Conic Assignment

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Problem

The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}$=1,and having centre at (0,3) is

Solution

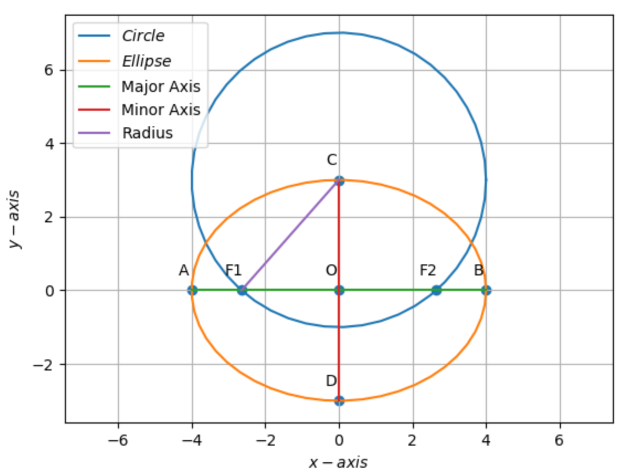

Figure 1: Ellpise with center O along with Circle C

Step1

Given, equation of an ellipse and centre of a circle passing through foci of the ellipse are,

$$9x^2 + 16y^2 - 144 = 0, \quad c = (0, 3) \tag{1}$$

The equation of a conic with diretrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and Focus $\mathbf{F}$ is given by,

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2}$$

So,equation of ellipse in (1) can be written in form of (2) and centre in vector form as,

$$\mathbf{x}^\top \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 & 0 \end{pmatrix} \mathbf{x} - 144 = 0, \quad \mathbf{C} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{3}$$

From this,

$$\mathbf{V} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \& \quad f = -144 \tag{4}$$

Since $\mathbf{V}$ is symmetric,The eigenvalue decomposition of a symmetric matrix $\mathbf{V}$ is given by

$$\mathbf{P}^\top \mathbf{V}\mathbf{P} = \mathbf{D} \tag{5}$$

$$where, \mathbf{P} = \mathbf{I}, \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{6}$$

So, On solving (5) using (6) and (4), we get

$$\mathbf{D} = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix} \tag{7}$$

This implies,

$$\lambda_1 = 9 \;\; and \;\; \lambda_2 = 16 \tag{8}$$

We have eccentricity,

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{9}$$

from (8),

$$e = 0.6614 \tag{10}$$

For e $\neq$ 1, we have

$$c = \frac{e\mathbf{u}^\top \mathbf{n} \pm \sqrt{e^2 \left(\mathbf{u}^\top \mathbf{n}\right)^2 - \lambda_2 \left(e^2 - 1\right) \left(\left\|\mathbf{u}\right\|^2 - \lambda_2 f\right)}}{\lambda_2 e \left(e^2 - 1\right)} \tag{11}$$

Normal vector of diretrix $\mathbf{n}$ is given by

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{P_1} \tag{12}$$

This gives,

$$\mathbf{n} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{13}$$

On solving equation(11), using (4),(8),(10) and (13), we get,

$$c = \pm 24.1911 \tag{14}$$

Focii of a conic are given by,

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{15}$$

On solving, it yields,

$$\mathbf{F} = \begin{pmatrix} \pm 2.6456 \\ 0 \end{pmatrix} \tag{16}$$

Therefore, focii of the ellipse are,

$$\mathbf{F_1} = \begin{pmatrix} -2.6456 \\ 0 \end{pmatrix} \quad \& \quad \mathbf{F_2} = \begin{pmatrix} 2.6456 \\ 0 \end{pmatrix} \tag{17}$$

Let the equation of circle passing through the ellipse be,

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u_1}^\top \mathbf{x} + f_1 = 0 \tag{18}$$

$$where, \quad \mathbf{V} = \mathbf{I} \, and \;\; \mathbf{u_1} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \tag{19}$$

Since circle is passing through $\mathbf{F_1}$,

$$\mathbf{F_1}^\top \mathbf{V}\mathbf{F_1} + 2\mathbf{u_1}^\top \mathbf{F_1} + f_1 = 0 \tag{20}$$

$$\begin{pmatrix} -2.6456 & 0 \end{pmatrix} \begin{pmatrix} -2.6456 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & -3 \end{pmatrix} \begin{pmatrix} -2.6456 \\ 0 \end{pmatrix} + f_1 = 0 \tag{21}$$

$$\implies f_1 = -6.99$$

Hence, Equation of the circle is given as,

$$\mathbf{x}^\top \mathbf{x} + 2 \begin{pmatrix} 0 & -3 \end{pmatrix} \mathbf{x} - 6.99 = 0 \tag{22}$$

Input parameters for this construction are:

Symbol	Value	Description
$\mathbf{O}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Center of Ellipse
$\mathbf{A}$	$\begin{pmatrix} -4 \\ 0 \end{pmatrix}$	Extreme point of major axis
$\mathbf{B}$	$\begin{pmatrix} 4 \\ 0 \end{pmatrix}$	Extreme point of major axis
$\mathbf{C}$	$\begin{pmatrix} 0 \\ 3 \end{pmatrix}$	Extreme point of minor axis and Center of circle (C)
$\mathbf{D}$	$\begin{pmatrix} 0 \\ -3 \end{pmatrix}$	Extreme point of minor axis
$\mathbf{F_1}$	$\begin{pmatrix} -2.6456 \\ 0 \end{pmatrix}$	Focus 1 of Ellipse
$\mathbf{F_2}$	$\begin{pmatrix} 2.6456 \\ 0 \end{pmatrix}$	Focus 2 of Ellipse

Table 1: Parameter's Table