Lines using Python

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Problem

The equations to the pair of opposite sides of a parallelogram are

$$x^2 - 5x + 6 = 0, \tag{1}$$
$$y^2 - 6y + 5 = 0. \tag{2}$$

Find the equations to the diagonals of the parallelogram.

Solution

On factorizing (1) and (2) we get,

$$(x-2)(x-3) = 0,$$
$$(y-1)(y-5) = 0.$$

So the x and y intercepts of the given lines are

$$\mathbf{X_1} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \mathbf{X_2} = \begin{pmatrix} 3 \\ 0 \end{pmatrix},$$
$$\text{and } \mathbf{Y_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{Y_2} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

The pair of straight lines represented by (1) are parallel to y-axis and the pair of straight lines represented by (2) are parallel to x-axis.

We can also say that the lines in (1) are normal to $\mathbf{e_1}$ and the lines in (2) are normal to $\mathbf{e_2}$, where, $\mathbf{e_1}$ and $\mathbf{e_2}$ are standard basis vectors, given by

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Equations of the sides of parallelogram in vector form,

$$L_1: \quad \mathbf{e_1}^\top (\mathbf{x} - \mathbf{X_1}) = 0, \tag{3}$$
$$L_2: \quad \mathbf{e_2}^\top (\mathbf{x} - \mathbf{Y_1}) = 0, \tag{4}$$
$$L_3: \quad \mathbf{e_1}^\top (\mathbf{x} - \mathbf{X_2}) = 0, \tag{5}$$
$$L_3: \quad \mathbf{e_2}^\top (\mathbf{x} - \mathbf{Y_2}) = 0. \tag{6}$$

Intersection of given sides yields the vertices of parallelogram. Let's find the point A, say, which is intersection of line L_1 and line L_2 .

On combining (3) and (4),

$$\begin{pmatrix} \mathbf{e_1} & \mathbf{e_2} \end{pmatrix}^\top \mathbf{A} = \begin{pmatrix} \mathbf{e_1}^\top \mathbf{X_1} \\ \mathbf{e_2}^\top \mathbf{Y_1} \end{pmatrix}$$
$$\text{Take, } \mathbf{n} = \begin{pmatrix} \mathbf{e_1} & \mathbf{e_2} \end{pmatrix}^\top \tag{7}$$
$$\implies \mathbf{A} = \mathbf{n}^{-1} \begin{pmatrix} \mathbf{e_1}^\top \mathbf{X_1} \\ \mathbf{e_2}^\top \mathbf{Y_1} \end{pmatrix}$$

Similarly, we can obtain other vertices as

$$\mathbf{B} = \mathbf{n}^{-1} \begin{pmatrix} \mathbf{e_1}^\top \mathbf{X_2} \\ \mathbf{e_2}^\top \mathbf{Y_1} \end{pmatrix},$$
$$\mathbf{C} = \mathbf{n}^{-1} \begin{pmatrix} \mathbf{e_1}^\top \mathbf{X_1} \\ \mathbf{e_2}^\top \mathbf{Y_2} \end{pmatrix},$$
$$\text{and } \mathbf{D} = \mathbf{n}^{-1} \begin{pmatrix} \mathbf{e_1}^\top \mathbf{X_2} \\ \mathbf{e_2}^\top \mathbf{Y_2} \end{pmatrix}.$$

where,

B is the intersection of lines L_2 and L_3,
C is the intersection of lines L_3 and L_4,
D is the intersection of lines L_4 and L_1.

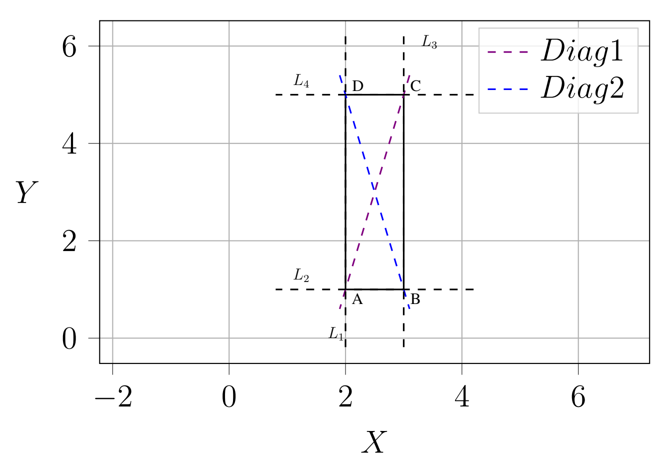

Fig. 1: Parallelogram and it's diagonals generated using python

Let the normal vectors to the diagonals $\overrightarrow{AC}$ and $\overrightarrow{BD}$ be $\mathbf{n_1}$ and $\mathbf{n_2}$ respectively,

$$\mathbf{n_1} = \mathbf{R}_{\frac{\pi}{2}} \left(\mathbf{A} - \mathbf{C}\right) \tag{8}$$

$$\mathbf{n_2} = \mathbf{R}_{\frac{\pi}{2}} \left(\mathbf{B} - \mathbf{D}\right) \tag{9}$$

where,

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Therefore, the equations of the diagonals passing through A and B in normal form,

$$\mathbf{n_1}^\top \left(\mathbf{x} - \mathbf{A}\right) = 0, \tag{10}$$

$$\mathbf{n_2}^\top \left(\mathbf{x} - \mathbf{B}\right) = 0. \tag{11}$$

On substituting the respective values, (10) and (11) becomes

$$\begin{pmatrix} 4 & -1 \end{pmatrix} \mathbf{x} = 7,$$

$$\begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} = 13.$$