Matrix Groups: Homework #7

Based on matrices over other fields cont.

Dr. Sachchidanand Prasad

Problem 1

In this problem set we will describe the isomorphism between the quotient $SU(2)/\{\pm I\} \cong SO(3)$ which was a problem in the previous homework (Homework 5, Problem 2 (iii)).

(i) The n-sphere is defined as

$$S^n = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} : x_0^2 + x_1^2 + ... + x_n^2 = 1\}.$$

Show that

$$S^3 = \{(w, x, y, z) \in \mathbb{R}^4 : w^2 + x^2 + y^2 + z^2 = 1\}$$

can be identified with the set of unit quaternions

$$\{q = w + xi + yj + zk \in \mathbb{H} w^2 + x^2 + y^2 + z^2 = 1\}.$$

Prove that S^3 is a subgroup of the multiplicative group of nonzero quaternions $\mathbb{H}^\times$.

(ii) In Homework 5, Problem 2 (ii) we proved that

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\}.$$

Show that $SU(2) \cong S^3$ as a group.

(iii) Let $\operatorname{Im} \mathbb{H} = \{xi + yj + zk : x, y, z \in \mathbb{R}\}$ be the set of pure quaternions, which we identify with $\mathbb{R}^3$.

For $q \in S^3$ and $v \in \operatorname{Im} \mathbb{H}$, define

$$T_q(v) = qvq^{-1}.$$

(a) Show that $T_q(v)$ is again a pure quaternion.

(b) Prove that T_q is $\mathbb{R}$-linear.

(c) Show that $|T_q(v)| = |v|$. Conclude that T_q is an orthogonal linear transformation of $\mathbb{R}^3$, i.e. $T_q \in O(3)$.

(iv) Prove that $\det(T_q) = 1$ for all $q \in S^3$. Conclude that $T_q \in SO(3)$.

(v) Thus we obtain a homomorphism

$$\Psi : SU(2) \cong S^3 \to SO(3), q \mapsto T_q.$$

Show that the kernel is $\{\pm I\}$ and hence conclude that $SU(2)/\{\pm I\} \cong SO(3)$.

In this problem set we will describe the isomorphism between the quotient

$$SU(2)/\{\pm I\} \quad \text{and} \quad SO(3).$$

We will proceed step by step, starting with quaternions and unitary matrices.

Problem 2

(a) The n-sphere is defined as

$$S^n = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

Show that

$$S^3 = \{(w, x, y, z) \in \mathbb{R}^4 : w^2 + x^2 + y^2 + z^2 = 1\}$$

can be identified with the set of unit quaternions

$$q = w + xi + yj + zk, \quad w^2 + x^2 + y^2 + z^2 = 1.$$

(b) Prove that S^3 is a subgroup of the multiplicative group of quaternions $\mathbb{H}^\times$.

Problem 3

In Homework 4 (Problem 3) we proved that

$$SU(2) = \{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1\}.$$

(a) Show that if $\alpha = a + bi$ and $\beta = c + di$ with $a, b, c, d \in \mathbb{R}$, then the above matrix can be associated with the quaternion

$$q = a + bi + cj + dk \in S^3.$$

(b) Verify that matrix multiplication in $SU(2)$ corresponds exactly to quaternion multiplication under this identification. Conclude that

$$SU(2) \cong S^3$$

as groups.

Problem 4

Let $\operatorname{Im} \mathbb{H} = \{xi + yj + zk : x, y, z \in \mathbb{R}\}$ be the set of pure quaternions, which we identify with $\mathbb{R}^3$.

For $q \in S^3$ and $v \in \operatorname{Im} \mathbb{H}$, define

$$T_q(v) = qvq^{-1}.$$

(a) Show that $T_q(v)$ is again a pure quaternion.

(b) Prove that T_q is $\mathbb{R}$-linear.

(c) Show that $|T_q(v)| = |v|$. Conclude that T_q is an orthogonal linear transformation of $\mathbb{R}^3$, i.e. $T_q \in O(3)$.

Problem 5

Prove that $\det(T_q) = 1$ for all $q \in S^3$. Conclude that

$$T_q \in SO(3).$$

Problem 6

Let

$$q = w + xi + yj + zk \in S^3.$$

By computing $T_q(e_1), T_q(e_2), T_q(e_3)$ (where $e_1 = i, e_2 = j, e_3 = k$), show that the associated 3×3 matrix of T_q is

$$Q(q) = \begin{pmatrix} 1 - 2y^2 - 2z^2 & 2xy - 2zw & 2xz + 2yw \\ 2xy + 2zw & 1 - 2x^2 - 2z^2 & 2yz - 2xw \\ 2xz - 2yw & 2yz + 2xw & 1 - 2x^2 - 2y^2 \end{pmatrix}.$$

Thus we obtain a homomorphism

$$\Psi : S^3 \to SO(3), \quad q \mapsto Q(q).$$

Problem 7

(a) Show that the kernel of Ψ is $\{\pm 1\}$.

(b) Show that Ψ is surjective onto $SO(3)$. (Hint: any rotation in $\mathbb{R}^3$ is rotation about some axis by some angle; construct a corresponding quaternion.)

(c) Conclude that

$$S^3/\{\pm 1\} \cong SO(3).$$

Problem 8

Using Problem 2, translate the above result into the language of matrices: prove that

$$SU(2)/\{\pm I\} \cong SO(3).$$

Final Conclusion. We have shown that the group $SU(2)$ is isomorphic to S^3, and the action of S^3 on $\mathbb{R}^3$ by conjugation gives a surjective homomorphism onto $SO(3)$ with kernel $\{\pm 1\}$. Therefore,

$$SU(2)/\{\pm I\} \cong SO(3).$$