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Conic Assignment

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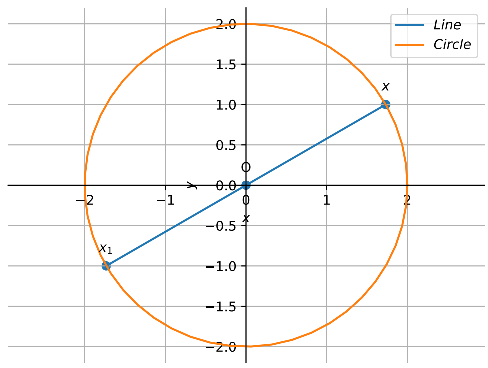
Problem Statement:

Find the area of the region in the first quadrant enclosed by x-axis, line $x = \sqrt{3}y$ and circle $x^2 + y^2 = 4$.

Figure:



Solution:

From the given information,

$$x^2 + y^2 = 4 \tag{1}$$

$$x - \sqrt{3}y = 0 \tag{2}$$

the above equations can be expressed in vector form as

$$\mathbf{x}^\top \mathbf{x} = r^2 \tag{3}$$

$$\mathbf{n}^\top \mathbf{x} = c \tag{4}$$

Construction

Symbol	Value	Description
r	2	radius of given circle
c	0	Line parameter
$\mathbf{n}$	$\begin{pmatrix} 1/\sqrt{3} \\ -1 \end{pmatrix}$	normal of the line
$\mathbf{m}$	$\begin{pmatrix} 1 \\ 1/\sqrt{3} \end{pmatrix}$	Direction vector of the line
$\mathbf{A}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	x-intercept of the line

The point of intersection of the line with the circle in the first quadrant is
Using the parameteric equation of the line

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{5}$$

Substituting the above in equation (3)

$$(\mathbf{A} + \lambda \mathbf{m})^\top (\mathbf{A} + \lambda \mathbf{m}) = r^2 \tag{6}$$

$$\implies \lambda^2 \|\mathbf{m}\|^2 + 2\lambda \mathbf{m}^\top \mathbf{A} + \|\mathbf{A}\|^2 - r^2 = 0 \tag{7}$$

yielding

$$\lambda = \frac{-\mathbf{m}^\top \mathbf{A} \pm \sqrt{(\mathbf{m}^\top \mathbf{A}^2) - \|\mathbf{m}\|^2 \left(\|\mathbf{A}\|^2 - r^2\right)}}{\|\mathbf{m}\|^2} \tag{8}$$

For this problem, the numerical values are

$$\mathbf{n} = \begin{pmatrix} 1/\sqrt{3} \\ -1 \end{pmatrix}, c = 0, \mathbf{m} = \begin{pmatrix} 1 \\ 1/\sqrt{3} \end{pmatrix}, \tag{9}$$

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, r^2 = 4 \tag{10}$$

Substituting the above values in equation (8) we get

$$\lambda = \sqrt{3} \tag{11}$$

By substituting the values in equation (5) the desired point of intersection is

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \sqrt{3} \begin{pmatrix} 1 \\ 1/\sqrt{3} \end{pmatrix} \tag{12}$$

$$\mathbf{x} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \tag{13}$$

we have $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\mathbf{x} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$ $\mathbf{p} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$
The direction vectors of lines Ox and Op are

$$\mathbf{X} = (\mathbf{O} - \mathbf{x}) = \begin{pmatrix} -\sqrt{3} \\ -1 \end{pmatrix} \tag{14}$$

$$\mathbf{P} = (\mathbf{O} - \mathbf{p}) = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{15}$$

The angle between two vectors is given by

$$\theta = \cos^{-1} \frac{\mathbf{X}^\top \mathbf{P}}{\|\mathbf{X}\| \|\mathbf{P}\|} \tag{16}$$

By substitting (14) and (15) in equation (16) we get

$$\theta = 30^\circ \tag{17}$$

The area of the sector is

$$\frac{\theta}{360} \pi r^2 \tag{18}$$

By substituting the values in above equation the desired region area is

$$\frac{\pi}{3} \tag{19}$$

Github link: Assignment-6.