# Deriving the equation of Ellipse with eccentricty, directrices and origin as its center Using Matrices
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## Contents



# 1 Problem statement

The eccentricity of an ellipse, with its center at the origin, is $\frac{1}{2}$. If one of the directrices is $\mathbf{x} = 4$, then find the equation of Ellipse.

# 2 Considerations

As per the statement, for the given Ellipse, the input parameters are described in the following table.

| Symbol | Value | Description |
|---|---|---|
| $\mathbf{O}$ | $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ | Origin |
| e | 0.5 | Eccentricity |
| $\mathbf{x}$ | $\mathbf{x} = 4$ | Directrix |

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# 3 Plotting the Ellipse with the given parameters

The plot of the Ellipse, with eccentricity $\mathbf{e} = 0.5$ and directrix ($\mathbf{x} = 4$) is shown in figure below.

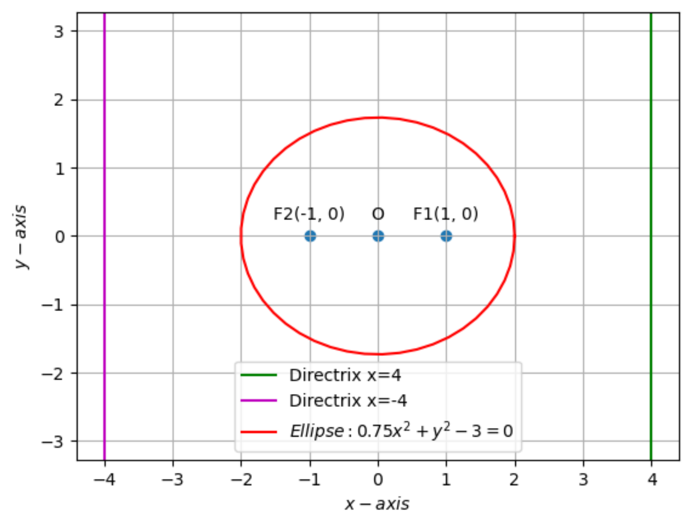


Figure 1: Ellipse with eccentricty e=0.5 and directrix x=4)

# 4 Solution

As per the statement, for the given Ellipse, the input parameters are,

Eccentricity of the Ellipse is,

$$\mathrm{e} = 0.5$$

And the Directrix of the Ellipse is,

$$\mathbf{x} = 4 \tag{4.0.1}$$

On comparing above equation (4.0.1) with, $\mathbf{n}^T\mathbf{x} = \mathbf{c}$ we get,

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{c} = 4$$

Therefore, the directrix of the Ellipse, $\mathbf{n}^T\mathbf{x} = \mathbf{c}$ can be written as,

(where $\mathbf{n}$ is normal vector of directix line)

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = 4 \tag{4.0.2}$$

## 4.1 Finding the Matrix V

The Matrix $\mathbf{V}$ can be expressed as,

$$\mathbf{V} = ||\mathbf{n}||^2\mathbf{I} - e^2\mathbf{n}\mathbf{n}^T \tag{4.1.1}$$

On submitting $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and e=$\frac{1}{2}$ in above equation, we get,

$$\mathbf{V} = (1^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \left(\tfrac{1}{2}\right)^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\implies \mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\implies \mathbf{V} = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & 1 \end{pmatrix} \tag{4.1.2}$$

## 4.2 Finding the Matrix u

As per the statement, Center of the Ellipse is origin,

$$\implies \mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.2.1}$$

And the center of conics is,

$$\mathbf{O} = -\mathbf{V}^{-1}\mathbf{u} \tag{4.2.2}$$

$$\implies \mathbf{u} = -\mathbf{V}\mathbf{O}$$

$$\implies \mathbf{u} = -\begin{pmatrix} \frac{3}{4} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\implies \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.2.3}$$

## 4.3 Finding the Focus point F

The Focus point $\mathbf{u}$ of the ellipse can be expressed as,

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{4.3.1}$$

On submitting c=4, $e = \frac{1}{2}$, $\mathbf{n}$, $\mathbf{u}$ and $\lambda_2 = 1$,

$$\mathbf{F} = \frac{4(\frac{1}{2})^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}}{1}$$

$$\implies \mathbf{F} = 4(\tfrac{1}{2})^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\implies \mathbf{F} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.3.2}$$

## 4.4 Finding the value of f

The expression for f is,

$$f = ||\mathbf{n}||^2||\mathbf{F}||^2 - c^2e^2 \tag{4.4.1}$$

On submitting c=4, $e = \frac{1}{2}$, $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{F} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then we get,

$$\implies f = (1)^2(1)^2 - (4)^2\left(\tfrac{1}{2}\right)^2$$

$$\implies f = -3 \tag{4.4.2}$$

## 4.5 Deriving the equation for Ellipse

The equation for Ellipse can be expressed as,

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{4.5.1}$$

On submitting the values of $\mathbf{V}$, $\mathbf{u}$ and f,

$$\mathbf{x}^T \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 0 & 0 \end{pmatrix} \mathbf{x} - 3 = 0 \tag{4.5.2}$$

Or,

$$\implies \mathbf{x}^T \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 3 = 0 \tag{4.5.3}$$

The above Ellipse equation can be expressed in general form as,

$$\frac{\mathbf{x}^2}{4} + \frac{\mathbf{y}^2}{3} = 1 \tag{4.5.4}$$

$$\implies 0.75\mathbf{x}^2 + \mathbf{y}^2 - 3 = 0 \tag{4.5.5}$$

# 5 Conclusion

1. At first, the Matrix $\mathbf{V}$ has been calculated from the given input parameters eccentricity, and n, and then, the Matrix $\mathbf{u}$ has been calculated.

2. Focus point $\mathbf{F}$ is calculated from the given input parameters, and then the value of $\mathbf{f}$ of the Ellipse has been calculated. It is found as $\mathbf{f} = -3$.

3. Finally, the equation of Ellipse has been derived as,

$$\implies \mathbf{x}^T \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 3 = 0$$

The above Ellipse equation can be expressed in general form as,

$$0.75\mathbf{x}^2 + \mathbf{y}^2 - 3 = 0$$