combinatorial Theory*, Series B, 68(2):255–276, 1996.
- [19] H. Becker and A. Kechris. The Descriptive Set Theory of Polish Group Actions. Number 232 in LMS Lecture Note Series. Cambridge University Press, 1996.
- [20] I. Ben Yaacov. Positive model theory and compact abstract theories. Journal of Mathematical Logic, 3(1):85–118, 2003.
- [21] B. Bennett. Spatial reasoning with propositional logics. In Proceedings of the International Conference on Knowledge Representation and Reasoning, pages 51–62. Morgan Kaufmann, 1994.
- [22] J. Berman, P. Idziak, P. Markovic, R. McKenzie, M. Valeriote, and R. Willard. Varieties with few subalgebras of powers. *Transactions of the American Mathematical Society*, 362(3):1445– 1473, 2010.

- [23] M. Bezem, R. Nieuwenhuis, and E. Rodríguez-Carbonell. The max-atom problem and its relevance. In Proceedings of the International Conference on Logic for Programming, Artificial Intelligence, and Reasoning (LPAR), pages 47–61, 2008.
- [24] M. Bodirsky. Constraint satisfaction with infinite domains. Dissertation, Humboldt-Universität zu Berlin, 2004.
- [25] M. Bodirsky. Cores of countably categorical structures. Logical Methods in Computer Science, 3(1):1–16, 2007.
- [26] M. Bodirsky. Constraint satisfaction problems with infinite templates. In H. Vollmer, editor, Complexity of Constraints (a collection of survey articles), volume 5250 of Lecture Notes in Computer Science, pages 196–228. Springer, 2008.
- [27] M. Bodirsky and H. Chen. Oligomorphic clones. Algebra Universalis, 57(1):109–125, 2007.
- [28] M. Bodirsky and H. Chen. Quantified equality constraints. SIAM Journal on Computing, 39(8):3682–3699, 2010. A preliminary version of the paper appeared in the proceedings of LICS'07.
- [29] M. Bodirsky, H. Chen, J. Kára, and T. von Oertzen. Maximal infinite-valued constraint languages. Theoretical Computer Science (TCS), 410:1684–1693, 2009. A preliminary version appeared at ICALP'07.
- [30] M. Bodirsky, H. Chen, and M. Pinsker. The reducts of equality up to primitive positive interdefinability. *Journal of Symbolic Logic*, 75(4):1249–1292, 2010.
- [31] M. Bodirsky and V. Dalmau. Datalog and constraint satisfaction with infinite templates. *Journal on Computer and System Sciences*, 79:79–100, 2013. A preliminary version appeared in the proceedings of the Symposium on Theoretical Aspects of Computer Science (STACS'05).
- [32] M. Bodirsky, V. Dalmau, B. Martin, and M. Pinsker. Distance constraint satisfaction problems. In P. Hlinený and A. Kucera, editors, Proceedings of Mathematical Foundations of Computer Science, Lecture Notes in Computer Science, pages 162–173. Springer Verlag, August 2010.
- [33] M. Bodirsky and M. Grohe. Non-dichotomies in constraint satisfaction complexity. In L. Aceto, I. Damgard, L. A. Goldberg, M. M. Halldórsson, A. Ingólfsdóttir, and I. Walukiewicz, editors, Proceedings of the International Colloquium on Automata, Languages and Programming (ICALP), Lecture Notes in Computer Science, pages 184 –196. Springer Verlag, July 2008.
- [34] M. Bodirsky, M. Hils, and A. Krimkevitch. Tractable set constraints. In T. Walsh, editor, Proceedings of International Joint Conferences on Artificial Intelligence (IJCAI), pages 510–515. AAAI, July 2011.
- [35] M. Bodirsky, M. Hils, and B. Martin. On the scope of the universal-algebraic approach to constraint satisfaction. *Logical Methods in Computer Science (LMCS)*, 8(3:13), 2012. An extended abstract that announced some of the results appeared in the proceedings of Logic in Computer Science (LICS'10).
- [36] M. Bodirsky, P. Jonsson, and T. V. Pham. The complexity of phylogeny constraint satisfaction. In Proceedings of the Symposium on Theoretical Aspects of Computer Science (STACS), 2016. Preprint arXiv:1503.07310.
- [37] M. Bodirsky, P. Jonsson, and T. von Oertzen. Horn versus full first-order: complexity dichotomies for algebraic constraint satisfaction. *Journal of Logic and Computation*, 22(3):643– 660, 2011.
- [38] M. Bodirsky, P. Jonsson, and T. von Oertzen. Essential convexity and complexity of semialgebraic constraints. Logical Methods in Computer Science, 8(4), 2012. An extended abstract about a subset of the results has been published under the title Semilinear Program Feasibility at ICALP'10.
- [39] M. Bodirsky and M. Junker. %0-categorical structures: interpretations and endomorphisms. *Algebra Universalis*, 64(3-4):403–417, 2011.
- [40] M. Bodirsky and J. Kára. The complexity of equality constraint languages. Theory of Computing Systems, 3(2):136–158, 2008. A conference version appeared in the proceedings of Computer Science Russia (CSR'06).
- [41] M. Bodirsky and J. Kára. The complexity of temporal constraint satisfaction problems. *Journal of the ACM*, 57(2):1–41, 2009. An extended abstract appeared in the Proceedings of the Symposium on Theory of Computing (STOC).
- [42] M. Bodirsky and J. Kára. A fast algorithm and Datalog inexpressibility for temporal reasoning. ACM Transactions on Computational Logic, 11(3), 2010.
- [43] M. Bodirsky and M. Kutz. Pure dominance constraints. In Proceedings of the Symposium on Theoretical Aspects of Computer Science (STACS), pages 287–298, 2002.
- [44] M. Bodirsky and M. Kutz. Determining the consistency of partial tree descriptions. Artificial Intelligence, 171:185–196, 2007.

- [45] M. Bodirsky and J. K. Mueller. Rooted phylogeny problems. Logical Methods in Computer Science, 7(4), 2011. An extended abstract appeared in the proceedings of ICDT'10.
- [46] M. Bodirsky and J. Nešetřil. Constraint satisfaction with countable homogeneous templates. In Proceedings of CSL, pages 44–57, Vienna, 2003.
- [47] M. Bodirsky and J. Nešetřil. Constraint satisfaction with countable homogeneous templates. Journal of Logic and Computation, 16(3):359–373, 2006.
- [48] M. Bodirsky, G. Nordh, and T. von Oertzen. Integer programming with 2-variable equations and 1-variable inequalities. *Information Processing Letters*, 109(11):572–575, 2009.
- [49] M. Bodirsky and D. Piguet. Finite trees are Ramsey with respect to topological embeddings. Technical report, arXiv:1002.1557, 2010.
- [50] M. Bodirsky and M. Pinsker. Reducts of Ramsey structures. AMS Contemporary Mathematics, vol. 558 (Model Theoretic Methods in Finite Combinatorics), pages 489–519, 2011.
- [51] M. Bodirsky and M. Pinsker. Minimal functions on the random graph. Israel Journal of Mathematics, 200(1):251–296, 2014.
- [52] M. Bodirsky and M. Pinsker. Schaefer's theorem for graphs. *Journal of the ACM*, 62(3):52 pages (article number 19), 2015. A conference version appeared in the Proceedings of STOC 2011, pages 655–664.
- [53] M. Bodirsky, M. Pinsker, and A. Pongrácz. Projective clone homomorphisms. Preprint arXiv:1409.4601, 2014.
- [54] M. Bodirsky, M. Pinsker, and T. Tsankov. Decidability of definability. *Journal of Symbolic Logic*, 78(4):1036–1054, 2013. A conference version appeared in the Proceedings of LICS 2011.
- [55] M. Bodirsky and S. Woelfl. RCC8 is tractable on instances of bounded treewidth. In T. Walsh, editor, Proceedings of International Joint Conferences on Artificial Intelligence (IJCAI), pages 756–761. AAAI, July 2011.
- [56] V. G. Bodnarčuk, L. A. Kalužnin, V. N. Kotov, and B. A. Romov. Galois theory for Post algebras, part I and II. Cybernetics, 5:243–539, 1969.
- [57] F. Börner, A. A. Bulatov, H. Chen, P. Jeavons, and A. A. Krokhin. The complexity of constraint satisfaction games and QCSP. *Information and Computation*, 207(9):923–944, 2009.
- [58] N. Bourbaki. General Topology, Volume 1. Springer, 1998.
- [59] M. Broxvall and P. Jonsson. Point algebras for temporal reasoning: Algorithms and complexity. Artificial Intelligence, 149(2):179–220, 2003.
- [60] M. Broxvall, P. Jonsson, and J. Renz. Disjunctions, independence, refinements. Artificial Intelligence, 140(1/2):153–173, 2002.
- [61] A. A. Bulatov. Tractable conservative constraint satisfaction problems. In Proceedings of the Symposium on Logic in Computer Science (LICS), pages 321–330, Ottawa, Canada, 2003.
- [62] A. A. Bulatov. H-coloring dichotomy revisited. Theoretical Computer Science, 349(1):31–39, 2005.
- [63] A. A. Bulatov. A dichotomy theorem for constraint satisfaction problems on a 3-element set. Journal of the ACM, 53(1):66–120, 2006.
- [64] A. A. Bulatov and V. Dalmau. A simple algorithm for Mal'tsev constraints. SIAM Journal on Computing, 36(1):16–27, 2006.
- [65] A. A. Bulatov and P. Jeavons. Algebraic structures in combinatorial problems. Technical report MATH-AL-4-2001, Technische Universität Dresden, 2001.
- [66] A. A. Bulatov, A. A. Krokhin, and P. G. Jeavons. Classifying the complexity of constraints using finite algebras. SIAM Journal on Computing, 34:720–742, 2005.
- [67] H. K. Büning and T. Lettmann. Propositional Logic: Deduction and Algorithms. Cambridge University Press, 1999.
- [68] S. N. Burris and H. P. Sankappanavar. A Course in Universal Algebra. Springer Verlag, Berlin, 1981
- [69] P. J. Cameron. Transitivity of permutation groups on unordered sets. Mathematische Zeitschrift, 148:127–139, 1976.
- [70] P. J. Cameron. Oligomorphic permutation groups. Cambridge University Press, Cambridge, 1999
- [71] G. Cantor. Über unendliche, lineare Punktmannigfaltigkeiten. Mathematische Annalen, 23:453–488, 1884.
- [72] H. Chen. A rendezvous of logic, complexity, and algebra. ACM Computing Surveys, 42(1),
- [73] H. Chen and V. Dalmau. (Smart) look-ahead arc consistency and the pursuit of CSP tractability. In Proceedings of CP, pages 182–196, 2004.
- [74] H. Chen and M. Grohe. Constraint satisfaction with succinctly specified relations. Journal of Computer and System Sciences, 76(8):847–860, 2010.

- [75] G. Cherlin, S. Shelah, and N. Shi. Universal graphs with forbidden subgraphs and algebraic closure. Advances in Applied Mathematics, 22:454–491, 1999.
- [76] D. Cohen, P. Jeavons, P. Jonsson, and M. Koubarakis. Building tractable disjunctive constraints. *Journal of the ACM*, 47(5):826–853, 2000.
- [77] D. A. Cohen, P. Jeavons, P. Jonsson, and M. Koubarakis. Building tractable disjunctive constraints. *Journal of the ACM*, 47(5):826–853, 2000.
- [78] T. Cornell. On determining the consistency of partial descriptions of trees. In Proceedings of the ACL, pages 163–170, 1994.
- [79] J. Covington. Homogenizable relational structures. Illinois Journal of Mathematics, 34(4):731–743, 1990.
- [80] N. Creignou, P. G. Kolaitis, and H. Vollmer, editors. Complexity of Constraints An Overview of Current Research Themes [Result of a Dagstuhl Seminar], volume 5250 of Lecture Notes in Computer Science. Springer, 2008.
- [81] B. Czákány. Minimal clones. Algebra Universalis, 54(1):73-89, 2005.
- [82] V. Dalmau. Linear datalog and bounded path duality of relational structures. Logical Methods in Computer Science, 1(1), 2005.
- [83] W. Deuber. A generalization of Ramsey's theorem for regular trees. Journal of Combinatorial Theory, Series B, 18:18–23, 1975.
- [84] W. F. Dowling and J. H. Gallier. Linear-time algorithms for testing the satisfiability of propositional Horn formulae. The Journal of Logic Programming, 1(3):267–284, 1984.
- [85] T. Drakengren and P. Jonsson. Reasoning about set constraints applied to tractable inference in intuitionistic logic. *Journal of Logic and Computation*, 8(6):855–875, 1998.
- [86] T. Drakengren and P. Jonsson. Computational complexity of temporal constraint problems. In Handbook of Temporal Reasoning in Artificial Intelligence, pages 197–218. Elsevier, 2005.
- [87] M. Droste. Structure of partially ordered sets with transitive automorphism groups. AMS Memoir, 57(334), 1985.
- [88] I. D. Duentsch. Relation algebras and their application in temporal and spatial reasoning. Artificial Intelligence Review, 23:315–357, 2005.
- [89] I. Düntsch, H. Wang, and S. McCloskey. A relation algebraic approach to the region connection calculus. Theoretical Computer Science, 255:63–83, 2001.
- [90] H.-D. Ebbinghaus and J. Flum. Finite Model Theory. Springer, Berlin, Heidelberg, New York, 1999. 2nd edition.
- [91] H.-D. Ebbinghaus, J. Flum, and W. Thomas. Mathematical Logic. Springer, Berlin, Heidelberg, New York, 1984.
- [92] L. Egri, B. Larose, and P. Tesson. Symmetric datalog and constraint satisfaction problems in logspace. In *Proceedings of LICS*, pages 193–202, 2007.
- [93] D. M. Evans. A closed oligomorphic permutation group without oligomorphic extremely amenable subgroups. Announced at the workshop "Homogeneous structures" in Banff, 2015.
- [94] G. Exoo. A lower bound for R(5,5). Journal of Graph Theory, 13:97–98, 1989.
- [95] T. Feder and M. Y. Vardi. The computational structure of monotone monadic SNP and constraint satisfaction: a study through Datalog and group theory. SIAM Journal on Computing, 28:57–104, 1999.
- [96] T. Feder and M. Y. Vardi. Homomorphism closed vs. existential positive. In *Proceedings of LICS*, pages 311–320, 2003.
- [97] R. Fraïssé. Sur l'extension aux relations de quelques propriétés des ordres. Annales Scientifiques de l'École Normale Supérieure, 71:363–388, 1954.
- [98] R. Fraïssé. Theory of Relations. Elsevier Science Ltd, North-Holland, 1986.
- [99] Z. Galil and N. Megiddo. Cyclic ordering is NP-complete. Theoretical Computer Science, 5(2):179–182, 1977.
- [100] S. Gao. Invariant Descriptive Set Theory. Pure and applied mathematics. Taylor and Francis, 2008.
- [101] M. Garey and D. Johnson. A guide to NP-completeness. CSLI Press, Stanford, 1978.
- [102] D. Geiger. Closed systems of functions and predicates. Pacific Journal of Mathematics, 27:95– 100, 1968.
- [103] A. Goerdt. On random ordering constraints. In Computer Science Russia (CSR), pages 105– 116, 2009.
- [104] R. L. Graham and B. L. Rothschild. Ramsey's theorem for n-parameter sets. Transactions of the AMS, 159:257–292, 1971.
- [105] R. L. Graham, B. L. Rothschild, and J. H. Spencer. Ramsey theory. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New York, 1990. Second edition.

- [106] E. E. Granirer. Extremely amenable semigroups 2. Mathematica Scandinavica, 20:93–113, 1967.
- [107] M. Grohe. The quest for a logic capturing PTIME. In Proceedings of LICS, pages 267–271, 2008
- [108] Y. Gurevich. Toward logic tailored for computational complexity. Computation and Proof Theory, pages 175–216, 1984.
- [109] W. Guttmann and M. Maucher. Variations on an ordering theme with constraints. In Proceedings of TCS, pages 77–90, 2006.
- [110] L. Haddad and I. G. Rosenberg. Finite clones containing all permutations. Canadian Journal of Mathematics, 46(5):951–970, 1994.
- [111] D. Haskell and D. Macpherson. Cell decompositions of C-minimal structures. Annals of Pure and Applied Logic, 66:113–162, 1994.
- [112] L. Heindorf. The maximal clones on countable sets that include all permutations. Algebra Universalis, 48:209–222, 2002.
- [113] P. Hell and J. Nešetřil. On the complexity of H-coloring. Journal of Combinatorial Theory, Series B, 48:92–110, 1990.
- [114] P. Hell and J. Nešetřil. The core of a graph. Discrete Mathematics, 109:117–126, 1992.
- [115] J. W. Helton and J. Nie. Sufficient and necessary conditions for semidefinite representability of convex hulls and sets. SIAM Journal on Optimization, 20(2):759-791, 2009.
- [116] C. W. Henson. Countable homogeneous relational systems and categorical theories. *Journal of Symbolic Logic*, 37:494–500, 1972.
- [117] R. Hirsch. Expressive power and complexity in algebraic logic. Journal of Logic and Computation, 7(3):309 351, 1997.
- [118] D. Hobby and R. McKenzie. The structure of finite algebras, volume 76 of Contemporary Mathematics. American Mathematical Society, 1988.
- [119] W. Hodges, Model theory, Cambridge University Press, 1993.
- [120] W. Hodges. A shorter model theory. Cambridge University Press, Cambridge, 1997.
- [121] J. Håstad, R. Manokaran, P. Raghavendra, and M. Charikar. Beating the random ordering is hard: Every ordering CSP is approximation resistant. SIAM Journal on Computing, 2011. To appear.
- [122] J. Hubička and J. Nešetřil. Homomorphism and embedding universal structures for restricted classes. ArXiv:0909.4939, 2009.
- [123] P. M. Idziak, P. Markovic, R. McKenzie, M. Valeriote, and R. Willard. Tractability and learnability arising from algebras with few subpowers. SIAM Journal on Computing, 39(7):3023– 3037, 2010.
- [124] K. Jänich. Topologie. Springer, 2000. In German; seventh edition.
- [125] P. G. Jeavons and M. C. Cooper. Tractable constraints on ordered domains. Artificial Intelligence, 79(2):327–339, 1995.
- [126] P. Jonsson and C. Bäckström. A unifying approach to temporal constraint reasoning. Artificial Intelligence, 102(1):143–155, 1998.
- [127] P. Jonsson and T. Drakengren. A complete classification of tractability in RCC-5. Journal of Artificial Intelligence Research, 6:211–221, 1997.
- [128] M. Junker and M. Ziegler. The 116 reducts of $(\mathbb{Q}, <, a)$. Journal of Symbolic Logic, 74(3):861–884, 2008.
- [129] P. C. Kanellakis, G. M. Kuper, and P. Z. Revesz. Constraint query languages. In Proceedings of Symposium on Principles of Database Systems (PODS), pages 299–313, 1990.
- [130] M. Karpinski, H. Kleine Büning, and P. H. Schmitt. On the computational complexity of quantified Horn clauses. In E. Börger, H. Kleine Büning, and M. M. Richter, editors, *Proceedings of CSL*, Lecture Notes in Computer Science, pages 129–137, 1987. 1st Workshop on Computer Science Logic, Karlsruhe, Germany.
- [131] R. Kaye and D. Macpherson, editors. Automorphisms of first-order structures. Oxford University Press, 1994.
- [132] A. Kechris, V. Pestov, and S. Todorcevic. Fraissé limits, Ramsey theory, and topological dynamics of automorphism groups. Geometric and Functional Analysis, 15(1):106–189, 2005.
- [133] J. Keisler. Reduced products and Horn classes. Transactions of the AMS, 117:307-328, 1965.
- [134] L. Khachiyan. A polynomial algorithm in linear programming. Doklady Akademii Nauk SSSR, 244:1093-1097, 1979.
- [135] P. G. Kolaitis and M. Y. Vardi. The decision problem for the probabilities of higher-order properties. In *Proceedings of the Symposium on Theory of Computing (STOC)*, pages 425– 435, 1987.
- [136] P. G. Kolaitis and M. Y. Vardi. Conjunctive-query containment and constraint satisfaction. In Proceedings of Symposium on Principles of Database Systems (PODS), pages 205–213, 1998.

- [137] S. Koppelberg. Projective boolean algebras. In *Handbook of Boolean Algebras*, volume 3, pages 741–773. North Holland, Amsterdam-New York-Oxford- Tokyo, 1989.
- [138] M. Koubarakis. Tractable disjunctions of linear constraints: Basic results and applications to temporal reasoning. Theoretical Computer Science, 266:311–339, 2001.
- [139] M. Krasner. Généralisations et analogues de la théorie de Galois. In Congrès de la Victoire, Association Française pour l'Avancement des Sciences, pages 54–58, 1945.
- [140] A. A. Krokhin, P. Jeavons, and P. Jonsson. Reasoning about temporal relations: The tractable subalgebras of Allen's interval algebra. *Journal of the ACM*, 50(5):591–640, 2003.
- [141] G. Kun. Constraints, MMSNP, and expander relational structures. Combinatorica, 33(3):335–347, 2013.
- [142] P. B. Ladkin and R. D. Maddux. On binary constraint problems. Journal of the Association for Computing Machinery, 41(3):435–469, 1994.
- [143] R. E. Ladner. On the structure of polynomial time reducibility. *Journal of the ACM*, 22(1):155–171, 1975.
- [144] B. Larose, C. Loten, and C. Tardif. A characterisation of first-order constraint satisfaction problems. Logical Methods in Computer Science, 3(4:6), 2007.
- [145] B. Larose and P. Tesson. Universal algebra and hardness results for constraint satisfaction problems. Theoretical Computer Science, 410(18):1629–1647, 2009.
- [146] B. Larose, M. Valeriote, and L. Zádori. Omitting types, bounded width and the ability to count. International Journal of Algebra and Computation, 19(5), 2009.
- [147] B. Larose and L. Zádori. Bounded width problems and algebras. Algebra Universalis, 56(3-4):439–466, 2007.
- [148] J.-L. Lassez and K. McAloon. Independence of negative constraints. In *International Joint Conference on Theory and Practice of Software Development (TAPSOFT)*, Volume 1, pages 19–27, 1989.
- [149] J.-L. Lassez and K. McAloon. A constraint sequent calculus. In *Proceedings of LICS*, pages 52–61, 1990.
- [150] G. Ligozat and J. Renz. What is a qualitative calculus? A general framework. In Proceedings of Pacific Rim International Conferences on Artificial Intelligence (PRICAI), pages 53–64, 2004
- [151] R. Lyndon. The representation of relational algebras. Annals of Mathematics, 51(3):707–729, 1950.
- [152] H. Machida and M. Pinsker. The minimal clones above the permutations. Semigroup Forum, 75:181–211, 2007.
- [153] D. Macpherson. A survey of homogeneous structures. Discrete Mathematics, 311(15):1599– 1634, 2011.
- [154] F. Madelaine. Constraint satisfaction problems and related logic. PhD-thesis, University of Leicester, 2003.
- [155] F. Madelaine and I. A. Stewart. Constraint satisfaction, logic and forbidden patterns. SIAM Journal on Computing, 37(1):132–163, 2007.
- [156] F. R. Madelaine. On the containment of forbidden patterns problems. In Proceedings of CP, pages 345–359, 2010.
- [157] F. R. Madelaine and I. A. Stewart. Some problems not definable using structure homomorphisms. Ars Combinatorica, 67:153–159, 2003.
- [158] D. Marker. Model Theory: An Introduction. Springer, New York, 2002.
- [159] M. Maróti and R. McKenzie. Existence theorems for weakly symmetric operations. Algebra Universalis, 59(3), 2008.
- [160] K. Marriott and M. Odersky. Negative Boolean constraints. Theoretical Computer Science, 160(1&2):365–380, 1996.
- [161] D. Marx. Tractable structures for constraint satisfaction with truth tables. Theory of Computing Systems, 48(3):444-464, 2011.
- [162] Y. V. Matiyasevich. Hilbert's Tenth Problem. MIT Press, Cambridge, Massachusetts, 1993.
- [163] K. R. Milliken. A Ramsey theorem for trees. Journal of Combinatorial Theory, Series A, 26(3):215 – 237, 1979.
- [164] R. H. Möhring, M. Skutella, and F. Stork. Scheduling with and/or precedence constraints. SIAM Journal on Computing, 33(2):393–415, 2004.
- [165] B. Nebel and H.-J. Bürckert. Reasoning about temporal relations: A maximal tractable subclass of Allen's interval algebra. *Journal of the ACM*, 42(1):43–66, 1995.
- [166] J. Nešetřil. Ramsey theory. Handbook of Combinatorics, pages 1331–1403, 1995.
- [167] J. Nešetřil. Ramsey classes and homogeneous structures. Combinatorics, Probability & Computing, 14(1-2):171–189, 2005.

- [168] J. Nešetřil and V. Rödl. Ramsey classes of set systems. Journal of Combinatorial Theory, Series A, 34(2):183–201, 1983.
- [169] J. Nešetřil and V. Rödl. The partite construction and Ramsey set systems. Discrete Mathematics, 75(1-3):327–334, 1989.
- [170] J. Opatrny. Total ordering problem. SIAM Journal on Computing, 8(1):111-114, 1979.
- [171] C. H. Papadimitriou. Computational Complexity. Addison-Wesley, 1994.
- [172] C. H. Papadimitriou and M. Yannakakis. Optimization, approximation, and complexity classes. Journal of Computer and System Sciences, 43:425–440, 1991.
- [173] M. Pinsker. Maximal clones on uncountable sets that include all permutations. Algebra Universalis, 54(2):129–148, 2005.
- [174] M. Pinsker. The number of unary clones containing the permutations on an infinite set. Acta Scientiarum Mathematicarum (Szeged), 71:461–467, 2005.
- [175] B. Poizat. A Course in Model Theory: An Introduction to Contemporary Mathematical Logic. Springer, 2000.
- [176] E. L. Post. The two-valued iterative systems of mathematical logic. Annals of Mathematics Studies, 5, 1941.
- [177] M. V. Ramana. An exact duality theory for semidefinite programming and its complexity implications. *Mathematical Programming*, 77:129–162, 1997.
- [178] O. Reingold. Undirected connectivity in log-space. Journal of the ACM, 55(4), 2008.
- [179] J. Renz and B. Nebel. On the complexity of qualitative spatial reasoning: A maximal tractable fragment of the region connection calculus. Artificial Intelligence, 108(1-2):69–123, 1999.
- [180] B. Rossman. Homomorphism preservation theorems. Journal of the ACM, 55(3), 2008.
- [181] D. Saracino. Model companions for ℵ₀-categorical theories. Proceedings of the AMS, 39:591–598, 1973.
- [182] T. J. Schaefer. The complexity of satisfiability problems. In Proceedings of the Symposium on Theory of Computing (STOC), pages 216–226, 1978.
- [183] U. Schöning. Logic for Computer Scientists. Springer, 1989.
- [184] J. Schreier and Stanisław Marcin Ulam. Über die Permutationsgruppe der natürlichen Zahlenfolge. Studia Mathematica, 4:134–141, 1933.
- [185] A. Schrijver. Theory of Linear and Integer Programming. Wiley Interscience Series in Discrete Mathematics and Optimization, 1998.
- [186] G. Schwandtner. Datalog on infinite structures. Dissertation, Humboldt-Universität zu Berlin, 2008.
- [187] M. H. Siggers. A strong Mal'cev condition for varieties omitting the unary type. Algebra Universalis. 64(1):15-20, 2010.
- [188] H. Simmons. Large and small existentially closed structures. Journal of Symbolic Logic, 41(2):379–390, 1976.
- [189] M. Steel. The complexity of reconstructing trees from qualitative characters and subtrees. Journal of Classification, 9:91–116, 1992.
- [190] A. Szendrei. Clones in universal algebra. Séminaire de Mathématiques Supérieures. Les Presses de l'Université de Montréal, 1986.
- [191] R. Tarjan. Depth-first search and linear graph algorithms. SIAM Journal on Computing, 1(2):146-160, 1972.
- [192] W. Taylor. Varieties obeying homotopy laws. Canadian Journal of Mathematics, 29:498–527, 1977
- [193] S. Thomas. Reducts of the random graph. Journal of Symbolic Logic, 56(1):176-181, 1991.
- [194] S. Thomas. Reducts of random hypergraphs. Annals of Pure and Applied Logic, 80(2):165–193, 1996.
- [195] T. Tsankov. Unitary representations of oligomorphic groups. Geometric and Functional Analysis, 22(2):528–555, 2012.
- [196] R. Willard. Testing expressibility is hard. In Proceedings of CP, pages 9-23, 2010.
- [197] H. Wolkowicz, R. Saigal, and L. Vandenberghe, editors. Handbook of semidefinite programming: theory, algorithms, and applications. Springer, 2000.
- [198] M. Wrona. Clausal descriptions of temporal relations. Personal Communication, 2008.