Matrix Problems
Straight Lines

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I. Problem Statement

The base of an equilateral triangle with side 2a lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.

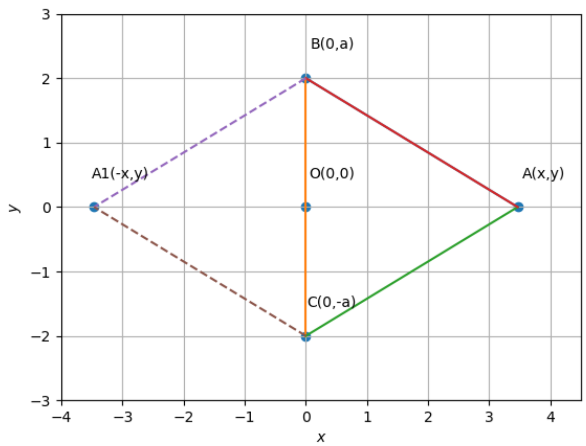

Fig. 1: Equilateral Triangle ABC

II. Construction

B and C are the inputs.

Symbol	Value	Description
B	$\begin{pmatrix} 0 \\ 2 \end{pmatrix}$	Vertex B
C	$\begin{pmatrix} 0 \\ -2 \end{pmatrix}$	Vertex C
A	$\begin{pmatrix} x \\ y \end{pmatrix}$	Vertex A
A1	$\begin{pmatrix} x1 \\ y1 \end{pmatrix}$	Vertex A

III. Solution

Given the base with 2a is lies on the y-axis with the mid-point of the base is at origin. The vertices of the two points on y-axis will be

$$\mathbf{B} = \begin{pmatrix} 0 \\ a \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ -a \end{pmatrix} \tag{1}$$

Given ΔABC is an equilateral triangle i.e

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{B}-\mathbf{C}\| = \|\mathbf{C}-\mathbf{A}\| = 2a \tag{2}$$

Consider, two sides of equilateral triangle be $\mathbf{A}$ and $\mathbf{B}$ then the third side will be $\mathbf{A}-\mathbf{B}$

$$\|\mathbf{A}\| = \|\mathbf{B}\| = \|\mathbf{A}-\mathbf{B}\| \tag{3}$$

$$\|\mathbf{A}\|^2 = \|\mathbf{B}\|^2 = \|\mathbf{A}-\mathbf{B}\|^2 \tag{4}$$

$$\|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{A}^T\mathbf{B} = \|\mathbf{A}\|^2 = \|\mathbf{B}\|^2 \tag{5}$$

$$\frac{\mathbf{A}^T\mathbf{B}}{\|\mathbf{A}\|^2} = \frac{\mathbf{A}^T\mathbf{B}}{\|\mathbf{B}^2\|} = \frac{1}{2} \tag{6}$$

Therefore, the triangle have all internal angles eaqual to 60^0

The angle between two vectors is given by

$$\theta = \cos^{-1}\frac{\mathbf{A}^\top\mathbf{B}}{\|A\|\,\|B\|} \tag{7}$$

$$(\mathbf{x}-\mathbf{B})^\top(\mathbf{x}-\mathbf{C}) = \|\mathbf{x}-\mathbf{B}\|\cdot\|\mathbf{x}-\mathbf{C}\|\cdot\cos\theta \tag{8}$$

$$\left(\mathbf{x}^\top\cdot\mathbf{x}\right)-\left(\mathbf{x}^\top\cdot\mathbf{C}\right)-\left(\mathbf{B}^\top\cdot\mathbf{x}\right)-\left(\mathbf{B}^\top\cdot\mathbf{C}\right) = 2a\cdot 2a\cos 60^0 \tag{9}$$

$$\|\mathbf{x}\|^2 - \mathbf{x}^\top(\mathbf{B}+\mathbf{C}) - \mathbf{B}^\top\cdot\mathbf{C} = 2a\cdot 2a\cdot\frac{1}{2} \tag{10}$$

$$\|\mathbf{x}\|^2 - \mathbf{x}^\top(0) - \begin{pmatrix} 0 \\ a \end{pmatrix}\begin{pmatrix} 0 & -a \end{pmatrix} = 4a^2 \tag{11}$$

$$\|\mathbf{x}\|^2 + a^2 = 4a^2 \tag{12}$$

$$\|\mathbf{x}\|^2 = 3a^2 \tag{13}$$

Considering, the line equation of $\mathbf{AB}$

$$\|\mathbf{x} - \mathbf{B}\|^2 = 4a^2 \tag{14}$$

$$(\mathbf{x} - \mathbf{B})^\top \cdot (\mathbf{x} - \mathbf{B}) = 4a^2 \tag{15}$$

$$\|\mathbf{x}\|^2 - 2 \cdot \mathbf{x}^\top \mathbf{B} + \|\mathbf{B}\|^2 = 4a^2 \tag{16}$$

$$3a^2 - 2 \cdot \mathbf{x}^\top \mathbf{B} + a^2 = 4a^2 \tag{17}$$

$$\mathbf{x}^\top \mathbf{B} = 0 \tag{18}$$

Since we can write,

$$\mathbf{B} = a \cdot \mathbf{e}_2 \tag{19}$$

$$\mathbf{x}^\top \cdot a \cdot \mathbf{e}_2 = 0 \tag{20}$$

$$\mathbf{x}^\top \cdot \mathbf{e}_2 = 0 \tag{21}$$

$$\mathbf{x} = \lambda \mathbf{e}_1 \tag{22}$$

From this its clearly concluded that third vertex will lie on x-axis. From the equation (13)

$$\mathbf{x} = \sqrt{3}a \tag{23}$$

Hence,the coordinates of the vertices of triangle are

$$\mathbf{A} = \begin{pmatrix} \pm\sqrt{3}a \\ 0 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ a \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ -a \end{pmatrix} \tag{24}$$

Get Python Code for image from

https://github.com/ManojChavva/FWC/blob/main/Matrix/line/code-py/triangle.py

Get LaTex code from

https://github.com/ManojChavva/FWC/blob/main/Matrix/line/line.tex