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Circle Assignment

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Problem Statement:

If a circle passes through the point (a,b) and cuts the circle $x^2 + y^2 = p^2$ orthogonally, then the equation of the locus of its centre is

Construction

Symbol	Value	Description
$\mathbf{U_1}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	center of given circle
$\mathbf{r_1}$	2	radius of given circle
$\begin{pmatrix} a & b \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \end{pmatrix}$	point on circle

Solution: With the given circle equation $x^2 + y^2 = p^2$, we can find out centre U_1 and radius r_1 of Circle-1

Centre of Circle-1,

$$\mathbf{U_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

Radius of Circle-1,

$$\mathbf{r_1} = \mathbf{p} \tag{2}$$

The general form of conic is

$$\mathbf{X}^\top \mathbf{V} \mathbf{X} + 2\mathbf{U}^\top \mathbf{X} + \mathbf{f} = 0 \tag{3}$$

For circle 1

$$\mathbf{X}^\top \mathbf{V_1} \mathbf{X} + 2\mathbf{U_1}^\top \mathbf{X} + \mathbf{f_1} = 0 \tag{4}$$

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + f1 = 0 \tag{5}$$

$$\tag{6}$$

as circles are orthogonal

$$\mathbf{r_1}^2 + \mathbf{r_2}^2 = \|\mathbf{U_1} - \mathbf{U_2}\|^{\mathbf{2}} \tag{7}$$

$$\|U_1\|^2 + \|U_2\|^2 - 2\mathbf{U_1}^\top \mathbf{U_2} = \mathbf{p}^2 + \mathbf{r_2}^2 \tag{8}$$

$$\mathbf{r_2}^2 = \|\mathbf{U_2}\|^2 - \mathbf{f_2} \tag{9}$$

By solving the equations (6) and (7)

$$\mathbf{f_2} = \mathbf{p}^2 \tag{10}$$

For circle 2

$$\mathbf{X}^\top \mathbf{V_2} \mathbf{X} + 2\mathbf{U_2}^\top \mathbf{X} + \mathbf{f_2} = 0 \tag{11}$$

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} -g & -t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + p^2 = 0 \tag{12}$$

$$\mathbf{x}^2 + \mathbf{y}^2 - \mathbf{2gx} - \mathbf{2ty} + \mathbf{p}^2 = 0 \tag{13}$$

By substituting (a,b) in equation (13)

$$\mathbf{a}^2 + \mathbf{b}^2 - \mathbf{2ga} - \mathbf{2tb} + \mathbf{p}^2 = 0 \tag{14}$$

The locus is

$$\mathbf{2ga} + \mathbf{2tb} - (\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{p}^2) = 0 \tag{15}$$