Quantum-Enhanced Verification of the AdS/CFT Correspondence via Variational Entropy Minimization on the Dual CFT Boundary

Pradhyumna Parthasarathy

October 2025

Abstract

This thesis presents a computational pipeline that combines Physics-Informed Neural Networks (PINNs) and the Variational Quantum Eigensolver (VQE) to verify predictions of the Anti-de Sitter / Conformal Field Theory (AdS/CFT) correspondence. The PINN component computes minimal surfaces in asymptotically AdS geometries to yield entanglement-entropy predictions via the Ryu–Takayanagi prescription. The VQE component uses a quantum-classical hybrid algorithm to find approximate ground states of a candidate boundary Hamiltonian (the Sachdev–Ye–Kitaev model, SYK). By comparing PINN geometric predictions with VQE-computed entanglement measures, we propose a practical verification pathway for holographic duality in regimes where classical computation is intractable.

Contents

1 Introduction

1.1 Motivation

The AdS/CFT correspondence conjectures an exact equivalence between a gravitational theory in a $(d+1)$-dimensional asymptotically Anti-de Sitter space and a conformal field theory living on its d-dimensional boundary. Verifying this duality computationally requires matching observables computed independently on both sides. Entanglement entropy is a privileged observable because Ryu and Takayanagi relate the entanglement entropy of a boundary region to the area of a bulk minimal surface anchored on that region. This work implements a two-pronged computational strategy: (1) PINNs solve the nonlinear PDEs governing minimal surfaces in the bulk, and (2) VQE computes the ground-state properties of a candidate boundary Hamiltonian (SYK) whose entanglement characteristics can be compared to the PINN geometry.

1.2 Contributions

- A robust, runtime-primitive-ready VQE pipeline for the SYK Hamiltonian, compatible with Qiskit 2.0.2 and IBM Quantum Runtime.
- A PINN-based solver template for minimal surface PDEs in asymptotically AdS geometries.
- A reproducible recipe for comparing PINN geometry outputs with VQE-computed entanglement-related observables, including code implementation and verification notes.

2 Background

2.1 AdS/CFT and Entanglement Entropy

We review the Ryu–Takayanagi formula: for a boundary region A, the holographic entanglement entropy is

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N}, \tag{1}$$

where γ_A is the minimal surface in the bulk anchored to ∂A.

2.2 SYK Model as a Boundary Theory

The SYK model is a quantum mechanical model of N randomly interacting Majorana fermions with all-to-all four-fermion couplings. It exhibits maximal chaos and has been proposed as a dual to AdS_2 gravity. Its Hamiltonian (q=4) reads

$$H_{\mathrm{SYK}} = \sum_{i<j<k<\ell} J_{ijkl}\, \chi_i \chi_j \chi_k \chi_\ell, \tag{2}$$

with Gaussian couplings J_{ijkl} satisfying $\mathbb{E}[J_{ijkl}^2] \propto J^2/N^3$.

3 Methods

3.1 PINNs for Bulk Minimal Surfaces

The PINN solves the Euler–Lagrange PDE for the area functional subject to boundary conditions. Architecturally, the network takes bulk coordinates (or a parameterization variable) and outputs embedding functions; the loss contains PDE residuals, boundary condition penalties and regularization terms. Implementation notes are in the Appendix.

3.2 VQE for Boundary Ground State

We use a parameterized circuit (ansatz) $|\psi(\theta)\rangle$ and minimize $\langle\psi(\theta)|H_{\mathrm{SYK}}|\psi(\theta)\rangle$ via a classical optimizer. We use Qiskit's modern pattern: an `Estimator` primitive (prefer Qiskit Runtime's runtime-backed Estimator; fallback to local primitives Estimator/Aer). The VQE implementation used is the `VQE` minimum-eigensolver from `qiskit_algorithms.minimum_eigensolvers`, constructed as:

```
VQE(ansatz, optimizer, estimator=estimator)
```

which will run with the Estimator provided.

4 Main idea abstract

The Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence links a quantum field theory (the CFT boundary) to a theory of quantum gravity (the AdS bulk). This paper synthesizes the computational strategies—Analytical, Numerical, and Machine Learning (PINNs and VQE)—used to verify this conjecture. We clarify the role of the gauge group rank N and define the specific regimes where Machine Learning provides a critical advantage over traditional methods, particularly by offering a quantum-mechanically grounded verification for the geometric calculations performed by PINNs.

5 Context: The Challenge of the Holographic Duality

The duality is fundamentally a strong-weak duality: the easy-to-solve side is often the computationally intractable side, and vice versa. The gauge group rank N controls this dichotomy, dictating the necessity of a hybrid computational approach:

- **Large N Limit (Classical AdS):** When $N \to \infty$, quantum gravity effects ($\sim 1/N^2$) vanish, making the AdS side a simple problem in classical General Relativity, often solvable via Partial Differential Equations (PDEs).
- **Intractable CFT:** This same large N limit makes the CFT side computationally impossible for classical computers, as the Hilbert space scales exponentially ($\mathcal{H} \sim 2^N$).

Our entire strategy is designed to bridge this gap, using PINNs to robustly solve the difficult geometry on the AdS side and VQE to gain the **Quantum Advantage** required to solve the CFT side at large N.

6 The Classical Side: PINN's Robustness in Geometry

The core calculation is finding the minimal surface area, which relates to the boundary entanglement entropy via the Ryu-Takayanagi formula:

$$S_{\text{boundary}} = \frac{\text{Area}(M)}{4G_N}$$

6.1 PINN vs. Traditional Numerical Methods

The Physics-Informed Neural Network (PINN) provides geometric robustness where traditional numerical methods fail:

1. **Mesh-Free Advantage:** PINNs are robustly mesh-free, avoiding the time-consuming and often failure-prone task of creating a custom computational grid (mesh) to conform to arbitrary boundary shapes (e.g., ellipses in AdS4).

2. **Complex Metrics:** PINNs are more stable when solving the PDEs in complex gravitational backgrounds, such as those with black hole metrics, where traditional solvers often encounter numerical instability.

7 The Quantum Side: VQE's Quantum Advantage

The Variational Quantum Eigensolver (VQE) provides the quantum advantage by offering the only known polynomial-time pathway to find the ground state energy E_0 (which dictates S_A) of the large-N Hamiltonian.

7.1 VQE's Advantage: Overcoming the 2^N Wall

VQE is crucial because it transforms the exponential problem into a tractable, polynomial-time hybrid process:

- **Memory Bypass:** The 2^N complex numbers defining the quantum state are physically encoded in only N qubits, bypassing the classical memory wall.
- **Time Efficiency:** The centuries-long task of classically diagonalizing the $2^N \times 2^N$ Hamiltonian is replaced by a polynomial number of variational measurements and classical optimization steps.

8 The Quantum Side: VQE's Quantum Advantage and Holographic Minimization

The Variational Quantum Eigensolver (VQE) provides the Quantum Advantage by offering the only known polynomial-time pathway to find the ground state energy E_0 (which dictates S_{boundary}) of the large-N Hamiltonian.

8.1 The VQE Process: Step-by-Step

1. **Problem Definition:** The complex CFT Hamiltonian H is expressed as a sum of measurable Pauli strings (e.g., Z_1Z_2, X_1). This Hamiltonian is the cost operator whose expectation value $\langle H \rangle$ must be minimized.
2. **Ansatz Design:** A parameterized quantum circuit (PQC), $|\psi(\theta)\rangle$, is designed. Its structure (e.g., MERA-inspired or hardware-efficient) is chosen to efficiently prepare the target state.
3. **Initialization:** A classical computer selects initial random values θ_{init} for the ansatz parameters.
4. **Quantum Measurement:** The quantum device executes the circuit and measures the expectation value:
$$C(\theta_k) = \langle \psi(\theta_k)|H|\psi(\theta_k)\rangle.$$
5. **Classical Optimization:** The optimizer (e.g., COBYLA, SLSQP, or a Kolmogorov–Arnold Network) updates parameters θ_{k+1} to reduce $C(\theta_k)$.
6. **Convergence:** Repeat until convergence to E_{min}, the approximated ground state energy of the CFT boundary.

8.2 The SYK Model and the AdS2 Duality

The Sachdev-Ye-Kitaev (SYK) model is the critical CFT toy model used for this verification:

- **Theory of the Duality:** SYK, a quantum mechanical system of N randomly coupled Majorana fermions, is dual to AdS2 gravity (specifically, Jackiw-Teitelboim gravity). This provides a simplified, tractable (yet still exponentially complex) setting to test the full AdS/CFT structure.
- **Holographic Goal:** Finding the SYK model's ground state energy (E_0) is equivalent to finding the minimal entanglement state of the boundary CFT. By the Ryu-Takayanagi principle, this state corresponds precisely to the minimal surface (the area of the wormhole throat) in the dual AdS2 bulk.

8.3 The VQE Process: Step-by-Step Minimization of Entanglement

The VQE process is a hybrid quantum-classical feedback loop designed to iteratively find the CFT state with the minimal entanglement entropy.

1. **Problem Definition:** The SYK Hamiltonian H is translated into a sum of measurable Pauli strings (the Cost Operator).
2. **Ansatz Design:** A parameterized quantum circuit ($|\psi(\theta)\rangle$) is designed to prepare the trial state efficiently.
3. **Quantum Measurement:** The quantum device measures the cost, $\text{Cost}(\theta_k) = \langle\psi(\theta_k)|H|\psi(\theta_k)\rangle$.
4. **Classical Optimization:** A classical optimizer updates parameters θ_{k+1} to minimize the measured cost.
5. **Convergence:** The process repeats until the energy converges to the minimum E_{min}, which is the approximated ground state energy (minimal entanglement state).

8.4 Classical Machine Learning Support: Robust Optimization

Classical ML (e.g., a KAN) is used to replace or augment the classical optimizer in VQE. This enhances robustness by learning the high-dimensional energy landscape, thereby preventing the VQE from becoming trapped in sub-optimal local minima.

9 The Three-Phase Research Strategy: Step-by-Step Verification

This strategy links all methods, ensuring that each step builds confidence in the overall verification.

1. **Phase I: Classical Ground Truth (Analytical Verification):** PINN results for simple, known geometries (e.g., AdS3) are first verified against the well-established **analytical CFT formulas** (e.g., Calabrese-Cardy). This establishes PINN's accuracy in the non-holographic limit.
2. **Phase II: Discovery (PINN Utility):** The PINN is then applied to complex, non-symmetric geometries (e.g., the AdS4 ellipse) where the CFT answer is unknown. The result is a bold, geometrically-derived prediction based on the assumption of the duality.

3. **Phase III: Quantum Validation (VQE Utility and Quantum Advantage):** VQE provides the final, essential verification. By calculating the large-N entanglement entropy S_A from the fundamental CFT Hamiltonian, VQE offers the **independent quantum ground truth**. The agreement between the VQE result and the PINN's geometric prediction constitutes the strongest experimental verification of the holographic principle.

10 Summary of Computational Methods

tabularx

—X—X—X—X—

Method	Nature of Solution	Primary Advantage	Example Role in Duality
Analytical	Exact Symbolic Formula	Perfect precision and physical insight.	Calabrese-Cardy formula (N $\to \infty$, weak coupling).
Numerical	Approximate Value (Discrete)	Wide applicability to complex problems.	Exact Diagonalization of small N Hamiltonians.
Machine Learning (PINN)	Function Approximation (Classical)	Robustness to geometric complexity and arbitrary inputs.	Solving the Minimal Area PDE for non-symmetric AdS boundaries.
Machine Learning (VQE)	Variational Minimization (Quantum)	Overcomes the exponential 2^N scaling (Quantum Advantage).	Calculating exact S_A for large N CFT systems (like SYK).

11 Implementation and Verified Code

11.1 Environment and Runtime

Target environment: `qiskit==2.0.2`, `qiskit-aer`, `qiskit-ibm-runtime`, `qiskit-algorithms`. The notebook includes a robust setup cell (installs missing packages in Colab) and a Runtime detection cell using `QiskitRuntimeService`.

11.2 SYK Hamiltonian Construction (Verified)

We implement the q=4 SYK Hamiltonian by mapping Majorana modes to qubit Pauli strings using Jordan–Wigner. The function `generate_syk_hamiltonian(n_qubits)` returns a `SparsePauliOp`. The important checks performed:

- Pauli labels are created in the correct qubit ordering for Qiskit (big-endian label conventions are handled).
- Multiplication of single-Majorana Pauli operators uses `SparsePauliOp` algebra (phases handled automatically).
- Coupling normalization follows $\mathrm{Var}(J_{ijkl}) \propto J^2/N^3$ scaling (implemented with a proper amplitude factor for sampling).

11.3 Ansatz and VQE Setup (Verified)

The ansatz in the notebook is a MERA-inspired layered circuit approximated with `TwoLocal` for flexibility. The VQE is constructed with the `estimator` primitive. Cost evaluation uses `Estimator.run(circuits=[ansatz], observables=[H], parameter_values=[params])` — note the nested list structure required by the API.

12 Results

12.1 Sample Run (n_qubits=3)

A small system (3 qubits = 6 Majoranas) is used for testing. The VQE converges to an estimated ground-state energy dependent on the random seed; the notebook fixes RNGs for reproducibility.

12.2 PINN vs VQE Comparison Strategy

To compare PINN predictions (area $\to S_A$ via Ryu–Takayanagi) with VQE results, compute the reduced density matrix ρ_A on the VQE ground state and evaluate the von Neumann entropy:

$$S(\rho_A) = -\mathrm{Tr}\,\rho_A \log \rho_A,$$

then compare to S_A from the bulk minimal surface. Finite-N corrections and noise must be accounted for.

13 Discussion and Limitations

Finite system sizes, noise on real hardware, and ansatz expressibility limit quantitative agreement. This pipeline is a proof-of-concept to bridge geometry and quantum state entanglement; scaling to large N requires improved ansatz (MERA proper) and error mitigation.

14 Conclusion and Future Work

We present a practical workflow to test holographic predictions: PINN geometry $\to$ VQE verification. Future work: integrate MERA circuits, scale SYK, and implement error-mitigation and readout calibration for hardware verification.

A Code appendix (selected verified excerpts)

```
# Majorana -> Pauli label
def majorana_to_pauli_label(i, n_qubits):
    qubit = i // 2
    label = ['I'] * n_qubits
    for k in range(qubit):
        label[k] = 'Z'
    label[qubit] = 'X' if (i % 2 == 0) else 'Y'
    return ''.join(label[::-1])

# generate_syk_hamiltonian returns SparsePauliOp
# VQE uses VQE(ansatz, optimizer, estimator=estimator) pattern
```

B How to reproduce

1. Upload the notebook `syk_vqe_pipeline_runtime_primitives_final.ipynb` to Google Colab.
2. Run the setup cell to install required packages.
3. Save your IBM token via `QiskitRuntimeService.save_account(...)` or set env vars.

4. Run the runtime detection cell; confirm `estimator` is runtime-backed or a local fallback.
5. Run the SYK generation cell, the ansatz cell, then the VQE cell.