Matrix Assignment - Circle

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Get Python code for the figure from

https://github.com/SurabhiSeetha/Fwciith2022/tree/main/Assignment%201/codes/src

Get LaTex code from

https://github.com/SurabhiSeetha/Fwciith2022/tree/main/avr%20gcc

1 Question-Class 10, Exercise 11.2, Q(5)

Draw a line segment AB of length 8cm. Taking A as centre,draw a circle of radius 4cm and taking B as centre,draw another circle of radius 3cm.Construct tangents to each circle from the centre of the circle.

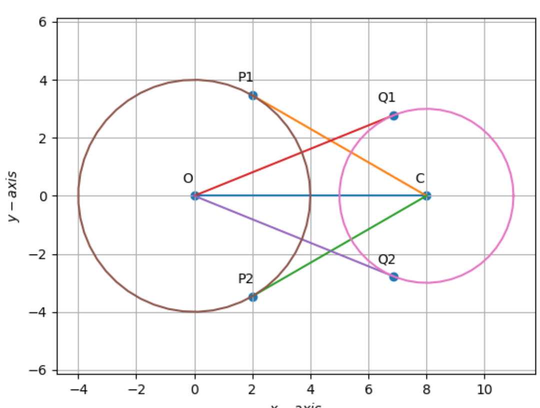

Figure- Circles A and B with tangents P1,P2,Q1,Q2

2 Construction

A circle with radius 4cm and another circle with radius 3cm are taken and their tangents are drawn with their point of contacts as P1,P2,Q1 and Q2 and their construction is done with the help of θ and α angles as shown in the table below.

Symbol	Value	Description
r_1	4cm	radius
r_2	3cm	radius
O	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Center O
d	8cm	Distance OC
e_1	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	Unit Vector
C	d x e_1	Centre C
θ	$\angle P_1OC$	Angle P_1OC
α	$\angle Q_1OC$	Angle Q_1OC
t	d $cos\alpha$	OQ_1
P_1	$r_1 \begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix}$	P.O.C
P_2	$r_1 \begin{pmatrix} cos\theta \\ -sin\theta \end{pmatrix}$	P.O.C
Q_1	t $\begin{pmatrix} cos\alpha \\ sin\alpha \end{pmatrix}$	P.O.C
Q_2	t $\begin{pmatrix} cos\alpha \\ -sin\alpha \end{pmatrix}$	P.O.C

3 Solution

3.1 Part-1:

Consider the circle A of radius 4cm and points O and C each at a distance 8cm from the centre. Two tangents can be drawn from point O onto the circle B with point of contacts Q1,Q2 and other two tangents from point C onto the circle A with the point of contacts P1,P2.

The point of intersection of line:

$$L : \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \tag{3.1.1}$$

with the conic section:

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{3.1.2}$$

is given by-

$$\mathbf{x_i} = \mathbf{q} + \mu_i \mathbf{m} \tag{3.1.3}$$

where,
$\mu_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}(-\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u})$

$$\pm\sqrt{(\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u}))^2-(\mathbf{q}^T+2\mathbf{u}^T\mathbf{q}+f)(\mathbf{m}^T\mathbf{V}\mathbf{m})} \tag{3.1.4}$$

If the line L touches the conic at exactly one point q
where q is the point of origin(0,0) which is A

$$\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u}=0) \tag{3.1.5}$$

In this case,the conic intercept has exactly one root Hence,

$$[\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u})]^2-(\mathbf{m}^T\mathbf{V}\mathbf{m})(\mathbf{q}^T\mathbf{V}\mathbf{q}+2\mathbf{u}^T\mathbf{q}+f)=0 \tag{3.1.6}$$

So,comparing the equation of our conic $x^2+y^2=4^2$ with the general Equation of a circle:

$$Ax^2+Bxy+Cy^2+Fx+Gy+f=0$$
$$x^2+y^2-16=0$$

we get,

$$\mathbf{V}=\begin{pmatrix}A & \frac{B}{2}\\ \frac{B}{2} & C\end{pmatrix} \qquad \mathbf{u}=\begin{pmatrix}\frac{F}{2}\\ \frac{G}{2}\end{pmatrix}$$
$$\mathbf{V}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \mathbf{u}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

Substituting the above values in eq.(3.1.2) we get,

$$\mathbf{x}^T\mathbf{V}\mathbf{x}+2(0\ \ 0)\mathbf{x}+4=0 \tag{3.1.7}$$

Let us consider the direction vector of L as m

$$\mathbf{m}=\begin{pmatrix}1\\ \lambda\end{pmatrix} \tag{3.1.8}$$

and q be the pont A,

$$\mathbf{q}=\begin{pmatrix}8\\ 0\end{pmatrix} \tag{3.1.9}$$

Substituting eq. (3.1.7), (3.1.8) and (3.1.9) in Eq.(3.1.6) we get,

$$[\mathbf{m}^T(\mathbf{I}\mathbf{q})]^2-(\mathbf{m}^T\mathbf{I}\mathbf{m})(\mathbf{q}^T\mathbf{I}\mathbf{q}+(-16))=0$$

$$[\begin{pmatrix}1 & \lambda\end{pmatrix}\begin{pmatrix}8\\ 0\end{pmatrix}]^2-(\begin{pmatrix}1 & \lambda\end{pmatrix}\begin{pmatrix}1\\ \lambda\end{pmatrix})(\begin{pmatrix}8 & 0\end{pmatrix}\begin{pmatrix}8\\ 0\end{pmatrix}-16)=0 \tag{3.1.10}$$

$$8^2-(1+\lambda^2)(64-16)=0$$

$$64-(1+\lambda^2)(48)=0$$
$$(1+\lambda^2)=\frac{64}{48}$$

$$\lambda=\pm\frac{1}{\sqrt{3}} \tag{3.1.11}$$

That is,

$$\mathbf{m}=\begin{pmatrix}1\\ \pm\frac{1}{\sqrt{3}}\end{pmatrix} \tag{3.1.12}$$

from (3.1.4) and (3.1.6)

$$\mu_i=\frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}(-\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u})) \tag{3.1.13}$$

$$\mu_i=\frac{1}{\begin{pmatrix}1\\ \frac{1}{\sqrt{3}}\end{pmatrix}I\begin{pmatrix}1 & \frac{1}{\sqrt{3}}\end{pmatrix}}(-(1\pm\frac{1}{\sqrt{3}}\begin{pmatrix}8\\ 0\end{pmatrix})$$

$$\mu_i=\frac{1}{1+\frac{1}{3}}(-8)$$

$$\mu_i=-6 \tag{3.1.14}$$

Now eq. (3.1.3) becomes,

$$\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}8\\ 0\end{pmatrix}+(-6)\begin{pmatrix}1\\ \pm\frac{1}{\sqrt{3}}\end{pmatrix} \tag{3.1.15}$$

$$\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}8\\ 0\end{pmatrix}+\begin{pmatrix}-6\\ \pm\frac{6}{\sqrt{3}}\end{pmatrix} \tag{3.1.16}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ \pm\frac{6}{\sqrt{3}} \end{pmatrix} \tag{3.1.17}$$

Therefore,

$$\mathbf{P_1} = \begin{pmatrix} 2 \\ \frac{6}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 2 \\ 3.464 \end{pmatrix}$$

$$\mathbf{P_2} = \begin{pmatrix} 2 \\ -\frac{6}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 2 \\ -3.464 \end{pmatrix}$$

3.2 Part-2:

Now for another circle B of radius 3cm which has its tangents drawn from circle A onto B with the point of contacts namely Q_1 and Q_2.

Let us consider the same line and conic equations as in Part 1. The line L touches the conic at exactly one point q.

$$\mathbf{m}^T(\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \tag{3.2.1}$$

Hence,

$$[\mathbf{m}^T(\mathbf{V}\mathbf{q} + \mathbf{u})]^2 - (\mathbf{m}^T\mathbf{V}\mathbf{m})(\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f) = 0 \tag{3.2.2}$$

The equation of conic is,

$$(x-8)^2 + y^2 = 3^2$$

$$x^2 + 64 - 16x + y^2 - 9 = 0$$

$$x^2 - 16x + y^2 + 55 = 0$$

Comparing it with general Eq. of a circle

$$Ax^2 + Bxy + Cy^2 + Fx + Gy + f = 0$$

We get,

$$\mathbf{V} = \begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix} \qquad \mathbf{u} = \begin{pmatrix} \frac{F}{2} \\ \frac{G}{2} \end{pmatrix}$$

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{u} = \begin{pmatrix} -8 \\ 0 \end{pmatrix}$$

Now Eq. (3.1.2) can be written as,

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x + 2\begin{pmatrix} -8 & 0 \end{pmatrix}\mathbf{x} + 3 = 0 \tag{3.2.3}$$

Let us consider the direction vector of $\mathbf{L}$ as m

$$\mathbf{m} = \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \tag{3.2.4}$$

and q be the pont A,

$$\mathbf{q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.2.5}$$

Substituting the above values in eq. (3.2.2) we get,

$$[\mathbf{m}^T(\mathbf{V}\mathbf{q} + \mathbf{u})]^2 - (\mathbf{m}^T\mathbf{V}\mathbf{m})(\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f) = 0$$

$$[\begin{pmatrix} 1 & \lambda \end{pmatrix}[I\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -8 \\ 0 \end{pmatrix}]]^2 - (\begin{pmatrix} 1 & \lambda \end{pmatrix} I \begin{pmatrix} 1 \\ \lambda \end{pmatrix})(55) = 0$$

$$[\begin{pmatrix} 1 & \lambda \end{pmatrix}\begin{pmatrix} -8 \\ 0 \end{pmatrix}]^2 - (1+\lambda^2)(55) = 0$$

$$64 - (1+\lambda^2)55 = 0$$

$$64 - 55 - 55\lambda^2 = 0$$

$$\lambda^2 = \frac{9}{55}$$

$$\lambda = \pm\frac{3}{\sqrt{55}} = \pm 0.4 \tag{3.2.6}$$

Now,

$$\mathbf{m} = \begin{pmatrix} 1 \\ \pm 0.4045 \end{pmatrix}$$

From eq. (3.1.4) and (3.1.6),

$$\mu_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}(-\mathbf{m}^T(\mathbf{V}\mathbf{q} + \mathbf{u}))$$

$$\mu_i =$$

$$\frac{1}{\begin{pmatrix} 1 \\ 0.4045 \end{pmatrix} I \begin{pmatrix} 1 & 0.4045 \end{pmatrix}}(-\begin{pmatrix} 1 & 0.4045 \end{pmatrix}(I\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -8 \\ 0 \end{pmatrix})$$

$$= \frac{1}{1+0.163}(-\begin{pmatrix} 1 & 0.4045 \end{pmatrix}\begin{pmatrix} -8 \\ 0 \end{pmatrix})$$

$$= \frac{1}{1.163} \times 8$$

Therefore,

$$\mu_\mathbf{i} = \frac{8}{1.163} = 6.875$$

Now eq. (3.1.3) becomes,

$$\mathbf{x_i} = \mathbf{q} + \mu_i \mathbf{m}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 6.875 \begin{pmatrix} 1 \\ \pm 0.4045 \end{pmatrix}$$

$$x_i = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6.875 \\ \pm 2.7810 \end{pmatrix}$$

Therefore,

$$\mathbf{Q_1} = \begin{pmatrix} 6.875 \\ 2.7810 \end{pmatrix}$$

$$\mathbf{Q_2} = \begin{pmatrix} 6.875 \\ -2.7810 \end{pmatrix}$$

Therefore, we have solved and gained the values of four point of contacts P1,P2,Q1,Q2 for two circles of radius 4cm and 3cm.