Conics Assignment

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***Problem Statement*:Through the vertex O of parabola y^2=4x, chords OP and OQ are drawn at right angles to one another show that for all positions of P,PQ cuts the axis of the parabola a fixed point. Also find the locus of the middle point of PQ**

Solution

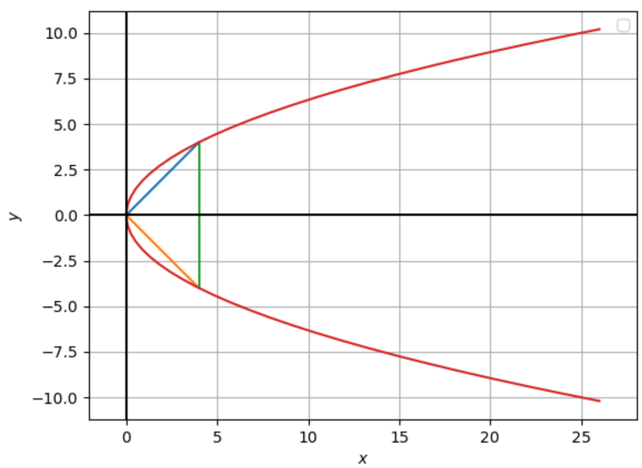

Figure 1: The intersection of PQ with x-axis is (4,0)

Construction

Symbol	value	Description
V	$\begin{pmatrix}0\\0\end{pmatrix}$	Vertex of parabola
F	$\begin{pmatrix}-2\\0\end{pmatrix}$	focus of parabola
n	$\begin{pmatrix}1\\0\end{pmatrix}$	normal of directrix

Table 1:

Proof:

The given equation of parabola $y^2 = 4x$ and vertex $\mathbf{O} = \begin{pmatrix}0\\0\end{pmatrix}$

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

the parametric coordinates of the parabola are P and Q

$$\mathbf{P} = \begin{pmatrix}at_1^2\\2at_1\end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix}at_2^2\\2at_2\end{pmatrix}$$

from above parabola equation a=1
let us consider $t_1 = 2$ and $t_2 = -2$,then

$$\mathbf{P} = \begin{pmatrix}4\\4\end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix}4\\-4\end{pmatrix}$$

the line equatin of OP and OQ is

$$\mathbf{A} = \mathbf{OP} = \mathbf{O} - \mathbf{P}$$
$$\mathbf{B} = \mathbf{OQ} = \mathbf{Q} - \mathbf{O}$$

Then,

$$\mathbf{A} = \begin{pmatrix}-4\\-4\end{pmatrix}$$
$$\mathbf{B} = \begin{pmatrix}4\\-4\end{pmatrix}$$

given that $\mathbf{OP} \perp \mathbf{OQ}$to find that

$$\theta = cos^{-1}(\frac{\mathbf{A^TB}}{\|\mathbf{A}\|\|\mathbf{B}\|})$$

on solving we get $\theta = 90$. Therefore, $\mathbf{A}$ and $\mathbf{B}$ are $\perp$ satisfy the given condition
the equation of PQ is

$$\mathbf{C} = \mathbf{PQ} = \mathbf{P} - \mathbf{Q}$$

the intersection of $\mathbf{PQ}$ line and X-axis gives the fixed point

$$\mathbf{X} = \begin{pmatrix}4\\0\end{pmatrix}$$

the mid point of $\mathbf{P}$ and $\mathbf{Q}$

$$\mathbf{R} = \frac{\mathbf{P}+\mathbf{Q}}{\mathbf{2}}$$
$$\mathbf{R} = \begin{pmatrix}4\\0\end{pmatrix}$$