Circle Assignment

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I. Question

Chapter 8, Section-B, Q(21): **The circle** $x^2+y^2 = 4x+8y+5$ **intersects the line** $3x-4y = m$ **at two distinct points if**

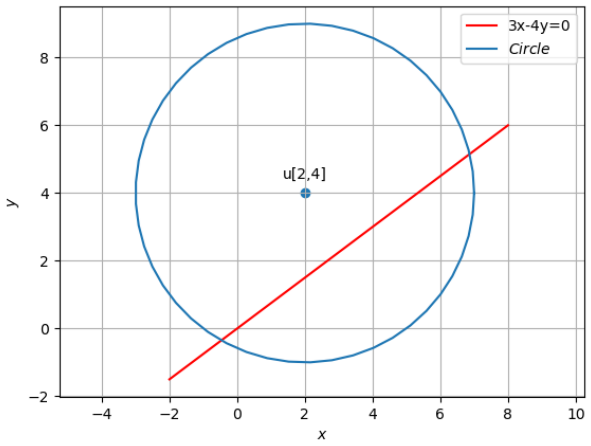

Figure 1: Circle with the line intersecting at two distinct points

II. Construction

Symbol	Value	Description
O	$\begin{pmatrix}2\\4\end{pmatrix}$	Center of Circle
r	5	Radius of Circle
n	$\begin{pmatrix}3\\-4\end{pmatrix}$	Normal vector
$\mathbf{P_1}$	$\begin{pmatrix}5\\0\end{pmatrix}$	Point of Contact P1
$\mathbf{P_2}$	$\begin{pmatrix}-1\\8\end{pmatrix}$	Point of Contact P2

III. Solution

The equation of a conic with directrix $\mathbf{n}^\intercal\mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{1}$$

where

$$\mathbf{V} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{2}$$

$$\mathbf{u} = \begin{pmatrix}-2\\-4\end{pmatrix} \tag{3}$$

$$f = -5 \tag{4}$$

If $\mathbf{V}^{-1}$ exists, given the normal vector $\mathbf{n}$, the tangent points of contact to $\mathbf{n}^\intercal\mathbf{x} = c$ are given by

$$\mathbf{q}_i = \mathbf{V}^{-1}\left(\kappa_i\mathbf{n} - \mathbf{u}\right), i = 1, 2 \tag{5}$$

$$\text{where } \kappa_i = \pm\sqrt{\frac{f_0}{\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n}}} \tag{6}$$

$$f_0 = \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} - f \tag{7}$$

Given equation of line,

$$3x - 4y = m \tag{8}$$

$$\begin{pmatrix}3 & -4\end{pmatrix}\mathbf{x} = m \tag{9}$$

Substituting (2), (3), (4) and (9) in (6) yields

$$\kappa = \pm 1 \tag{10}$$

By substituting (10) in (5), we get points of contact $\mathbf{P_1}$ and $\mathbf{P_2}$

For $\kappa = 1$

$$\mathbf{P_1} = \begin{pmatrix}5\\0\end{pmatrix} \tag{11}$$

For $\kappa = -1$

$$\mathbf{P_2} = \begin{pmatrix}-1\\8\end{pmatrix} \tag{12}$$

Substituiting $\mathbf{P_1}$ and $\mathbf{P_2}$ in (9) yields

$$\begin{pmatrix}3 & -4\end{pmatrix}\begin{pmatrix}5\\0\end{pmatrix} = m_1 \tag{13}$$

$$m_1 = 15 \tag{14}$$

$$\begin{pmatrix}3 & -4\end{pmatrix}\begin{pmatrix}-1\\8\end{pmatrix} = m_2 \tag{15}$$

$$m_2 = -35 \tag{16}$$

Therefore, the range of m is

$$-35 < m < 15$$

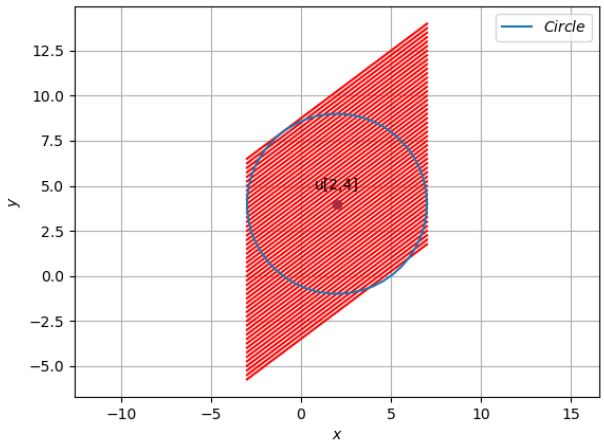

Figure 2: Circle with the line 3x-4y=m

Get the python code from

https://github.com/SinkonaChinthamalla/fwc/blob/main/matrix/line/codes