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Assignment-6

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Problem Statement:

Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$

SOLUTION:

0.1 Without numericals

Given:

The general equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

The general equation of line is

$$\frac{x}{a} + \frac{y}{b} = 1 \tag{2}$$

To Find

To find the intersection points and area of shaded region shown in figure

STEP-1

The given ellipse can be expressed as conics with parameters,

$$\mathbf{V} = \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \tag{3}$$

$$\mathbf{u} = 0 \tag{4}$$

$$f = -(a^2 b^2) \tag{5}$$

STEP-2

the given line equation can be written as

$$\mathbf{x} = \begin{pmatrix} a \\ 0 \end{pmatrix} + k \begin{pmatrix} \frac{1}{b} \\ -\frac{1}{a} \end{pmatrix} \tag{6}$$

STEP-3

The points of intersection of the line,

$$L: \quad \mathbf{x} = \mathbf{q} + \kappa \mathbf{m} \quad \kappa \in \mathbb{R} \tag{7}$$

with the conic section,

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{8}$$

are given by

$$\mathbf{x}_i = \mathbf{q} + \kappa_i \mathbf{m} \tag{9}$$

where,

$$\kappa_i = \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \Big(-\mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u}) \pm \sqrt{\left[\mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u})\right]^2 - \left(\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f\right)\left(\mathbf{m}^T \mathbf{V} \mathbf{m}\right)} \Big) \tag{10}$$

On substituting

$$\mathbf{q} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{11}$$

$$\mathbf{m} = \begin{pmatrix} \frac{1}{b} \\ -\frac{1}{a} \end{pmatrix} \tag{12}$$

With the given ellipse as in eq(3),(4),(5),
The value of κ ,

$$\kappa = 0, -6 \tag{13}$$

by substituting eq(13) in eq(6)we get the points of intersection of line with ellipse

$$\mathbf{A} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{14}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ b \end{pmatrix} \tag{15}$$

0.2 Substituting the numericals according to the problem

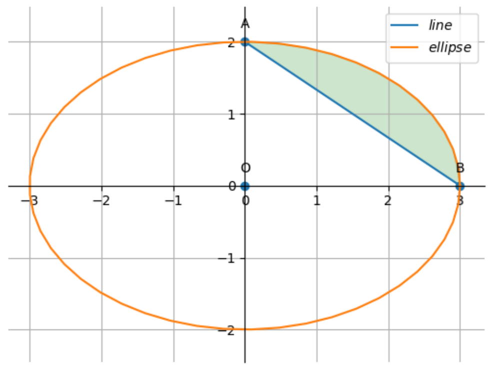
$$a = 3, b = 2 \tag{16}$$

$$\mathbf{V} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}, \mathbf{u} = 0, f = -36 \tag{17}$$

$$\mathbf{q} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{3} \end{pmatrix}, \tag{18}$$

$$\mathbf{A} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ o \end{pmatrix} \tag{19}$$

Result



From the figure,
Total area of portion is given by,
Total Area=(area of ellipse in first quadrant)-(area of a triangle**AOB**)

Area of ellipse

$$\implies A2 = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx \tag{20}$$
$$= \int_0^3 \frac{2}{3}\sqrt{9 - x^2}\,dx \tag{21}$$

by solving the above equation we get area of ellipse $\frac{3\pi}{2}$

Download code

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https://github.com/SivaKrishna/blob/main/conics/code/conic.py
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Area of triangle

$$\implies A1 = \int_0^a \frac{b}{a}(a - x)\,dx \tag{22}$$
$$= \int_0^3 \frac{2}{3}(3 - x)\,dx \tag{23}$$

by solving the above equation we get area of triangle 3 square units.
The total area is

$\implies A = \frac{3\pi}{2} - 3$

The area of the smaller region is ,

$$A = 3(\frac{\pi}{2} - 1) squareunits \tag{24}$$

Construction

Points	**coordinates**
B	$\begin{pmatrix} a \\ 0 \end{pmatrix}$
A	$\begin{pmatrix} 0 \\ b \end{pmatrix}$