→ Prasad Netalkar

→ PPN 5

5.5) Find the dual function of the LP

$$\text{minimize} \quad c^T x$$
$$\text{subject to} \quad Gx \le h$$
$$Ax = b$$

The lagrangian is given by

$$L(x, \lambda, \nu) = c^T x + \lambda^T (Gx - h) + \nu^T (Ax - b)$$

$$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu)$$

$$= \inf_x \{ c^T x + \lambda^T (Gx - h) + \nu^T (Ax - b) \}$$

$$= \inf_x \{ (c^T + \lambda^T G + \nu^T A) x - \lambda^T h - \nu^T b \}$$

↳ Affine, unbounded.

$$g(\lambda, \nu) = \begin{cases} -\lambda^T h - \nu^T b & \text{if } c^T + \lambda^T G + \nu^T A = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

The dual problem is ⇒

$$\max \ g(\lambda, \nu)$$
$$\text{subject to} \quad \lambda \ge 0$$

⇒

$$\max \ \lambda^T h - \nu^T b$$
$$\text{subject to} \quad c + \lambda^T G + \nu^T A = 0$$
$$\lambda \ge 0$$

{Implicit constraint explicit}

S.12 Analytic Centering.

$$\text{minimize} \quad -\sum_{i=1}^{m} \log(b_i - a_i^T x)$$

with domain $\{x \mid a_i^T x < b_i,\ i = 1 \ldots m\}$

Introducing new variable $y_i = b_i - a_i^T x$

original problem $\Rightarrow$ $\quad \text{minimize} \ -\sum_{i=1}^{m} \log y_i$

Subject to $\quad y = b - Ax \qquad y > 0$

$A \in R^{m \times n}$

The Lagrangian is $L(x, y, v) = -\left\{\sum_{i=1}^{m} \log y_i + v^T(y - b - Ax)\right\}$

$$g(v) = \inf_{x,y} \left\{ -\sum_{i=1}^{m} \log y_i + v^T(y - b - Ax) \right\}$$

$v^T A x$ is unbounded below, unless $v^T A = 0$

Also $v^T y$ is unbounded if $v < 0$

$\therefore$ The dual function achieve minimum when $\nabla_y g(v) = 0$

$$\Rightarrow \quad -\sum_{i=1}^{m} \frac{1}{y_i} + v = 0$$

$$\Rightarrow \quad v_i = \frac{1}{y_i}$$

$\therefore$ The dual problem is $\Rightarrow$

$$g(v) = \begin{cases} \sum_{i=1}^{m} \log v_i + m - v^T b & \text{if } A^T v = 0,\ v > 0 \\ -\infty & \text{otherwise.} \end{cases}$$

$$\text{maximize} \quad \sum_{i=1}^{m} \log v_i + m - v^T b$$

Subject to $\quad A^T v = 0$