Conic Assignment

Namrath Pinnamaneni FWC22042

September 2022

***Problem Statement* - The normal to the curve $x_2 = 4y$ passing (1,2) is:**

(a)x+y=3 **(b)x-y=3**
(c)x+y=1 **(d)x-y=1**

Solution

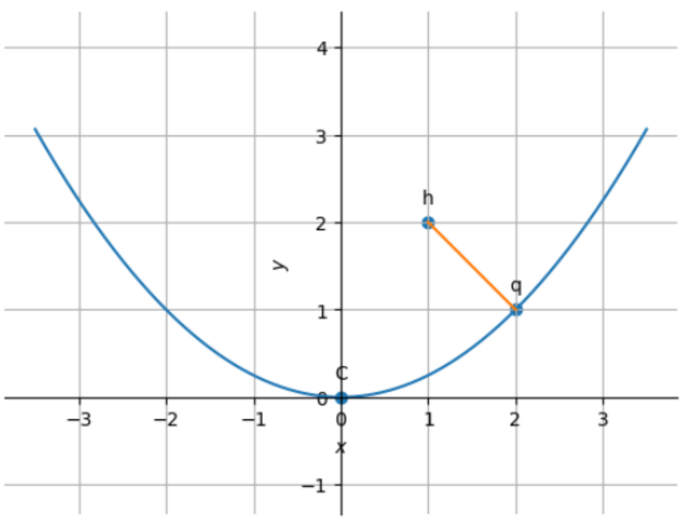

Figure 1: Tangents from A to circle through B, C and D

The given equation of parabola $x^2 = 4y$ can be written in the general quadratic form as

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \tag{3}$$

$$f = 0 \tag{4}$$

Let the point from which normals are drawn be $\mathbf{h}$. Then, the equation of the normal can be written as

$$\mathbf{x} = \mathbf{h} + \lambda \mathbf{m} \tag{5}$$

Say the point of intersection of (5) with the conic is $\mathbf{q}$. A tangent drawn at $\mathbf{q}$ satisfies the equation

$$\mathbf{n}^\top(\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \tag{6}$$

Where $\mathbf{n}$ is the direction vector of the tangent and is perpendicular to $\mathbf{m}$ in (5).

In general, the parameter values for points of intersection of a line given by (5) with a conic is given by

$$\lambda_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^T(\mathbf{V}\mathbf{h}+\mathbf{u}) \pm \sqrt{\left[\mathbf{m}^T(\mathbf{V}\mathbf{h}+\mathbf{u})\right]^2 - \left(\mathbf{h}^T\mathbf{V}\mathbf{h} + 2\mathbf{u}^T\mathbf{h} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\right) \tag{7}$$

Using (7) and (5), the intersection point $\mathbf{q}$ can be written as

$$\mathbf{q} = \mathbf{h} + \lambda_i \mathbf{m} \tag{8}$$

Substituting (8) in (6),

$$\mathbf{n}^\top(\mathbf{V}(\mathbf{h} + \lambda_i\mathbf{m}) + \mathbf{u}) = 0 \tag{9}$$

$$\implies \lambda_i \mathbf{n}^\top\mathbf{V}\mathbf{m} = -\mathbf{n}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) \tag{10}$$

Substituting value of λ_i from (7) in (10)

$$\frac{1}{\mathbf{m}^\top\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) \pm \sqrt{\left[\mathbf{m}^T(\mathbf{V}\mathbf{h}+\mathbf{u})\right]^2 - \left(\mathbf{h}^T\mathbf{V}\mathbf{h} + 2\mathbf{u}^T\mathbf{h} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\right)\mathbf{n}^\top\mathbf{V}\mathbf{m} = -\mathbf{n}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) \tag{11}$$

Rearranging the terms,

$$\pm\sqrt{\left[\mathbf{m}^T(\mathbf{V}\mathbf{h}+\mathbf{u})\right]^2 - \left(\mathbf{h}^T\mathbf{V}\mathbf{h} + 2\mathbf{u}^T\mathbf{h} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\left(\mathbf{n}^\top\mathbf{V}\mathbf{m}\right) = (\mathbf{V}\mathbf{h}+\mathbf{u})^\top\left(\left(\mathbf{n}^\top\mathbf{V}\mathbf{m}\right)\mathbf{m} - \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\mathbf{n}\right) \tag{12}$$

Squaring on both sides

$$\left[\left[\mathbf{m}^T(\mathbf{V}\mathbf{h}+\mathbf{u})\right]^2 - \left(\mathbf{h}^T\mathbf{V}\mathbf{h} + 2\mathbf{u}^T\mathbf{h} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)\right]\left(\mathbf{n}^\top\mathbf{V}\mathbf{m}\right)^2 = \left[(\mathbf{V}\mathbf{h}+\mathbf{u})^\top\left(\left(\mathbf{n}^\top\mathbf{V}\mathbf{m}\right)\mathbf{m} - \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\mathbf{n}\right)\right]^2 \tag{13}$$

If $\mathbf{n}$ is taken as $\begin{pmatrix} -\mu \\ 1 \end{pmatrix}$, then $\mathbf{m}$ is $\begin{pmatrix} -1 \\ -\mu \end{pmatrix}$. Substituting these values in (13) and solving for μ, the different possible normals passing through $\mathbf{h}$ are obtained.

Thus after solving we get the following values for μ = -1, 1/2 - sqrt(3)*I/2, 1/2 + sqrt(3)*I/2

Taking μ=1 we get,

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

By calculating λ_i from (10), we get

$$\lambda_i = -1$$

We find out $\mathbf{q}$ from (8),

where $\mathbf{h} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{m} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\lambda_i = -1$

$$\mathbf{q} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (-1) \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Thus $\mathbf{q}$ satisfies Option(a) i.e. $x + y = 3$

Construction

Symbol	**Value**	**Description**
$\mathbf{h}$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	Given point through which Normal is passing
$\mathbf{q}$	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$	Foot of Normal
$\mathbf{m}$	$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$	Direction Vector of Normal
$\mathbf{n}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	Direction Vector of Tangent at $\begin{pmatrix} q \end{pmatrix}$