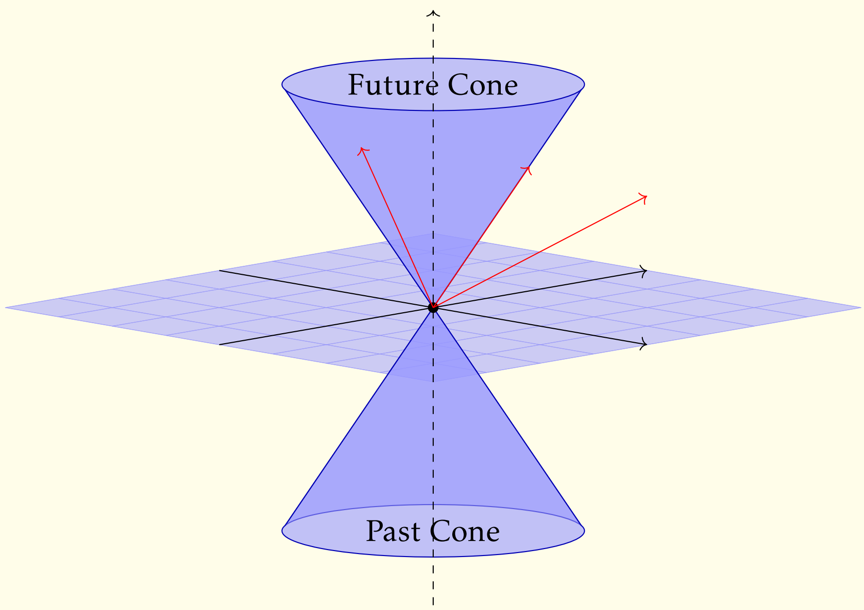
LECTURE NOTES

Lorentzian and Semi-Riemannian Geometry



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Contents

Course information

- **Course name** : Lorentzian and semi-Riemannian Geometry
- **Instructor** : Sachchidanand Prasad
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- **Office** - 2.110 (Institute for Numerical and Applied Mathematics, Lotzestraße, 37073 Göttingen)
- **Classroom** : SZ (Mathematisches Institut)
- **Time** : Friday 12:15 - 13:45
- **Course webpage** : Link to the course website
- **References** :

  [1] *Global Lorentzian Geometry*, by John K. Beem and Paul E. Ehrlich.

  [2] *Semi-Riemannian Geometry with Applications to Relativity*, by Barrett O'Neill.

  [3] *Techniques of Differential Topology in Relativity*, by Roger Penrose.

- **Evaluation** : The evaluation will be done by presentation of a project. A paper will be assigned to a group of students and they will present it in the class. The project will be evaluated based on the presentation and the report (a detailed writeup of the project) submitted.
- **Papers** : The following papers/projects will be assigned to the students for presentation:
  - **Existence of Lorentzian Metric on a Smooth Manifold** [O'N83]
  - **Warped products** [O'N83]
    Study of the wrapped product of Lorentzian manifolds.
  - **Paracompactness of Lorentzian Manifolds**
    A smooth Hausdorff manifold admitting a Lorentzian metric is paracompact.
    A Condition for Paracompactness of a Manifold[Mar73]
    K. B. Marathe

- **Timelike Cut Locus**
  Study of the timelike cut locus in space-time geometry.
  The Space-Time Cut Locus [BE79]
  J. K. Beem and P. E. Ehrlich

- **Null Cut Locus**
  Exploration of the null cut locus in Lorentzian geometry.
  The Space-Time Cut Locus[BE79]
  J. K. Beem and P. E. Ehrlich

- **Morse-Index Theorem for Null Geodesics**
  A theorem relating the Morse index to null geodesics in space-time.
  A Morse Index Theorem for Null Geodesics[Bee75]
  J. K. Beem

- **Comparison Theorems in Lorentzian Geometry**
  Cut points, conjugate points, and their role in comparison theorems.
  Cut Points, Conjugate Points, and Lorentzian Comparison Theorems [BE76]
  J. K. Beem and P. E. Ehrlich

- **Geodesic completeness**
  Geodesic completeness in submanifolds of Minkowski space.
  Geodesic Completeness of Submanifolds in Minkowski Space [BE80]
  J. K. Beem and P. E. Ehrlich

1 Scalar Product Space

Lecture–1

1.1 Introduction

Definition 1.1. *A* Riemannian manifold (M_0, g_0) *is a smooth paracompact manifold with a positive definite inner product*

$$g_0\big|_p : T_pM_0 \times T_pM_0 \to \mathbb{R}$$

on each tangent space T_pM_0.

In addition, if X, Y *are smooth vector fields on* M_0*, then the function*

$$M_0 \to \mathbb{R}, \quad p \mapsto g_0(X,Y)(p) = g_0\big|_p(X_p, Y_p)$$

is smooth.

- The Riemannian structure $g_0 : TM_0 \times TM_0 \to \mathbb{R}$, then defines the Riemannian distance function
$$d_0 : M_0 \times M_0 \to [0,\infty),$$
as follows:

Length

Let $\Omega_{p,q}$ is the set of piecewise smooth curves in M_0 from p to q. Given a curve $\gamma \in \Omega_{p,q}$ with $\gamma : [0,1] \to M_0$, there is a finite partition

$$0 = t_0 < t_1 < t_2 < \ldots < t_k = 1$$

such that $\gamma\big|_{[t_i,t_{i+1}]}$ is smooth for each i. Then the *Riemannian arc length* of γ is give by

$$L_0(\gamma) := \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \sqrt{g_0(\dot\gamma(t), \dot\gamma(t))}\, dt$$

Distance

The *Riemannian distance* will be

$$d_0(p,q) = \inf\left\{L_0(\gamma) : \gamma \in \Omega_{p,q}\right\}.$$

- The Riemannian distance satisfies the following properties:

1. $d_0(p,q) = d_0(q,p), \quad p,q \in M_0$.
2. $d_0(p,q) = 0$ if and only if $p = q$.
3. $d_0(p,q) \leq d_0(p,r) + d_0(r,q), \quad p,q,r \in M_0$.
4. $d_0 : M_0 \times M_0 \to [0,\infty)$ is continuous, and the metric balls

$$\{B(p,\epsilon) : p \in M_0, \epsilon > 0\}$$

forms a basis for the given manifold topology, where $B(p,\epsilon) = \{q \in M_0 : d_0(p,q) < \epsilon\}$.

- Since (M_0, d_0) is a metric space, now we can talk about its completeness and for that we have Hopf-Rinow theorem.

Theorem 1.2. *For any Riemannian manifold (M_0, g_0), the following are equivalent:*

1. *(M_0, d_0) is a complete metric space.*
2. *For any $\mathbf{v} \in TM_0$, the geodesic $\gamma(t)$ in M_0 with $\dot{\gamma}(0) = v$ is defined for all $t \in \mathbb{R}$.*
3. *For some $p \in M_0$, the exponential map $\exp_p$ is defined on the entire tangent space T_pM_0 to M_0 at p.*
4. *Every subset $N \subseteq M_0$ that is d_0 bounded, that is, $\sup\{d_0(p,q) : p,q \in N\}$ has compact closure. Furthermore, if one of (1)-(4) holds, then*
5. *given any $p,q \in N$, there exists a smooth geodesic segment γ joining p to q such that $L_0(\gamma) = d_0(p,q)$.*

Unfortunately, none of these statements is valid for arbitrary Lorentzian manifolds.

- We know that every smooth manifold is a Riemannian manifold. Does there exists a complete Riemannian metric? This was first answered by Nomizu and Ozeki in [NO61].

Recall

Two Riemannian metrics f and g are conformal if there exists a smooth function $f : M_0 \to \mathbb{R}$ such that

$$f = e^{2u} g.$$

Theorem 1.3. *[NO61] For any Riemannian metric g_0 on M_0, there exists a* *complete Riemannian metric which is conformal to g.*

Theorem 1.4. *[NO61] For any Riemannian metric g_0 on M_0, there exists a bounded Riemannian metric which is conformal to g.*

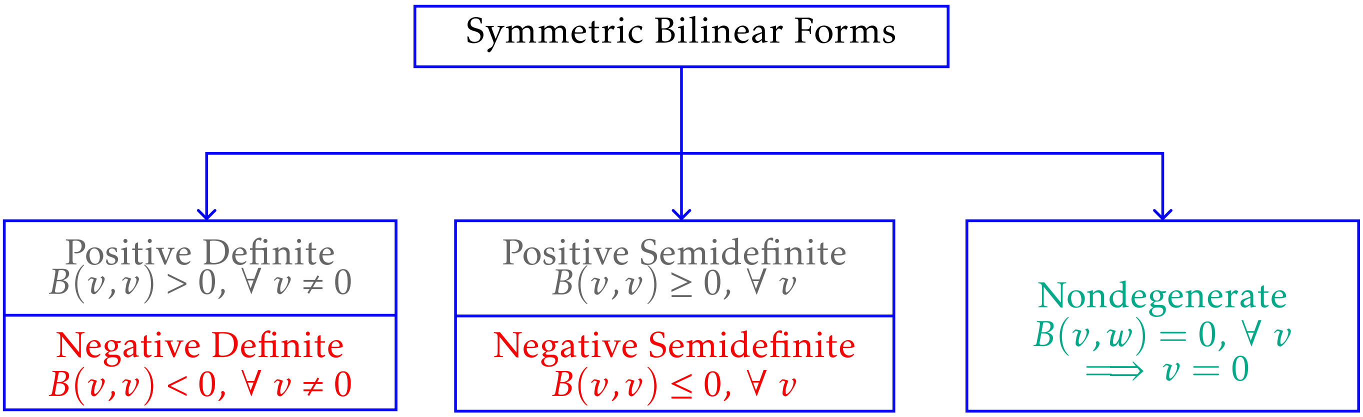
1.2 Bilinear Form

We will start with recalling the symmetric bilinear forms.

- A *bilinear form* on a vector space V is a bilinear map

$$B : V \times V \to \mathbb{R}.$$

It is *symmetric* if $B(v,w) = B(w,v)$ for $v,w \in V$.



- We will call B to be *(semi)definite* if it is either positive or negative (semi)definite.

Note. For a given symmetric bilinear form B on V, we note that

$$B \text{ is definite} \iff B \text{ is semidefinite and nondegenerate.}$$

Proof. If B is definite, then it is clear that B is semidefinite and nondegenerate. For the other part, let B be semidefinite and nondegenerate. To prove that B is definite, let us assume that $B(v,v) = 0$. Then for any $w \in V$,

$$\begin{aligned} B(v+w, v+w) &= 2B(v,w) + B(w,w) \geq 0 \\ B(v-w, v-w) &= -2B(v,w) + B(w,w) \geq 0. \end{aligned}$$

Using these two equations, for any $w \in V$, we get

$$\begin{aligned} 2|B(v,w)| \leq B(w,w) &\implies 2|B(v,w)| \leq \lambda B(w,w) \quad \forall\ \lambda > 0 \\ &\implies B(v,w) = 0 \implies v = 0. \end{aligned}$$

□

For a vector subspace $W \leqslant V$, and symmetric bilinear form on B on V, it is clear that the restriction $B\big|_{W\times W} := B\big|_W$ of B to W is again a symmetric bilinear form. It also preserves the semi(definite) property on the restriction.

Definition 1.5. *Let B be a symmetric bilinear form on a vector space V. The index of B is defined as*

$$\operatorname{ind}(B) := \max\left\{\dim W : W \leqslant V \text{ and } B\big|_{W\times W} \text{ is neg. def. }\right\}$$

Remark. Let $\dim V = n$. It is clear that

- $0 \leq \operatorname{ind}(B) \leq n$
- $\operatorname{ind}(B) = 0 \iff B$ is positive semidefinite.
- $\operatorname{ind}(B) = n \iff B$ is negative definite.

Given a symmetric bilinear form B, we define the *quadratic form associated with B* as a function

$$Q : V \to \mathbb{R}, \quad Q(v) = B(v,v) \quad \forall\, v \in V.$$

By the polarization identity, we can recover the bilinear form from Q as

$$B(v,w) = \frac{1}{2}\left[Q(v+w) - Q(v) - Q(w)\right].$$

Therefore, all the information of B are enclosed in Q.

Let $\mathcal{B} = \{e_1, e_2, \ldots, e_n\}$ be a basis of V, then the matrix of B with respect to $\mathcal{B}$ is given by

$$[B]_{ij} := \left[B(e_i, e_j)\right]_{1\leq i,j\leq n}.$$

It is clearly symmetric and completely determines B. We can characterize the nondegeneracy of B by its matrix with respect to any basis.

Lemma 1.6. *A symmetric bilinear form is nondegenerate if and only if its matrix with respect to one (and hence every) basis is invertible.*

Proof. Let B be a symmetric bilinear form on a vector space V and $v \in V$. Let $B(v,w) = 0$ for $w \in V$. Let $\mathcal{B} = \{e_1, e_2, \ldots, e_n\}$ be a basis of V. Then, for each $1 \leq j \leq n$,

$$B(v, e_j) = 0 \implies B\left(\sum_i v_i e_i, e_j\right) = 0$$

$$\implies \sum_i v_i \cdot B(e_i, e_j) = 0.$$

Thus,

$$\begin{aligned} B \text{ is nondegenerate} \iff v = 0 &\iff (v_1, v_2, \ldots, v_n) = 0 \\ &\iff \ker B = \{0\} \\ &\iff B \text{ is invertible.} \end{aligned}$$

□

Lecture–2

1.3 Scalar product space

Definition 1.7. *A scalar product g on a vector space V is a nondegenerate symmetric bilinear form. We will call (V, g) a scalar product space. An inner product is a positive definite scalar product.*

Example 1.8. (i) The standard dot product on $\mathbb{R}^n$,

$$\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} v_i w_i,$$

is an example of an inner product.

(ii) Changing one sign in the definition of the dot product on $\mathbb{R}^2$ gives the simplest example of an indefinite scalar product. We call this space *two-dimensional Minkowski space*, $\mathbb{R}^2_1$. The scalar product on $\mathbb{R}^2_1$ is defined as

$$g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \quad g(\mathbf{v}, \mathbf{w}) = -v_1 w_1 + v_2 w_2. \tag{1.1}$$

It is clear that g is symmetric and bilinear. To prove that it is nondegenerate, let for any $\mathbf{w} \in \mathbb{R}^2$, $g(\mathbf{v}, \mathbf{w}) = 0$. Set $\mathbf{w} = (1, 0)$ and then $(0, 1)$, we get

$$g(\mathbf{v}, (1,0)) = 0 \text{ and } g(\mathbf{v}, (0,1)) = 0 \implies v_1 = v_2 = 0 \implies \mathbf{v} = 0.$$

To see that g is indefinite note that

$$g((1,0),(1,0)) = -1, \quad g((0,1),(0,1)) = 1 > 0, \quad g((1,1),(1,1)) = 0.$$

The corresponding quadratic form is $Q(\mathbf{v}) = -v_1^2 + v_2^2$.

Let (V, g) be a (finite dimensional) vector space with g being a scalar product. A vector $\mathbf{v} \neq 0$ is called a *null vector* if $Q(\mathbf{v}) = 0$. Null vectors exist iff g is indefinite. In $\mathbb{R}_1^2$, for any $\alpha > 0$, the set $Q = \alpha$ and $Q = -\alpha$ are hyperbolas asymptotic to the the null lines($Q = 0$) (Figure 1.1).

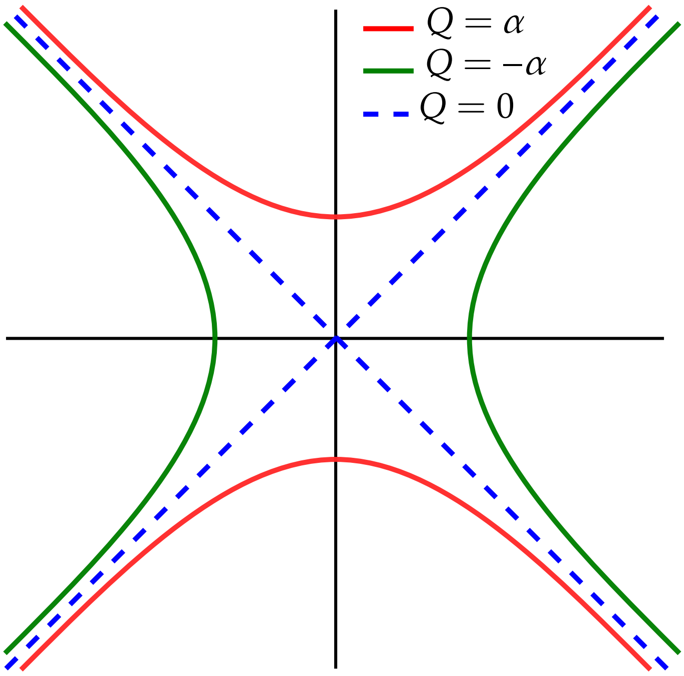


Figure 1.1: Q in 2-dimensional Minkowski space

Two vector $\mathbf{v}, \mathbf{w} \in V$ are *orthogonal*, written $\mathbf{v} \perp \mathbf{w}$, if $g(\mathbf{v}, \mathbf{w}) = 0$. Analogously, we call subspaces U and W of V are orthogonal, if $g(\mathbf{u}, \mathbf{w}) = 0$ for any $\mathbf{u} \in U$ and $\mathbf{w} \in W$.

When the scalar product is indefinite, two vectors that are orthogonal need not to be at right angles to one another as the following example illustrates.

Example 1.9. Let $\mathbf{w} = (1,1) = \mathbf{w}'$, $\mathbf{u} = (1,0), \mathbf{u}' = (0,1)$ and $\mathbf{v} = (1, v), \mathbf{v}' = (v, 1), v > 0$. Then $\mathbf{w} \perp \mathbf{w}', \mathbf{u} \perp \mathbf{u}'$ and $\mathbf{v} \perp \mathbf{v}'$ (see Figure 1.2).

In the above example the null vectors $\mathbf{w}, \mathbf{w}'$ are orthogonal which illustrates the fact that a nonzero null vector is orthogonal to each itself. If W is a subspace of V, let

$$W^{\perp} := \{\mathbf{v} \in V : v \perp w, \ \forall \ w \in W\}. \tag{1.2}$$

It is clear that $W^{\perp}$ is a subspace of V.

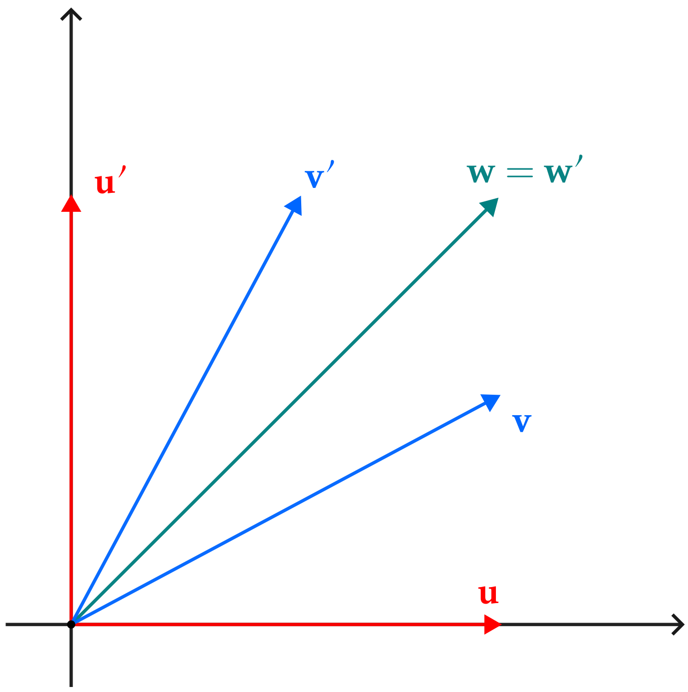


Figure 1.2: Orthogonal vectors in $\mathbb{R}^2_1$

We cannot call $W^\perp$ the orthogonal complement of W since, in general, $W + W^\perp \neq V$. For example, if $W = \operatorname{span}\{\mathbf{w}\}$ in Example 1.9, then we have $W = W^\perp$.

However, the following properties hold for $W^\perp$.

Exercise 1.10. Let W be a subspace of a scalar product space V, then

(i) $\dim W + \dim W^\perp = \dim V$

(ii) $(W^\perp)^\perp = W$.

Note that a symmetric bilinear form g on V is nondegenerate if and only if $V^\perp = \{0\}$. A subspace W of (V, g) is called *nondegenerate* if $g\big|_W$ is nondegenerate. When V is an inner product space, then any subspace W is again an inner product space and hence nondegenerate. However, when g is definite, then there always exists a degenerate subspace; for example, $W = \operatorname{span}\{\mathbf{w}\}$, where $\mathbf{w}$ is a null vector. Hence a subspace of a scalar product space need not be a scalar product space. We now give a simple characterization of nondegeneracy for subspaces.

Exercise 1.11 (Characterization of nondegenerate subspaces)**.** A subspace W of a scalar product space V is nondegenerate if and only if $V = W \oplus W^\perp$.

Exercise 1.12. A subspace W of a scalar product space V is nondegenerate if and only if $W^{\perp}$ is nondegenerate.

We will now talk about norm of a vector. Since Q can take negative values, we define the *norm* of any vector as

$$\|\mathbf{v}\| = |g(\mathbf{v},\mathbf{v})|^{\frac{1}{2}}. \tag{1.3}$$

A vector $\mathbf{v}$ is called a *unit vector* if its norm is 1, that is, $g(\mathbf{v},\mathbf{v}) = \pm 1$. In $\mathbb{R}^2_1$, the unit circle will be

$$S^1 = \left\{(v_1, v_2) \in \mathbb{R}^2 : -v_1^2 + v_2^1 = \pm 1\right\}$$

A family of pairwise orthogonal unit vectors is called *orthonormal*. Observe that the

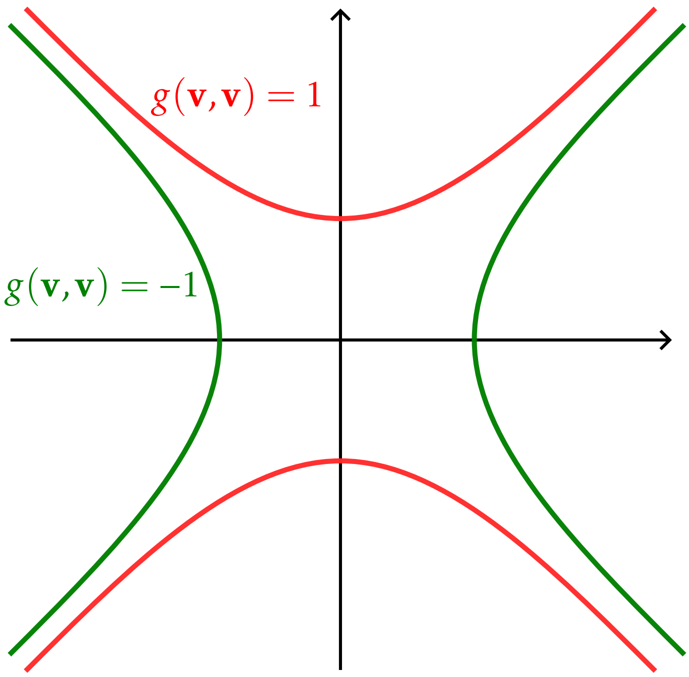


Figure 1.3: Unit circle in $\mathbb{R}^2_1$

set of $n = \dim V$ orthonormal vectors in V is necessarily a basis for V. The following results guarantee that any scalar product space has an orthonormal basis (ONB).

Lemma 1.13. *A scalar product space $V \neq 0$ has an orthonormal basis.*

Proof. We will show this by the method of induction on dimension of V, say n. If $n = 1$, choose $0 \neq \mathbf{v} \in V$ such that $g(\mathbf{v},\mathbf{v}) \neq 0$ (this is possible because g is nondegenerate. To see this, if $g(\mathbf{v},\mathbf{v}) = 0$ for every $\mathbf{v} \in V$, then by using polarization identity for any $\mathbf{w} \in V$, $g(\mathbf{v},\mathbf{w}) = 0$, which implies $\mathbf{v} = 0$, a contradiction). Let $\mathbf{e}_1 = \mathbf{v}/\|\mathbf{v}\|$. Then $\{\mathbf{e}_1\}$ is an ONB for V. Suppose that for k-dimensional space we have an ONB, and we want to show that for $k+1$ dimensional space such an ONB exists. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\}$ be an orthonormal basis for a k-dimensional subspace, say W. This implies (by Lemma 1.6), W is nondegenerate and so is $W^\perp$ (by Exercise 1.12). Let e_{k+1} be a unit vector in $W^\perp$ (same argument as for $\dim = 1$). Then an ONB of V is $\{\mathbf{e}_1, \ldots, \mathbf{e}_{k+1}\}$. □

- An ONB for $\mathbb{R}^2_1$ can be given by

$$\{(1,0),(0,1)\}, \quad \left\{(1,\sqrt{2}),(\sqrt{2},1)\right\}.$$

- The matrix of g relative to any orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for V is diagonal, more precisely,

$$g(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}\epsilon_j, \quad \text{where } \epsilon_j = \pm 1.$$

- We shall order the vectors in an ONB in such a way that in the so called *signature* $(\epsilon_1, \ldots, \epsilon_n)$ the negative signs come first.

Exercise 1.14. The following are easy properties for ONB.

(i) Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be an ONB for V. Then any $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = \sum_{i=1}^{n} \epsilon_i g(\mathbf{v}, \mathbf{e}_i)\mathbf{e}_i.$$

(ii) For any ONB $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for V the number of negative signs in the signature $(\epsilon_1, \ldots, \epsilon_n)$ is the index of V.

(iii) Let W be a nondegenerate subspace of V. Then $\mathrm{ind}(V) = \mathrm{ind}(W) + \mathrm{ind}(W^\perp)$.

Let (V, g) and (W, h) be two scalar product spaces. A linear map $T : V \to W$ is said to *preserve scalar products* if $h(T\mathbf{v}_1, T\mathbf{v}_2) = g(\mathbf{v}_1, \mathbf{v}_2)$. A linear isomorphism $T : V \to W$ that preserves scalar products is called a *linear isometry*.

Exercise 1.15. Scalar product space V and W have the same dimension and index if and only if there exists a linear isometry from V to W.

1.4 Causality

A scalar product with index 0 is called a *Riemannian scalar product* and a vector space with a Riemannian scalar product is called *Riemannian scalar product space*. A scalar product with index 1 is called a *Lorentz scalar product* and a vector space with a Lorentz scalar product is called *Lorentz scalar product space*.

If V is an n-dimensional Riemannian scalar product space, then there is a linear isometry from V to $\mathbb{R}^n$. If V is an n-dimensional Lorentz scalar product space, then there is a linear isometry from V to $\mathbb{R}^n_1$.

Definition 1.16. *Let (V, g) be a Lorentz scalar product space. Then a vector $\mathbf{v} \in V$ is said to be*

(a) timelike *if $g(\mathbf{v}, \mathbf{v}) < 0$,*

(b) spacelike *if $g(\mathbf{v}, \mathbf{v}) > 0$ or $\mathbf{v} = 0$,*

(c) lightlike *or* null *if $g(\mathbf{v}, \mathbf{v}) = 0$ and $\mathbf{v} \neq 0$.*

(d) causal *if $\mathbf{v}$ is timelike or lightlike.*

The classification of a vector $\mathbf{v} \in V$ according to the above is called the causal character *of the vector $\mathbf{v}$.*

This terminology matters because it connects to the idea of causality in physics, which is about how events can affect one another. In special relativity, nothing can travel faster than light, setting a speed limit for information. Picture γ as a path in Minkowski spacetime—like the trail of something moving, such as a particle, a spacecraft, or a beam of light. The speed of this object compared to light depends on the "causal character" of $\dot{\gamma}$, the tangent vector showing its direction and speed in this space. If $\dot{\gamma}$ is timelike, the object moves slower than light; if $\dot{\gamma}$ is lightlike, it moves exactly at light speed; and if $\dot{\gamma}$ is spacelike, it would imply moving faster than light, which isn't possible for physical objects but can describe mathematical paths.

In Minkowski spacetime, a vector $\mathbf{v} = (v_0, \vec{v}) \in \mathbb{R}^{n+1}$, where $\vec{v} \in \mathbb{R}^n$, is measured using the expression $g(\mathbf{v}, \mathbf{v}) = -(v_0)^2 + |\vec{v}|^2$. Here, $|\vec{v}|$ is the usual length of the spatial part $\vec{v}$, like the distance in regular space. We classify $\mathbf{v}$ based on this value: it's *timelike* if $|v_0| > |\vec{v}|$, meaning the time part dominates; *lightlike* if $|v_0| = |\vec{v}| \neq 0$, so they balance perfectly; and *spacelike* if $|v_0| < |\vec{v}|$ or $\mathbf{v} = 0$, where the spatial part is larger or the vector is zero. The timelike vectors split into two groups: those with $v_0 > 0$ (pointing toward the future) and those with $v_0 < 0$ (pointing toward the past). Picking one of these groups decides what we call the "future" and "past"—this choice is known as the *time orientation*. Below, we'll explain these ideas further and define the time orientation more clearly.

Definition 1.17. *Let (V, g) be a scalar product space and $W \subseteq V$ be a subspace. Then W is said to be* spacelike *if $g\big|_W$ is positive definite, that is, if $g\big|_W$ is nondegenerate of index 0. Moreover, W is said to be* lightlike *if $g\big|_W$ is degenerate. Finally, W is said to be* timelike *if $g\big|_W$ is nondegenerate of index 1.*

By using Exercise 1.12, we can conclude that

Exercise 1.18. Let (V, g) be a Lorentzian scalar product space and $W \subseteq V$ be a subspace. Then W is timelike if and only if $W^\perp$ is spacelike.

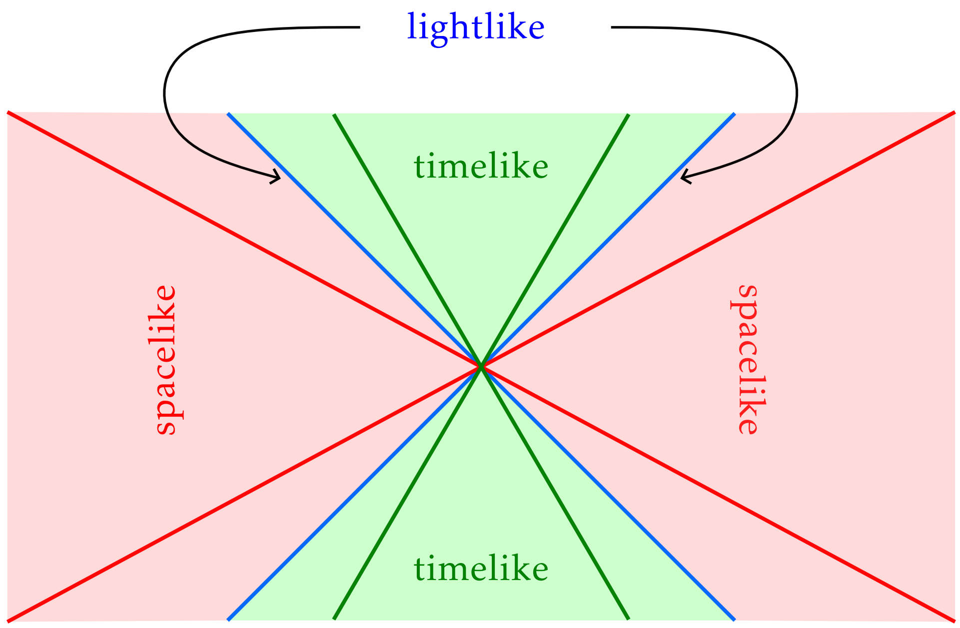


Figure 1.4: Causal Subspaces

Exercise 1.19. Some More problems from [O'N83].

1. Let B be a symmetric bilinear form on a vector space V. The *nullspace* of B is $N = \{\mathbf{v} : B(\mathbf{v}, \mathbf{w}) = 0,\ \forall\ \mathbf{w} \in V\}$. The *nullcone* of B is the set Λ of all null vectors in V. Let $A = \Lambda \cup \{0\}$. Prove

   (a) N is a subspace of V, but A is not unless $A = \{0\}$ or $A = V$.

   (b) B is nondegenerate if and only if $N = 0$; B is definite if and only if $A = \{0\}$.

   (c) B is semidefinite if and only if $N = A$.

2. Let g be a scalar product of index k on an n-dimensional vector space V. Prove that there exists a subspace W of dimension $\min\{k, n-k\}$, and no larger,

on which $g = 0$.

3. Let V have indefinite scalar product g, and let B be a symmetric bilinear form on V with corresponding quadratic form Q. Show that the following conditions are equivalent.

   (a) $B = cg$ for some $c \in \mathbb{R}$,

   (b) $Q = 0$ on null vectors,

   (c) $|Q|$ is bounded on timelike unit vectors,

   (d) $|Q|$ is bounded on spacelike unit vectors.

Lecture–3

1.5 Timelike cones

Let (V, g) be a Lorentzian scalar product space of dimension $n \geq 2$ with a Lorentzian scalar product g.

Proposition 1.20. *The subset of the timelike vectors (resp. causal; lightlike if $n > 2$) has two connected parts.*

Proof. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an orthonormal basis of V, and $\mathbf{v} \in V$ such that $\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i$. Then it is clear that

$$\mathbf{v} \text{ is lightlike} \iff |v_1| = \sqrt{\sum_{i=2}^{n} v_i^2}, \quad v_1 \neq 0$$

$$\mathbf{v} \text{ is timelike} \iff |v_1| > \sqrt{\sum_{i=2}^{n} v_i^2}$$

$$\mathbf{v} \text{ is causal} \iff |v_1| \geq \sqrt{\sum_{i=2}^{n} v_i^2}, \quad v_1 \neq 0.$$

Therefore, in each case there exist two connected parts, the corresponding to $v_1 < 0$ and the corresponding to $v_1 > 0$. $\square$

Definition 1.21. *A* time orientation *of Lorentzian vector space is a choice of one of the two timelike cones (or, equivalently, of one of the causal or lightlike cones). The* *chosen cone will be called* future*, and the other one,* past*.*

From now on, the vectors in the future (resp. past) cone wil be called *future directed* or *future-pointing* (resp. *pat-directed* or *past-pointing*) vectors.

Proposition 1.22. *Two timelike vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *lie in the same timelike cone if and only if* $g(\mathbf{v},\mathbf{w}) < 0$.

Proof. Without loss of generality, let us assume that $\|\mathbf{v}\| = 1$, and $\mathbf{v}$ can be completed to an orthonormal basis $\{\mathbf{v},\mathbf{e}_2,\mathbf{e}_3,\ldots,\mathbf{e}_n\}$. Observe that

$$\mathbf{w} = -g(\mathbf{v},\mathbf{w})\mathbf{v} + \sum_{i=2}^{n} g(\mathbf{e}_i,\mathbf{w})\,\mathbf{e}_i.$$

Then using the proof of Proposition 1.20, $\mathbf{v}$ and $\mathbf{w}$ are in the same cone if and only if $-g(\mathbf{v},\mathbf{w}) > 0$. □

Proposition 1.23. *If* $\mathbf{v},\mathbf{w}$ *are timelike vectors in the same cone, then so is* $a\mathbf{v}+b\mathbf{w}$ *for any* $a,b > 0$. *In particular, each timelike cone is convex.*

Proof. Using Proposition 1.22, we have $g(\mathbf{v},\mathbf{w}) < 0$. Consider,

$$g(a\mathbf{v}+b\mathbf{w},a\mathbf{v}+b\mathbf{w}) = a^2 g(\mathbf{v},\mathbf{v}) + 2ab\,g(\mathbf{v},\mathbf{w}) + b^2 g(\mathbf{w},\mathbf{w}) < 0 \tag{1.4}$$

$$g(\mathbf{v},a\mathbf{v}+b\mathbf{w}) = a g(\mathbf{v},\mathbf{v}) + b g(\mathbf{v},\mathbf{w}) < 0. \tag{1.5}$$

The Equation (1.4) implies the vector $a\mathbf{v}+b\mathbf{w}$ is timelike and Equation (1.5) implies it belongs to the same cone. □

Reverse Inequalities

Theorem 1.24 (Reverse Cauchy-Schwarz Inequality)**.** *If* $\mathbf{v},\mathbf{w} \in V$ *are timelike vectors, then*

(i) $|g(\mathbf{v},\mathbf{w})| \geq \|\mathbf{v}\|_g \cdot \|\mathbf{w}\|_g$. *Moreover, the equality holds if and only if* $\mathbf{v},\mathbf{w}$ *are colinear.*

(ii) If $\mathbf{v}$ *and* $\mathbf{w}$ *lie in the same cone, then there exists a unique* $\phi \geq 0$, *called the* hyperbolic angle between $\mathbf{v}$ and $\mathbf{w}$ *such that*

$$g(\mathbf{v},\mathbf{w}) = -\|\mathbf{v}\|_g \cdot \|\mathbf{w}\|_g \cosh(\phi).$$

Proof. (i) Note that the inequality remains true if we take $a\mathbf{v}$ for $a > 0$. So, without loss of generality, we assume that $g(\mathbf{v},\mathbf{v}) = -1$. Recall from Exercise 1.11, a subspace W of V is nondegenerate if and only if $V = W \oplus W^{\perp}$, so we can write $V = \mathbb{R}\mathbf{v} \oplus \{\mathbf{v}\}^{\perp}$. Let $a \in \mathbb{R}$ and $\mathbf{w}_0$ be such that $\mathbf{w} = a\mathbf{v} + \mathbf{w}_0$ with $\mathbf{w}_0 \perp \mathbf{v}$ (that means $\mathbf{w}_0$ is a spacelike vector, see Exercise 1.12). Then

$$g(\mathbf{v},\mathbf{w}) = g(\mathbf{v}, a\mathbf{v} + \mathbf{w}_0) = ag(\mathbf{v},\mathbf{v}) + g(\mathbf{v},\mathbf{w}_0) = -a.$$

Since $g(\mathbf{w},\mathbf{w}) < 0$ and $g(\mathbf{w}_0,\mathbf{w}_0) \geq 0$, consider,

$$\begin{aligned} g(\mathbf{w},\mathbf{w}) &= a^2 g(\mathbf{v},\mathbf{v}) + g(\mathbf{w}_0,\mathbf{w}_0) = -[g(\mathbf{v},\mathbf{w})]^2 + g(\mathbf{w}_0,\mathbf{w}_0) \\ \implies &-[g(\mathbf{v},\mathbf{w})]^2 = -|g(\mathbf{w},\mathbf{w})| - g(\mathbf{w}_0,\mathbf{w}_0) \\ \implies &[g(\mathbf{v},\mathbf{w})]^2 = |g(\mathbf{w},\mathbf{w})| + g(\mathbf{w}_0,\mathbf{w}_0) \geq |g(\mathbf{w},\mathbf{w})| \\ \implies &|g(\mathbf{v},\mathbf{w})|^2 \geq \|\mathbf{w}\|_g = \|\mathbf{w}\|_g \cdot \|\mathbf{v}\|_g, \end{aligned}$$

since $\|\mathbf{v}\|_g = 1$.

It is clear that the equality holds if and only if $g(\mathbf{w}_0,\mathbf{w}_0) = 0$, that is, $\mathbf{w} = a\mathbf{v}$, that is, the vectors $\mathbf{v}$ and $\mathbf{w}$ are colinear.

(ii) If $\mathbf{v},\mathbf{w}$ lie in the same cone, then $g(\mathbf{v},\mathbf{w}) < 0$ (Proposition 1.22). So the reversed Cauchy-Schwarz inequality gives,

$$\frac{-g(\mathbf{v},\mathbf{w})}{\|\mathbf{v}\|_g \cdot \|\mathbf{w}\|_g} \geq 1.$$

Since cosh is a bijection from $[0,\infty)$ to $[1,\infty)$, so we get unique $\phi \geq 0$ such that

$$g(\mathbf{v},\mathbf{w}) = -\|\mathbf{v}\|_g \cdot \|\mathbf{w}\|_g \cosh(\phi).$$

□

Theorem 1.25 (Reversed Triangle Inequality)**.** *Let $\mathbf{v},\mathbf{w} \in V$ are timelike vectors in the same cone, then*

$$\|\mathbf{v}+\mathbf{w}\|_g \geq \|\mathbf{v}\|_g + \|\mathbf{w}\|_g$$

and the equality holds if and only if $\mathbf{v},\mathbf{w}$ are colinear.

Proof. Using Theorem 1.24, we observe that

$$\begin{aligned} \|\mathbf{v}+\mathbf{w}\|_g^2 &= -g(\mathbf{v}+\mathbf{w},\mathbf{v}+\mathbf{w}) \quad \text{(Proposition 1.23)} \\ &= -g(\mathbf{v},\mathbf{v}) - 2g(\mathbf{v},\mathbf{w}) - g(\mathbf{w},\mathbf{w}) \\ &= \|\mathbf{v}\|_g^2 + \|\mathbf{w}\|_g^2 + 2|g(\mathbf{v},\mathbf{w})| \quad \text{(Proposition 1.22)} \\ &\geq \|\mathbf{v}\|_g^2 + \|\mathbf{w}\|_g^2 + \|\mathbf{v}\|_g \cdot \|\mathbf{w}\|_g \quad \text{(Theorem 1.24)} \\ &= (\|\mathbf{v}_g + \|\mathbf{w}\|_g\|)^2. \end{aligned}$$

Moreover, the equality holds if and only if $|g(\mathbf{v},\mathbf{w})| = \|\mathbf{v}\|_g \cdot \|\mathbf{w}\|_g$, which holds, by Theorem 1.24, if and only if $\mathbf{v}$ and $\mathbf{w}$ are colinear. □

We now discuss some analogous properties of lightlike and causal cones.

Proposition 1.26. *If $\mathbf{u},\mathbf{w} \in V$ are lightlike vectors, then*

$$\{\mathbf{u},\mathbf{w}\} \text{ are linearly dependent } \iff g(\mathbf{u},\mathbf{w}) = 0.$$

Proof. Since for any lightlike vector $\mathbf{v}$, $g(\mathbf{v},\mathbf{v}) = 0$, so it is clear that if the vectors are linearly dependent then $g(\mathbf{u},\mathbf{w}) = 0$. For the other side, let us assume that $g(\mathbf{u},\mathbf{v}) = 0$. Take a unit timelike vector $\mathbf{v}$ and decompose V as $V = \mathbb{R}\mathbf{v} \oplus \{\mathbf{v}\}^{\perp}$. Write

$$\mathbf{u} = a\mathbf{v} + \mathbf{u}_0 \quad \text{and} \quad \mathbf{w} = b\mathbf{v} + \mathbf{w}_0,$$

for unique $a, b \in \mathbb{R}$ and $\mathbf{w}_0, \mathbf{u}_0 \in \{\mathbf{v}\}^{\perp}$. Then,

$$\begin{aligned} g(\mathbf{u},\mathbf{w}) = 0 &\implies g(a\mathbf{v} + \mathbf{u}_0, b\mathbf{v} + \mathbf{w}_0) = 0 \\ &\implies abg(\mathbf{v},\mathbf{v}) + g(\mathbf{u}_0,\mathbf{u}_0) = 0 \\ &\implies g(\mathbf{u}_0,\mathbf{w}_0) = ab. \end{aligned}$$

Similarly,

$$g(\mathbf{u}_0,\mathbf{u}_0) = a^2 \quad \text{and} \quad g(\mathbf{w}_0,\mathbf{w}_0) = b^2.$$

Since, $g(a\mathbf{w}_0 - b\mathbf{u}_0, \mathbf{v}) = 0$, so $a\mathbf{w}_0 - b\mathbf{u}_0 \in \{v\}^{\perp}$ and hence spacelike. Now,

$$\begin{aligned} g(a\mathbf{w}_0 - b\mathbf{u}_0, a\mathbf{w}_0 - b\mathbf{u}_0) &= a^2 g(\mathbf{w}_0,\mathbf{w}_0) - 2abg(\mathbf{w}_0\mathbf{u}_0) + b^2 g(\mathbf{u}_0,\mathbf{u}_0) \\ &= a^2b^2 - 2a^2b^2b^2a^2 = 0. \end{aligned}$$

This implies, $a\mathbf{w}_0 - b\mathbf{u}_0 = 0$. Note that $a \neq 0$ and $b \neq 0$ since $\mathbf{u}$ and $\mathbf{v}$ are lightlike. Thus,

$$a\mathbf{w} - b\mathbf{u} = ab\mathbf{v} + a\mathbf{w}_0 - ab\mathbf{v} - b\mathbf{u}_0 = a\mathbf{w}_0 - b\mathbf{u}_0 = 0,$$

and hence, $\{\mathbf{u},\mathbf{w}\}$ are linearly independent. □

Exercise 1.27. Show the following.

1. If $\mathbf{u},\mathbf{w} \in V$ are two linearly independent vectors, then

$$\mathbf{u},\mathbf{w} \text{ are in the same causal cone } \iff g(\mathbf{u},\mathbf{w}) < 0.$$

2. The causal cones are convex.

Proposition 1.28. *If $W < V$, with $\dim W \geq 2$, the following conditions are equivalent.*

(i) W is timelike,

(ii) W contains two linearly independent lightlike vectors,

(iii) W contains one timelike vector.

2 Semi-Riemannian Metrics

Lecture–4

Definition 2.1. *A semi-Riemannian metric tensor (or metric, for short) on a smooth manifold M is a symmetric nondegenerate $(0,2)$-tensor field g on M of constant index.*

In other words, g smoothly assigns to each point $p \in M$ a symmetric nondegenerate bilinear form $g(p) \equiv g_p : T_pM \times T_pM \to \mathbb{R}$ such that the index r_p of g_p is the same for all p. We call this common value r_p the *index* r of the metric g. We clearly have $0 \le r \le n = \dim M$. In case $r = 0$, all g_p are inner products on T_pM and we call g a Riemannian metric. In case $r = 1$, and $n \ge 2$, we call g *Lorentzian metric*.

Remark. The requirement that the index r is chosen constant must be taken into account only when M is not connected. Indeed, one can show that if g is degenerate at every point, then the index is locally constant (see Lemma 2.2 below).

Lemma 2.2. *Let M be a semi-Riemannian manifold with g be a symmetric $(0,2)$-tensor field on M. Then the set of all points where g is nondegenerate with index r, $0 \le r \le n$ is open.*

Proof. Let
$$U = \{p \in M : g_p \text{ is nondegenerate of index } r\}.$$
If $U = \emptyset$, then there is nothing to prove. Let us assume that $U \neq \emptyset$. Let $p \in U$ and g_p is nondegenerate of index r. Choose an orthonormal basis for (T_pM, g_p), say $\mathcal{B}_p = \{\mathbf{e}_1, \ldots, \mathbf{e}_r, \mathbf{e}_{r+1}, \ldots, \mathbf{e}_n\}$ such that

$$g_p(\mathbf{e}_i, \mathbf{e}_i) = \begin{cases} -1, & \text{if } 1 \le i \le r; \\ 1, & \text{if } r+1 \le i \le n. \end{cases}$$

Extend $\mathcal{B}_p$ to a local frame $\{X_1, \ldots, X_r, X_{r+1}, \ldots, X_n\}$ on a neighborhood U of p (that is, $X_i(p) = \mathbf{e}_i$) and let g_{ij} be functions on U defined by $g_{ij} = g(X_i, X_j)$ for $1 \le i, j \le n$. Since g_p is nondegenerate at p, so $\det(g_{ij}(p)) \neq 0$ at p and hence in a neighborhood of p. By shrinking U to this neighborhood, if necessary, we may assume that g_q is nondegenerate of some index r_q at each point $q \in U$. Since g_{ij} are continuous functions on U, we let $\epsilon_+ = \frac{1}{n-r+2}$. So, there exists a neighborhood U_+ of p such that
$$|g_{ij}(q) - g_{ij}(p)| < \epsilon_+ \quad \forall\ r+1 \le i \le n, q \in U_+$$
and
$$|g_{ij}(q)| < \epsilon_+ \quad i \neq j,\ r+1 \le i \le n,\ q \in U_+.$$

Let $W_{+_q} = \operatorname{span}\{X_{r+1}(q),\ldots,X_n(q)\}$. We claim that for any $\mathbf{x}(\neq 0) \in W_{+_q}$ $g(\mathbf{x},\mathbf{x}) > 0$. Let us write $\mathbf{x} = \sum_{i=r+1}^{n} \lambda_i X_i(q)$. Then

$$\begin{aligned}
g(\mathbf{x},\mathbf{x}) &= g\left(\sum_{i=r+1}^{n} \lambda_i X_i(q), \sum_{i=r+1}^{n} \lambda_i X_i(q)\right) \\
&= \sum_{i,j=r+1}^{n} \lambda_i \lambda_j g(X_i(q), X_j(q)) \\
&= \sum_{i=r+1}^{n} \lambda_i^2 g_{ii}(q) + 2 \sum_{r+1 \le i<j \le n} \lambda_i \lambda_j g_{ij}(q) \\
&> \sum_{i=r+1}^{n} \lambda_i^2 (1-\epsilon_+) - 2 \sum_{r+1 \le i<j \le n} |\lambda_i||\lambda_j|\epsilon_+ \\
&= \sum_{i=r+1}^{n} \lambda_i^2 - \epsilon_+ \sum_{i=r+1}^{n} \lambda_i^2 - 2\epsilon_+ \sum_{i,j} |\lambda_i||\lambda_j| \\
&\ge \sum_{i} \lambda_i^2 - \epsilon_+ \sum_{i} \lambda_i^2 - \epsilon_+ \sum_{i,j} (\lambda_i^2 + \lambda_j^2) \\
&= \sum_{i=r+1}^{n} \lambda_i^2 - \epsilon_+ \left(2\lambda_{r+1}^2 + 3\lambda_{r+2}^2 + \cdots + (n-r+1)\lambda_n^2\right) > 0.
\end{aligned}$$

Similarly, one can show that there exists a neighborhood U_- of p such that $g(\mathbf{x},\mathbf{x}) < 0$ for $\mathbf{x} \in W_{-_q} = \operatorname{span}\{X_1(q),\ldots,X_r(q)\}$. Let $U' = U_+ \cap U_-$. Then on U', g_q is positive definite on W_{+_q} and negative definite on W_{-_q}. Thus, $n - r_q \ge n - r$ and $r_q \ge r$. This implies $r_q = r$, for $q \in U'$. □

Definition 2.3. *A semi-Riemannian manifold is a pair (M,g), where g is a metric tensor on M. In case g is Riemannian or Lorentzian we call (M,g) a Riemannian manifold or Lorentzian manifold, respectively.*

- If (U,ϕ) is a chart of M with coordinates $\phi = \left(x^1, x^2, \ldots, x^n\right)$ and natural basis vector fields $\partial_i \equiv \frac{\partial}{\partial x^i}$, we write,

$$g_{ij} = \left\langle \partial_i, \partial_j \right\rangle, \quad 1 \le i,j \le n \tag{2.1}$$

  for the local components of g on V.

- Since g is nondegenerate, at each point of U, the matrix $\left(g_{ij}(p)\right)$ is invertible (by Lemma 1.6) and its inverse matrix is denoted by $\left(g^{ij}(p)\right)$. By the inversion formula, it is clear that $(g^{ij}(p))$ is smooth on U and by symmetry of g we have $g^{ij} = g^{ji}$ for all i and j.

☞ Denoting the dual basis covector fields of ∂_i by $\mathrm{d}x_i$ we have

$$g\big|_U = \sum_{i,j} g_{ij}\mathrm{d}x_i \otimes \mathrm{d}x_j.$$

Example 2.4. (i) Let $M = \mathbb{R}^n$. For each $p \in \mathbb{R}^n$, there is a canonical linear isomorphism from $\mathbb{R}^n$ to T_pM that, in terms of natural coordinates, sends $\mathbf{v}$ to $\mathbf{v}_p = \sum_i v_i \partial_i$. This induces a metric tensor on M which we denote by

$$\left\langle \mathbf{v}_p, \mathbf{w}_p \right\rangle = \mathbf{v} \cdot \mathbf{w} = \sum_i v_i w_i.$$

Henceforth we will always consider $\mathbb{R}^n$ equipped with this Riemannian metric.

(ii) For any integer $0 \le r \le n$,

$$\left\langle \mathbf{v}_p, \mathbf{w}_p \right\rangle = -\sum_{i=1}^{r} v_i w_i + \sum_{j=r+1}^{n} v_j w_j$$

defines a metric on $\mathbb{R}^n$ of index r. We will denote $\mathbb{R}^n$ with this metric tensor by $\mathbb{R}^n_r$.

(a) If $r = 0$, then it is the Euclidean space.

(b) For $n \ge 2$, $\mathbb{R}^n_1$ is called n-dimensional *Minkowski space*.

(c) If $n = 4$, it is the simplest example of a spacetime in the sense of Einstein's general relativity.

Setting $\epsilon_i = \begin{cases} -1, & \text{if } 1 \le i \le r; \\ 1, & \text{if } r+1 \le i \le n. \end{cases}$, the metric of $\mathbb{R}^n_r$ takes the form

$$g = \sum_i \epsilon_i \mathrm{d}x_i \otimes \mathrm{d}x_i.$$

All the properties that we have studied on scalar products can be applied to every tangent space (T_pM, g_p).

- A Lorentzian manifold (M, g) is said to be *time oroiented* if M admits a continuous, nowhere vanishing vector field X.
- This vector field is used to separate the nonspacelike vectors at each point into two classes called *future directed* and *past directed* vector fields.

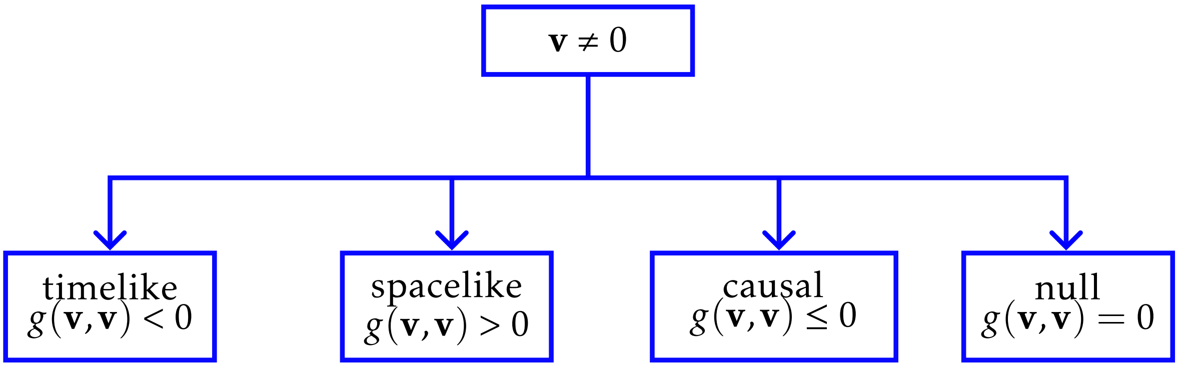


Figure 2.1: lorentzianVectors

- A space-time is a Lorentzian manifold (M, g) together with a choice of time orientation.

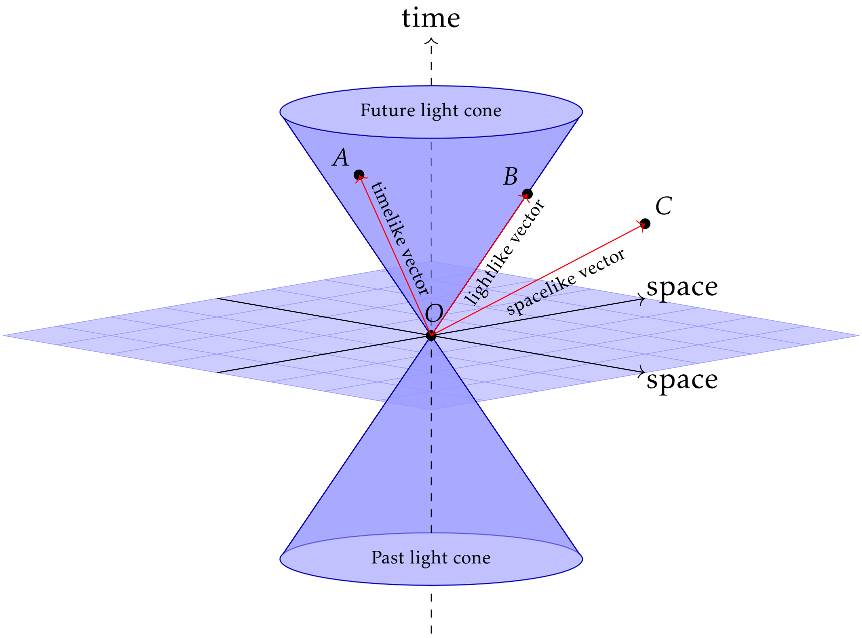


Figure 2.2: Light Cone in 2d Space plus a Time Dimension

Given a way to get new smooth manifolds from old, there is often a corresponding wat to derive a metric tensor on the new manifold from metric tensor on the old.

Let N be a submanifold of a Riemannian manifold (M, g) with embedding $j : N \hookrightarrow M$. Then the pull back j^*g of the metric g to the submanifold N is given by

$$(j^*g)_p(\mathbf{v}, \mathbf{w}) = g_{j(p)}(\mathrm{d}j_p(\mathbf{v}), \mathrm{d}j_p(\mathbf{w})) = g_p(\mathbf{v}, \mathbf{w}),$$

where in the final equality we have identified $\mathrm{d}j_p(T_pN)$ with T_pN.Hence j^*g_p is just the restriction of g_p to the subspace T_pN of T_pM. Since g is a Riemannian metric, this restriction is positive definite ans so j^*g turns N into a Riemannian manifold.

However, if M is only semi-Riemannian manifold then the $(0,2)$-tensor field j^*g on N need not be a metric. Indeed j^*g is a metric and hence (N, j^*g) a semi-Riemannian manifold if and only if every T_pN is nondegenerate in T_pM and the index of T_pN is the same for all $p \in N$. Of course, this index can be different from the index of g. These considerations lead to the following definition.

Definition 2.5. *A submanifold N of a semi-Riemannian manifold (M,g) is called a semi-Riemannian submanifold if j^*g is a metric on N.*

We now consider the product manifolds.

Lemma 2.6. *Let M and N be semi-Riemannian manifolds with metric g_m and g_N. If π and σ are the projections of $M \times N$ onto M and N, respectively, let*

$$g = \pi^*(g_M) + \sigma^*(g_N).$$

Then g is a metric on $M \times N$ making it a semi-Riemannian product manifold.

Exercise 2.7. Proof Lemma 2.6.

Isometries

Definition 2.8. *Let (M, g_M) and (N, g_N) be semi-Riemannian manifolds and $\phi : M \to N$ be a diffeomorphism. Then we call ϕ an isometry if ϕ preserves the metric, that is, $\phi^*(g_N) = g_M$. We call M and N are isometric.*

More explicitly,

$$\langle d\phi(\mathbf{v}), d\phi(\mathbf{w})\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$$

for all $\mathbf{v}, \mathbf{w} \in T_pM, p \in M$. Since ϕ is a diffeomorphism, every differential map $d\phi_p : T_pM \to T_{\phi(p)}N$ is a linear isometry.

Remark. (i) It is easy to see that the identity map of semi-Riemannian manifold is an isometry. A composition of isometries is an isometry. The inverse map of an isometry is an isometry.

(ii) If V is an n-dimensional scalar product space with $\mathrm{ind}(V) = r$, then V as a semi-Riemannian manifold is isometric to $\mathbb{R}^n_r$.

Lecture–5

2.1 Levi-Civita Connection

Let us recall the directional derivatives in Euclidean space. Let $x_1, x_2, \ldots, x_n$ be natural coordinates of $\mathbb{R}^n_r$. Suppose $X = \sum_i a_i \dfrac{\partial}{\partial x_i}$ and X_p is a tangent vector at a point $p = (p_1, p_2, \ldots, p_n)$ in $\mathbb{R}^n_r$ and f is a C^∞ function in a neighborhood of p in $\mathbb{R}^n_r$. To compute the directional derivative of f at p in the direction X_p, we first write down a set of parametric equations for the line through p in the direction of X_p:

$$x_i = p_i + ta_i, \quad 1 \leq i \leq n.$$

Let $a = (a_1, \ldots, a_n)$. Then the directional derivative $D_{X_p} f$ is

$$\begin{aligned} D_{X_p} f &= \lim_{t \to 0} \frac{f(p + ta) - f(p)}{t} = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f(p + ta) \\ &= \sum_i \frac{\partial f}{\partial x_i}\bigg|_p \cdot \frac{\mathrm{d}x_i}{\mathrm{d}t}\bigg|_{t=0} \qquad \text{(by chain rule)} \\ &= \sum_i \frac{\partial f}{\partial x_i}\bigg|_p a_i = \left(\sum_i a_i \frac{\partial}{\partial x_i} \right) f = X_p(f). \end{aligned}$$

Similarly, the directional derivative at p of a smooth vector field $Y = \sum_i b_i \dfrac{\partial}{\partial x_i}$ on $\mathbb{R}^n_r$ in the direction X_P is defined as (note that here we are using global chart for $\mathbb{R}^n_r$)

$$D_{X_p} Y = \sum_i X_p(b_i) \frac{\partial}{\partial x_i}\bigg|_p. \tag{2.2}$$

The directional derivative in $\mathbb{R}^n$ gives a map

$$D : \mathfrak{X}(\mathbb{R}^n_r) \times \mathfrak{X}(\mathbb{R}^n_r) \to \mathfrak{X}(\mathbb{R}^n_r),$$

which is called the *covariant derivative* of Y with respect to X.

Exercise 2.9. Let $\mathcal{F}$ be the ring of smooth functions on $\mathbb{R}^n_r$. For $X, Y \in \mathfrak{X}(\mathbb{R}^n_r)$, the covariant derivative $D_X Y$ satisfies the following properties:

(i) $D_X Y$ is $\mathcal{F}$-linear and $\mathbb{R}$ linear in Y.

(ii) $D_X Y$ satisfies *Leibniz rule*, that is, if $f \in C^\infty(\mathbb{R}^n)$, then

$$D_X(fy) = X(f)Y + fD_X Y.$$

On an arbitrary manifold M, we can define the covariant derivative (or directional derivative) of a C^∞ function f in the direction $X_p \in T_PM$ in the same way as in the Euclidean case:

$$D_{X_p} f = X_p(f).$$

However, there is no longer a canonical way to define the directional derivative of a vector field Y. The formula (2.2) fails because unlike in $\mathbb{R}^n_r$, we do not have a canonical basis for the tangent space T_pM. We, therefore, begin by putting the properties which is satisfied by the operator D. Let $\mathfrak{X}(M)$ denotes the set of all vector fields on M.

Definition 2.10. *A (linear) connection on a C^∞ manifold M is a map*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \ (X,Y) \mapsto \nabla_X Y$$

such that

$\nabla 1)$ *$\nabla_X Y$ is $C^\infty(M)$-linear in X. That is, for any $X_1, X_2 \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, we have*

$$\nabla_{X_1 + f X_2} Y = \nabla_{X_1} Y + f \nabla_{X_2} Y.$$

$\nabla 2)$ *$\nabla_X Y$ is $\mathbb{R}$-linear in Y, that is, for $\alpha \in \mathbb{R}$ and $Y_1, Y_2 \in \mathfrak{X}(M)$,*

$$\nabla_X (Y_1 + \alpha Y_2) = \nabla_X Y_1 + \alpha \nabla_X Y_2.$$

$\nabla 3)$ *$\nabla_X Y$ satisfies the Leibniz rule, that is, for any $f \in C^\infty(M)$,*

$$\nabla_X (fY) = X(f)Y + f\nabla_X Y.$$

We call $\nabla_X Y$ the covariant derivative of Y in the direction of X with respect to the connection ∇.

Exercise 2.11. Show that ∇ is local, that is, for any $p \in M$ and U neighborhood of p if $X\big|_U = \tilde{X}\big|_U$ and $Y\big|_U = \tilde{Y}\big|_U$, then $\nabla_X Y\big|_U = \nabla_{\tilde{X}} \tilde{Y}\big|_U$.

Our next goal is to show that on every semi-Riemannian manifold, there exists a unique connection, (that will be called Levi-Civita connection) satisfying some extra properties. Let $\Omega^k(M)$ denotes the set of all k-forms on M.

Proposition 2.12. *Let M be a semi-Riemannian manifold and $X \in \mathfrak{X}(M)$. Let*

$X^\flat \in \Omega^1(M)$ such that

$$X^\flat(Y) = \langle X, Y\rangle, \quad Y \in \mathfrak{X}(M). \tag{2.3}$$

The function $X \mapsto X^\flat$ is a $C^\infty(M)$-linear isomorphism from $\mathfrak{X}(M)$ to $\Omega^1(M)$.

Proof. Let us denote the map by $\phi : \mathfrak{X}(M) \to \Omega^1(M)$. It is easy to see that ϕ is $C^\infty(M)$-linear. We will now show that ϕ is an isomorphism.

Injectivity: Let $\phi(X) = \phi(Y)$. This implies,

$$\forall\, Z \in \mathfrak{X}(M), \quad \langle X, Z\rangle = \langle Y, Z\rangle \implies \langle X - Y, Z\rangle = 0.$$

We claim that if $\langle X, Y\rangle = 0$ for any $Y \in \mathfrak{X}(M)$, then $X = 0$. Since $\langle X, Y\rangle = 0$, so for any $p \in M$, $\left\langle X_p, Y_p\right\rangle = 0$. Let $\mathbf{v} \in T_pM$ then we have

$$\left\langle X_p, \mathbf{v}\right\rangle = 0 \overset{g_p \text{ is nondegenerate}}{\Longrightarrow} X_p = 0.$$

Since p is arbitrary, $X = 0$.

Surjectivity: Let $\omega \in \Omega^1(M)$. Then we need to show that there exists $X \in \mathfrak{X}(M)$ such that $\phi(X) = \omega$. At first we will deal this locally. Let $(\varphi = (x_1, \ldots, x_n), U)$ is a chart of M. So write

$$\omega\big|_U = \sum_{i=1}^n w_i \mathrm{d}x_i.$$

Define

$$X_U := \sum_{i,j=1}^n g^{ij} w_i \frac{\partial}{\partial x_j} \in \mathfrak{X}(U).$$

Then,

$$\begin{aligned}
\left\langle X_U, \frac{\partial}{\partial x_k}\right\rangle &= \sum_{i,j} g^{ij} w_i \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right\rangle \\
&= \sum_{i,j} w_i g^{ij} g_{jk} \\
&= \sum_i w_i \delta_{ik} = \omega_k = \omega\big|_U \left(\frac{\partial}{\partial x_k}\right).
\end{aligned}$$

Thus, by $C^\infty(M)$-linearity, $\phi\left(X_U\right) = \omega\big|_U$.

Also, note that this is well-behaved with the change of charts. That is, for any charts $(\phi = (x_1, \ldots, x_n), U)$ and $\psi = (y_1, \ldots, y_n), V$ if $U \cap V \neq \emptyset$, then $X_U\big|_{U\cap V} = X_V\big|_{U\cap V}$. Write,

$$\omega\big|_U = \sum_i w_i \mathrm{d}x_i \quad \text{and} \quad \omega\big|_V = \sum_j \bar{w}_j \mathrm{d}y_j$$

$$g\big|_U = \sum_{i,j} g_{ij} \mathrm{d}x_i \otimes \mathrm{d}x_j \quad \text{and} \quad g\big|_V = \sum_{i,j} \bar{g}_{ij} \mathrm{d}y_i \otimes \mathrm{d}y_j.$$

At first we show that $\sum_{i,j} g^{ij} w_i \frac{\partial}{\partial x_j} = \sum_{i,j} \bar{g}^{ij} \bar{w}_i \frac{\partial}{\partial y_j}$. Recall that $dx_j = \sum_i \frac{\partial x_j}{\partial y_i} \mathrm{d}y_i$. So,

$$\omega\big|_{U\cap V} = \sum_j w_j \mathrm{d}x_j = \sum_{i,j} w_j \frac{\partial x_j}{\partial y_i} \mathrm{d}y_i = \sum_i \bar{w}_i \mathrm{d}y_i \implies \bar{w}_i = \sum_m w_m \frac{\partial x_m}{\partial y_i}.$$

We also recall that $\frac{\partial}{\partial y_i} = \sum_k \frac{\partial x_k}{\partial y_i} \frac{\partial}{\partial x_k}$. This,

$$\begin{aligned} \bar{g}_{ij} &= g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) = g\left(\sum_k \frac{\partial x_k}{\partial y_i}\frac{\partial}{\partial x_k}, \sum_l \frac{\partial x_l}{\partial y_j}\frac{\partial}{\partial x_l}\right) \\ &= \sum_{k,l} \frac{\partial x_k}{\partial y_i}\frac{\partial x_l}{\partial y_j} g\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right) \\ &= \sum_{k,l} \frac{\partial x_k}{\partial y_i}\frac{\partial x_l}{\partial y_j} g_{kl}. \end{aligned}$$

So, by setting $A = (a_{ki}) = \left(\frac{\partial x_k}{\partial y_i}\right)$, we obtain

$$\left(\bar{g}_{ij}\right) = A^t \left(g_{ij}\right) A \implies \left(\bar{g}^{ij}\right) = A^{-1}\left(g^{ij}\right)\left(A^{-1}\right)^t \implies \bar{g}^{ij} = \sum_{k,l} \frac{\partial y_i}{\partial x_k} g^{kl} \frac{\partial y_j}{\partial x_l}.$$

Finally, we obtain

$$\begin{aligned} \sum_{i,j} \bar{g}^{ij} \bar{w}_i \frac{\partial}{\partial y_j} &= \sum_{k,l,m,n} \frac{\partial y_i}{\partial x_k} g^{kl} \frac{\partial y_j}{\partial x_l} w_m \frac{\partial x_m}{\partial y_i}\frac{\partial x_n}{\partial y_j}\frac{\partial}{\partial x_n} \\ &= \sum_{m,n} g^{mn} w_m \frac{\partial}{\partial x_n}. \end{aligned}$$

Therefore, $X_U\big|_{U\cap V} = X_V\big|_{U\cap V}$.

Finally, we will use partition of unity to patch them up. Choose a cover $\mathcal{U} = \{U_i : i \in I\}$ of M by charts neighborhoods and a subordinate partition of unity $(\chi_i)_i$ such that $\operatorname{supp}(\chi_i) \subset U_i$. For any $Y \in \mathfrak{X}(M)$, we then have

$$\begin{aligned} \langle X, Y\rangle &= \left\langle X, \sum_i \chi_i Y\right\rangle = \sum_i \langle X, \chi_i Y\rangle = \sum_i \left\langle X\big|_{U_i}, \chi_i Y\right\rangle \\ &= \sum_i \omega\big|_{U_i}(\chi_i Y) = \sum_i \omega(\chi_i Y) = \omega\left(\sum_i \chi_i Y\right) = \omega(Y). \end{aligned}$$

□

Thus, in semi-Riemannian geometry, we can identify a vector field into a one-form and vice-versa. We now will prove the existence of a special connection.

Theorem 2.13 (Fundamental Theorem of semi-Riemannian Geometry). *Let (M,g) be a semi-Riemannian manifold. Then there exists a unique connection ∇ on M such that ∇ satisfies $(\nabla 1)-(\nabla 3)$ and for any $X,Y,Z \in \mathfrak{X}(M)$,*

$\nabla 4)$ $[X,Y] = \nabla_X Y - \nabla_Y X$ (Torsion Free)

$\nabla 5)$ $Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$ (Metric Compatibility).

The connection ∇ is called Levi-Civita connection *of (M,g). It is uniquely determined by the Koszul formula*

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ &\quad - \langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle. \end{aligned} \tag{2.4}$$

Proof. $\boxed{\text{Uniquenss}}$ Let

$$\begin{aligned} F(X,Y,Z) := X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ - \langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle. \end{aligned}$$

Using Koszul formula (2.4), and $\nabla 4$ & $\nabla 5$, we have

$$\begin{aligned} F(X,Y,Z) &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle + \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \\ &\quad - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle - \langle X, \nabla_Y Z\rangle + \langle X, \nabla_Z Y\rangle \\ &\quad + \langle Y, \nabla_Z X\rangle - \langle Y, \nabla_X Z\rangle + \langle Z, \nabla_X Y\rangle - \langle Z, \nabla_Y X\rangle \\ &= 2\langle \nabla_X Y, Z\rangle. \end{aligned}$$

Now by the injectivity of ϕ (in Proposition 2.12), $\nabla_X Y$ is uniquely determined.

$\boxed{\text{Existence}}$ Given $X,Y \in \mathfrak{X}(M)$, the mapping $\omega : Z \mapsto F(X,Y,Z)$ is $C^\infty(M)$-linear. Thus, $\omega \in \Omega^1(M)$ and by Equation (2.4), there exists a unique vector field which we call $\nabla_X Y$ such that

$$2\langle \nabla_X Y, Z\rangle := F(X,Y,Z), \quad \forall\, Z \in \mathfrak{X}(M).$$

Now it remains to show that $\nabla_X Y$ satisfies $(\nabla 1)-(\nabla 5)$. □

Exercise 2.14. Check $\nabla_X Y$ satisfies $(\nabla 1)-(\nabla 5)$.

Lemma 2.15. *Let $U \subseteq M$ be open and $X,Y,X_1,X_2,Y_1,Y_2 \in \mathfrak{X}(M)$. Then we have*

(i) $X_1\big|_U = X_2 = \big|_{U_2} \implies \left(\nabla_{X_1} Y\right)\big|_U = \left(\nabla_{X_2} Y\right)\big|_U.$

(ii) $Y_1\big|_U = Y_2 = \big|_{U_2} \implies (\nabla_X Y_1)\big|_U = (\nabla_X Y_2)\big|_U$.

2.2 Christoffel Symbol

Definition 2.16. *Let $(\varphi = (x_1, \ldots, x_n), U)$ be a chart of a semi-Riemannian manifold M. The* Christoffel symbols *with respect to φ are the C^∞-functions $\Gamma^i_{jk} : U \to \mathbb{R}$ defined by*

$$\nabla_{\partial_i}\partial_j =: \sum_{k=1}^{n} \Gamma^k_{ij}\partial_k, \quad 1 \le i, j \le n.$$

Note. Since

$$\nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i = \left[\partial_i, \partial_j\right] = 0 \implies \Gamma^k_{ij} = \Gamma^k_{ji}.$$

Proposition 2.17. *Let M be a semi-Riemannian manifold and $(\varphi = (x_1, \ldots, x_n), U)$ be a chart of M. Let $Z = \sum_{i=1}^n z_i\partial_i \in \mathfrak{X}(U)$. Then*

(i) $\nabla_{\partial_i} \sum_{j=1}^n z_j\partial_j = \sum_{k=1}^n \left(\frac{\partial z_k}{\partial x_i} + \sum_{j=1}^n \Gamma^k_{ij} z_j\right)\partial_k$,

(ii) $\Gamma^k_{ij} = \frac{1}{2}\sum_{m=1}^n g^{km}\left(\frac{\partial g_{jm}}{\partial x_i} + \frac{\partial g_{im}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m}\right)$.

Proof. (i) This is immediate from Leibniz rule ($\nabla 3$).

(ii) Apply the Koszul formula (Equation (2.4)) by taking $X = \partial_i, Y = \partial_j$ and $Z = \partial_m$. Then brackets are zero and hence,

$$2\left\langle \nabla_{\partial_i}\partial_j, \partial_m\right\rangle = \frac{\partial g_{jm}}{\partial x_i} + \frac{\partial g_{im}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m}.$$

By the definition of Christoffel symbols,

$$2\left\langle \nabla_{\partial_i\partial_j,\partial_m}\right\rangle = 2\left\langle \sum_{a=1}^n \Gamma^a_{ij}\partial_a, \partial_m\right\rangle \implies 2\left\langle \nabla_{\partial_i\partial_j,\partial_m}\right\rangle = 2\sum_{a=1}^n \Gamma^a_{ij} g_{am}.$$

Multiplying by $\sum_m g^{mk}$ leads to the required result, that is,

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{m=1}^n g^{km}\left(\frac{\partial g_{jm}}{\partial x_i} + \frac{\partial g_{im}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m}\right).$$

□

Lecture–6

Lemma 2.18. *For $X, Y \in \mathfrak{X}(\mathbb{R}^n_r)$ with $Y = \sum_i y_i \partial_i$, let*

$$\nabla_X Y = \sum_i X(y_i)\partial_i.$$

Then ∇ is the Levi-Civita connection on $\mathbb{R}^n_r$ and in natural coordinates on $\mathbb{R}^n_r$, we have

(i) $g_{ij} = \delta_{ij}\epsilon_j$, *where* $\epsilon_j = \begin{cases} -1, & \text{if } 1 \le j \le r; \\ 1, & \text{if } r+1 \le j \le n, \end{cases}$

(ii) $\Gamma^k_{ij} = 0$

for all $1 \le i, j, k \le n$.

Definition 2.19. *A vector field X on (M, g) is said to be* parallel *if $\nabla_Y X = 0$ for all $Y \in \mathfrak{X}(M)$.*

Example 2.20. The coordinate vector fields in $\mathbb{R}^n_r$ are parallel. For any $Y = \sum_i y_i \partial_i$

$$\nabla_Y \partial_j = \sum_i y_i \nabla_{\partial_i}\partial_j = 0.$$

Exercise 2.21. In $\mathbb{R}^n_r$, a vector field is parallel if and only if it is constant, that is,

$$\nabla_Y X = 0 \quad \forall\ Y \iff X = \text{ constant }.$$

Example 2.22 (Cylindrical Coordinates in $\mathbb{R}^3$)**.** Let $(x, y, z) = (r\cos\varphi, r\sin\varphi, z)$ be the usual cylindrical coordinates in $\mathbb{R}^3$. Actually, the above one is a chart on $\mathbb{R}^3 \setminus \{x \ge 0, y = 0\}$ with an inverse defined by $(r, \varphi, z) \mapsto (r\cos\varphi, r\sin\varphi, z)$. Hence, we have

$$\partial_r = \cos\varphi\partial_x + \sin\varphi\partial_y,$$

$$\partial_\varphi = rU, \text{ where } U = -\sin\varphi\partial_x + \cos\varphi\partial_y,$$
$$\partial_z = \partial_z.$$

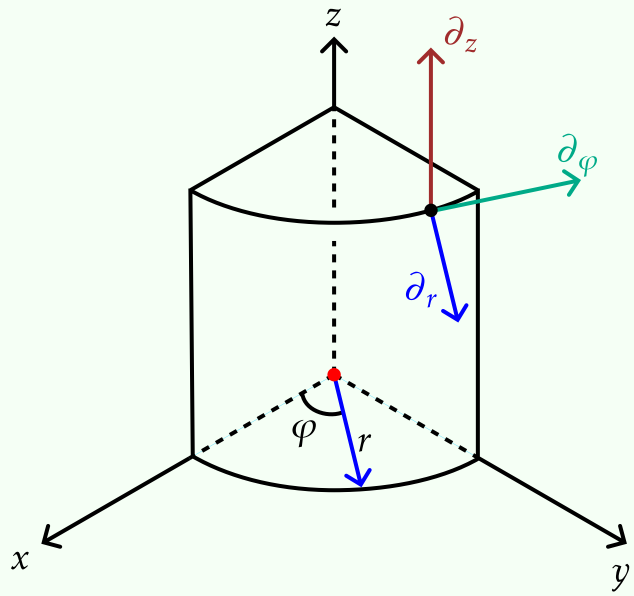


Figure 2.3: Cylindrical Coordinates

Setting $y_1 = r, y_2 = \varphi, y_3 = z$, we obtain

$$\begin{aligned} g_{11} &= \langle \partial_r, \partial_r \rangle = 1, \\ g_{22} &= \left\langle \partial_\varphi, \partial_\varphi \right\rangle = r^2 \\ g_{33} &= \langle \partial_z, \partial_z \rangle = 1, \\ g_{ij} &= 0, \quad \text{for all } i \neq j. \end{aligned}$$

So, we have

$$(g_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad (g^{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and hence,

$$g = \sum_{i,j} g_{ij} dy_i \otimes dy_j = dr \otimes dr + r^2 d\varphi \otimes d\varphi + dz \otimes dz.$$

By looking at the matrix (g_{ij}), we have $\{\partial_r, \partial_\varphi, \partial_z\}$ is orthogonal and hence (r, φ, z) is an orthogonal coordinate system. For the Christoffel symbols we find

$$\Gamma^1_{22} = -r, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r},$$

and other $\Gamma_{ij}^k = 0$. Hence we have $\nabla_{\partial_i}\partial_j = 0$ for all i,j except

$$\nabla_{\partial_\varphi}\partial_\varphi = -r\partial_r, \text{ and } \nabla_{\partial_\varphi}\partial_r = \nabla_{\partial_r}\partial_\varphi = \frac{1}{r}\partial_\varphi = U.$$

By Figure 2.3, we see that ∂_r and ∂_φ are parallel if one moves in the z-direction. We hence expect that $\nabla_{\partial_z}\partial_\varphi = 0 = \nabla_{\partial_z}\partial_r$, which also verified from our calculations. Moreover, ∂_z is parallel since it is a coordinate vector field in the natural basis of $\mathbb{R}^3$.

2.3 Vector field along a curve

Let M be a semi-Riemannian manifold. Given a smooth curve $\gamma : I \to M$, a *vector field along γ* is a continuous map $X : I \to TM$ such that $X(t) \in T_{\gamma(t)}M$ for every $t \in I$. It is *smooth vector field along γ* if it is smooth as a map from I to TM. We will denote the set of all smooth vector fields along γ by $\mathfrak{X}(\gamma)$. In particular, $t \mapsto \dot{\gamma} \equiv \gamma'(t)$ is an element of $\mathfrak{X}(\gamma)$. For any $f \in C^\infty(I)$, we have

$$\gamma'(t)(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_t (f \circ \gamma)$$

and therefore, in coordinates $\phi = (x_1, \dots, x_n)$ the local expression of the velocity vector takes the form

$$\gamma'(t) = \sum_{i=1}^n \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_t (x_i \circ \gamma) \left.\frac{\partial}{\partial x_i}\right|_{\gamma(t)}.$$

Another example of a vector field along a curve is: if γ is a curve in $\mathbb{R}^2$, let $N(t) = R\gamma'(t)$, where R is counterclockwise rotation by $\frac{\pi}{2}$, so $N(t)$ is normal to $\gamma'(t)$. In standard coordinates, $N(t) = (-\dot{\gamma}_2(t), \dot{\gamma}_1(t))$, so N is a smooth vector field along γ.

A general example is provided by the following construction: suppose $\gamma : I \to M$ is a smooth curve and $\tilde{X}$ is a smooth vector field on an open subset of M containing the image of γ. Define $V : I \to TM$ by setting $X(t) = \tilde{X}(\gamma(t))$. A smooth vector field X along γ is said to be *extendible* if there exists a smooth vector field $\tilde{X}$ on a neighborhood of the image of γ such that $X(t) = \tilde{X}\big|_{\gamma(t)}$ for $t \in I$, otherwise the curve is not extendible. For example, if γ is a curve such that $\gamma(t_1) = \gamma(t_2)$ but $\gamma'(t_1) \neq \gamma'(t_2)$, then γ' is not extendible. This is clear as if X is an extension of γ' then for any $t \in I$ then $X_{\gamma(t)} = \gamma'(t)$ which implies $X_{\gamma(t_1)} = \gamma'(t_1)$ and $X_{\gamma(t_2)} = \gamma'(t_2)$, which is a not possible.

On any semi-Riemannian manifold with a connection ∇, we can define the covariant derivative along a curve γ.

Proposition 2.23 (Covariant derivative along a curve). *Let $\gamma : I \to M$ be a smooth curve into a semi-Riemannian manifold M with a connection ∇. Then there exists a unique mapping*

$$D_t : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma), \quad X \mapsto X' = D_t(X) \equiv \frac{\nabla X}{\mathrm{d}t}$$

called the covariant derivative along γ such that the following properties holds.

(i) $D_t(aX_1 + bX_2) = aD_tX_1 + bD_tX_2$ *for* $a, b \in \mathbb{R}$.

(ii) $D_t(fX) = \frac{\mathrm{d}f}{\mathrm{d}t}X + fD_tX$, *for* $f \in C^\infty(I)$.

(iii) If $X \in \mathfrak{X}(\gamma)$ is extendible, then for every extension $\tilde{X}$ of X, we have

$$D_tX(t) = D_t(X(\gamma(t))) = \nabla_{\gamma'(t)}\tilde{X}.$$

Furthermore, if the connection is Levi-Civita, then

(iv) $\frac{\mathrm{d}}{\mathrm{d}t}\langle X_1, X_2\rangle = \langle D_tX_1, X_2\rangle + \langle X_1, D_tX_2\rangle$.

For a proof, please see [O'N83, Proposition 18, pp. 65] or Lee's book on Riemannian manifold [Lee18, Theorem 4.24].

Using the above proposition (Proposition 2.23), we can write D_tX in terms of Christoffel symbols. In a chart $(\phi = (x_1, \dots, x_n), U)$ we have

$$\nabla_{\gamma'(t)}\partial_i = \sum_{j=1}^{n} \nabla_{\frac{\mathrm{d}(x_j(\gamma))}{\mathrm{d}t}\partial_j}\partial_i = \sum_{j=1}^{n} \frac{\mathrm{d}(x_j(\gamma))}{\mathrm{d}t}\nabla_{\partial_j}\partial_i$$
$$= \sum_{j,k=1}^{n} \frac{\mathrm{d}(x_j(\gamma))}{\mathrm{d}t}\Gamma_{ij}^k\partial_k.$$

Write

$$Z(t) = \sum_{i=1}^{n} Z(t)(x_i)\partial_i\big|_{\gamma(t)} =: \sum_{i=1}^{n} Z_i(t)\partial_i\big|_{\gamma(t)}$$

Thus using Proposition 2.23 (ii), we get

$$D_tZ(t) = \sum_{i}^{n} \frac{\mathrm{d}Z_k}{\mathrm{d}t}(t)\partial_i\big|_{\gamma(t)} + Z_iD_t\partial_i\big|_{\gamma(t)}.$$

Using (iii) and use the fact that each ∂_i is extendible, we deduce that

$$D_tZ(t) = \sum_{i}^{n} \left[\frac{\mathrm{d}Z_i}{\mathrm{d}t}(t)\partial_i\big|_{\gamma(t)} + Z_i(t)\nabla_{\gamma'(t)}\partial_i\big|_{\gamma(t)}\right].$$

Hence,

$$D_t Z(t) = \sum_{k=1}^{n} \left[\frac{\mathrm{d}Z_k}{\mathrm{d}t} + \sum_{i,j} \Gamma_{ij}^{k} \frac{\mathrm{d}(x_i \circ \gamma)(t)}{\mathrm{d}t} Z_j(t) \right] \partial_k \Big|_{\gamma(t)}. \tag{2.5}$$

In the special case that $Z = \gamma'$, we call $D_t Z = \gamma''$ the *acceleration* of γ. Also we call a vector field $Z \in \mathfrak{X}(\gamma)$ *parallel* if $D_t Z = 0$. The above formula (2.5) shows that being parallel can be expressed by a system of linear ODEs of first order. if we provide an initial condition, then we can solve this system uniquely and hence the following result.

Proposition 2.24. *Let $\gamma : I \to M$ be a smooth curve. Let $a \in I$ and $z \in T_{\gamma(a)}M$. Then there exists an unique parallel vector field $Z \in \mathfrak{X}(\gamma)$ such that $Z(a) = z$.*

This result gives rise to the following notion.

Definition 2.25. *Let $\gamma : I \to M$ be a smooth curve. Let $a, b \in I$ with $\gamma(a) = p$ and $\gamma(b) = q$. For $z \in T_pM$ and Z denote the vector field as in Proposition 2.24 with $Z(a) = z$. Then we call the mapping*

$$P = P_a^b(\gamma) : T_pM \to T_qM, \quad z \mapsto Z(b)$$

the parallel transport along γ from p to q.

Lemma 2.26. *Parallel transport is a linear isometry.*

Proof. Let $v, w \in T_pM$ with parallel vector field V and W. Using the linearity of D_t $V + W$ and λV are also parallel for any $\lambda \in \mathbb{R}$, so we have

$$P(v + w) = (V + W)(b) = V(b) + W(b) = P(v) + P(w) \text{ and}$$
$$P(\lambda v) + (\lambda V)(b) = \lambda(V(b)) = \lambda P(v).$$

Thus, P is linear.

Now we will show that P is an isomorphism. Note that if $v \in \ker P$ then $P(v) = 0$. Since the zero vector field $O \in \mathfrak{X}(M)$ is parallel and $O(b) = 0$. So by the uniqueness (Proposition 2.24) of parallel vector field, we get $v = V(a) = O(a) = 0$.

Finally to see it is an isometry, note that

$$D_t \langle V, W \rangle = \langle D_t V, W \rangle + \langle V, D_t W \rangle = 0,$$

and hence $\langle V, W \rangle$ is a constant vector field along γ. Thus,

$$\langle P(v), P(w) \rangle = \langle V(b), W(b) \rangle = \langle V(a), W(a) \rangle = \langle v, w \rangle.$$

□

3 Geodesics

In this section we will generalize the Euclidean notion of straight line in a semi-Riemannian manifold. In Euclidean space a curve is a straight line if and only if its acceleration is zero. Analogously, we define in a semi-Riemannian manifold a smooth curve γ is said to be a *geodesic* with respect to the connection ∇ if its acceleration is zero, that is, $D_t\gamma' \equiv 0$. Locally this condition translates into a system of nonlinear ODEs of second order.

Proposition 3.1. *Let $(\phi = (x_1, \ldots, x_n), U)$ be a chart of M and let $\gamma : I \to U$ be a smooth curve. Then γ is a geodesic if and only if the coordinate functions $x_i \circ \gamma$ satisfy*

$$\frac{\mathrm{d}^2(x_k \circ \gamma)}{\mathrm{d}t^2} + \sum_{i,j=1}^{n} \Gamma_{ij}^k(\gamma) \frac{\mathrm{d}(x \circ \gamma)}{\mathrm{d}t} \cdot \frac{\mathrm{d}(x_j \circ \gamma)}{\mathrm{d}t} = 0, \quad 1 \le k \le n. \tag{3.1}$$

Proof. Since γ is a geodesic so $D_t\gamma' \equiv 0$. We have $\gamma'(t) = \frac{\mathrm{d}(x_i \circ \gamma)}{\mathrm{d}t}$. Using the local expression of $D_t Z$ (Proposition 2.24) we obtain the geodesic equations (3.1). $\square$

It is most common to abbreviate the coordinate functions of γ as γ_i rather than $x_i \circ \gamma$. Using this notation the geodesic equations (3.1) become

$$\frac{\mathrm{d}^2\gamma_k}{\mathrm{d}t^2} + \sum_{i,j} \Gamma_{ij}^k \frac{\mathrm{d}\gamma_i}{\mathrm{d}t} \frac{\mathrm{d}\gamma_j}{\mathrm{d}t} = 0, \quad 1 \le n. \tag{3.2}$$

The existence and uniqueness theorem for ODEs give the existence and uniqueness of geodesics.

Lemma 3.2. *Let $p \in M$ and $\mathbf{v} \in T_pM$. Then there exists a unique geodesic $\gamma : I \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = \mathbf{v}$.*

In the last equation, such a geodesic, from now onward, will be called as *geodesic starting at p with initial velocity* $\mathbf{v}$. The following lemma asserts that we can not have two geodesics with same initial point and same velocity.

Lemma 3.3. *Let $\gamma_1, \gamma_2 : I \to M$ be two geodesics. If there is a number $a \in I$ such that $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1'(a) = \gamma_2'(a)$, then $\gamma_1 = \gamma_2$.*

Proof. To the contrary, let us assume that $\gamma_1 \neq \gamma_2$, so there exists $t_0 \in I$ such that $\gamma_1(t_0) \neq \gamma_0(t_0)$ and $t_0 > a$. Therefore, the set $G = \{t \in I : \gamma_1(t) \neq \gamma_2(t) \text{ and } t > a\}$ is bounded below. Let $b = \inf G$. We claim that $\gamma_1'(b) = \gamma_2'(b)$. If $a = b$, then we are done. Otherwise, $b > a$ and hence $\gamma_1 = \gamma_2$ on (a,b). Since γ_1 and γ_2 are smooth curves, we have

$$\gamma_1'(b) = \lim_{t\to b-} \gamma_1'(t) = \lim_{t\to b-} \gamma_2'(t) = \gamma_2'(b).$$

Note that $\gamma_i(t+b)$ for $i = 1,2$ are also geodesics and Lemma 3.2 shows that $\gamma_1 = \gamma_2$ on some neighbourhood of b which implies there exists $t > b$ such that $\gamma_1(t) = \gamma_2$, which is a contradiction that b is the greatest lower bound of B. □

Proposition 3.4. *Let $\mathbf{v} \in T_pM$. Then there exists a unique geodesic $\gamma_{\mathbf{v}}$ in M such that*

(i) $\gamma_{\mathbf{v}}(0) = p$ and $\gamma_{\mathbf{v}}'(0) = \mathbf{v}$.

(ii) The domain I of $\gamma_{\mathbf{v}}$ is maximal, that is, if $\eta : J \to M$ is a geodesic with $\eta(0) = p$ and initial velocity $\mathbf{v}$, then $J \subseteq I$ and $\eta = \gamma_{\mathbf{v}}\big|_J$.

Proof. Let

$$\Omega(p,\mathbf{v}) = \{\gamma : 0 \in I_\gamma \to M \mid \gamma(0) = p \text{ and } \gamma'(0) = \mathbf{v}\}.$$

Using Lemma 3.2, the set $\Omega(p,\mathbf{v}) \neq \emptyset$. Also using Lemma 3.3 for any two curves $\eta_1, \eta_2 \in \Omega(p,\mathbf{v})$, the they matches on $I_{\eta_1} \cap I_{\eta_2}$ and hence the collection $\Omega(p,\mathbf{v})$ consistently defines a single curve $\gamma_{\mathbf{v}}$ on the interval $I = \cup I_\gamma$. Clearly $\gamma_{\mathbf{v}}$ satisfies (i) and (ii). □

Definition 3.5 (Geodesically Completeness). *A semi-Riemannian manifold M is said to be geodesically complete if domain of any maximal geodesic is $\mathbb{R}$.*

Example 3.6. (i) $\mathbb{R}^n_r$ is geodesically complete.

(ii) $\mathbb{R}^n_r \setminus \{0\}$ is not geodesically complete.

Lecture–7

We now turn to the study of the causal character of geodesics. We begin with the following definition.

Definition 3.7. *A curve γ in a semi-Riemannian manifold M is called* spacelike*, (*timelike *or* null*) if for all t its velocity $\gamma'(t)$ is spacelike (timelike or null). We call γ* causal *if it is timelike or null. These properties of c are commonly referred to as its* causal character.

In general a curve need not to have a causal character, i.e., its velocity vector could change its causal character along the curve. However, geodesics do have a causal character. Indeed, if γ is a geodesic, then by definition γ' is parallel vector field along γ. So,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\gamma'(t),\gamma'(t)\rangle = 2\langle\gamma''(t),\gamma(t)\rangle = 0.$$

Hence, $\langle\gamma'(t),\gamma'(t)\rangle$ is constant for all t.

Lemma 3.8. *Let $\gamma : I \to M$ be a nonconstant geodesic. A reparametrization $\gamma \circ h : J \to M$ of γ is a geodesic iff h is of the form $h(t) = at + b$ for some $a, b \in \mathbb{R}$.*

Exercise 3.9. Prove Lemma 3.8.

This result shows that the geodesic parametrization have geometric significance. If a curve has a reparametrization as a geodesic we call it a *pregeodesic*.

If a system of second-order ODEs is given by smooth functions, then its solutions are smooth not just in the parameter but simultaneously in the parameter, initial values, and initial first derivatives. Applying this fact to geodesic equations (Equation (3.2)) give the following result.

Lemma 3.10. *Let $\mathbf{v} = (p, \mathbf{v}) \in TM$. Then there exists a neighborhood $\mathcal{N}$ of $\mathbf{v}$ in TM and an interval I around 0 such that the mapping*

$$\mathcal{N} \times I \ni (\mathbf{w}, s) \mapsto \gamma_{\mathbf{w}}(s) \in M$$

is smooth.

A semi-Riemannian submanifold N of M is called *totally geodesic* if a curve γ in N is an N-geodesic if and only if it is an M-geodesic.

Exercise 3.11. Show that the following are equivalent.

(i) M is totally geodesic.

(ii) Let $p \in N$, $\mathbf{v} \in T_pN \subseteq T_pM$. Then the N-geodesic $\gamma_{\mathbf{v}}$ lies in M initially, that is, there exists an open interval $I \ni 0$ such that $\gamma_{\mathbf{v}}(t) \in N$ for all $t \in I$.

Lemma 3.12. *Let φ be an isometry of a semi-Riemannian manifold (M,g) and*

$$F = \{x \in M : \varphi(x) = x\}$$

be the set of fixed points of φ. If F is C^∞-submanifold, then it is totally geodesic.

Proof. The proof follows from previous exercise (part (ii)). $\square$

3.1 Exponential Map

At each point p of a semi-Riemannian manifold M we collect the geodesics starting at p into a single mapping.

Definition 3.13. *Let $p \in M$ be a point in a semi-Riemannian manifold M and let $\mathcal{D}_p = \{\mathbf{v} \in T_pM : \gamma_{\mathbf{v}}$ is at least defined on $[0,1]\ \}$. The exponential map of M at p is defined as*

$$\exp_p : \mathcal{D}_p \to M, \quad \exp_p(\mathbf{v}) := \gamma_{\mathbf{v}}(1). \tag{3.3}$$

Observe that $\mathcal{D}_p$ is the largest subset of T_PM on which $\exp_p$ can be defined. If M is geodesically complete, then $\mathcal{D}_p = T_PM$ for any $p \in M$.

Let $\mathbf{v} \in T_pM$ and $t \in \mathbb{R}$. We want to see where does a straight line in T_pM passing through 0 maps under the exponential map $\exp_p$. Let $\mathbf{v} \mapsto \gamma_{\mathbf{v}}(1)$. Then the geodesic $s \mapsto \gamma_{\mathbf{v}}(ts)$ has initial velocity $t\gamma_{\mathbf{v}}'(0) = t\mathbf{v}$ and so we have $\gamma_{t\mathbf{v}}(s) = \gamma_{\mathbf{v}}(ts)$ for all t,s for which one and hence both sides are defined. This implies the exponential map

$$\exp_p(t\mathbf{v}) = \gamma_{t\mathbf{v}}(1) = \gamma_{\mathbf{v}}(t).$$

Thus the exponential map $\exp_p$ maps straight lines $t \mapsto t\mathbf{v}$ through the origin in T_PM to geodesics $\gamma_{\mathbf{v}}(t)$ through $p \in M$ (see Figure 3.1).

Theorem 3.14. *For each $p \in M$ there exist neighborhoods $\tilde{U}$ of $0 \in T_pM$ and U of p in M such that $\exp_p : \tilde{U} \to U$ is a diffeomorphism.*

Exercise 3.15. Use Lemma 3.10 to show that $\exp_p$ is smooth on a suitable neighborhood of $0 \in T_pM$ and then apply inverse function theorem to prove Theo-

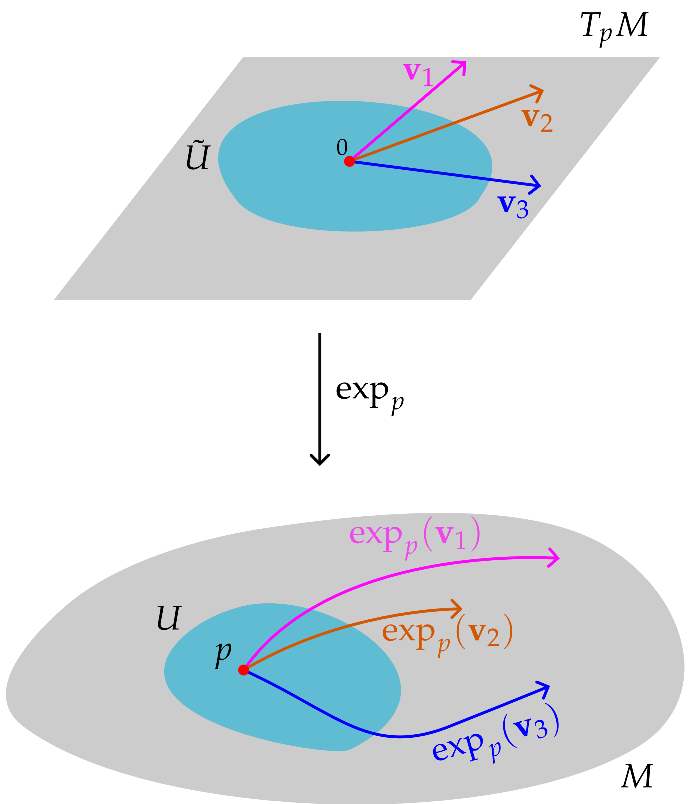


Figure 3.1: $\exp_p$ maps straight lines through $0 \in T_PM$ to geodesics through p

rem 3.14.

A subset S of a vector space is called *star shaped* (around 0) if $\mathbf{v} \in S$ and $t \in [0,1]$ implies $t\mathbf{v} \in S$. That means the line segment joining 0 and $\mathbf{v}$ will also lies in S. If U and $\tilde{U}$ are as in Theorem 3.14 and $\tilde{U}$ is star shaped, then we call U a normal neighborhood of p. Using Theorem 3.14, we see that normal neighborhoods exist around each point by restricting $\exp_p$ to a small star-shaped open neighborhood of $0 \in T_pM$.

Proposition 3.16. *Let U be a normal neighborhood of p. Then for each $q \in U$ there exists a unique geodesic $\sigma : [0,1] \to U$ from p to q called the radial geodesic from p to q. Moreover, we have $\sigma'(0) = \exp_p^{-1}(q) \in \tilde{U}$.*

We will now see a very important result in (semi)-Riemannian geometry, the Gauss lemma. For any $p \in M$ the tangent space T_pM is a finite dimensional vector space and for any $x \in T_pM$, we may identify $T_x(T_pM)$ with T_pM itself. Hence, if $\mathbf{v}_x \in T_x(T_pM)$, we will view $\mathbf{v}_x$ also as an element of T_pM. We call $\mathbf{v}_x$ *radial* if it is multiple of x.

Theorem 3.17. *Let M be a semi-Riemannian manifold and let $p \in M$, $0 \neq x \in \mathcal{D}_p \subseteq$*

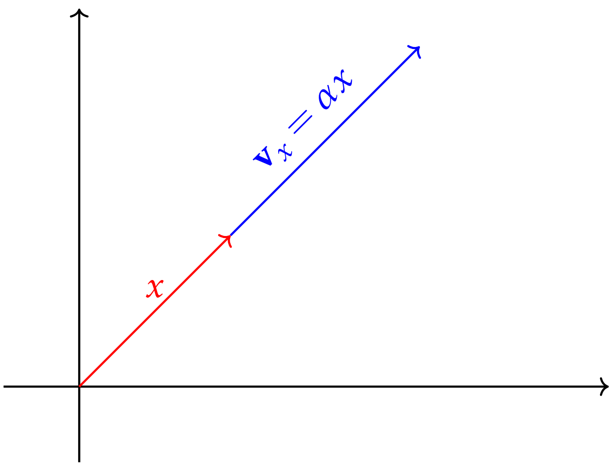


Figure 3.2: $\mathbf{v}_x$ is a radial vector

T_pM. Then for any $\mathbf{v}_x, \mathbf{w}_x \in T_x(T_pM)$ with $\mathbf{v}_x$ being radial, we have

$$\left\langle d(\exp_p)_x(\mathbf{v}_x), d(\exp_p)_x(\mathbf{w}_x)\right\rangle = \langle \mathbf{v}_x, \mathbf{w}_x\rangle. \tag{3.4}$$

Proof. Since $\mathbf{v}_x$ is radial and Equation 3.4 is linear, we may assume (after identifying T_pM with $T_x(T_pM)$) that $\mathbf{v}_x = x$ and let us write $\mathbf{v}_x = \mathbf{v}$ and $\mathbf{w}_x = \mathbf{w}$.

$$f(t,s) := \exp_p\left(t(\mathbf{v} + s\mathbf{w})\right).$$

The map satisfies,

$$\frac{\partial f}{\partial t}(1,0) = d(\exp_p)_x(\mathbf{v}) \text{ and } \frac{\partial f}{\partial s}(1,0) = d(\exp_p)_x(\mathbf{w}).$$

Thus, we need to show that

$$\left\langle \frac{\partial f}{\partial t}(1,0), \frac{\partial f}{\partial s}(1,0)\right\rangle = \langle \mathbf{v}, \mathbf{w}\rangle.$$

For simplicity, write $\frac{\partial f}{\partial s} = f_s$ and $\frac{\partial f}{\partial t} = f_t$. Note that for each s, the curve $t \mapsto f(t,s)$ is a geodesic passing through p with initial velocity $\mathbf{v} + s\mathbf{w}$ and therefore,

$$D_t(f_t) \equiv 0 \implies \langle f_t, f_t\rangle = \text{ constant } = \langle \mathbf{v} + s\mathbf{w}, \mathbf{v} + s\mathbf{w}\rangle.$$

Now consider,

$$\begin{aligned}\frac{\partial}{\partial t}\langle f_t, f_s\rangle &= \langle D_t(f_t), f_s\rangle + \langle f_t, D_t f_s\rangle \\ &= \langle f_t, D_t f_s\rangle = \langle f_t, D_s f_t\rangle \\ &= \frac{1}{2}\frac{\partial}{\partial s}\langle f_t, f_t\rangle = \frac{1}{2}\frac{\partial}{\partial s}\langle \mathbf{v} + s\mathbf{w}, \mathbf{v} + s\mathbf{w}\rangle \\ &= \langle \mathbf{v}, \mathbf{w}\rangle + s\langle \mathbf{w}, \mathbf{w}\rangle.\end{aligned}$$

Therefore, for $\phi(t) = \langle f_t(t,0), f_s(t,0)\rangle$, we obtain $\phi'(t) = \langle \mathbf{v}, \mathbf{w}\rangle$. Also, since $f(0,s) = p$, implies $f_s(0,0) = 0$. Therefore, $\phi(0) = 0$. Integrating ϕ', we obtain,

$$\phi(t) = t\langle \mathbf{v}, \mathbf{w}\rangle \implies \phi(1) = \langle f_t(1,0), f_s(1,0)\rangle = \langle \mathbf{v}, \mathbf{w}\rangle.$$

$\square$

Lecture–8

4 Lorentzian Manifolds

We recall the following from lorentz scalar product space. To differentiate the special Minkowski inner product, we will denote this by $\langle\langle\cdot,\cdot\rangle\rangle$.

(i) $\mathbb{R}^{n+1}_1$ is the 1-Minkowski space (Lorentz space) with the scalar product defined by
$$\langle\langle \mathbf{x},\mathbf{y}\rangle\rangle = -x_0y_0 + \sum_{i=1}^{n} x_iy_i.$$

(ii) Light cone: $C := \left\{\mathbf{x}\in\mathbb{R}^{n+1}_1 : \langle\langle \mathbf{x},\mathbf{x}\rangle\rangle\right\} = 0.$

(iii) Future/past light cone: $C^{\pm} := \{\mathbf{x}\in\mathbb{R}^{n+1}_1 : \pm x_0 \geq 0\}$.

(iv) Timelike vectors: $I := \left\{\mathbf{x}\in\mathbb{R}^{n+1}_1 : \langle\langle \mathbf{x},\mathbf{x}\rangle\rangle\right\} < 0.$

(v) The set of timelike vector I consists of two connected components and we choose a time-orientation by picking one component I^+ and call its member future directed. The members of the other component I^- are called past directed.

(vi) Future/past timelike vectors: $I^{\pm} = \{\mathbf{x}\in\mathbb{R}^{n+1}_1 : \pm x_0 > 0\}$.

(vii) Causal: $J := C\cup I = \{\mathbf{x}\in\mathbb{R}^{n+1}_1 : \langle\langle \mathbf{x},\mathbf{x}\rangle\rangle \leq 0\}$. Similarly, we have $J^{\pm} = C^{\pm}\cup I^{\pm}$.

(viii) Note that $C^{\pm} = \partial I^{\pm}$.

(ix) $I^{\pm}, J^{\pm}$ and $C^{\pm}$ are convex sets.

4.1 Another example: de Sitter Space

This is analogs of spheres in $\mathbb{R}^n$. Consider the following function
$$f : \mathbb{R}^{n+1}_1 \to \mathbb{R}, \mathbf{x} \mapsto \langle\langle \mathbf{x},\mathbf{x}\rangle\rangle.$$

Then f is C^∞ and
$$df_{\mathbf{x}} = -2x_0\mathrm{d}x_0 + 2\sum_{i=1}^{n} x_i\mathrm{d}x_i.$$

Note that for $\mathbf{x}\neq 0$, the derivative $df_{\mathbf{x}}\neq 0$ and hence every $r\in\mathbb{R}\setminus\{0\}$ is a regular value of f. [Recall that if $f : M\to N$ be a smooth map between two manifolds M and N then $y\in N$ is said to be regular value of f if for all $x\in f^{-1}(y)$ the derivative map $df_x : T_xM \to T_yN$ is of full rank. For any regular value y of f the set $f^{-1}(y)$ is a submanifold of X].

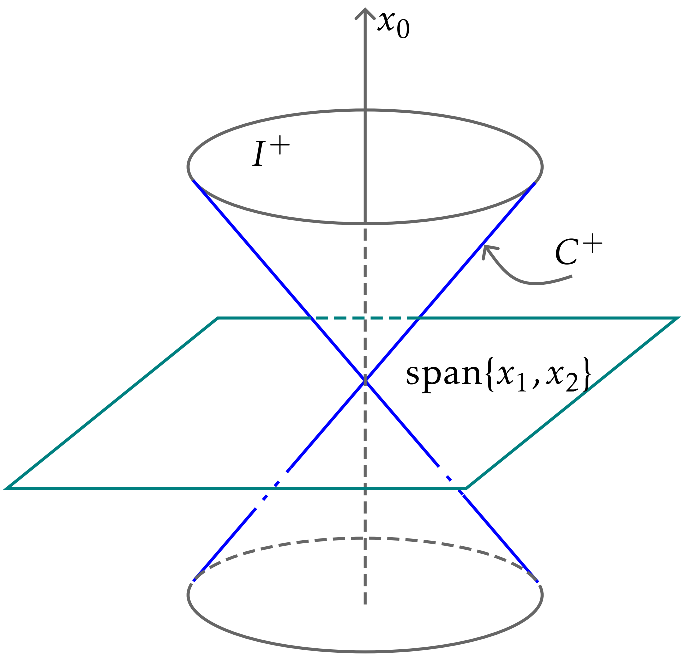


Figure 4.1: Causal Character

Definition 4.1. *For $r > 0$, the hypersurface*

$$S_1^n(r) := f^{-1}(r^2)$$

is called n-dimensional de Sitter space.

Exercise 4.2. Show that as a differentiable manifold $S_1^n(r)$ is diffeomorphic to $\mathbb{R} \times \mathbb{S}^{n-1}$ via the following map. Let $\hat{x} = (x_1, \ldots x_n)$ and $\mathbf{x} = (x, \hat{x})$.

$$S_1^n(r) \to \mathbb{R} \times S^{n-1}, (x_0, \hat{x}) \mapsto \left(x_0, \frac{\hat{x}}{\sqrt{r^2 + x_0^2}} \right).$$

Project: Determine the geodesics of $S_1^n(r)$.

4.2 Causality

In an arbitrary manifold, on each tangent space we have an orientation, but we don't "how to patch this up in a continuous manner" to get an orientation on the manifold. For that we will define time orientation.

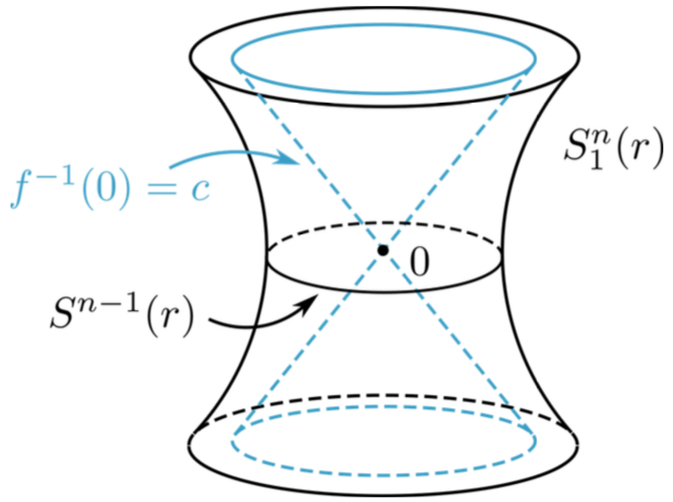


Figure 4.2: de Sitter space

Definition 4.3. *Let M be a Lorentzian manifold and $\mathcal{P}(TM)$ be the subsets of the tangent bundle TM. A* time-orientation *of M is a function*

$$\tau : M \to \mathcal{P}(M), \quad p \mapsto \tau_p$$

such that for all $p \in M$

(i) τ_p is one of the connected components of the the of timelike vectors;

(ii) there exists chart $(\phi = (x_0, \ldots, x_n), U)$ of p such that the vector field $\frac{\partial}{\partial x_0}(q) \in \tau_q$ for any $q \in U$.

We call the pair (M, τ) a time-oriented Lorentzian manifold *and M* time-orientable *if it admits a time orientation.*

Proposition 4.4. *For a Lorentzian manifold (M, g) the following are equivalent.*

(i) M admits a smooth timelike vector field.

(ii) M admits a continuous timelike vector field.

(iii) M is time-orientable.

Proof. (i) $\implies$ (ii) Clear.

(ii) $\implies$ (iii). Let X be a continuous timelike vector field, that is, $X : M \to TM$ is a continuous map such that $g(X(p), X(p)) < 0$ for any $p \in M$. We need to define a time-orientation τ. Define $\tau(p)$ as the connected component containing

$X(p)$. Choose a chart $(\phi = (x_0, \dots, x_n), U)$ at p such that $\frac{\partial}{\partial x_0}$ is timelike on U and $\frac{\partial}{\partial x_0}(p) \in \tau(p)$. Since $\frac{\partial}{\partial x_0}$ and $X(p)$ are timelike vectors lying in the same connected component, so by Proposition 1.22, we have $g\left(X(p), \frac{\partial}{\partial x_0}(p)\right) < 0$. By shrinking U, if needed, and using the continuity of X and $\frac{\partial}{\partial x_0}$, we conclude that

$$g\left(X(q), \frac{\partial}{\partial x_0}(q)\right) < 0 \quad \forall\, q \in U,$$

which implies $\frac{\partial}{\partial x_0}(q) \in \tau(q)$ for any $q \in U$. Thus τ is a time-orientation and hence M is time-orientable.

(iii) $\Longrightarrow$ (i). Let τ be a time-orientation of M and $\mathcal{U} = \{(\phi_\alpha = (x_0^\alpha, \dots, x_n^\alpha), U_\alpha) : \alpha \in A\}$ be a covering of M by charts such that $\frac{\partial}{\partial x_0^\alpha}(q) \in \tau(q)$ for $q \in U_\alpha$ and $\alpha \in A$. Let $\{f_\alpha : \alpha \in A\}$ be a smooth partition of unity subordinate to the covering $\mathcal{U}$. Set $X = \sum_{\alpha \in A} f_\alpha \frac{\partial}{\partial x_0^\alpha}$. Then

- $X \in \mathfrak{X}(M)$, and
- using the convexity of time cones, for any $p \in M$,

$$X(p) = \sum_{\alpha \in A} f_\alpha(p) \frac{\partial}{\partial x_0^\alpha}(p) \in \tau(p).$$

  Hence, X is timelike.

□

Example 4.5. (i) $\mathbb{R}_1^{n+1}$ is time-orientable, one choice for the time-orientation is $\frac{\partial}{\partial x_0} \in \mathfrak{X}(\mathbb{R}_1^{n+1})$.

(ii) $S_1^n(r)$ is time-orientable. If $\frac{\partial}{\partial x_0} = \partial_0$ is the timelike natural coordinate vector field on $\mathbb{R}_1^{n+1}$, then $X = \tan \partial_0$ is a timelike vector field on S_1^n.

Remark. Unlike orientability, time-orientability not only depends on the topological space M but also on the metric g. Moreover, there is no connection between both orientability properties as the following examples show (see Figure 4.3).

Exercise 4.6. Consider the infinite cylinder $\mathbb{R} \times S^1$ and take as coordinates (x, θ) where $\theta \sim \theta + 2\pi$.

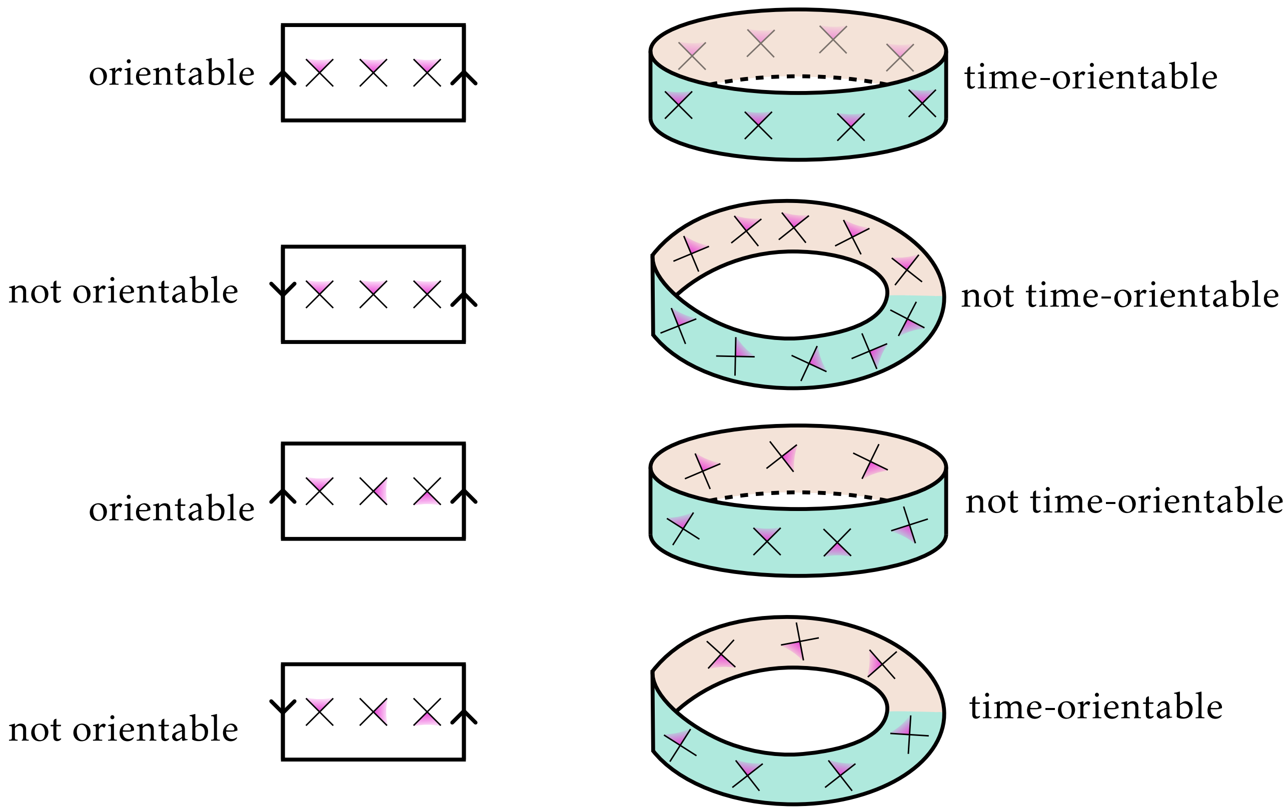


Figure 4.3: timeOrientability

(i) Show that when M is equipped with the Lorentzian metric

$$g = -\mathrm{d}x^2 + \mathrm{d}\theta^2,$$

is time orientable.

(ii) Now consider the metric

$$g = -\cos\theta \mathrm{d}x^2 + 2\sin\theta \mathrm{d}x\mathrm{d}\theta + \cos\theta \mathrm{d}\theta^2.$$

(a) Show that the vector fields defined for $\theta \in [0, 2\pi)$ by

$$X_0 = \cos\frac{\theta}{2}\partial_x - \sin\frac{\theta}{2}\partial_\theta, \quad X_1 = \sin\frac{\theta}{2}\partial_x + \cos\frac{\theta}{2}\partial_\theta,$$

satisfy

$$g(X_0, X_0) = -1, \quad g(X_0, X_1) = 0, \quad g(X_1, X_1) = 1.$$

Deduce that g is a Lorentzian metric.

(b) Let us denote the point $x = 0, \theta = 0$ by p. Suppose that there exists a nowhere vanishing timelike vector field T, and without loss of generality assume that $g(X_0(p), T(p)) < 0$. Show that if $\gamma : [0,1] \to \mathbb{R} \times [0, 2\pi)$ is any smooth curve with $\gamma(0) = p$, then $g(X_0, T)\big|_\gamma < 0$.

(c) By considering the curve $\gamma : s \mapsto (0, 2\pi s)$, deduce that M is not time-orientable.

Let (M, g) be a Lorentzian manifold which is not time-orientable. We will construct a time-orientable Lorentzian manifold $\tilde{M}$ and a map $\pi : \tilde{M} \to M$ such that π is a two sheeted covering map.

Fix a base point $p \in M$. Give a time-orientation to T_pM by choosing a timelike vector $\mathbf{v}_0 \in T_pM$ and defining a nonspacelike $\mathbf{w} \in T_pM$ to be future (or past) directed if $g(\mathbf{v}_0, \mathbf{w}) < 0$ (or $g(\mathbf{v}_0, \mathbf{w}) > 0$). Let $q \in M$. Define the path space

$$\Omega_{pq} = \{\gamma : [0,1] \to M \mid \gamma \text{ is piecewise smooth}, \gamma(0) = p, \gamma(1) = q\}.$$

The above set can be divided into two equivalence classes as follows: given $\gamma_1, \gamma_2 \in \Omega_{pq}$, let V_1 (respectively, V_2) be the unique parallel vector field along γ_1 (respectively, γ_2) with $V_1(0) = V_2(0) = \mathbf{v}_0$. We say that γ_1 is equivalent to γ_2 if $g(V_1(1), V_2(1)) < 0$. Let $[\gamma]$ denote the equivalence class of γ. Let $\tilde{M}$ consists of all such equivalence classes of piecewise smooth curves $\gamma : [0,1] \to M$ with $\gamma(0) = p$. Define $\pi : \tilde{M} \to M$ by $\pi([\gamma]) = \gamma(1)$. If (M, g) is time orientable, then $M = \tilde{M}$, otherwise $\pi : \tilde{M} \to M$ is a two-sheeted covering (see [BE96, Theorem 3.3]).

Note. (i) From now on (M, τ) always be connected and time-oriented Lorentzian manifold, which we will simply denote by M. Such Lorentzian manifolds are called *sapcetimes*.

(ii) A curve γ in M is always considered to be continuous and piecewise smooth.

(iii) We call a causal curve *future* or *past* directed if for all t, $\gamma'(t) \in \overline{\tau_{\gamma(t)}}$ or $\gamma'(t) \in -\overline{\tau_{\gamma(t)}}$.

For any $p, q \in M$, we will use the following notations:

- $p \ll q$ if there exists a future directed timelike curve joining p to q;
- $p < q$ if there exists a future directed causal curve joining p to q;
- $p \leqslant q$ if either $p = q$ or $p < q$.
- $I^+(p) = \{q \in M : p \ll q\}$ = *Chronological future of p*;
- $J^+(p) = \{q \in M : p \leqslant q\}$ = *Causal future of p*.
- Analogously, one can define $I^-(p)$ and $J^-(p)$ which will be called *chronological past of p* and *causal past of p*, respectively.
- For a subset $A \subseteq M$, we define

$$I^{\pm}(A) := \bigcup_{p \in A} I^{\pm}(p) \quad \text{and} \quad J^{\pm}(A) = \bigcup_{p \in A} J^{\pm}(p).$$

For example, take $M = \mathbb{R}^2_1$, then the chronological past/future and causal past and future are shown in Figure 4.4.

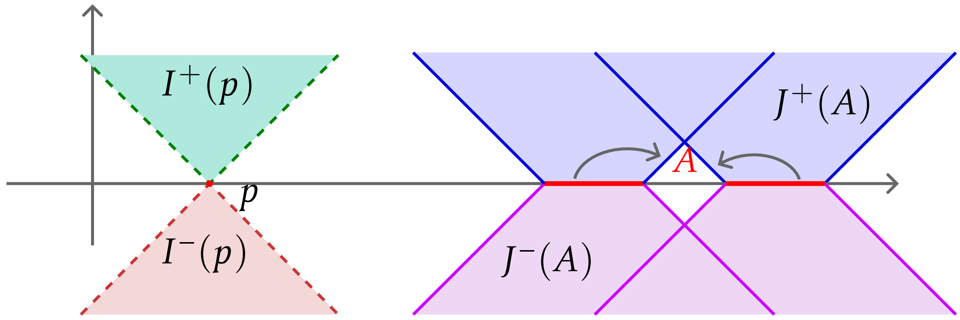


Figure 4.4: chronological future of p and causal past of A

If $M = \mathbb{R}^2_1/\mathbb{Z}e_0 \cong (S^1 \times \mathbb{R})$ with the same metric described in Exercise 4.6, that is $-dx^2 + d\theta^2$. Then due to closed curves, $I^+(p) = M = I^- = J^+ = J^-$.

Lecture–9

Remark. Since one can concatenate piecewise C^∞ curves, we have

$$p \leqslant q \text{ and } q \leqslant r \implies p \leqslant r$$
$$p \ll q \text{ and } q \ll r \implies p \ll r.$$

We even have a stronger form of this transitivity.

Proposition 4.7. *In a spacetime (M,g) for all $p,q,r \in M$, we have*

(i) $p \ll q$ and $q \leqslant r \implies p \ll r$

(ii) $p \leqslant q$ and $q \ll r \implies p \ll r$.

The following proof is based on [Pen72, Proposition 2.18]. A simple region N is simply convex open subset of a spacetime (M,g) such that $\bar{N}$ is compact and is contained in simply convex open set.

Lemma 4.8. *[Pen72, Lemma 2.16] Let N be a simple region and let $p,q,r \in \bar{N}$ be such that γ_{pq} and γ_{qr} be future directed causal curves joining p to q and q to r, respectively. Assume either of the following two conditions holds:*

(i) Both γ_{pq} and γ_{qr} are null, but their directions at the point q are not the same;

(ii) There exists at least one timelike curve from p to r in $\bar{N}$.

Then the curve γ_{pr} is a future directed timelike curve joining p to r.

Proof of Proposition 4.7. We will prove (i) and the the other is similar. If $q = r$, then it is done. So let us assume that $q \neq r$. Let γ_{pq} be future directed timelike curve joining p to q and γ_{qr} be future directed causal curve joining q to r. The issue here is that γ_{qr} may be lightlike at times, while we want to prove that there is always a curve that is not lightlike at any point (except possibly the endpoint). Since any compact subset of M can be covered by finitely many simple regions, so we can cover the image of γ_{qr} by finitely many simple covers, say $N_1, N_2, \ldots, N_k$. Set $x_0 = p \in N_{i_0}$. Let x_1 be the future endpoint of $\gamma_{qr} \cap \overline{N}_{i_0}$ from x_0. Choose $y_1 \in N_{i_0} \cap \gamma_{qr}$ with $y_1 \neq x_0$ (see Figure 4.5). Thus, we have two future directed causal curves $\gamma_{y_1x_0}, \gamma_{x_0x_1}$ and thus using Lemma 4.8, the curve $\gamma_{y_1x_1}$ is future directed timelike curve. If $x_1 = r$, then we are done or $x_1 \notin N_{i_0}$, whence $x_1 \in N_{i_1}$, say.

In the latter case, let x_2 be the future endpoint of $\gamma_{qr} \cap \overline{N}_{i_1}$ from x_1. Choose $y_2 \in N_{i_1}$ on $\gamma_{y_1x_1}$ with $y_2 \neq x_1$. Then either $x_2 = r$, in which case we are done, or we can repeat the argument. The process must eventually terminate, for the cover is finite. □

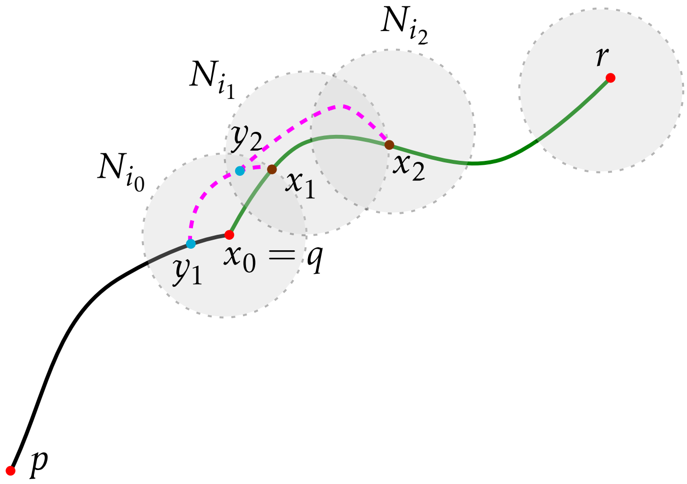


Figure 4.5: The construction of timelike curve joining p and r

Now we will proceed to the topology of the sets $I^{\pm}(p)$ and $J^{\pm}(p)$. Before that we have an easy but important result which can be proved using Gauss lemma (Theorem 3.17).

Lemma 4.9. *Let (M, g) be a spacetime and $p \in M$. Let $\gamma : [0, b] \to T_pM$ be a curve with $\gamma(0) = 0$ such that it is entirely contained in the domain of $\exp_p$. If $\eta = \exp_p(\gamma) : [0, b] \to M$ is a future directed timelike curve, then $\gamma(t) \in I^+(0) \subseteq T_pM$ for all $t \in (0, b]$.*

As an application to the above lemma, we have the following corollary:

Corollary 4.10. *Let $p \in M$ and U be a normal neighbourhood of p such that $\exp_p : \tilde{U} \to U$ is a diffeomorphism. Then*

$$\exp_p\left(I^{\pm}(0) \cap \tilde{U}\right) = I^{\pm}(p) \cap U \text{ and } \exp_p\left(J^{\pm}(0) \cap \tilde{U}\right) = J^{\pm}(p) \cap U.$$

We will be writing $I^{\pm}(p) \cap U =: I^{\pm}_U(p)$ and similarly for J.

Proposition 4.11. *Let (M, g) be a spacetime. Then the relation '$\ll$' is an open relation. That is, $p \ll q$ implies there exists open neighborhoods $U_p \ni p$ and $V_p \ni q$ such that for any $p' \in U_p$ and $q' \in U_q$ we have $p' \ll q'$.*

Proof. Let $q \in I^+(p)$. This means there exists a future directed timelike curve $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$ to $\gamma(1) = q$. Choose $\epsilon > 0$ such that there exists an open set Ω such that $\gamma[0, \epsilon] \subseteq \Omega$. Set $\tilde{p} = \gamma(\epsilon) \in \Omega$. Shrink Ω if possible so that $\exp_{\tilde{p}} : \Omega' \to \Omega$ is a diffeomorphism for some star shaped neighborhood Ω' of 0 in

$T_{\tilde{p}}M$. Now set $U_p = I^-_{\Omega}(\tilde{p})$. Similarly choose $\tilde{q}$ and U_q. Now note that for any $p' \in U_p$ and $q' \in U_q = I^-_{\Omega'}(\tilde{q})$, (see Figure 4.6) we have

$$p' \ll \tilde{p} \ll p \ll q \ll \tilde{q} \ll q'.$$

□

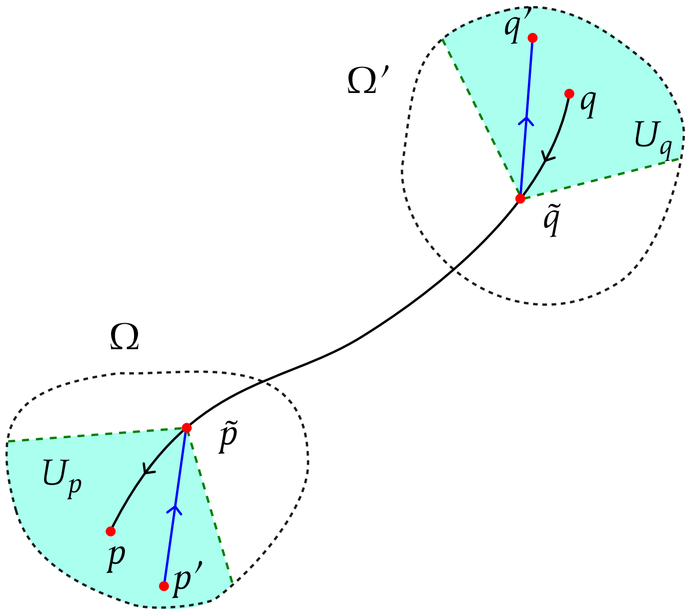


Figure 4.6: The relation $\ll$ is an open relation

Remark. 1. Since for any $A \subseteq M$, the chronological past/future $I^{\pm}(A) = \cup_a I^{\pm}(a)$, and union of open sets is open, so $I^{\pm}(A)$ is open.

2. What about $J^{\pm}(A)$? Is it closed? Not necessarily, for example, consider $M = \mathbb{R}^2 \setminus \{(1,1)\}$. Then $J^+(0)$ is neither open nor closed.

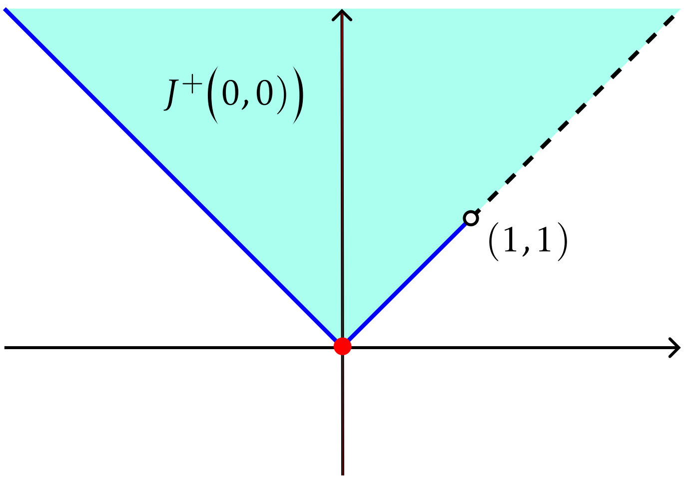


Figure 4.7: $J^+(0,0)$ is not closed

Proposition 4.12. *Let $A \subseteq M$. Then*

(i) $I^{\pm}(A) = \operatorname{int}^{\pm}(A)$, where $\operatorname{int}(U)$ denotes the interior of the set U.

(ii) $J^{\pm}(A) \subseteq \overline{I^{+}(A)}$, with equality if and only if $J^{+}(A)$ is closed.

Proof. We will show for the future events and the past will similarly follows.

(i) $I^{\pm}(A) \subseteq \operatorname{int}(J^{+}(A))$.

Since $I^{+}(A) \subseteq J^{+}(A) \implies \operatorname{int}(I^{+}(A)) \subseteq \operatorname{int}(J^{+}(A))$. Since $I^{+}(A)$ is open, we have $I^{+}(A) \subseteq J^{+}(A)$.

$\operatorname{int}(J^{+}(A)) \subseteq I^{+}(A)$.

Let $q \in \operatorname{int}(J^{+}(A))$. Then there exists an open set U_q containing q such that $x \in J^{+}(A)$ for any $x \in U_q$. We now choose $p \in U_q \cap I^{-}(q)$.

> Why can we choose such a p? Take a past directed timelike vector $\mathbf{v}$ at q and consider the geodesic $\gamma_{\mathbf{v}} : [0,1] \to M$ such that $\gamma_{\mathbf{v}}(0) = q$. By the continuity of γ, there exists $\epsilon > 0$ such that $\gamma[0,\epsilon] \subseteq U_q$ and hence $p := \gamma(\epsilon) \neq q$ is in the chronological past of q.

Therefore, there exists $a \in A$ such that

$$a \leqslant p \ll q \overset{\text{Proposition 4.7}}{\implies} a \ll q \implies \operatorname{int}(J^{+}(A)) \subseteq I^{+}(A).$$

(ii) Exercise.

□

The following exercise is the summary of the above results for chronological and causal future of any point.

Exercise 4.13. Let (M,g) be a time-oriented spacetime and $p \in M$. Show that:

(a) $I^{+}(p)$ is open;

(b) $J^{+}(p)$ is not necessarily closed;

(c) $J^{+}(p) \subset \overline{I^{+}(p)}$;

(d) $I^{+}(p) = \operatorname{int}(J^{+}(p))$;

(e) If $r \in J^{+}(p)$ and $q \in I^{+}(r)$, then $q \in I^{+}(p)$;

(f) If $r \in I^{+}(p)$ and $q \in J^{+}(r)$, then $q \in I^{+}(p)$;

(g) It may happen that $I^+(p) = M$.

Proposition 4.14. *Any compact spacetime contains a closed timelike curve.*

Proof. Let (M,g) be compact space time. Since for every $p \in M$, the chronological future of p, $I^+(p)$ is an open set. Thus, $\{I^+(p) : p \in M\}$ forms an open cover for M. Due to compactness, it must have a finite subcover. Let

$$M = \bigcup_{i=1}^{n} I^+(p_i).$$

Also without loss of generality, we assume that $I^+(p_i) \subsetneq I^+(p_j)$ for $i \neq j$ (otherwise remove $I^+(p_i)$). Now we claim that $p_i \in I^+(p_i)$. If $p_1 \in I^+(p_i)$ for some $i \geq 2$, then for any $q \in I^+(p_1)$, we have $p_i \ll p_1 \ll q$ and thus $I^+(p_1) \subseteq I^+(p_i)$, which is a contradiction. Thus, $p_1 \in I^+(p_1)$, which means p_1 can be joined with p_1 by a future directed timelike curve which is closed. □

Definition 4.15. *A spacetime M is called*

- chronological, *if there do not exist closed timelike curves;*
- causal, *if there do not exist closed casual curves;*
- strongly causal *if for every $p \in M$ and for every neighborhood U of p there exists a neighborhood V such that any causal curve starting and ending in V has to remain in U.*

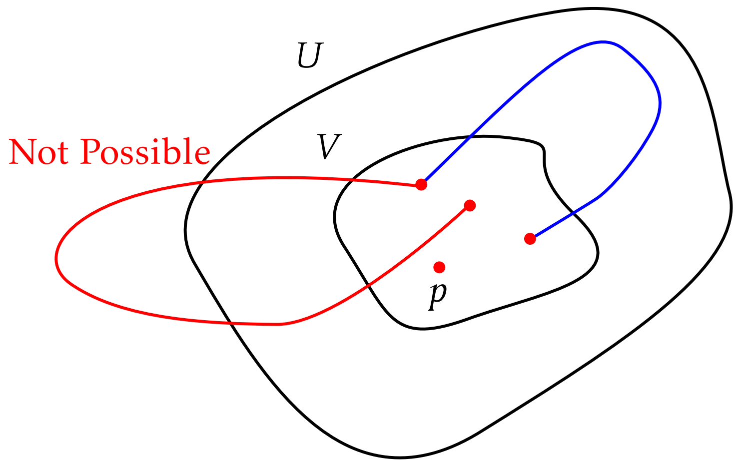


Figure 4.8: Strongly causal space

Remark. For any spacetime M

$$\text{strongly causal} \implies \text{causal} \implies \text{chronological}$$

however,

$$\text{chronological} \not\Longrightarrow \text{causal} \not\Longrightarrow \text{strongly causal}.$$

To see the above let

1. $M = \mathbb{R}^2_1/\mathbb{Z}\cdot(1,-1)$ with metric and time-orientation induced by the Minkowski space $\mathbb{R}^2_1$. This is chronological but not causal since the ray $t. \mapsto t\cdot(1,-1)$ yields a closed lightlike curve.

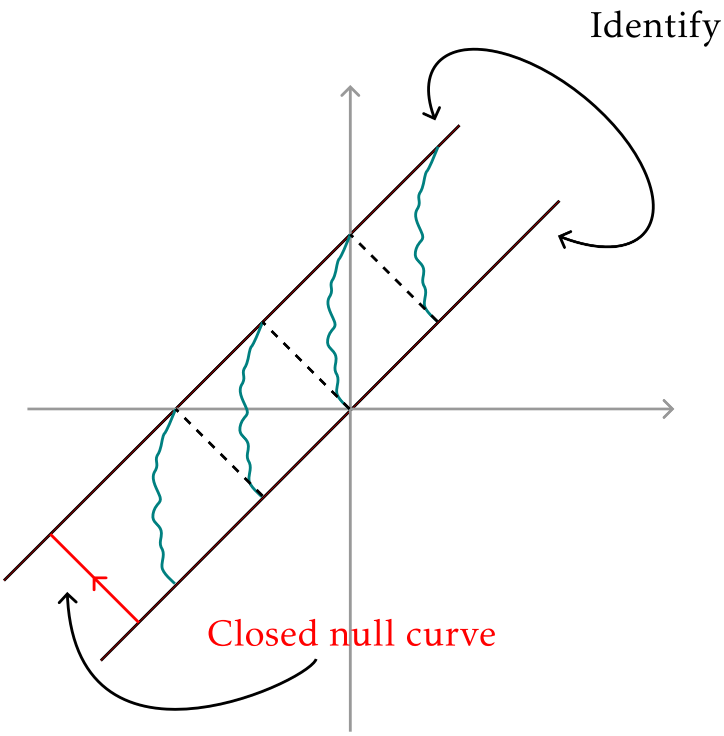


Figure 4.9: Chronological does not imply causal

2. Take $M = \left\{\mathbb{R}^2/\mathbb{Z}\cdot(1,0)\right\} \setminus (G_1 \cup G_2)$, where $G_1 = \left\{\left(\frac{1}{8}, s\right) : s \geq -\frac{1}{8}\right\}$ and $G_2 = \left\{\left(-\frac{1}{8}, s\right) : s \leq \frac{1}{8}\right\}$. Then M is causal but not strongly causal.

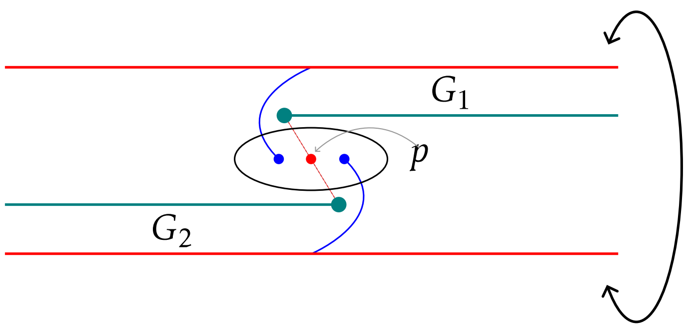


Figure 4.10: Causal does not imply strongly causal

Lecture–10

4.3 Lorentzian Distance

We now introduce the analogue to the Riemannian distance function.

Definition 4.16. *The* length of a curve $\gamma : [a,b] \to M$ *is defined as*

$$L(\gamma) := \int_a^b \sqrt{\left|g\left(\dot{\gamma}(t),\dot{\gamma}(t)\right)\right|}\,\mathrm{d}t$$

Remark. Since for lightlike (null) curves $g\left(\dot{\gamma}(t),\dot{\gamma}(t)\right) = 0$, so the length of any lightlike curve is zero.

Definition 4.17. *For $p,q \in M$, the function*

$$d(p,q) := \begin{cases} \sup\{L(\gamma) : \gamma \textit{ is future directed causal curve from } p \textit{ to } q\}, & \textit{if } q \in J^+(p); \\ 0, & \textit{if } q \notin J^+(p). \end{cases}$$

is called time difference between p and q *or* lorentzian distance between p and q.

Remark. 1. From the definition it is clear that

$$d(p,q) > 0 \iff q \in I^+(p).$$

2. The Lorentzian distance function is not symmetric.

3. In Lorentzian cylinder $\mathbb{R}^2_1/\mathbb{Z}$, we have $d(p,q) = \infty$ for any $p,q \in M$.

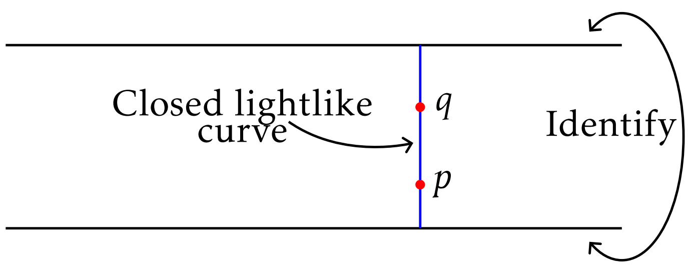


Figure 4.11: $d(p,q) = \infty$

Example 4.18 (Minkowski space)**.** For $p < q$,

$$d(p,q) = \sqrt{|\langle\langle p-q, p-q\rangle\rangle|}.$$

Indeed, the curve

$$g(t) = qt + (1-t)p, \quad 0 \le t \le 1,$$

is a future directed causal curve from p to q and we have

$$d(p,q) \geq L(\gamma) = \int_0^1 |\langle\langle q-p, q-p\rangle\rangle|^{\frac{1}{2}}\, \mathrm{d}t$$
$$= \sqrt{|\langle\langle p-q, p-q\rangle\rangle|}.$$

If $p-q$ is lightlike, then without loss of generality let $p=(0,0,\dots,0)$ and $q \in C^+(0)$. Then $\sqrt{|\langle\langle p-q,p-q\rangle\rangle|} = 0$. Then all causal curves connecting p and q are necessarily lightlike and hence of length 0, that is $d(p,q)=0$.

If $p-q$ is timelike, then without loss of generality we can assume that $p=(0,0,\dots,0)$ and $q=(T,0,0,\dots,0)$ for some $T>0$. Let γ be future directed causal curve from p to q. Then $\dot{\gamma}_0 > 0$. After reparametrization, $\dot{\gamma}(t) = t$ and so $\gamma(t) = (t, \hat{\gamma}(t))$ for some $\hat{\gamma} : [0,T] \to \mathbb{R}^n$. Then,

$$L(\gamma) = \int_0^T \left|\langle\langle (1,\dot{\hat{\gamma}}), (1,\dot{\hat{\gamma}})\rangle\rangle\right|^{\frac{1}{2}}\, \mathrm{d}t$$
$$= \int_0^T \sqrt{1 - \left\|\dot{\hat{\gamma}}\right\|^2}\mathrm{d}t$$
$$\leq \int_0^T 1\mathrm{d}t = T = \sqrt{|\langle\langle p-q, p-q\rangle\rangle|}.$$

Proposition 4.19. *In a spacetime (M,g) we have*

1. $d(p,q) > 0$ *if and only if* $q \in I^+(p)$.
2. *For* $q \in J^+(p)$ *and* $r \in J^+(q)$, *we have the reverse triangle inequality*
$$d(p,q) + d(q,r) \leq d(p,r).$$
3. $d : M \times M \to \mathbb{R}$ *is lower semicontinuous, that is, for any* $p,q \in M$
$$\forall\ \epsilon > 0 \Big(\exists\ U_p, V_q \Big(d(p',q') > d(p,q) - \epsilon \quad \forall\ p' \in U_p \text{ and } q' \in V_q\Big)\Big).$$

1. Add the example for d being not continuous.

5 Appendix

Let M is a smooth manifold and $p \in M$. Let M has a chart $(\phi = (x_1, x_2, \ldots, x_n), U)$, that is $x_i : U \to \mathbb{R}$. One of the charts that will be used for sphere is the following:

$$U_\pm = S^n \setminus \{\pm(1,0,0,\ldots,0)\}$$
$$\varphi_\pm : U_\pm \to \mathbb{R}^n, (x_0, x_1, \ldots, x_n) \mapsto \frac{1}{\pm 1 - x_0}(x_1, \ldots, x_n).$$

The charts are (ϕ_+, U_+) and (ϕ_-, U_-).

➲ **Tangent Space:** The tangent space T_pM at p is the vector space of all tangent vectors to M at p. A *tangent vector* is defined via a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$. Two curves γ_1, γ_2 are equivalent if, in a local chart (U, ϕ) around p, their derivatives satisfy $\frac{d}{dt}\phi(\gamma_1(t))\big|_{t=0} = \frac{d}{dt}\phi(\gamma_2(t))\big|_{t=0}$. The tangent space T_pM consists of all such equivalence classes, with vector space operations defined via the chart.

Alternatively, T_pM can be defined as the vector space of all derivations at p. A *derivation at p* is a linear map $D : C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule: $D(fg) = f(p)D(g) + g(p)D(f)$ for all $f, g \in C^\infty(M)$, where $C^\infty(M)$ is the algebra of smooth functions on M. Each derivation corresponds to a tangent vector, and T_pM is the set of all such derivations.

1. $T_p\mathbb{R}^n \cong \mathbb{R}^n$. (Ex: Give a natural isomorphism from $T_p\mathbb{R}^n$ into $\mathbb{R}^n$)
2. $T_p\mathbb{S}^n = \{\mathbf{v} \in \mathbb{R}^{n+1} : p \cdot \mathbf{v} = 0\}$.
3. If $fF : M \to N$ is a smooth map, then it induces a map on the tangent space:

   $$dF_p : T_pM \to T_{F(p)}N,$$

   called the *differnetial of f at p*. Given $\mathbf{v} \in T_pM$, we let $dF_p(\mathbf{v})$ be the derivation at $F(p)$ that acts on $f \in C^\infty(N)$ by the rule $dF_p(\mathbf{v})(f) = \mathbf{v}(f \circ F)$.
4. *The differential of a map between Euclidean spaces.* Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth function and $p \in \mathbb{R}^n$. Let $x_1, \ldots, x_n$ be the coordinates on $\mathbb{R}^n$ and $y_1, \ldots, y_m$ the coordinates on $\mathbb{R}^m$. Then the tangent vectors $\left\{\frac{\partial}{\partial x_1}(p), \ldots, \frac{\partial}{\partial x_n}(p)\right\}$ forms a basis for the tangent space $T_p\mathbb{R}^n$ and $\{\frac{\partial}{\partial y_1}(F(p)), \ldots, \frac{\partial}{\partial y_m}(F(p))\}$ forms a basis for the tangent space $T_{F(p)\mathbb{R}^m}$. The linear map $dF_p = F_* : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ is described by a matrix $[a_{ij}]$ relative to these two bases:

   $$dF_p\left(\frac{\partial}{\partial x_j}(p)\right) = \sum_k a_{jk} \frac{\partial}{\partial y_k}(F(p)).$$

5. *The chain rule.* let $F : N \to M$ and $G : M \to P$ be smooth maps of manifolds, and $p \in N$. The differential are

$$T_pN \xrightarrow{dF_p} T_{F(p)}M \xrightarrow{dG_{F(p)}} T_{G(F(p))}P.$$

Then,

$$d(G \circ F)_p = dg_{F(p)} \circ dF_p.$$

6. *Bases for the tangent space at a point.* If $(\phi = (x_1, \ldots, x_n), U)$ is a chart on M containing p, then the tangent space T_pM has basis $\left\{\frac{\partial}{\partial x_1}(p), \ldots, \frac{\partial}{\partial x_n}(p)\right\}$. To understand this locally, let $(r_1, \ldots, r_n)$ is the standard coordinate chart on $\mathbb{R}^n$. Since $\phi : U \to \mathbb{R}^n$ is a diffeomorphism, the differential

$$d\phi_p : T_pM \to T_{\phi(p)}\mathbb{R}^n$$

is a vector space isomorphism and $d\phi\left(\frac{\partial}{\partial x_i}(p)\right) = \frac{\partial}{\partial r_i}(\phi(p))$. Since $\{\frac{\partial}{\partial r_1}, \ldots, \frac{\partial}{\partial r_n}\}$ is a basis for the tangent space $T_{\phi(p)}\mathbb{R}^n$ and $d\phi_p$ is a vector space isomorphism, $\{\frac{\partial}{\partial x_1}(p), \ldots, \frac{\partial}{\partial x_n}(p)\}$ is a basis for T_pM.

➲ **Tangent Bundle:** The *tangent bundle* of a manifold M is the union of all the tangent spaces of M:

$$TM := \bigcup_{p \in M} T_pM.$$

If M is smooth manifold of dimension n, then TM is a smooth manifold of dimension $2n$.

➲ **Vector bundles and section:** Any surjective map $\pi : E \to M$ of manifolds is said to be *locally trivial of rank r* if

(i) each fiber $\pi^{-1}(p)$ has the structure of a vector space of dimension r;

(ii) for each $p \in M$, there exists an open neighborhood U of p and a fiber-preserving diffeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ such that for any $q \in U$, the restriction

$$\phi\big|_{\pi^{-1}(q)} : \pi^{-1}(q) \to \{q\} \times \mathbb{R}^r$$

is a vector space isomorphism.

A C^∞ *vector bundle of rank r* is a triple (E, M, π) consisting of manifolds E, M and a surjective map $\pi : E \to M$ that is locally trivial of rank r. The tangent bundle is a vector bundle over M. A *section* of a vector bundle $\pi : E \to M$ is a map $s : M \to E$ such that $\pi \circ s = id_M$, the identity map on M.

➲ **Partition of Unity:** A C^∞ *partition of unity* on a manifold is a collection of nonnegative C^∞ functions $\{\chi_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$ such that

(i) the collection of supports, $\{\operatorname{supp}\chi_\alpha\}_{\alpha\in A}$, locally finite,

(ii) $\sum_\alpha \chi_\alpha = 1$.

Given an open cover $\{U_\alpha\}_{\alpha\in A}$ of M, we say that a partition of unity $\{\chi_\alpha\}_{\alpha\in A}$ is *subordinate to the open cover* $\{U_\alpha\}$ if $\operatorname{supp}\chi_\alpha \subseteq U_\alpha$ for every $\alpha \in A$.

- **Vector Field:** A vector field X on a manifold M is the assignment of a tangent vector $X_p \in T_pM$ to each point $p \in M$.More formally, a vector field on M is a section of the tangent bundle TM of M. A vector field is *smooth* if the map $X : M \to TM$ is smooth as a section of the tangent bundle. In a coordinate chart $(\phi = (x_1,\dots,x_n), U)$ on M,

$$X(p) = X_p := \sum_{i=1}^{n} a_i(p)\frac{\partial}{\partial x_i}(p),$$

where $a_i : U \to \mathbb{R}$. A vector field X on U is smooth iff the coefficient functions a_i are all smooth on U. By the derivation definition of the tangent space, given any smooth function f and a vector field X on M, we define Xf to be the function

$$(Xf)(p) = X_p(f) = \sum_i a_i(p)\frac{\partial f}{\partial x_i}, \quad p \in M.$$

The vector field X is smooth for every smooth function f on M, the function Xf is smooth.

- **The Lie Bracket:** Let X and Y be two smooth vector field on an open subset U of a manifold M. We view X and Y as derivations on $C^\infty(U)$. We define their *Lie bracket* $[X,Y]$ at p to be

$$[X,Y]_p f = (X_pY - Y_pX)f, \quad f \in C^\infty(U).$$

- **Differential 1-Forms:** Let M be a smooth manifold and $p \in M$. The *cotangent space* of M at p, denoted by T_p^*M, defined to be the dual space of the tangent space T_pM, that is,

$$T_p^*M = \{f : T_pM \to \mathbb{R} : f \text{ is linear }\}.$$

An element of the cotangent space T_p^*M is called a *covector* at p. Thus, a covector ω_p at p is a linear function $\omega_p : T_pM \to \mathbb{R}$. A *differential 1-form*, or simply a 1-form on M is a function ω that assigns to each point $p \in M$ a covector ω_p at p. If f is a C^∞ real-valued function on a manifold M, its *differential* is defined to be the 1-form df on M such that for any $p \in M$ and $X_p \in T_pM$,

$$(df)_p(X_p) = X_p f.$$

If (ϕ, U) is a coordinate chart on M, then the differentials $dx_1,\dots,dx_n$ are 1-forms on U and the covectors $(dx_1)_p,\dots,(dx_n)_p$ form a basis for the cotangent space T_p^*M dual to the basis $\frac{\partial}{\partial x_1}(p), \ldots, \frac{\partial}{\partial x_n}(p)$ for the tangent space. A local expression for df can be given as

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

Similar to the tangent bundle, we have the *cotangent bundle* T^*M defined as

$$T^*M := \bigcup_{p \in M} T_p^*M.$$

In terms of the cotangent bundle, a 1-form on M is simply a section of the cotangent bundle. In a coordinate chart (ϕ, U),

$$\omega_p = \sum_{i=1}^{n} a_i(p)(dx_i)_p.$$

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