Complex Variables: Homework #2

Based on algebra of complex numbers

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Problem 1

Represent the following complex number in the polar form. Let me show an example that you need to do. For example, consider the complex number $z = 1 + \sqrt{3}\iota$. Here

$$\begin{aligned} r &= \sqrt{1^2 + \sqrt{3}^2} = \sqrt{4} = 2 \\ \arg(z) &= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} + 2n\pi, \quad n \in \mathbb{Z} \\ \mathrm{Arg}(z) &= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}. \end{aligned}$$

The polar form of z will be

$$z = 2\left(\cos\left(\frac{\pi}{6}\right) + \iota\sin\left(\frac{\pi}{6}\right)\right).$$

Note that one can also write the polar form as

$$z = 2\left(\cos\left(\frac{13\pi}{6}\right) + \iota\sin\left(\frac{13\pi}{6}\right)\right).$$

When we write the polar form, it is not necessary to write the principal argument.

1. $-4 + 4\iota$
2. -5
3. $\frac{12}{\sqrt{3}+\iota}$
4. $1 - \iota$
5. 2ι, -2ι
6. $-2 - 2\sqrt{3}\iota$

Problem 2

In the following problems, write the complex number in the form of $a + \iota b$.

1. $z = 10\left(\cos\frac{\pi}{3} + \iota\sin\frac{\pi}{3}\right)$
2. $z = 5\left(\cos\frac{7\pi}{6} + \iota\sin\frac{7\pi}{6}\right)$
3. $z = 8\sqrt{2}\left(\cos\left(11\frac{\pi}{4}\right) + \iota\frac{\sin(11\pi)}{4}\right)$

Problem 3

In the following problems find $z_1 z_2$ and $\frac{z_1}{z_2}$.

1. $z_1 = 2\left(\cos\frac{\pi}{8} + \iota\sin\frac{\pi}{8}\right)$ and $z_2 = 4\left(\cos\frac{3\pi}{8} + \iota\sin\frac{3\pi}{8}\right)$
2. $z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + \iota\sin\frac{\pi}{4}\right)$ and $z_2 = \sqrt{3}\left(\cos\frac{\pi}{12} + \iota\sin\frac{\pi}{12}\right)$

Problem 4

Determine the argument and principal argument of the following complex numbers.

1. $z = -1 - \iota$
2. $\frac{\iota}{-2-2\iota}$
3. $\left(\sqrt{3} - \iota\right)^6$
4. $\left(\sqrt{3} + \iota\right)^7$

Problem 5

Simplify

$$\frac{(\cos 3\theta + \iota \sin 3\theta)^4 (\cos 4\theta - \iota \sin 4\theta)^5}{(\cos 4\theta + \iota \sin 4\theta)^3 (\cos 5\theta + \iota \sin 5\theta)^{-5}}.$$

Problem 6

Show that

$$(1 + \cos\theta + \iota \sin\theta)^n + (1 + \cos\theta - \iota \sin\theta)^n = 2^{n+1} \cos^n\left(\frac{\theta}{2}\right) \cdot \left(\cos\frac{n\theta}{2}\right).$$

Problem 7

Find the four fourth roots of $z = 1 + \iota$.

Problem 8

In the following problems compute all roots.

1. $(8)^{\frac{1}{3}}$
2. $(-\iota)^{\frac{1}{3}}$
3. $(3 + 4\iota)^{\frac{1}{2}}$
4. $\left(\frac{16\iota}{1+\iota}\right)^{\frac{1}{8}}$
5. $\left(\frac{1+\iota}{\sqrt{3}+\iota}\right)^{\frac{1}{6}}$

Problem 9

Find all solutions of $z^4 + 1 = 0$.