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FWC22025

MATRICES-CONICS

Contents

1 Problem

Find a point on the curve

$$y = (x-2)^2$$

at which a tangent is parallel to the chord joining the points (2,0) and (4,4).

2 Solution

The equation of the conic can be represented as,

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -2 & \frac{-1}{2} \end{pmatrix} \mathbf{x} + 4 = 0 \tag{1}$$

So,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{u}^\top = \begin{pmatrix} -2 & \frac{-1}{2} \end{pmatrix}$$

$$f = 4$$

The equation of the line passing through (2,0) and (4,4) is y=2x-4 can be represented as,

$$\frac{x-2}{1} = \frac{y-0}{2}$$

So, the direction vector can be given as,

$$\mathbf{m} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The normal vector $\mathbf{n}$ can be given as,

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m}$$

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

If $\mathbf{V}$ is not invertible, given the normal vector $\mathbf{n}$, the point of contact to parabola is given by the matrix equation

$$\begin{pmatrix} (\mathbf{u} + \kappa \mathbf{n})^\top \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \tag{2}$$

$$\text{where } \kappa = \frac{\mathbf{p}_1^\top \mathbf{u}}{\mathbf{p}_1^\top \mathbf{n}}, \quad \mathbf{V}\mathbf{p}_1 = 0 \tag{3}$$

If $\mathbf{V}$ is non-invertible, it has a zero eigenvalue. If the corresponding eigenvector is $\mathbf{p}_1$, then,

$$\mathbf{V}\mathbf{p}_1 = 0$$

Let, the eigenvector be

$$\mathbf{p}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$x_1 = 0$$

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$$

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_2$$

The eigenvector is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Now, the κ can be given as,

$$\kappa = \frac{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ \frac{-1}{2} \end{pmatrix}}{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}}$$

$$\kappa = \frac{\frac{-1}{2}}{-1}$$

$$\kappa = \frac{1}{2}$$

Substitute κ in (2), we get

$$\begin{pmatrix} \left[\begin{pmatrix} -2 \\ \frac{-1}{2} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right]^{\top} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ \frac{1}{2}\begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ \frac{-1}{2} \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}$$

As, the last row elements are all zero, we can eliminate that row

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

For applying row reduction method the augmented matrix is written as

$$\left(\begin{array}{cc|c} -1 & -1 & -4 \\ 1 & 0 & 3 \end{array}\right)$$
$$\xleftrightarrow{R_1 \leftarrow R_1 + 2R_2} \left(\begin{array}{cc|c} 1 & -1 & 2 \\ 1 & 0 & 3 \end{array}\right)$$
$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 1 & 1 \end{array}\right)$$
$$\xleftrightarrow{R_1 \leftarrow R_1 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right)$$
$$\implies \mathbf{q} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

The point of contact of tangent to parabola is $\mathbf{q} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

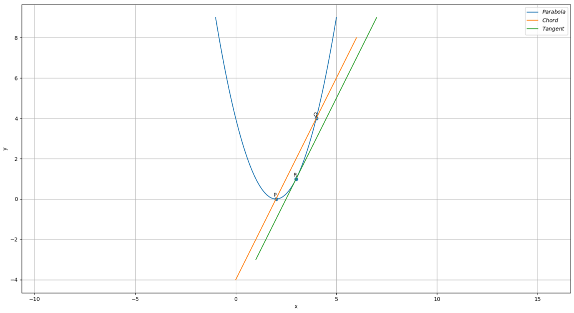

Figure
3 Construction

The parabola and tangent can be constructed using,

Symbol	Co-ordinates	Description
m	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	direction vector of PQ
P	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$	point vector P
Q	$\begin{pmatrix} 4 \\ 4 \end{pmatrix}$	point vector Q
R	$\mathbf{q}$	point of contact

The figure above is generated using python code provided in the below source code link.

https://github.com/madind5668
/FWC/blob/main/matrices/conics
/codes/main.py