Conic Assignment

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September 2022

***Problem Statement* - The normal at the point (1,1) on the curve $2y + x^2 = 3$:**
(a)x+y=0 **(b)x-y=0**
(c)x+y+1=0 **(d)x-y=1**

Solution

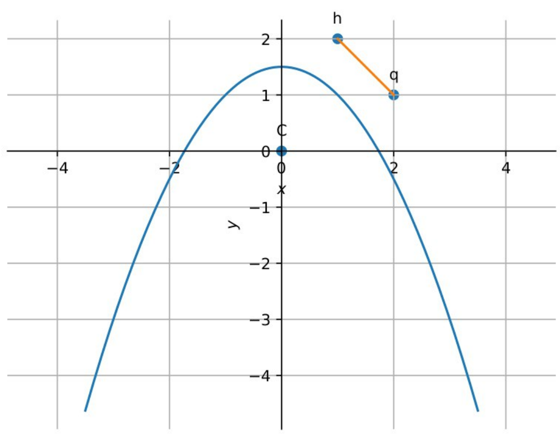

Figure 1: Tangents from A to circle through B, C and D

The given equation of parabola 2y+x^2 = 3 can be written in the general quadratic form as

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \tag{3}$$

$$f = 3 \tag{4}$$

The parabola in (**??**) can be expressed in standard form (center/vertex at origin, major-axis - x axis) as

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = -2\eta \mathbf{e}_1^\top \mathbf{y} \qquad |V| = 0 \tag{5}$$

where

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad \text{(Affine Transformation)} \tag{6}$$

$$\mathbf{P}^\top \mathbf{V}\mathbf{P} = \mathbf{D}. \quad \text{(Eigenvalue Decomposition)} \tag{7}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{8}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \quad \mathbf{P}^\top = \mathbf{P}^{-1}, \tag{9}$$

$$\eta = \mathbf{u}^\top \mathbf{p}_1 \tag{10}$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{11}$$

To find $\mathbf{c}$ which is the center of the parabola in (**??**), substitute (**??**) in (**??**)

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^T \mathbf{V}(\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^T(\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0, \tag{12}$$

yielding

$$\mathbf{y}^T\mathbf{P}^T\mathbf{V}\mathbf{P}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{P}\mathbf{y} + \mathbf{c}^T\mathbf{V}\mathbf{c} + 2\mathbf{u}^T\mathbf{c} + f = 0 \tag{13}$$

From (**??**) and (**??**),

$$\mathbf{y}^T\mathbf{D}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^T \mathbf{P}\mathbf{y} + \mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0 \tag{14}$$

For a parabola $|\mathbf{V}| = 0, \lambda_1 = 0$ and

$$\mathbf{V}\mathbf{p}_1 = 0, \mathbf{V}\mathbf{p}_2 = \lambda_2\mathbf{p}_2. \tag{15}$$

where $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors of $\mathbf{V}$ such that (**??**)

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \tag{16}$$

Substituting (**??**) in (**??**),

$$\begin{aligned} &\mathbf{y}^T\mathbf{D}\mathbf{y} + 2(\mathbf{c}^T\mathbf{V} + \mathbf{u}^T)\begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}\mathbf{y} \\ &+ \mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0 \\ &\implies \mathbf{y}^T\mathbf{D}\mathbf{y} \\ &+ 2\left((\mathbf{c}^T\mathbf{V} + \mathbf{u}^T)\mathbf{p}_1 \; (\mathbf{c}^T\mathbf{V} + \mathbf{u}^T)\mathbf{p}_2\right)\mathbf{y} \\ &+ \mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0 \\ &\implies \mathbf{y}^T\mathbf{D}\mathbf{y} \\ &+ 2\begin{pmatrix} \mathbf{u}^T\mathbf{p}_1 & (\lambda_2\mathbf{c}^T + \mathbf{u}^T)\mathbf{p}_2 \end{pmatrix}\mathbf{y} \\ &+ \mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0 \text{ from } (\mathbf{??}) \end{aligned}$$

$$\implies \lambda_2 y_2^2 + 2(\mathbf{u}^T\mathbf{p}_1) y_1 + 2y_2(\lambda_2\mathbf{c} + \mathbf{u})^T \mathbf{p}_2 + \mathbf{c}^T(\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^T\mathbf{c} + f = 0$$

which is the equation of a parabola. Thus, (??) can be expressed as (??) by choosing

$$\eta = \mathbf{u}^T \mathbf{p}_1 \tag{17}$$

and $\mathbf{c}$ in (??) such that

$$\mathbf{P}^T \left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) = \eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{18}$$

$$\mathbf{c}^T \left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^T \mathbf{c} + f = 0 \tag{19}$$

Multiplying (??) by $\mathbf{P}$ yields

$$(\mathbf{V}\mathbf{c} + \mathbf{u}) = \eta \mathbf{p}_1, \tag{20}$$

which, upon substituting in (??) results in

$$\eta \mathbf{c}^T \mathbf{p}_1 + \mathbf{u}^T \mathbf{c} + f = 0 \tag{21}$$

(??) and (??) can be clubbed together to obtain (??).

$$\begin{pmatrix} \mathbf{u}^\top + \eta \mathbf{p}_1^\top \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \qquad |V| = 0 \tag{22}$$

Substituting appropriate values from (??), (??), (??), (??), and (??) into (??), the below matrix equation is obtained

$$\begin{pmatrix} 0 & -4 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{23}$$

$$\tag{24}$$

The augmented matrix for (??) can be expressed as

$$\left(\begin{array}{cc|c} 0 & -4 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \tag{25}$$

$$\xleftrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right) \tag{26}$$

$$\xleftrightarrow{-\frac{R_2}{4} \leftarrow R_2} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \tag{27}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{28}$$

Let the point from which normals are drawn be $\mathbf{h}$. Then, the equation of the normal can be written as

$$\mathbf{x} = \mathbf{h} + \lambda \mathbf{m} \tag{29}$$

Say the point of intersection of (??) with the conic is $\mathbf{q}$. A tangent drawn at $\mathbf{q}$ satisfies the equation

$$\mathbf{n}^\top (\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \tag{30}$$

Where $\mathbf{n}$ is the direction vector of the tangent and is perpendicular to $\mathbf{m}$ in (??).

In general, the parameter values for points of intersection of a line given by (??) with a conic is given by

$$\lambda_i = \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^T \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right) \pm \sqrt{\left[\mathbf{m}^T \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\right]^2 - \left(\mathbf{h}^T \mathbf{V} \mathbf{h} + 2\mathbf{u}^T \mathbf{h} + f\right)\left(\mathbf{m}^T \mathbf{V} \mathbf{m}\right)}\right) \tag{31}$$

Using (??) and (??), the intersection point $\mathbf{q}$ can be written as

$$\mathbf{q} = \mathbf{h} + \lambda_i \mathbf{m} \tag{32}$$

Substituting (??) in (??),

$$\mathbf{n}^\top (\mathbf{V}(\mathbf{h} + \lambda_i \mathbf{m}) + \mathbf{u}) = 0 \tag{33}$$

$$\implies \lambda_i \mathbf{n}^\top \mathbf{V}\mathbf{m} = -\mathbf{n}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \tag{34}$$

Substituting value of λ_i from (??) in (??)

$$\frac{1}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right) \pm \sqrt{\left[\mathbf{m}^T \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\right]^2 - \left(\mathbf{h}^T \mathbf{V} \mathbf{h} + 2\mathbf{u}^T \mathbf{h} + f\right)\left(\mathbf{m}^T \mathbf{V} \mathbf{m}\right)}\right) \mathbf{n}^\top \mathbf{V} \mathbf{m} = -\mathbf{n}^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right) \tag{35}$$

Rearranging the terms,

$$\pm \sqrt{\left[\mathbf{m}^T \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\right]^2 - \left(\mathbf{h}^T \mathbf{V} \mathbf{h} + 2\mathbf{u}^T \mathbf{h} + f\right)\left(\mathbf{m}^T \mathbf{V} \mathbf{m}\right)} \left(\mathbf{n}^\top \mathbf{V} \mathbf{m}\right) = \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)^\top \left(\left(\mathbf{n}^\top \mathbf{V} \mathbf{m}\right) \mathbf{m} - \left(\mathbf{m}^\top \mathbf{V} \mathbf{m}\right) \mathbf{n}\right) \tag{36}$$

Squaring on both sides

$$\left[\left[\mathbf{m}^T \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\right]^2 - \left(\mathbf{h}^T \mathbf{V} \mathbf{h} + 2\mathbf{u}^T \mathbf{h} + f\right)\left(\mathbf{m}^T \mathbf{V} \mathbf{m}\right)\right] \left(\mathbf{n}^\top \mathbf{V} \mathbf{m}\right)^2 = \left[\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)^\top \left(\left(\mathbf{n}^\top \mathbf{V} \mathbf{m}\right) \mathbf{m} - \left(\mathbf{m}^\top \mathbf{V} \mathbf{m}\right) \mathbf{n}\right)\right]^2 \tag{37}$$

If $\mathbf{n}$ is taken as $\begin{pmatrix} -\mu \\ 1 \end{pmatrix}$, then $\mathbf{m}$ is $\begin{pmatrix} -1 \\ -\mu \end{pmatrix}$. Substituting these values in (??) and solving for μ, the different possible normals passing through $\mathbf{h}$ are obtained.

Thus after solving we get the following values for μ = -1, 1/2 - sqrt(3)*I/2, 1/2 + sqrt(3)*I/2

Taking μ=1 we get,

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

By calculating λ_i from (??), we get

$$\lambda_i = -1$$

We find out $\mathbf{q}$ from (??),

$$\text{where } \mathbf{h} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \lambda_i = -1$$

$$\mathbf{q} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (-1) \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Thus $\mathbf{q}$ satisfies Option(a) i.e. $x + y = 3$

Construction

Symbol	Value	Description
$\mathbf{h}$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	Given point through which Normal is passing
$\mathbf{q}$	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$	Foot of Normal
$\mathbf{m}$	$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$	Direction Vector of Normal
$\mathbf{n}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	Direction Vector of Tangent at $\begin{pmatrix} q \end{pmatrix}$
$\mathbf{P}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	eigenvectors of $\mathbf{V}$