

Conic Assignment

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Find the equation of the normal to curve $x^2 = 4y$ which passes through the point (1, 2).

Solution

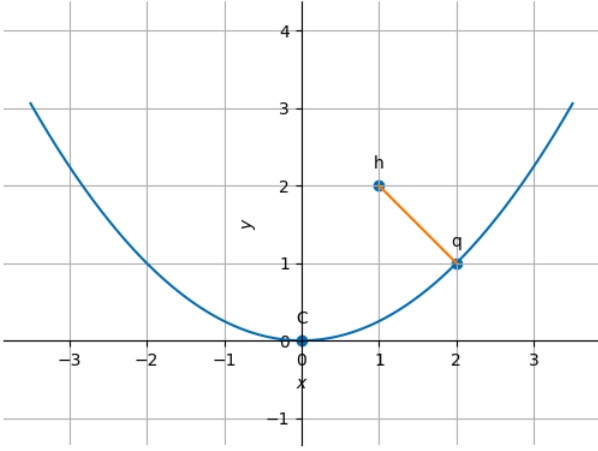


Figure 1: Tangents from A to circle through B, C and D

The given equation of parabola $x^2 = 4y$ can be written in the general quadratic form as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1)$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (2)$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \quad (3)$$

$$f = 0 \quad (4)$$

Let the point from which normals are drawn be \mathbf{h} . Then, the equation of the normal can be written as

$$\mathbf{x} = \mathbf{h} + \lambda \mathbf{m} \quad (5)$$

Say the point of intersection of (5) with the conic is \mathbf{q} . A tangent drawn at \mathbf{q} satisfies the equation

$$\mathbf{n}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) = 0 \quad (6)$$

Where \mathbf{n} is the direction vector of the tangent and is perpendicular to \mathbf{m} in (5).

In general, the parameter values for points of intersection of a line given by (5) with a conic is given by

$$\lambda_i = \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^T (\mathbf{V} \mathbf{h} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^T (\mathbf{V} \mathbf{h} + \mathbf{u})]^2 - (\mathbf{h}^T \mathbf{V} \mathbf{h} + 2\mathbf{u}^T \mathbf{h} + f) (\mathbf{m}^T \mathbf{V} \mathbf{m})} \right) \quad (7)$$

Using (7) and (5), the intersection point \mathbf{q} can be written as

$$\mathbf{q} = \mathbf{h} + \lambda_i \mathbf{m} \quad (8)$$

Substituting (8) in (6),

$$\mathbf{n}^T (\mathbf{V} (\mathbf{h} + \lambda_i \mathbf{m}) + \mathbf{u}) = 0 \quad (9)$$

$$\Rightarrow \lambda_i \mathbf{n}^T \mathbf{V} \mathbf{m} = -\mathbf{n}^T (\mathbf{V} \mathbf{h} + \mathbf{u}) \quad (10)$$

Substituting value of λ_i from (7) in (10)

$$\frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^T (\mathbf{V} \mathbf{h} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^T (\mathbf{V} \mathbf{h} + \mathbf{u})]^2 - (\mathbf{h}^T \mathbf{V} \mathbf{h} + 2\mathbf{u}^T \mathbf{h} + f) (\mathbf{m}^T \mathbf{V} \mathbf{m})} \right) \mathbf{n}^T \mathbf{V} \mathbf{m} = -\mathbf{n}^T (\mathbf{V} \mathbf{h} + \mathbf{u}) \quad (11)$$

Rearranging the terms,

$$\pm \sqrt{[\mathbf{m}^T (\mathbf{V} \mathbf{h} + \mathbf{u})]^2 - (\mathbf{h}^T \mathbf{V} \mathbf{h} + 2\mathbf{u}^T \mathbf{h} + f) (\mathbf{m}^T \mathbf{V} \mathbf{m})} (\mathbf{n}^T \mathbf{V} \mathbf{m}) = (\mathbf{V} \mathbf{h} + \mathbf{u})^T ((\mathbf{n}^T \mathbf{V} \mathbf{m}) \mathbf{m} - (\mathbf{m}^T \mathbf{V} \mathbf{m}) \mathbf{n}) \quad (12)$$

Squaring on both sides

$$[\mathbf{m}^T (\mathbf{V} \mathbf{h} + \mathbf{u})]^2 - (\mathbf{h}^T \mathbf{V} \mathbf{h} + 2\mathbf{u}^T \mathbf{h} + f) (\mathbf{m}^T \mathbf{V} \mathbf{m}) (\mathbf{n}^T \mathbf{V} \mathbf{m})^2 = [(\mathbf{V} \mathbf{h} + \mathbf{u})^T ((\mathbf{n}^T \mathbf{V} \mathbf{m}) \mathbf{m} - (\mathbf{m}^T \mathbf{V} \mathbf{m}) \mathbf{n})]^2 \quad (13)$$

If \mathbf{n} is taken as $\begin{pmatrix} -\mu \\ 1 \end{pmatrix}$, then \mathbf{m} is $\begin{pmatrix} -1 \\ -\mu \end{pmatrix}$. Substituting these values in (13) and solving for μ , the different possible normals passing through \mathbf{h} are obtained.

Thus after solving we get the following values for $\mu = -1, \frac{1}{2} - \frac{\sqrt{-3}}{2}, \frac{1}{2} + \frac{\sqrt{-3}}{2}$

Taking $\mu=1$ we get,

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

By calculating λ_i from (10), we get

$$\lambda_i = -1$$

We find out \mathbf{q} from (8),

$$\text{where } \mathbf{h} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \lambda_i = -1$$

$$\mathbf{q} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (-1) \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Equation of normal is given by

$$\mathbf{n}^\top \mathbf{x} = \mathbf{c} \quad (14)$$

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = c \quad (15)$$

We get $c=3$

Equation of normal is,

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 3 \quad (16)$$

Construction

Symbol	Value	Description
\mathbf{h}	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	Given point through which Normal is passing
\mathbf{q}	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$	Foot of Normal
\mathbf{m}	$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$	Direction Vector of Normal
\mathbf{n}	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	Direction Vector of Tangent at \mathbf{q}