Day 5 : 19th January, 2026

5.1. Wedge sum and smash product The disjoint union of two spaces is geometrically exactly what it sounds like: you just imagine the two spaces side by side.

Definition 5.1: (*Disjoint union*)

Let X and Y be two topological space. The *disjoint union*, $X \sqcup Y$, is defined by

$$X \sqcup Y := \text{disjoint union of } X \text{ and } Y.$$

A subset $U \subseteq X \sqcup Y$ is open if $U \cap X$ and $U \cap Y$ are open in X and Y, respectively.

Exercise 5.2: (*Disjoint union on any indexed set*)

Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of topological space indexed by Λ. Let

$$X := \bigsqcup_{\alpha \in \Lambda} X_\alpha$$

denote the disjoint union of X_α's. For each $\alpha \in \Lambda$, we have the natural inclusion map

$$f_\alpha : X_\alpha \hookrightarrow X.$$

Show that the above defined topology is the strongest topology on X such that each f_α is continuous.

Definition 5.3: (*Based or pointed space*)

A *based space* or *pointed space* is a pair (X, x_0), where X is a topological and $x_0 \in X$ is a chosen *basepoint*.

If (X, x_0) and (Y, y_0) are based spaces, then a continuous map $f : (X, x_0) \to (Y, y_0)$ is called a *based map* if $f(x_0) = y_0$.

Definition 5.4: (*Wedge sum*)

Let (X, x_0) and (Y, y_0) be two based spaces. Then the *wedge sum*, denoted by $X \vee Y$, is defined as the quotient space of the disjoint union of X and Y by the identification $x_0 \sim y_0$.

$$X \vee Y := (X \sqcup Y)/\sim, \text{where } x_0 \sim y_0.$$

Similarly, the wedge sum of collection of pointed spaces is defined as

$$\bigvee_{\alpha \in \Lambda} (X_\alpha, x_\alpha) := \left(\bigsqcup_{\alpha \in \Lambda} (X_\alpha, x_\alpha) \right) / \sim .$$

Definition 5.5: (*Smash product*)

Let (X, x_0) and (Y, y_0) be two based spaces. The *smash product*, denoted by $X \wedge Y$, is defined as the quotient space of $X \times Y$ with the identifications $(x, y_0) \sim (x_0, y)$ for all $x \in X$ and $y \in Y$.

Note that the wedge sum of X and Y be viewed as $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$ and hence

$$X \wedge Y = \frac{X \times Y}{X \vee Y}.$$

5.2. Category theory

Definition 5.6: (*Category*)

A *category* $\mathscr{C}$ consists of following:

i) A collection of *objects* denoted by $\mathrm{Ob}(\mathscr{C})$.
ii) For any two objects $A, B \in \mathrm{Ob}(\mathscr{C})$, a set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ called a *morphism*.
iii) For any three objects $A, B, C \in \mathrm{Ob}(\mathscr{C})$, we have a *composition*

$$\circ : \mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(B, C) \to \mathrm{Hom}_{\mathscr{C}}(A, C),$$
$$(f, g) \mapsto g \circ f$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

satisfying the following properties.

a) (**Associativity**) Let $A, B, C, D \in \mathrm{Ob}(\mathscr{C})$. For morphisms $f \in \mathrm{Hom}_{\mathscr{C}}(A, B), g \in \mathrm{Hom}_{\mathscr{C}}(B, C)$ and $h \in \mathrm{Hom}_{\mathscr{C}}(C, D)$, we have
$$h \circ (g \circ f) = (h \circ g) \circ f.$$
b) (**Identity**) For each object A, there is a morphism $1_A \in \mathrm{Hom}_{\mathscr{C}}(A, A)$ such that for any $g \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ and any $h \in \mathrm{Hom}_{\mathscr{C}}(C, A)$, we have
$$g \circ 1_A = g, \quad 1_A \circ h = h.$$

When the context is clear, we will write $\mathrm{Hom}(A, B)$ instead of $\mathrm{Hom}_{\mathscr{C}}(A, B)$.

Remark 5.7:

i) The only restriction on Hom is that it be a set. In particular, $\mathrm{Hom}(A, B) = \emptyset$ is allowed, although the identity axiom shows that $\mathrm{Hom}(A, A) \neq \emptyset$ as it contains 1_A.
ii) Instead of writing $f \in \mathrm{Hom}(A, B)$, we usually write $f : A \to B$.

Example 5.8: (*Examples of category*)

Categories are everywhere in mathematics (and even in computer science). We will list some of them here.

i) $\mathscr{C} = \mathbf{Sets}$. Here

$$
\begin{aligned}
\mathrm{Ob}(\mathscr{C}) &= \text{all sets,} \\
\mathrm{Hom}(A,B) &= \{\text{all functions } f : A \to B\} \\
\circ &= \text{composition is the usual function composition}
\end{aligned}
$$

ii) $\mathscr{C} = \mathbf{Top}$. Here

$$
\begin{aligned}
\mathrm{Ob}(\mathscr{C}) &= \text{all topological spaces,} \\
\mathrm{Hom}(X,Y) &= \{\text{all continuous functions } f : X \to Y\} \\
\circ &= \text{composition is the usual function composition}
\end{aligned}
$$

iii) $\mathscr{C} = \mathbf{Top}_*$. Here

$$
\begin{aligned}
\mathrm{Ob}(\mathscr{C}) &= \{(X, *_X) : X \text{ is a topological space and } *_X \in X\} \\
&\quad *_X \in X \text{ is called the basepoint,} \\
\mathrm{Hom}((X,*_X),(Y,*_Y)) &= \{\text{all continuous functions } f : (X,*_X) \to (Y,*_Y) \text{ with } f(*_X) = *_Y\} \\
&\quad \text{this is called basepoint preserving functions} \\
\circ &= \text{composition is the usual function composition}
\end{aligned}
$$

iv) $\mathscr{C} = \mathbf{Groups}$: Category of groups and group homomorphisms.

v) $\mathscr{C} = \mathbf{Ab}$: Category of abelian groups and group homomorphisms.

Example 5.9: (*Discrete Category*)

Let X be a nonempty set. Consider a category $\mathscr{C}$,

$$
\begin{aligned}
\mathrm{Ob}(\mathscr{C}) &= \text{elements of } X \\
\mathrm{Hom}(x,y) &= \begin{cases} \emptyset & \text{if } x \neq y \\ 1_x & \text{if } x = y. \end{cases}
\end{aligned}
$$

This is called a *discrete category*.