Algebraic Topology I: Homework #1

Based on review of point set topology

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Theory :

1. Given a set X, a *relation* on it is a subset $\mathcal{R} \subset X \times X$. We say $\mathcal{R}$ is an *equivalence relation* if the following holds.
   a) **(Reflexive)** For each $x \in X$ we have $(x, x) \in \mathcal{R}$.
   b) **(Symmetric)** If $(x, y) \in \mathcal{R}$, then $(y, x) \in \mathcal{R}$.
   c) **(Transitive)** If $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(x, z) \in \mathcal{R}$.

   For any $x \in X$, the *equivalence class* (with respect to the equivalence relation $\mathcal{R}$) is defined as the set

   $$[x] := \{y \in X \mid (x, y) \in \mathcal{R}\}.$$

   We shall denote $x \sim_{\mathcal{R}} y$ (sometimes also denoted $x\mathcal{R}y$, or simply $x \sim y$) whenever $(x, y) \in \mathcal{R}$. The collection of equivalence classes are sometimes denoted as $X/_{\sim}$.
2. Given a set X, a *topology* on X is a collection $\mathcal{T}$ of subsets of X (i.e., $\mathcal{T} \subset \mathcal{P}(X)$), such that the following holds.
   a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
   b) $\mathcal{T}$ is closed under arbitrary unions. That is, for any collection of elements $U_\alpha \in \mathcal{T}$ with $\alpha \in \mathcal{J}$, an indexing set, we have $\bigcup_{\alpha \in \mathcal{J}} U_\alpha \in \mathcal{T}$.
   c) $\mathcal{T}$ is closed under finite intersections. That is, for any finite collection of elements $U_1, \ldots, U_n \in \mathcal{T}$, we have $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$.

   The tuple $(X, \mathcal{T})$ is called a topological space.

   Given any set X we always have two standard topologies on it.
   a) **(Discrete Topology)** $\mathcal{T}_0 = \mathcal{P}(X)$.
   b) **(Indiscrete Topology)** $\mathcal{T}_1 = \{\emptyset, X\}$.

   They are distinct whenever X has at least 2 points.
3. Given a topological space $(X, \mathcal{T})$, a subset $U \subset X$ is called an *open set* if $U \in \mathcal{T}$, and a subset $C \subset X$ is called a *closed set* if $X \setminus C \in \mathcal{T}$ (i.e., if $X \setminus C$ is open).
4. Given a topological space $(X, \mathcal{T})$, a *basis* for it is a sub-collection $\mathcal{B} \subset \mathcal{T}$ of open sets such that every open set $U \in \mathcal{T}$ can be written as the union of some elements of $\mathcal{B}$.
5. Given a topological space $(X, \mathcal{T})$ and a subset $A \subset X$, the *subspace topology* on A is defined as the collection

   $$\mathcal{T}_A := \{U \subset A \mid U = A \cap O \text{ for some } O \in \mathcal{T}\}.$$

   We say $(A, \mathcal{T}_A)$ is a subspace of $(X, \mathcal{T})$.
6. Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a function $f : X \to Y$ is said to be *continuous* if $f^{-1}(U) \in \mathcal{T}_X$ for any $U \in \mathcal{T}_Y$ (i.e., pre-image of open sets are open).

Problem 1

i) Given an equivalence relation $\mathcal{R}$ on X, check that any two equivalence classes are either disjoint or equal (i.e., they cannot have nontrivial intersection).

ii) Suppose X is a given set, and $A \subset X$ is a nonempty subset. Define the relation $\mathcal{R} \subset X \times X$ as follows.

$$\mathcal{R} := \{(x, x) \mid x \in X \setminus A\} \cup \{(a, b) \mid a, b \in A\}.$$

a) Check that $\mathcal{R}$ is an equivalence relation.
b) Identify the equivalence classes. We shall denote the collection of equivalence classes as X/A.
c) What is X/X ?

Problem 2

i) Given any set X, verify that both the discrete and the indiscrete topologies are indeed topologies, that is, check that they satisfy the axioms.

ii) Given X, suppose $\mathcal{C} \subset \mathcal{P}(X)$ is a collection of subsets that satisfy the following.
   a) $\emptyset \in \mathcal{C}, \ X \in \mathcal{C}$.
   b) $\mathcal{C}$ is closed under arbitrary intersections.
   c) $\mathcal{C}$ is closed under finite unions.

   Define the collection,

$$\mathcal{T} := \{U \subset X \mid X \setminus U \in \mathcal{C}\}.$$

   Prove that $\mathcal{T}$ is a topology on X.

iii) On any set X, consider the following collections of subsets.
   a) $\mathcal{T}_1 := \{A \subset X \mid X \setminus A \text{ is finite}\} \cup \{\emptyset\}$.
   b) $\mathcal{T}_2 := \{A \subset X \mid X \setminus A \text{ is countable}\} \cup \{\emptyset\}$.

   Show that $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, respectively called the *cofinite* and the *cocountable* topologies.

   Now, suppose X is uncountable (say, $X = \mathbb{R}$), and consider the collection

$$\mathcal{T}_3 := \{A \subset X \mid X \setminus A \text{ is uncountable}\}.$$

   Is $\mathcal{T}_3$ a topology on X?

iv) On the real line $\mathbb{R}$, consider the collection of subsets

$$\mathcal{T}_{\leftarrow} := \{\emptyset, \mathbb{R}\} \cup \{(-\infty, a) \mid a \in \mathbb{R}\}.$$

   Show that $\mathcal{T}_{\leftarrow}$ is a topology on $\mathbb{R}$.

Problem 3

i) ***(Necessary condition for basis)*** Suppose $(X, \mathcal{T})$ is a topological space, and consider a basis $\mathcal{B} \subset \mathcal{T}$. Then, the following holds.
   a) [**(B1)**] For any $x \in X$, there exists some $U \in \mathcal{B}$ such that $x \in U$.
   b) [**(B2)**] For any $U, V \in \mathcal{B}$ and any element $x \in U \cap V$, there exists some $W \in \mathcal{B}$ such that $x \in W \subset U \cap V$.

ii) Suppose $\mathcal{B} \subset \mathcal{P}(X)$ is a collection of subsets of X satisfying (B1) and (B2). Consider $\mathcal{T}$ to be the collection of all possible unions of elements of $\mathcal{B}$. Show that $\mathcal{T}$ is a topology on X and $\mathcal{B}$ is a basis for it.

Problem 4

i) Suppose $U \subset X$ is an open set. What are the open subsets of U in the subspace topology? What are the closed sets?

ii) Suppose $\mathbb{R}$ is equipped with the Euclidean topology (that is topology genrated by the open intervals). Consider $\mathbb{Q}$ with the subspace topology.
   - Is the set $\left(0, \sqrt{2}\right) \cap \mathbb{Q}$ open or closed in $\mathbb{Q}$?
   - Is the set $(0, 3] \cap \mathbb{Q}$ open or closed in $\mathbb{Q}$?

Problem 5

i) Show that $f : X \to Y$ is continuous if and only if preimage of closed sets of Y is closed in X.

ii) Suppose $(X, \mathcal{T})$ is a topological space. Show that the following are equivalent.
   a) X has the discrete topology, i.e., $\mathcal{T} = \mathcal{P}(X)$.
   b) Given any space Y, any function $f : X \to Y$ is continuous.
   c) The map $\mathrm{Id} : (X, \mathcal{T}) \to (X, \mathcal{P}(X))$ is continuous.

iii) Suppose $(X, \mathcal{T})$ is a space, and some $A \subset X$ is equipped with the subspace topology $\mathcal{T}_A$.
   a) Show that the inclusion map $\iota : A \hookrightarrow X$ is continuous.
   b) Suppose $\mathcal{S}$ is some topology on A such that the inclusion map $\iota : (A, \mathcal{S}) \hookrightarrow (X, \mathcal{T})$ is continuous. Show that $\mathcal{S}$ is finer than $\mathcal{T}_A$.