CRITICAL SUBMANIFOLDS OF DIFFERENTIABLE MAPPINGS

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1. **The problems and definitions.** There is a general type of problem which contains critical point theory at one extreme, and immersion theory at another. The problems of interest to us lie between these two theories. A glance into their nature is afforded by a simple example to be given following some definitions. Let N^n and M^m denote two differentiable manifolds-with-boundary (perhaps empty) of dimensions n and m respectively, and let $f\colon N\to M$ be a continuous function with sufficient differentiability at any stage to allow the discussion to proceed. The *deficiency* of f at a point x of N is defined by (minimum (n, m)-rank f at x). Then x is said to be an *ordinary point* of f if f has deficiency zero at x; otherwise x is called a *critical point* of f. If each point of N is an ordinary point of f, we shall simply say f is *ordinary*. Note that if f is ordinary and $n\leqq m$ then f is just an immersion, while if $n\geqq m$ then (in terms of suitable coordinate systems) f is locally a projection.

To proceed with the example, let S^n denote the unit sphere in the $(n+1)$-dimensional euclidean space R^{n+1}, and consider the map $f\colon S^n\to R^r$ (induced in this instance by the natural projection $R^{n+1}\to R^r$), $r\leqq n$. Then we observe that: (a) the set of critical points of f is confined to the submanifold S^{r-1} of S^n; (b) $f|(S^n-S^{r-1})$, the restriction of f to the complement of S^{r-1} in S^n, and $f|S^{r-1}$ are ordinary; and (c) there exists a map $g\colon R^r\to R$ (here the natural projection $R^r\to R^1$) such that gf and $(gf)|S^{r-1}$ are Morse functions having the same number of critical points. Now if one attempts to replace S^n in the above by a compact manifold N^n and S^{r-1} by a submanifold K of N, one is immediately faced with the questions of which pairs (N, K) are admissible and what types of singularities to expect? Should it be possible to find an $f\colon N\to R^r$ satisfying the modified (a) and (b), $N-K$ must for instance admit r linearly independent vector fields and K must be immersible in R^r; while the addition of (c) would require that the Euler characteristics of K and N be congruent modulo two, since the number of critical points of a Morse function defined on a compact manifold is congruent modulo two to the Euler characteristic. These are some aspects of problems which we consider.

In this paper we give a condition of a local nature for the set of critical points of f in the deficiency 1 case to be (not just to be confined to) a submanifold of N, and conclude with a section concerning the effect on the structure of N of the existence of a function $f\colon N\to R^r$ subject to conditions weaker than (a) and (b) above. The results in this section depend largely on the behavior of $f|(N-K)$ and $f|K$, and make no essential use of the crucial behavior of f in a neighborhood of K. We shall take up this latter question in a subsequent publication.

We conclude this section with a historical note. The classical critical point theory of Morse is concerned with the case $r=1$ and $M=R$. The remaining case for $r=1$, namely $f\colon N^n\to S^1$, has been discussed for a compact manifold N by one of the writers [3]. The attack consists of lifting f, with greatest economy, to a covering map g in the diagram

$$\begin{tikzcd} W^n \arrow[r, "g"] \arrow[d] & R \arrow[d, "h"] \\ N \arrow[r, "f"] & S^1 \end{tikzcd}$$

The function hg is invariant under the appropriate factor group of $\pi_1(N)$ and ordinary critical point theory can be applied to g on a suitable fundamental domain.

The case $r=n$, which will be seen to be of special significance, has been treated by Tucker [4]. Fiberings with singularities have been discussed in various terms, for instance [2]. There is also a general spectral theory of maps by Fary [1].

2. **Deficiency 1.** An example of deficiency 1 is the map $(x^1, \cdots, x^n)\to(Q(x), x^2, \cdots, x^r)$, where Q is a nondegenerate quadratic form and $r\leqq n$. If Q is a definite form, this is intuitively a "fold" about the plane $Q_{x^1}=0$, $Q_{x^{r+1}}=0, \cdots, Q_{x^n}=0$. The term "fold" is most intuitive when $r=n$.

THEOREM 1. *If x_0 is a critical point of $f\colon R^n\to R^r$, $n\geqq r$, of deficiency 1 and if the critical point of $F=\lambda_i f^i$, with multipliers $\lambda\neq 0$, at x_0 is nondegenerate, then the critical points of f near x_0 form a manifold of dimension $r-1$.*

PROOF. A change in coordinates in R^n and R^r reduces the problem to the case in which $x_0=0$, $f^r_{x^j}(0)=0$, $|f^r_{x^p x^q}(0)|\neq 0$ with $p, q=1, 2, \cdots, r-1$, and there is no solution except $(c)=(0)$ for the system $c_p f^p_{x^j}(0)=0$. Then the equations

$$v_p f^p_{x^j} + u f^r_{x^j} = 0,$$
$$v_p v_p + u^2 = 1$$

admit solutions $x^j = \phi^j(v)$, (and also, for reference, $u = \psi(v)$), by virtue of the implicit function theorem, with (x, u, v) near $(0, 1, 0)$. Further the solution defines a manifold as required. To see this, note that $v_p f^p_{x^j}(\phi) + \psi f^r_{x^j}(\phi) = 0$ so that

$$f^q_{x^j}(\phi) + v_p f^p_{x^j x^h}\phi^h_{v_q} + \psi_{v_h} f^r_{x^j}(\phi) + \psi f^r_{x^j x^h}\phi^h_{v_q} = 0.$$

At the initial solution $f^q_{x^j}(0) + f^r_{x^j x^h}(0)\phi^h_{v_q}(0) = 0$. If there were numbers $(c) \neq (0)$ such that $\phi^h_{v_q}(0)c_q = 0$, it would follow that $c_q f^q_{x^j}(0) = 0$, contrary to hypothesis.

3. **Relationship to Stiefel-Whitney classes.** The following conventions will be used throughout. Let N denote an n-dimensional compact connected differentiable manifold, let K denote a compact k-dimensional differentiable submanifold-with-boundary of N, and let $N-K$ denote the complement of K in N. Unless the contrary is implied, we shall use the singular cohomology theory with coefficient domain Z_2. If V is an n-plane bundle over X and Y is a subspace of X we shall denote by $V|\,Y$ the restriction of V to Y. As usual $w_i(V)$ and $\bar{w}_i(V)$ will respectively denote the ith Stiefel-Whitney class and the dual ith Stiefel-Whitney class of V; while $w(V)$ and $\bar{w}(V)$ will denote the corresponding total classes. If M is a differentiable manifold with boundary, $\tau(M)$ will denote the tangent bundle of M; and $w_i(M)$ will denote $w_i(\tau(M))$ the ith Stiefel class of M, etc. Finally P^m will denote the real m-dimensional projective space, and R^r will denote the r-dimensional euclidean space. We will always assume that $n \geqq r$.

For the purposes of the following theorem let L be a disjoint union of compact submanifolds-with-boundary of N having maximum dimension k.

THEOREM 2. *With L and N as above, assume that there exists an ordinary mapping $f\colon N-L \to R^r$. Then $w_j(N) = 0$ for all j satisfying $n-r < j < n-k$.*

PROOF. The fact that f is ordinary implies that $\tau(N-L)$ is the Whitney sum of an $(n-r)$-plane bundle and a trivial r-plane bundle. Hence $w_t(N-L) = 0$ for $t > n-r$. Moreover it follows from Poincaré duality that $H^t(N, N-L) = 0$ for $t < n-k$, so that $i^*\colon H^t(N) \to H^t(N-L)$ is a monomorphism in this range. The proposition follows since $w_j(N-L) = i^*(w_j(N))$.

COROLLARY. *Suppose that* n, j, k *and* r *are integers satisfying* $n-r<j<n-k$ *and that the binomial coefficient* $C_{n+1,j}$ *is odd.* (*For example for* $n=2^a-2$ *and* $k\leqq r-2$ *any* j *strictly between* $n-r$ *and* $n-k$ *will do*). *Then there exists no ordinary mapping* $f\colon (P^n-L)\to R^r$.

For the next theorem recall that if K is immersible in R^r, then $\bar{w}_i(K)=0$ for $i>r-k$.

THEOREM 3. *Let* K *be a compact* (*not necessarily connected*) *k-dimensional submanifold-with-boundary of* N, *and assume that*:

(1) $w_1(N)=\cdots=w_{n-r}(N)=0$ *if* $n>r$,

(2) $w_{n-k}(N)=0$ *if* $k<n$,

(3) $\bar{w}_i(K)=0$ *for all positive* $i>n-k$,

(4) *there exists an ordinary mapping* $f\colon (N-K)\to R^r$.

Then the characteristic ring of N *is trivial* (*i.e.* $w_s(N)\cdot w_t(N)=0$ *for all* $s>0$ *and* $t>0$).

PROOF. Fixing a Riemannian metric on N, let W be the normal bundle of K in N, and write $\tau(N)|K$ as the Whitney sum $\tau(K)\oplus W$. Then (1), (2), (4), Theorem 2, and the naturality of the w_i's imply that $w(W)w(K)=1+$terms of degree greater than $n-k$. Since W is an $(n-k)$-plane bundle this implies in view of (3) that $\bar{w}(K)=w(W)$ and hence $w(\tau(N)|K)=1$. Thus $i^*(w_i(N))=0$ for $i\geqq 1$, where $i^*\colon H^i(N)\to H^i(K)$ is induced by inclusion. Let T be a small compact tubular neighbourhood of K in N, (if $k=n$ let $T=K$), and let C be the closure of $N-T$. Now suppose integers s and t exist which contradict the conclusion of the theorem. Since the inclusion $K\to T$ is a homotopy equivalence we conclude from the last equation that there exists an element a_s in $H^s(N,T)$ mapping onto $w_s(N)$ under the map $H^s(N,T)\to H^s(N)$ induced by inclusion. Next, as in the proof of Theorem 2, the fact that $f|C$, more correctly $f|(C\cap(N-K))$, is ordinary implies that $w_i(C)=0$ for $i>n-r$; and since $\tau(C)=\tau(N)|C$ it follows from (1) that $w(C)=1$. Again there is an element a_t in $H^t(N,C)$ mapping onto $w_t(N)$ under the natural map $H^t(N,C)\to H^t(N)$. However $a_s\cdot a_t\in H^{s+t}(N,T\cup C)=0$, which contradicts $w_s(N)\cdot w_t(N)\neq 0$ by the naturality of cup products.

COROLLARY. *Suppose that* K *is an* $(n-1)$*-dimensional compact submanifold of* P^n *where* n *is an odd integer not of the form* 2^a-1, a *an integer. Then there exists no differentiable mapping* $f\colon P^n\to R^n$ *such that* $f|(P^n-K)$ *and* $f|K$ *are ordinary.*

The proof of the following theorem is similar to that of Theorem 3 and is more straightforward. Note that if a compact orientable $(r-1)$-dimensional manifold M is immersible in R^r then $w(M)=1$.

THEOREM 4. *Let K be a k-dimensional compact submanifold-with-boundary of N and assume further that*: (1) $w(K)=1$, *and* (2) *for some integers s and t satisfying $s\geqq n-r+1$ and $t\geqq n-k+1$ we have* $w_s(N)\cdot w_t(N)\neq 0$. *Then there exists no ordinary mapping* $f\colon (N-K)\rightarrow R^r$.

COROLLARY. *Suppose n has the form 2^a-2, $a>2$; and suppose that K is a compact orientable $(r-1)$-dimensional submanifold of P^n where $2r\geqq n+3$. Then there exists no differentiable mapping $f\colon P^n\rightarrow R^r$ such that $f|(N-K)$ and $f|K$ are ordinary.*

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