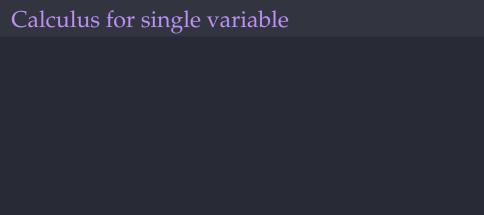
### Limits

**Engineering Mathematics-I** 

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SPNREC, Araria

October 6, 2024



▶ Intermediate Form

- Intermediate Form
- ► L'Hospital Rule

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- ► Rolle's Theorem

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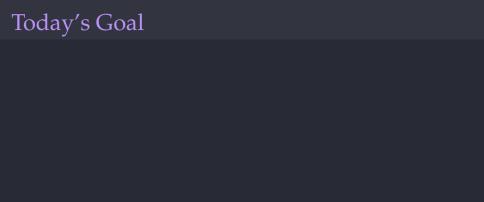
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- Expansion of function
- ► Taylor and Maclaurin series
- Riemann Integration
- Riemann Sum
- Improper integral
- Beta and Gamma functions and their properties



# Today's Goal

► Intermediate Form

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- ► L'Hospital Rule
- ▶ Rolle's Theorem

► When we evaluate the limit, we encounter the following forms:

 $\frac{0}{0}$ 

$$\frac{0}{0} \qquad \lim_{x \to 0} \frac{x^2 - 1}{x - 1}$$

$$\frac{0}{0} \qquad \lim_{x \to 0} \frac{x^2 - 1}{x - 1}$$

$$\frac{\infty}{\infty}$$

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 $\infty - \infty$ 

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$$\infty - \infty \qquad \lim_{x \to \infty} \left( x - \sqrt{x^2 + x} \right)$$

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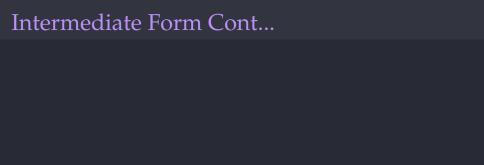
$$0 \times \infty$$

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$$\frac{\infty}{\infty} \qquad \lim_{x \to \infty} \frac{x^2 - 1}{x - 1}$$

$$\infty - \infty \qquad \lim_{x \to \infty} \left(x - \sqrt{x^2 + x}\right)$$

$$0 \times \infty \qquad \lim_{x \to \infty} \frac{2x}{x^3 - 1} \cdot \ln x$$



 $0^0$ 

$$0^0 \qquad \lim_{x \to 0} x^x$$

$$\lim_{x \to 0} x^x$$

 $1^{\infty}$ 

$$\begin{array}{ll}
0^0 & \lim_{x \to 0} x^x \\
1^\infty & \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x
\end{array}$$

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$$\begin{array}{ll}
0^0 & \lim_{x \to 0} x^x \\
1^\infty & \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x \\
\infty^0 & \lim_{x \to \frac{\pi}{2}} (\tan x)^{x - \frac{\pi}{2}}
\end{array}$$

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For these types of limits, we will use L'Hospital Rule.

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Theorem (L'Hospital Rule)

For a limit  $\lim_{x\to a} \frac{f(x)}{g(x)}$  of the indeterminate form  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ ,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if 
$$\lim_{x\to a} \frac{f'(x)}{g'(x)}$$
 exists or equals to  $\pm\infty$ .

# Examples

1.  $\lim_{x\to\pi}\frac{x^2-\pi^2}{\sin x}$ 

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$$\lim_{x \to \pi} \frac{x^2 - \pi^2}{\sin x} = -2\pi$$

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$$2. \lim_{x \to \infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1}$$

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$$\lim_{x \to \pi} \frac{x^2 - \pi^2}{\sin x} = -2\pi$$

2. 
$$\lim_{x \to \infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = 2.$$

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3. 
$$\lim_{x \to 0} \frac{\sec x - x}{\sin x}$$

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$$3. \lim_{x \to 0} \frac{\sec x - x}{\sin x} = 0$$

4. 
$$\lim_{x \to 1} \frac{x^3 - x^2 \ln x + \ln x - 1}{x^2 - 1}$$

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5. If 
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, then find the values of  $a$  and  $b$ .

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**6.** 
$$\lim_{t \to 0} \left( t + \frac{1}{t} \right) \left( (4 - t)^{3/2} - 8 \right)$$

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6. 
$$\lim_{x\to 0} 2x \tan\left(\frac{\pi}{2} - x\right)$$

Consider

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$$= \lim_{x \to \infty} \left[ \left( x - \sqrt{x^2 + x} \right) \times \frac{\left( x + \sqrt{x^2 + x} \right)}{\left( x + \sqrt{x^2 + x} \right)} \right]$$

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$$= \lim_{x \to \infty} \frac{x^2 - (x^2 + x)}{\left( x + \sqrt{x^2 + x} \right)}$$

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We can also write this in terms of  $\frac{\infty}{\infty}$ .

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