MATHEMATICS

FOCAL POINTS OF CLOSED SUBMANIFOLDS OF RIEMANNIAN SPACES

BY

ROBERT HERMANN [1])

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1. *Introduction*

Let M be a C^∞, connected, complete Riemannian manifold, let N be a closed connected regularly imbedded submanifold, and let $N^\perp$ be the normal bundle of N, i.e. the set of all tangent vectors to M at points of N that are perpendicular to N. Let Exp: $N^\perp \to M$ be the usual exponential map of $N^\perp$ onto M. Let V_N be the set of $v \in N^\perp$ such that there are no focal points of N on the geodesic $t \to \operatorname{Exp}(tv)$ for $0 \leqslant t \leqslant 1$. Let V_N' be boundary of V_N in $N^\perp$. Thus, Exp (V_N') may be considered as the "focal locus" of N in M, generalizing the well-known concept of conjugate locus in the case where N is a point.

In this paper, we deal with two types of results concerning the focal locus. In the first, we look for sufficient conditions that the map Exp restricted to V_N be a covering map on Exp (V_N). For example, if N is a point and if $V_N = N^\perp$, i.e. there are no focal (in this case, conjugate) points, it is known that Exp is a covering map. In [6] we proved, in case the curvature of M is non-positive and N is totally geodesic, that $V_N = N^\perp$ and that Exp is a covering map [2]). However, further conditions of this type do not seem to have been considered before, despite their a-priori importance for global Riemannian geometry, and we will give such conditions here.

Second, we will give estimates of the location of focal points and the Morse indices in terms of the curvature of M and the second fundamental form of N, using techniques due to Morse and very successfully applied by numerous authors in the last years in case N is a point. It may be possible to refine these estimates using Rauch comparison techniques, but we will defer that work to a later time. As application, we can in some favorable cases apply Morse theory and obtain information on the topology of N. We will now fix notations and describe our main results in more detail.

Theorem A. *Suppose M, N, $N^\perp$,* Exp, *V_N and V_N' are as described*

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[2]) In [6] we did not know whether the facts proved there proved that Exp was a covering map, but we will fill in that gap in this paper.

above. Suppose in addition that $\mathrm{Exp}\,(V_N')$ *is contained in the boundary of* $\mathrm{Exp}\,(V_N)$. *Then,* $\mathrm{Exp}\colon V_N \to \mathrm{Exp}\,(V_N)$ *is a covering map.*

To state the next theorem, we need to recall some concepts from Riemannian geometry (HELGASON's book [5] will serve as a basic reference). For $p \in M$, let M_p be the tangent space to M at p. Let $\langle\ ,\ \rangle$ be the positive definite inner product on M_p defined by the metric. For $u \in M_p$, let $\|u\| = \langle u, u\rangle^{1/2}$. If $p \in N$, identify N_p with a subspace of M_p using the differential of the map defining the imbedding of N in M. If $\sigma\colon [0, 1] \to M$ is a curve in M (usually parameterized proportionally to arc-length), for $0 \leqslant t \leqslant 1$ let $\sigma'(t)$ be the tangent vector to σ at t, an element of $M_{\sigma(t)}$. If $u, v \in M_p$ for some $p \in M$, let $K(u, v)$ be the sectional curvature of M in the plane spanned by u and v. For $p \in N$, let $N_p^\perp$ be the space of vectors in M_p perpendicular to N_p, and let $N^\perp = \bigcup_{p \in N} N_p^\perp$. For $p \in N$, $u \in N_p^\perp$, let $S_u(\ ,\ )$ be the second fundamental form of N, evaluated on v. Thus, for each u, S_u is a real-valued symmetric bilinear form on N_p [1, 2]. A geodesic $\sigma\colon [0, 1] \to N$ is said to be perpendicular to N (at $t = 0$) if $\sigma(0) \in N$ and $\sigma'(0) \in N^\perp_{\sigma(0)}$. A vector field $v\colon t \to v(t) \in M_{\sigma(t)}$ on σ is said to be *transversal to* N if:

$$1.1 \qquad \begin{cases} \text{a)} \quad v(0) \in N_{\sigma(0)} \\ \text{b)} \quad \langle \nabla v(0), u\rangle = -S_{\sigma'(0)}(v(0), u) \ \text{ for all } \ u \in N_p \end{cases}$$

($t \to \nabla v(t)$ is the covariant derivative vector field along σ).

For $p \in M$, let $R(\ ,\ )(\ )$ be the Riemann curvature tensor evaluated at p. Thus, for $u_1, u_2 \in M_p$, $R(u_1, u_2)$ is a linear transformation of $M_p \to M_p$, whose value on a $u \in M_p$ is $R(u_1, u_2)(u)$. A vector field $t \to v(t)$ on σ is a *Jacobi field* on σ if it is C^∞ and if:

$$\nabla\nabla v(t) + R(v(t), \sigma'(t))(\sigma'(t)) = 0 \ \text{ for } \ 0 \leqslant t \leqslant 1.$$

If $\sigma\colon [0, 1] \to M$ is a perpendicular geodesic to N, a point $\sigma(t_0)$ on σ, for $t_0 \in (0, 1]$, is a *focal point* of N with respect to σ if there is a Jacobi field on σ that is not identically zero, that vanishes at $t = t_0$, and that is transversal to N, i.e. satisfies 1.1.

Theorem B. *Let* $\sigma\colon [0, 1] \to M$ *be a geodesic perpendicular to* N. *Suppose that* $e_1, \ldots, e_n$ *are the eigenvalues of* $S_{\sigma'(0)/\|\sigma'(0)\|}$. *For* $0 \leqslant t \leqslant 1$, *let* $N_t \subset M_{\sigma(t)}$ *be the subspace obtained by parallel translating* $N_{\sigma(0)}$ *along* σ *to* $\sigma(t)$. *Suppose that* $\sigma(1)$ *is a focal point of* N *(with respect to* σ*), but that there are no focal points on* $(0, 1)$. *For* $1 \leqslant i \leqslant n$, *let* λ_i *be the smallest positive root of the equation:*

$$\lambda \cot \lambda = -e_i \,(\text{length } \sigma).$$

Suppose δ_1 *and* δ_2 *are positive numbers such that*

$$0 < \delta_1 \leqslant K(\sigma'(t), u) \quad \textit{for all } u \in N_t,\ 0 \leqslant t \leqslant 1, \textit{ and}$$
$$0 < K(\sigma'(t), u) \leqslant \delta_2 \quad \textit{for all } u \in M_{\sigma(t)},\ o \leqslant t \leqslant 1.$$

Then,

$$\frac{1}{\sqrt{\delta_2}} \min_{1 \leqslant i \leqslant n} \{\lambda_i, \pi\} \leqslant \text{length } \sigma \leqslant \frac{1}{\sqrt{\delta_1}} \min_{1 \leqslant i \leqslant n} \{\lambda_i\}.$$

Theorem C. *Suppose that M is a complete, simply connected Riemannian manifold of non-positive sectional curvature. Let N be a closed submanifold of M of* $\dim n$. *Suppose that k is an integer such that, for each $u \in N^\perp$, at least k (counted according to multiplicity) of the eigenvalues of S_u are non-positive. Then,*

$$0 = H_{n-k+1}(N) = \ldots = H_n(N),$$

where the indicated homology groups of N are taken with any field as coefficients.

Using algebraic results of Otsuki [10), conjectured by Chern and Kuiper [3], conditions will be given later on that involve only the curvatures of M and the induced metric on N and that imply the hypotheses of Theorem C. Note that the results proved by Chern, Kuiper and Otsuki are concerned with proving that the nth Betti number of N is zero, hence that N cannot be compact.

2. *Proof of Theorem A*

The notations will be those of the introduction. In addition, all manifolds will be C^∞, paracompact and connected, and all maps, curves, tensor-fields, etc. will be C^∞ unless mentioned otherwise. If P and Q are manifolds $\varphi\colon P \to Q$ a map, $p \in P$, $\varphi_*\colon P_p \to Q_{\varphi(p)}$ denotes the linear map induced by φ on tangent vectors.

Lemma 2.1. *Suppose that P and Q are connected manifolds of the same dimension and that $\varphi\colon P \to Q$ is a map which has everywhere non-zero Jacobian. Suppose further that there is a point $p \in P$ such that the map induced by φ on curves beginning at p is onto the space of curves beginning at $\varphi(p)$. Then, φ is a covering map of P on Q.*

Proof. By a theorem due to Nomizu and Ozeki [9], Q has a complete Riemannian metric. This metric pulled back to P by φ defines a metric on P such that φ is a local isometry. The lifting of a geodesic of Q beginning at $\varphi(p)$ is a geodesic of P, hence the metric on P is complete, by the Hopf–Rinow theorem, hence φ is a covering map [5, p. 74], q.e.d.

We now proceed to apply Lemma 2.1 to the case $P = V_N \subset N^\perp$, $Q = \operatorname{Exp}(V_N) \subset M$. A well-known alternate characterization of focal points goes as follows: Given $v \in N^\perp$, $\operatorname{Exp}(v)$ is a focal point of N with respect to the geodesic $t \to \operatorname{Exp}(tv)$ if and only if the map Exp has Jacobian zero at v. Thus, Exp restricted to V_N is an open map, and $\operatorname{Exp}(V_N)$ is open in $V_N^\perp$. Let p_0 be a fixed point of N, and let $\gamma\colon [0, 1] \to$ $\to \operatorname{Exp}(V_N)$ be a curve starting at p_0, parameterized by s, $0 \leqslant s \leqslant a$. We attempt to lift γ by Exp to a curve beginning at the point $(p_0, 0) \subset N^\perp$. Since Exp restricted to V_N is an open map, this can be lifted for sufficiently small s. This lifting will succeed in lifting γ unless there is an "obstruction" number $a \in (0, 1]$ encountered such that a lifted curve exists over $0 \leqslant s < a$, but over no larger interval. Suppose such an obstacle is encountered. Suppose the lifted curve is of the form $(\gamma_0(s), u(s))$, with $\gamma_0(s) \in N$, $u(s) \in N^\perp_{\gamma_0(s)}$ $\gamma(s) = \mathrm{Exp}\,(u(s))$, for $0 \leqslant s < a$. Let δ: $[0, a) \times \times [0, 1] \to M$ be a homotopy such that:

$$\delta(s, t) = \mathrm{Exp}\,(tu(s)) \text{ for } 0 \leqslant t \leqslant 1,\ 0 \leqslant s < a.$$

Let $\partial_s\delta(s, t)$ and $\partial_t\delta(s, t)$ be the elements of $M_{\delta(s,t)}$ defined as follows: $\partial_s\delta(s, t)$ (resp. $\partial_t\delta(s, t)$) is the tangent vector to the curve $\lambda \to \delta(s, t)$ (resp. $\lambda \to \delta(s, \lambda)$) at $\lambda = s$ (resp. $\lambda = t$). Thus, δ has the following properties:

$$\|\partial_t\delta(s, t)\| = \text{the length of the curve } t \to \delta(s, t), \text{ for fixed } s$$
$$\partial\delta(s, 0) = u(s) \in N^\perp_{\delta(s,t)} \text{ for } 0 \leqslant s < a$$
$$\delta(s, 0) \in N \text{ for } 0 \leqslant s < a.$$

For fixed s, the curve $t \to \delta(s, t)$ is a geodesic, perpendicular to N, i.e. δ is a geodesic deformation. The vector field $t \to \partial_s\delta(s, t)$ along this geodesic is a Jacobi vector field transversal to N.

$$\delta(s, 1) = \gamma(s) \text{ for } 0 \leqslant s < a.$$

Lemma 2.2. *If the homotopy* $\delta(s, t)$, $0 \leqslant s < a$, $0 \leqslant t \leqslant 1$, *has the properties listed above, and if* $L(s)$ = *the length of the curve* $t \to \delta(s, t)$ *for* $0 \leqslant t \leqslant 1$, *then*:

$$|L(a) - L(0)| \leqslant \textit{length of } \gamma(s) \textit{ for } 0 \leqslant s < a.$$

Proof. The classical first variation formula implies that:

$$L(s)\frac{dL}{ds} = \langle \gamma'(s), \partial_t\delta(s, 1)\rangle$$

(since $\langle \partial_s\delta(s, 0), \partial_t\delta(s, 0) = 0$, hence

$$\begin{aligned} |L(s) - L(0)| &\leqslant \int_0^s \frac{|\langle \gamma'(\lambda), \partial_t\delta(\lambda, 1)\rangle|}{L(\lambda)}\, d\lambda \\ &\leqslant \int_0^s \frac{\|\gamma'(\lambda)\|\, \|\partial_t\delta(\lambda, 1)\|}{L(\lambda)}\, d\lambda \\ &= \int_0^s \|\gamma'(\lambda)\|\, d\lambda \leqslant \text{length } \gamma. \end{aligned}$$

Lemma 2.3. $\|\partial_s\delta(s, 0)\|$ *is bounded as* $s \to a$.

Proof. Suppose otherwise, i.e. there is a sequence of numbers, $s_j \in (0, a)$, $j = 1, 2, \ldots$, such that:

$$\lim_j s_j = a,\ \lim_j \|\partial_s\delta(s_j, 0\| = \infty.$$

By Lemma 2.2, all points $\delta(s_j, 0)$ lie within a bounded distance from $\gamma(a) = \delta(a, 1)$. Since M is complete and N is closed in N, we can suppose, taking subsequences if necessary, that:

$$\lim_j \delta(s_j, 0) = q \in N, \text{ and } \lim_j \partial_t \delta(s_j, 0) = u \in N_q^\perp.$$

Thus, the geodesic $t \to \delta(s_j, t)$ converges uniformly to the geodesic $t \to \operatorname{Exp}(tu)$ as $j \to \infty$.

For each j, let the Jacobi vector fields $v_j(t)$ and $w_j(t)$ be defined on the geodesic $t \to \delta(s_j, t)$ as follows:

$$v_j(t) = \partial_s \delta(s_j, t)$$

$$w_j(t) = \frac{v_j(t)}{\|v_j(0)\| + \|\nabla v_j(0)\|}.$$

Now, since $v_j(0) = \partial_s \delta(s_j, 0)$, we have $\lim_{j \to \infty} \|v_j(0)\| = \infty$. But, $\|w_j(0)\| \leqslant 1$ and $\|\nabla w_j(0)\| \leqslant 1$. Hence, taking subsequences if necessary, we can suppose that:

$$\lim_j w_j(0) = w_0 \in N_q, \ \lim_j \nabla w_j(0) = w_1 \in M_q.$$

Further, since each $w_j(t)$ is a vector field that is transversal to N, if $w(t)$ is the Jacobi field on the geodesic $t \to \operatorname{Exp}(tu)$ such that $w(0) = w_0$, $\nabla w() = w_1$, then w is also transversal to N. Further, since the initial conditions converge, we must have:

$$\lim_j w_j(t) = w(t) \text{ uniformly for } 0 \leqslant t \leqslant 1.$$

w cannot be identically zero, since $\|w(0)\| + \|\Delta w(0)\| = 1$. Now, $v_j(1) = \partial_s \delta(s_j, 1) = \gamma'(s_j)$, which does have a limit as $j \to \infty$, by hypothesis. Thus, $w(1) = 0$, hence $\gamma(a)$ is a focal point of N with respect to the geodesic $t \to \operatorname{Exp}(tu)$, hence $\gamma(a) \in \operatorname{Exp}(V_N) \cap \operatorname{Exp}(V_N')$, contradiction.

Now that lemma 2.3 is proved, we see that the length of the curve $s \to \delta(s, 0)$ for $0 \leqslant s < a$ is finite, hence by completeness of M, the limit as $s \to a$ of $\delta(s, 0)$ exists, and equals, say, q again. Since N is closed in M, $q \in N$. By lemma 2.2, we can again choose a sequence of numbers (s_j) such that:

$$\lim_j s_j = a, \ \lim_j \partial_t \delta(s_j, 0) = u.$$

Then, $\operatorname{Exp}(u) = \gamma(a)$. Again, $\gamma(a)$ cannot be a focal point for N with respect to the geodesic $t \to \operatorname{Exp}(tu)$, since $\operatorname{Exp}(V_N') \subset (\operatorname{Exp}(V_N))'$. Thus, Exp is a diffeomorphism on some neighborhood of u. This diffeomorphism can be used to extend the curve $(\gamma_0(s), u(s))$ whose image under Exp is γ up to and a little beyond $s = a$, which is a contradiction to our assumption that $s = a$ was the obstacle to lifting γ. Lemma 2.1 can now be applied to infer that Exp restricted to V_N is a covering map, and proves Theorem A.

3. *Comparison theorems for focal points*

Suppose that M, N, $N^\perp$, Exp, S, ∇, $\langle\ ,\ \rangle$, $R(\ ,\ )(\ )$ and $K(\ ,\ )$ are as defined in the introduction. We now recall the second variation formula [1, 2, 8]. Suppose that $\sigma(t)$, $0 \leqslant t \leqslant 1$, is a geodesic (parameterized proportionally to arc-length) that is perpendicular to N at $t=0$. Suppose that $\delta(s, t)$, $0 \leqslant s, t \leqslant 1$, is a homotopy such that:

$\delta(0, t) = \sigma(t)$, i.e. δ is a deformation of σ.
$\delta(s, 1) = \sigma(1)$, i.e. δ has a fixed end-point at $t=1$.
$\delta(s, 0) \in N$ and $\partial_t \delta(s, 0) \in N^\perp$ for $0 \leqslant s \leqslant 1$.

For fixed s, the curve $t \to \delta(s, t)$ is parameterized proportionally to arc-length, i.e.

$$\|\partial_t \delta(s, t)\| = \text{length of the curve } t \to \delta(s, t) \underset{\text{def}}{\equiv} L(s).$$

Then,

$$3.1 \quad \begin{cases} L(0)\dfrac{d^2L}{ds^2}(0) = -S_{\sigma'(0)}(v(0), v(0)) \\ \qquad + \int\limits_0^1 [\|\triangle v(t)\|^2 - \|v(t)\|^2 L(0)^2 \sin^2\theta(t)\, K(v(t), \sigma'(t))]\, dt \\ \qquad \underset{\text{def}}{\equiv} I(v) \end{cases}$$

where $v\colon t \to v(t) = \partial_s \delta(0, t)$ is the infinitesimal deformation vector field on σ, and where $\theta(t)$ is the angle between $v(t)$ and $\sigma'(t)$. If $v(t)$ is transversal to N, i.e. satisfies 1.1, then an integration by parts and using the relation between sectional curvature and the Riemann curvature tensor gives:

$$3.2 \qquad I(v) = -\int\limits_0^1 \langle v(t), \nabla\nabla v(t) + R(v(t), \sigma'(t))\,(\sigma'(t)\rangle\, dt,$$

$$3.3 \quad \langle v(t), R(v(t), \sigma'(t))\,(\sigma'(t))\rangle = \|\sigma'(t)\|^2\, \|v(t)\|^2 \sin^2\theta(t)\, K(v(t), \sigma'(t)).$$

$I(v)$ is the *Morse index form.* We want to apply the Morse index theorem, a generalization of the Sturm oscillation theorem. With a view towards later applications, we shall describe without proofs a more general formulation of Morse's results.

Let $\sigma\colon [0, 1] \to M$ be a curve (not necessarily a geodesic) that will be fixed throughout the discussion. A *boundary condition* at $t=0$ will be defined by an ordered pair (W, Q), consisting of a subspace $W \subset M_{\sigma(0)}$ and a bilinear, symmetric form $(u, v) \to Q(u, v) \in R$ defined for $u, v \in W$. We say that a vector field v: $t \to v(t) \in M_{\sigma(t)}$ defined on σ *satisfies the boundary condition* (W, Q) if:

$$3.3 \quad \begin{cases} \text{a)} \quad v(0) \in W. \\ \text{b)} \quad \langle \nabla v(0), w\rangle = -Q(v(0), w) \text{ for all } w \in W. \end{cases}$$

Suppose that we are given such a boundary condition and a tensor field $t \to T_t$, $0 \leqslant t \leqslant 1$, along σ, where, for each $t \in [0, 1]$, T_t is a linear transformation:

$M_{\sigma(t)} \to M_{\sigma(t)}$ that is symmetric with respect to $\langle\ ,\ \rangle$, i.e.

$$\langle T_t(u), v\rangle = \langle u, T_t(v)\rangle \text{ for } u, v \in M_{\sigma(t)},\ 0 \leqslant t \leqslant 1.$$

Associated with T_t, we can construct the ordinary differential operator $\nabla^2 + T_t$ acting on vector fields defined on σ. Explicitly, if $v\colon t \to v(t) \in M_{\sigma(t)}$ is a vector field on σ,

$u = (\nabla^2 + T_t)(v)$ is a vector field on σ such that $u(t) = \nabla\nabla v(t) + T_t(v(t))$.

A point $\sigma(t_0)$ is said to be a *focal point* of $\sigma(0)$ with respect to the boundary condition (W, Q) and the operator $\nabla^2 + T_t$ if there is at least one (C^∞) vector field $v(t)$, $0 \leqslant t \leqslant 1$, on σ such that:

3.4
$$\begin{cases} \text{a) } v \text{ is not identically zero.} \\ \text{b) } v \text{ satisfies } (\nabla^2 + T_t)(v) = 0. \\ \text{c) } v \text{ satisfies 3.3, i.e. the boundary condition } (Q, W) \text{ at } t = 0. \\ \text{d) } v \text{ satisfies } v(t_0) = 0. \end{cases}$$

The following result, after taking account of differences in notation, can be found in [1], [2] or [8].

Theorem 3.1. *Given (W, Q) and $t \to T_t$ as above,suppose that σ contains no focal points with respect to the boundary condition (W, Q) and the operator $\nabla^2 + T_t$. Let $u(t)$, $0 \leqslant t \leqslant 1$, be a continuous, piecewis) C^1 vector field on σ satisfying the boundary condition (W, Q) at $t = 0$. Let $v(t)$ be a C^∞ vector field that satisfies $(\nabla^2 + T_t)(v) = 0$, the (W, Q)-boundary condition at $t = 0$, and $v(1) = u(1)$. Then,*

3.5
$$\begin{cases} -Q(v(0), v(0)) + \int_0^1 \|\nabla v(t)\|^2 - \langle v(t), T_t(v(t))\rangle\, dt \\ \qquad \leqslant -Q(u(0), u(0)) + \int_0^1 \|\nabla u(t)\|^2 - \langle u(t), T_t(u(t))\rangle\, dt. \end{cases}$$

Equality can hold only if $u(t) = v(t)$ for $0 \leqslant t \leqslant 1$.

To apply this result, suppose that we are given two such operators, $\nabla^2 + T_t$ and $\nabla^2 + T_t^*$. Suppose that the geodesic σ is free of focal points of the operator $\nabla^2 + T_t$, but that the vector field $u(t)$, $0 \leqslant t \leqslant 1$, satisfies the (W, Q)-boundary condition at $t = 0$ and:

$$\nabla^2 u(t) + T_t^*(u(t)) \text{ for } 0 \leqslant t \leqslant 1.$$

For each $s \in (0, 1]$, let $s_s(t)$, $0 \leqslant t \leqslant 1$, be the unique C^∞ vector field on σ such that:

$$\nabla\nabla v_s(t) + T_t(v_s(t)) = 0.$$

$$t \to v_s(t) \text{ satisfies the } (W, Q)\text{-boundary condition at } t = 0.$$

$$v_s(s) = u(s) \text{ for } 0 < s \leqslant 1.$$

The fundamental estimate is now:

Corollary to Theorem 3.1. *With the above notations,*

3.6
$$\begin{cases} \dfrac{1}{2}\dfrac{d}{ds}\|u(s)\|^2 \geqslant \langle \nabla v_s(s), v_s(s)\rangle \\ \qquad\qquad + \displaystyle\int_0^s \langle u(t), T_t(u(t)) - T_t^*(u(t))\rangle\, dt. \end{cases}$$

Equality holds only if $u(t)=v(t)$ *for* $0 \leqslant t \leqslant s$.

To prove 3.6, it suffices to deal with the case $s=1$. It then follows by substituting u and v_s in 3.5, integrating by parts, and applying the boundary conditions and differential equations satisfied by u and v.

The Morse index theorem, towards which we now turn, describes, at least in qualitative terms, what happens beyond the first focal point. Let Ω be the real vector space of continuous, piecewise C^2 vector fields $v: t \to v(t) \in M_{\sigma(t)}$ on σ satisfying the (W, Q) boundary condition at $t=0$ and such that $v(1)=0$. The *index* of the interval $[0, 1]$ (with respect to σ, T_t, (W, Q), of course) is the maximal dimension of a subspace of Ω on which the real-valued form

$$v \to -\int_0^1 \langle v(t), \nabla\nabla v(t) + T_t(v(t))\rangle\, dt$$

is negative definite. If $t_0 \in [0, 1]$ is a focal point of the operator $\nabla^2 + T_t$ with respect to (W, Q), then the *index* of t_0 is the dimension of the subspace of Ω composed of C^2 vector fields annihilated by $\nabla^2 + T_t$.

Theorem 3.2 (*Morse index theorem*). *The index of* $[0, 1]$ *is finite and equal to the sum of indices of the focal points on the interval* $(0, 1)$. *The index of* $[0, 1]$ *is also equal to the maximal number of linear independent* C^2 *elements of* Ω *that are eigenfunctions of* $\nabla^2 + T_t$ *with positive eigenvalues.*

Notice that the following fact follows at once from the definition of the index given above. Its implication for focal points follows from Theorem 3.2.

Suppose that $\nabla^2 + T_t^*$ *is another such operator such that*

3.7 $\qquad \langle u, T_t(u)\rangle \geqslant \langle u, T_t^*(u)\rangle$ *for all* $u \in M_{\sigma(t)}$, $0 \leqslant t \leqslant 1$.

Then, the index of $[0, 1]$ *with respect to* $\nabla^2 + T_t^*$ *is greater or equal to the index with respect to* $\nabla^2 + T_t$.

These results suggest proving estimates about focal points of an "unknown" operator $\nabla^2 + T_t$ by "pinching" it between "known" operators $\nabla^2 + T_t^{**}$ and $\nabla^2 + T_t^*$ such that:

$$\langle u, T_t^{**}(u)\rangle \geqslant \langle u, T_t(u)\rangle \geqslant \langle u, T_t^*(u)\rangle \text{ for all } u \in M_{\sigma(t)},\ 0 \leqslant t \leqslant 1.$$

Since the simplest candidates for T_t^* and T_t^{**} are operators which are invariant under parallel translation along σ, we will retreat from generalities to consider the indices of these operators.

Theorem 3.3. *Suppose that* ∇^2+T_t *is an operator such that the covariant derivative of* $t \to T_t$ *along* σ *is zero. Let* P *be the symmetric linear transformation*: $W \to W$ *such that*:

3.8 $$Q(w_1, w_2) = \langle w_1, P(w_2)\rangle \text{ for all } w_1, w_2 \in W.$$

Then, the index of $[0, 1]$ *with respect to this operator and the boundary condition* (W, Q) *is equal to the maximal number of linearly independent solutions* $w \in W$ *of the equations*

3.9 $$\sum_{j=0}^{\infty} (\lambda^2 I - T_0)^j \left(\frac{I}{2j!} - \frac{P}{(2j+1)!}\right)(w) = 0,$$

for some $\lambda > 0$, *where* $I\colon M_{\sigma(0)} \to M_{\sigma(0)}$ *is the identity transformation.* (In other words, the index is equal to the union of the intersections with W of the kernel of the linear transformations

$$\sum_{j=0}^{\infty} (\lambda^2 I - T_0)^j \left(\frac{I}{2j!} - \frac{P}{(2j+1)!}\right),$$

for all $\lambda > 0$.)

Suppose now that

3.10 $T_0(W) \subset W$, *and* T_0 *restricted to* W *commutes with* P. W *then has a basis composed of simultaneous eigenvectors of* P *and* T_0.

Let $C_1 \leqslant \dots C_m$ $(m = \dim W)$ *be the eigenvalues of* T_0 (repeated if necessary), *and let* $e_1, \dots, e_m$ *be the corresponding eigenvalues of* P. *For* $1 \leqslant k \leqslant m$, *let* $\Lambda(k)$ *be the number of positive solutions* λ *of either of the following equations*:

3.11 $$\begin{cases} \text{a)} & \sqrt{\lambda^2 - C_k}\,\coth\sqrt{\lambda^2 - C_k} = e_k, \text{ when } \lambda^2 - C_k \geqslant 0 \\ \text{or} & \\ \text{b)} & \sqrt{C_k - \lambda^2}\,\cot\sqrt{C_k - \lambda^2} = e_k, \text{ when } C_k - \lambda^2 > 0. \end{cases}$$

Then, the index of $[0, 1]$ *is the sum* $\Lambda(1) + \dots + \Lambda(m)$.

Proof. It is easily seen that a vector field $v\colon t \to v(t)$ along σ satisfies the (W, Q)-boundary condition if and only if:

The projection on W of $\nabla v(0) = -P(v_0)$.

Thus, if $v(t)$ satisfies: $(\nabla^2 + T_t - \lambda^2)(v) = 0$, these differential equations can be solved explicitly, by power series, with the result that:

$v(t)$ is the parallel translate along σ of

3.12 $$\sum_{j=0}^{\infty} (\lambda^2 I - T_0)^j \left(\frac{t^{2j}}{2j!} - \frac{t^{2j+1}}{(2j+1)!} P\right)(v(0)).$$

Requiring that $v(1) = 0$ gives 3.9. The rest of the results, in case P and T_0 commute, are obtained by substituting for $v(0)$ the eigenvectors of P and T_0.

Of course, $\Lambda(k)$ can in principle be computed, but the following qualitative information will be sufficient for our purposes.

Lemma 3.4. *Suppose x is a real variable restricted to $x \geqslant 0$. Consider the following equations for x:*

3.12 $$x \coth x = e$$

3.13 $$x \cot x = e.$$

If $e < 1$, 3.12 has no roots; if $e \geqslant 1$, precisely one, say $x(e)$. $x(e)$ is monotone increasing for $e \geqslant 1$. 3.13 has an infinite number of roots, say $x_1(e) < x_2(e) < \dots$, all monotone decreasing functions of e. Then,

$$x_1(1) = 0. \ 2\pi > x_2(1) > \pi, \ 3\pi > x_3(1) > 2\pi, \ etc.$$

If $e \leqslant 1$, $0 \leqslant x_1(e) < \pi$, $x_2(1) \leqslant x_2(e)$, etc. If $e > 1$, $\pi < x_1(e) < x_2(1)$, $2\pi < x_2(e) < < x_3(1)$, etc.

Proof. First treat $f(x) = x \coth x$. Note that:

$$f(0) = 1,$$

$$f'(x) = \frac{\frac{1}{2}\sinh 2x - x}{\sinh^2 x},$$

$$\frac{d}{dx}\left(\tfrac{1}{2}\sinh 2x - x\right) = \cosh 2x - 1 > 0 \text{ if } x > 0.$$

The listed properties of $x(e)$ in this case follow.

Now, let $g(x) = x \cot x$.

$$g(0) = 1,$$

$$g'(x) = \frac{\frac{1}{2}\sin 2x - x}{\sin^2 x} < 0 \text{ if } x > 0.$$

$$g(\pi -) = g(2\pi -) = \dots = -\infty, \ g(\pi +) = g(2\pi +) = \dots = +\infty.$$

The listed properties of the roots follow in this case also from these facts.

Corollary to Theorem 3.3. *Suppose that P commutes with T_0. Then, any one of the following conditions is sufficient to guarantee that* $[0, 1]$ *has no focal points of the operator $\nabla^2 + T_t$ with respect to the boundary condition (W, Q):*

3.14 $$C_k \leqslant 0 \text{ and } e_k < 1 \text{ for } k = 1, \dots, m.$$

3.15 $$\begin{cases} C_k \leqslant \delta, \text{ with } \delta \geqslant 0 \\ e_k < 1, \text{ but } x_1(e_k) \geqslant \sqrt{\delta} \text{ for each } k = 1, \dots, m, \text{ where } x_1(e) \text{ is the} \\ \text{smallest positive root of } x \cot x = e. \end{cases}$$

The proof of Theorem B stated in the introduction follows from these results. Suppose that $\sigma(t)$, $0 \leqslant t \leqslant 1$, is a geodesic of M parameterized proportionally to arc-length. Let $R(\ ,\ )(\ )$ be the curvature tensor of M, let T_t be the linear transformation $v \to R(v, \sigma'(t))(\sigma'(t))$. For $v \in M_{\sigma(t)}$ which satisfies $\langle v, \sigma'(t)\rangle = 0$,

$$\langle v, R(v, \sigma'(t))(\sigma'(t))\rangle = \|v\|^2 \,(\text{length } \sigma)^2\, K(v, \sigma'(t)).$$

$$\langle (\text{length } \sigma)^2\, \delta_1 v, v\rangle \leqslant \langle v, T_t v\rangle \leqslant \langle (\text{length } \sigma)^2\, \delta_2 v, v\rangle$$

if $\delta_1 \leqslant K(v, \sigma'(t)) \leqslant \delta_2$.

Suppose that N is a submanifold of M, with $\sigma(0) \in N$ and $\sigma'(0) \in N_{\sigma(0)}$. Then the boundary conditions (W, Q) can be chosen as follows:

$$W = N_{\sigma(0)}, \text{ and } Q(u, v) = S_{\sigma'(0)}(u, v),$$

where $S_{(\ )}(\ ,\ )$ is the second fundamental form of N. The focal points for the operator $\nabla^2 + T_t$ with boundary condition (W, Q), for this choice of T_t, W and Q, are now just the focal points of N along σ in the usual sense. Theorem 3.3 and its corollary now apply to prove Theorem B.

Turn to Theorem C, i.e. suppose that M is simply connected, that the sectional curvature of M is always $\leqslant 0$, and that, for all $p \in N$, all $v \in N_p^\perp$, at least k (counted according to multiplicity) of the eigenvalues of S_v are $\leqslant 0$. Let $p \in M - N$ be a point that is not a focal point of N, (chosen using Sard's theorem, i.e. as a regular image point for the mapping $\text{Exp}\colon N^\perp \to M$). Let $\sigma\colon [0, 1) \to M$ be a geodesic with $\sigma(0) \in N$, $\sigma'(0) \in N^\perp_{\sigma(0)}$, $\sigma(1) = p$. By comparison with the operator ∇^2, with boundary conditions at $t = 0$ given by $Q(u, v) = S_{\sigma'(0)}(u, v)$, $W = N_{\sigma(0)}$, we conclude that the index of σ with respect to the Jacobi operator $\nabla^2 + R(\ , \sigma'(t))(\sigma'(t))$ is no greater than the operator ∇^2, i.e. no greater than $n - k$, where $n = \dim N$. Let $\mathbf{P}(p, N)$ be the space of all continuous, piecewise C^∞ curves starting at p, ending at N. Morse theory now applies [8, Theorem 14.2], to prove that $H_i(\mathbf{P}(p, N)) = 0$ for $n - k < i \leqslant n = \dim N$, where the homology groups are taken with respect to any field as coefficients. However, since M is simply connected and carries a complete Riemannian metric of non-positive curvature, it is diffeomorphic to Euclidean space, implying that $\mathbf{P}(p, N)$ has the same homotopy type as N.

Now we investigate further the sufficient conditions that the hypotheses of Theorem C be satisfied. For $p \in N$, $u, v \in N_p$, let $K(u, v)$ and $K_N(u, v)$ be the sectional curvature of the plane spanned by u and v with respect to, respectively, the given metric on M and the induced metric on N. Recall the following classical formula:

3.16 $$K_N(u, v) - K(u, v) = \sum_{j=n+1}^{m} S_{w_j}(u, u)\, S_{w_j}(v, v) - (S_{w_j}(u, v))^2,$$

where $w_{n+1}, \ldots, w_m$ is any orthonormal basis of $N_p^\perp$, and where $S_{w_j}(\ ,\ )$ is the second fundamental form evaluated at w_j, $n+1 \leqslant j \leqslant m = \dim M$.

The following algebraic lemma was conjectured by CHERN and KUIPER [3], and proved by OTSUKI [10] [1]).

[1]) T. A. SPRINGER also proved this independently. His proof is given in N. H. Kuiper. Ausgewählte Kapitel der Riemannschen Geometric. Bonn Math. Institut 1957.

Lemma 3.4 *Let V be a real vector space of dimension d. Suppose that $Q_1, \dots, Q_{d-1}$ are symmetric bilinear real-valued forms on V such that:*

$$\sum_{j=1}^{d-1} Q_j(u, u)\, Q_j(v, v) - (Q_j(u, v))^2 \leqslant 0 \text{ for all } u, v \in V.$$

Then, there is at least one non-zero vector $v \in V$ such that:

$$Q_j(v, v) = 0 \text{ for } 1 \leqslant j \leqslant d-1.$$

Combining this lemma with Theorem C, we have:

Theorem 3.5. *Suppose that N is a closed submanifold of a complete simply connected Riemannian manifold M of non-positive sectional curvature. Suppose that:*

3.17 $K(u, v) \geqslant K_N(u, v)$ *for all $p \in N$, all $u, v \in N_p$, where K and K_N are the sectional curvature of M and the induced metric on N, respectively. Then,*

$$H_i(N) = 0 \text{ for } i > (\dim M - \dim N).$$

Proof. Let $m = \dim M$, $n = \dim N$, $k = 2\dim N - \dim M$. We will show that, for each $p \in N$, each $w \in N_p^{\perp}$, at least k eigenvectors of $S_w(\,,\,)$ are $\leqslant 0$. Theorem C will then apply. In 3.16, we are free to choose the orthonormal basis (w_j) of $N_p^{\perp}$, $n+1 \leqslant j \leqslant m$. Suppose first it is chosen so that $w_{n+1} = w$. 3.16, Lemma 3.4 and 3.17 imply that S_w has at least one eigenvector v_1 with a non-positive eigenvalue. Let V_1 be the orthonormal complement of v_1 in N_p. The right hand side of 3.16 is also $\leqslant 0$ for (u, v) restricted to V_1, of course. If $2(m-n) \geqslant \dim V_1 = n-1$, then $k \leqslant 1$, and we are through. If $2(m-n) < \dim V_1 = n-1$, then Lemma 3.4 applies to $S_{w_{n+1}}, \dots, S_{w_m}$ again, to infer the existence of another eigenvector v_2 of S_w in V_1 with a non-positive eigenvalue. Let V_2 be the orthogonal complement of v_1 and v_2 in N_p. If $2(m-n) \geqslant \dim V_2 = n-2$, then $k \leqslant 2$, and we are finished. Otherwise, Lemma 3.4 applies again, etc. We see that we end up with the required number of non-positive eigenvalues of S_w, q.e.d.

4. *Applications and extensions of Theorem A*

Suppose that $N, M, N^{\perp}$, Exp, $V_N \subset N^{\perp}$ are as defined in the introduction.

Theorem 4.1. *Let U_N be the set of all $v \in V_N$ so that $t \to \operatorname{Exp}(tv)$, $0 \leqslant t \leqslant 1$, is the unique curve of minimum length joining* $\operatorname{Exp}(v)$ *to N. Then,*

a) *U_N is an open subset of $N^{\perp}$.*

b) Exp *restricted to U_N is a diffeomorphism of U_N with* $\operatorname{Exp}(U_N)$.

c) *The closure of* $\operatorname{Exp}(U_N)$ *is all of M.*

Proof. Suppose that $v \in U_N$. Since Exp_* has non-zero Jacobian at v, there is a neighborhood U of v in $N^{\perp}$ such that Exp is one-one on U,

$\operatorname{Exp}(U)$ is open in M, and contains no focal points of geodesics whose initial tangent vectors lie in U. We assert that taking U sufficiently small implies that U lies in U_N. Otherwise, there is a sequence of points $y_j \in \operatorname{Exp}(U)$, and unequal vectors $u_j, v_j \in N^\perp$, and $v_j \in U$ such that: i) $\operatorname{Exp}(u_j) = \operatorname{Exp}(v_j)$, ii) $v_j \to v$ as $j \to \infty$, iii) length of $t \to \operatorname{Exp}(tu_j)$, $0 \leqslant t \leqslant 1$, is no greater than length of $t \to \operatorname{Exp}(tv_j)$, $0 \leqslant t \leqslant 1$. Since every bounded closed subset of M is compact, we can suppose, by taking subspaces if necessary, that $u_j \to u$ as $j \to \infty$, where $u \in N^\perp$. Then, $\operatorname{Exp}(u) = \operatorname{Exp}(v)$, by continuity of the Exp mapping, and $u = v$ by definition of $v \in U_N$. Then, $u_j \in U$ if j is sufficiently large, which contradicts that Exp is one-one restricted to U, hence proves a). Note that Exp restricted to U_N is one-one. To show that it is a diffeomorphism, it suffices to show that it is a covering map.

It suffices again to show that we can lift curves. Suppose that $\gamma(s)$, $0 \leqslant s \leqslant 1$, is a curve in $\operatorname{Exp}(U_N)$, and that $(\gamma_0(s), u(s))$, $\gamma_0(s) \in N$, $u(s) \in N^\perp_{\gamma_0(s)}$ is a curve in U_N for $0 \leqslant s < a$ which is mapped into γ. We must show that the lifting can be continued beyond a. Otherwise, going through the details used to prove Theorem A, we see that there will be a sequence of numbers (p_j), $j = 1, 2, \ldots$, such that:

$$\lim_j s_j = a, \ \lim_j \gamma_0(s_j) = q, \ \lim_j u(s_j) = u \in N^\perp_q,$$

$\gamma(a) = \operatorname{Exp}(u)$, and such that $\gamma(a)$ is a focal point of N with respect to the geodesic $t \to \operatorname{Exp}(tu)$. Since $\gamma(s) \in U_N$ for $0 \leqslant s \leqslant 1$, the distance from $\gamma(s_j)$ to N must equal $\|u(s_j)\|$ for all j, hence the distance from $\gamma(a)$ to N must equal $\|u\|$. Now, $u \notin V_N$. But, since $\gamma(a) \in U_N$, there is at least one other vector $w \in N^\perp$ with $\|w\| = \|u\|$ and $\operatorname{Exp}(w) = \gamma(a)$, which is a contradiction of the definition of U_N. This finishes the proof of b).

We now show that the closure of $\operatorname{Exp}(U_N)$ is M. Let $p \in M$. Using the Hopf–Rinow theorem and the fact that N is closed in M, there is a $u \in N^\perp$ such that $\operatorname{Exp}(u) = p$ and such that $t \to \operatorname{Exp}(tu)$ is a geodesic of minimal length joining p to N. It can contain no focal points for $0 \leqslant t < 1$, for otherwise it would not be minimizing at $t = 1$. For $t_0 \in (0, 1)$, there can be no other $w \in N^\perp$ with $\operatorname{Exp}(t_0 w) = \operatorname{Exp}(t_0 u)$ and $\|w\| = \|u\|$, for otherwise the corner between $\operatorname{Exp}(tw)$ and $\operatorname{Exp}(tu)$ at $t = t_0$ could be cut across to give a shorter curve joining p to N. Hence, $tu \in U_N$ for $0 \leqslant t < 1$, and $p \in \overline{\operatorname{Exp}(U_N)}$.

Remark. In case N is a point, say p_0, U_{p_0} is just the inside of the cut locus of p_0 [7, 13]. J. H. C. Whitehead proved Theorem 3.5 in this case, so our result can be considered as a generalization, and gives another proof when specialized. It is then natural to ask about generalizations of Klingenberg's results on the cut locus. We plan further work on this at a later time, but we can present the following result here, generalizing Lemma 1 of [7].

Theorem 4.2. *Suppose that M is a complete Riemannian manifold and that N is a submanifold. Suppose that $p, q \in N$, that $u_0 \in N_p^\perp$, $v_0 \in N_q^\perp$ satisfy:*

a) $\operatorname{Exp}(u_0) = \operatorname{Exp}(v_0)$, $\|u_0\| = \|v_0\|$.

b) *If u, v are any pair of vectors in $N/$ with* $\operatorname{Exp}(u) = \operatorname{Exp}(v)$, $\|u\| = \|v\|$, *but $u \neq v$, then $\|u\| \geqslant \|u_0\|$.*

c) $\operatorname{Exp}(u_0)$ *is not a focal point of N with respect to either of the geodesics $t \to \operatorname{Exp}(tu_0)$ or $t \to \operatorname{Exp}(tv_0)$.*

Then, the geodesics $t \to \operatorname{Exp}(tu_0)$ and $t \to \operatorname{Exp}(tv_0)$ meet at an angle of π at $t = 1$, i.e. without a corner.

Proof. Suppose the geodesics do not meet at an angle of π. For each real number r, let

$$C_r = \{v \in N^\perp : \|v\| = r\},$$

i.e. C_r is the "cylinder" of radius r about N in $N^\perp$. Let $r_0 = \|u_0\| = \|v_0\|$. It is easily seen, using c), that Exp restricted to C_{r0} is a submanifold map in a neighborhood of u_0 and v_0, say U_0 and U_1 respectively. By Gauss' lemma [1], geodesics $t \to \operatorname{Exp}(tu_0)$ and $t \to \operatorname{Exp}(tv_0)$ are perpendicular to, respectively, $\operatorname{Exp}(U_0 \cap C_{r0})$ and $\operatorname{Exp}(U_1 \cap C_{r0})$ at $t = 1$. Hence, if σ_0 and σ_1 meet in a corner at $t = 1$, the submanifolds $\operatorname{Exp}(U_0 \cap C_{r0})$ and $\operatorname{Exp}(U_1 \cap C_{r0})$ meet in general position at $\operatorname{Exp}(u_0)$. Using the implicit function theorem, $\operatorname{Exp}(U_0 \cap C_r)$ and $\operatorname{Exp}(U_1 \cap C_r)$ must meet if r is sufficiently close to r_0, but with, say, $r < r_0$. This contradicts hypotheses b), q.e.d.

Now we prove a series of results meant to elucidate the geometric significance of the hypotheses and conclusion of Theorem A.

Theorem 4.3. *Let $\varphi : P \to Q$ be a mapping between manifolds of the same dimension. Let D be a connected open set of P such that φ restricted to D has maximal rank.*

a) *If φ restricted to D is a covering map, then φ maps the boundary of D into the boundary of $\varphi(D)$.*

b) *If φ maps the boundary of D into the boundary of $\varphi(D)$, and if the closure of D in M is compact, then, for each $q \in \varphi(D)$, $\varphi^{-1}(q) \cap D$ is finite.*

Proof. Suppose that $p \in M$ lies on the boundary of D, but that $\varphi(p) = q'$ lies in $\varphi(D)$. Let U be a neighborhood of q lying on $\varphi(D)$. $\varphi^{-1}(U)$ contains points of D that are arbitrarily close to p, hence that can be connected to p by a curve in $\varphi^{-1}(U)$. The projection of such a curve in $\varphi(D)$ would then be a curve that could not be lifted to D via φ, contradicting that φ restricted to D is a covering map. This proves a).

To prove b), notice that otherwise $\varphi^{-1}(q) \cap D$ would contain an infinite sequence of points having a limit point. This limit point would have to lie on the boundary of D, for φ restricted to D is locally one-one, and would also map into $\varphi(D)$, contradiction.

Theorem 4.4. *Suppose that N is a closed submanifold of a complete Riemannian manifold M. Let V_N be the set of all $v \in N/$ such that there are no focal points of N along the geodesic*

$$t \to \operatorname{Exp} tv, \quad 0 \leqslant t \leqslant 1.$$

Let U_N be the subset of V_N consisting of those $v \in V_N$ such that $\operatorname{Exp} tv$, $0 \leqslant t \leqslant 1$, *is the unique geodesic of minimal length joining N to* $\operatorname{Exp} v$. *Suppose that* $\operatorname{Exp} V_N \to V_N$ *is a covering map.*

a) *If the closure of V_N in $N^\perp$ is compact, if $\pi_1(N)$ is finite, and if* $\dim (M - \operatorname{Exp} V_N) \leqslant \dim M - 2$, *then $\pi_1(M)$ is finite.*

b) *If $M - \operatorname{Exp} V_N$ is the union of a locally finite family of connected regularly imbedded submanifolds of M that all have codimension no less than three, and if $\pi_1(M) = 0$, then $U_N = V_N$.*

Proof. If the hypotheses of a) are satisfied, the inclusion map sends $\pi_1(\operatorname{Exp} V_N)$ onto $\pi_1(M)$ [5, p. 277]. Theorem 4.3 implies that φ restricted to V_N is a finite covering. This proves a). The hypotheses of b) imply that the inclusion map injects $\pi_1(\operatorname{Exp} V_N)$ in $\pi_1(M)$ [5, p. 278], hence that $\pi_1(\operatorname{Exp} V_N) = 0$, hence that φ restricted to V_N is a diffeomorphism. We must then show that $V_N \subset U_N$. Suppose that $v \in V_N$, but that there exists a $v_1 \in N^\perp$ whose length is no greater than that of v, but that $\operatorname{Exp} v = \operatorname{Exp} v_1$, $v \neq v_1$. We can suppose without loss of generality that $\operatorname{Exp} tv_1$, $0 \leqslant t \leqslant 1$, is a geodesic of minimal length joining $\operatorname{Exp} v$ to N. v_1 cannot belong to V_N, but there can be no focal points on $\operatorname{Exp} tv$ for $0 \leqslant t < 1$, hence v_1 must belong to the boundary of V_N, contradicting Theorem 4.3.

Finally, note in case N is a point, M is a simply connected globally symmetric space, that Crittenden has proved that $U_N = V_N$. It is expected that further work will extend this to focal point situations. It seems likely that Whitehead's results on the local structure of the conjugate locus [13] can be extended to the focal point situation. As a slight variant of Theorem A, proved with similar lifting-of-curve techniques, we present the following result.

Theorem 4.5. *Suppose that N is a closed submanifold of a complete Riemannian manifold M. Let $S \subset M$ be the singular image set of the mapping* $\operatorname{Exp}\colon N^\perp \to M$, *i.e. a point $p \in M$ lies on S if and only if p is a focal point of N with respect to some geodesic joining p to N. Now, if S is closed in M, then* Exp *restricted to* $\operatorname{Exp}^{-1}(M - S)$ *is a covering map.*

Northwestern University

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