Circle Assignment

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Problem

Let $x^2 + y^2 - 4x - 2y - 11 = 0$ be a circle.A pair of tangents from point(4,5) with pair of radii form a quadrilateral of area.........

Solution

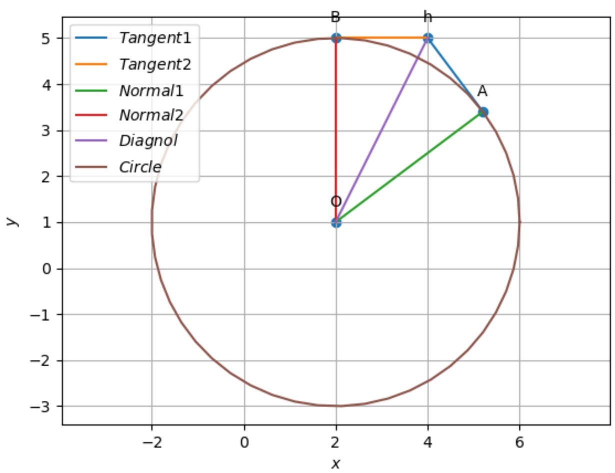

Figure 1: Circle with center O and points A,B & h

Symbol	Value	Description
$\mathbf{h}$	$\begin{pmatrix}4\\5\end{pmatrix}$	external point
$\mathbf{O}$	$\begin{pmatrix}2\\1\end{pmatrix}$	centre of circle
$\mathbf{A}$	$\begin{pmatrix}5.201\\3.403\end{pmatrix}$	point of contact
$\mathbf{B}$	$\begin{pmatrix}2\\5.008\end{pmatrix}$	point of contact

Step 1

Given,equation of circle and point are,

$$x^2 + y^2 - 4x - 2y - 11 = 0, h = (4, 5) \tag{1}$$

The equation of a conic section is given by,

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2}$$

So,equation of circle in (1) can be written in form of equation(2) and point P in vector form as,

$$\mathbf{x}^\top \mathbf{x} + 2\begin{pmatrix}-2 & -1\end{pmatrix}\mathbf{x} - 11 = 0, \mathbf{h} = \begin{pmatrix}4\\5\end{pmatrix} \tag{3}$$

From this,

$$\mathbf{V} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\mathbf{u} = \begin{pmatrix}-2\\-1\end{pmatrix}, \mathbf{O} = \begin{pmatrix}2\\1\end{pmatrix}, f = -11 \tag{4}$$

If $\mathbf{V}^{-1}$ exists, given normal vector $\mathbf{n}$, the tangent points of contact to equation(2) are given by,

$$\mathbf{q_i} = \mathbf{V}^{-1}(k_i\mathbf{n} - \mathbf{u})^T \tag{5}$$

$$where, k_i = \pm\sqrt{\frac{f_0}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{6}$$

$$f_0 = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{7}$$

(8)

The normal vectors of tangents from a point $\mathbf{h}$ to the conic(2) are given by'

$$\mathbf{n_1} = \mathbf{P}\begin{pmatrix}\sqrt{\lambda_1}\\\sqrt{\lambda_2}\end{pmatrix}, \mathbf{n_2} = \mathbf{P}\begin{pmatrix}\sqrt{\lambda_1}\\-\sqrt{\lambda_2}\end{pmatrix} \tag{9}$$

where $\lambda_i, \mathbf{P}$ are eigen parameters of

$$\sum = (\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^T - \mathbf{V}(\mathbf{h}^T\mathbf{V}\mathbf{h} + 2\mathbf{u}^T h + f) \tag{10}$$

So, by solving above equation,

$$\mathbf{n_1} = \begin{pmatrix}1.333\\1\end{pmatrix}, \mathbf{n_2} = \begin{pmatrix}0\\1\end{pmatrix} \tag{11}$$

By solving equation(6), using (4),(7) and (11), we get,

$$k = \pm 0.8968 \tag{12}$$

Solving equation(5), using (4),(11) and (12), we get,

$$\mathbf{A} = \begin{pmatrix}5.201\\3.403\end{pmatrix} \ and \ \mathbf{B} = \begin{pmatrix}2\\5.008\end{pmatrix} \tag{13}$$

Now, Area of $\triangle\mathbf{OBh}$ is given by,

$$ar(\triangle\mathbf{OBh}) = \frac{1}{2}\|\mathbf{BO} \times \mathbf{Bh}\| \tag{14}$$

$$ar(\triangle\mathbf{OBh}) = \frac{1}{2}\|(\mathbf{B}-\mathbf{O}) \times (\mathbf{B}-\mathbf{h})\|$$

$$ar(\triangle\mathbf{OBh}) = \frac{1}{2}\left\|\left(\begin{pmatrix}2\\5.008\end{pmatrix} - \begin{pmatrix}2\\1\end{pmatrix}\right) \times \left(\begin{pmatrix}2\\5.008\end{pmatrix} - \begin{pmatrix}4\\5\end{pmatrix}\right)\right\|$$

$$\implies \quad ar(\triangle\mathbf{OBh}) = 4squ \tag{15}$$

Area of Quadrilateral $\mathbf{OBhA}$ is given by,

$$ar(\mathbf{OBhA}) = 2ar(\triangle\mathbf{OBh}) \tag{16}$$

Therefore,

$$ar(\mathbf{OBhA}) = 8squ \tag{17}$$