Circle Assignment

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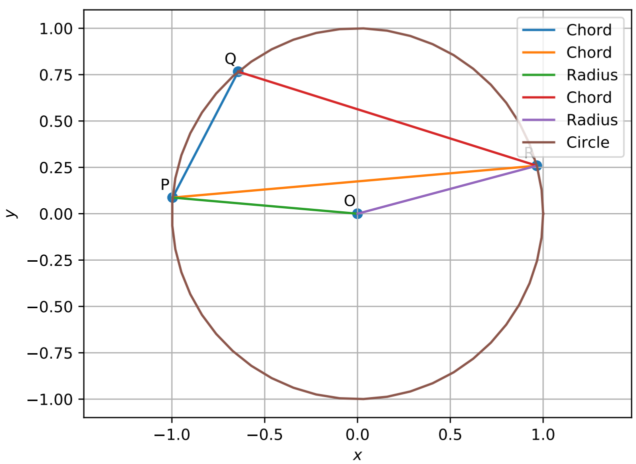
***Problem Statement* - Let ∠PQR = 100° where P,Q and R are points on a circle with centre O.Find ∠OPR**

Solution



Given ∠PQR=100°

Construction

Symbol	Value	Description
O		Centre
∠PQR	100°	Angle between vectors P and R
∠OPR	??	Angle b/w vectors O and R w.r.to P

Proof:

From assumptions the vector points P,Q,R be

$$\mathbf{P} = \begin{pmatrix} cos\theta_1 \\ sin\theta_1 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} cos\theta_2 \\ sin\theta_2 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} cos\theta_3 \\ sin\theta_3 \end{pmatrix}, \mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1)$$

$$cos(\angle PQR) = \frac{(P-Q)^T(R-Q)}{\|P-Q\|\,\|R-Q\|} \quad (2)$$

Where

$$\mathbf{P-Q} = \begin{pmatrix} cos\theta_1 - cos\theta_2 \\ sin\theta_1 - sin\theta_2 \end{pmatrix}, \mathbf{R-Q} = \begin{pmatrix} cos\theta_3 - cos\theta_2 \\ sin\theta_3 - sin\theta_2 \end{pmatrix} \quad (3)$$

$$(P-Q)^T(R-Q) = \begin{pmatrix} cos\theta_1 - cos\theta_2 sin\theta_1 - sin\theta_2 \end{pmatrix} \begin{pmatrix} cos\theta_3 - cos\theta_2 \\ sin\theta_3 - sin\theta_2 \end{pmatrix}$$

$$= (\cos\theta_1 - \cos\theta_2)(\cos\theta_3 - \cos\theta_2) + (\sin\theta_1 - \sin\theta_2)(\sin\theta_3 - \sin\theta_2)$$

$$= -2\sin\frac{\theta_1-\theta_2}{2}\sin\frac{\theta_1+\theta_2}{2}\cdot(-2)\sin\frac{\theta_3-\theta_2}{2}\sin\frac{\theta_3+\theta_2}{2} + 2\cos\frac{\theta_1+\theta_2}{2}\sin\frac{\theta_1-\theta_2}{2}\cdot 2\cos\frac{\theta_2+\theta_3}{2}\sin\frac{\theta_3-\theta_2}{2}$$

$$= 4\sin\frac{\theta_1-\theta_2}{2}\sin\frac{\theta_3-\theta_2}{2}(\sin\frac{\theta_1+\theta_2}{2}\sin\frac{\theta_3+\theta_2}{2} + \cos\frac{\theta_1+\theta_2}{2}\cos\frac{\theta_3+\theta_2}{2})$$

$$= 4\sin\frac{\theta_1-\theta_2}{2}\sin\frac{\theta_3-\theta_2}{2}\cos\left(\frac{\theta_1+\theta_2}{2} - \frac{\theta_3+\theta_2}{2}\right)$$

$$= 4\sin\frac{\theta_1-\theta_2}{2}\sin\frac{\theta_3-\theta_2}{2}\cos\frac{\theta_1-\theta_3}{2} \quad (4)$$

$$\|P-Q\|^2\|R-Q\|^2 = ((\cos\theta_1 - \cos\theta_2)^2 + (\sin\theta_1 - \sin\theta_2)^2)((\cos\theta_3 - \cos\theta_2)^2 + (\sin\theta_3 - \sin\theta_2)^2)$$

$$= (2 - 2\cos\theta_1\cos\theta_2 - 2\sin\theta_1\sin\theta_2)(2 - 2\cos\theta_3\cos\theta_2 - 2\sin\theta_3\sin\theta_2)$$

$$= 16\sin^2\frac{\theta_1-\theta_2}{2}\sin^2\frac{\theta_3-\theta_2}{2}$$

$$\|P-Q\|\,\|R-Q\| = 4\sin\frac{\theta_1-\theta_2}{2}\sin\frac{\theta_3-\theta_2}{2} \quad (5)$$

Substituting (4) and (5) in (2),

$$cos(\angle PQR) = \frac{4sin\frac{\theta_1-\theta_2}{2}sin\frac{\theta_3-\theta_2}{2}cos\frac{\theta_1-\theta_3}{2}}{4\sin\frac{\theta_1-\theta_2}{2}\sin\frac{\theta_3-\theta_2}{2}}$$

$$cos(\angle PQR) = cos\frac{\theta_1-\theta_3}{2} \quad (6)$$

$$\angle PQR == \frac{\theta_1 - \theta_3}{2} = 100^\circ$$

$$cos(\angle OPR) = \frac{(P-R)^T(P-O)}{\|P-R\|\,\|P-O\|} \tag{7}$$

$$\mathbf{P}-\mathbf{R} = \begin{pmatrix} cos\theta_1 - cos\theta_3 \\ sin\theta_1 - sin\theta_3 \end{pmatrix}, \mathbf{P}-\mathbf{O} = \begin{pmatrix} cos\theta_1 \\ sin\theta_1 \end{pmatrix} \tag{8}$$

$$(P-R)^T(P-O) = \begin{pmatrix} cos\theta_1 - cos\theta_3 \\ sin\theta_1 - sin\theta_3 \end{pmatrix} \begin{pmatrix} cos\theta_1 sin\theta_1 \end{pmatrix}$$

$$= \ (\cos\theta_1 \ - \ \cos\theta_3)(\cos\theta_1) \ + \ (\sin\theta_1 \ - \ \sin\theta_3)(\sin\theta_1)$$

$$= \ (\cos^2\theta_1 \ - \ \cos\theta_3\cos\theta_1) \ + \ (\sin^2\theta_1 \ - \ \sin\theta_1\sin\theta_3)$$

$$= 1 - (\cos\theta_3\cos\theta_1 + \sin\theta_1\sin\theta_3)$$

$$= 1 - (cos(\theta_3 - \theta_1)) \tag{9}$$

$$\|P-R\|^2\,\|P-O\|^2 = ((\cos\theta_1 - \cos\theta_3)^2 + (\sin\theta_1 - \sin\theta_3)^2)$$
$$((\cos\theta_1)^2 + (\sin\theta_1)^2)$$

$$= 2 - 2\cos\theta_1\cos\theta_3 - 2\sin\theta_1\sin\theta_3$$

$$= 2(1 - \cos(\theta_1 - \theta_3))$$

$$\|P-R\|\,\|P-O\| = \sqrt{2(1-\cos(\theta_1 - \theta_3))} \tag{10}$$

$$cos(\angle OPR) = \frac{(1-\cos(\theta_1 - \theta_3))}{\sqrt{2(1-\cos(\theta_1 - \theta_3))}}$$

$$\cos(\angle OPR) = 0.98$$

$$\angle OPR = 10^\circ \tag{11}$$