Line Assignment

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Problem

Show that the diagonals of a square are equal and bisect each other at right angles

Solution

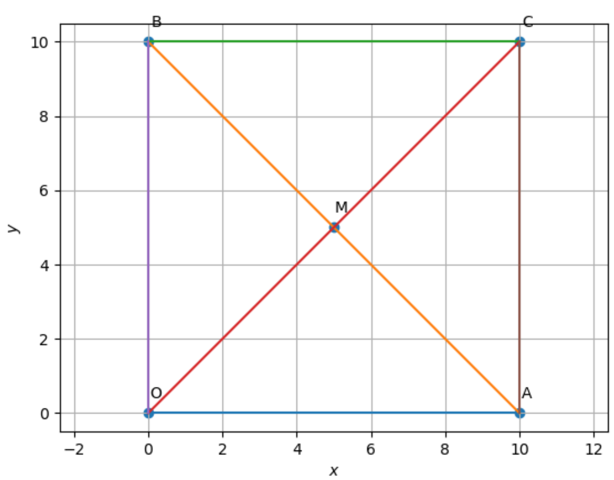

Figure 1: Square generated using python

Construction

Inputs taken for the construction of the Square is 'a' , which is the side length of the square.

Symbol	Value	Description
a	10	length of OA
O	(0,0)	point O
A	(a,0)	point A
B	(0,a)	point B
C	A+B	point C
M	$\frac{C}{2}$	point M

Let OABC is a Square. Length of all sides are equal for a square and all interior angles equal to 90°. O at the origin and vectors A, B & C represent other vertices of the square.

$$\|\boldsymbol{OA}\| = \|\boldsymbol{OB}\| = \|\boldsymbol{BC}\| = \|\boldsymbol{AC}\| \tag{1}$$

$$\angle OAC = \angle OBC = \angle BCA = \angle AOB = 90° \tag{2}$$

Here, D_1 and D_2 are the diagonals of the square and we can compute D_1 and D_2 as

$$\boldsymbol{D_1} = (\boldsymbol{A} + \boldsymbol{B}) \tag{3}$$

$$\boldsymbol{D_2} = (\boldsymbol{A} - \boldsymbol{B}) \tag{4}$$

To prove that the diagonals of the square are equal, we can find the length of the two diagonals and compare. Hence,

$$\|\boldsymbol{D_1}\| = \|\boldsymbol{A} + \boldsymbol{B}\| \tag{5}$$

$$\|\boldsymbol{D_2}\| = \|\boldsymbol{A} - \boldsymbol{B}\| \tag{6}$$

For finding length of D_1, we can write from equation (5),

$$\|\boldsymbol{A} + \boldsymbol{B}\| = \sqrt{\|\boldsymbol{A}\|^2 + \|\boldsymbol{B}\|^2 + 2\boldsymbol{A}^T\boldsymbol{B}} \tag{7}$$

But, for a square we know that length of all sides are equal.

$$\|\boldsymbol{A}\| = \|\boldsymbol{B}\| \tag{8}$$

and, the angle between two adjacent sides is 90°. The dot product of two vectors which are seperated by 90°angle is always '0'.

$$\boldsymbol{A}^T\boldsymbol{B} = 0 \tag{9}$$

So the equation (7) becomes

$$\|\boldsymbol{A} + \boldsymbol{B}\| = \sqrt{2}\,\|\boldsymbol{A}\| \tag{10}$$

$$\|\boldsymbol{D_1}\| = \sqrt{2}\,\|\boldsymbol{A}\| \tag{11}$$

Similarly, for finding the length of D_2

$$\|\boldsymbol{A}-\boldsymbol{B}\| = \sqrt{\|\boldsymbol{A}\|^2 + \|\boldsymbol{B}\|^2 - 2\boldsymbol{A}^T\boldsymbol{B}} \tag{12}$$

But, from (8) and (9)

$$\|\boldsymbol{A}-\boldsymbol{B}\| = \sqrt{2}\,\|\boldsymbol{A}\| \tag{13}$$
$$\|\boldsymbol{D_2}\| = \sqrt{2}\,\|\boldsymbol{A}\| \tag{14}$$

So, from the equations (11) and (14), we can say that the lengths of diagonals $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ are equal

$$\|\boldsymbol{D_1}\| = \|\boldsymbol{D_2}\| \tag{15}$$

We know that, if the dot product of two vectors is zero then the vectors are perpendicular to each other.
So, by taking the dot product of $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$

$$\boldsymbol{D_1}.\boldsymbol{D_2} = \boldsymbol{D_1}^T\boldsymbol{D_2} \tag{16}$$
$$\boldsymbol{D_1}.\boldsymbol{D_2} = (\boldsymbol{A}+\boldsymbol{B})^T(\boldsymbol{A}-\boldsymbol{B}) \tag{17}$$
$$\boldsymbol{D_1}.\boldsymbol{D_2} = \|\boldsymbol{A}\|^2 - \|\boldsymbol{B}\|^2 \tag{18}$$

From the equation (8),

$$\boldsymbol{D_1}.\boldsymbol{D_2} = \|\boldsymbol{A}\|^2 - \|\boldsymbol{A}\|^2 \tag{19}$$
$$\boldsymbol{D_1}.\boldsymbol{D_2} = 0 \tag{20}$$

as the dot product of the diagonals is equal to 0, we can say that both diagonals are perpendicular to each other.

Let diagonals $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ intersect at a point $\boldsymbol{M}$. We have to prove that $\boldsymbol{M}$ is the mid point of $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$, in order to say that both diagonals bisect eachother.

$$\boldsymbol{OM} = x\boldsymbol{D_1} \tag{21}$$
$$\boldsymbol{MA} = y\boldsymbol{D_2} \tag{22}$$

From the equations (3) and (4), the above equations can be written as

$$\boldsymbol{OM} = x\boldsymbol{A} + \boldsymbol{B} \tag{23}$$
$$\boldsymbol{MA} = y\boldsymbol{A} - \boldsymbol{B} \tag{24}$$

Now, if we consider

$$\boldsymbol{OA} = \boldsymbol{OM} + \boldsymbol{MA} \tag{25}$$
$$\boldsymbol{A} = x(\boldsymbol{A}+\boldsymbol{B}) + y(\boldsymbol{A}-\boldsymbol{B}) \tag{26}$$
$$\boldsymbol{A} = x\boldsymbol{A} + x\boldsymbol{B} + y\boldsymbol{A} - y\boldsymbol{B} \tag{27}$$
$$\boldsymbol{A} = (x+y)\boldsymbol{A} + (x-y)\boldsymbol{B} \tag{28}$$

Equating the co-efficient of $\boldsymbol{A}$ and $\boldsymbol{B}$, we get

$$x+y=1, x-y=0 \tag{29}$$
$$2x = 1 \tag{30}$$
$$x = \frac{1}{2} \tag{31}$$
$$y = \frac{1}{2} \tag{32}$$

now we can say that

$$\boldsymbol{OM} = \frac{1}{2}\boldsymbol{D_1} \tag{33}$$
$$\boldsymbol{MA} = \frac{1}{2}\boldsymbol{D_2} \tag{34}$$

Hence, M is the mid point of diagonals D_1 and D_2 and we can say that both diagonals bisect eachother.

we have proved that diagonals of a square are equal in length and bisect eachother at right angles.