Power of a Point

Prasanna Ramakrishnan

TTMO 2019 Camp: Geometry Lecture III

1 Introduction

In our last lecture, we saw how sine rule, similar triangles and cyclic quadrilaterals can help us relate angles to lengths and find relationships that we can't get just by angle chasing. Power of a point is another useful tool that helps us do exactly the same thing. We also saw in a few examples how a perpendicular bisector of a segment is the set of points that are equidistant from the segment's vertexes.

This general idea is referred to as a *locus* – the set of points that satisfy a fixed property. Thus, what we saw about perpendicular bisectors can be summarized as the following theorem.

Theorem 1.1. For any two points A, B, the locus of points that are equidistant to A and B is the perpendicular bisector of AB.

The above gives us an if and only if statement, so it's very strong. If you want some good practice with geometry fundamentals, see if you can prove the above from scratch.

In this lecture, we'll also see another useful locus – the Radical Axis. Keep in mind the way that we used perpendicular bisectors for context.

For this lecture, we'll mainly rely on Yufei Zhao's lecture notes. I've curated a list of problems (many of which also come from that same handout) below.

2 Problems

2.1 Warm ups

1. Let Γ_1 and Γ_2 be two intersecting circles. Let a common tangent to Γ_1 and Γ_2 touch Γ_1 at A and Γ_2 at B. Suppose that Γ_1 and Γ_2 intersect at C and D. Show that the line CD bisects the segment AB.

2. Let C be a point on a semicircle of diameter AB and let D be the midpoint of arc AC. Let E be the projection of D onto the line BC and F the intersection of line AE with the semicircle. Prove that BF bisects the line segment DE.

3. Let A, B, C be three points on a circle Γ with $AB = BC$. Let the tangents at A and B meet at D. Let DC meet Γ again at E. Prove that the line AE bisects segment BD.

4. Let ABC be an acute triangle. Let the line through B perpendicular to AC meet the circle with diameter AC at points P and Q, and let the line through C perpendicular to AB meet the circle with diameter AB at points R and S. Prove that P, Q, R, S are concyclic.

2.2 IMO Problems

(2007/4) In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

(1995/1) Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

(2000/1) Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

(2008/1) Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2.

Prove that the six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic.

(2009/2) Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ. respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

(2013/4) Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively.

Denote by ω_1 is the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the point such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.