Conics Assignment

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Problem Statement **- The equation of the tangent to the parabola y^2=8x is x-y+2=0.The point on this line from which other tangent to the parabola is perpendicular to the given tangent is :**

1. **The equation of parabola is y^2=8x**

2. **The equation of tangent is x-y+2=0**

Solution

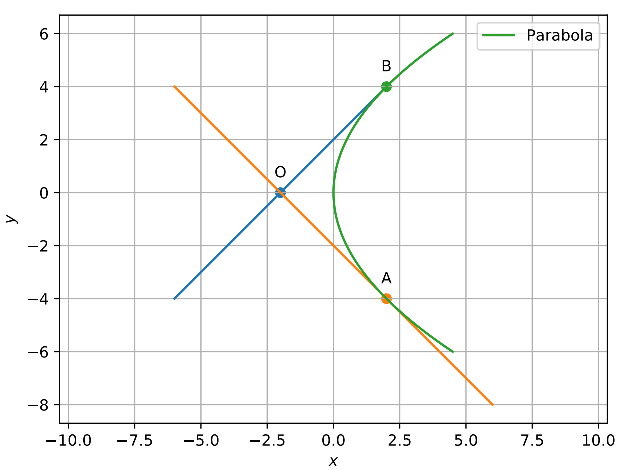

Figure 1: Two tangent is drawn to the circle and parabola

Solution

Part 1

Construction

The input parameters are equation of the curve and the point of contacts

Symbol	Value	Description
O	$\begin{pmatrix} x \\ x+2 \end{pmatrix}$	The required point
P1	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	eigen vector
a	2	Given value of a
q	$\begin{pmatrix} 2 \\ 4 \end{pmatrix}$	point of contact of parabola
q_1	$\begin{pmatrix} 2 \\ -4 \end{pmatrix}$	point of contact of circle

Part 2

The standard equation of the parabola is given as :

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

The directrix of parabola is given as:

$$n_1^T x = c \tag{2}$$

where

$$\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \tag{3}$$

$$\mathbf{n_1} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \tag{4}$$

$$f = 0 \tag{5}$$

$$c = -a \tag{6}$$

The equation of a parabola with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{7}$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top, \tag{8}$$

$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \tag{9}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{10}$$

$$e = 1 \tag{11}$$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{12}$$

$$\mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{13}$$

$$f = 0 \tag{14}$$

Consider the equation of circle:

$$\mathbf{x}^\top \mathbf{V_1} \mathbf{x} + 2\mathbf{u_1}^\top \mathbf{x} + f_1 = 0 \tag{15}$$

If $\mathbf{V}$ is not invertible, given the normal vector $\mathbf{n}$, the point of contact to the circle is given by the matrix equation

$$\begin{pmatrix} \mathbf{u} + \kappa \mathbf{n}^\top \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \tag{16}$$

$$\text{where } \kappa = \frac{\mathbf{p_1^\top u}}{\mathbf{p_1^\top n_1}}, \quad \mathbf{V}\mathbf{p}_1 = 0 \tag{17}$$

the normal vector is obtained by the equation of the circle is :

$$\kappa \mathbf{n_1} = \mathbf{V}\mathbf{q_1} + \mathbf{u} \tag{18}$$

The normal vector of tangent is :

$$\mathbf{n_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{19}$$

yielding k we get,

$$\kappa \mathbf{q_1}^\top \mathbf{n_1} + \mathbf{q_1}^\top \mathbf{u} + f = 0 \tag{20}$$

By solving the above equation The point of contact of tangent to circle with normal vector $\mathbf{n_1}$ is given by

$$\mathbf{q_1} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \tag{21}$$

The Directional vector of tangent to parabola is :

$$\mathbf{q_1} - \mathbf{q} \tag{22}$$

Now the normal vector of the other tangent which is perpendicular to the given tangent is give by :

$$\mathbf{n_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{23}$$

$$\kappa \mathbf{n_2} = \mathbf{V}\mathbf{q_2} + \mathbf{u} \tag{24}$$

$$\kappa \mathbf{q_2}^\top \mathbf{n_2} + \mathbf{q_2}^\top \mathbf{u} + f = 0 \tag{25}$$

By solving the above equation the point of contact of tangent to circle with normal vector $\mathbf{n_2}$ is given by :

$$\mathbf{q_2} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} \tag{26}$$

The Directional vector of other tangent to the parabola is :

$$\mathbf{q} - \mathbf{q_2} \tag{27}$$

In order to get the required point on the tangent :

$$(\mathbf{q_1} - \mathbf{q})^{\mathbf{T}} (\mathbf{q} - \mathbf{q_2}) = 0 \tag{28}$$

$$\mathbf{q} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{29}$$