PYTHON PROGRAMMING ON MATRICES

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IITH Future Wireless Communication (FWC)

FWC22011 Matrix:conic

Contents

1 Problem

An arch is in the form of a semi-ellipse. It is 8 m wide and 2 m high at the centre.Find the height of the arch at a point 1.5 m from one end.

2 Construction

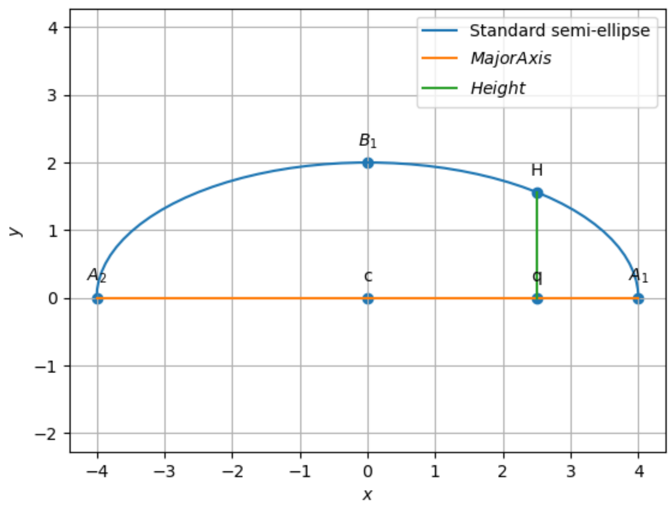

Figure of construction

3 Solution

Ellipse equation:

$$\frac{x^2}{16}+\frac{y^2}{4}=1 \tag{1}$$

The standard equation of the conics is given as :

$$\mathbf{x}^\top\mathbf{V}\mathbf{x}+2\mathbf{u}^\top\mathbf{x}+f=0 \tag{2}$$

The given ellipse can be expressed in conics as

$$\mathbf{u}=\begin{pmatrix}0\\0\end{pmatrix}, f=-1 \tag{3}$$

The input parameters for this construction are

Symbol	Value	Description
a	4	Length of semi major axis
b	2	Length of semi minor axis
$\mathbf{e_1}$	$\begin{pmatrix}1\\0\end{pmatrix}$	Standard basis vector along X-axis
$\mathbf{m}$	$\begin{pmatrix}0\\1\end{pmatrix}$	Directional vector along Y-axis

The steps for constructing above figure are :

1. Generate semi-ellipse with semi major axis and semi minor axis lengths equal to $\mathbf{a}$ and $\mathbf{b}$ respectively.
2. Locate center $\mathbf{c}$ and vertices $\mathbf{A_1}$ and $\mathbf{A_2}$.
3. Locate point $\mathbf{q}$ on the major axis.
4. Find the height of ellipse at $\mathbf{q}$.

For the standard ellipse, the length of the major axis and minor axis are:

$$2\sqrt{\left|\frac{f_0}{\lambda_1}\right|} \tag{4}$$

$$2\sqrt{\left|\frac{f_0}{\lambda_2}\right|} \tag{5}$$

Given ,The major axis and minor axis are 8m and 4m in length respectively.

$*f_0=\mathbf{u}^\top\mathbf{v}^{-1}\mathbf{u}-f=1$

Equation (5) $\implies 2\sqrt{\left|\frac{f_0}{\lambda_1}\right|}=8$

$\implies \lambda_1=1/16$

Equation (6) $\implies 2\sqrt{\left|\frac{f_0}{\lambda_2}\right|}=4$

$\implies \lambda_2=1/4$

$$\implies \mathbf{v}=\begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}=\begin{pmatrix}1/16 & 0\\0 & 1/4\end{pmatrix} \tag{6}$$

vertices: $\mathbf{v}=\pm a\mathbf{e_1}$

$$\mathbf{v}=\pm a\begin{pmatrix}1\\0\end{pmatrix}=\pm\begin{pmatrix}4\\0\end{pmatrix} \tag{7}$$

$$Let, \mathbf{A_1}=\begin{pmatrix}4\\0\end{pmatrix}, \mathbf{A_2}=\begin{pmatrix}-4\\0\end{pmatrix} \tag{8}$$

To find the height of ellipse at a point 1.5m from end,

$$\implies \|\mathbf{A_1}-\mathbf{q}\|^2 = (1.5)^2 \tag{9}$$
$$(\mathbf{A_1}-\mathbf{q})^\top(\mathbf{A_1}-\mathbf{q}) = (1.5)^2 \tag{10}$$
$$\left\|\mathbf{A_1}^2\right\| + \left\|\mathbf{q}^2\right\| - 2\mathbf{A_1}^\top\mathbf{q} = (1.5)^2 \tag{11}$$
$$\|\mathbf{q}\|^2 - 2\mathbf{A}^\top\mathbf{q} + 13.75 = 0 \tag{12}$$
$$\mathbf{e_2}^\top\mathbf{q} = 0 \tag{13}$$
$$\implies \mathbf{q} = \lambda\mathbf{e_1} \tag{14}$$

substitute (14) in (12); $\implies \lambda^2 - 8\lambda + 13.75 = 0$
$\implies \lambda = \frac{5}{2}, \frac{11}{2}$
The length of semi major axis is 4m,we need to find height of ellipse at a point 1.5m from one end.
$\therefore$ the possible solution is $\lambda = \frac{5}{2}$
λ lies on x-axis.

$$\implies \mathbf{q} = \begin{pmatrix} \frac{5}{2} \\ 0 \end{pmatrix} \tag{15}$$

Directional vector m:
The unit vector along $\mathbf{Y}$-axis become the directional vector along $\mathbf{Y}$-axis.

$$\implies \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{16}$$

Theorem: The points of intersection of the line

$$L: \quad \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \quad \mu \in \mathbb{R} \tag{17}$$

with the conic section in (2) are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i\mathbf{m} \tag{18}$$

where μ_i is given by

$$\mu_i = \frac{1}{\mathbf{m}^\top\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^\top(\mathbf{V}\mathbf{q}+\mathbf{u}) \pm Z\right) \tag{19}$$

Z $= \sqrt{[\mathbf{m}^\top(\mathbf{V}\mathbf{q}+\mathbf{u})]^2 - (\mathbf{q}^\top\mathbf{V}\mathbf{q} + 2\mathbf{u}^\top\mathbf{q} + f)(\mathbf{m}^\top\mathbf{V}\mathbf{m})}$

By substituting the vectors $\mathbf{m}$,$\mathbf{q}$,$\mathbf{v}$,$\mathbf{u}$ and constant f in (19) results intersection points on the conic section .Consider absolute value ,say $\mathbf{H}$.

$\mathbf{H}$ gives height of ellipse at point $\mathbf{q}$.

(or)

$\|\mathbf{H}-\mathbf{q}\|$ results the same.
$\therefore$ Height of ellipse at $\mathbf{q}$=1.56.

termux commands :

bash conic.sh............using shell command

Below python code realizes the above construction :

https://github.com/FWC_module1/blob/main/matrices/conic/conic.py