Matrix Groups: Homework #5

Based on matrices over other fields

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Problem 1

- (i) Let p be a prime. Prove that if $p \mid ab$, then p divides either a or b.
- (ii) In a field \mathbb{F} , prove that if ab = 0, then either a = 0 or b = 0.

Problem 2

Let

$$U(n) = \{ A \in GL_n(\mathbb{C}) : AA^* = I_n = A^*A \},$$

where A^* is the conjugate transpose. For example,

$$A = \begin{pmatrix} 2 + \iota & 1 \\ 1 - \iota & 2\iota \end{pmatrix}$$
, then $A^* = \begin{pmatrix} 2 - \iota & 1 + \iota \\ 1 & -2\iota \end{pmatrix}$.

This is called *unitary group* (analogus to set of orthogonal group). Similarly, we have *special unitary group* which is

$$SU(n) = \{ A \in U(n) : \det A = 1 \}.$$

- (i) Can you identify the groups U(1) and SU(1)?
- (ii) Prove that $SU(2)=\Big\{\Big(egin{array}{cc} a & b \ -\overline{b} & \overline{a} \ \Big): a,b\in\mathbb{C} \ \ {
 m and} \ \ |a|^2+|b|^2=1\Big\}.$
- (iii) Show that $SU(2)/\{\pm I\} \cong SO(3)$.

Problem 3

Determine the groups $GL_1(\mathbb{C}), SL_1(\mathbb{C}), O_1(\mathbb{C})$ and $SO_1(\mathbb{C}).$

Problem 4

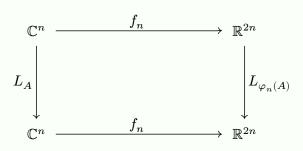
This problem involve calculations for matrix group over \mathbb{Z}_p .

- (i) How many elements are there in the group $GL_2(\mathbb{Z}_3)$?
- (ii) How many are there in $SL_2(\mathbb{Z}_3)$?
- (iii) Find the inverse of the matrix $\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$ in $GL_2(\mathbb{Z}_7)$.

Problem 5

We want to define an injective homomorphism $\varphi_n:M_n(\mathbb{C})\to M_{2n}(\mathbb{R}).$ Given any $A\in M_n(\mathbb{C})$, we have a corresponding linear map $L_A:\mathbb{C}^n\to\mathbb{C}^n.$ Also, we have a canonical map $f_n:\mathbb{C}^n\to\mathbb{R}^{2n},\,(a+\iota b_1,...,a_n+\iota b_n)\mapsto (a_1,b_1,...,a_n,b_n).$

Given $A\in M_{n(\mathbb{C})}$, we need to determine $B=\varphi_n(A)\in M_{2n}(\mathbb{R})$, equivalently, we need to find a linear map $L_{\varphi_n(A)}:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ so that the following diagram commutes.



That is, $f_n\circ L_A=L_{\varphi_n(A)}\circ f_n.$

Consider

$$\varphi_1: M_1(\mathbb{C}) \to M_2(\mathbb{R}), \quad a + \iota b \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Show that with this definition of φ_1 the above diagram is commutative.