Matrices Assignment - Conic

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Get Python code for the figure from

https://github.com/dukkipativijay/Fwciith2022/tree/main/Assignment%201/Codes/src

Get LaTex code from

https://github.com/dukkipativijay/Fwciith2022/tree/main/Assignment%201%20−%20Assembly/Codes

1 Question

Class 12-1, Exercise 6.3,Q(27)

The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point?

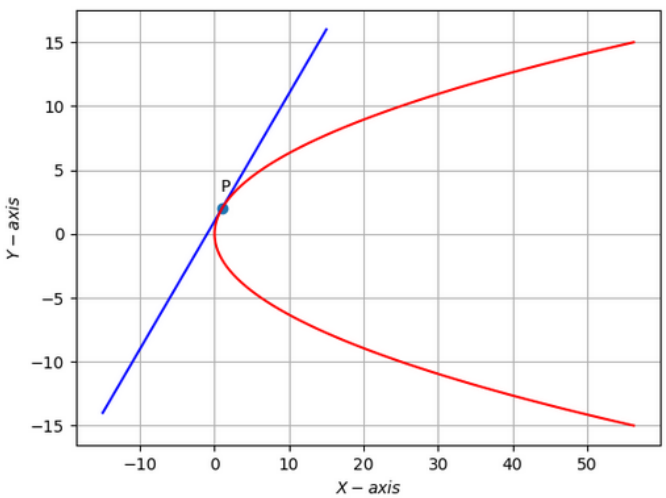

Figure 1 - Conic with Tangent at P

2 Construction

Symbol	Value	Description
L	$y = x + 1$	Line L
C	$y^2 = 4x$	Conic C
P	$x_i = q + \mu_i m$	Point of Contact P

Table 1: Parameters Table

3 Solution

Let P be the point of Contact of the Line $y = x + 1$ to the parabola $y^2 = 4x$

The point of intersection of line

$$L : \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \quad \mu\epsilon\mathbb{R} \tag{1}$$

With the conic section

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2}$$

is given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i\mathbf{m} \tag{3}$$

Where

$$\mu_i = \frac{1}{\mathbf{m^TVm}}(-\mathbf{m^T}(\mathbf{Vq}+\mathbf{u}) \pm \sqrt{[\mathbf{m^T}(\mathbf{Vqu})]^2 - (\mathbf{q}^T\mathbf{Vq} + 2\mathbf{u^Tq} + f)(\mathbf{m^TVm})}\ ) \tag{4}$$

If the line L touches the conic at exactly one point,the conic intercept has exactly one root. Hence,

$$[\mathbf{m^T}(\mathbf{Vqu})]^2 - (\mathbf{m^TVm})(\mathbf{q}^T\mathbf{Vq} + 2\mathbf{u^Tq} + f) = 0 \tag{5}$$

The equation of our Conic (which here is a parabola) is,

$$y^2 = 4x \tag{6}$$

Comparing it with the General Equation of a Conic,

$$Ax^2 + Bxy + Cy^2 + Fx + Gy + f = 0 \tag{7}$$

We Have,

A = 0, B = 0, C = 1, F = -4, G = 0, f = 0

We Know That,

$$V = \begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix} \qquad u = \begin{pmatrix} \frac{F}{2} \\ \frac{G}{2} \end{pmatrix}$$

So we get,

$$V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad u = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

Hence, the Eq. (6) of our Parabola can be written in the form of Conic Eq. (2) as,

$$\mathbf{x^T}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} -2 & 0 \end{pmatrix}\mathbf{x} + 0 = 0 \tag{8}$$

Let us consider the direction vector of **L** as m,

$$\mathbf{m} = \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \qquad (9)$$

and **q** be any point on the Line $y = x + 1$,

$$\mathbf{q} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (10)$$

The Equation of the given line can be re-written as,

$$x - y = -1$$

Comparing it with the normal form of the line,

$$\mathbf{n}^T\mathbf{x} = c$$

We Get,

$$\mathbf{n}^T = \begin{pmatrix} 1 & -1 \end{pmatrix} \ \ and \ \ c = -1$$

Hence,

$$m = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

From Eq. (4) and (5)

$$\mu_i = \frac{1}{\mathbf{m^T V m}}(-\mathbf{m^T}(\mathbf{Vq} + \mathbf{u}))$$

$$\mu_i = \frac{1}{\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}(-\begin{pmatrix} 1 & 1 \end{pmatrix}(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix}))$$

$$\mu_i = \frac{1}{\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}(-\begin{pmatrix} 1 & 1 \end{pmatrix}(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix}))$$

$$\mu_i = \frac{1}{1}(-\begin{pmatrix} 1 & 1 \end{pmatrix}(\begin{pmatrix} -2 \\ 1 \end{pmatrix}))$$

$$\mu_i = 1$$

Now Eq. (3) becomes,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + (1)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Therefore,

$$P = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Is the required point of contact of the given line and the conic.