Annals of Mathematics, **108** (1978), 347-372

# Scattering of geodesic fields, I

By HERMAN GLUCK and DAVID SINGER

Dedicated to the memory of our friend, George Cooke

We study here the scattering of geodesic fields on a Riemannian manifold, and introduce an obstruction whose vanishing is necessary and sufficient for deforming the metric so as to deflect one geodesic field into another. This is analogous to the inverse scattering problems of physics and to the design of lenses with prescribed distortion. As an application, we prove that every smooth manifold of dimension $\geqq 2$ can be given a Riemannian metric with a nontriangulable cut locus.

In Part II, we obtain by different methods an explicit inverse scattering theorem for surfaces of revolution. Applying this again to the study of cut loci, we exhibit a closed surface of revolution in three-space which is strictly convex, yet has nontriangulable cut loci from an *open* set of points.

This example shows that triangulability of the cut locus can not in general be achieved by slight perturbation of the reference point, contrary to the suggestion by Thom in [8]. It can however, be achieved by slight perturbation of the metric, as shown by Wall [19].

Announcements of these results have appeared in [4] and [16], and alternate proofs of some of the theorems can be found in [5]. We thank Professors Eugenio Calabi, Richard Hamilton, Albert Nijenhuis, David Tischler and Alan Weinstein for many insights gained in conversation with them. We also thank the National Science Foundation for support.

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0003-486X/78/0108-0002 $01.30

## 1. Introduction

On each geodesic from the point $p$ on the compact Riemannian manifold $M$, the *cut point* is the last point to which the geodesic minimizes distance, and the *cut locus* $C(p)$ is the set of these.

This notion was introduced by Poincaré [14] in 1905 in his study of closed geodesics on convex surfaces, to which he was led by his work on celestial mechanics and the three body problem. The idea was taken up again in much greater detail by J.H.C. Whitehead [21] in 1935 for $n$-dimensional manifolds. At the same time, the cut locus on a two-dimensional manifold was studied by Myers [13]. Starting in the late fifties, these ideas began to play an important role in global differential geometry, especially through the work of Klingenberg [7], [8] and Berger [1]. A good survey is provided by the article of Kobayashi [9].

The question of triangulability of the cut locus goes back to Myers, who proved in [13] that on a compact real analytic surface, the cut locus of any point can be triangulated as a finite graph. Recently this has been extended to analytic manifolds of arbitrary dimension by the work of Buchner [3] and Hironaka [6]. Combining the theses of Buchner [2] and Looijenga [11], Wall [19] has shown that on any smooth compact manifold, there is a residual set of Riemannian metrics for which the cut locus of a point $p$ is triangulable.

So in a "statistical" sense, cut loci are triangulable. By contrast, we prove

THEOREM A. *For any smooth manifold $M$ of dimension $\geqq 2$ and any point $p$ in $M$, there is a Riemannian metric with nontriangulable cut locus $C(p)$.*

We illustrate this in Figure 1. The simplest candidate $C$ for a non-

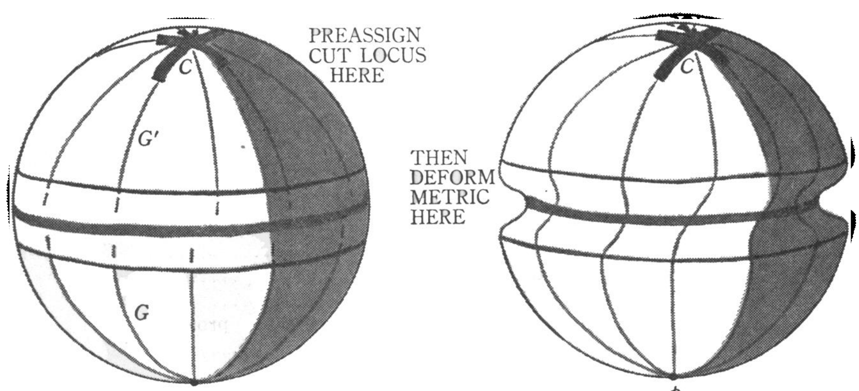


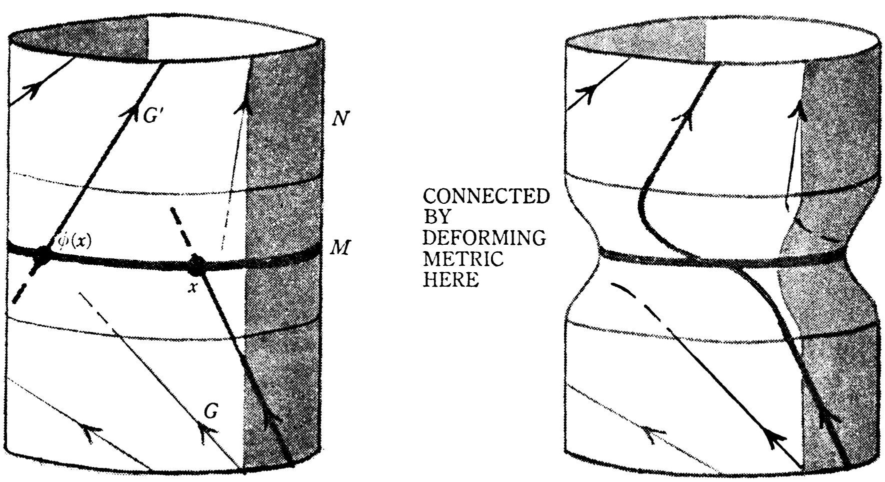

FIGURE 1

triangulable cut locus on a sphere consists of infinitely many arcs, sharing a common endpoint and having lengths approaching zero. Start with the round sphere on the left and try to preassign such a set $C$ as the cut locus of the south pole, $p$. Draw out from $p$ the family $G$ of geodesics heading north and from $C$ a family $G'$ of geodesics heading roughly south. It is appealing to try to deform the round metric in a neighborhood of the equator so as to deflect the geodesic field $G$ into $G'$. Indeed, a similar construction was made by Weinstein [20] in order to turn any disc in a Riemannian manifold into a unit disc about some interior point.

This approach to the problem of preassigning a cut locus on a manifold leads in a natural way to the study of scattering of geodesic fields. Recasting in more general terms, the round sphere has been replaced in Figure 2 by a Riemannian manifold $N$, and the equator by a compact hypersurface $M$ which separates $N$. Two geodesic fields $G$ and $G'$ are given on $N$ (or at least on a neighborhood of $M$), both crossing $M$ transversally. This is shown as "BEFORE" in the figure, while as "AFTER" we show the metric on $N$ deformed in a neighborhood $U$ of $M$ so to produce a field $G''$ of geodesics connecting $G$ below $M$ to $G'$ above. The region $U$, with its deformed metric, can be thought of as a "lens" which "focuses" $G$ into $G'$.

As shown in the "AFTER" portion, for each point $x$ of $M$ there is a unique $y \in M$ such that the geodesic of $G$ which crosses $M$ at $x$ is connected by a geodesic of $G''$ to the geodesic of $G'$ which crosses $M$ at $y$. We thus get a map $\phi\colon M \to M$. We say that $G$ has been *connected to $G'$ according to $\phi$.*

PROBLEM. Given $G$, $G'$ and $\phi$, when is this possible?




BEFORE AFTER

FIGURE 2

Let $V$ denote the unit vector field tangent to $G$ and $\omega$ the dual one-form on $N$ defined by $\omega(U) = U \cdot V$. Similarly define $V'$ and $\omega'$ relative to $G'$, orienting $V'$ so that $G'$ crosses $M$ from the same side as does $G$.

THEOREM 1. *Necessary and sufficient conditions for deforming the metric on $N$ near $M$ so as to produce a field of geodesics connecting $G$ to $G'$ according to $\phi$ are*:

(a) *The map $\phi: M \to M$ is a diffeomorphism, concordant to the identity*; *and*

(b) *The one-form on $M$, $\omega|_M - \phi^*(\omega'|_M)$ is exact.*

We use the phrase "near $M$" to mean "within an arbitrarily small neighborhood of $M$".

In studying these scattering problems, the special case of "integrable" geodesic fields occurs repeatedly and naturally. A field $G$ of geodesics on $N^n$ is *integrable* if the orthogonal $(n-1)$plane distribution is integrable in the usual sense of being tangent to a foliation (which is then a "Riemannian foliation" [15]). In this case $d\omega = 0$, so that $\omega$ determines a cohomology class $[\omega]$ in $H^1(N; R)$. If $i: M \to N$ is the inclusion, we may also consider the restriction $i^*[\omega] = [\omega|_M]$ in $H^1(M; R)$.

Then Theorem 1 simplifies to

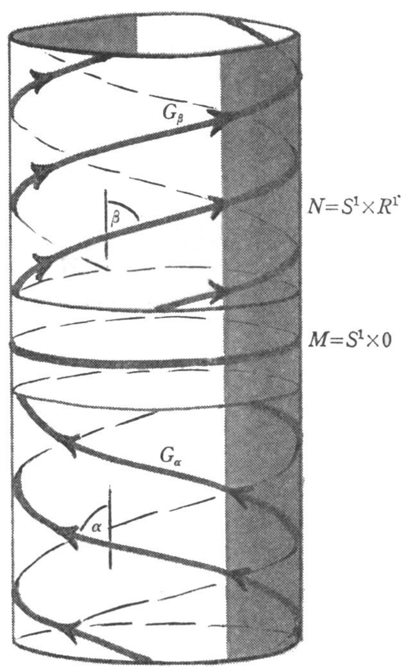



FIGURE 3

THEOREM 2. *Necessary and sufficient conditions for deforming the metric on $N$ near $M$ so as to produce a field of geodesics connecting the integrable geodesic fields $G$ to $G'$ according to $\phi$ are*:

(a) *The map $\phi$ is a diffeomorphism, concordant to the identity*; *and*

(b) $[\omega|_M] = [\omega'|_M]$ *in* $H^1(M; R)$.

A simple illustration of Theorem 2: let $N = S^1 \times R^1$ and $M = S^1 \times 0$. Let $G_\alpha$ be the rising family of helices making an angle $\alpha$ with the vertical, $-\pi/2 < \alpha < \pi/2$. Then we easily conclude from this theorem:

(1) Any rising field of geodesics on a neighborhood of $S^1 \times 0$ can be connected to some $G_\alpha$ by deformation of metric near $S^1 \times 0$.

(2) But $G_\alpha$ can *not* be so connected to $G_\beta$ if $\alpha \neq \beta$. See Figure 3.

Note that in Theorem 2, conditions (a) and (b) are now disconnected, so that if the required deformation exists for one $\phi$ concordant to the identity, then it exists for any such $\phi$. If we weaken Theorem 2 by relinquishing the right to choose $\phi$, then we can strengthen it in another direction.

THEOREM 3. *If condition* (b) *of Theorem* 2 *is satisfied, then for some $\phi$ the deformed metric on $N$ can be chosen pointwise conformal to the original metric.*

Recall that in optics, pointwise conformality is the mathematical counterpart of an isotropic lens. The proof of Theorem 3 is given in the Appendix, because it dovetails better with the alternative proof of Theorem 1 given there.

Theorems 1, 2 and 3 admit relative versions, in which the separating hypersurface $M$ is only required to be closed, not compact. In that case the geodesic fields $G$ and $G'$ agree outside some compact subset of $M$, in a neighborhood of which the metric is to be deformed so as to connect $G$ to $G'$. What happens here is that conditions (a) and (b), suitably relativized, are still necessary but no longer sufficient. A third condition appears, which is easiest to appreciate once we have gone through the proof in the absolute case. So we delay this material until Section 6.

## 2. Two handy lemmas

We will make repeated use of the following simple facts. The first tells how to recognize when a field of curves on a Riemannian manifold consists of geodesics.

LEMMA 2.1. *Let $G$ be a field of curves on a Riemannian manifold $N^n$ and let $V$, $\{g_t\}$ and $\omega$ be the corresponding unit tangent vector field, flow and dual one-form. Let $\Theta$ be the $(n-1)$plane distribution on $N$ orthogonal* *to $G$. Then the following conditions are equivalent*:

(1) *The curves of $G$ are geodesics.*

(2) *The one-form $\omega$ is invariant under the flow $\{g_t\}$, or in other words, $L_V\omega = 0$, where $L_V$ denotes Lie differentiation with respect to $V$.*

(3) *The orthogonal distribution $\mathcal{O}$ is invariant under the flow $\{g_t\}$.*

Conditions (2) and (3) are easily seen to be equivalent, and it is a standard result of differential geometry that they are consequences of condition (1). It is a straightforward exercise to check that such a proof is reversible.

LEMMA 2.2. *Let $G$ be a field of geodesics on a Riemannian manifold $N$, and let $V$, $\{g_t\}$ and $\omega$ be as above. Let $C$: $[0, 1] \to N$ be a smooth curve in $N$, and $t$: $[0, 1] \to R^1$ a smooth function such that the curve $C'(s) = g_{t(s)}(C(s))$ is defined. Then*

$$\int_{C'}\omega = \int_C \omega + t(1) - t(0) .$$

*In particular, if $t(0) = t(1)$ (a natural condition if $C$ is a closed curve), then $\int_{C'}\omega = \int_C\omega$ .*

Using the notation $dC/ds$ for the tangent vector $C_*(\partial/\partial s)$ to the curve $C$, and likewise for $C'$, we have by the "Chain Rule"

$$\frac{dC'}{ds} = g_{t(s)*}\left(\frac{dC}{ds}\right) + \frac{dt}{ds}V_{C'(s)} . \tag{2.3}$$

Hence

$$\begin{aligned}\int_{C'}\omega &= \int_0^1\omega\left(\frac{dC'}{ds}\right)ds = \int_0^1\omega\left(g_{t(s)*}\frac{dC}{ds}\right)ds + \int_0^1\omega\left(\frac{dt}{ds}V_{C'(s)}\right)ds \\ &= \int_0^1\omega\left(\frac{dC}{ds}\right)ds + \int_0^1\frac{dt}{ds}\,ds ,\end{aligned}$$

where the first integral has been simplified by applying Lemma 2.1, and the second simplified because $\omega(V) = 1$.

Therefore

$$\int_{C'}\omega = \int_C\omega + t(1) - t(0) ,$$

proving the lemma.

## 3. Proof of necessity for Theorem 1

We start with a Riemannian manifold $N$ and a compact hypersurface $M$ which separates it into components $A$ and $A'$. On $N$ we have geodesic fields $G$ and $G'$, both crossing $M$ transversally. Furthermore, we are given a map $\phi$: $M \to M$.

We assume that the metric on $N$ can be deformed inside an arbitrarily small neighborhood $U$ of $M$, and a geodesic field $G''$ with respect to the new metric found which agrees with $G$ on $A - U$, agrees with $G'$ on $A' - U$, and connects $G$ to $G'$ according to $\phi$. We want to verify conditions (a) and (b) of Theorem 1.

First we must show that $\phi$ is a diffeomorphism of $M$, concordant to the identity. Recall that $\phi$ is *concordant* to the identity if there exists a diffeomorphism $F: M \times [0, 1] \to M \times [0, 1]$ such that $F(x, 0) = (x, 0)$ and $F(x, 1) = (\phi(x), 1)$.

Let $\{g_t\}$ and $\{g'_t\}$ be the flows associated with the unit vector fields $V$ and $V'$ tangent to $G$ and $G'$. By compactness of $M$, there is a number $\varepsilon > 0$ such that $g_t(x)$ and $g'_t(x)$ are defined for all $x$ in $M$ and all $t$ in $[-\varepsilon, \varepsilon]$. The map $x \mapsto g_{-\varepsilon}(x)$ pushes $M$ down along the geodesics of $G$ to the submanifold $M_-$ of $N$, while the map $x \mapsto g'_\varepsilon(x)$ pushes $M$ up along the geodesics of $G'$ to $M_+$. Since the neighborhood $U$ of $M$ in $N$ is arbitrarily small, we may assume that it lies in the region between $M_-$ and $M_+$.

Notice that if we push $M$ down to $M_-$ along the geodesics of $G$, then up to $M_+$ along the geodesics of $G''$ and finally back down to $M$ along $G'$, then the composite map of $M \to M$ is just $\phi$. This makes it intuitively clear that $\phi$ is a diffeomorphism of $M$, concordant to the identity. The verification is a routine exercise using the fundamental technical lemma about concordance found in Munkres [12, p. 59].

To check condition (b), we must show that the one-form $\omega|_M - \phi^*(\omega'|_M)$ is exact, and we do this by showing that its integral over any closed curve $C$ in $M$ vanishes. Equivalently, we must show that $\int_C \omega = \int_{C'} \omega'$, where $C' = \phi(C)$.

Let $C_-$ be the curve on $M_-$ obtained by pushing $C$ down to $M_-$ along the geodesics of $G$. Push $C_-$ up along $G''$ to get the curve $C_+$ on $M_+$, and push this down along $G'$ to get $C'$ on $M$ again.

Now,

$$\begin{aligned}
\int_C \omega &= \int_{C_-} \omega && \text{by Lemma 2.2, pushing down along } G,\\
&= \int_{C_-} \omega'' && \text{because } \omega = \omega'' \text{ there},\\
&= \int_{C_+} \omega'' && \text{pushing up along } G'',\\
&= \int_{C_+} \omega' && \text{because } \omega'' = \omega' \text{ there},\\
&= \int_{C'} \omega' && \text{pushing down along } G',
\end{aligned}$$

completing the proof of necessity for Theorem 1.

## 4. Proof of sufficiency for Theorem 1

We start as before with the Riemannian manifold $N$ and the compact hypersurface $M$ separating it into components $A$ and $A'$. On $N$ we have geodesic fields $G$ and $G'$, both crossing $M$ transversally. This time we are given a diffeomorphism $\phi: M \to M$ which is concordant to the identity, such that the one-form $\omega|_M - \phi^*(\omega'|_M)$ is exact. We must modify the metric within a preassigned neighborhood $U$ of $M$ and exhibit a field $G''$ of geodesics in this new metric which connects $G$ to $G'$ according to $\phi$.

The proof begins by using the concordance between $\phi$ and $1_M$ to connect $G$ and $G'$ by a family $G''$ of curves which will be geodesics in the yet-to-be-chosen metric on $N$. This metric is then constructed in several steps, during which the exactness of $\omega|_M - \phi^*(\omega'|_M)$ serves as the integrability condition for two overdetermined problems: finding the lengths of the curves of $G''$ and finding their orthogonal subspaces in the new metric.

*Step* 1. *Construction of* $G''$. Let $\{g_t\}$ and $\{g'_t\}$ be the flows associated with the unit vector fields $V$ and $V'$ tangent to $G$ and $G'$. As in the proof of necessity, we invoke the compactness of $M$ to find a number $\varepsilon > 0$ such that $g_t(x)$ and $g'_t(x)$ are defined for all $x$ in $M$ and all $t$ in $[-\varepsilon, \varepsilon]$.

Using these flows, we define two tubular neighborhoods of $M$ in $N$,

$$H \text{ and } H': M \times [-\varepsilon, \varepsilon] \longrightarrow N$$

by

$$H(x, t) = g_t(x) \quad \text{and} \quad H'(x, t) = g'_t(x) .$$

We choose $\varepsilon$ small enough so that both these neighborhoods lie within the preassigned neighborhood $U$ of $M$.

Then, using Munkres' lemma [12, p. 59], $H$ and $H'$ can be pieced together to yield

$$\bar{H}: M \times [-\varepsilon, \varepsilon] \longrightarrow N$$

such that

$$\bar{H}(x, t) = \begin{cases} H(x, t) & \text{for } -\varepsilon \leqq t \leqq -\varepsilon/2 \\ H'(x, t) & \text{for } \varepsilon/2 \leqq t \leqq \varepsilon . \end{cases}$$

Since $\phi$ is concordant to the identity, we can use Munkres' lemma once more to produce a diffeomorphism

$$F: M \times [-\varepsilon, \varepsilon] \longrightarrow M \times [-\varepsilon, \varepsilon]$$

such that

$$F(x,\ t) = \begin{cases} (x,\ t) & \text{for } -\varepsilon \leqq t \leqq -\varepsilon/2 \\ (\phi(x),\ t) & \text{for } \varepsilon/2 \leqq t \leqq \varepsilon\ . \end{cases}$$

Combining these, we get a smooth embedding

$$\bar{H} \circ F : M \times [-\varepsilon,\ \varepsilon] \longrightarrow \text{Image of } \bar{H} \subset N$$

such that

$$\bar{H} \circ F(x,\ t) = \begin{cases} H(x,\ t) & \text{for } -\varepsilon \leqq t \leqq -\varepsilon/2 \\ H'(\phi(x),\ t) & \text{for } \varepsilon/2 \leqq t \leqq \varepsilon\ . \end{cases}$$

The curve of $G$ passing through $g_{-\varepsilon}(x)$ unites with the arc $\bar{H} \circ F\big(x \times [-\varepsilon, \varepsilon]\big)$ and then with the curve of $G'$ passing through $g'_\varepsilon\big(\phi(x)\big)$ to form a curve of $G''$, and in this way the field $G''$ is obtained.

*Step 2. Determination of the length (in the yet-to-be-chosen metric) of each curve of $G''$.* In this step we will specify, for each point $x$ in $M$, the length $L(x)$ of the arc $\bar{H} \circ F\big(x \times [-\varepsilon,\ \varepsilon]\big)$ of $G''$ in the new metric. We will see that the choice of $L(x_0)$ for a single point $x_0$ forces the choice of $L(x)$ for all $x$ in $M$, and that the consistency of this forcing procedure depends precisely on the exactness of the one-form $\omega|_M - \phi^*(\omega'|_M)$.

Thus let $x_0$ be a specified point of $M$ and $C$ a curve on $M$ from $x_0$ to some point $x$. Assume for the moment that the metric on $N$ can indeed be deformed within the preassigned neighborhood $U$ of $M$ so as to make the curves of $G''$ geodesics.

LEMMA 4.1. *Under these conditions, and with the above definition of $L$, we have*

$$L(x) = L(x_0) - \int_C \omega|_M - \phi^*(\omega'|_M)\ .$$

Push $C$ down along the geodesics of $G$ to the curve $C_-$ on $g_{-\varepsilon}(M) = M_-$, then up along $G''$ to the curve $C_+$ on $g'_\varepsilon(M) = M_+$, and finally down along $G'$ to $C' = \phi(C)$ back on $M$. Let $\omega''$ and $\{g''_t\}$ denote the dual one-form and flow corresponding to the unit tangent vector field $V''$ to $G''$ in the new metric. Then note that $C_+(s) = g''_{L(C(s))}\big(C_-(s)\big)$. Applying Lemma 2.2, we get

$$L(x) = L(x_0) + \int_{C_+} \omega'' - \int_{C-} \omega''\ . \tag{4.2}$$

Furthermore, by the already familiar application of Lemma 2.2,

$$\int_{C_+} \omega'' = \int_{C_+} \omega' = \int_{C'} \omega' = \int_C \phi^*(\omega'|_M)\ ,$$

and

$$\int_{C_-} \omega'' = \int_{C_-} \omega = \int_C \omega|_M\ .$$

Making these substitutions in (4.2), we get the lemma.

This shows how $L(x_0)$ "forces" $L(x)$, since $\omega$, $\omega'$ and $\phi$ are known in advance. The consistency of this forcing procedure depends on showing that for any *closed* curve $C$ on $M$, $\int_C \omega|_M - \phi^*(\omega'|_M) = 0$, which follows from the exactness of the integrand. In this way, a choice of $L(x_0)$ leads to a consistent choice of $L(x)$ for all $x$ in $M$. Clearly this function $L$ is smooth.

Any choice for $L$ must be bounded from below by compactness of $M$, so by adding a suitable constant we can (and do) make $L > 0$.

*Note*. It will *not* be possible to do this in the proof of the relative deformation theorems, and a third condition, involving an appropriate inequality, will reveal itself.

We record for future use the differential form of (4.2):

$$\frac{dL}{ds} = V''\cdot\frac{dC_+}{ds} - V''\cdot\frac{dC_-}{ds}\,, \tag{4.3}$$

the "first variation of arc length" formula.

*Step* 3. *Selection of the new metric in the directions tangent to $G''$.* Let

$$H_1\colon M\times[-\varepsilon, \varepsilon] \longrightarrow \{(x, t)\in M\times R\colon |t| \leqq L(x)/2\}$$

be a diffeomorphism which fixes the first coordinate and which is length preserving with respect to the second on neighborhoods of $M\times -\varepsilon$ and $M\times\varepsilon$. Note that we are using the fact that $L > 0$.

Then define

$$H_2 = \bar{H}\circ F\circ H_1^{-1}\colon \{(x, t)\in M\times R\colon |t| \leqq L(x)/2\} \longrightarrow \operatorname{Im}\bar{H}\ .$$

On the neighborhood $\operatorname{Im}\bar{H}$ of $M$ in $N$, $V'' = H_{2*}(\partial/\partial t)$ is a smooth vector field tangent to $G''$ and equal to $V$ near its bottom and to $V'$ near its top. Thus $V''$ can be extended over all of $N$ via $V$ and $V'$.

The new metric in the direction of $G''$ is now specified by requiring that it make $V''$ a *unit* vector field. The curves of $G''$ then acquire lengths in this new metric as designated in the previous step.

*Step* 4. *Selection of the $(n-1)$ plane distribution orthogonal to $G''$.* So far we have the curves $G''$ which are to be geodesics in the new metric, as well as the vector field $V''$ tangent to them which is to be of unit length. Let $\{g_t''\}$ be the corresponding flow.

For any point $x$ on $M$, let $x_- = g_{-\varepsilon}(x)$ on $M_-$ and $x_+ = g_\varepsilon'(\phi(x))$ on $M_+$. In the tangent space to $N$ at $x_-$, let $\mathcal{O}_{x_-}$ be the $(n-1)$ dimensional subspace orthogonal to $G$ in the original metric. The new metric agrees with the old in a neighborhood of $x_-$, and $G'' = G$ there. Hence $\mathcal{O}_{x_-}$ will also be orthogonal to $G''$ in the new metric.

Now if the curves of $G''$ are to be geodesics in the new metric, then the flow $\{g''_t\}$ must carry $\mathcal{O}_{x_-}$ to orthogonal subspaces all along the curve of $G''$ through $x_-$, according to Lemma 2.1. Thus we do not "select" these orthogonal subspaces: what we know so far about the new metric forces them.

Indeed, the process is overdetermined. For the flow $\{g''_t\}$ carries $x_-$ to $x_+$, and here the new metric must also agree with the old, and $G''$ with $G'$, so that $\mathcal{O}_{x_+}$ is likewise known in advance. Thus the consistency of this construction depends on showing that

$$g''_{L(x)*}(\mathcal{O}_{x_-}) = \mathcal{O}_{x_+} \tag{4.4}$$

for any $x$ in $M$.

Since $V''$ is invariant under its own flow, and equals $V$ at $x_-$ and $V'$ at $x_+$, we have

$$g''_{L(x)*} V_{x_-} = V'_{x_+} \,. \tag{4.5}$$

This has two simple consequences:

(1) First, (4.4) is equivalent to

$$(g''_{L(x)*} U_{x_-}) \cdot V'_{x_+} = U_{x_-} \cdot V_{x_-} \tag{4.6}$$

for all tangent vectors $U_{x_-}$ to $N$ at $x_-$, where the inner products are unambiguous since the old and new metrics agree at $x_-$ and at $x_+$.

(2) Second, (4.6) need only be verified for vectors $U_{x_-}$ tangent to $M_-$ at $x_-$, since by (4.5) it is true for $U_{x_-} = V_{x_-}$. But if $U_{x_-}$ is tangent to $M_-$, we may write $U_{x_-} = dC_-/ds$ for some curve $C_-$ on $M_-$. We then compute

$$\begin{aligned} g''_{L(x)*}\left(\frac{dC_-}{ds}\right) &= \frac{dC_+}{ds} - \frac{dL}{ds} V'_{x_+} && \text{by (2.3)} \\ &= \frac{dC_+}{ds} + \left(\frac{dC_-}{ds} \cdot V_{x_-} - \frac{dC_+}{ds} \cdot V'_{x_+}\right) V'_{x_+} && \text{by (4.3)}\,. \end{aligned}$$

Taking the inner product with $V'_{x_+}$ now yields (4.6), and with it, (4.4).

Thus the (n − 1) dimensional subspaces $\mathcal{O}$ which will be orthogonal to the curves of $G''$ in the new metric can be "selected" so that they are invariant under the flow $\{g''_t\}$ and so that they agree with the old orthogonal subspaces where the two metrics agree.

*Step* 5. *Selection of the new metric in the directions orthogonal to* $G''$. On each $(n-1)$ dimensional subspace $\mathcal{O}_x$ orthogonal to $G''$, we just use the old metric, and this completes the description of the new metric on $N$. It agrees with the old metric outside the neighborhood Im $\bar{H}$ of $M$, and hence certainly outside the preassigned neighborhood $U$.

*Step* 6. *Verification that $G''$ is a field of geodesics in the new metric.* We arranged in Step 4 that the $(n-1)$ plane distribution $\Theta$, which is orthogonal to $G''$ in the new metric, be invariant under the corresponding unit speed flow $\{g''_t\}$. By Lemma 2.1, this guarantees that the curves of $G''$ are geodesics in the new metric.

This completes the proof of Theorem 1.

## 5. Integrable geodesic fields

In this section we prove Theorem 2. Let $G$ be a field of curves (not necessarily geodesics) on the Riemannian manifold $N^n$, $V$ a unit vector field tangent to $G$ and $\omega$ the dual one-form. By the Frobenius theorem, the $(n-1)$ plane distribution orthogonal to $G$ will be tangent to a foliation if and only if $d\omega \wedge \omega = 0$. If this condition holds, then in fact $d\omega = \omega \wedge L_V\omega$.

If, in addition, $G$ happens to be a field of geodesics, then by Lemma 2.1, $L_V\omega = 0$. But then $d\omega = 0$, and as remarked in Section 1, an integrable field of geodesics determines in this fashion a cohomology class $[\omega]$ in $H^1(N; R)$, and by restriction to the hypersurface $M$, a cohomology class $[\omega|_M]$ in $H^1(M; R)$.

Note that a geodesic field streaming out orthogonally from a submanifold (of any dimension) of $N$ will be integrable. In particular, the geodesics streaming out from a point form an integrable field, and hence the relevance of this notion to the study of the cut locus.

Let us check that Theorem 2 follows immediately from Theorem 1. If $\omega$ and $\omega'$ are closed forms, then $\omega|_M - \phi^*(\omega'|_M)$ is exact if and only if $[\omega|_M] = \phi^*[\omega'|_M]$ in $H^1(M; R)$. But if $\phi$ is concordant to the identity, then it is certainly homotopic to the identity, in which case $\phi^* = 1$ on the cohomology level. Hence condition (b) of Theorem 1 reduces to $[\omega|_M] = [\omega'|_M]$, that is, condition (b) of Theorem 2.

## 6. Relative scattering theorems

We turn now to formulating and proving relative versions of Theorems 1, 2 and 3. In them, the separating hypersurface $M$ is only required to be closed, not necessarily compact. On $M$ we are given $K$, a compact submanifold-with-boundary, of the same dimension as $M$ and smoothly embedded. The geodesic fields $G$ and $G'$ on $N$ cross $M$ transversally and agree outside $K$; that is, if $x$ is a point of $\overline{M-K}$, then the geodesics of $G$ and $G'$ crossing $M$ at $x$ coincide. The map $\phi\colon M \to M$ is the identity on $\overline{M-K}$.

THEOREM 1′. *Necessary and sufficient conditions for deforming the metric on $N$ in an arbitrarily small neighborhood of $K$ so as to produce a* *field of geodesics connecting $G$ to $G'$ according to $\phi$ are:*

(a) *The map $\phi$: $M \to M$ is a diffeomorphism, concordant to the identity relative to $\overline{M - K}$;*

(b) *The one-form on $M$, $\omega|_M - \phi^*(\omega'|_M)$, is exact* rel $\overline{M - K}$; *and*

(c) *For any curve $C$ on $M$ beginning in $\overline{M - K}$,*

$$\int_C \omega|_M - \phi^*(\omega'|_M) \leqq 0 \ .$$

*If the words "arbitrarily small neighborhood" are replaced by "neighborhood of width $2\varepsilon$", then the integral in* (c) *is only required to be less than $2\varepsilon$.*

*Comments.* (1) We explain the language of the theorem. To obtain a small neighborhood of $K$ in $N$, first choose a small compact neighborhood $W$ of $K$ on $M$. Then there is an $\varepsilon > 0$ such that $g_t(x)$ and $g'_t(x)$ are defined for all $x$ in $W$ and all $t$ in $[-\varepsilon, \varepsilon]$. Using these flows, we define tubular neighborhoods $H$ and $H'$: $W \times [-\varepsilon, \varepsilon] \to N$ as in the absolute case, and mean $H(W \times [-\varepsilon, 0]) \cup H'(W \times [0, \varepsilon])$ as a neighborhood of $K$ of width $2\varepsilon$ in the statement of the theorem. Furthermore, a deformation of metric on this neighborhood will be understood to restrict to the identity on the geodesic arcs of $G = G'$ passing through points of $\overline{W - K}$.

A concordance $F$: $M \times [0, 1] \to M \times [0, 1]$ between the identity and $\phi$ is said to be *relative to* $\overline{M - K}$ if $F$ is the identity on $\overline{M - K} \times [0, 1]$. A one-form on $M$ which vanishes on $\overline{M - K}$ is said to be exact rel $\overline{M - K}$ if it is the differential of a function on $M$ which vanishes on $\overline{M - K}$.

(2) Notice that if $M$ is a compact manifold and we choose $K = M$, then Theorem 1′ reduces to Theorem 1, and in particular, condition (c) disappears.

*Proof of necessity.* With only obvious changes, the argument given in Section 3 that conditions (a) and (b) are necessary in the absolute case works here also. So we turn to condition (c).

Recall Lemma 4.1:

$$L(x) = L(x_0) - \int_C \omega|_M - \phi^*(\omega'|_M) \ .$$

Here we choose $x$ in $W$ and $x_0$ in $W - K$, and $C$ a curve on $W$ connecting $x_0$ to $x$. As before, $L(x)$ denotes the length of the arc of $G''$ running from $g_{-\varepsilon}(x)$ to $g'_\varepsilon(\phi(x))$, measured in the new metric.

But $L(x) > 0$ and $L(x_0) = 2\varepsilon$, so the integral above must be $< 2\varepsilon$. This is under the assumption that the deformation of metric takes place within a tubular neighborhood of $K$ of width $2\varepsilon$. If it can take place within an arbitrarily small neighborhood of $K$, the integral must be $\leqq 0$. Thus condition (c) is indeed necessary.

*Proof of sufficiency.* Again with only obvious changes, the arguments given in the absolute case will carry us as far as Lemma 4.1. But now $L(x_0) = 2\varepsilon$ if $x_0$ is chosen in $W - K$, and we are no longer free to add a constant to $L$ to make it positive. Instead we invoke condition (c). The rest of the proof continues as in the absolute case and we have Theorem 1′.

*Comment.* Think again of the tubular neighborhood of $K$, together with its deformed metric, as the "lens" which "focuses" $G$ into $G'$. In this context, we think of

$$f(x) = \int_C \omega|_M - \phi^*(\omega'|_M)$$

as the "divergent power" required of the lens, measured along any curve $C$ from the "edge" $M - K$ of the lens to $x$. Condition (c) may then be paraphrased as saying: *"the divergent power of a lens is less than the width at its edge".*

In the integrable case, Theorem 1′ simplifies to the following relative version of Theorem 2.

THEOREM 2′. *Necessary and sufficient conditions for deforming the metric on $N$ in an arbitrarily small neighborhood of $K$ so as to produce a field of geodesics connecting the integrable geodesic fields $G$ to $G'$ according to $\phi$ are:*

(a) *Same as in Theorem* 1′;

(b) $[\omega|_M - \omega'|_M] = 0$ *in* $H^1(M, \overline{M - K}; R)$;

(c) *Same as in Theorem* 1′.

*If we use a neighborhood of $K$ of width $2\varepsilon$, then the integral in* (c) *is only required to be less than* $2\varepsilon$.

Condition (b) of Theorem 1′ reduces in the integrable case to $[\omega|_M - \phi^*(\omega'|_M)] = 0$ in $H^1(M, \overline{M - K}; R)$. But if $\phi$ is concordant to the identity rel $\overline{M - K}$, then it is certainly homotopic to the identity rel $\overline{M - K}$, and this implies that $[\phi^*(\omega'|_M) - \omega'|_M] = 0$ in $H^1(M, \overline{M - K}; R)$. Adding, we get condition (b) of Theorem 2′.

Theorem 3 relativizes to

THEOREM 3′. *If conditions* (b) *and* (c) *are satisfied in Theorem* 2′, *then for some $\phi$ the deformed metric on $N$ can be chosen pointwise conformal to the original metric.*

The proof of Theorem 3, given in the appendix, extends easily to the relative case.

## 7. Preassigning cut loci on spheres

In this section we prove

THEOREM 7.1. *The round metric on the sphere $S^n$, $n \geqq 2$, can be deformed in a neighborhood of the equator so as to cause the south pole to have a nontriangulable cut locus. Furthermore, the deformed metric can be chosen pointwise conformal to the original one.*

We begin with an "atomic construction": the round metric on the two-sphere is modified in a neighborhood of the equator so as to cause the south pole to have a cut locus $C$ which is an arc with one end at the north pole. Next we assemble finitely or infinitely many of these constructions to produce more complicated cut loci. Finally we mention the changes necessary to do all this on the $n$-sphere.

The atomic construction is carried out in three steps:

(1) We describe a $C^\infty$ field $G'$ of geodesics on the round two-sphere which coalesces along the preassigned arc $C$.

(2) Comparing $G'$ with the field $G$ of geodesics streaming out from the south pole, we apply Theorem 3 to deform the round metric near the equator so as to connect $G$ to $G'$ by a field $G''$ of geodesics in the new metric.

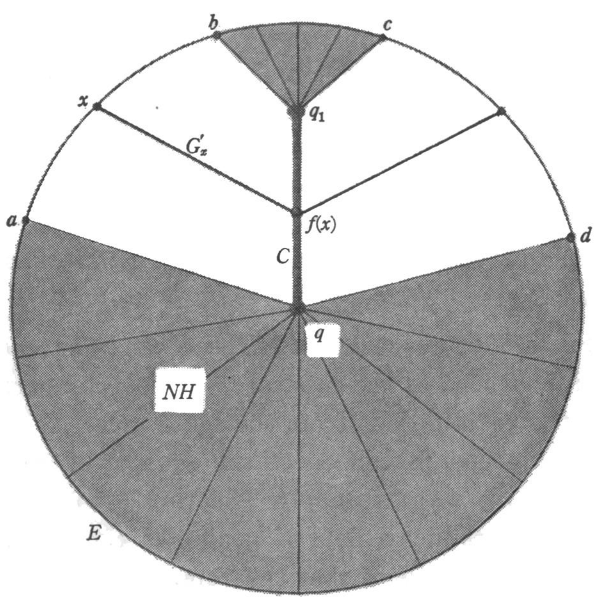



FIGURE 4

(3) We check that in this new metric, $C$ is really the cut locus $C(p)$ of the south pole.

*Step* 1. Let $NH$ denote the northern hemisphere of the two-sphere. We want a field $G'$ of geodesics on $NH - C$ which has the appearance shown in Figure 4. In particular:

(1) Each point $x$ on the equator $E$ is connected by a geodesic $G'_x$ of $G'$ to a corresponding point $f(x)$ on $C$.

(2) For each $x$ on the large arc $da$ of $E$, $f(x)$ is the north pole, $q$.

(3) For each $x$ on the small arc $bc$ of $E$, $f(x)$ is the other end $q_1$ of $C$.

(4) As $x$ moves along $E$ from $a$ to $b$, $f(x)$ moves along $C$ from $q$ to $q_1$.

(5) Similarly, as $x$ moves along $E$ from $d$ to $c$.

(6) The field $G'$ is invariant under reflection in the great semicircle containing $C$.

It is clear that this can be done. In addition, however, we want the field $G'$ to be of class $C^\infty$, and this can be accomplished by making the function $f: E \to C \subset S^2$ a $C^\infty$ function. We use a standard $C^\infty$ function $\alpha: R \to R$ such that $\alpha(x) = 0$ for $x \leqq 0$, $\alpha(x)$ increases monotonically from 0 to 1 as $x$ increases from 0 to 1, and $\alpha(x) = 1$ for $x \geqq 1$. The function $f$ on the arcs $ab$ and $dc$ of $E$ is then obtained from $\alpha$ by linear change of variables.

At each point $x$ of $E$, consider the unit vector tangent to $G'_x$, that is, pointing in the direction of $f(x)$. Because $f$ is a $C^\infty$ function, this is a $C^\infty$ vector field on $E$. Since the curves of $G'$ do not cross on $NH - C$, it follows that $G'$ is a $C^\infty$ field of geodesics there.

*Comment*. Given a point $p$ on the compact Riemannian manifold $M$, there is an analogue of $f$ which maps the unit sphere in the tangent space to $M$ at $p$ onto the cut locus $C(p)$, simply by following the appropriate geodesic from $p$ to $C(p)$. Such an $f$ need not be differentiable, as can be seen easily in the case of a flat torus. We make it differentiable in the above construction simply as a device for deducing the differentiability of the geodesic field $G'$.

*Step* 2. The field $G'$ of geodesics on $NH - C$ can be extended backwards a little over a neighborhood of the equator. We then compare this with the field $G$ of geodesics streaming out from the south pole.

Let $N$ be a tubular neighborhood of the equator $E$, on which both $G$ and $G'$ are defined. We want to apply Theorem 3 to deform the round metric near $E$ so as to produce a field of geodesics connecting $G$ to $G'$. So we must verify the hypotheses of that theorem.

A field $G$ of geodesics streaming out from a point $p$ is always integrable.

Since we are working in dimension 2, we get the integrability of $G'$ as well. The one-form $\omega|_E$ corresponding to $G$ is zero, since the geodesics of $G$ meet $E$ orthogonally. The closed one-form $\omega'|_E$ is not zero, but the symmetry of $G'$ (see item 6 of Step 1) implies that $\int_E \omega' = 0$. Hence the cohomology class $[\omega'|_M] = 0$ in $H^1(E; R)$.

Now Theorem 3 can be applied, and we get a pointwise conformal deformation of metric on $N$ yielding a field of geodesics connecting $G$ to $G'$. This deformed metric agrees with the round metric outside a neighborhood of the equator, and hence extends via the round metric over all of $S^2$.

In this deformed metric on $S^2$, the geodesics emanating from the south pole coalesce along the set $C$, making $C$ a candidate for the cut locus $C(p)$. But it is easy to see from examples that this alone does not guarantee that $C$ is the cut locus of $p$.

*Step* 3. For each $x$ in $E$, let $G''_x$ be the geodesic arc in the deformed metric which begins at $p$, eventually connects to $G'_x$, and ends at $f(x)$ on $C$. In general, $G''_x$ will *not* pass through $x$. Let $L(x)$ denote the length of $G''_x$ in the new metric. From the first variation formula for arc length (4.3), we easily obtain the following properties of $L$:

(1) $L(x)$ is constant along the large arc $da$ of $E$, decreases monotonically as $x$ moves from $a$ to $b$, remains constant as $x$ moves from $b$ to $c$, and then increases monotonically as $x$ moves from $c$ to $d$.

(2) $L$ is invariant under reflection of $E$ in the great semicircle containing $C$.

We want to check now that $C$ is the cut locus $C(p)$ of $p$ in the new metric. First we observe

(7.2) *For each $x$ in $E$, the cut point of $p$ along the geodesic arc $G''_x$ occurs no later than at $f(x)$.*

If $x$ lies on the large arc $da$ or on the small arc $bc$ of $E$, then $f(x)$ is conjugate to $p$ along $G''_x$, and by a standard theorem [9], the cut point can occur no later.

Otherwise, given $x$ in $E$, let $y$ denote its reflection through the great semicircle containing $C$. Then $f(x)=f(y)$ by item 6 of Step 1, and $L(x)=L(y)$ by item 2 above. So the geodesic arcs $G''_x$ and $G''_y$ meet at $f(x) = f(y)$ and have the same length. Hence neither can be minimizing if extended beyond this point, so again the cut point to $p$ along $G''_x$ can occur no later than at $f(x)$.

By construction, we have

(7.3) *The open arcs $G''_x$ between $p$ and $f(x)$ are disjoint from one another.*

Finally,

(7.4) *$C$ is the cut locus $C(p)$ of $p$ in the new metric.*

This follows immediately from (7.2) and (7.3). Suppose for some $x$ in $E$, the cut point of $p$ along $G''_x$ occurs before $f(x)$, say at $r$. Then this geodesic is no longer minimizing beyond $r$. Pick a point $r'$ between $r$ and $f(x)$ on $G''_x$. *Some* minimizing geodesic must go from $p$ to $r'$. But by (7.2), each minimizing geodesic from $p$ is a subset of some $G''_y$. By (7.3), these are all disjoint from the open arc $G''_x$, hence could not possibly hit $r'$.

Hence the cut point of $p$ along $G''_x$ cannot occur before $f(x)$, and since by (7.2) it cannot occur later, it is $f(x)$.

Summarizing so far, we have started with the round metric on $S^2$ and by a pointwise conformal deformation of metric in a neighborhood of the equator, arranged that the cut locus of the south pole be an arc with one end at the north pole. This is the atomic construction referred to at the beginning of the section.

There is no trouble in assembling finitely many such constructions so as to make the cut locus of the south pole a finite union of great circle arcs meeting only at the north pole. Consider therefore how to make the cut locus an *infinite* union of such arcs, say $qq_n$, $1 \leqq n < \infty$, meeting only at the

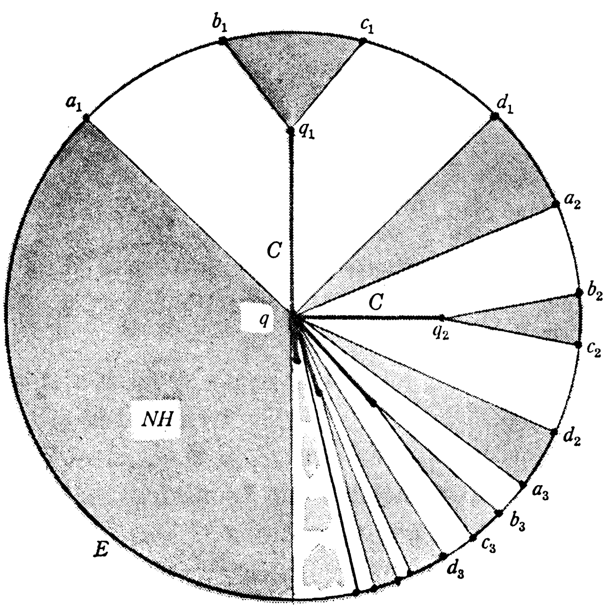



FIGURE 5

north pole $q$ and having lengths $\to 0$ as $n \to \infty$, as illustrated in Figure 5.

The function $f\colon E \to C$ is constructed from the individual functions $f_n\colon E \to C$ by the obvious process of restriction and union. It will clearly be continuous, but unlike the finite case, its $C^\infty$ character is now in doubt. Indeed, whether or not $f$ is $C^\infty$ depends on the relation between the length of the arc $a_n b_n$ of $E$ and the length of the arc $qq_n$ of $C$. We draw these arcs so as to satisfy

(7.5) *The length of the arc $qq_n$ of $C$ is to be the $n^{\text{th}}$ power $(A_n)^n$ of the length $A_n$ of the arc $a_n b_n$ of $E$.*

In such a case, we can parametrize the arc $a_n b_n$ by the interval $[0, A_n]$ of $R$ and the arc $qq_n$ by the interval $[0, A_n^n]$. The function $f_n$ will then be given by the formula

$$f_n(x) = A_n^n \ \alpha(x/A_n) ,$$

where the fixed $C^\infty$ function $\alpha$ was described in Step 1.

Taking $k^{\text{th}}$ derivatives, we have

$$f_n^{(k)}(x) = A_n^{n-k} \ \alpha^{(k)}(x/A_n) .$$

Since the $k^{\text{th}}$ derivative of $\alpha$ is bounded, we see by fixing $k$ and letting $n \to \infty$ that the $k^{\text{th}}$ derivative of $f_n$ approaches 0. Hence the $k^{\text{th}}$ derivative of $f$ approaches 0 as $x$ approaches the point $x_0$ on $E$ where the differentiability of $f$ was in question. It now follows easily that $f$ is of class $C^\infty$.

The rest of the construction parallels the atomic case, and we end up with a metric on $S^2$, obtained via Theorem 3 by pointwise conformal deformation of the round metric in a neighborhood of the equator, in terms of which the cut locus of the south pole is the nontriangulable set $C$ displayed above in Figure 5. In other words, we have proved Theorem 7.1 in the case $n = 2$.

*Comment* 7.6. In this deformed metric on $S^2$, let $\gamma_n$ be a minimizing geodesic from the south pole $p$ to the north pole $q$, crossing the equator anywhere in the arc $d_{n-1}a_n$. It follows easily from Lemma 4.1 that all the $\gamma_n$ have the same length. By contrast, consider the minimizing geodesic from $p$ to $q_n$ which crosses the equator somewhere in the arc $b_n c_n$ and when continued to the north pole via the geodesic arc $qq_n$ has no corner at $q_n$. It will be a nonminimizing geodesic, say $\lambda_n$, from $p$ to $q$. Thus, interleaved with the geodesics $\gamma_n$, all of the same length, are the geodesics $\lambda_n$, all longer. This observation will be used in Part II.

To complete the proof of Theorem 7.1, we must consider the case $n > 2$. Again we begin with an atomic construction, which deforms the round metric on the $n$-sphere $S^n$ so as to make the cut locus of the south pole a great circle arc $C$ with one end at the north pole. The details parallel the two-dimensional case, except that now we want $f: E \to C$ to be invariant under all rotations of $S^n$ which fix the great circle containing $C$. To accomplish this, just rotate the two-dimensional construction about that great circle.

The conditions for the application of Theorem 3 are somewhat different. We do not get the integrability of the geodesic field $G'$ freely anymore. But if we observe the curves orthogonal to $G'$ in the two-dimensional case, then during rotation about $C$ they will trace out hypersurfaces orthogonal to $G'$, so that integrability will not be lost. Since the equator $E$ is now at least two-dimensional, $H^1(E; R) = 0$, and there is no cohomology condition to check. Theorem 3 can then be applied to give the desired deformation of metric, and we obtain a field $G''$ of geodesics in this new metric which stream out from the south pole $p$ and coalesce along the arc $C$. The verification that $C$ is indeed the cut locus $C(p)$ of $p$ is identical to that in the two-dimensional case, which completes the atomic construction.

The assembly of finitely or infinitely many such atomic constructions is no different from before, and in this way Theorem 7.1 is proved in general.

## 8. Proof of Theorem A

We now show how to produce a Riemannian metric with a nontriangulable cut locus on any smooth manifold $M^n$, $n \geqq 2$. The construction is summarized in Figure 6 below.

We start with a round metric on the $n$-sphere $S^n$, and a nontriangulable

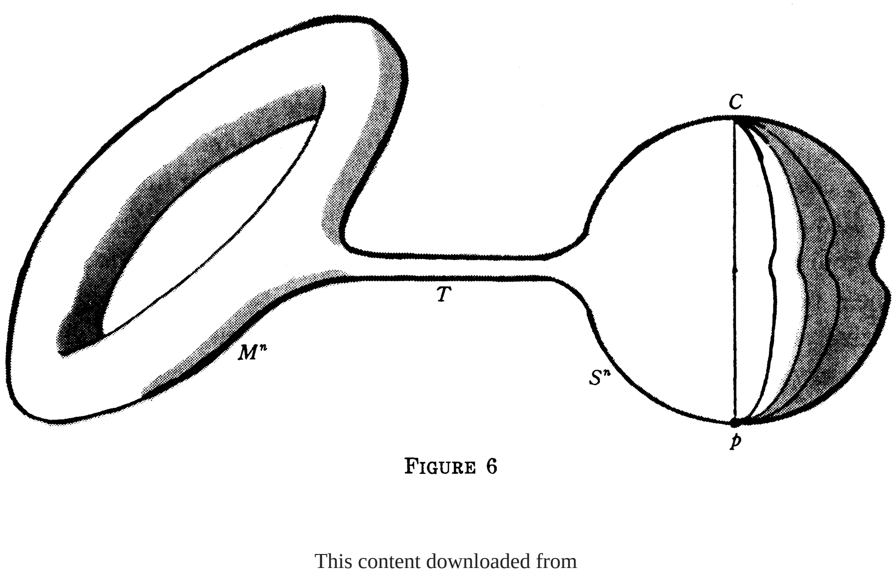



FIGURE 6

set $C$ consisting of infinitely many great circle arcs meeting at the north pole and pointing into the eastern hemisphere. By the results of the previous section, we can deform the round metric in a neighborhood of the equator so as to make $C$ the cut locus $C(p)$ of the south pole $p$. If we apply Theorem 3′ (rather than Theorem 3) we can restrict the deformation of metric to the eastern hemisphere, so that the western hemisphere remains round.

We then attach a copy of the manifold $M^n$ to the western hemisphere of $S^n$ by a long thin tube $T$. We can easily do this so that no path $\gamma$ which starts on $S^n$, enters $T$ and then returns to $S^n$, is shorter than the great circle route between its endpoints. It will follow that a geodesic $\gamma$ on the manifold $M^n \# S^n$ which begins at the south pole $p$, enters and leaves the tube $T$ and then leaves the western hemisphere at a point $r$ distinct from the north pole $q$ must be longer than the great circle route from $p$ to $r$. So it is no longer minimizing and must have passed its cut point.

Let $C_M(p)$ and $C_S(p)$ denote the cut locus of $p$ on $M^n \# S^n$ and on $S^n$ (with the deformed metric), respectively. We have just seen that $C_M(p)$ meets the eastern hemisphere of $S^n$ precisely in the set $C_S(p)$. But then the non-triangulability of $C_S(p)$ certainly implies the same for $C_M(p)$: for example $C_M(p)$ is disconnected into infinitely many components by removal of the north pole $q$.

Thus we have constructed a Riemannian manifold $M^n \# S^n$ which is diffeomorphic to $M^n$ and which contains a point $p$ with nontriangulable cut locus. Since the diffeomorphism with $M^n$ could take $p$ to any point of $M^n$, we have proved Theorem A.

## 9. Appendix: Alternative proof of Theorem 1 and proof of Theorem 3

In this section we give an alternative proof of sufficiency for Theorem 1, and then prove Theorem 3.

*Proof of Theorem* 1. The setting is the same as at the beginning of Section 4. The proof begins, as before, by using the concordance between $\phi$ and $1_M$ to connect $G$ and $G'$ by a family $G''$ of curves which will be geodesics in the yet-to-be-chosen metric. But then, using the exactness of $\omega|_M - \phi^*(\omega'|_M)$, we interpolate between $\omega$ and $\omega'$ a one-form $\omega''$ and define a tangent vector field $V''$ to $G''$ so that $\omega''(V'')=1$. Using this data, we define the new metric on $N$ and check that the curves of $G''$ become geodesics.

*Step* 1. *Construction of* $G''$. This is accomplished as before. In addition, we extend $\phi$ to a diffeomorphism

$$\Phi\colon \operatorname{Im} H \longrightarrow \operatorname{Im} H'$$

by
$$\Phi(g_t(x)) = g'_t(\phi(x)) \ ,$$
that is,
$$\Phi = H' \circ (\phi \times 1) \circ H^{-1} \ .$$
Thus $\Phi$ takes an arc of the geodesic of $G$ through $x$ to an arc of the geodesic of $G'$ through $\phi(x)$.

Finally we define a diffeomorphism
$$\Psi \colon \operatorname{Im} H \longrightarrow \operatorname{Im} \bar{H}$$
by
$$\Psi = \bar{H} \circ F \circ H^{-1} \ .$$
Note that $\Psi =$ identity on $H(M \times [-\varepsilon, -\varepsilon/2])$ and $\Psi = \Phi$ on $H(M \times [\varepsilon/2, \varepsilon])$. $\Psi$ takes an arc of a geodesic of $G$ to an arc of $G''$.

*Step* 2. *Construction of* $\omega''$. We know by hypothesis that $\omega|_M - \phi^*(\omega'|_M)$ is exact, so let $f \colon M \to R$ be a smooth function such that $\omega|_M - \phi^*(\omega'|_M) = df$. Now extend $f$ to a function $f \colon \operatorname{Im} H \to R$ by defining $f(H(x, t)) = f(x)$; in other words, $f$ is constant along the geodesics of $G$. Then we have

(9.1) $\omega - \Phi^*(\omega')$ *is exact on* Im $H$ *and equals* $df$ *there.*

To see this, let $C$ be a curve in Im $H$, running from $H(x, s)$ to $H(y, t)$. Then
$$\int_C \omega - \Phi^*(\omega') = \int_C \omega - \int_{\Phi(C)} \omega' \ .$$
Push $C$ along the geodesics of $G$ to the curve $C_0$ on $M$, running from $x$ to $y$. Then
$$\int_C \omega = \int_{C_0} \omega + t - s$$
by Lemma 2.2, and similarly
$$\int_{\Phi(C)} \omega' = \int_{\phi(C_0)} \omega' + t - s \ ,$$
if we push $\Phi(C)$ along the geodesics of $G'$ to $\phi(C_0)$ on $M$. Therefore,
$$\begin{aligned}\int_C \omega - \Phi^*(\omega') &= \int_C \omega - \int_{\Phi(C)} \omega' = \int_{C_0} \omega - \int_{\phi(C_0)} \omega' \\ &= \int_{C_0} \omega|_M - \phi^*(\omega'|_M) = f(y) - f(x) \\ &= f(H(y, t)) - f(H(x, s)) \ .\end{aligned}$$
It follows that $\omega - \Phi^*(\omega') = df$ on Im $H$.

Next, let $\rho \colon \operatorname{Im} H \to R$ be a smooth function depending only on the $t$

coordinate, which vanishes for $-\varepsilon \leqq t \leqq -\varepsilon/2$, equals 1 for $\varepsilon/2 \leqq t \leqq \varepsilon$, and is monotonically increasing in between.

Now define a one-form $\bar{\omega}$ on Im $H$ by the equation

$$\bar{\omega} = \omega - d(\rho f) .$$

Note that for $-\varepsilon \leqq t \leqq -\varepsilon/2$, $\rho = 0$ and hence $\bar{\omega} = \omega$. Similarly, for $\varepsilon/2 \leqq t \leqq \varepsilon$, $\rho = 1$ and so $\bar{\omega} = \omega - df = \Phi^*(\omega')$. We assert

(9.2) $\bar{\omega}(V) > 0$, *and hence* $\bar{\omega}$ *is nondegenerate.*

Now

$$\begin{aligned} \bar{\omega}(V) &= \omega(V) - d(\rho f)(V) = \omega(V) - V(\rho f) \\ &= \omega(V) - \rho V(f) - f V(\rho) . \end{aligned}$$

But $\omega(V) - \Phi^*\omega'(V) = df(V) = V(f)$, and using this value for $V(f)$, we get

$$\bar{\omega}(V) = (1 - \rho)\omega(V) + \rho\Phi^*\omega'(V) - fV(\rho) .$$

But $\omega(V) = 1$ and $\Phi^*\omega'(V) = \omega'(\Phi_* V) = \omega'(V') = 1$, so

$$\bar{\omega}(V) = 1 - f\frac{d\rho}{dt} , \tag{9.3}$$

where we have written $d\rho/dt$ in place of $V(\rho)$. Since $d\rho/dt > 0$, (9.2) would certainly follow if $f \leqq 0$. Since Im $H$ is compact, we can simply add a constant to $f$ to make it a negative function, without altering the relation $\omega - \Phi^*(\omega') = df$. We do this, and (9.2) follows.

*Note.* Once again we observe that it will not be possible to do this in the proof of the relative deformation theorems. As before, condition (c) reveals itself at this point.

Finally we define the one-form $\omega''$ on Im $\bar{H}$ by

$$\omega'' = (\Psi^{-1})^*\bar{\omega} .$$

Note that $\omega'' = \omega$ on $H(M\times[-\varepsilon, -\varepsilon/2])$ because $\bar{\omega} = \omega$ there and $\Psi^{-1}$ is the identity there. Also, $\omega'' = \omega'$ on $H'(M\times[\varepsilon/2, \varepsilon])$ because $\bar{\omega} = \Phi^*\omega'$ on $H(M\times[\varepsilon/2, \varepsilon])$ and $\Psi = \Phi$ there. Since $\bar{\omega}$ is nondegenerate, so is $\omega''$.

*Step* 3. *Construction of the new metric.* So far we have the family $G''$ of curves and the nondegenerate one-form $\omega''$.

Since $\Psi$: Im $H \to$ Im $\bar{H}$ takes curves of $G$ to curves of $G''$, the vector field $\Psi_* V$ is tangent to $G''$. Moreover,

$$\omega''(\Psi_* V) = (\Psi^{-1})^*\bar{\omega}(\Psi_* V) = \bar{\omega}(V) > 0 ,$$

by (9.2). Hence multiplying $\Psi_* V$ by a smooth positive function, we get a tangent vector field $V''$ to $G''$ such that

$$\omega''(V'') = 1 .$$

It is easy to check that $V'' = V$ on $H(M\times[-\varepsilon, -\varepsilon/2])$ and that $V'' = V'$ on $H'(M\times[\varepsilon/2, \varepsilon])$.

Now we define a Riemannian metric on Im $\bar{H}$ by stipulating:

(1) $V''$ is to be a unit vector field.

(2) ker $\omega''$ is to be perpendicular to $V''$.

(3) The original metric is to be used on ker $\omega''$.

Since $V'' = V$ and $\omega'' = \omega$ on $H(M\times[-\varepsilon, -\varepsilon/2])$, the new metric agrees with the old metric there. Similarly, $V''= V'$ and $\omega''=\omega'$ on $H'(M\times[\varepsilon/2, \varepsilon])$, so the new and old metrics agree there. Hence the new metric can be extended via the old metric over the rest of $N$.

*Step* 4. *Verification that $G''$ is a field of geodesics in the new metric.* Invoking Lemma 2.1, we will do this by showing that the Lie derivative $L_{V''}\omega'' = 0$. Now

$$L_{V''}\omega'' = V'' \lrcorner\, d\omega'' + d(V'' \lrcorner\, \omega'') , \tag{9.4}$$

where $\lrcorner$ indicates left interior multiplication [17, p. 102]. Since $V'' \lrcorner\, \omega''= \omega''(V'') = 1$, the second term on the right of (9.4) is certainly zero. Next observe that

$$d\omega'' = (\Psi^{-1})^* d\bar{\omega} = (\Psi^{-1})^* d\omega .$$

Therefore

$$\begin{aligned} V'' \lrcorner\, d\omega'' &= V'' \lrcorner\, (\Psi^{-1})^* d\omega \\ &= k\,\Psi_* V \lrcorner\, (\Psi^{-1})^* d\omega \\ &= k\,(\Psi^{-1})^*(V \lrcorner\, d\omega) . \end{aligned} \qquad \begin{pmatrix}\text{because } V'' \text{ is some}\\ \text{multiple } k \text{ of } \Psi_* V\end{pmatrix}$$

However,

$$L_V(\omega) = V \lrcorner\, d\omega + d(V \lrcorner\, \omega) ,$$

and the left side must vanish by Lemma 2.1, while $d(V \lrcorner\, \omega) = d(1) = 0$. It follows that $V \lrcorner\, d\omega = 0$, and inserting this above, we get $V'' \lrcorner\, d\omega'' = 0$. But then by (9.4) $L_{V''}\omega'' = 0$, so the curves of $G''$ are indeed geodesics in the new metric, completing the alternative proof of Theorem 1.

*Proof of Theorem* 3. Here it is given that the closed one-forms $\omega$ and $\omega'$, corresponding to the integrable geodesic fields $G$ and $G'$, satisfy $[\omega|_M]= [\omega'|_M]$ in $H^1(M; R)$. Without loss of generality, we can assume that $N$ is a tubular neighborhood of $M$. In that case, the inclusion $M \hookrightarrow N$ is a homotopy equivalence, hence induces an isomorphism $H^1(N; R) \to H^1(M; R)$. From this we conclude that $[\omega] = [\omega']$ in $H^1(N; R)$, and so there is a smooth function $f: N \to R$ such that $\omega - \omega' = df$.

As in Step 2 of the preceding proof of Theorem 1 (but this time more directly) we obtain a nondegenerate one-form

$$\omega'' = \omega - d(\rho f) ,$$

which coincides with $\omega$ somewhat below $M$ and with $\omega'$ somewhat above. Notice that the field $G''$ of curves has yet to be constructed.

Now let $V''$ be the vector field orthogonal in the old metric to the $(n-1)$plane ker $\omega''$ at each point of $N$, satisfying $\omega''(V'') = 1$, and oriented so that it coincides with $V$ somewhat below $M$ and with $V'$ somewhat above. In between, however, it may not be a unit vector field. There is a unique new metric on $N$, pointwise conformal to the old one, making $V''$ a unit vector field everywhere. This new metric coincides with the old one outside a neighborhood of $M$.

Finally, let $G''$ be the field of curves tangent to $V''$. Clearly, $G''$ coincides with $G$ somewhat below $M$ and with $G'$ somewhat above. But notice that we have lost control over which curves of $G$ and $G'$ are connected by $G''$.

It remains to check that the curves of $G''$ are geodesics in the new metric. We do this by showing that $L_{V''}\omega'' = 0$, again by using formula (9.4):

$$L_{V''}\omega'' = V'' \lrcorner\, d\omega'' + d(V'' \lrcorner\, \omega'') .$$

But this time $\omega''$ is a closed form (because $\omega$ is), so we see directly that the first term on the right vanishes. And the second term vanishes as before, since $V'' \lrcorner\, \omega'' = 1$. So the curves of $G''$ are indeed geodesics, completing the proof of Theorem 3.

UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA
CASE WESTERN RESERVE UNIVERSITY, CLEVELAND, OHIO

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(Received July 27, 1977)