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Indian Institute of Technology Hyderabad

FWC22025

MATRICES-CIRCLES

Contents

1 Problem

Construct a tangent to a circle of radius 4cm from a point on the concentric circle of radius 6cm and measure its length.Also verify the measurement by actual calculation.

2 Solution

Consider a point P on the circle of radius 6 cm is at the origin and the center of circle is at d distance from P (where d=6).

Two tangents can be drawn from point P on to the circle of radius 4 cm which is concentric to the circle of radius 6 cm and let the point of contacts be Q1 and Q2.

The point of intersection of line

$$L: \quad \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \quad \mu \in \mathbb{R} \tag{1}$$

with the conic section

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2}$$

is given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \tag{3}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \Big(-\mathbf{m}^\top (\mathbf{V}\mathbf{q}+\mathbf{u}) \pm \sqrt{[\mathbf{m}^\top (\mathbf{V}\mathbf{q}+\mathbf{u})]^2 - (\mathbf{q}^\top \mathbf{V}\mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f)(\mathbf{m}^\top \mathbf{V}\mathbf{m})}\Big) \tag{4}$$

If the line L touches the conic at exactly one point $\mathbf{q}$,

$$\mathbf{m}^\top (\mathbf{V}\mathbf{q}+\mathbf{u}) = 0 \tag{5}$$

In this case, the conic intercept has exactly one root.Hence,

$$\left[\mathbf{m}^\top (\mathbf{V}\mathbf{q}+\mathbf{u})\right]^2 - \left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right)\left(\mathbf{q}^\top \mathbf{V}\mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f\right) = 0 \tag{6}$$

So, the equation of conic $(x-6)^2 + y^2 = 16$ can be written in the form of eq (2) as,

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -6 & 0 \end{pmatrix} \mathbf{x} + f = 0 \tag{7}$$

Let us conisder the direction vector m as,

$$\mathbf{m} = \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \tag{8}$$

and $\mathbf{q}$ be the point P,

$$\mathbf{q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{9}$$

Substituting (7),(8) and (9) in eq (6), we get

$$\left[\mathbf{m}^\top (\mathbf{u})\right]^2 - f\left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right) = 0$$

$$\left[\begin{pmatrix} 1 & \lambda \end{pmatrix} \begin{pmatrix} -6 \\ 0 \end{pmatrix}\right]^2 - 20\begin{pmatrix} 1 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \lambda \end{pmatrix} = 0$$

$$36 - 20\left(\begin{pmatrix} 1 & \lambda \end{pmatrix} \begin{pmatrix} 1 \\ \lambda \end{pmatrix}\right) = 0$$

$$36 - 20\left(1+\lambda^2\right) = 0$$

$$16 = 20\lambda^2$$

$$\lambda = \pm 2/\sqrt{5}$$

Let m1 and m2 be direction vectors of two tangents PQ1 and PQ2, then

$$\mathbf{m1} = \begin{pmatrix} 1 \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$\mathbf{m2} = \begin{pmatrix} 1 \\ \frac{-2}{\sqrt{5}} \end{pmatrix}$$

Substituting (6) in (4), we get

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \left(-\mathbf{m}^\top (\mathbf{V}\mathbf{q}+\mathbf{u})\right) \tag{10}$$

Substituting m1 and m2 in (10),we get

$$\mu_1 = \frac{10}{3} \; and \; \mu_2 = \frac{10}{3}$$

Hence,

$$\mathbf{x}_1 = \mathbf{q} + \mu_1 \mathbf{m1} \tag{11}$$

$$\mathbf{x}_2 = \mathbf{q} + \mu_2 \mathbf{m2} \tag{12}$$

Solving above equations, we get

$$\mathbf{x}_1 = \begin{pmatrix} \frac{10}{3} \\ \frac{20}{3\sqrt{5}} \end{pmatrix} \implies \mathbf{x}_1 = \begin{pmatrix} 3.33 \\ 2.98 \end{pmatrix}$$

$$\mathbf{x}_2 = \begin{pmatrix} \frac{10}{3} \\ \frac{-20}{3\sqrt{5}} \end{pmatrix} \implies \mathbf{x}_2 = \begin{pmatrix} 3.33 \\ -2.98 \end{pmatrix}$$

Thus,

$$\mathbf{Q1} = \mathbf{x}_1 \; and \; \mathbf{Q2} = \mathbf{x}_2 \tag{13}$$

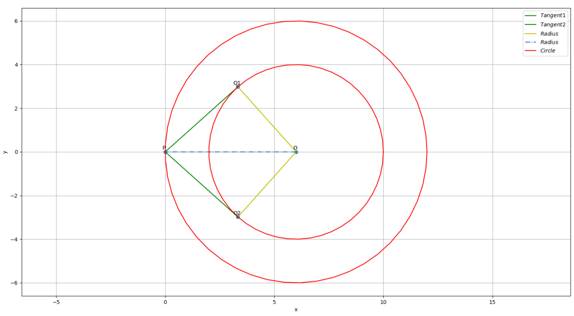

Figure
3 Construction

The concentric circles and tangents are constructed with,

Symbol	Co-ordinates	Description
r1	4	radius
r2	6	radius
d	6	OP
m1	$\begin{pmatrix} 1 \\ \frac{2}{\sqrt{5}} \end{pmatrix}$	direction vector of PQ1
m2	$\begin{pmatrix} 1 \\ \frac{-2}{\sqrt{5}} \end{pmatrix}$	direction vector of PQ2
μ_1	$\frac{10}{3}$	root
μ_2	$\frac{10}{3}$	root
P	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	point vector P
O	$\begin{pmatrix} d \\ 0 \end{pmatrix}$	point vector O
Q1	$\mu_1 \mathbf{m1}$	point of contact 1
Q2	$\mu_2 \mathbf{m2}$	point of contact 2

The figure above is generated using python code provided in the below source code link.

https://github.com/madind5668/FWC/blob/main/matrices/circles/codes/main.py