

Limits

Engineering Mathematics-I

Dr. (PhD) Sachchidanand Prasad

SPNREC, Araria

October 6, 2024

Calculus for single variable

Calculus for single variable

- ▶ Intermediate Form

Calculus for single variable

- ▶ Intermediate Form
- ▶ L'Hospital Rule

Calculus for single variable

- ▶ Intermediate Form
- ▶ L'Hospital Rule
- ▶ Rolle's Theorem

Calculus for single variable

- ▶ Intermediate Form
- ▶ L'Hospital Rule
- ▶ Rolle's Theorem
- ▶ Mean Value Theorem

Calculus for single variable

- ▶ Intermediate Form
- ▶ L'Hospital Rule
- ▶ Rolle's Theorem
- ▶ Mean Value Theorem
- ▶ Expansion of function

Calculus for single variable

- ▶ Intermediate Form
- ▶ L'Hospital Rule
- ▶ Rolle's Theorem
- ▶ Mean Value Theorem
- ▶ Expansion of function
- ▶ Taylor and Maclaurin series

Calculus for single variable

- ▶ Intermediate Form
- ▶ L'Hospital Rule
- ▶ Rolle's Theorem
- ▶ Mean Value Theorem
- ▶ Expansion of function
- ▶ Taylor and Maclaurin series
- ▶ Riemann Integration

Calculus for single variable

- ▶ Intermediate Form
- ▶ L'Hospital Rule
- ▶ Rolle's Theorem
- ▶ Mean Value Theorem
- ▶ Expansion of function
- ▶ Taylor and Maclaurin series
- ▶ Riemann Integration
- ▶ Riemann Sum

Calculus for single variable

- ▶ Intermediate Form
- ▶ L'Hospital Rule
- ▶ Rolle's Theorem
- ▶ Mean Value Theorem
- ▶ Expansion of function
- ▶ Taylor and Maclaurin series
- ▶ Riemann Integration
- ▶ Riemann Sum
- ▶ Improper integral

Calculus for single variable

- ▶ Intermediate Form
- ▶ L'Hospital Rule
- ▶ Rolle's Theorem
- ▶ Mean Value Theorem
- ▶ Expansion of function
- ▶ Taylor and Maclaurin series
- ▶ Riemann Integration
- ▶ Riemann Sum
- ▶ Improper integral
- ▶ Beta and Gamma functions and their properties

Today's Goal

- Understand the importance of the **customer** in business
- Understand the importance of the **employee** in business
- Understand the importance of the **owner** in business
- Understand the importance of the **community** in business
- Understand the importance of the **environment** in business
- Understand the importance of the **technology** in business
- Understand the importance of the **innovation** in business
- Understand the importance of the **competition** in business
- Understand the importance of the **market** in business
- Understand the importance of the **product** in business
- Understand the importance of the **service** in business
- Understand the importance of the **brand** in business
- Understand the importance of the **reputation** in business
- Understand the importance of the **trust** in business
- Understand the importance of the **loyalty** in business
- Understand the importance of the **customer satisfaction** in business
- Understand the importance of the **employee satisfaction** in business
- Understand the importance of the **owner satisfaction** in business
- Understand the importance of the **community satisfaction** in business
- Understand the importance of the **environmental satisfaction** in business
- Understand the importance of the **technological satisfaction** in business
- Understand the importance of the **innovative satisfaction** in business
- Understand the importance of the **competitive satisfaction** in business
- Understand the importance of the **market satisfaction** in business
- Understand the importance of the **product satisfaction** in business
- Understand the importance of the **service satisfaction** in business
- Understand the importance of the **brand satisfaction** in business
- Understand the importance of the **reputation satisfaction** in business
- Understand the importance of the **trust satisfaction** in business
- Understand the importance of the **loyalty satisfaction** in business
- Understand the importance of the **customer satisfaction** in business
- Understand the importance of the **employee satisfaction** in business
- Understand the importance of the **owner satisfaction** in business
- Understand the importance of the **community satisfaction** in business
- Understand the importance of the **environmental satisfaction** in business
- Understand the importance of the **technological satisfaction** in business
- Understand the importance of the **innovative satisfaction** in business
- Understand the importance of the **competitive satisfaction** in business
- Understand the importance of the **market satisfaction** in business
- Understand the importance of the **product satisfaction** in business
- Understand the importance of the **service satisfaction** in business
- Understand the importance of the **brand satisfaction** in business
- Understand the importance of the **reputation satisfaction** in business
- Understand the importance of the **trust satisfaction** in business
- Understand the importance of the **loyalty satisfaction** in business

Today's Goal

- ▶ Intermediate Form

Today's Goal

- ▶ Intermediate Form
- ▶ L'Hospital Rule

Today's Goal

- ▶ Intermediate Form
- ▶ L'Hospital Rule
- ▶ Rolle's Theorem

Intermediate Form

- ▶ When we evaluate the limit, we encounter the following forms:

Intermediate Form

- ▶ When we evaluate the limit, we encounter the following forms:

$$\frac{0}{0}$$

Intermediate Form

- ▶ When we evaluate the limit, we encounter the following forms:

$$\frac{0}{0} \qquad \lim_{x \rightarrow 0} \frac{x^2 - 1}{x - 1}$$

Intermediate Form

- ▶ When we evaluate the limit, we encounter the following forms:

$$\frac{0}{0} \qquad \lim_{x \rightarrow 0} \frac{x^2 - 1}{x - 1}$$

$$\frac{\infty}{\infty}$$

Intermediate Form

- ▶ When we evaluate the limit, we encounter the following forms:

$$\frac{0}{0} \qquad \lim_{x \rightarrow 0} \frac{x^2 - 1}{x - 1}$$

$$\frac{\infty}{\infty} \qquad \lim_{x \rightarrow \infty} \frac{x^2 - 1}{x - 1}$$

Intermediate Form

- When we evaluate the limit, we encounter the following forms:

$$\frac{0}{0} \qquad \lim_{x \rightarrow 0} \frac{x^2 - 1}{x - 1}$$

$$\frac{\infty}{\infty} \qquad \lim_{x \rightarrow \infty} \frac{x^2 - 1}{x - 1}$$

$$\infty - \infty$$

Intermediate Form

- When we evaluate the limit, we encounter the following forms:

$$\frac{0}{0} \qquad \lim_{x \rightarrow 0} \frac{x^2 - 1}{x - 1}$$

$$\frac{\infty}{\infty} \qquad \lim_{x \rightarrow \infty} \frac{x^2 - 1}{x - 1}$$

$$\infty - \infty \qquad \lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + x} \right)$$

Intermediate Form

- When we evaluate the limit, we encounter the following forms:

$$\frac{0}{0} \qquad \lim_{x \rightarrow 0} \frac{x^2 - 1}{x - 1}$$

$$\frac{\infty}{\infty} \qquad \lim_{x \rightarrow \infty} \frac{x^2 - 1}{x - 1}$$

$$\infty - \infty \qquad \lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + x} \right)$$

$$0 \times \infty$$

Intermediate Form

- When we evaluate the limit, we encounter the following forms:

$$\frac{0}{0} \qquad \lim_{x \rightarrow 0} \frac{x^2 - 1}{x - 1}$$

$$\frac{\infty}{\infty} \qquad \lim_{x \rightarrow \infty} \frac{x^2 - 1}{x - 1}$$

$$\infty - \infty \qquad \lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + x} \right)$$

$$0 \times \infty \qquad \lim_{x \rightarrow \infty} \frac{2x}{x^3 - 1} \cdot \ln x$$

Intermediate Form Cont...

Intermediate Form Cont...

$$0^0$$

Intermediate Form Cont...

$$0^0$$

$$\lim_{x \rightarrow 0} x^x$$

Intermediate Form Cont...

$$0^0$$

$$\lim_{x \rightarrow 0} x^x$$

$$1^\infty$$

Intermediate Form Cont...

$$0^0$$

$$\lim_{x \rightarrow 0} x^x$$

$$1^\infty$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$$

Intermediate Form Cont...

$$0^0$$

$$\lim_{x \rightarrow 0} x^x$$

$$1^\infty$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$$

$$\infty^0$$

Intermediate Form Cont...

$$0^0$$

$$\lim_{x \rightarrow 0} x^x$$

$$1^\infty$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$$

$$\infty^0$$

$$\lim_{x \rightarrow \frac{\pi}{2}} (\tan x)^{x - \frac{\pi}{2}}$$

Limit of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

For these types of limits, we will use **L'Hospital Rule**.

Limit of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

For these types of limits, we will use **L'Hospital Rule**.

Theorem (L'Hospital Rule)

For a limit $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ of the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

if $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists or equals to $\pm\infty$.

Examples

1. $\lim_{x \rightarrow \pi} \frac{x^2 - \pi^2}{\sin x}$

Examples

$$1. \lim_{x \rightarrow \pi} \frac{x^2 - \pi^2}{\sin x} = -2\pi$$

Examples

1. $\lim_{x \rightarrow \pi} \frac{x^2 - \pi^2}{\sin x} = -2\pi$

2. $\lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1}$

Examples

$$1. \lim_{x \rightarrow \pi} \frac{x^2 - \pi^2}{\sin x} = -2\pi$$

$$2. \lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = 2.$$

Examples

1. $\lim_{x \rightarrow \pi} \frac{x^2 - \pi^2}{\sin x} = -2\pi$

2. $\lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = 2.$

3. $\lim_{x \rightarrow 0} \frac{\sec x - x}{\sin x}$

Examples

$$1. \lim_{x \rightarrow \pi} \frac{x^2 - \pi^2}{\sin x} = -2\pi$$

$$2. \lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = 2.$$

$$3. \lim_{x \rightarrow 0} \frac{\sec x - x}{\sin x} = 0$$

Examples

$$1. \lim_{x \rightarrow \pi} \frac{x^2 - \pi^2}{\sin x} = -2\pi$$

$$2. \lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = 2.$$

$$3. \lim_{x \rightarrow 0} \frac{\sec x - x}{\sin x} = 0$$

$$4. \lim_{x \rightarrow 1} \frac{x^3 - x^2 \ln x + \ln x - 1}{x^2 - 1}$$

Examples

$$1. \lim_{x \rightarrow \pi} \frac{x^2 - \pi^2}{\sin x} = -2\pi$$

$$2. \lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = 2.$$

$$3. \lim_{x \rightarrow 0} \frac{\sec x - x}{\sin x} = 0$$

$$4. \lim_{x \rightarrow 1} \frac{x^3 - x^2 \ln x + \ln x - 1}{x^2 - 1} = \frac{3}{2}$$

Examples

1. $\lim_{x \rightarrow \pi} \frac{x^2 - \pi^2}{\sin x} = -2\pi$

2. $\lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = 2.$

3. $\lim_{x \rightarrow 0} \frac{\sec x - x}{\sin x} = 0$

4. $\lim_{x \rightarrow 1} \frac{x^3 - x^2 \ln x + \ln x - 1}{x^2 - 1} = \frac{3}{2}$

5. If $\lim_{x \rightarrow \infty} \left(\frac{x^2 - 1}{x + 1} - ax - b \right) = 0$, then find the values of a and b .

Examples

$$1. \lim_{x \rightarrow \pi} \frac{x^2 - \pi^2}{\sin x} = -2\pi$$

$$2. \lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = 2.$$

$$3. \lim_{x \rightarrow 0} \frac{\sec x - x}{\sin x} = 0$$

$$4. \lim_{x \rightarrow 1} \frac{x^3 - x^2 \ln x + \ln x - 1}{x^2 - 1} = \frac{3}{2}$$

$$5. \text{ If } \lim_{x \rightarrow \infty} \left(\frac{x^2 - 1}{x + 1} - ax - b \right) = 0, \text{ then find the values of } a \text{ and } b.$$

$$6. \lim_{t \rightarrow 0} \left(t + \frac{1}{t} \right) \left((4 - t)^{3/2} - 8 \right)$$

Limit of the form $0 \times \infty$

Limit of the form $0 \times \infty$

We can transform these limits in either of the above two forms, that is, either $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Limit of the form $0 \times \infty$

We can transform these limits in either of the above two forms, that is, either $\frac{0}{0}$ or $\frac{\infty}{\infty}$. If

$$\lim_{x \rightarrow a} f(x) = 0 \text{ and } \lim_{x \rightarrow a} g(x) = \infty,$$

Limit of the form $0 \times \infty$

We can transform these limits in either of the above two forms, that is, either $\frac{0}{0}$ or $\frac{\infty}{\infty}$. If

$$\lim_{x \rightarrow a} f(x) = 0 \text{ and } \lim_{x \rightarrow a} g(x) = \infty,$$

then

$$\lim_{x \rightarrow a} f(x)g(x) = \lim_{x \rightarrow a} \frac{f(x)}{\frac{1}{g(x)}}$$

Limit of the form $0 \times \infty$

We can transform these limits in either of the above two forms, that is, either $\frac{0}{0}$ or $\frac{\infty}{\infty}$. If

$$\lim_{x \rightarrow a} f(x) = 0 \text{ and } \lim_{x \rightarrow a} g(x) = \infty,$$

then

$$\lim_{x \rightarrow a} f(x)g(x) = \lim_{x \rightarrow a} \frac{f(x)}{\frac{1}{g(x)}}$$

$$\underline{\underline{\text{Or}}} \lim_{x \rightarrow a} \frac{g(x)}{\frac{1}{f(x)}}.$$

Limit of the form $0 \times \infty$

We can transform these limits in either of the above two forms, that is, either $\frac{0}{0}$ or $\frac{\infty}{\infty}$. If

$$\lim_{x \rightarrow a} f(x) = 0 \text{ and } \lim_{x \rightarrow a} g(x) = \infty,$$

then

$$\lim_{x \rightarrow a} f(x)g(x) = \lim_{x \rightarrow a} \frac{f(x)}{\frac{1}{g(x)}}$$

$$\underline{\underline{\text{Or}}} \lim_{x \rightarrow a} \frac{g(x)}{\frac{1}{f(x)}}.$$

$$6. \lim_{x \rightarrow 0} 2x \tan \left(\frac{\pi}{2} - x \right)$$

Limit of the form $\infty - \infty$

Limit of the form $\infty - \infty$

Consider

$$\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + x} \right).$$

We can rationalize this,

Limit of the form $\infty - \infty$

Consider

$$\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + x} \right).$$

We can rationalize this,

$$\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + x} \right)$$

Limit of the form $\infty - \infty$

Consider

$$\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + x} \right).$$

We can rationalize this,

$$\begin{aligned} & \lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + x} \right) \\ &= \lim_{x \rightarrow \infty} \left[\left(x - \sqrt{x^2 + x} \right) \times \frac{\left(x + \sqrt{x^2 + x} \right)}{\left(x + \sqrt{x^2 + x} \right)} \right] \end{aligned}$$

Limit of the form $\infty - \infty$

Consider

$$\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + x} \right).$$

We can rationalize this,

$$\begin{aligned} & \lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + x} \right) \\ &= \lim_{x \rightarrow \infty} \left[\left(x - \sqrt{x^2 + x} \right) \times \frac{\left(x + \sqrt{x^2 + x} \right)}{\left(x + \sqrt{x^2 + x} \right)} \right] \\ &= \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 + x)}{\left(x + \sqrt{x^2 + x} \right)} \end{aligned}$$

Limit of the form $\infty - \infty$

Consider

$$\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + x} \right).$$

We can rationalize this,

$$\begin{aligned} & \lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + x} \right) \\ &= \lim_{x \rightarrow \infty} \left[\left(x - \sqrt{x^2 + x} \right) \times \frac{\left(x + \sqrt{x^2 + x} \right)}{\left(x + \sqrt{x^2 + x} \right)} \right] \\ &= \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 + x)}{\left(x + \sqrt{x^2 + x} \right)} = \lim_{x \rightarrow \infty} \frac{-x}{x + \sqrt{x^2 + x}} \end{aligned}$$

Limit of the form $\infty - \infty$

Consider

$$\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + x} \right).$$

We can rationalize this,

$$\begin{aligned} & \lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + x} \right) \\ &= \lim_{x \rightarrow \infty} \left[\left(x - \sqrt{x^2 + x} \right) \times \frac{\left(x + \sqrt{x^2 + x} \right)}{\left(x + \sqrt{x^2 + x} \right)} \right] \\ &= \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 + x)}{\left(x + \sqrt{x^2 + x} \right)} = \lim_{x \rightarrow \infty} \frac{-x}{x + \sqrt{x^2 + x}} \\ &= \lim_{x \rightarrow \infty} \frac{-x/x}{x/x + (\sqrt{x^2 + x})/x} \end{aligned}$$

Limit of the form $\infty - \infty$

Consider

$$\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + x} \right).$$

We can rationalize this,

$$\begin{aligned} & \lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + x} \right) \\ &= \lim_{x \rightarrow \infty} \left[\left(x - \sqrt{x^2 + x} \right) \times \frac{\left(x + \sqrt{x^2 + x} \right)}{\left(x + \sqrt{x^2 + x} \right)} \right] \\ &= \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 + x)}{\left(x + \sqrt{x^2 + x} \right)} = \lim_{x \rightarrow \infty} \frac{-x}{x + \sqrt{x^2 + x}} \\ &= \lim_{x \rightarrow \infty} \frac{-x/x}{x/x + (\sqrt{x^2 + x})/x} = -\frac{1}{1 + 1} = -\frac{1}{2} \end{aligned}$$

Limit of the form 1^∞

If

$$\lim_{x \rightarrow a} f(x) = 1 \text{ and } \lim_{x \rightarrow a} g(x) = \infty,$$

then

Limit of the form 1^∞

If

$$\lim_{x \rightarrow a} f(x) = 1 \text{ and } \lim_{x \rightarrow a} g(x) = \infty,$$

then

$$\lim_{x \rightarrow a} f(x)^{g(x)} = \exp \left(\lim_{x \rightarrow a} (g(x) \ln f(x)) \right)$$

Limit of the form 1^∞

If

$$\lim_{x \rightarrow a} f(x) = 1 \text{ and } \lim_{x \rightarrow a} g(x) = \infty,$$

then

$$\begin{aligned} \lim_{x \rightarrow a} f(x)^{g(x)} &= \exp \left(\lim_{x \rightarrow a} (g(x) \ln f(x)) \right) \\ &= \exp \left(\lim_{x \rightarrow a} \frac{\ln f(x)}{\frac{1}{g(x)}} \right) \end{aligned}$$

Limit of the form 1^∞

If

$$\lim_{x \rightarrow a} f(x) = 1 \text{ and } \lim_{x \rightarrow a} g(x) = \infty,$$

then

$$\begin{aligned} \lim_{x \rightarrow a} f(x)^{g(x)} &= \exp \left(\lim_{x \rightarrow a} (g(x) \ln f(x)) \right) \\ &= \exp \left(\lim_{x \rightarrow a} \frac{\ln f(x)}{\frac{1}{g(x)}} \right) \end{aligned}$$

We can also write this in terms of $\frac{\infty}{\infty}$.

Limit of the form 1^∞

If

$$\lim_{x \rightarrow a} f(x) = 1 \text{ and } \lim_{x \rightarrow a} g(x) = \infty,$$

then

$$\begin{aligned} \lim_{x \rightarrow a} f(x)^{g(x)} &= \exp \left(\lim_{x \rightarrow a} (g(x) \ln f(x)) \right) \\ &= \exp \left(\lim_{x \rightarrow a} \frac{\ln f(x)}{\frac{1}{g(x)}} \right) \end{aligned}$$

We can also write this in terms of $\frac{\infty}{\infty}$.

$$\lim_{x \rightarrow a} f(x)^{g(x)} = \exp \left(\lim_{x \rightarrow a} (g(x) \ln f(x)) \right)$$

Limit of the form 1^∞

If

$$\lim_{x \rightarrow a} f(x) = 1 \text{ and } \lim_{x \rightarrow a} g(x) = \infty,$$

then

$$\begin{aligned} \lim_{x \rightarrow a} f(x)^{g(x)} &= \exp \left(\lim_{x \rightarrow a} (g(x) \ln f(x)) \right) \\ &= \exp \left(\lim_{x \rightarrow a} \frac{\ln f(x)}{\frac{1}{g(x)}} \right) \end{aligned}$$

We can also write this in terms of $\frac{\infty}{\infty}$.

$$\begin{aligned} \lim_{x \rightarrow a} f(x)^{g(x)} &= \exp \left(\lim_{x \rightarrow a} (g(x) \ln f(x)) \right) \\ &= \exp \left(\lim_{x \rightarrow a} \frac{g(x)}{\frac{1}{\ln f(x)}} \right) \end{aligned}$$