Day 6 : 21st January, 2026

Definition 6.1: (*Subcategory*)

Let $\mathscr{C}$ be a category. A *subcategory* $\mathscr{S}$ of $\mathscr{C}$ is a category such that

i) $\mathrm{Ob}(\mathscr{S}) \subseteq \mathrm{Ob}(\mathscr{C})$;

ii) for any $A, B \in \mathrm{Ob}(\mathscr{S})$, $\mathrm{Hom}_{\mathscr{S}}(A, B) \subseteq \mathrm{Hom}_{\mathscr{C}}(A, B)$;

iii) composition in $\mathscr{S}$ is the same as composition in $\mathscr{C}$.

The third condition says that the function $\mathrm{Hom}_{\mathscr{S}}(A, B) \times \mathrm{Hom}_{\mathscr{S}}(B, C) \to \mathrm{Hom}_{\mathscr{S}}(A, C)$ is the restriction of the corresponding composition with subscripts $\mathscr{C}$.

Example 6.2: (*Examples of subcategory*)

i) Let $\mathscr{C} = \mathbf{Top}$. Then the following are subcategories of $\mathscr{C}$.
   a) The objects are subspaces of $(\mathbb{R}^n, \mathscr{T}_{\mathrm{Euc}})$ and morphisms are continuous maps between spaces.
   b) The objects are Hausdorff topological spaces.
   c) The objects are compact topological spaces.

ii) Let $\mathscr{C} = \mathbf{Groups}$ be the category of groups. Then $\mathbf{Ab}$, the category of abelian groups is a subcategory. Category of rings $\mathbf{Rings}$ is also a subcategory.

iii) Consider the category $\mathscr{C} = \mathbf{Top^2}$ with

$$\begin{aligned} \mathrm{Ob}(\mathscr{C}) &= \text{ordered pairs } (X, A) \text{ where } X \text{ is a topological space} \\ &\quad \text{and } A \text{ is a subspace of } X. \\ \mathrm{Hom}((X, A), (Y, B)) &= \text{set of continuous functions } f : X \to Y \text{ with } f(A) \subseteq B. \end{aligned}$$

Then $\mathbf{Top}_*$ is a subcategory of $\mathbf{Top^2}$.

Exercise 6.3:

i) Let $\mathscr{C}$ be a category and let $A \in \mathrm{Ob}(\mathscr{C})$. Prove that $\mathrm{Hom}(A, A)$ has a unique identity 1_A.

ii) If $\mathscr{S}$ is a subcategory of $\mathscr{C}$, and if $A \in \mathrm{Ob}(\mathscr{S})$, then the identity of A in $\mathrm{Hom}_{\mathscr{S}}(A, A)$ is the identity $1_A \in \mathrm{Hom}_{\mathscr{C}}(A, A)$.

iii) Show that one may regard $\mathbf{Top}$ as a subcategory of $\mathbf{Top^2}$ if one identifies a space X with the pair $(X, \emptyset)$.

Definition 6.4: (*Commutative Diagram*)

A *diagram* in a category $\mathscr{C}$ is a directed graph whose vertices are labeled by objects of $\mathscr{C}$ and whose directed edges are labeled by morphisms in $\mathscr{C}$. A *commutative diagram* in $\mathscr{C}$ is a diagram in which for each pair of vertices, every two paths (composites) between them are equal as morphisms.

Example 6.5: (*A. commutative diagram*)

THe following is a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f'"'] & B \arrow[d, "g"] \\ C \arrow[r, "g'"] & D \end{tikzcd}$$

In addition, if $g \circ f = g' \circ f'$, then the diagram is commutative.

Definition 6.6: (*Congruence*)

A *congruence* on a category $\mathscr{C}$ is an equivalence relation $\sim$ on the class $\bigcup_{(A,B)} \mathrm{Hom}(A, B)$ of all morphisms in $\mathscr{C}$ such that

i) $f \in \mathrm{Hom}(A, B)$ and $f \sim f'$ implies $f' \in \mathrm{Hom}(A, B)$;
ii) $f \sim f'$, $g \sim g'$ and the composition $g \circ f$ exists imply that $g \circ f \sim g' \circ f'$.

Theorem 6.7: (*Quotient Category*)

Let $\mathscr{C}$ be a category with congruence $\sim$, and let $[f]$ denote the equivalence class of a morphism f. Define $\mathscr{C}'$ by

$$\begin{aligned} \mathrm{Ob}(\mathscr{C}') &= \mathrm{Ob}(\mathscr{C}); \\ \mathrm{Hom}_{\mathscr{C}'}(A, B) &= \{[f] : f \in \mathrm{Hom}_{\mathscr{C}}(A, B)\}; \\ [g] \circ [f] &= [g \circ f]. \end{aligned}$$

Then $\mathscr{C}'$ is a category. It is called a *quotient category*.

One usually denotes $\mathrm{Hom}_{\mathscr{C}'}(A, B)$ by $[A, B]$.

Exercise 6.8:

Show that $\mathscr{C}'$ is a category.

Exercise 6.9: (*Category of conjugacy classes*)

Consider the category of groups $\mathbf{Groups}$. Let G, H be two groups and $f, g \in \mathrm{Hom}(G, H)$. Define

$$f \sim g \Leftrightarrow \text{there exists } \; a \in H \text{ such that } \; f(x) = ag(x)a^{-1}, \text{for all } \; x \in G.$$

i) Show that $\sim$ is an equivalence relation on each $\mathrm{Hom}(G, H)$.
ii)