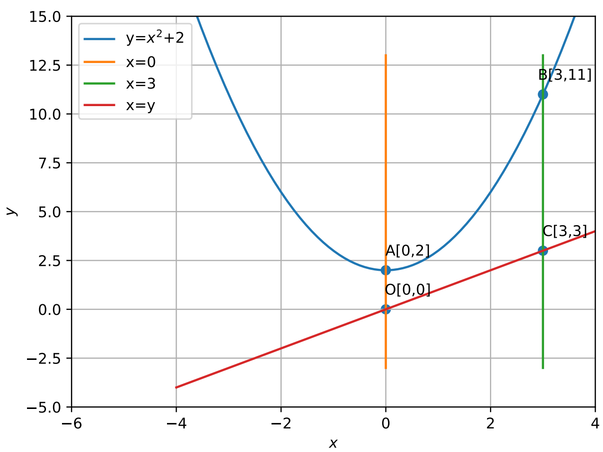
Conics Assignment

Sinkona Chinthamalla - FWC22054

I. Question

Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.



II. Construction

Symbol	Value	Description
A	$\begin{pmatrix} 0 \\ 2 \end{pmatrix}$	$\mathbf{q_1}$
B	$\begin{pmatrix} 3 \\ 11 \end{pmatrix}$	$\mathbf{q_2}$
C	$\begin{pmatrix} 3 \\ 3 \end{pmatrix}$	$\mathbf{q_3}$
O	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\mathbf{q_4}$

III. Solution

The equation of a conic with directrix $\mathbf{n}^\intercal\mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{1}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} \tag{3}$$

$$f = 2 \tag{4}$$

Finding Points of Intersection

1. Consider line

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = 0 \tag{5}$$

$$\mathbf{q_1} = y_1\mathbf{e_2} \tag{6}$$

To find the point of intersection of the Parabola with (5), substitute (6) in (1)

$$\mathbf{q_1}^\top\mathbf{V}\mathbf{q_1} + 2\mathbf{u}^\top\mathbf{q_1} + f = 0 \tag{7}$$

$$y_1^2\mathbf{e_2}^\top\mathbf{V}\mathbf{e_2} + 2y_1\mathbf{u}^\top\mathbf{e_2} + f = 0 \tag{8}$$

The value of y_1 is given by

1. when $\mathbf{e_2}^\top\mathbf{V}\mathbf{e_2} \neq 0$

$$y_1 = \frac{-b_1 \pm \sqrt{b_1^2 - 4a_1c_1}}{2a_1} \tag{9}$$

where,

$$a_1 = \mathbf{e_2}^\top\mathbf{V}\mathbf{e_2}$$
$$b_1 = 2\mathbf{u}^\top\mathbf{e_2}$$
$$c_1 = f$$

2. when $\mathbf{e_2}^\top\mathbf{V}\mathbf{e_2} = 0$

$$y_1 = \frac{-f}{2\mathbf{u}^\top\mathbf{e_2}} \tag{10}$$

From (2) and $\mathbf{e_2}$, $\mathbf{e_2}^\top\mathbf{V}\mathbf{e_2} = 0$
Therefore, y_1 is obtained by substituting (3), (4) and $\mathbf{e_2}$ in (10).

2. Now consider line

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = 3 \tag{11}$$

$$\mathbf{q_2} = y_2\mathbf{e_2} + 3\mathbf{e_1} \tag{12}$$

To find the point of intersection of the Parabola with (11), substitute (12) in (1)

$$\mathbf{q_2}^\top \mathbf{V}\mathbf{q_2} + 2\mathbf{u}^\top \mathbf{q_2} + f = 0 \tag{13}$$

$$y_2^2 \mathbf{e_2}^\top \mathbf{V}\mathbf{e_2} + 3y_2\mathbf{e_2}^\top \mathbf{V}\mathbf{e_1} + 3y_2\mathbf{e_1}^\top \mathbf{V}\mathbf{e_2} + 9\mathbf{e_1}^\top \mathbf{V}\mathbf{e_1}$$
$$+2y_2\mathbf{u}^\top \mathbf{e_2} + 6\mathbf{u}^\top \mathbf{e_1} + f = 0 \tag{14}$$

The value of y_2 is given by

1. when $\mathbf{e_2}^\top \mathbf{V}\mathbf{e_2} \neq 0$

$$y_2 = \frac{-b_2 \pm \sqrt{b_2^2 - 4a_2c_2}}{2a_2} \tag{15}$$

where,

$$a_2 = \mathbf{e_2}^\top \mathbf{V}\mathbf{e_2}$$
$$b_2 = 3\mathbf{e_2}^\top \mathbf{V}\mathbf{e_1} + 3\mathbf{e_1}^\top \mathbf{V}\mathbf{e_2} + 2\mathbf{u}^\top \mathbf{e_2}$$
$$c_2 = 9\mathbf{e_1}^\top \mathbf{V}\mathbf{e_1} + 6\mathbf{u}^\top \mathbf{e_1} + 2$$

2. when $\mathbf{e_2}^\top \mathbf{V}\mathbf{e_2} = 0$

$$y_2 = \frac{-f - 9\mathbf{e_1}^\top \mathbf{V}\mathbf{e_1}}{2\mathbf{u}^\top \mathbf{e_2}} \tag{16}$$

From (2) and $\mathbf{e_2}$, $\mathbf{e_2}^\top \mathbf{V}\mathbf{e_2} = 0$
Therefore, d_2 is obtained by substituting (2), (3), (4), $\mathbf{e_1}$ and $\mathbf{e_2}$ in (16).

Therefore, the Points of Intersection of (5) and (11) with the given parabola are

$$\mathbf{q_1} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{17}$$

and

$$\mathbf{q_2} = \begin{pmatrix} 3 \\ 11 \end{pmatrix} \tag{18}$$

respectively.

The Points of Intersection of (11) and (5) with the line

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{19}$$

are

$$\mathbf{q_3} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \tag{20}$$

$$\mathbf{q_4} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{21}$$

and
respectively.

Finding area of the bounded region
From Fig. 1, the area covered by the parabola is given by

$$\int_0^3 (x^2+2)dx = \frac{x^3}{3} + 2x\Big|_0^3 \tag{22}$$
$$= 15 \tag{23}$$

The area covered by (19) is given by

$$\int_0^3 x dx = \frac{x^2}{2}\Big|_0^3 \tag{24}$$
$$= \frac{9}{2} \tag{25}$$

Thus, the desired area is the bounded region in Fig. 1, and is given by

$$\frac{21}{2} \quad sq.units \tag{26}$$