Conics Assignment

lakshmi kamakshi

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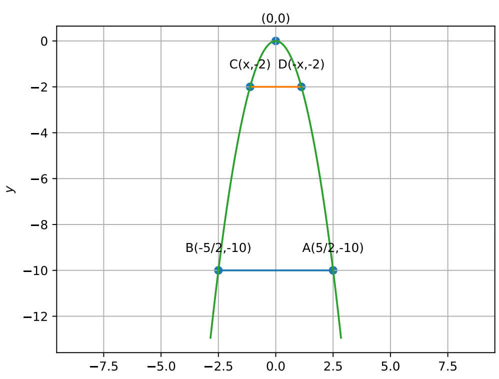
Problem Statement **-An arch is in the form of a parabola with its axis vertical.The arch is 10m high and 5m wide at the base.How wide is it 2m from the vertex of the parabola?**

Solution

Given, the axis of parabola is vertical,
Let the equation of the axis be y-axis:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{x} = 0 \tag{1}$$

The above quadratic equation can be written in the general quadratic form as:



$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{2}$$

where,

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3}$$

$$u = \begin{pmatrix} 0 \\ 2a \end{pmatrix} \tag{4}$$

$$f = 0 \tag{5}$$

Given arch is 10m high and 5m wide at the base. So the point $(\frac{5}{2}, -10)$ lies on the parabola

$$(X_1) = \begin{pmatrix} \frac{5}{2} \\ -10 \end{pmatrix} \tag{6}$$

Substitute the point X_1

$$(X_1)^T (V) (X_1) + 2 (u^T) (X_1) = 0 \tag{7}$$

$$a = \frac{5}{32} \tag{8}$$

Now , the matrix u will be,

$$(u) = \begin{pmatrix} 0 \\ \frac{5}{16} \end{pmatrix} \tag{9}$$

We need to find the width of parabola at a height of $2m$ from the vertex.So,the line parallel to x axis and passing through point $(0, -2)$ intersects the conic at 2 places C and D.
The line parallel to x-axis and passing through point $(0, -2)$ is

$$(X) \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T = (2) \tag{10}$$

The points of intersection of the line

$$L: \quad \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbf{R} \tag{11}$$

with the conic section are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \tag{12}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} - \mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u}) \pm \sqrt{\mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u})^2 - \left(\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f\right) \left(\mathbf{m}^T \mathbf{V} \mathbf{m}\right)} \tag{13}$$

Substituting the line in the conic

$$(\mathbf{q} + \mu \mathbf{m})^T \mathbf{V} (\mathbf{q} + \mu \mathbf{m}) \tag{14}$$

$$+2\mathbf{u}^T (\mathbf{q} + \mu \mathbf{m}) + f = 0 \tag{15}$$

$$\implies \mu^2 \mathbf{m}^T \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u}) \tag{16}$$

$$+\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f = 0 \tag{17}$$

Solving the above quadratic equations yeilds the roots.Let the point of intersections of line and curve be C and D.

$$C = q + \mu_1 m \tag{18}$$

$$D = q + \mu_2 m \tag{19}$$

The line CD will be

$$C - D = m(\mu_1 - \mu_2) \tag{20}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{21}$$

The required width of parabola is the norm of the line CD.

$$||\boldsymbol{C}-\boldsymbol{D}|| = 2\sqrt{\mathbf{m}^T\left(\mathbf{Vq}+\mathbf{u}\right)^2-\left(\mathbf{q}^T\mathbf{Vq}+2\mathbf{u}^T\mathbf{q}+f\right)} \quad (22)$$

substitute the values of m,q,V and u

$$\frac{1}{2}||\mathbf{C}-\mathbf{D}||^2 = \begin{pmatrix}1 & 0\end{pmatrix}\left(\mathbf{V}\begin{pmatrix}0\\-2\end{pmatrix}+\mathbf{u}\right)\left(\mathbf{V}\begin{pmatrix}0\\-2\end{pmatrix}+\mathbf{u}\right)^T - \quad (23)$$

$$\left(\begin{pmatrix}0 & -2\end{pmatrix}\mathbf{V}\begin{pmatrix}0\\-2\end{pmatrix}+2\mathbf{u}^T\begin{pmatrix}0\\-2\end{pmatrix}\right)$$

$$\implies \begin{pmatrix}1 & 0\end{pmatrix}\left(\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0\\-2\end{pmatrix}+\begin{pmatrix}0\\\frac{5}{16}\end{pmatrix}\right)\left(\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0\\-2\end{pmatrix}+\begin{pmatrix}0\\\frac{5}{16}\end{pmatrix}\right)^T -$$

$$\left(\begin{pmatrix}0 & -2\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0\\-2\end{pmatrix}+2\begin{pmatrix}0 & \frac{5}{16}\end{pmatrix}\begin{pmatrix}0\\-2\end{pmatrix}-0\right)$$
$$\left(\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$\implies \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\\frac{5}{16}\end{pmatrix}^2-\left(\begin{pmatrix}0\\0\end{pmatrix}+2\begin{pmatrix}0\\\frac{5}{-8}\end{pmatrix}-0\right)(1) \quad (24)$$

$$= \begin{pmatrix}5\\2\end{pmatrix} \quad (25)$$

The width of the Parabola at $2m$ height is the length of the line CD.

$$||\mathbf{C}-\mathbf{D}|| = (\sqrt{5})\mathrm{m} \quad (26)$$

Construction

The input parameters are V,u,X_1,y_2

Symbol	Value	Description
$X_1=\begin{pmatrix}x_1\\y_1\end{pmatrix}$	$\begin{pmatrix}\frac{5}{2}\\-10\end{pmatrix}$	point at base
y_2	-2	height of point C
$\mathbf{P}$	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	eigenvectors of $\mathbf{V}$
$\mathbf{c}$	$\begin{pmatrix}0\\0\end{pmatrix}$	center of parabola
η	$\mathbf{u}^\top\mathbf{p}_1$	from Eq11
λ_2	$\mathbf{e}_2^\top D\mathbf{e}_2$	from Eq9
$(\mathbf{A},\mathbf{B})$	$\begin{pmatrix}x_1 & -x_1\\y_1 & y_1\end{pmatrix}$	points at the base
$(\mathbf{C},\mathbf{D})$	$\begin{pmatrix}\sqrt{\frac{5y_2}{8}} & \sqrt{\frac{-5y_2}{8}}\\2 & 2\end{pmatrix}$	points at 2m height