Day 6 : 21st January, 2026

Definition 6.1: (*Subcategory*)

Let $\mathscr{C}$ be a category. A *subcategory* $\mathscr{S}$ of $\mathscr{C}$ is a category such that

i) $\mathrm{Ob}(\mathscr{S}) \subseteq \mathrm{Ob}(\mathscr{C})$;
ii) for any $A, B \in \mathrm{Ob}(\mathscr{S})$, $\mathrm{Hom}_{\mathscr{S}}(A, B) \subseteq \mathrm{Hom}_{\mathscr{C}}(A, B)$;
iii) composition in $\mathscr{S}$ is the same as composition in $\mathscr{C}$.

The third condition says that the function $\mathrm{Hom}_{\mathscr{S}}(A, B) \times \mathrm{Hom}_{\mathscr{S}}(B, C) \to \mathrm{Hom}_{\mathscr{S}}(A, C)$ is the restriction of the corresponding composition with subscripts $\mathscr{C}$.

Example 6.2: (*Examples of subcategory*)

i) Let $\mathscr{C} = \mathbf{Top}$. Then the following are subcategories of $\mathscr{C}$.
   a) The objects are subspaces of $(\mathbb{R}^n, \mathscr{T}_{\mathrm{Euc}})$ and morphisms are continuous maps between spaces.
   b) The objects are Hausdorff topological spaces.
   c) The objects are compact topological spaces.
ii) Let $\mathscr{C} = \mathbf{Groups}$ be the category of groups. Then $\mathbf{Ab}$, the category of abelian groups is a subcategory. Category of rings $\mathbf{Rings}$ is also a subcategory.
iii) Consider the category $\mathscr{C} = \mathbf{Top^2}$ with

$$\begin{aligned} \mathrm{Ob}(\mathscr{C}) &= \text{ordered pairs } (X, A) \text{ where } X \text{ is a topological space} \\ &\quad \text{and } A \text{ is a subspace of } X. \\ \mathrm{Hom}((X, A), (Y, B)) &= \text{set of continuous functions } f : X \to Y \text{ with } f(A) \subseteq B. \end{aligned}$$

Then $\mathbf{Top}_*$ is a subcategory of $\mathbf{Top^2}$.

Exercise 6.3:

i) Let $\mathscr{C}$ be a category and let $A \in \mathrm{Ob}(\mathscr{C})$. Prove that $\mathrm{Hom}(A, A)$ has a unique identity 1_A.
ii) If $\mathscr{S}$ is a subcategory of $\mathscr{C}$, and if $A \in \mathrm{Ob}(\mathscr{S})$, then the identity of A in $\mathrm{Hom}_{\mathscr{S}}(A, A)$ is the identity $1_A \in \mathrm{Hom}_{\mathscr{C}}(A, A)$.
iii) Show that one may regard $\mathbf{Top}$ as a subcategory of $\mathbf{Top^2}$ if one identifies a space X with the pair $(X, \emptyset)$.

Definition 6.4: (*Commutative Diagram*)

A *diagram* in a category $\mathscr{C}$ is a directed graph whose vertices are labeled by objects of $\mathscr{C}$ and whose directed edges are labeled by morphisms in $\mathscr{C}$. A *commutative diagram* in $\mathscr{C}$ is a diagram in which for each pair of vertices, every two paths (composites) between them are equal as morphisms.

Example 6.5: (*A. commutative diagram*)

THe following is a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f'"'] & B \arrow[d, "g"] \\ C \arrow[r, "g'"] & D \end{tikzcd}$$

In addition, if $g \circ f = g' \circ f'$, then the diagram is commutative.

Definition 6.6: (*Congruence*)

A *congruence* on a category $\mathscr{C}$ is an equivalence relation $\sim$ on the class $\bigcup_{(A,B)} \mathrm{Hom}(A, B)$ of all morphisms in $\mathscr{C}$ such that

i) $f \in \mathrm{Hom}(A, B)$ and $f \sim f'$ implies $f' \in \mathrm{Hom}(A, B)$;
ii) $f \sim f'$, $g \sim g'$ and the composition $g \circ f$ exists imply that $g \circ f \sim g' \circ f'$.

Theorem 6.7: (*Quotient Category*)

Let $\mathscr{C}$ be a category with congruence $\sim$, and let $[f]$ denote the equivalence class of a morphism f. Define $\mathscr{C}'$ by

$$\begin{aligned} \mathrm{Ob}(\mathscr{C}') &= \mathrm{Ob}(\mathscr{C}); \\ \mathrm{Hom}_{\mathscr{C}'}(A, B) &= \{[f] : f \in \mathrm{Hom}_{\mathscr{C}}(A, B)\}; \\ [g] \circ [f] &= [g \circ f]. \end{aligned}$$

Then $\mathscr{C}'$ is a category. It is called a *quotient category*.

One usually denotes $\mathrm{Hom}_{\mathscr{C}'}(A, B)$ by $[A, B]$.

Exercise 6.8:

Show that $\mathscr{C}'$ is a category.

Exercise 6.9: (*Category of conjugacy classes*)

Consider the category of groups $\mathbf{Groups}$. Let G, H be two groups and $f, g \in \mathrm{Hom}(G, H)$. Define

$$f \sim g \Leftrightarrow \text{there exists } \ a \in H \text{ such that } \ f(x) = ag(x)a^{-1}, \text{for all } \ x \in G.$$

i) Show that $\sim$ is an equivalence relation on each $\mathrm{Hom}(G, H)$.
ii)

Definition 6.10: (*Functor*)

Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A *covariant functor* $F : \mathscr{C} \to \mathscr{D}$ consists of the following data.

i) For each object $A \in \mathscr{C}$, there is an object $F(A) \in \mathscr{D}$.
ii) For each morphism $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, there is a morphism $F(f) \in \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$ such that
 a) F preserves the identity, that is, for any object $A \in \mathscr{C}$,

$$F(1_A) = 1_{F(A)}.$$

 b) F preserves the composition, that is, if $f : A \to B$ and $g : B \to C$, then

$$F(g \circ f) = F(g) \circ F(f).$$

We say $F : \mathscr{C} \to \mathscr{D}$ is a *contravariant functor* if it reverses the morphism and the composition, that is, for $f : A \to B$ and $g : B \to C$ in $\mathscr{C}$

$$\begin{aligned} &F(f) : F(B) \to F(A) \text{ in } \mathscr{D} \quad \text{and} \\ &F(g \circ f) = F(f) \circ F(g). \end{aligned}$$

Example 6.11: (*Some functors*)

1. Let $\mathscr{C}$ be any category. Then we always have the identity functor $\mathrm{Id} : \mathscr{C} \to \mathscr{C}$.
2. The easiest example of functors are *Forgetful functors*. For any mathematical object defined as a set with some additional structure, we can “forget” the extra structure and get a forgetful functor.
 i) $F : \mathbf{Top} \to \mathbf{Sets}$ defined as follows. For any topological space X, $F(X)$ is the underlying set and if $f : X \to Y$ is a continuous function, then $F(f)$ is the same function. So F forgets the topological structure of X and also forgets the continuous functions between topological spaces.
 ii) Similarly, there is a functor $\mathbf{Groups} \to \mathbf{Sets}$ forgetting the group structure on groups and a functor $\mathbf{Rings} \to \mathbf{Sets}$ forgetting the ring structure on rings.
 iii) We can also have forgetful functor which does not forget all the structures. For example, if $\mathbf{Ab}$ is the category of abelian groups, then a functor from $\mathbf{Ab} \to \mathbf{Groups}$ forgets the abelian structure in groups and remember just underlying group structure. Similarly, we can have functors $\mathbf{Rings} \to \mathbf{Ab}$ which forgets the multiplicative structure and remembers the underlying additive group.
3. Let X and Y be two sets and $f : X \to Y$ be a function. We have a functor between the two discrete categories and conversely. That is, given a function $f : X \to Y$ one can give a functor and given a functor $F : \mathbf{Disc(X)} \to \mathbf{Disc(Y)}$ one can define $f : X \to Y$.
4. Let M be a fixed topological space. Then

$$P_M : \mathbf{Top} \to \mathbf{Top}, \quad X \mapsto X \times M$$

 is a functor.
5. Hom *functors.*
 i) Let $\mathscr{C}$ be any category and A be an object in $\mathscr{C}$. Then we have the following functor

$$\mathrm{Hom}(A, \_) : \mathscr{C} \to \mathbf{Sets}, \quad B \mapsto \mathrm{Hom}(A, B).$$

 For any $f : X \to Y$ in $\mathscr{C}$,

$$f_* := \mathrm{Hom}(A, f) : \mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y), \quad g \mapsto f \circ g.$$

 This is a covariant functor.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, dashed, "f \circ g"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

ii) We can also define a contravariant functor as follows.

$$\mathrm{Hom}(\_, A) : \mathscr{C} \to \mathbf{Sets}, \quad B \mapsto \mathrm{Hom}(B, A).$$

For any $f : X \to Y \text{ in } \mathscr{C}$,

$$f^* := \mathrm{Hom}(f, A) : \mathrm{Hom}(X, A) \to \mathrm{Hom}(Y, A), \quad g \mapsto g \circ f.$$

This is a contravariant functor.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dashed, "g \circ f"'] & Y \arrow[d, "g"] \\ & A \end{tikzcd}$$

6. Let $\mathbf{Vect}$ be the category of vector space over a field $\mathbb{F}$. Here morphisms are $\mathbb{F}$-linear maps. For any two vector spaces V and W, $\mathrm{Hom}(V, W)$ is a vector space over $\mathbb{F}$. We have two Hom functors,

$$\mathrm{Hom}(V, \_) : \mathbf{Vect} \to \mathbf{Vect}$$
$$\mathrm{Hom}(\_, V) : \mathbf{Vect} \to \mathbf{Vect}.$$