Circle Assignment

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***Problem Statement* - Find equation of the circle which touches the line $2x + 3y + 1$ at the point (1, -1) and cuts orthogonal the circle which has the line segment joining (0, 3) and (-2, -1) as a diameter.**

Solution

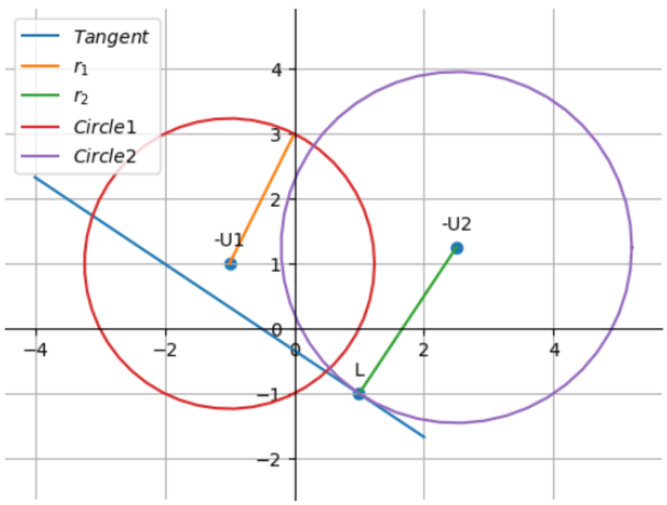

Figure 1: Tangents from A to circle through B, C and D

With the given end vertices A(0,3) and B(-2,-1), we can find out centre U_1 and radius r_1 of Circle-1. We know that tangent $2x + 3y + 1 = 0$ touches Circle-2 at L(1,-1) hence using mentioned we will solve for centre and radius of Circle-2 i.e. U_2 and r_2 using 3 equations as follows.

STEP-1

Calculating $\mathbf{U_1}$ using $\mathbf{A}\begin{pmatrix}0\\3\end{pmatrix}$ and $\mathbf{B}\begin{pmatrix}-2\\-1\end{pmatrix}$

$$\mathbf{U_1} = \frac{\mathbf{A}+\mathbf{B}}{2} \tag{1}$$

Centre of Circle-1, $\mathbf{U_1} = \begin{pmatrix}-1\\1\end{pmatrix}$

Calculating r_1 using $\mathbf{A}\begin{pmatrix}0\\3\end{pmatrix}$ and $\mathbf{U_1}\begin{pmatrix}-1\\1\end{pmatrix}$

$$r_1 = \|\mathbf{U_1} - \mathbf{A}\| \tag{2}$$

Radius of Circle-1, $r_1 = \sqrt{5}$

STEP-2

As both the circles are orthogonal, we get:

$$\|\mathbf{U_2} - \mathbf{U_1}\|^2 = r_1^2 + r_2^2 \tag{3}$$

$$\implies \|\mathbf{U_2}\|^2 + \|\mathbf{U_1}\|^2 - 2\mathbf{U_1}^\top \mathbf{U_2} = r_1^2 + r_2^2 \tag{4}$$

Equation when tangent touches a Circle at a point L

$$\mathbf{m}^\top (\mathbf{L} + \mathbf{U_2}) = 0 \tag{5}$$

$$\implies \mathbf{m}^\top \mathbf{U_2} = -\mathbf{m}^\top \mathbf{L} \tag{6}$$

We get $\mathbf{m}$ by

$$\mathbf{m} = \mathbf{In} \tag{7}$$

$$\text{where } \mathbf{I} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}, \mathbf{n} = \begin{pmatrix}2\\3\end{pmatrix}$$

Finally we also know that,

$$\|\mathbf{L} - (-\mathbf{U_2})\|^2 = r_2^2 \tag{8}$$

$$\implies \|\mathbf{L}\|^2 + \|\mathbf{U_2}\|^2 + 2\mathbf{L}^\top \mathbf{U_2} = r_2^2 \tag{9}$$

STEP-3

After substituting **(9)** in **(4)** we get,

$$2\left(\mathbf{L} + \mathbf{U_1}\right)^\top \mathbf{U_2} = \|\mathbf{U_1}\|^2 - \|\mathbf{L}\|^2 - r_1^2 \tag{10}$$

By using **(6)** and **(10)**,

$$2\left(\mathbf{L} + \mathbf{U_1}\right)^\top \mathbf{U_2} = \|\mathbf{U_1}\|^2 - \|\mathbf{L}\|^2 - r_1^2$$

$$\mathbf{m}^\top \mathbf{U_2} = -\mathbf{m}^\top \mathbf{L}$$

yielding,

$$\implies \begin{pmatrix}2\left(\mathbf{L} + \mathbf{U_1}\right)^\top\\ \mathbf{m}^\top\end{pmatrix} \mathbf{U_2} = \begin{pmatrix}\|\mathbf{U_1}\|^2 - \|\mathbf{L}\|^2 - r_1^2\\ \mathbf{m}^\top \mathbf{L}\end{pmatrix} \tag{11}$$

solving we get,

$$\mathbf{U_2} = \begin{pmatrix}2.5\\1.25\end{pmatrix} \tag{12}$$

therefore we get r2,

$$r_2 = \|\mathbf{U_2} - \mathbf{L}\| \tag{13}$$

$$r_2 = 2.704$$

Construction

Symbol	Value	Description
$\mathbf{A}, \mathbf{B}$	$\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \end{pmatrix}$	Given diametric points of U_1
$\mathbf{U_1}$	$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$	Centre of Circle U_1
r_1	$\sqrt{5}$	Radius of Circle U_1
$\mathbf{L}$	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	Point at which Tangent touches U_2
$\mathbf{m}$	$\begin{pmatrix} -3 \\ 2 \end{pmatrix}$	Direction vector of Tangent
$\mathbf{U_2}$	$\begin{pmatrix} 2.5 \\ 1.25 \end{pmatrix}$	Centre of Circle U_2
r_2	2.704	Radius of Circle U_2