

# IMO 1s/4s since 2000

TTMO 2019 Camp\*

## Algebra

(2023/4) Let  $x_1, x_2, \dots, x_{2023}$  be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \dots + x_n) \left( \frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \right)}$$

is an integer for every  $n = 1, 2, \dots, 2023$ . Prove that  $a_{2023} \geq 3034$ .

(2019/1) Let  $\mathbb{Z}$  be the set of integers. Determine all functions  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  such that, for all integers  $a$  and  $b$ ,

$$f(2a) + 2f(b) = f(f(a+b)).$$

(2014/1) Let  $a_0 < a_1 < a_2 \dots$  be an infinite sequence of positive integers. Prove that there exists a unique integer  $n \geq 1$  such that

$$a_n < \frac{a_0 + a_1 + a_2 + \dots + a_n}{n} \leq a_{n+1}.$$

(2012/4) Find all functions  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  such that, for all integers  $a, b, c$  that satisfy  $a + b + c = 0$ , the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

(Here  $\mathbb{Z}$  denotes the set of integers.)

(2010/1) Find all function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that for all  $x, y \in \mathbb{R}$  the following equality holds

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

where  $\lfloor a \rfloor$  is greatest integer not greater than  $a$ .

(2008/4) Find all functions  $f : (0, \infty) \mapsto (0, \infty)$  (so  $f$  is a function from the positive real numbers) such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers  $w, x, y, z$ , satisfying  $wx = yz$ .

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(2007/1) Real numbers  $a_1, a_2, \dots, a_n$  are given. For each  $i$ , ( $1 \leq i \leq n$ ), define

$$d_i = \max\{a_j \mid 1 \leq j \leq i\} - \min\{a_j \mid i \leq j \leq n\}$$

and let  $d = \max\{d_i \mid 1 \leq i \leq n\}$ .

(a) Prove that, for any real numbers  $x_1 \leq x_2 \leq \dots \leq x_n$ ,

$$\max\{|x_i - a_i| \mid 1 \leq i \leq n\} \geq \frac{d}{2}. \quad (*)$$

(b) Show that there are real numbers  $x_1 \leq x_2 \leq \dots \leq x_n$  such that the equality holds in (\*).

(2004/4) Let  $n \geq 3$  be an integer. Let  $t_1, t_2, \dots, t_n$  be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \dots + t_n) \left( \frac{1}{t_1} + \frac{1}{t_2} + \dots + \frac{1}{t_n} \right).$$

Show that  $t_i, t_j, t_k$  are side lengths of a triangle for all  $i, j, k$  with  $1 \leq i < j < k \leq n$ .

## Combinatorics

(2022/1) The Bank of Oslo issues two types of coin: aluminum (denoted A) and bronze (denoted B). Marianne has  $n$  aluminum coins and  $n$  bronze coins arranged in a row in some arbitrary initial order. A chain is any subsequence of consecutive coins of the same type. Given a fixed positive integer  $k \leq 2n$ , Gilberty repeatedly performs the following operation: he identifies the longest chain containing the  $k^{\text{th}}$  coin from the left and moves all coins in that chain to the left end of the row. For example, if  $n = 4$  and  $k = 4$ , the process starting from the ordering  $AABBBABA$  would be  $AABBBABA \rightarrow BBBAAABA \rightarrow AAABBBBA \rightarrow BBBBAAAA \rightarrow \dots$

Find all pairs  $(n, k)$  with  $1 \leq k \leq 2n$  such that for every initial ordering, at some moment during the process, the leftmost  $n$  coins will all be of the same type.

(2020/4) There is an integer  $n > 1$ . There are  $n^2$  stations on a slope of a mountain, all at different altitudes. Each of two cable car companies,  $A$  and  $B$ , operates  $k$  cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The  $k$  cable cars of  $A$  have  $k$  different starting points and  $k$  different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for  $B$ . We say that two stations are linked by a company if one can start from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed). Determine the smallest positive integer  $k$  for which one can guarantee that there are two stations that are linked by both companies.

(2018/4) A site is any point  $(x, y)$  in the plane such that  $x$  and  $y$  are both positive integers less than or equal to 20.

Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to  $\sqrt{5}$ . On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone.

Find the greatest  $K$  such that Amy can ensure that she places at least  $K$  red stones, no matter how Ben places his blue stones.

(2015/1) We say that a finite set  $\mathcal{S}$  of points in the plane is balanced if, for any two different points  $A$  and  $B$  in  $\mathcal{S}$ , there is a point  $C$  in  $\mathcal{S}$  such that  $AC = BC$ . We say that  $\mathcal{S}$  is centre-free if for any three different points  $A$ ,  $B$  and  $C$  in  $\mathcal{S}$ , there is no points  $P$  in  $\mathcal{S}$  such that  $PA = PB = PC$ .

(a) Show that for all integers  $n \geq 3$ , there exists a balanced set consisting of  $n$  points.

(b) Determine all integers  $n \geq 3$  for which there exists a balanced centre-free set consisting of  $n$  points.

(2011/4) Let  $n > 0$  be an integer. We are given a balance and  $n$  weights of weight  $2^0, 2^1, \dots, 2^{n-1}$ . We are to place each of the  $n$  weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed.

Determine the number of ways in which this can be done.

- (2003/1) Let  $A$  be a 101-element subset of the set  $S = \{1, 2, \dots, 1000000\}$ . Prove that there exist numbers  $t_1, t_2, \dots, t_{100}$  in  $S$  such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \quad j = 1, 2, \dots, 100$$

are pairwise disjoint.

- (2002/1) Let  $n$  be a positive integer. Each point  $(x, y)$  in the plane, where  $x$  and  $y$  are non-negative integers with  $x + y < n$ , is coloured red or blue, subject to the following condition: if a point  $(x, y)$  is red, then so are all points  $(x', y')$  with  $x' \leq x$  and  $y' \leq y$ . Let  $A$  be the number of ways to choose  $n$  blue points with distinct  $x$ -coordinates, and let  $B$  be the number of ways to choose  $n$  blue points with distinct  $y$ -coordinates. Prove that  $A = B$ .
- (2001/4) Let  $n$  be an odd integer greater than 1 and let  $c_1, c_2, \dots, c_n$  be integers. For each permutation  $a = (a_1, a_2, \dots, a_n)$  of  $\{1, 2, \dots, n\}$ , define  $S(a) = \sum_{i=1}^n c_i a_i$ . Prove that there exist permutations  $a \neq b$  of  $\{1, 2, \dots, n\}$  such that  $n!$  is a divisor of  $S(a) - S(b)$ .
- (2000/4) A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card. A member of the audience draws two cards from two different boxes and announces the sum of numbers on those cards. Given this information, the magician locates the box from which no card has been drawn.

How many ways are there to put the cards in the three boxes so that the trick works?

## Geometry

(2022/4) Let  $ABCDE$  be a convex pentagon such that  $BC = DE$ . Assume that there is a point  $T$  inside  $ABCDE$  with  $TB = TD, TC = TE$  and  $\angle ABT = \angle TEA$ . Let line  $AB$  intersect lines  $CD$  and  $CT$  at points  $P$  and  $Q$ , respectively. Assume that the points  $P, B, A, Q$  occur on their line in that order. Let line  $AE$  intersect  $CD$  and  $DT$  at points  $R$  and  $S$ , respectively. Assume that the points  $R, E, A, S$  occur on their line in that order. Prove that the points  $P, S, Q, R$  lie on a circle.

(2020/1) Consider the convex quadrilateral  $ABCD$ . The point  $P$  is in the interior of  $ABCD$ . The following ratio equalities hold:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC$$

Prove that the following three lines meet in a point: the internal bisectors of angles  $\angle ADP$  and  $\angle PCB$  and the perpendicular bisector of segment  $AB$ .

(2018/1) Let  $\Gamma$  be the circumcircle of acute triangle  $ABC$ . Points  $D$  and  $E$  are on segments  $AB$  and  $AC$  respectively such that  $AD = AE$ . The perpendicular bisectors of  $BD$  and  $CE$  intersect minor arcs  $AB$  and  $AC$  of  $\Gamma$  at points  $F$  and  $G$  respectively. Prove that lines  $DE$  and  $FG$  are either parallel or they are the same line.

(2017/4) Let  $R$  and  $S$  be different points on a circle  $\Omega$  such that  $RS$  is not a diameter. Let  $\ell$  be the tangent line to  $\Omega$  at  $R$ . Point  $T$  is such that  $S$  is the midpoint of the line segment  $RT$ . Point  $J$  is chosen on the shorter arc  $RS$  of  $\Omega$  so that the circumcircle  $\Gamma$  of triangle  $JST$  intersects  $\ell$  at two distinct points. Let  $A$  be the common point of  $\Gamma$  and  $\ell$  that is closer to  $R$ . Line  $AJ$  meets  $\Omega$  again at  $K$ . Prove that the line  $KT$  is tangent to  $\Gamma$ .

(2016/1) Triangle  $BCF$  has a right angle at  $B$ . Let  $A$  be the point on line  $CF$  such that  $FA = FB$  and  $F$  lies between  $A$  and  $C$ . Point  $D$  is chosen so that  $DA = DC$  and  $AC$  is the bisector of  $\angle DAB$ . Point  $E$  is chosen so that  $EA = ED$  and  $AD$  is the bisector of  $\angle EAC$ . Let  $M$  be the midpoint of  $CF$ . Let  $X$  be the point such that  $AMXE$  is a parallelogram. Prove that  $BD, FX$  and  $ME$  are concurrent.

(2015/4) Triangle  $ABC$  has circumcircle  $\Omega$  and circumcenter  $O$ . A circle  $\Gamma$  with center  $A$  intersects the segment  $BC$  at points  $D$  and  $E$ , such that  $B, D, E$ , and  $C$  are all different and lie on line  $BC$  in this order. Let  $F$  and  $G$  be the points of intersection of  $\Gamma$  and  $\Omega$ , such that  $A, F, B, C$ , and  $G$  lie on  $\Omega$  in this order. Let  $K$  be the second point of intersection of the circumcircle of triangle  $BDF$  and the segment  $AB$ . Let  $L$  be the second point of intersection of the circumcircle of triangle  $CGE$  and the segment  $CA$ .

Suppose that the lines  $FK$  and  $GL$  are different and intersect at the point  $X$ . Prove that  $X$  lies on the line  $AO$ .

(2014/4) Let  $P$  and  $Q$  be on segment  $BC$  of an acute triangle  $ABC$  such that  $\angle PAB = \angle BCA$  and  $\angle CAQ = \angle ABC$ . Let  $M$  and  $N$  be the points on  $AP$  and  $AQ$ , respectively, such that  $P$  is the midpoint of  $AM$  and  $Q$  is the midpoint of  $AN$ . Prove that the intersection of  $BM$  and  $CN$  is on the circumference of triangle  $ABC$ .

- (2013/4) Let  $ABC$  be an acute triangle with orthocenter  $H$ , and let  $W$  be a point on the side  $BC$ , lying strictly between  $B$  and  $C$ . The points  $M$  and  $N$  are the feet of the altitudes from  $B$  and  $C$ , respectively. Denote by  $\omega_1$  is the circumcircle of  $BWN$ , and let  $X$  be the point on  $\omega_1$  such that  $WX$  is a diameter of  $\omega_1$ . Analogously, denote by  $\omega_2$  the circumcircle of triangle  $CWM$ , and let  $Y$  be the point such that  $WY$  is a diameter of  $\omega_2$ . Prove that  $X, Y$  and  $H$  are collinear.
- (2012/1) Given triangle  $ABC$  the point  $J$  is the centre of the excircle opposite the vertex  $A$ . This excircle is tangent to the side  $BC$  at  $M$ , and to the lines  $AB$  and  $AC$  at  $K$  and  $L$ , respectively. The lines  $LM$  and  $BJ$  meet at  $F$ , and the lines  $KM$  and  $CJ$  meet at  $G$ . Let  $S$  be the point of intersection of the lines  $AF$  and  $BC$ , and let  $T$  be the point of intersection of the lines  $AG$  and  $BC$ . Prove that  $M$  is the midpoint of  $ST$ .
- (The excircle of  $ABC$  opposite the vertex  $A$  is the circle that is tangent to the line segment  $BC$ , to the ray  $AB$  beyond  $B$ , and to the ray  $AC$  beyond  $C$ .)
- (2010/4) Let  $P$  be a point interior to triangle  $ABC$  (with  $CA \neq CB$ ). The lines  $AP$ ,  $BP$  and  $CP$  meet again its circumcircle  $\Gamma$  at  $K$ ,  $L$ , respectively  $M$ . The tangent line at  $C$  to  $\Gamma$  meets the line  $AB$  at  $S$ . Show that from  $SC = SP$  follows  $MK = ML$ .
- (2009/4) Let  $ABC$  be a triangle with  $AB = AC$ . The angle bisectors of  $\angle CAB$  and  $\angle ABC$  meet the sides  $BC$  and  $CA$  at  $D$  and  $E$ , respectively. Let  $K$  be the incentre of triangle  $ADC$ . Suppose that  $\angle BEK = 45^\circ$ . Find all possible values of  $\angle CAB$ .
- (2009/2) Let  $ABC$  be a triangle with circumcentre  $O$ . The points  $P$  and  $Q$  are interior points of the sides  $CA$  and  $AB$  respectively. Let  $K, L$  and  $M$  be the midpoints of the segments  $BP, CQ$  and  $PQ$ , respectively, and let  $\Gamma$  be the circle passing through  $K, L$  and  $M$ . Suppose that the line  $PQ$  is tangent to the circle  $\Gamma$ . Prove that  $OP = OQ$ .
- (2008/1) Let  $H$  be the orthocenter of an acute-angled triangle  $ABC$ . The circle  $\Gamma_A$  centered at the midpoint of  $BC$  and passing through  $H$  intersects the sideline  $BC$  at points  $A_1$  and  $A_2$ . Similarly, define the points  $B_1, B_2, C_1$  and  $C_2$ .  
Prove that the six points  $A_1, A_2, B_1, B_2, C_1$  and  $C_2$  are concyclic.
- (2007/4) In triangle  $ABC$  the bisector of angle  $BCA$  intersects the circumcircle again at  $R$ , the perpendicular bisector of  $BC$  at  $P$ , and the perpendicular bisector of  $AC$  at  $Q$ . The midpoint of  $BC$  is  $K$  and the midpoint of  $AC$  is  $L$ . Prove that the triangles  $RPK$  and  $RQL$  have the same area.
- (2006/1) Let  $ABC$  be triangle with incenter  $I$ . A point  $P$  in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that  $AP \geq AI$ , and that equality holds if and only if  $P = I$ .

- (2005/1) Six points are chosen on the sides of an equilateral triangle  $ABC$ :  $A_1, A_2$  on  $BC$ ,  $B_1, B_2$  on  $CA$  and  $C_1, C_2$  on  $AB$ , such that they are the vertices of a convex hexagon  $A_1A_2B_1B_2C_1C_2$  with equal side lengths.

Prove that the lines  $A_1B_2$ ,  $B_1C_2$  and  $C_1A_2$  are concurrent.

- (2004/1) Let  $ABC$  be an acute-angled triangle with  $AB \neq AC$ . The circle with diameter  $BC$  intersects the sides  $AB$  and  $AC$  at  $M$  and  $N$  respectively. Denote by  $O$  the midpoint of the side  $BC$ . The bisectors of the angles  $\angle BAC$  and  $\angle MON$  intersect at  $R$ . Prove that the circumcircles of the triangles  $BMR$  and  $CNR$  have a common point lying on the side  $BC$ .
- (2003/4) Let  $ABCD$  be a cyclic quadrilateral. Let  $P, Q, R$  be the feet of the perpendiculars from  $D$  to the lines  $BC, CA, AB$ , respectively. Show that  $PQ = QR$  if and only if the bisectors of  $\angle ABC$  and  $\angle ADC$  are concurrent with  $AC$ .
- (2001/1) Consider an acute-angled triangle  $ABC$ . Let  $P$  be the foot of the altitude of triangle  $ABC$  issuing from the vertex  $A$ , and let  $O$  be the circumcenter of triangle  $ABC$ . Assume that  $\angle C \geq \angle B + 30^\circ$ . Prove that  $\angle A + \angle COP < 90^\circ$ .
- (2000/1) Two circles  $G_1$  and  $G_2$  intersect at two points  $M$  and  $N$ . Let  $AB$  be the line tangent to these circles at  $A$  and  $B$ , respectively, so that  $M$  lies closer to  $AB$  than  $N$ . Let  $CD$  be the line parallel to  $AB$  and passing through the point  $M$ , with  $C$  on  $G_1$  and  $D$  on  $G_2$ . Lines  $AC$  and  $BD$  meet at  $E$ ; lines  $AN$  and  $CD$  meet at  $P$ ; lines  $BN$  and  $CD$  meet at  $Q$ . Show that  $EP = EQ$ .

## Number Theory

(2023/1) Determine all composite integers  $n > 1$  that satisfy the following property: if  $d_1, d_2, \dots, d_k$  are all the positive divisors of  $n$  with  $1 = d_1 < d_2 < \dots < d_k = n$ , then  $d_i$  divides  $d_{i+1} + d_{i+2}$  for every  $1 \leq i \leq k - 2$ .

(2021/4) Let  $n \geq 100$  be an integer. Ivan writes the numbers  $n, n + 1, \dots, 2n$  each on different cards. He then shuffles these  $n + 1$  cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

(2019/4) Find all pairs  $(k, n)$  of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \dots (2^n - 2^{n-1}).$$

(2017/1) For each integer  $a_0 > 1$ , define the sequence  $a_0, a_1, a_2, \dots$  for  $n \geq 0$  as

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise.} \end{cases}$$

Determine all values of  $a_0$  such that there exists a number  $A$  such that  $a_n = A$  for infinitely many values of  $n$ .

(2016/4) A set of positive integers is called fragrant if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let  $P(n) = n^2 + n + 1$ . What is the least possible positive integer value of  $b$  such that there exists a non-negative integer  $a$  for which the set

$$\{P(a + 1), P(a + 2), \dots, P(a + b)\}$$

is fragrant?

(2013/1) Assume that  $k$  and  $n$  are two positive integers. Prove that there exist positive integers  $m_1, \dots, m_k$  such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right) \dots \left(1 + \frac{1}{m_k}\right).$$

(2011/1) Given any set  $A = \{a_1, a_2, a_3, a_4\}$  of four distinct positive integers, we denote the sum  $a_1 + a_2 + a_3 + a_4$  by  $s_A$ . Let  $n_A$  denote the number of pairs  $(i, j)$  with  $1 \leq i < j \leq 4$  for which  $a_i + a_j$  divides  $s_A$ . Find all sets  $A$  of four distinct positive integers which achieve the largest possible value of  $n_A$ .

(2009/1) Let  $n$  be a positive integer and let  $a_1, a_2, a_3, \dots, a_k$  ( $k \geq 2$ ) be distinct integers in the set  $1, 2, \dots, n$  such that  $n$  divides  $a_i(a_{i+1} - 1)$  for  $i = 1, 2, \dots, k - 1$ . Prove that  $n$  does not divide  $a_k(a_1 - 1)$ .

(2006/4) Determine all pairs  $(x, y)$  of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

(2005/4) Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1, \quad n \geq 1.$$

(2002/4) Let  $n \geq 2$  be a positive integer, with divisors  $1 = d_1 < d_2 < \dots < d_k = n$ . Prove that  $d_1 d_2 + d_2 d_3 + \dots + d_{k-1} d_k$  is always less than  $n^2$ , and determine when it is a divisor of  $n^2$ .