Similar Triangles and Cyclic Quadrilaterals

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TTMO 2019 Camp: Geometry Lecture I/II

1 Introduction

Olympiad students often don't like geometry because they don't know where to start with a problem. They can sometimes chase a few angles, but there doesn't seem to be any way to get close to proving the desired result. That's because in most high school mathematics, we're used to working forwards or backwards to get the result, but in geometry it's easy to get stuck at a point where it's hard to make any progress.

The reason this happens is that we come across things that we want to show, that are hard to show directly (i.e., just by computing angles or lengths). The game of Olympiad geometry (and in fact of a lot of Mathematics) is to find ways to reduce the things that are hard to show directly, into a bunch of things that are easier to show directly. Here are some examples of both, in the case of IMO Problem 1/4 from the last 18 years:
Hard to show directly:

- Concurrency: show that three lines are concurrent, three points are collinear, or two points intersect on a circle, or two circles intersect on a line
- Show that a line is tangent to a circle
- Show that two lengths are equal/two angles are equal

Easy to show directly

- Show that two triangles are similar
- Show that a quadrilateral is cyclic

These lectures aim to show how reduce the results from the first list to results in the second.

2 Sine Rule

Sine rule is a powerful tool for showing that triangles are similar, and in particular to **relating angles to sides**. Sometimes, if you're struggling to get anywhere with angle chasing alone, and the problem says that some lengths are equal, sine rule can help in bridging the two.

Theorem 2.1. (Sine rule) In any triangle ABC with angles α, β, γ at vertexes A, B, C, respectively, and with side lengths a, b, c opposite A, B, C, respectively, we have

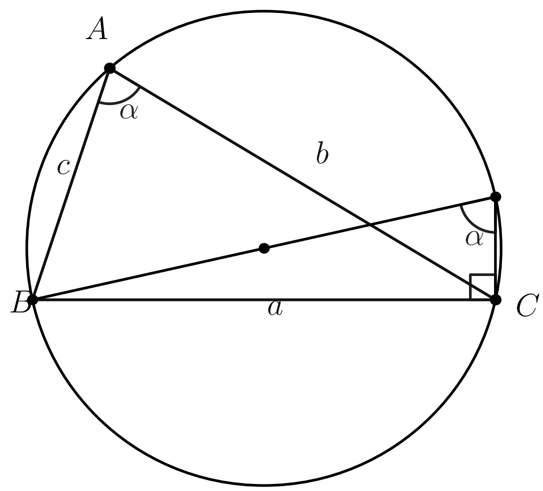
$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R$$

where R is the radius of the circumcircle of ABC.

Proof. Let D be the point diametrically opposite B in the circumcircle. By our basic circle geometry, we know that $\angle BDC = \angle BAC = \alpha$ and $\angle BCD = 90°$. Hence,

$$\sin \alpha = \frac{BC}{BD} = \frac{a}{2R} \implies \frac{a}{\sin \alpha} = 2R.$$

Applying the same for the other vertexes, we get the desired result.



□

Here are a couple of corollaries to sine rule which sometimes can be handy:

Corollary 2.1. Let ABC be a triangle, and let D be a point on the line BC. Then

$$\frac{BD}{CD} = \frac{AB}{AC} \cdot \frac{\sin \angle BAD}{\sin \angle CAD}$$

Proof. Applying sine rule to $\triangle ABD$, we have $\frac{\sin \angle BAD}{BD} = \frac{\sin \angle ADB}{AB}$, and with $\triangle ACD$ we get $\frac{\sin \angle CAD}{CD} = \frac{\sin \angle ADC}{AC}$. Now, $\angle ADB + \angle ADC = 180°$, so the sines of these two angles are the same. Hence,

$$AB \cdot \frac{\sin \angle BAD}{BD} = AC \cdot \frac{\sin \angle CAD}{CD} \implies \frac{BD}{CD} = \frac{AB}{AC} \cdot \frac{\sin \angle BAD}{\sin \angle CAD}$$

as desired.

□

This previous corollary rarely comes in handy in the easier IMO geometry problems, but it can be useful on some of the harder ones. It does however have a famous corollary itself, which can come in handy:

Corollary 2.2. (Angle Bisector Theorem) Let ABC be a triangle, and let D be the intersection of the angle bisector (internal or external) of $\angle A$ with BD. Then

$$\frac{BD}{CD} = \frac{AB}{AC}.$$

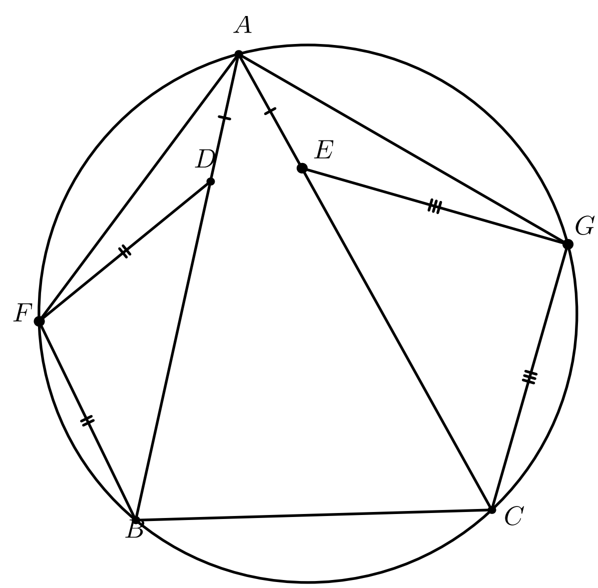
Proof. We just apply Corollary 2.1. For the internal angle bisector we have $\angle BAD = \sin \angle CAD$, and for the external angle bisector, we have $\angle BAD + \sin \angle CAD = 180°$. Thus, in either case, $\frac{\sin \angle BAD}{\sin \angle CAD} = 1$, and the result follows. □

People often leave out the "$2R$" part of sine rule, but it is very useful. For instance, if two triangles share a circumcircle, then the sine rule ratio will be the same for *both* triangles.

To demonstrate this, we'll look at a possible first step of a recent IMO problem:

Example 2.1. (IMO 2018/1) Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of BD and CE intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or they are the same line.

We will show that $\angle AFD = \angle AGE$, and leave the rest of the problem as an exercise.



Proof. Let R be the radius of Γ. Since F lies on the perpendicular bisector of DB, we know that $FD = FB$. Then applying sine rule to $\triangle ADF$ and $\triangle ABF$, we have

$$\frac{\sin \angle AFD}{AD} = \frac{\sin \angle FAD}{FD} = \frac{\sin \angle FAB}{FB} = 2R.$$

Similarly, applying sine rule to $\triangle AEG$ and $\triangle AEG$, we get that $\frac{\sin \angle AFD}{AE} = 2R$. Since $AD = AE$, we get that $\sin \angle AFD = \sin \angle AGE$. This means that either the angles are equal, or they sum to $180°$.

But observe that $\angle ADF, \angle AEG > 90°$ since F and G are on the perpendicular bisectors of BD and CE respectively. This means that $\angle AFD, \angle AGE < 90°$, so they can't sum to $180°$. Thus, $\angle AFD = \angle AGE$ as claimed. $\square$

This also illustrates the way we can use sine rule to switch between side lengths and angles. We used it to take the rather difficult to work with condition that $AD = AE$, into one about angles on the circle, which ends up being more useful for the problem. We also see how if two triangles share an angle and a side length opposite (for instance $\triangle ADF$ and $\triangle ABF$), we can also use sine rule to relate them.

Exercise 2.1. For a triangle ABC with the usual notation, show that

$$\text{Area of } ABC = \frac{1}{2}ah = \frac{1}{2}bc\sin\alpha = \frac{abc}{4R}$$

where h is the distance from A to BC (you can assume the first one if you want, but it's pretty easy to prove).

3 Similar Triangles

The next powerful tool for making progress on Olympiad geometry problems is finding similar triangles. Similar triangles are useful, because they take a handful of angle/side relationships, and create many more. If you run into a situation where you think "gosh I really wish these two angles were equal!", maybe try to find a pair of similar triangles that might make it so.

It's also important to note that similar triangles come hand in hand with ideas from cyclic quadrilaterals, since the latter often gives you a lot of useful angle relationships. Problems often have a back and forth between similar triangles and cyclic quadrilaterals, with instance of each breeding instances of the other.

First, the basics:

Theorem 3.1. For triangles ABC and XYZ, the following are equivalent:

(i) $\triangle ABC \sim \triangle XYZ$

(ii) (SSS) $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}$

(iii) (AA) $\angle A = \angle X$, $\angle B = \angle Y$

(iv) (SAS) $\angle A = \angle X$, and $\frac{AB}{XY} = \frac{CA}{ZX}$.

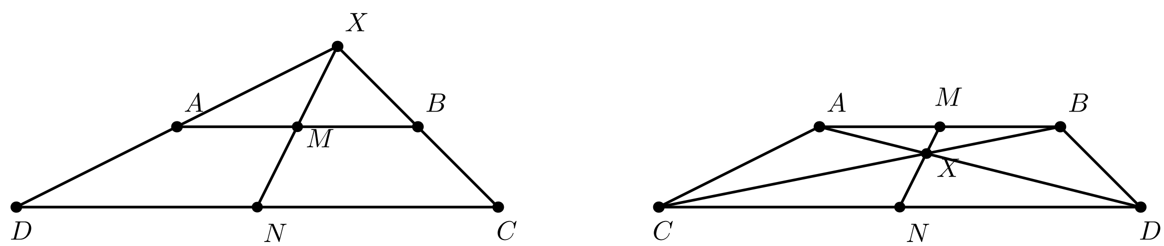
The proof is straightforward: any of these properties fix the triangle up to scaling.

A few important notes. For (iv), the angle that's equal must be *between* the pairs of sides in the ratio, which is why it is written as SAS rather than SSA or ASS. For (i), it is convention to write the triangle names out in the order that the angles are equal to each other, and doing otherwise will mean something different (though $\triangle ABC \sim \triangle XYZ$ and $\triangle BAC \sim \triangle YXZ$ are the same). This is useful because it's easy to write out (ii) with this convention, since each ratio picks out two vertexes in each triangle in the same relative order in (i).

If $\triangle ABC \sim \triangle XYZ$, then *everything* is similar about them. As a silly example, If O_1 and O_2 are the circumcenters of the two triangles, and I_1 and I_2 are the incenters of AO_1B and XO_2Z, then $\triangle AO_1I_1 \sim \triangle XO_2I_2$ as well. The key point to remember is that the two triangles are *exactly* the same up to scaling.

To illustrate this, consider the following example problem:

Example 3.1. Let $ABCD$ be a quadrilateral with $AB \parallel CD$. Let M and N be the midpoints of AB and CD respectively. Then the lines AD, BC, and MN are concurrent.

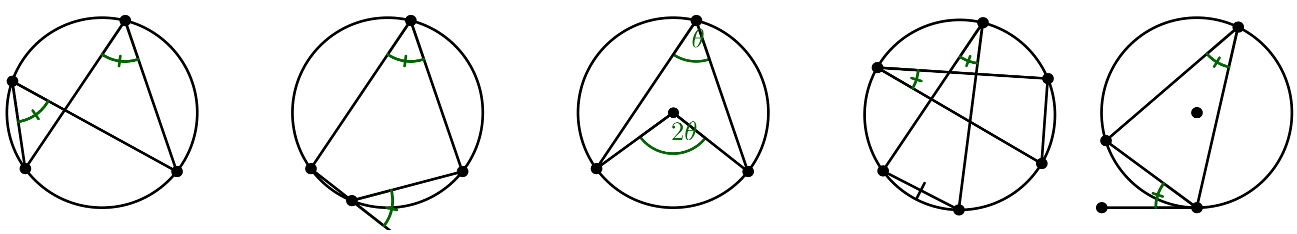


Proof. Notice that this is one of those statements that's hard to prove directly. We'll instead define X to be the intersection of AD and BC, and show that $\angle XMA = \angle XND$. Since $AB \parallel CD$, this must imply that X, M, N are collinear, which is the same as saying that the three desired lines are concurrent.

The parallel lines imply that $\angle XAB = \angle XDC$ and $\angle XBA = \angle XCD$, so $\triangle XAB \sim \triangle XDC$. This means that $\triangle XAM \sim \triangle XDN$ as well, which means exactly that $\angle XMA = \angle XND$. $\square$

4 Circles and Cyclic Quadrilaterals

Let's review some basic circle geometry first, since these results will be the cornerstone of our study of cyclic quadrilaterals.



Theorem 4.1. For points A, B on a circle Γ, and any points C, D on the same AB arc on the circle, we have $\angle ACB = \angle ADB$.

Theorem 4.2. For points A, B on a circle Γ, and any points C, D on the opposite AB arcs on the circle, we have $\angle ACB + \angle ADB = 180°$.

Theorem 4.3. For points A, B on a circle Γ with center O, and a point C on the major arc AB, $\angle AOB = 2\angle ACB$.

Corollary 4.1. If O lies on AB, then $\angle ACB = 90°$ for any points C on the circle.

Theorem 4.4. If AB and CD are arcs of Γ whose inscribed angles are the same size, then $AB = CD$.

Theorem 4.5. Let A, B, C be points on a circle Γ, and let D be a point on the tangent to Γ at A so that the line AB separates C and D. Then $\angle DAB = \angle ACB$.

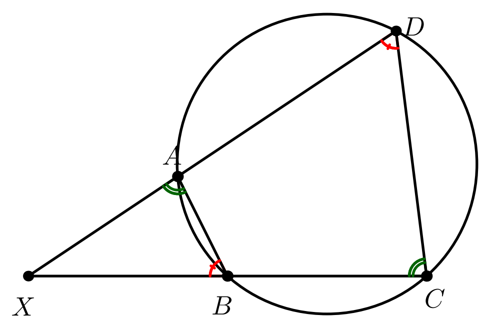
Corollary 4.2. For any point T outside a circle Γ, if A and B are on Γ so that TA and TB are tangent to Γ, then $TA = TB$.

In relation to cyclic quadrilaterals, the key usefulness of these results is that they work both ways. For instance, if you have a quadrilateral $ABCD$, and you've shown that $\angle ACB = \angle ADB$, or maybe that they sum to $180°$, then that quadrilateral is cyclic.

Another useful tool is to note that since a circle is determined by 3 points, if $ABCD$ is cyclic, and $BCDE$ is cyclic, then in fact, all five points lie on the same circle. Using ideas like this can open up a number of different angle relationships through the theorems above, and doing so can help illuminate the problem.

We also have the following very useful connection between similar triangles and cyclic quadrilaterals:

Theorem 4.6. Let $ABCD$ be a quadrilateral, and let AB and CD intersect at a point X. Then $ABCD$ is cyclic if and only if $\triangle XAD \sim \triangle XCB$.



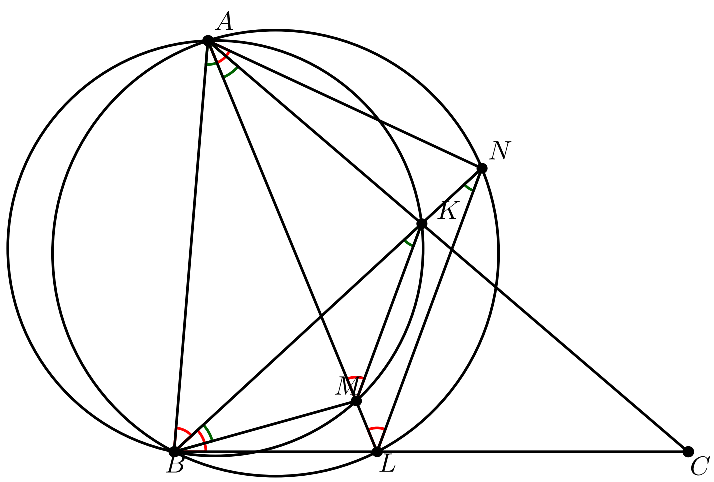
Proof. Follows directly from Theorem 4.2 when X is outside the circle, and Theorem 4.1 when inside. $\square$

And here is another rather straightforward, but surprisingly useful fact that is good to remember:

Theorem 4.7. Isosceles trapezoids are cyclic.

Once again, we'll go through an example problem that does a good job of illustrating the different ways one can use cyclic quadrilaterals to get useful angle relationships.

Example 4.1. (JBMO[1] 2010) Let AL and BK be angle bisectors in the non-isosceles triangle ABC (L lies on the side BC, K lies on the side AC). The perpendicular bisector of BK intersects the line AL at point M. Point N lies on the line BK such that LN is parallel to MK. Prove that $LN = NA$.



Proof. The key is to spot cyclic quadrilaterals. First, recall that in any triangle ABC, the perpendicular bisector of BC and the angle bisector of $\angle A$ intersect at the midpoint of the arc BC on the circumcircle[2]. This means that $ABMK$ is cyclic. Angle chasing, we get that $\angle BAL = \angle BKM = \angle BNL$. This means that $BANL$ is cyclic. Then, angle chasing again we get that $\angle LAN = \angle LBN = \angle ABK = \angle AMK = \angle ALN$. This gives us that $LN = NA$. □

The next key tool that cyclic quadrilaterals lend themselves nicely to is Power of a Point, and the Radical Axis theorem. This will be left to the next lecture.

5 Conclusion

The ideas in this lecture are by far the most useful for solving IMO geometry problems 1 or 4. A good chunk of them need only maybe a couple of ideas from this lecture, and some angle chasing. The best way to get mastery of these ideas is to practice them *a lot*. Try your best to get through as many of the problems below.

[1] Junior Balkan MO

[2] If you haven't seen this before, this is a fact worth remembering!!

6 Problems

6.1 Warm ups

1. (Miquel's Theorem) If ABC is a triangle, and we are given points A', B', C' on the sides opposite A, B, C respectively, then the circumcircles of $AB'C', A'BC', A'B'C$ intersect at a common point.

2. Let ABC be a triangle, and let E and F be the feet of the altitudes from B and C respectively. If M is the midpoint of BC, show that ME and MF are tangent to the circumcircle of $\triangle AEF$.

3. Let ABC be an isosceles triangle with $CA = CB$. Let P be a point on the side AC. Suppose that the tangent at P to the circumcircle of ABP intersects the circumcircle of CBP at D. Show that $AB \parallel CD$.

4. Let ABC be a triangle. Let D be the altitude from A to BC. Further let E and F be the feet of the perpendiculars from D to AC and AB respectively.

   (a) If $\angle A = 90°$, show that $\frac{DF}{DE} = \frac{AB}{AC}$.

   (b) Show that $BCEF$ is cyclic (not assuming anything about $\angle A$).

5. (Simson Line) Let ABC be a triangle, and let P be a point on its circumcircle. Let the feet of the altitudes from P to BC, CA, AB be P, Q, R respectively. Show that P, Q, R are collinear.

6. Let Ω and ω be circles that are internally tangent at a point A. Suppose that ω is the smaller circle, and suppose that X, Y are points on Ω so that XY is tangent to ω at a point T. Show that AT is the angle bisector of $\angle XAY$.

6.2 IMO Problems

1. (IMO 2012/1) Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

   (The excircle of ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C.)

2. (IMO 2014/4) Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on AP and AQ, respectively, such that P is the midpoint of AM and Q is the midpoint of AN. Prove that the intersection of BM and CN is on the circumference of triangle ABC.

3. (IMO 2003/4) Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

4. (IMO 2004/1) Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

5. (IMO 2018/1) Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of BD and CE intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or they are the same line.

6. (IMO 2005/1) Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths.

   Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

7. (IMO 2017/4) Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.