Two-parameter Langlands Correspondence

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Abstract

In [1] we established, for every simply-laced Lie algebra \mathfrak{g} , a canonical isomorphism between the spaces of deformed conformal blocks of the deformed \mathcal{W} -algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ and the quantum affine algebra of $\widehat{\mathfrak{g}}$, which we view as a q-deformation of the quantum Langlands correspondence. This was done by realizing the deformed conformal blocks of these algebras via the quantum K-theory of the Nakajima quiver varieties. We also linked this isomorphism to a duality emerging from the 6d little string theory. Here, we give a brief survey of these results and propose an extension to the non-simply laced case, which exhibits a Langlands-type duality.*

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1 Introduction

The Langlands Program was launched by Robert Langlands in late 1960s with the goal of relating certain questions in number theory and harmonic analysis. It was subsequently generalized to the geometric setting of complex algebraic curves, commonly referred to as the geometric Langlands correspondence. Recently, a sheaf-theoretic, categorical Langlands correspondence has been proved in [18], following the seminal paper [6]. In addition, a function-theoretic, analytic Langlands correspondence, has been introduced in [9].

In our paper [1], we proposed a two-parameter deformation of a specific manifestation of the geometric Langlands correspondence; namely, an isomorphism between two spaces of conformal blocks. We proved it in the simply-laced case. In this paper, we give a survey of these results and discuss a generalization to the non-simply laced case.

Let \mathfrak{g} be a simple Lie algebra over \mathbb{C} and ${}^L\mathfrak{g}$ its Langlands dual.¹ In the geometric Langlands correspondence, a crucial role is played by the center of the affine Kac–Moody algebra $\widehat{L}_{\mathfrak{g}}$ at the critical level ${}^Lk=-{}^Lh^\vee$ and its isomorphism with the classical \mathcal{W} -algebra $\mathcal{W}_{\infty}(\mathfrak{g})$, established in [11] (see also [12]).

The level Lk of $\widehat{}^L\mathfrak{g}$ may be deformed away from the critical value, and at the same time $\mathcal{W}_{\infty}(\mathfrak{g})$ may be deformed to the quantum \mathcal{W} -algebra $\mathcal{W}_{\beta}(\mathfrak{g})$.

The algebra $W_{\beta}(\mathfrak{g})$ is obtained by the quantum Drinfeld–Sokolov reduction [10, 7, 11] of the affine algebra $\widehat{\mathfrak{g}}$ at level k, where $\beta = m(k + h^{\vee})$, m being the lacing number of \mathfrak{g} .²

Hence one is naturally led to a quantum deformation of the geometric Langlands correspondence. Many interesting structures have been studied in this framework (see the references in [1]). In addition, it has been linked in [20] to S-duality of maximally supersymmetric 4d gauge theories with gauge groups being the compact forms of Langlands dual groups G and LG with the Lie algebras $\mathfrak g$ and $^L\mathfrak g$, respectively.

Let's focus on the following feature of the quantum geometric Langlands correspondence, an isomorphism between the spaces of conformal blocks of certain modules of two chiral algebras:

$$\widehat{L}_{\mathfrak{g}_{L_k}}$$
-blocks \longleftrightarrow $\mathcal{W}_{\beta}(\mathfrak{g})$ -blocks, (1.1)

if the parameters are related by the formula (here ${}^Lh^{\vee}$ is the dual Coxeter number of ${}^L\mathfrak{g}$)

$$\beta = \frac{1}{{}^{L}k + {}^{L}h^{\vee}} \tag{1.2}$$

¹We follow the notation of [1], which is opposite to what has become the standard notation in the subject.

 $^{{}^{2}\}mathcal{W}_{\beta}(\mathfrak{g})$ agrees with the notation of [16], and it is $\mathcal{W}_{k}(\mathfrak{g})$ of [14, 13], where $\beta = m(k+h^{\vee})$.

and are *generic*, meaning that β is not a rational number. (For an explanation of how this isomorphism of the spaces of conformal blocks follows from the quantum geometric Langlands correspondence, see Section 6.4 of [1].)

In [1], we proved a stronger result in the case of simply-laced \mathfrak{g} and genus zero curve \mathcal{C} : a canonical isomorphism between the spaces of conformal blocks of the two algebras if we modify the relation (1.2) to

$$\beta - m = \frac{1}{L(k+h^{\vee})} \tag{1.3}$$

The shift by -m in formula (1.2) is essential (see [1], Section 6.4, for more details). In the simply-laced case m=1 and $^Lh^{\vee}=h$ but we will need formula (1.3) in the non-simply laced case (see Section 5).

Remark 1.1. The relation between the corresponding chiral algebras and their conformal blocks may be viewed as a strong/weak coupling transformation well-known in quantum physics. Indeed, if we define $\tau = \beta/m$ and $^L\tau = -^L(k+h^\vee)$, then (1.3) becomes

$$\tau - 1 = -1/(m^L \tau),\tag{1.4}$$

and so small values of $^L\tau$ correspond to large values of τ . The parameters τ and $^L\tau$ are related to the complexified coupling constants of the two S-dual 4d Yang-Mills theories. The shift of β by -m (passing from (1.2) to (1.3)) corresponds to the shift $\tau \mapsto \tau - 1$ is a shift of the theta-angle (see [1] for more details).

The canonical isomorphism between the spaces of conformal blocks established in [1] arises only after we introduce one more parameter and perform one more deformation, which was the main idea of [1]. Namely, we introduced there a two-parameter deformation of the geometric Langlands correspondence, a q-deformation in addition to the deformation away from the critical level.

To define this two-parameter deformation, we replace the above algebras with their deformed counterparts. The first is the quantum affine algebra of level Lk , which we denote by $U_{\hbar}(\widehat{}^L\mathfrak{g})_{L_k}$. The second algebra is the deformed \mathcal{W} -algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ introduced in [16], which is a deformation of the conformal \mathcal{W} -algebra $\mathcal{W}_{\beta}(\mathfrak{g})$. As shown in [16], the latter arises in the limit of $\mathcal{W}_{q,t}(\mathfrak{g})$ as $q \to 1$ and we impose the condition $t = q^{\beta}$.

In [1], we introduced the spaces of deformed conformal blocks of these two algebras for a curve C with the insertion of vertex operators at finitely many marked points. We believe

that these spaces can only be defined in the case that the curve \mathcal{C} is an infinite cylinder, or a plane, or a torus. But a plane can be obtained from an infinite cylinder by taking the radius of the cylinder to infinity, while the case of the torus should follow from the case of the infinite cylinder by imposing periodic identifications of marked points.

Suppose now that ${}^L k \neq -{}^L h^{\vee}$ (we will discuss the case ${}^L k = -{}^L h^{\vee}$ in Section 5). Let q and t in $\mathcal{W}_{q,t}(\mathfrak{g})$ be given by the formulas

$$q = \hbar^{-L(k+h^{\vee})}, \qquad t = q^{m + \frac{1}{L(k+h^{\vee})}}.$$
 (1.5)

Note that if we define β by our relation (1.3), then the formulas in (1.5) give

$$\hbar = tq^{-m}, \qquad t = q^{\beta}. \tag{1.6}$$

In [1], we proved, if \mathfrak{g} is simply-laced, a canonical isomorphism between the spaces of deformed conformal blocks of $U_{\hbar}(\widehat{L}_{\mathfrak{g}})_{L_{k}}$ and $\mathcal{W}_{q,t}(\mathfrak{g})$,

$$U_{\hbar}(\widehat{\mathfrak{lg}})_{L_k}$$
-blocks \longleftrightarrow $\mathcal{W}_{q,t}(\mathfrak{g})$ -blocks, (1.7)

where q and t are expressed in terms of \hbar and ^{L}k by formula (1.5), and \hbar and ^{L}k are generic.

The case of non-simply laced \mathfrak{g} is more subtle. We will discuss it in Section 5 below.

The above identification of the deformed conformal blocks is what we refer to as a two-parameter Langlands correspondence in the title of the present paper (equivalently, the quantum q-Langlands correspondence in the title of [1]).

Remark 1.2. As we explained in [1], the physical setting for this correspondence is a sixdimensional string theory, called the (2,0) little string theory. This theory was introduced in [25, 21] and is a one-parameter deformation of the famous 6d (2,0) superconformal theory (see e.g. [26]) – the parameter is the characteristic size of the string. This deforms the relevant chiral algebras, associated to $\widehat{\mathfrak{g}}$ and $\mathcal{W}_{\beta}(\mathfrak{g})$, into the quantum affine algebra and the deformed \mathcal{W} -algebra, respectively. The little string interpretation of the deformed \mathcal{W} algebra is due to [3].

In the case of an infinite cylinder \mathcal{C} , the space of conformal blocks of $U(\widehat{L\mathfrak{g}})_{L_k}$ can be realized as the space of solutions of the Knizhnik-Zamolodchikov (KZ) equations. We define the space of deformed conformal blocks of the quantum affine algebra $U_{\hbar}(\widehat{L\mathfrak{g}})_{L_k}$ similarly, following [17], as the space of solutions of the quantum Knizhnik-Zamolodchikov (qKZ) equations. In both cases, there is a particular fundamental solution of the equations which

comes from sewing chiral vertex operators. This solution is given by formula (2.1) in the case of deformed conformal blocks of $U_{\hbar}(\widehat{L}\widehat{\mathfrak{g}})_{L_k}$.

On the other hand, the appropriate notion of the space of deformed conformal blocks for the deformed W-algebra $W_{q,t}(\mathfrak{g})$ was introduced by us in [1]. These blocks are defined using the free field realization of the vertex operators of $W_{q,t}(\mathfrak{g})$ algebra in (2.3) as well as the screening operators, both defined in [16]. In addition, one has to specify the contours of integration for the screening charges.

Thus, in [1] we defined the space of deformed conformal blocks of $W_{q,t}(\mathfrak{g})$ and we also introduced analogues of the qKZ equations for these blocks. Our key insight was the *geometric interpretation* of these objects in terms of the (quantum) K-theory of a Nakajima quiver variety X [24], whose quiver diagram is based on the Dynkin diagram of \mathfrak{g} .

This paper is organized as follows. In Section 2, we define the deformed conformal blocks on both sides of the two-parameter Langlands correspondence and state the main result, Theorem 1. In Section 3, we explain the geometry behind this correspondence; namely, that the deformed conformal blocks can be realized as vertex functions in the equivariant quantum K-theory of a Nakajima quiver variety. In Section 4, we give an outline of the proof of Theorem 1 following [1]. Finally, in Section 5, we discuss a generalization to the non-simply laced case.

2 Statement of the correspondence

Let x be a coordinate on $\mathcal{C} \cong \mathbb{C}^{\times}$. Fix a finite collection of distinct points on \mathcal{C} , with coordinates a_i . We propose, and prove in the simply-laced case, a correspondence between the following two types of deformed conformal blocks on \mathcal{C} .

2.1 Electric side

On the *electric* side, we consider the quantum affine algebra $U_{\hbar}(\widehat{L}_{\mathfrak{g}})_{L_{k}}$ blocks [17]

$$\langle \lambda' | \prod_{i} \Phi_{L_{\rho_{i}}}(a_{i}) | \lambda \rangle$$
 (2.1)

where $\Phi_{L_{\rho}}(x)$ is a chiral vertex operator corresponding to a finite-dimensional $U_{\hbar}(\widehat{L_{\mathfrak{g}}})_{L_{k}}$ module L_{ρ} . The state $|\lambda\rangle$ is the highest weight vector in a level L_{k} Verma module. Its
weight $\lambda \in L_{\mathfrak{h}}^{*}$ is an element of the dual of the Cartan subalgebra for $L_{\mathfrak{g}}$. This is illustrated
in Figure 1.

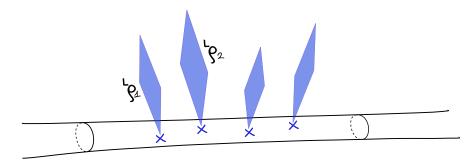


Figure 1: The cylinder \mathcal{C} with the insertions of vertex operators corresponding to finite-dimensional $U_{\hbar}(\widehat{\mathfrak{lg}})_{L_k}$ -modules ${}^L\rho_i$ at the points $a_i \in \mathcal{C}$. Boundary conditions at infinity are the highest weight vectors $\langle \lambda' |$ and $|\lambda \rangle$.

In [1], we focused on the vertex operators corresponding to the fundamental representations, with highest weights being the fundamental weights Lw_a . All others may be generated from them (by fusion). The block (2.1) takes values in a weight subspace of

$$\otimes_i ({}^{L}\rho_i) = \otimes_a ({}^{L}\rho_a)^{\otimes m_a},$$

namely, it has

weight
$$= \lambda' - \lambda$$

$$= \sum_{a} m_a^L w_a - \sum_{a} d_a^L e_a, \quad d_a \ge 0.$$
(2.2)

Here, the ${}^{L}e_{a}$, $a=1,\ldots,\operatorname{rk}(\mathfrak{g})$, are the simple positive roots of ${}^{L}\mathfrak{g}$.

2.2 Magnetic side

On the magnetic side, we consider matrix elements of vertex and screening operators of $W_{q,t}(\mathfrak{g})$ algebra. They have the form

$$\langle \mu' | \prod_{i} V_{i}^{\vee}(a_{i}) \prod_{a} \left(Q_{a}^{\vee} \right)^{d_{a}} | \mu \rangle.$$
 (2.3)

where $V_a^{\vee}(x)$ and Q_a^{\vee} are the vertex and the screening charge operators, respectively, defined in [16]. They are labeled by the coroots and coweights of \mathfrak{g} , respectively. The screening charge operators are defined as integrals of screening current vertex operators $Q_a^{\vee} = \int dx \ S_a^{\vee}(x)$, so (2.3) is in fact an integral formula, in which contours of integration must be specified (and this is a non-trivial problem).

The coweights of \mathfrak{g} labeling the $V_a^{\vee}(x)$ are the highest weights of the fundamental representations of $L_{\mathfrak{g}}$. The operator $V_i^{\vee}(a_i)$, inserted at a point on \mathcal{C} with the coordinate a_i , is associated to the same representation of $L_{\mathfrak{g}}$ as the corresponding vertex operator in (2.1).

The state $|\mu\rangle$, labeled by an element $\mu \in \mathfrak{h}$ of the Cartan subalgebra of \mathfrak{g} , generates an irreducible Fock representation of the $W_{q,t}(\mathfrak{g})$ algebra [16]. The (co)weights μ and μ' are determined by λ and λ' (the exact formula depends on the chosen normalization).

2.3 Main result

The main result of [1] is the following theorem:

Theorem 1. Let \mathfrak{g} be a simply-laced simple Lie algebra. The deformed conformal blocks of $U_{\hbar}(\widehat{L}_{\mathfrak{g}})_{L_k}$ in (2.1) and the deformed conformal blocks of $W_{q,t}(\mathfrak{g})$ in (2.3), whose parameters are generic and are related by equation (1.5), are canonically identified by the formula

specific covector
$$\times U_{\hbar}(\widehat{L}_{\mathfrak{g}})_{L_k}$$
-block $\leftrightarrow W_{q,t}(\mathfrak{g})$ -block. (2.4)

The covector in (2.4), as well other ingredients of Theorem 1, are explained in [1] in geometric terms; namely, in terms of the (quantum) K-theory of a Nakajima quiver variety X (see the next section). Specifically, the covector in question corresponds to the insertion of the identity $\mathscr{O}_X \in K_{\mathsf{T}}(X)$ in a certain enumerative problem.

3 Geometry behind the correspondence

The central ingredient of our proof is that for Lie algebras of simply-laced type, i.e.

$$^{L}\mathfrak{a}=\mathfrak{a}.$$

we can realize the deformed conformal blocks (2.1) and (2.3) as vertex functions in the equivariant quantum K-theory of a certain holomorphic symplectic variety X. The variety X is the Nakajima quiver variety with

quiver
$$Q = Dynkin diagram of \mathfrak{g}$$
.

3.1 Nakajima quiver variety

The Nakajima quiver variety X is a hyper-Kähler quotient (or a holomorphic symplectic reduction)

$$X = T^* \operatorname{Rep} \mathcal{Q} / \! / \! / G_{\mathcal{O}}, \tag{3.1}$$

where

$$\operatorname{Rep} \mathcal{Q} = \bigoplus_{a \to b} \operatorname{Hom}(V_a, V_b) \oplus_a \operatorname{Hom}(V_a, W_a)$$
(3.2)

and

$$G_{\mathcal{Q}} = \prod_{a} GL(V_a), \qquad G_W = \prod_{a} GL(W_a).$$
 (3.3)

The arrows in (3.2) are the arrows of the quiver. The dimensions of the vector spaces V_a and W_a correspond as follows

$$\dim V_a = d_a \,, \quad \dim W_a = m_a$$

to the weight space data in (2.2).

The quotient in (3.1) involves a geometric invariant theory (GIT) quotient, which depends on a choice of stability conditions. As a result, vertex functions also depend on a stability condition. This stability condition makes them analytic in a certain region of the Kähler moduli space of X. The transition matrix between the vertex functions and the deformed conformal blocks also depend on the stability condition.

3.2 Equivariant variables

Most variables in (2.1) and (2.3) become *equivariant* in their geometric interpretation. We have

$$G_W \times \mathbb{C}_{\hbar}^{\times} \subset \operatorname{Aut}(X)$$

where $\mathbb{C}_{\hbar}^{\times}$ rescales the cotangent directions in (3.1) with the weight \hbar^{-1} . This gives the symplectic form on X weight \hbar under $\mathbb{C}_{\hbar}^{\times}$. We fix a maximal torus $A \subset G_W$ and denote

$$T = A \times \mathbb{C}_{\hbar}^{\times}$$
.

The coordinates a_i of A are the positions at which the vertex operators are inserted in (2.1) and (2.3), while \hbar is the quantum group deformation parameter in $U_{\hbar}(\widehat{\mathfrak{lg}})_{L_k}$.

A multiplicative group \mathbb{C}_q^{\times} acts on quasimaps $\mathbb{P}^1 \dashrightarrow X$ by automorphisms of the domain \mathbb{P}^1 . The coordinate $q \in \mathbb{C}_q^{\times}$ is the q-difference parameter from the title of the paper.

In [23], Nakajima identified $K_T(X)$ with a space of weight (2.2) in a $U_{\hbar}(\widehat{L}_{\mathfrak{g}})_{L_k}$ -module.

3.3 Vertex functions

The basic object of the theory of [24] is the vertex function V. The vertex function is an equivariant K-theoretic count of quasimaps from \mathbb{C} to X of all possible degrees. It is an analog of Givental's I-function. The variables z in this generating function are called $K\ddot{a}hler$ parameters. They are related to the choice of the Fock vacuum $|\lambda\rangle$ in (2.1) and $|\mu\rangle$ in (2.3). For its definition and basic properties, see [1], Section 3.2.

A key geometric property of the vertex functions are the q-difference equations that they satisfy, as functions of both equivariant and Kähler variables (see [24], Section 8, for an introduction). In particular, the q-difference equations in the variables a_i were identified in Section 10 of [24] with the qKZ equations [17]. In [17], these were introduced as the q-difference equations that determine the deformations of conformal blocks corresponding to $\widehat{L_g}$ in (2.1).

More precisely, the fundamental solutions of the qKZ equations are vertex functions counting maps from \mathbb{C}^{\times} to X together with relative insertions at $0 \in \mathbb{C}$ [24].

4 Outline of the proof of the main theorem

The proof of Theorem 1 follows by establishing connections between (2.1), (2.3), and the vertex functions. Here's an outline.

4.1 Electric side

On the electric side, i.e. the $U_{\hbar}(\widehat{L}_{\mathfrak{g}})_{L_k}$ -algebra side, we have a characterization of deformed conformal blocks in (2.1) as solutions of the qKZ equations that they satisfy. Vector vertex functions provide a different basis of solutions of the same qKZ equations. The difference manifests itself through difference analytic dependence on the equivariant variables a_i and the Kähler variables z. Like any two bases of meromorphic solutions to the same difference equations, the vector vertex functions and the $U_{\hbar}(\widehat{L}_{\mathfrak{g}})_{L_k}$ -blocks are connected by a q-periodic transition matrix. This q-periodic transition matrix may be called the *pole subtraction matrix* because it cancels unwanted poles in one set of variables at the expense of introducing poles in another set of variables, see [4] for a detailed discussion.

This pole subtraction matrix was identified geometrically in [4] as the elliptic cohomology version of the stable envelopes of the Nakajima variety X. Stable envelopes in equivariant cohomology were introduced in [22]. They are the main geometric input in the construction of quantum group actions suggested there, see Section 9 of [24] for an overview. This notion has a natural lift to equivariant K-theory, derived categories of coherent sheaves, and, as shown in [4], also to the equivariant elliptic cohomology.

In parallel to cohomology and K-theory, elliptic stable envelopes produce an action of a quantum group, namely an elliptic quantum group. The analysis of [4] relates the monodromy of the qKZ equations to the braiding for this elliptic quantum group. First steps

towards such identification were taken already in [17], with many subsequent developments, as discussed in [4].

In the enumerative problem, elliptic stable envelopes are inserted via the the evaluation map at infinity of \mathbb{C}^{\times} , away from the point 0 where the relative conditions have been inserted. They appear as elliptic functions multiplying the measure of integration (see [1], Section 2.2). In either interpretation, they map vector vertex functions to $U_{\hbar}(\widehat{^L}\mathfrak{g})_{L_k}$ -blocks.

4.2 Magnetic side

On the magnetic, i.e. W-algebra side, we prove in [1], Theorem 3.1 that the vertex functions \mathbf{V} of X, counting quasimaps

$$\mathbb{C} \dashrightarrow X, \tag{4.1}$$

equal the integrals (2.3) for a specific choices of contours of integration. The integral formulas for vertex functions of X arise as follows.

K-theoretic computations on a GIT-quotient by a reductive group G may be expressed as G-invariants in a G-equivariant computation on the prequotient. The projection onto G-invariants may be recast, by the Weyl integration formula, as an integral over a suitable cycle in a maximal torus in G. In [1], we show that for K-theoretic computations on the moduli spaces of quasimaps to a GIT-quotient, there are similar integral formulas.

4.3 The match of deformed conformal blocks

To establish the match, it suffices to recognize in these formulas the integral formulas of [16] for the free field correlators of $W_{a,t}(\mathfrak{g})$.

The same dichotomy arises in the discussion of the magnetic deformed conformal blocks. Vertex functions are analytic as $z \to 0$, while the natural requirement for the $W_{q,t}(\mathfrak{g})$ -blocks is to be analytic in regions of the form

$$|a_5| \gg |a_1| \gg |a_3| \gg \dots,$$
 (4.2)

Importantly, the very same elliptic stable envelopes transform the z-series into functions with the right analyticity in the a-variables. The geometry of the correspondence is tautologically the same, as the insertion of the elliptic stable envelope happens at infinity, away from the point 0, which distinguishes vertex functions from their vector analogs. In integral formulas, stable envelopes appear as elliptic functions multiplying the measure of integration.

Informally, vertex functions are a special case of the vector vertex functions, namely the one corresponding to no insertion at 0. Since the moduli spaces in questions are not really identical, the correct technical way to see this it is via the degeneration formula as in [1], Section 4.1. Then applying elliptic stable envelopes to both sides, we obtain the statement of Theorem 1. See [1] for more details.

5 Non-simply laced case

If \mathfrak{g} is non-simply laced, the statement of Theorem 1 should be modified in a non-trivial way. On the magnetic side of the isomorphism (2.4) we still have the deformed conformal blocks of $W_{q,t}(\mathfrak{g})$ (given by formula (2.3)) but what should appear on the electric side is *not* the deformed conformal blocks of $U_{\hbar}(\widehat{\mathfrak{lg}})_{L_k}$ (given by formula (2.1)).

To see that, consider the critical level limit of $U_{\hbar}(\widehat{L}_{\mathfrak{g}})_{L_k}$, i.e. the limit in which its level L_k tends to $-L_k$. In this limit, the first formula in (1.5) yields q = 1, and the first formula in (1.6) yields $\hbar = t$.

It is known that in this limit, the solutions of the qKZ equations of $U_{\hbar}(\widehat{L}\mathfrak{g})_{L_k}$ give rise to the Bethe eigenvectors of the XXZ-type model associated to the quantum affine algebra $U_{\hbar}(\widehat{L}\mathfrak{g})$. Moreover, the difference operators of the qKZ equations become in this limit the transfer-matrices, i.e. the Hamiltonians, of this model (see the Appendix of [15] for details). Thus, the system of qKZ equations can be viewed as a deformation of this model.

If \mathfrak{g} is simply-laced, then $W_{1,t}(\mathfrak{g})$ is isomorphic to the algebra of transfer-matrices of the XXZ-type model associated to $U_t(\widehat{\mathfrak{g}}) = U_{\hbar}(\widehat{L}_{\mathfrak{g}})$, so everything is consistent. However, if \mathfrak{g} is not simply-laced, the algebra $W_{1,t}(\widehat{\mathfrak{g}})$ is not isomorphic to the algebra of transfer-matrices of the XXZ-type model associated to $U_t(\widehat{L}_{\mathfrak{g}})$. Rather, $W_{1,t}(\widehat{\mathfrak{g}})$ is isomorphic to the algebra of transfer-matrices of a different model, the folded quantum integrable model associated to \mathfrak{g} , which was defined in [15] by "folding" the XXZ-type model associated to $U_t(\widehat{\mathfrak{g}}')$, where \mathfrak{g}' is the simply-laced simple Lie algebra equipped with an automorphism of order 2 or 3 whose invariant Lie subalgebra is \mathfrak{g} .

This suggests that for non-simply laced \mathfrak{g} the two-parameter Langlands correspondence is more subtle. Namely, it follows from the preceding paragraph that the q-difference equations appearing in this correspondence are not the usual qKZ equations associated to $U_{\hbar}(\widehat{L}_{\mathfrak{g}})_{L_k}$ (if they were, then in the critical level limit we would recover the eigenvectors of the XXZ-type model associated to $U_{\hbar}(\widehat{L}_{\mathfrak{g}}) = U_t(\widehat{L}_{\mathfrak{g}})$, but this would be inconsistent with the limit

on the other side of the correspondence, which yields $W_{1,t}(\mathfrak{g})$). Rather, it must be a new q-deformation of the KZ equation of $\widehat{L}_{\mathfrak{g}_{L_k}}$, yet to be defined.

We are going to address this issue in our forthcoming paper [2].

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