Independent sets in Hasse diagrams

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Abstract

In this article, we survey bounds on the minimum independence number of Hasse diagrams of partially ordered sets on n elements. Following recent progress on this question, we discuss three constructions that yield $O(n\frac{\log\log\log n}{\log n})$, $O(\frac{n}{\log n})$, and $O(n^{3/4})$ upper bounds. We discuss another interesting construction that yields at least a $o(n)$ bound as well that may be of independent interest due to its connections to additive combinatorics. Lastly, we also discuss bounds on the special case of dimension 2 posets, giving the proof of the best known upper bound.

1 Introduction

A partially ordered set (or *poset*) is a set together with a binary relation that is reflexive, antisymmetric, and transitive (often denoted $\leq$, with an irreflexive counterpart $<$). Given a finite poset $P = (X, \leq)$ the *Hasse diagram* $H(P)$ is the directed graph whose vertex set is X and whose edges are the pairs $(x, y) \in X^2$ such that x is an *immediate predecessor* of y, i.e. $x < y$ but there does not exist any $z \in X$ with $x < z < y$. The Hasse diagram $H(P)$ can be thought of as a minimal graph representation of the structure that $\leq$ places on P. We will call the underlying undirected graph of $H(P)$ the *covering graph* of P.

For a graph G, let $\alpha(G)$ denote the size of its largest independent set. In this article, we seek to understand how small $\alpha(H(P))$ can be for n element posets P. In particular we study the quantity $f(n)$, the minimum possible value of $\alpha(H(P))$ over all n element posets P. Note that for the purpose of considering independent sets, the distinction between Hasse diagrams and covering graphs is unimportant.

Understanding $\alpha(G)$ for graphs G of a certain family is often helpful in understanding the extremal properties of that family, and so problems of this kind are common in extremal graph theory. For example, bounding the minimal value of $\alpha(G)$ over graphs without a triangle (or other small clique) is equivalent to bounding off-diagonal Ramsey numbers, a key focus of Ramsey Theory (see [Kim95, AKS80]). Similarly, $f(n)$ is a lens into understanding the extremal structure of posets.

In addition to this motivation, $f(n)$ is closely related to some other problems of interest. For example, independence numbers of Hasse diagrams are related to minimal *conflict-free colorings* in the plane, introduced by Even et al. [ELRS03]. Given a set X of n points in the plane, a conflict-free coloring of X is an assignment of one of k colors to each of the points such that any axis-parallel rectangular region that contains at least one of the points of X has some color that is represented only once in the region. The goal is to use the minimum number of colors, denoted $k(X)$. This quantity can be bounded below by the chromatic number of the *Delaunay graph* of X, denoted $GD(X)$, whose vertex set is X and whose edges $(x, y) \in X$ consist of pairs such that there is an axis parallel rectangle whose intersection with X is exactly $\{x, y\}$. Equivalently, the rectangle that has diagonally opposite corners x and y contains no other points of X. Delaunay graphs are closely related to Hasse diagrams, since $GD(X)$ can be represented as the union of two Hasse diagrams of posets on the set X; $(X, \leq_1)$ where $(x_1, y_1) \leq_1 (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$, and $(X, \leq_2)$ where $(x_1, y_1) \leq_2 (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$. An example is pictured in Figure 1. More details of the connections between $f(n)$ and these problems can be found in [MP06].

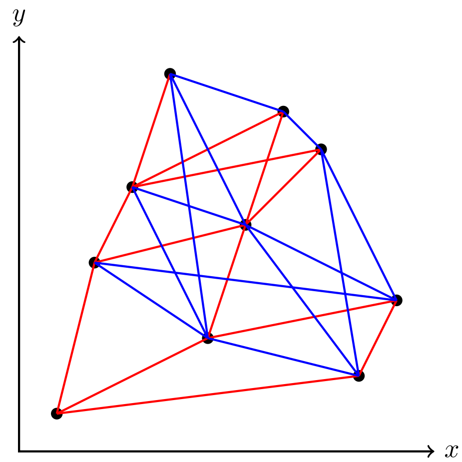

Figure 1. The Delaunay graph of a set of points X in the plane. The red edges are the Hasse diagram of $(X, \leq_1)$, and the blue edges are the Hasse diagram of $(X, \leq_2)$.

In light of these applications to planar point sets, there is also interest in bounding $f_2(n)$, the minimum possible value of $\alpha(H(P))$ over all n element *dimension 2* posets P (see Definition 2.2).

Until recently, few bounds on $f(n)$ were known. The best lower bound is a corollary of a result of Kim that any triangle free graph without an independent set of size t must have at most $O(t^2/\log t)$ vertices [Kim95]. Using the fact that covering graphs are necessarily triangle free (see Corollary 2.5), it follows that a Hasse diagram on n vertices must have an independent set of size at least $\Omega(\sqrt{n \log n})$. Thus, $f(n) = \Omega(\sqrt{n \log n})$. For upper bounds, in 1991, Brightwell and Nešetřil showed using a probabilistic construction that $f(n) = O(n \frac{\log \log \log n}{\log n})$ [BN91]. Nearly thirty years later, this was improved marginally to $f(n) = O(\frac{n}{\log n})$ independently by Tomon and Pach [PT19] and Mani and Ramakrishnan [MR19] using a more refined probabilistic construction. Shortly afterwards, Suk and Tomon [ST20] showed that $f(n) = O(n^{3/4})$, giving the first polynomial improvement to the trivial upper bound. The problem of closing the gap between $\Omega(\sqrt{n \log n})$ and $O(n^{3/4})$ remains open.

On the more restricted problem of bounding $f_2(n)$, Chen et al. [CPST09] showed that $f_2(n) = O(n \frac{\log^2 \log n}{\log n})$ by considering random configurations of points in the plane. Like $f(n)$, no improvement on the lower bound $f_2(n) = \Omega(\sqrt{n \log n})$ has been made.

Outline

In Section 2, we discuss some preliminaries on the structure of posets and Hasse diagrams that are important for the constructions later on. Section 3 discusses various constructions that give upper bounds on $f(n)$, including those of [BN91, MR19, ST20], and another construction that gives $f(n) = o(n)$ that is based on number theory. These cover many of the different techniques that can be used to construct Hasse diagrams with small independent sets. Lastly, in Section 4, we discuss the best known upper bound on $f_2(n)$, due to [CPST09].

Acknowledgments

I would like to thank Nitya Mani for collaborating with me on the results in Sections 3.2 and 3.4, and Jacob Fox for proposing this problem, suggesting Constructions 3.6 and 3.22 to investigate, and providing guidance along the way.

Thanks as well to István Tomon and János Pach for helpful discussions, and Sarah Peluse for helping us find reference [LM09] and answering some background questions. I'm also greatful for Andy Chen for helpful comments on an earlier draft of this paper.

2 Structure in posets and Hasse diagrams

In this section, we review a few key definitions and structural properties of posets and Hasse diagrams that will be useful later on. In the discussion below, we consider a poset $P = (X, \leq)$.

Definition 2.1. A subset $S \subseteq X$ is a *chain* if every pair of elements in S is comparable, and an *antichain* if no pair of elements is comparable.

The size of the longest chain in P is often called its *height*, and the size of the longest antichain is often called its *width*. We note that large chains and antichains immediately imply large independent sets in $H(P)$, since every antichain is an independent set, and if $S = \{x_1, x_2, ..., x_k\}$ is a chain where $x_1 < x_2 < \cdots < x_k$, then both $\{x_1, x_3, x_5, ...\}$ and $\{x_2, x_4, x_6, ...\}$ are independent sets. It follows that $\alpha(H(P))$ is at least the width of P and half the height of P.

Bounds on the height and width of a poset are fairly well known. A classical theorem of Dilworth [Dil50] implies that the product of the height and width of an n element poset P is at least n, so it follows easily that $f(n) = \Omega(\sqrt{n})$. This already is within a logarithmic factor of best known lower bound.

Definition 2.2. The *dimension* $\dim(P)$ is the is the minimum number of total orders whose intersection is P.

Equivalently, if $\dim(P) = d$ then every member of P can be represented as a point in $\mathbb{R}^d$, such that $a \leq b$ if and only if $a_i \leq b_i$ for every coordinate $1 \leq i \leq d$ (and $\dim(P)$ is the smallest d for which this is possible).

Next, we give a characterization of Hasse diagrams which will be vital in Section 3 for showing that the directed graphs we construct are indeed Hasse diagrams of some poset. For this we need two key definitions:

Definition 2.3. In a directed graph $G = (V, E)$, a *cycle* is a sequence of edges of the form

$$(v_1, v_2), (v_2, v_3), ..., (v_{k-1}, v_k), (v_k, v_1)$$

and a *bypass* is a sequence of edges of the form

$$(v_1, v_2), (v_2, v_3), ..., (v_{k-1}, v_k), (v_1, v_k)$$

for $k \geq 3$. For a bypass with the labeling above, we say that its *start* is v_1 and its *end* is v_k.

Theorem 2.4. *A directed graph $G = (V, E)$ is the Hasse diagram of some poset $P = (V, \leq)$ if and only if G has no cycles and no bypasses.*

Proof. First, we note that for a poset $P = (X, \leq)$, if $H(P)$ has any cycles, then this would violate the transitive property of $\leq$, and if it had a bypass $(v_1, v_2), (v_2, v_3), ..., (v_{k-1}, v_k), (v_1, v_k)$, then $v_1 < v_2 < v_k$, so this contradicts the assumption that v_1 is an immediate predecessor of v_k (by the existence of the edge (v_1, v_k)). Hence, every Hasse diagram has no cycles or bypasses, which establishes the reverse direction of the theorem.

Now, suppose that $G = (V, E)$ has no cycles or bypasses. Then consider the partial order $\leq$ on V where $u \leq v$ if and only if v is reachable from u. Then it is easy to see that $\leq$ is reflexive and transitive, and $\leq$ is antisymmetric by the assumption that G has no cycles. Hence, $P = (V, \leq)$ is a valid poset. Now we show that $H(P) = G$. For every edge $(u, v) \in E$, there is no w such that $u < w < v$, since this would imply that the union of the edges in the path from u to w, the path from w to v, and (u, v) forms a bypass. Hence, u is an immediate predecessor of v in P, which means that every edge of G is indeed an edge of $H(P)$.

Similarly, if (u, v) is an edge of $H(P)$ then the fact that $u < v$ means that there is a path from u to v. However, if there is a vertex w besides u and v on the path, then it follows that $u < w < v$, and so u is not an immediate predecessor of v. This is a contradiction, so the only vertices on the path from u to v are u and v. This means exactly that $(u, v) \in E$. Thus, every edge of $H(P)$ is also an edge of the G, so it follows that $H(P)$ and G are indeed the same. This establishes the forward direction of the theorem, and completes the proof. ■

Noting that any assignment of directions to the edges of a triangle either results in a cycle or a bypass, Theorem 2.4 has the following corollary.

Corollary 2.5. *Covering graphs are triangle free.*

3 Upper bounds on the independence number of Hasse diagrams

In this section, we analyze several constructions of Hasse diagrams with no large independent sets, with the goal of establishing upper bounds on $f(n)$. Sections 3.1 and 3.2 consider randomized constructions of partially ordered sets that with positive probability have independence number $O(n^{\frac{\log \log \log n}{\log n}})$ and $O(\frac{n}{\log n})$ respectively. Section 3.3 studies a construction based on geometric graph theory, showing that $f(n) = O(n^{3/4})$. Finally, in Section 3.4, we study a number theoretic construction of a Hasse diagram which we can show has independence number $o(n)$, and may have independence number $O(n^{2/3})$. This construction may be of future interest because of its close connections to problems in additive combinatorics.

3.1 $f(n) = O(n\frac{\log\log\log n}{\log n})$

We show the following theorem of Brightwell and Nešetřil [BN91]:

Theorem 3.1. *For each n, there exists a Hasse diagram on n vertices with independence number $O(n\frac{\log\log\log n}{\log n})$.*

The main idea is to probabilistically construct a directed acyclic graph that with high probability has no large independent sets and in expectation has few bypasses. Then if we remove one edge from each bypass, the resulting graph is a Hasse diagram (by Theorem 2.4), and the independence number is increased only by a small amount.

Construction 3.2. *Let $G_{k,t,p}$ be a directed graph on kt vertices constructed as follows. The vertices of $G_{k,t,p}$ are split into of k sets $V_1, ..., V_k$ of t vertices each. Then, for each pair of vertices u, v for $u \in V_i$ and $v \in V_j$ and $i < j$, we add the directed edge (u, v) to the graph with probability p, with each choice made independently.*

$G_{k,t,p}$ can be visualized as a $k \times t$ grid of vertices with an edge from a vertex v to any vertex in a row below it with probability p. An example is shown in Figure 2.

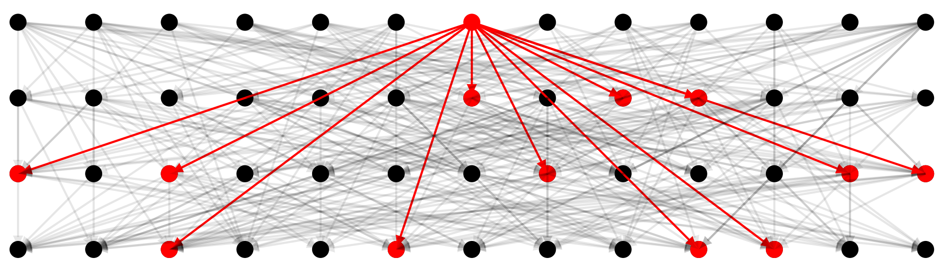

Figure 2. An example $G_{k,t,p}$ with $k = 4$, $t = 13$ and $p = \frac{4}{13}$. The ith row from the top represents the vertices of V_i. The neighborhood of one vertex in V_1 is shown in red.

We make two quick notes about $G_{k,t,p}$. First, $G_{k,t,p}$ is clearly acyclic since the vertices of any path come from sets with increasing indexes (i.e., as per the visualization in Figure 2, every edge is directed downward). However, $G_{k,t,p}$ may still have bypasses, so it is not necessarily a Hasse diagram. Second, each set V_i is an independent set, so we already know that $\alpha(G_{k,t,p}) \geq t$. The goal is to choose p and t appropriately as functions of k so that in fact $\alpha(G_{k,t,p}) = O(t)$, and we can remove the bypasses of $G_{k,t,p}$ without increasing its independence number by more than $O(t)$. The smaller we can make t, the better the bound on $f(n)$ will be.

Lemma 3.3. *If $p \geq \frac{3\log k}{t}$, then for sufficiently large k, t we have $\alpha(G_{k,t,p}) \leq 2t$ with high probability.*

Proof. Given a fixed set S of $2t$ vertices, the number of edges that can possibly exist between vertices of S is at least t^2, since this quantity is minimized when $S = V_i \cup V_j$ for some $i \neq j$. S is an independent set if and only if none of these edges exists, which happens with probability at most $(1-p)^{t^2}$, since each edge exists independently with probability p. Thus, union bounding over all $\binom{kt}{2t}$ choices of S, the probability that there is an independent set of size $2t$ is at most

$$\binom{kt}{2t}(1-p)^{t^2} \leq \left(\frac{ek}{2}\right)^{2t} \cdot e^{-pt^2} = \left(\frac{e^2k^2}{4e^{pt}}\right)^t \leq \left(\frac{e^2}{4k}\right)^t$$

using the well known approximations $1-x \leq e^{-x}$ and $\binom{n}{a} \leq (\frac{en}{a})^a$. Clearly then, for sufficiently large k and t, this probability is $o(1)$. Hence, with high probability there is no independent set of size $2t$, which means exactly that $\alpha(G_{k,t,p}) \leq 2t$. ∎

Lemma 3.4. *If $p = \frac{3\log k}{t}$, and $t \geq (6\log k)^k$, then the expected number by bypasses in $G_{k,t,p}$ is at most t.*

Proof. Once again, since the vertices of any path come from sets with increasing indexes, the vertices of a bypass must come from distinct sets. Hence, the number of bypasses with g vertices is at most $\binom{k}{g}t^g$, since there are $\binom{k}{g}$ choices of sets, and within each set there are t choices of vertices. Each of these potential bypasses exists in $G_{k,t,p}$ with probability p^g, since each of the g edges exists independently with probability p. Hence, the expected number of bypasses in $G_{k,t,p}$ is at most

$$\sum_{g=3}^{k}\binom{k}{g}p^g t^g \leq (1+pt)^k \leq (2pt)^k = (6\log k)^k \leq t$$

as desired. ∎

These two lemmas give us the pieces necessary to prove Theorem 3.1.

Proof of Theorem 3.1. Consider the graph $G_{k,t,p}$ with

$$k = \left\lfloor \frac{\log n}{2\log\log\log n} \right\rfloor \quad \text{and} \quad t = \left\lceil \frac{n}{k} \right\rceil = \Theta\left(n\frac{\log\log\log n}{\log n}\right)$$

and $p = \frac{3\log k}{t}$. Note that $kt \geq n$. The choice of p implies that Lemma 3.3 applies to $G_{k,t,p}$, which means that for sufficiently large n, $\alpha(G_{k,t,p}) \leq 2t$ with probability at least $3/4$.

Next, we show that Lemma 3.4 also applies to $G_{k,t,p}$. Noting that $\log k \leq \log\log n$, $(\log\log n)^{\frac{\log n}{2\log\log\log n}} = n^{\frac{1}{2}}$, and $6^{\frac{\log n}{2\log\log\log n}} = n^{\frac{\log 6}{\log\log\log n}} = n^{o(1)}$, we have

$$(6\log k)^k \leq (6\log\log n)^{\frac{\log n}{2\log\log\log n}} \leq n^{\frac{1}{2}+o(1)} \leq t$$

for sufficiently large n, since $t = n^{1-o(1)}$. Thus, Lemma 3.4 indeed applies. Hence, by Markov's inequality, with probability less than $1/2$, the number of bypasses in $G_{k,t,p}$ is greater than $2t$.

Since the probability that $\alpha(G_{k,t,p}) > 2t$ is at most $1/4$, it follows by the union bound that with positive probability, we have both that $\alpha(G_{k,t,p}) \leq 2t$ and the number of bypasses is at most $2t$. Suppose that G is one such realization of $G_{k,t,p}$, with one edge removed from each bypass, and some $kt-n$ vertices removed so that the total number of vertices is n. Then G has no bypasses or cycles, so by Theorem 2.4, G is a Hasse digram. Furthermore, deleting $2t$ edges can only increase the independence number of the graph by at most $2t$ and deleting vertices cannot increase the independence number, so it follows that $\alpha(G) \leq 4t = O(n^{\frac{\log\log\log n}{\log n}})$. Thus, G is a Hasse digram on n vertices with independence number $O(n^{\frac{\log\log\log n}{\log n}})$, as desired. ■

To motivate our improvements to Construction 3.2 in Section 3.2, we discuss some of its bottlenecks.

The key inefficiency of Construction 3.2 is that p is the same for all pairs of vertices. Intuitively, if the same p is used for every pair, then for a given vertex $v \in V_i$, the number of vertices in V_{i+d} reachable from v is at least about $(pt)^d$ (just considering edges that go from V_j to V_{j+1}). This means that for there not to be too many bypasses in expectation, we need for $(pt)^d$ to be much smaller than t, otherwise it's likely that the neighbors of v in V_{i+d} are also reachable from v via a path of length d. Since Lemma 3.3 forces us to have $pt = \Omega(\log k)$, these two constraints make it impossible for us to make t smaller than $(\log k)^{\Theta(k)}$ (considering the worst case $d \approx k$). However, to improve our bound to $f(n) = O(\frac{n}{\log n})$, we would need $t = 2^{O(k)}$.

As we'll see in Section 3.2, if we carefully make vertices in nearby rows less likely to be adjacent than vertices in distant rows, we can make the graph on average much denser (and so, giving a similar guarantee to Lemma 3.3), while still ensuring that the expected number of bypasses is small.

3.2 $f(n) = O(\frac{n}{\log n})$

In this section, we present the following improvement to Theorem 3.1, shown independently by Tomon and Pach, and Mani and Ramakrishnan [PT19, MR19].

Theorem 3.5. *For each n, there exists a Hasse diagram on n vertices with independence number $O(\frac{n}{\log n})$.*

We consider the following random directed graph construction that will enable us to give the desired upper bound.

Construction 3.6. *Let $Q_{k,t}$ be a directed graph on kt vertices constructed as follows. The vertices of $Q_{k,t}$ are split into of k sets $V_1, ..., V_k$ of t vertices each. Then, each vertex $v \in V_i$ is adjacent to 2^{j-i} uniformly randomly chosen elements from V_j for each $j > i$, with all edges oriented from V_i to V_j.*

Once again, we can visualize $Q_{k,t}$ as a $k \times t$ grid of vertices, with the ith row representing the set V_i. An example is pictured in Figure 3. Since this grid representation is especially helpful for understanding Construction 3.6, we'll often refer to V_i as row i.

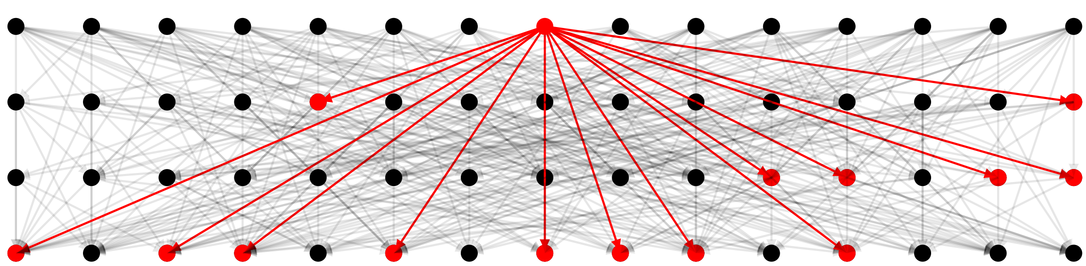

Figure 3. An example $Q_{k,t}$ with $k = 4$ and $t = 15$. The neighborhood of one vertex in V_1 is shown in red.

Like Construction 3.2, $Q_{k,t}$ is necessarily acyclic, and has independence number at least t. We will show that when $t \geq 10^k$ (any sufficiently large constant can replace 10 without changing the analysis), with high probability, $\alpha(Q_{k,t}) = O(t)$, and $Q_{k,t}$ can be made into a Hasse diagram without deleting too many vertices.

The argument is similar in structure to the proof of Theorem 3.1:

In Lemma 3.12 we show that with high probability, $\alpha(Q_{k,t}) = O(t)$. The edge density between pairs of rows in $Q_{k,t}$ increases exponentially in the distance between the rows. Consequently, a vertex subset of $Q_{k,t}$ with vertices spread across many rows of $Q_{k,t}$ is very unlikely to be an independent set. As a result, the vertex subsets that are most likely to be independent sets are concentrated on a few rows. However, among all the vertex subsets of a fixed size, only a small fraction are concentrated on a few rows, while many are spread out. By balancing these two cases, we can attain the desired bound on $\alpha(Q_{k,t})$.

In Lemma 3.13 we show that in expectation, $Q_{k,t}$ can be made into a Hasse diagram by deleting $o(t)$ vertices. This is done via a probabilistic argument similar to the proof of Lemma 3.4. Combining these two results yields Theorem 3.5.

We study vertex subsets S of size ct for some sufficiently large constant $c > 0$, with the goal of showing that with positive probability none of these sets are independent sets. We group these different vertex subsets by *configurations*, which behave equivalently in our probabilistic setting.

Definition 3.7. A vertex subset $S \subset Q_{k,t}$ has *configuration* $(a_1, \ldots, a_k)$ if it has a_i elements in row i. For a fixed configuration $(a_1, \ldots, a_k)$ let $x_i := a_i/t$, so $x_i \in [0, 1]$.

For each configuration, we bound the probability that a set of vertices in that configuration is an independent set.

Lemma 3.8. *Let $S \subset Q_{k,t}$ be a subset of vertices in configuration $(a_1, ...a_n)$. Then the probability that S is an independent set is bounded above by*

$$\exp\left(-t\sum_{i<j} 2^{j-i}x_i x_j\right).$$

Proof. We consider the probability that a fixed vertex in row i is adjacent to any vertex in row j. Let $v \in S$ be a vertex in row i. Recall that v is adjacent to 2^{j-i} elements of row j chosen uniformly at random. Hence, the probability that no vertex in row j is adjacent to v is

$$\frac{\binom{t-a_j}{2^{j-i}}}{\binom{t}{2^{j-i}}} \le \left(1 - \frac{a_j}{t}\right)^{2^{j-i}}.$$

Repeating this analysis over all a_i vertices in row i and all rows $i < j$ gives the following upper bound on the probability that S is an independent set

$$\Pr(S \text{ is an independent set}) \le \prod_{i<j}\left(1 - \frac{a_j}{t}\right)^{2^{j-i}a_i} \le \exp\left(-\sum_{i<j} 2^{j-i}\frac{a_i a_j}{t}\right) = \exp\left(-t\sum_{i<j} 2^{j-i}x_i x_j\right),$$

as desired. ∎

From Lemma 3.8, we observe that a set of vertices in configuration $(a_1, ..., a_k)$ is less likely to to be an independent set when $\sum_{i<j} 2^{j-i}x_i x_j$ is large. The following notion allows us to categorize configurations by the order of magnitude of this expression.

Definition 3.9. A configuration $(a_1, ..., a_k)$ is *of type* ℓ if ℓ is the smallest positive integer such that there exist ℓ consecutive rows $A, A+1, ..., B$ (with $B = A + \ell - 1$) such that for each $r \geq 0$,

$$x_{A-r} \leq \frac{1}{2^r}, x_{B+r} \leq \frac{1}{2^r}.$$

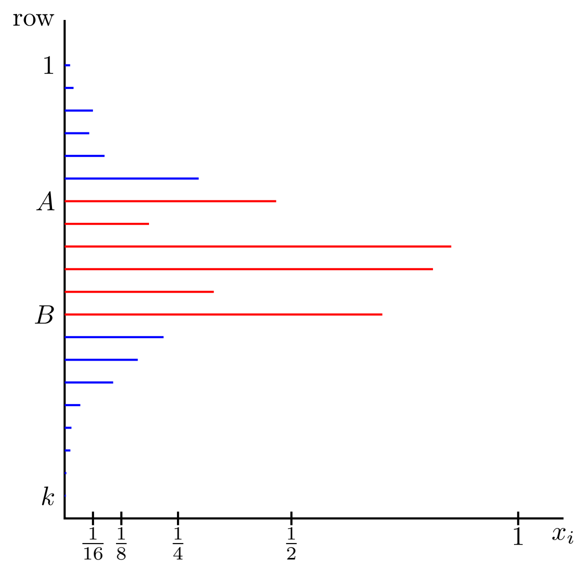

Figure 4. The values of x_i for each row in an example configuration of type 6

Intuitively, a configuration is of type ℓ when there are ℓ rows where elements can be concentrated, but the density of the other rows decreases exponentially as these rows get farther from the concentrated region. This is depicted in Figure 4. Configurations of higher type are less likely to be independent sets, as captured by the following lemma.

Lemma 3.10. *If $S \subset Q_{k,t}$ with $|S| = ct$ for $c > 3$ has a configuration $(a_1, ..., a_k)$ of type ℓ, then*

$$\sum_{i<j} 2^{j-i} x_i x_j \geq 2^{\ell-3}.$$

Proof. Since $|S| = ct$, we have $\sum_{i=1}^{k} a_i = ct$, and so $\sum_{i=1}^{k} x_i = c$. Note that

$$\sum_{r=1}^{A-1} x_{A-r} + \sum_{r=1}^{k-B} x_{B+r} \leq \sum_{r=1}^{A-1} \frac{1}{2^r} + \sum_{r=1}^{k-B} \frac{1}{2^r} < 2.$$

Recall that $x_i \leq 1$ for each x_i. Thus, we have that

$$\ell \geq \sum_{i=A}^{B} x_i \geq c - \left(\sum_{r=1}^{A-1} x_{A-r} + \sum_{r=1}^{k-B} x_{B+r} \right) > c - 2,$$

and in particular, $A < B - 1$. We claim that by the minimality of ℓ in Definition 3.9, there must exist $r, s \geq 0$

such that

$$\frac{1}{2^{r+1}} \leq x_{A-r} \leq \frac{1}{2^r} \quad \text{and} \quad \frac{1}{2^{s+1}} \leq x_{B+s} \leq \frac{1}{2^s}.$$

Suppose this is not the case. Then, either for all $r \geq 0$, $x_{A-r} < \frac{1}{2^{r+1}}$ or for all $s \geq 0$, $x_{B+s} < \frac{1}{2^{s+1}}$. Since $x_{A+1}, x_{B-1} \leq 1$, the first case we can replace A with $A+1$ in Definition 3.9 and in the second case we can replace B with $B-1$. Thus, $(a_1, \ldots, a_k)$ is of type at most $\ell - 1$, which is a contradiction. This proves the claim, and yields

$$\sum_{i<j} 2^{j-i} x_i x_j \geq 2^{\ell-1+r+s} x_{A-r} x_{B+s} \geq 2^{\ell-3}$$

as desired. ∎

Next, we bound the total number of vertex sets S of size ct in a configuration of type ℓ.

Lemma 3.11. *The number of sets S with $|S| = ct$ in a configuration of type ℓ is $\exp((4+\ell)t + o(t))$.*

Proof. There are $k - \ell + 1$ ways to choose the rows A, B with $B - A = \ell - 1$ satisfying the conditions of Definition 3.9. Given A and B, we consider the number of ways to choose the vertices in rows $A, A+1, \ldots, B$. The total number of vertices in these rows is βt where $(c-2)t \leq \beta t \leq ct$. Thus, by Stirling's approximation, we can bound the total number of ways to choose the A, B, and the vertices in rows $A, A+1, \ldots, B$ by

$$(k - \ell + 1) \sum_{\beta=(c-2)t}^{ct} \binom{\ell t}{\beta} \leq 2t \cdot k \cdot \binom{\ell t}{\ell t/2} \leq 2t \cdot k \cdot 4^{\ell t/2} \leq \exp(\ell t \log 2 + o(t)).$$

From each row $A - r$ for $r \geq 1$ we choose most $t/2^r$ vertices. Hence, the number of ways to choose the elements of row $A - r$ is

$$\sum_{i=0}^{t/2^r} \binom{t}{i} \leq \left(\frac{t}{2^r} + 1\right) \binom{t}{t/2^r} \leq t \cdot (e \cdot 2^r)^{t/2^r} \leq \exp\left(\frac{t}{2^r} + \frac{rt}{2^r} \log 2 + \log t\right).$$

The number of ways to choose the vertices from rows $A-1, A-2, \ldots, 1$ is at most

$$\prod_{r=1}^{A-1} \exp\left(\frac{t}{2^r} + \frac{rt}{2^r} \log 2 + \log t\right) \leq \exp\left(t \sum_{r=1}^{A-1} \frac{1}{2^r} + t \log 2 \sum_{r=1}^{A-1} \frac{r}{2^r} + k \log t\right) \leq \exp\left(2t + o(t)\right).$$

We can apply an identical argument to show that the number of ways to pick vertices from rows $B+1, B+2, \ldots, k$ is at most $\exp\left(2t + o(t)\right)$.Thus, the total number of ways to choose a subset S of ct elements with configuration of type ℓ is at most $\exp\left((4+\ell)t + o(t)\right)$ as desired. ∎

We now have all the necessary pieces to bound the independence number of $Q_{k,t}$.

Lemma 3.12. *With high probability $\alpha(Q_{k,t}) = O(t)$.*

Proof. Lemmas 3.8 and 3.10 show that a vertex set S of ct elements in a configuration of type ℓ is an independent set with probability at most

$$\exp\left(-t \sum_{i<j} 2^{j-i} x_i x_j\right) \leq \exp\left(-2^{\ell-3} t\right).$$

Using Lemma 3.11 and the union bound, the probability that any such set is an independent set is at most $\exp\big((4+\ell)t - 2^{\ell-3}t + o(t)\big)$. Noting that ℓ must be between $c-2$ and k, it follows that the probability that there are any independent sets of size ct is at most

$$\sum_{\ell=c-2}^{k} \exp\big((4+\ell)t - 2^{\ell-3}t + o(t)\big).$$

We can choose c sufficiently large such that for any $\ell \geq c-2$, we have $(4+\ell)t - 2^{\ell-3}t + o(t) < -t\ell$. Then,

$$\sum_{\ell=c-2}^{k} \exp\big((4+\ell)t - 2^{\ell-3}t + o(t)\big) \leq \sum_{\ell=c-2}^{k} \exp(-\ell t) \leq \frac{e^{-t}}{1-e^{-t}} = \frac{1}{e^t - 1},$$

which can be made arbitrarily small by modifying k. Thus, with high probability, there are no independent sets of size ct in $Q_{k,t}$ as desired. ■

Since the graph $Q_{k,t}$ has high independence number with high probability, it remains to show that this graph can be made into a Hasse diagram without affecting the independence number asymptotically.

Lemma 3.13. *If $t \geq 10^k$, the expected number of vertices in $Q_{k,t}$ that are the start of some bypass is $o(t)$.*

Proof. For a vertex v in row i, and a row $j = i + d$, say that the pair (v, j) is *bad* if there is a bypass starting with v and ending with some vertex in row j, and let $p_{v,j}$ be the probability that (v, j) is bad. Finally, let X_B the expected number of bad pairs (v, j). We aim to bound X_B.

We consider the total number of paths $v, u_1, u_2, \ldots, u_m$ where u_m is in row j. Suppose that $u_1, u_2, ..., u_{m-1}$ are from some fixed rows $i_1, i_2, ..., i_{m-1}$. For ease we also define $i_0 := i$ and $i_m := j$. Since each vertex in row i_m is adjacent to $2^{i_{m+1}-i_m}$ elements in row i_{m+1}, the total number of such paths is

$$\prod_{m=0}^{m-1} 2^{i_{m+1}-i_m} = 2^{j-i} = 2^d.$$

Next, the total number of ways to choose the rows $i_1, i_2, ..., i_{m-1}$ is $2^{d-1} - 1$, since we can pick any subset of the rows $i+1, i+2, ..., j-1$ besides the empty set. Thus, the total number of paths starting with v and ending in row j is $2^d(2^{d-1} - 1) \leq 4^d$.

There is a bypass starting at v and ending in row j if and only if v is adjacent to one of the at most 4^d ends of these paths. Since v is adjacent to a random set of 2^d vertices in row j, we have

$$p_{v,j} \leq 1 - \frac{\binom{t-4^d}{2^d}}{\binom{t}{2^d}} \leq 1 - \left(\frac{t - 4^d - 2^d - 1}{t - 2^d + 1}\right)^{2^d} = 1 - \left(1 - \frac{4^d}{t - 2^d + 1}\right)^{2^d}.$$

Recalling that $1 - x \geq \exp(\frac{-x}{1-x})$, and using the fact that $1 - e^{-x} \leq x$, we have,

$$p_{v,j} \leq 1 - \exp\left(-\frac{8^d}{t - 4^d - 2^d + 1}\right) \leq \frac{8^d}{t - 4^d - 2^d + 1}.$$

Since $t = 10^k$, for k sufficiently large, the above expression is maximized when $d = k$. Overall, there are kt

choices for v and k choices for j, so we can bound

$$\mathbf{E}[X_B] = \sum_{v,j} p_{v,j} \leq k^2 t \frac{8^k}{t - 4^k - 2^k + 1} =\leq k^2 8^k \frac{t}{t - 4^k - 2^k + 1} \leq 2k^2 8^k = o(t)$$

for k sufficiently large. The number of vertices that are the starts of bypasses is bounded above by X_B, so the lemma follows. ■

Now we have all the pieces necessary to prove Theorem 3.5.

Proof of Theorem 3.5. We construct a Hasse diagram on n vertices as follows. We start with an instance of $Q_{k,t}$ with

$$k = \left\lfloor \frac{1}{2} \log_{10} n \right\rfloor \quad \text{and} \quad t = \left\lceil \frac{2n}{k} \right\rceil = \Theta\left(\frac{n}{\log n}\right).$$

Note that $10^k \leq n^{1/2} \leq t$ for sufficiently large n, since $t = n^{1-o(1)}$. This means that we can apply Lemmas 3.13 and 3.12 to show that with positive probability, $Q_{k,t}$ has independence number $O(t)$ and at most $o(t)$ vertices are the start of a bypass.

Then, we consider deleting those vertices that are the start of a bypass. The resulting graph then has no cycles or bypasses, so by Theorem 2.4, the graph is a Hasse diagram. Furthermore, the total number of vertices is $kt - o(t) \geq kt/2 \geq n$ for sufficiently large n. Then, we can remove any additional vertices to ensure that the total number of vertices is n. Since deleting vertices cannot increase the independence number, the result is a Hasse diagram on n vertices and independence number $O(t) = O(\frac{n}{\log n})$ as desired. ■

3.3 $f(n) = O(n^{3/4})$

In this section, we study a deterministic construction that Suk and Tomon used to show the following improvement to Theorems 3.1 and 3.5.

Theorem 3.14 ([ST20])**.** *For each n, there exists a Hasse diagram on n vertices with independence number $O(n^{3/4})$.*

Their strategy is to build a poset using the incidence structure of points and lines in the plane, and then bound the independence number of this poset by the number of points and lines. This approach is similar to that used to find explicit constructions of triangle free graphs with small independence number, such as in [CPR00, KPR10]. The key to getting a tight bound is to find constructions of points and lines with a large number of incidences, which is a well understood problem in combinatorial geometry.

Definition 3.15. For a point p and a line ℓ in the plane, let $x(p)$ denote the x-coordinate of p, and let $s(\ell)$ denote the slope of ℓ.

Definition 3.16. For a set of points P and lines L in the plane, let the *incidence set* $\mathcal{I}(P, L)$ denote the set of pairs $(p, \ell) \in P \times L$ such that p is on ℓ.

Note that given a set of points P and lines L, we can adjust the points and lines slightly so that the x-coordinates of the points and slopes of the lines are distinct, without changing $\mathcal{I}(P, L)$. Henceforth, we assume that such an adjustment is made for any set of points and lines that we consider.

Given this, we construct the following directed graph on the incidence set of P and L:

Construction 3.17. *Given a set of points P and lines L in the plane, let $G_{P,L}$ be the directed graph whose vertex set is $\mathcal{I}(P,L)$, and whose edges are the pairs $((p,\ell),(p',\ell'))$ such that $x(p) < x(p')$, $s(\ell) < s(\ell')$, and p' is on ℓ.*

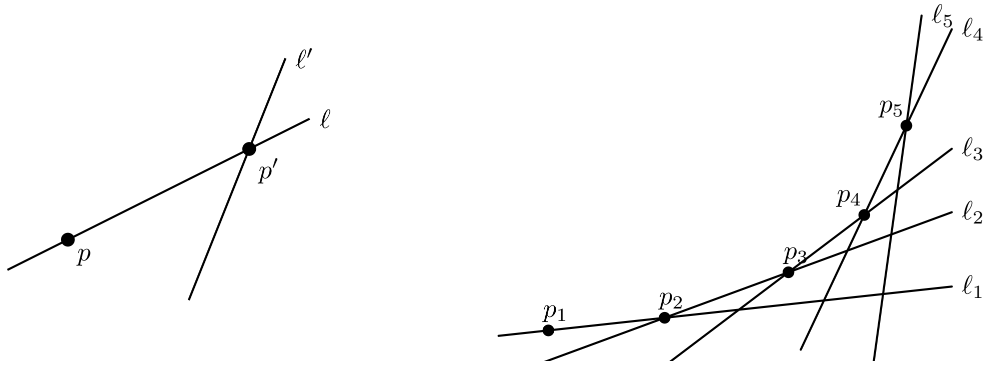

Figure 5. On the left, a configuration of points $p, p' \in P$ and lines $\ell, \ell' \in L$ such that $((p,\ell),(p',\ell'))$ is an edge of $G_{P,L}$. On the right, the incidences (p_i, ℓ_i) for $1 \leq i \leq 5$ form a path.

Lemma 3.18. *For any set of points P and lines L in the plane, $G_{P,L}$ is a Hasse diagram.*

Proof. By Theorem 2.4, it suffices to show that $G_{P,L}$ has no cycles and no bypasses.

For any path $(p_1,\ell_1), ..., (p_k,\ell_k)$ in the graph with $k \geq 3$, the sequence $x(p_1), ..., x(p_k)$ is strictly increasing, so $x(p_1) < x(p_k)$. This means that it is not possible that $((p_k,\ell_k),(p_1,\ell_1))$ is an edge of the graph, and so $G_{P,L}$ has no cycles.

Similarly, the fact that the sequence $s(\ell_1), ..., s(\ell_k)$ is strictly increasing means that the points $p_1, p_2, ..., p_k$ form a convex polygon. In particular, this means that no three of these points are collinear. Hence, it is impossible for $((p_1,\ell_1),(p_k,\ell_k))$ to be an edge of the graph, since this would imply that p_1, p_2, and p_k all lie on ℓ_1. This means that $G_{P,L}$ has no bypasses, so it is indeed a Hasse diagram. ■

Next, we bound the size of independent sets in $G_{P,L}$.

Lemma 3.19. *We have $\alpha(G_{P,L}) \leq |P| + |L|$.*

Proof. For ease, let $|P| = N$ and $|L| = M$. Suppose towards a contradiction that $S \subseteq \mathcal{I}(P,L)$ is a set of $M + N + 1$ incidences that form an independent set in $G_{P,L}$.

Let P_S be the set of points such that there exists ℓ for which $(p,\ell) \in S$. For each point $p \in P_S$ let ℓ_p be the line such that $(p,\ell_p) \in S$ and $s(\ell_p)$ is maximal. Let $S' = S \setminus \{(p,\ell_p) : p \in P_S\}$.

Since $|P_S| \leq N$, we know that $S' \geq M+1$. Hence, by the pigeonhole principle, there exists a line ℓ that is involved in two incidences of S'. Suppose that $(p,\ell),(p',\ell) \in S'$, and $x(p) < x(p')$. But then the incidences, (p,ℓ) and $(p',\ell_{p'})$ are both in S, and satisfy $x(p) < x(p')$, $s(\ell) < s(\ell_{p'})$, and p' is on ℓ. This means that $((p,\ell),(p',\ell_{p'}))$ is an edge of $G_{P,L}$, which contradicts the assumption that S is an independent set. ■

Lemmas 3.18 and 3.19 imply that we can construct Hasse diagrams with small independence number if we can construct configurations in the plane with a small number of points and lines, but a large number of incidences. This is a well studied problem in combinatorial geometry, and the following theorem of Szemerédi and Trotter [ST83] tells us the best possible bound:

Theorem 3.20 (Szemerédi-Trotter)**.** *Given N points and M lines in the plane, the total number of point-line incidences is at most $O((MN)^{2/3} + M + N)$.*

For example, if $|L| = M$, $|P| = O(M)$ (which gives the best bound in Lemma 3.19), the maximum number of incidences we can get is $O(M^{4/3})$. One such construction that meets this bound is the following.

Construction 3.21. *We construct a set of $M = k^3$ lines and $2M$ points as follows*

$$P = \{(a, b) : a \in [1, k], b \in [1, 2k^2]\}$$
$$L = \{y = ax + b : a \in [1, k], b \in [1, k^2]\}$$

In the above construction, each line $y = ax + b$ in L is incident to exactly the k points $(x, ax + b)$ for $x \in [1, k]$. Each of these points is in P since $x \in [1, k]$ and $1 \leq ax + b \leq 2k^2$. Hence, the total number of incidences is $k^4 = M^{4/3}$ as claimed.

This gives us all of the pieces necessary to prove Theorem 3.14.

Proof of Theorem 3.14. We construct a Hasse diagram on n vertices with independence number $O(n^{3/4})$. Let $k = \lceil n^{1/4} \rceil$, and construct P, L as in Construction 3.21. Note that $|P|, |L| = O(k^3)$. Consider the directed graph $G_{P,L}$. Then $G_{P,L}$ is a graph on $k^4 \geq n$ vertices. By Lemmas 3.18 and 3.19, we know that $G_{P,L}$ is a Hasse diagram, and $\alpha(G_{P,L}) = O(k^3) = O(n^{3/4})$. Then we can remove any $k^4 - n$ vertices from $G_{P,L}$ to get a Hasse diagram on n vertices with independence number $O(n^{3/4})$, as desired. ∎

3.4 Another construction

In this section, we present a number theoretic construction of a poset whose Hasse diagram has small independence number. While the bounds we show do not give any improvement on the previous upper bounds we've shown for $f(n)$, it remains possible that future work may give such an improvement.

Construction 3.22. *Let $P_m = (X_m, <)$ be a poset on m^3 elements with $X_m = [m^2] \times [m]$, and ordering $<$ where*

$$(x_1, y_1) < (x_2, y_2) \text{ if and only if } y_1 < y_2 \text{ and } |x_2 - x_1| \leq (y_2 - y_1)^2.$$

It is easy to see that for P_m, the operation $<$ is irreflexive and antisymmetric. To see that it is transitive, suppose that $(x_1, y_1) < (x_2, y_2)$ and $(x_2, y_2) < (x_3, y_3)$. Then we know that $y_1 < y_2$ and $y_2 < y_3$, so $y_1 < y_3$. By the triangle inequality, we have

$$|x_3 - x_1| \leq |x_2 - x_1| + |x_3 - x_2| \leq (y_2 - y_1)^2 + (y_3 - y_2)^2 \leq (y_3 - y_1)^2$$

since $a^2 + b^2 \leq (a + b)^2$ when $a, b \geq 0$, and we can take $a = y_2 - y_1$ and $b = y_3 - y_2$. Hence, P_m is indeed a valid poset.

We claim that $H(P_m)$ has the following description.

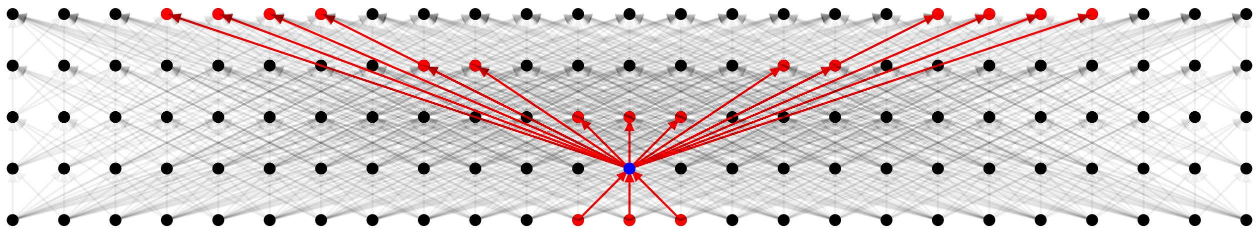

Figure 6. The Hasse diagram $H(P_5)$. The in and out neighbors of the blue vertex are shown in red.

Claim 3.23. *For an element* $(x, y) \in X_m$*,* $H(P_m)$ *has the edge* $((x, y), (x + d_x, y + d_y))$ *if and only if* $(x + d_x, y + d_y) \in X_m$ *and*

- $(d_x, d_y) \in \{(-1, 1), (0, 1), (1, 1)\}$*, or*
- $d_y > 1$ *and* $(d_y - 1)^2 + 1 < |d_x| \leq d_y^2$*.*

Proof. The claim is easy to verify for the first bullet point when $d_y = 1$. Next, we consider $d_y > 1$, and due to symmetry, it suffices to consider $d_x \geq 0$. By definition, if $d_x > d_y^2$, then $((x, y), (x + d_x, y + d_y))$ cannot be in the Hasse diagram. Similarly, if $d_x \leq (d_y - 1)^2 + 1$, then $d_x - 1 \leq (d_y - 1)^2$ and so we have

$$(x, y) < (x + 1, y + 1) < (x + d_x, y + d_y).$$

This means that (x, y) is not an immediate predecessor of $(x + d_x, y + d_y)$, and so once again the edge is not in the Hasse diagram. This is depicted in Figure 7.

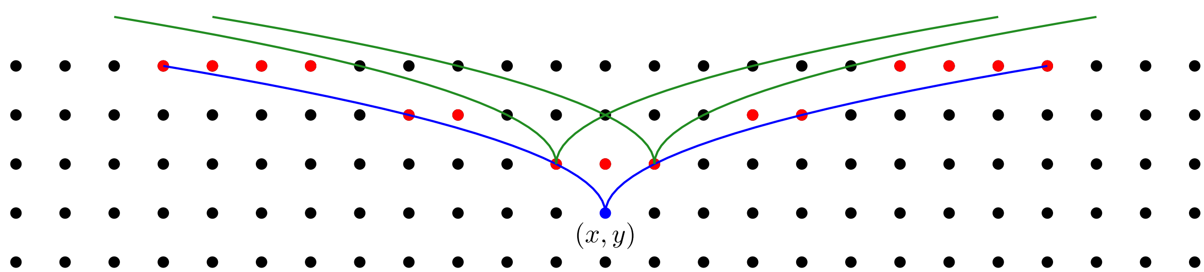

Figure 7. The points above the blue curves are greater than (x, y). The points above the green curves are greater than $(x - 1, y + 1)$ and $(x + 1, y + 1)$. The red points are those that are adjacent to (x, y) in the Hasse diagram, via out edges from (x, y).

It follows that the only possible edges in the Hasse diagram are those described in the claim. We show that these edges are indeed in the Hasse diagram. Suppose that $d_y > 1$ and $(d_y - 1)^2 + 1 < d_x \leq d_y^2$, but there exists r_x and r_y such that

$$(x, y) < (x + r_x, y + r_y) < (x + d_x, y + d_y). \tag{3.1}$$

By using the fact that the $<$ operator is invariant under translation, we also have that

$$(x, y) < (x + r_y^2, y + r_y) < (x + r_y^2 - r_x + d_x, y + d_y).$$

However, the maximum value of x' such that $(x + r_y^2, y + r_y) < (x', y + d_y)$ is $x' = x + r_y^2 + (d_y - r_y)^2$. It follows that

$$r_y^2 + (d_y - r_y)^2 \geq r_y^2 - r_x + d_x \geq d_x > (d_y - 1)^2 + 1$$

using the fact that $r_y^2 \geq r_x$. However, the inequality $a^2+b^2 > (a+b-1)^2+1$ is equivalent to $0 > 2(a-1)(b-1)$, which is false if $a, b \geq 1$. Applying this with $a = r_y$, and $b = d_y - r_y$, we get a contradiction since (3.1) implies that $r_y > 0$ and $d_y > r_y$. ∎

While it is difficult to bound $\alpha(H(P_m))$ directly, if we consider the subgraph of $H(P_m)$ with just the edges of the form $((x, y), (x+d^2, y+d))$, then we can apply the following result of Lyall and Magyar [LM09]:

Theorem 3.24. *If $B \subseteq [1, N]^k$ and $(d, d^2, ..., d^k) \notin B - B$ for any $d \neq 0$, then we necessarily have*

$$\frac{|B|}{N^k} \ll C \left(\frac{\log \log N}{\log N}\right)^{\frac{1}{k-1}}$$

for some constant C depending on k.

Corollary 3.25. *If $n = |X_m| = m^3$, then $\alpha(H(P_m)) \leq O(n\frac{\log \log n}{\log n})$*

Proof. Using $k = 2$ and $N = m$, it follows that any subset of the vertices if $H(P_m)$ that avoids edges of the form $((x, y), (x + d^2, y + d))$ must have density at most $O(\frac{\log \log m}{\log m}) = O(\frac{\log \log n}{\log n})$ (since it must have this density for any $m \times m$ subgrid). It follows that $\alpha(H(P_m)) \leq O(n\frac{\log \log n}{\log n})$. ∎

While qualitatively, this is not a better bound than even Theorem 3.1, we note that its proof uses a subgraph of $H(P_m)$ with a $o(1)$ fraction of the edges. This suggests that a much better bound is possible, and Corollary 3.25 should be taken as a proof of concept that $H(P_m)$ has independence number $o(n)$. We note that the trivial lower bound of $\alpha(H(P_m)) = \Omega(n^{2/3})$ leaves room for this construction to improve upon the $O(n^{3/4})$ bound of Suk and Tomon.

4 Hasse diagrams of dimension-2 posets

While Section 3 shows a considerable amount of progress in upper bounding $f(n)$ for general Hasse diagrams, the constructions we studied are of posets that are typically not low dimensional. Thus, they say nothing about $f_2(n)$, the minimum independence number of Hasse diagrams restricted to posets of dimension 2, which is of particular interest because of its connection to conflict-free colorings and Delaney graphs.

In this section, we show the best nontrivial bound that is known, due to Chen, Pach, Szegedy, and Tardos [CPST09].

Theorem 4.1. *For each n, there exists a poset P on n elements such that $\dim(P) = 2$ and $\alpha(H(P)) \leq O(n\frac{\log^2 \log n}{\log n})$.*

Note that this bound is weaker than even the $O(n\frac{\log \log \log n}{\log n})$ bound from Theorem 3.1, which is due to the fact that upper bounds on $f_2(n)$ consider constructions on a more restricted set of graphs with less freedom.

Proof. The main idea is to construct a set X of n points in the square $[0,1] \times [0,1]$ randomly, and consider the poset $P = (X, \leq)$ where $(x, y) \leq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$. We show that with positive probability, $H(P)$ has independence number $O(n^{\frac{\log^2 \log n}{\log n}})$.

Suppose that $X = \{(\frac{i}{n}, y_i) : 1 \leq i \leq n\}$, where each y_i is chosen independently and uniformly at random from $[0,1]$. Let $p_i = (\frac{i}{n}, y_i)$. Note then that $p_i < p_j$ if and only if $i < j$ and $y_i < y_j$.

For ease of analysis, we suppose that each y_i is generated in the following way. For some $L \geq 2$ to be chosen later, we consider y_i represented in base L:

$$y_i = (0.d_i^{(1)} d_i^{(2)} \cdots)_L.$$

Then we imagine that the digits of each y_i are one chosen at time in a series of stages, where in the tth stage, $d_i^{(t)}$ is chosen uniformly at random from $\{0, 1, ..., L-1\}$ for each i. Let $y_i^{(t)}$ denote the partial value of y_i just before stage t, i.e.,

$$y_i^{(t)} = (0.d_i^{(1)} d_i^{(2)} \cdots d_i^{(t-1)})_L.$$

The key observation is the following. Suppose that for some time t, and some $i < j$ the following three conditions are true:

1. $y_i^{(t)} = y_j^{(t)}$,
2. $y_k^{(t)} \neq y_i^{(t)}$ for all $i < k < j$, and
3. $d_i^{(t)} < d_j^{(t)}$.

Then we claim that (p_i, p_j) is an edge of $H(P)$. Conditions (1) and (3) imply that $p_i < p_j$. Suppose towards a contradiction that there exists some p_k such that $p_i < p_k < p_j$. Then, by condition (1), p_i and p_j agree on first $t-1$ digits, so p_k must also agree on those digits. However, this is prohibited by condition (2), so the claim is true.

Say that the edge (p_i, p_j) is *forced at stage* t if the above conditions occur, and let this event be denoted $A_{i,j}^{(t)}$. Our goal is to show that any sufficiently large subset of X will eventually have a forced edge with high probability. We note that $H(P)$ may have edges besides forced edges, but for the purpose of showing that large independent sets are unlikely, considering forced edges is enough. In fact, we'll even only consider edges that are forced at even stages t, since this makes the analysis a little easier.

Suppose we have a set $I \subseteq [n]$ of indexes. We'll bound the probability that $X|_I = \{p_i : i \in I\}$ is an independent set in $H(P)$.

Stage $t-1$ is the key stage which determines if the desired conditions take place in stage t (since in this stage, $y_i^{(t)}$ is determined for each i). Just before stage $t-1$, each value $y_i^{(t-1)}$ can take on one of L^{t-2} values, since $t-2$ digits have been chosen for each p_i at this point. For each possible value y, let

$$H_y^{(t-1)} = \{i \in [n] : y_i^{(t-1)} = y_i\}.$$

These sets partition $[n]$ (and by extension, I) into at most L^{t-2} parts. Say that $i, j \in I$ are *neighbors* if for some y, $i, j \in H_y^{(t-1)}$, and for each $k \in I$ such that $i < k < j$, $k \notin H_y^{(t-1)}$. For neighbors i, j we also let

$$S_{i,j}^{(t-1)} = \{k \in H_y^{(t-1)} : i < k < j\}.$$

We say that i and j are *close* neighbors if $|S_{i,j}^{(t-1)}| \leq L$. Intuitively, i and j are close neighbors if the edge (p_i, p_j) is comparatively likely to be forced in stage t.

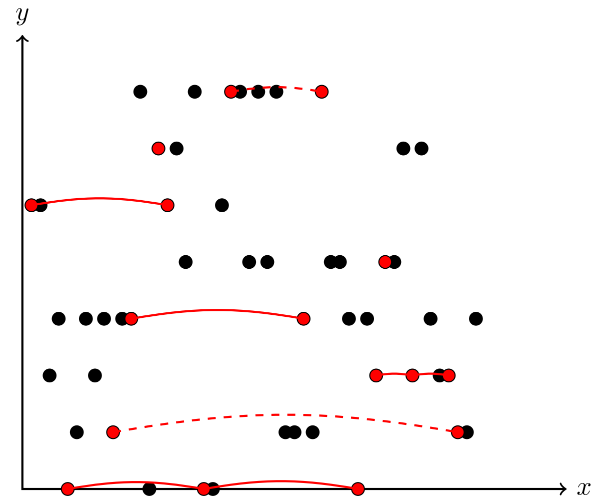

Figure 8. The above is an example state just before stage 4, with $n = 50$ and $L = 2$. Each set of points on the same horizontal line represents one set $H_y^{(4)}$, of which there are 8. The members of $X|_I$ are shown in red, neighbors that are not close are connected by dashed lines, and close neighbors are connected by solid lines.

If i and j are close neighbors, then for (p_i, p_j) to be forced in stage t the following three random events need to occur. First, in stage $t-1$, we need that $d_i^{(t-1)} = d_j^{(t-1)}$. Second, in stage $t-1$, we need that $d_k^{(t-1)} \neq d_i^{(t-1)}$ for each $k \in S_{i,j}^{(t-1)}$. Third, we need that in stage t, $d_i^{(t)} < d_j^{(t)}$. These three events occur with probability $\frac{1}{L}$, $\left(1-\frac{1}{L}\right)^{|S_{i,j}^{(t-1)}|}$, and $\frac{1}{2}\left(1-\frac{1}{L}\right)$ respectively, so overall the edge (p_i, p_j) is forced in stage t with probability

$$\Pr(A_{i,j}^{(t)}) = \frac{1}{2L}\left(1-\frac{1}{L}\right)^{|S_{i,j}^{(t-1)}|+1} \geq \frac{1}{2L}\left(1-\frac{1}{L}\right)^{L+1} \geq \frac{1}{16L}$$

since $L \geq 2$. Now suppose that at the start of stage $t-1$, we have m pairs of close neighbors i, j. Since each i can have at most two neighbors j, j' where $j' < i < j$, we can choose some set S^* of half of the m pairs of neighbors such that each $i \in I$ is only part of one pair (say, by sorting the pairs by the smaller element, and then taking every other pair in sorted order). Then, conditioning on the outcomes of stages $t' < t-1$, we have that for each $(i,j) \in S^*$, the events $A_{i,j}^{(t)}$ are independent since they depend on disjoint sets of digits. It follows that

$$\Pr\left(\bigcap_{(i,j)\in S^*} \overline{A_{i,j}^{(t)}} \;\middle|\; \text{outcomes of stages } t' < t-1\right) \leq \left(1-\frac{1}{16L}\right)^{m/2} \leq \exp\left(-\frac{m}{32L}\right). \tag{4.1}$$

i.e., the probability that it is still possible that $X|_I$ is an independent set after stage t given the outcomes of the stages before $t-1$ is at most $\exp\left(-\frac{m}{32L}\right)$.

Now we bound the number of close neighbors at the start of stage $t-1$. For each $i \in I$, as long as $i \neq \max H_y^{(t-1)}$ for some y, i has some neighbor $j > i$. The sets $S_{i,j}^{(t-1)}$ are disjoint for each pair of neighbors

i, j, so at most $\frac{n}{L}$ of them can be of size greater than L. Hence, the first statement eliminates at most L^{t-2} elements of I, and the second eliminates at most $\frac{n}{L}$ elements of I, so there are at least $|I| - \frac{n}{L} - L^{t-2}$ elements $i \in I$ that has a close neighbor $j > i$. i.e.,

$$m \geq |I| - \frac{n}{L} - L^{t-2}.$$

Thus, if we consider only $t \leq \log_L n$, and $|I| = \frac{3n}{L}$, we have $m \geq \frac{n}{L}$. Plugging this into (4.1), it follows that the probability that it is still possible that $X|_I$ is an independent set after stage t given the outcomes of the stages before $t-1$ is at most $\exp\left(-\frac{n}{32L^2}\right)$. Considering this for each even stage $t \leq \log_L n$, it follows that the probability that $X|_I$ is an independent set is at most

$$\exp\left(-\frac{n}{32L^2} \cdot \frac{\log_L n}{2}\right) = \exp\left(-\frac{1}{64} \cdot \frac{n \log n}{L^2 \log L}\right).$$

Union bounding over all possible sets $I \subseteq [n]$ of size $\frac{3n}{L}$, of which there are at most

$$\binom{n}{\frac{3n}{L}} \leq \left(\frac{eL}{3}\right)^{\frac{3n}{L}} \leq L^{\frac{3n}{L}}$$

we have that the probability that there is an independent set of size $\frac{3n}{L}$ is at most

$$\exp\left(\frac{3n \log L}{L} - \frac{1}{64} \cdot \frac{n \log n}{L^2 \log L}\right).$$

This is less than 1 as long as $192 L \log^2 L < \log n$, which is clearly true if we choose $L = \left\lfloor \frac{\log n}{200 \log^2 \log n} \right\rfloor$. Hence, with this choice, we have no independent set of size $\frac{3n}{L} = O(n \frac{\log^2 \log n}{\log n})$ with positive probability, as desired. ■

5 Conclusion

The recent work on bounding the independence numbers of Hasse diagrams has demonstrated why posets are such central objects in combinatorics. Just within Section 3, we saw examples of posets that had connections to probabilistic combinatorics, combinatorial geometry, and additive combinatorics.
Still, a number of questions remain open.

Question 5.1. Can the lower bound $f(n) = \Omega(\sqrt{n \log n})$ be improved?

Question 5.2. Can the upper bound $f(n) = O(n^{3/4})$ be improved?

Question 5.3. Can the lower bound $f_2(n) = \Omega(\sqrt{n \log n})$ be improved?

Question 5.4. Can the upper bound on $f_2(n)$ be improved to $O(\frac{n}{\log n})$ or better?

Question 5.5. For the graph $H(P_m)$ defined in Section 3.4, is $\alpha(H(P_m)) = O(n^{1-\varepsilon})$ for some $\varepsilon > 0$?

Since Question 5.3 is inherently easier than Question 5.1, it is the most tractable direction on the lower bound front. In particular, it would be interesting to see a result that uses the specific structure of dimension-2 posets, since the current best lower bounds only use the triangle freeness of covering graphs.

On the other hand, the recent progress on improving the upper bounds on $f(n)$ covered in Section 3 give hope that one could make progress on Question 5.4.

We are hopeful that some of these questions will be answered with future work.

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