Day 3 : 12th January, 2026

3.1. Quotient Space Let X be a set and $\sim$ be an equivalence relation on X. Let $X/\sim$ be the set of al equivalence classes of $\sim$. We have a natural map, called the *quotient map*

$$\pi : X \to X/\sim, \quad x \mapsto [x].$$

If we further assume that X is a topological space, we then want to introduce a topology on the quotient space $X/\sim$ such that the quotient map π is continuous. One choice for the same is indiscrete topology on $X/\sim$. However, we would like to have the largest possible topology on $X/\sim$ such that π is continuous. If $\mathcal{T}$ is such a topology on $X/\sim$, then for any $V \in \mathcal{T}$, the set $\pi^{-1}(V)$ must be open in X. This suggests the following.

Definition 3.1: (*Quotient topology*)

With the notation above, we define the *quotient topology* on $X/\sim$ as

$$\mathcal{T} = \{V \subseteq X/\sim : \pi^{-1}(V) \text{ is open in } X\}.$$

The map $\pi : X \to X/\sim$ is called a *quotient map*. The space $(X/\sim, \mathcal{T})$ is called the *quotient space*.

Proposition 3.2: (*Quotient topology is a topology*)

Let X be a topological space and $\sim$ is an equivalence relation on X. Then the quotient topology on $X/\sim$ is the largest topology for which the natural quotient map $\pi : X \to X/\sim$ is continuous.

Theorem 3.3: (*Universal Mapping Property*)

Let $\pi : X \to X/\sim$ be a quotient map. A map $f : X \to X/\sim$ is continuous if and only if $f \circ \pi$ is continuous.

Proof : Since π is continuous, so continuity of f implies $f \circ \pi$ is continuous. For the other way, if $f \circ \pi$ is continuous, then we need to show that f is continuous. For that, if $V \subseteq X/\sim$ is an open set, then we need to show that $f^{-1}(V)$ is open in X. But V is open in $X/\sim$ implies $\pi^{-1}(V)$ is open in X and

$$\begin{aligned}(f \circ \pi)^{-1}(V) = \pi^{-1}(f^{-1}(V)) &\Rightarrow \pi^{-1}(f^{-1}V) \text{ is open} \\ &\Rightarrow f^{-1}(V) \text{ is open in } X.\end{aligned}$$

□

The next theorem tells us how to generate quotient space.

Theorem 3.4: (*Generate a quotient space*)

Let X, Y be two topological space and $f : X \to Y$ be a continuous function. Define an equivalence relation $\sim$ on X as $x_1 \sim x_2 \Leftrightarrow f(x_1) = f(x_2)$. Then there exists a continuous function $g : X/\sim \to Y$ such that $f = g \circ \pi$.

Exercise 3.5:

Prove the above theorem.

Caution 3.6: (*A different description*)

Many a times, we define a quotient topology directly on a set.

Definition 3.7: (*Quotient map*)

Given a space $(X, \mathcal{T})$ and a function $f : X \to Y$ to a set Y, the *quotient topology* on Y is defined as

$$\mathcal{T}_f := \{U \mid f^{-1}(U) \in \mathcal{T}\}.$$

The map $f : (X, \mathcal{T}) \to (Y, \mathcal{T}_f)$ is called a *quotient map*. In other words, f is a quotient map if $U \subset Y$ is open if and only if $f^{-1}(U) \subset X$ is open.

- The quotient topology $\mathcal{T}_f$ is indeed a topology on Y, and $f : (X, \mathcal{T}) \to (Y, \mathcal{T}_f)$ is continuous.

Combining all, we have the following.

Theorem 3.8: (*Universal property of quotient topology*)

Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are given. Then, for any set function, $q : X \to Y$, the following are equivalent.

1. $\mathcal{T}_Y$ is the quotient topology induced by q (in other words, q is a quotient map).
2. $\mathcal{T}_Y$ is the finest (i.e., largest) topology for which q is continuous.
3. $\mathcal{T}_Y$ is the unique topology having the following property : for any space $(Z, \mathcal{T}_Z)$ and any set map $f : Y \to Z$, we have f is continuous if and only if $f \circ q$ is continuous

Remark 3.9: (*Quotient map and surjectivity*)

Suppose $f : X \to Y$ is a quotient map. Assume that f is *not* surjective. Then, for any $y \in Y \setminus f(X)$ we have $f^{-1}(y) = \emptyset \subset X$ open, and hence, $\{y\}$ is open in Y. In other words, $Y \setminus f(X)$ has the discrete topology. Also, $f(X) \subset Y$ is both an open and closed set. Hence, the open and closed sets of $f(X)$ in the subspace topology are precisely the same in the actual (quotient) topology on Y. For these reasons, we can (and usually we do) assume that a quotient map is surjective.

Remark 3.10: (*Surjective map and equivalence relation*)

Suppose $f : X \to Y$ is a surjective map. Then, the collection $\bigsqcup_{y \in Y} f^{-1}(y)$ is a partition on X, and hence, induces an equivalence relation. Indeed, we can define $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$. Conversely, given any equivalence relation $\sim$ on X, we see that $q : X \to X/_\sim$, is a surjective map, where $X/_\sim$ is the collection of all equivalence classes under the relation $\sim$.

Given a set map $f : X \to Y$, a subset $S \subset X$ is called *saturated* (or *f-saturated*) if $S = f^{-1}(f(S))$ holds.

Exercise 3.11: (*Saturated open set*)

Given a quotient map $q : X \to Y$, a set $U \subset X$ is q-saturated if and only if it is the union of the equivalence classes of its elements (i.e, $U = \bigcup_{x \in U} [x]$).

Example 3.12: ($[0,1]/\{0,1\}$ *is* S^1)

Consider the quotient space obtained from $[0,1]$ got by identifying the end points 0 and 1. That is, $X := [0,1]/\sim$, where

$$x \sim y \Leftrightarrow x = y \text{ or } x, y \in \{0,1\}.$$

Show that X is homeomorphic to circle S^1.

Proof : Consider the map $f : [0,1] \to S^1$ given by $f(t) = e^{2\pi\iota t} = (\cos 2\pi t, \sin 2\pi t)$. Clearly, f is continuous and surjective. Also, $f(0) = (1,0) = f(1)$. Passing to the quotient $X = [0,1]/\{0,1\}$, we get a map $\tilde{f} : X \to S^1$ defined by $\tilde{f}([x]) = f(x)$. It is easy to see that $\tilde{f}$ is well-defined, and hence, by the property of the quotient topology, $\tilde{f}$ is continuous. Now, $\tilde{f}$ is surjective (as f was), and moreover, it is injective.

In order to show $\tilde{f}$ is open, we consider the two cases.

i) Suppose $V \subset X$ is an open set, such that $[0] = [1] = \{0,1\} \notin V$. Then, $q^{-1}(V) \subset [0,1]$ is an open set, which is actually contained in $(0,1)$. In particular, $q^{-1}(V)$ is a union of open intervals. Observe that (by drawing picture or otherwise) f maps such open intervals to open arcs of S^1 (which are open in S^1). Then, $\tilde{f}(V) = f(q^{-1}(V))$ is open.

ii) Suppose $V \subset X$ is an open set, such that $[0] = [1] = \{0,1\} \in V$. Then, $q^{-1}(V)$ is the union of open intervals of $(0,1)$, as well as, $[0, \varepsilon_1) \cup (1 - \varepsilon_2, 1]$ for some $\varepsilon_1, \varepsilon_2 > 0$. We have already seen that any open intervals get mapped to open arcs. Also, $f([0, \varepsilon_1) \cup (1 - \varepsilon_2, 2])$ is another open arc in S^1 containing the point $(0,1)$. Thus, $\tilde{f}(V) = f(q^{-1}(V))$ is open in S^1.

Hence, $\tilde{f} : X \to S^1$ is a homeomorphism. $\square$

The following result allows us to identify quotient spaces with other concrete spaces.

Theorem 3.13: (*Identify quotient spaces with other space*)

Let X and Y be compact topological spaces. Assume further that Y is Hausdorff. Let $f : X \to Y$ be surjective continuous map. Define an equivalence relation $\sim$ be declaring $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$. Then $X/\sim$ is homeomorphic to Y.

With the above theorem it is easy to show that $[0,1]/\sim$ is S^1.

Exercise 3.14:

i) Show that the quotient space got by identifying two of the opposite sides of a rectangle is homeomorphic to a cylinder.

ii) Let $X = [0,1] \times [0,1]$. Identify

$$\{(0,t) : t \in [0,1]\} \text{ with } \{(1,t) : t \in [0,1]\}$$
$$\{(s,0) : s \in [0,1]\} \text{ with } \{(s,1) : s \in [0,1]\}.$$

Show that the quotient space obtained from the above identification is a torus, that is, $S^1 \times S^1$.

From any space X and a subset $A \subseteq X$, the space X/A stands for the quotient space of X with respect to the equivalence relation

$$x_1 \sim x_2 \Leftrightarrow x_1 = x_2 \text{ or } x_1, x_2 \in A.$$

Thus, X/A is the space obtained from X by collapsing A to a single point.

Exercise 3.15: ($\mathbb{R}/\mathbb{Z}$ *is* S^1)

Consider the quotient space $X = \mathbb{R}/\mathbb{Z}$, where the equivalence relation is given as $a \sim b$ if and only $a - b \in \mathbb{Z}$. Show that X is homeomorphic to the circle S^1.