

Day 4 : 14th January, 2026

Exercise 4.1: (Cone of a space)

Given a space X , consider the quotient space (known as the *cone* of X)

$$CX := \frac{X \times [0, 1]}{X \times \{0\}}.$$

Draw the cones for the following spaces.

- i) $X = \{0, 1\}$.
- ii) $X = \{0, 1, 2\}$.
- iii) $X = [0, 1]$.
- iv) $X = S^1$.
- v) Show that for any sphere S^n , the cone is homeomorphic to $(n + 1)$ -disc, that is, $CS^n \cong D^{n+1}$.

Exercise 4.2: (Suspension of sphere)

Given a space X , consider the quotient space (known as the *(unreduced) suspension space* of X)

$$\Sigma X := \frac{X \times [0, 1]}{X \times \{0, 1\}}.$$

Show that for any $n \geq 0$, we have a homeomorphism $S^{n+1} \cong \Sigma S^n$, where $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$ is the n -sphere as a subspace in \mathbb{R}^{n+1} .

Exercise 4.3: (Attaching map)

$5 + (3 + 2) + 5 = 15$

- a) Suppose $A \subset X$ is a subspace and $f : A \rightarrow Y$ is a continuous map. On the disjoint union $X \sqcup Y$, consider the relation $u \sim v$ if and only if
 - i) $u = v$, or
 - ii) $u, v \in A$, $f(u) = f(v)$, or
 - iii) $u \in A$, $v = f(u)$, or
 - iv) $v \in A$, $u = f(v)$, or

Check that \sim is an equivalence relation on $X \sqcup Y$.

- b) Let us denote the quotient space under the equivalence relation as $X \cup_f Y$, which is called the *attaching space* obtained by the *attaching map* f . Check that the maps

$$\begin{aligned} p : X &\rightarrow X \cup_f Y & q : Y &\rightarrow X \cup_f Y \\ x &\mapsto [x], & \text{and} & y &\mapsto [y] \end{aligned}$$

are continuous. Moreover, check that $p|_A = q \circ f$.

- c) Suppose we have maps $\varphi : X \rightarrow Z$ and $\psi : Y \rightarrow Z$ such that the outer square in the diagram commutes (i.e., $\varphi|_A = \psi \circ f$): Then, show that there exists a unique continuous map $h : X \cup_f Y \rightarrow Z$ making the triangles commutative, i.e., $h \circ p = \varphi$ and $h \circ q = \psi$.

Definition 4.4: (Group action)

Let G be a group and X be a topological space. We say that G acts on X if there exists a map $\varphi : G \times X \rightarrow X$ such that

- i) For any $x \in X$, $\varphi(e, x) = x$;
- ii) For any $g, h \in G$ and $x \in X$, $\varphi(g, \varphi(h, x)) = \varphi(g \cdot h, x)$.

From now onward, we will write $\varphi(g, x) = g \cdot x$.

We say X is a G -space if an action of G on X is given. On any G -space, we have a natural equivalence relation defined as

$$x \sim y \Leftrightarrow \exists g \in G \text{ such that } y = g \cdot x.$$

The equivalence classes are called *orbits* of G . The corresponding quotient space is denoted by X/G .

Exercise 4.5: (Based on quotient space via a group)

- i) Let $X = \mathbb{R}$ and $G = \mathbb{Z}$. Let G acts on X by $n \cdot x = x + n$, for $x \in X$ and $n \in G$. The quotient space \mathbb{R}/\mathbb{Z} is homeomorphic to S^1 .
- ii) Take $X = \mathbb{R}^2$ and $G = \mathbb{Z}$. The action is given by

$$n \cdot (x, y) = (x + n, y).$$

Show that X/G is homeomorphic to infinite cylinder $\{(x, y, z) : x^2 + y^2 = 1, z \in \mathbb{R}\}$.

- iii) Take $X = S^n$ and $G = \mathbb{Z}_2 = \{\pm 1\}$. The action is defined as $-1 \cdot x = -x$ for $x \in S^n$. Then $X/G \cong \mathbb{RP}^n$.

Exercise 4.6: (Lens Spaces)

Consider $S^3 \subseteq \mathbb{C}^2$ as

$$S^3 := \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}.$$

Let p, q be co-prime numbers. Let a generator $g \in \mathbb{Z}_p$ acts on S^3 by

$$g \cdot (z_1, z_2) = (e^{(2\pi i)/p} z_1, e^{(2\pi i)/q} z_2).$$

The quotient space is denoted by $L(p, q)$ and is called a *Lens space*.

- i) Show that $L(2, 1)$ is homeomorphic to \mathbb{RP}^3 .
- ii) If p divides $q - q'$, then $L(p, q) \cong L(p, q')$.

4.1. Hausdorff Quotient Spaces

Definition 4.7: (Open equivalence relation)

Let \sim be an equivalence relation on a topological space X . For any set $A \subseteq X$, we define

$$[A] := \{x \in X : \exists a \in A \text{ such that } a \sim x\}.$$

The equivalence is called *open* if $[A]$ is open whenever A is open in X .

Exercise 4.8: (*Open equivalence relation and the quotient map*)

An equivalence relation \sim is open if and only if the quotient map $\pi : X \rightarrow X / \sim$ is open.

Proposition 4.9: (*Quotient space is Hausdorff*)

Let \sim be an open equivalence relation on a space X . Then

$$R = \{(x, y) : x \sim y\} \subseteq X \times X$$

is closed if and only if the quotient space X / \sim is Hausdorff.

Proof: Let X / \sim is Hausdorff. We will show that $R^c = (X \times X) \setminus R$ is open in $X \times X$. Let $(x, y) \in R^c$, that is, $x \not\sim y$. This implies $[x] \neq [y]$ and hence there exists open neighborhoods U and V of $[x]$ and $[y]$, respectively, such that $U \cap V = \emptyset$. Let $\pi : X \rightarrow X / \sim$ be the quotient map. Then $\tilde{U} := \pi^{-1}(U)$ and $\tilde{V} := \pi^{-1}(V)$ are open neighborhoods of x and y , respectively. We will show that $\tilde{U} \times \tilde{V} \subseteq R^c$. Suppose $(a, b) \in (\tilde{U} \times \tilde{V}) \cap R$, then $(a, b) \in R$ implies $a \sim b$ which implies $U \ni [a] = [b] \in V$, a contradiction. Thus, $\tilde{U} \times \tilde{V} \subseteq R^c$ and hence R is closed.

On the other hand, let R be closed in $X \times X$. We need to show that X / \sim is Hausdorff. Let $[x] \neq [y] \in X / \sim$. This implies $x \not\sim y$ which implies $(x, y) \in R^c$. Since R^c is open, there exists basic open sets $U \times V$ such that $(x, y) \in U \times V \subseteq R^c$. Since the relation is open, $\tilde{U} := \pi(U)$ and $\tilde{V} := \pi(V)$ are open subsets of X / \sim . Since $U \times V \in R^c$, so $\tilde{U} \cap \tilde{V} = \emptyset$ and hence X / \sim is Hausdorff. \square

Exercise 4.10: (*Real projective space is Hausdorff*)

Show that \mathbb{RP}^n is Hausdorff.