Circle Assignment

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Problem Statement

- Prove that the angle between the two tangents drawn from an external point to a circle is supplementary to the angle subtended by the line-segment joining the points of contact at the centre.

Solution

Construction

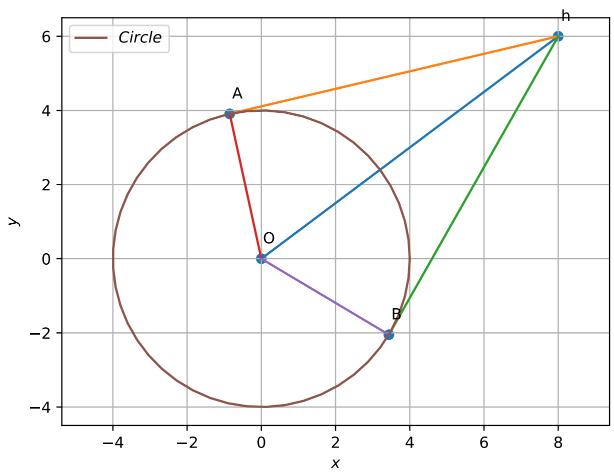

Figure 1: Figure

The dimensions of the figure is taken as below

symbol	value
Origin	(0,0)
r	4
h	(8,6)

TO PROVE :

$$\angle \boldsymbol{AOB} + \angle \boldsymbol{AhB} = \mathbf{180^\circ} \tag{1}$$

The equation of a conic with directrix $\mathbf{n^T x} = c$, eccentricity e anf focus $\boldsymbol{f}$ is given by

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T + f = 0 \tag{2}$$

for circle eccentricity e = 0 then,

$$\mathbf{V} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f = -r^2 \tag{3}$$

Point q on conic is given by

$$\mathbf{q} = \mathbf{V}^{-1}(\mathbf{n} - \mathbf{u}) \tag{4}$$

where, $\mathbf{n}$ is the normal vectors of the tangents from a point h to the conic are given by

$$\mathbf{n} = \frac{\mathbf{e_1}}{\mathbf{e_1^T} h} + \mu_i \mathbf{Rh} \tag{5}$$

where μ_i 's are given by the following equation

$$\mu_i = \frac{1}{\mathbf{m^T V m}}(-\mathbf{m^T}(\mathbf{Vq} + \mathbf{u}) \tag{6}$$

$$\pm\sqrt{[\mathbf{m^T}(\mathbf{Vq} + \mathbf{u})]^2 - (\mathbf{q^T V q} + \mathbf{2u^T} + f)(\mathbf{m^T V m})})$$

μ_i 's are obtained by substituting the following in equation 6

$$\mathbf{m} = \mathbf{Rh} = \begin{pmatrix} -2 \\ 8 \end{pmatrix}; \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \mathbf{q} = \frac{\mathbf{e_1}}{\mathbf{e_1^T} h} \tag{7}$$

R = $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$

The obtained μ_i's are substituted in equation 5 and equation 5 is substituted in equation 6 the required points on conic A and B are obtained.

Calculation Part

By Solving equation number 6 using equation number 7 parameters we will get two μ_i values
Therefore ,

$\mu_i = \pm 0.488525$

n1 is obtained by substituting μ_i = 0.488525
n2 is obtained by substituting μ_i = -0.488525

$$\boldsymbol{n1} = \frac{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} 8 \\ 6 \end{pmatrix}} + \mu_1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 8 \\ 6 \end{pmatrix} = \begin{pmatrix} -0.8 \\ 3.9 \end{pmatrix} \tag{8}$$

$$\boldsymbol{n2} = \frac{\begin{pmatrix}1\\0\end{pmatrix}}{\begin{pmatrix}1\\0\end{pmatrix}^T\begin{pmatrix}8\\6\end{pmatrix}} + \mu_2\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}8\\6\end{pmatrix} = \begin{pmatrix}3.43\\-2.04\end{pmatrix} \tag{9}$$

$$\boldsymbol{A} = \left(\boldsymbol{V}\right)^{-1}\left(\boldsymbol{n_1} - \boldsymbol{u}\right) = \begin{pmatrix}-0.8\\3.9\end{pmatrix} \tag{10}$$

$$\boldsymbol{B} = \left(\boldsymbol{V}\right)^{-1}\left(\boldsymbol{n_2} - \boldsymbol{u}\right) = \begin{pmatrix}3.43\\-2.04\end{pmatrix} \tag{11}$$

Now the point A and B are formed and tangents are drawn

To find the angle between AOB and AhB use inner product method

$$\angle AOB = cos^{-1}\frac{(\mathbf{A}-\mathbf{O})^T(\mathbf{B}-\mathbf{O})}{\|(\mathbf{A}-\mathbf{O})\|\,\|(\mathbf{B}-\mathbf{O})\|} \tag{12}$$

$$\angle AOB = cos^{-1}\frac{\left(\begin{pmatrix}-0.8\\3.9\end{pmatrix}-\begin{pmatrix}0\\0\end{pmatrix}\right)^T\left(\begin{pmatrix}3.43\\-2.04\end{pmatrix}-\begin{pmatrix}0\\0\end{pmatrix}\right)}{\left\|(A-O)\right\|\,\left\|(B-O)\right\|} \tag{13}$$

$$\angle AOB = 2.32radians = 2.32 * \frac{180}{\pi} = 133^\circ \tag{14}$$

$$\angle AhB = cos^{-1}\frac{(\mathbf{h}-\mathbf{A})^T(\mathbf{h}-\mathbf{B})}{\|(\mathbf{h}-\mathbf{A})\|\,\|(\mathbf{h}-\mathbf{B})\|} \tag{15}$$

$$\angle AhB = cos^{-1}\frac{\left(\begin{pmatrix}8\\6\end{pmatrix}-\begin{pmatrix}-0.8\\3.9\end{pmatrix}\right)^T\left(\begin{pmatrix}8\\6\end{pmatrix}-\begin{pmatrix}3.43\\-2.04\end{pmatrix}\right)}{\left\|(h-A)\right\|\,\left\|(h-B)\right\|} \tag{16}$$

$$\angle AhB = 0.82radians = 0.82 * \frac{180}{\pi} = 47^\circ \tag{17}$$

If $\angle AOB + \angle AhB = 180^\circ$ then
Angle AOB and angle AhB form a supplementary angle. Therefore,

$$\boldsymbol{\angle AOB + \angle AhB = 180^\circ} \tag{18}$$

$$\boldsymbol{133^\circ + 47^\circ = 180^\circ} \tag{19}$$