Geometry Potpourri

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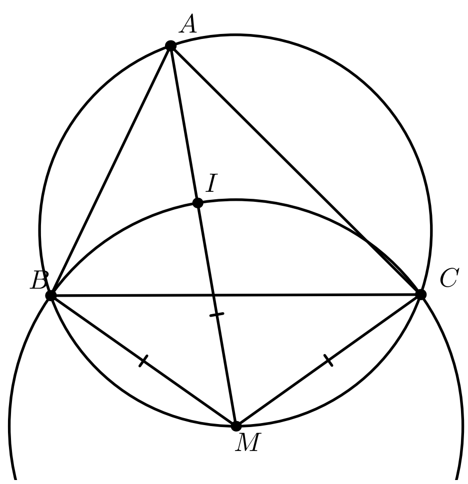
TTMO 2019 Camp: Geometry Lecture IV

1 Pras' Favorite Lemma

We saw in the first and second lectures that for a triangle ABC, the perpendicular bisector of BC and the angle bisector of A intersect on the circumcircle of ABC, specifically at the midpoint of the arc AB.

But even beyond that, this point has more that it wants to say. It just so happens that this beautiful little lemma finds its way into a good number of Olympiad problems as well.

Theorem 1.1. (PFL) Let ABC be a triangle, with incenter I. If M is the midpoint of the arc BC, then $MB = MC = MI$.



Proof. Recall that M is the intersection of the perpendicular bisector of of BC and the angle bisector of A, so from this we get $MB = MC$. Some basic angle chasing also shows that $\angle MBI = \angle MIB = \alpha/2 + \beta/2$, so this gives us $MB = MI$. Thus the result is proven. $\square$

There are a couple of alternate ways to think about this. One is that M is the circumcenter of $\triangle BIC$. The other more useful perspective is that if we are given M, then the intersection of the circle centered at M and the angle bisector of A is the incenter.

Normally PFL shows up as a somewhat hidden substructure of a problem. However, any time you have a cyclic quad which has an isosceles triangle in it, it's worth seeing if PFL can be applied.

I should also note that it's kind of rare for PFL to be able to be applied directly. Normally it shows up after some amount of angle chasing or finding cyclic quads, but keep a look out for this kind of substructure.

1.1 Problems

1. Show that I_A, the excenter opposite A, is also on the circumcircle of BIC.

2. (IMO 2006/1) Let ABC be triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

   Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

3. (IMO 2002 SL G3) The circle S has centre O, and BC is a diameter of S. Let A be a point of S such that $\angle AOB < 120°$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incenter of the triangle CEF.

4. (APMO 2007/2) Let ABC be an acute angled triangle with $\angle BAC = 60°$ and $AB > AC$. Let I be the incenter, and H the orthocenter of the triangle ABC . Prove that $2\angle AHI = 3\angle ABC$.

5. (IMO 2018/1) Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of BD and CE intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or they are the same line.

6. (IMO 2016/1) Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen so that $DA = DC$ and AC is the bisector of $\angle DAB$. Point E is chosen so that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram. Prove that BD, FX and ME are concurrent.

2 Apollonius Circles

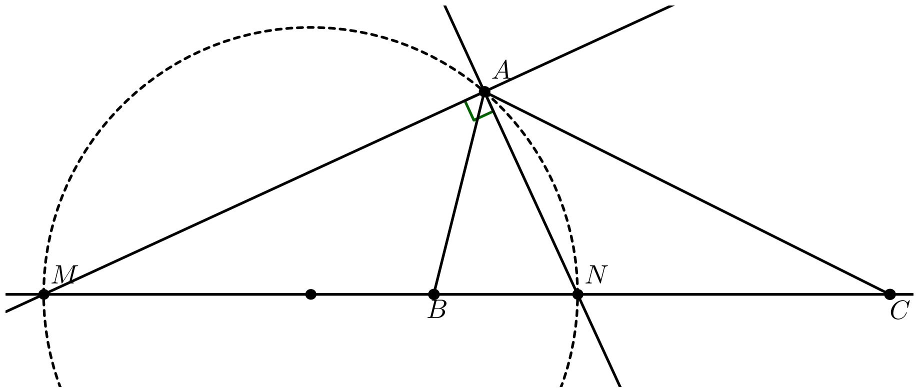
Theorem 2.1. Let ABC be a triangle. The set of points P such that $\frac{PB}{PC} = \frac{AB}{AC}$ forms a circle (called the A-Apollonius Circle of the triangle).

Proof. Recall that by the angle bisector theorem, if M and N are the points where the internal and external angle bisectors meet BC respectively, then both M and N are on this set of points.

Now suppose we have a point P that is also on this set. Then by the angle bisector theorem again, PM and PN are the internal and external angle bisectors of $\angle BPC$. Hence, $\angle MPN = 90°$, which means that P must lie on the circle with diameter MN.

Now we show that if a point P is on the circle with diameter MN, then $\frac{PB}{PC} = \frac{AB}{AC}$. Let $\angle BPM = \theta$ and let $\angle CPM = \phi$. By Cosmin's lemma, we have

$$\frac{BM}{MC} = \frac{AB}{AC} \cdot \frac{\sin\theta}{\sin\phi} \qquad \text{and} \qquad \frac{BN}{NC} = \frac{AB}{AC} \cdot \frac{\sin(90+\theta)}{\cos(90-\phi)} = \frac{AB}{AC} \cdot \frac{\cos\theta}{\cos\phi}.$$



This means that $\sin\theta\cos\phi = \sin\phi\cos\theta$, which means that $\sin(\theta - \phi) = 0$. This means θ and ϕ are either the same or their difference is $180°$. The latter is nonsense, so we're done.

□

Exercise 2.1. Show that the center of the A-Apollonius circle is the intersection of the tangent at A to the circumcircle of ABC, and the line BC.

Exercise 2.2. Show that the three Apollonius circles of a triangle intersect at the same two points.

2.1 Problem

1. (IMO 2010/4) Let P be a point interior to triangle ABC (with $CA \neq CB$). The lines AP, BP and CP meet again its circumcircle Γ at K, L, respectively M. The tangent line at C to Γ meets the line AB at S. Show that from $SC = SP$ follows $MK = ML$.

3 Some Theorems

These are theorems that you're extremely unlikely to need for IMO problem 1s/4s, but sometimes they can be useful for motivating ideas.

Theorem 3.1. (Pascal's theorem) In a cyclic hexagon $ABCDEF$, the points $AB \cap ED$, $BC \cap EF$, and $CD \cap FA$ are collinear.

Note that the points don't need to appear in this order on the circle, and they don't even need to be distinct!

Theorem 3.2. (Miquel's Theorem) If ABC is a triangle, and we are given points A', B', C' on the sides opposite A, B, C respectively, then the circumcircles of $AB'C', A'BC', A'B'C$ intersect at a common point.

Try to show this!

Theorem 3.3. (Miquel Point) For any four lines in the plane (no two parallel), if we consider the four triangles we get from choosing any three of the lines, then their circumcircles share a common point.

Theorem 3.4. If $ABCD$ is cyclic, then its Miquel point lies on the line QR where $Q = AB \cap CD$ and $R = AD \cap BC$.

Theorem 3.5. Suppose that two circles Γ_1 and Γ_2 intersect at O and X. Suppose that X lies on the lines AC and BD where A, B are on Γ_1 and C, D are on Γ_2. Then $\triangle OAB \sim \triangle OCD$ and $\triangle OAC \sim \triangle OBD$.

This is referred to as a spiral similarity (a combination of a rotation and dilation). The theorem works in the opposite way as well. If $\triangle OAB \sim \triangle OCD$, and AC and BD intersect at X, then the quads $OXAB$ and $OXCD$ are cyclic.