Day 1 : 5th January, 2026

1.1. Equivalence Relation

Definition 1.1: (*Relation*)

Given a set X, a *relation* on it is a subset $\mathcal{R} \subset X \times X$. We say $\mathcal{R}$ is an *equivalence relation* if the following holds.

a) **(Reflexive)** For each $x \in X$ we have $(x, x) \in \mathcal{R}$.
b) **(Symmetric)** If $(x, y) \in \mathcal{R}$, then $(y, x) \in \mathcal{R}$.
c) **(Transitive)** If $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(x, z) \in \mathcal{R}$.

For any $x \in X$, the *equivalence class* (with respect to the equivalence relation $\mathcal{R}$) is defined as the set

$$[x] := \{y \in X \mid (x, y) \in \mathcal{R}\}.$$

We shall denote $x \sim_{\mathcal{R}} y$ (sometimes also denoted $x\mathcal{R}y$, or simply $x \sim y$) whenever $(x, y) \in \mathcal{R}$. The collection of equivalence classes are sometimes denoted as $X/_{\sim}$.

Exercise 1.2:

i) Given an equivalence relation $\mathcal{R}$ on X, check that any two equivalence classes are either disjoint or equal (i.e., they cannot have nontrivial intersection).
ii) Suppose X is a given set, and $A \subset X$ is a nonempty subset. Define the relation $\mathcal{R} \subset X \times X$ as follows.

$$\mathcal{R} := \{(x, x) \mid x \in X \setminus A\} \bigcup \{(a, b) \mid a, b \in A\}.$$

a) Check that $\mathcal{R}$ is an equivalence relation.
b) Identify the equivalence classes. We shall denote the collection of equivalence classes as X/A.
c) What is X/X ?

Definition 1.3: (*Partition*)

Given a set X, a *partition of X* is a collection of subsets $X_\alpha \subset X$ for some indexing set $\alpha \in \mathcal{J}$, such that the following holds.

- $X_\alpha \cap X_\beta = \emptyset$ for any $\alpha, \beta \in \mathcal{J}$ with $\alpha \neq \beta$.
- $X = \bigcup_{\alpha \in \mathcal{J}} X_\alpha$.

Exercise 1.4: (*Partitions and equivalence relations*)

Given an equivalence relation $\mathcal{R}$ on a set X, show that the collection of equivalence classes is a partition of X. Conversely, given any partition of X, show that there exists a unique equivalence relation which gives that partition.

1.2. Topology

Definition 1.5: (*Topology*)

Given a set X, a *topology* on X is a collection $\mathcal{T}$ of subsets of X (i.e., $\mathcal{T} \subset \mathcal{P}(X)$), such that the following holds.

a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
b) $\mathcal{T}$ is closed under arbitrary unions. That is, for any collection of elements $U_\alpha \in \mathcal{T}$ with $\alpha \in \mathcal{J}$, an indexing set, we have $\bigcup_{\alpha \in \mathcal{J}} U_\alpha \in \mathcal{T}$.
c) $\mathcal{T}$ is closed under finite intersections. That is, for any finite collection of elements $U_1, \ldots, U_n \in \mathcal{T}$, we have $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$.

The tuple $(X, \mathcal{T})$ is called a topological space.

Example 1.6:

Given any set X we always have two standard topologies on it.

a) **(Discrete Topology)** $\mathcal{T}_0 = \mathcal{P}(X)$.
b) **(Indiscrete Topology)** $\mathcal{T}_1 = \{\emptyset, X\}$.

They are distinct whenever X has at least 2 points.

Exercise 1.7:

Given any set X, verify that both the discrete and the indiscrete topologies are indeed topologies, that is, check that they satisfy the axioms.

Definition 1.8: (*Open and closed sets*)

Given a topological space $(X, \mathcal{T})$, a subset $U \subset X$ is called an *open set* if $U \in \mathcal{T}$, and a subset $C \subset X$ is called a *closed set* if $X \setminus C \in \mathcal{T}$ (i.e., if $X \setminus C$ is open).

Exercise 1.9: (*Topology defined by closed sets*)

Given X, suppose $\mathcal{C} \subset \mathcal{P}(X)$ is a collection of subsets that satisfy the following.

a) $\emptyset \in \mathcal{C},\ \ X \in \mathcal{C}$.
b) $\mathcal{C}$ is closed under arbitrary intersections.
c) $\mathcal{C}$ is closed under finite unions.

Define the collection,

$$\mathcal{T} := \{U \subset X \mid X \setminus U \in \mathcal{C}\}.$$

Prove that $\mathcal{T}$ is a topology on X.

1.3. Basis of a topology

Definition 1.10: (*Basis of a topology*)

Given a topological space $(X, \mathcal{T})$, a *basis* for it is a sub-collection $\mathcal{B} \subset \mathcal{T}$ of open sets such that every open set $U \in \mathcal{T}$ can be written as the union of some elements of $\mathcal{B}$.

Example 1.11: (*Usual topology on* $\mathbb{R}$)

The collection of all open intervals $\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}\}$ is a basis for the usual topology on the real line $\mathbb{R}$.

Proposition 1.12: (*Necessary condition for basis*)

Suppose $(X, \mathcal{T})$ is a topological space, and consider a basis $\mathcal{B} \subset \mathcal{T}$. Then, the following holds.

a) [**(B1)**] For any $x \in X$, there exists some $U \in \mathcal{B}$ such that $x \in U$.
b) [**(B2)**] For any $U, V \in \mathcal{B}$ and any element $x \in U \cap V$, there exists some $W \in \mathcal{B}$ such that $x \in W \subset U \cap V$.

Exercise 1.13:

Prove the above proposition.

Example 1.14:

Consider the collection

$$\mathcal{B} = \{(a, \infty) \mid a \in \mathbb{R}\}.$$

This is a subcollection of open sets of $\mathbb{R}$ (in the usual topology), and moreover, $\mathcal{B}$ satisfies both B1 and B2 (Check!). But $\mathcal{B}$ is *not* a basis for the usual topology on $\mathbb{R}$. Thus, B1 and B2 is not a sufficient condition for $\mathcal{B}$ to be a basis.

Exercise 1.15: (*Topology generated by a basis*)

Suppose $\mathcal{B} \subset \mathcal{P}(X)$ is a collection of subsets of X satisfying (B1) and (B2). Consider $\mathcal{T}$ to be the collection of all possible unions of elements of $\mathcal{B}$. Show that $\mathcal{T}$ is a topology on X and $\mathcal{B}$ is a basis for it.

1.4. Fine and coarse topology

Definition 1.16: (*Fine and coarse topology*)

Given two topologies $\mathcal{T}_1, \mathcal{T}_2$ on a set X, we say that $\mathcal{T}_1$ is *finer (stronger)* than $\mathcal{T}_2$ (and $\mathcal{T}_2$ is said to be *coarser (weaker)* than $\mathcal{T}_1$) if $\mathcal{T}_1 \supset \mathcal{T}_2$.

1.5. Limit points and closure

Definition 1.17: (*Limit point*)

Given a space X and a subset $A \subset X$, a point $x \in X$ is called a *limit point* (or *cluster point*, or *point of accumulation*) of A if for any open set $U \subset X$, with $x \in U$, we have $A \cap U$ contains a point other than x.

Exercise 1.18:

Show that if A is a closed set of X, then A contains all of its limit points. Give an example of a space X and a subset $A \subset X$, such that

a) there is a limit point x of A which is not an element of A, and
b) there is an element $a \in A$ which is not a limit point of A.

Definition 1.19: (*Adherent and isolated points*)

Given a subset $A \subset X$, a point $x \in X$ is called an *adherent point* (or *points of closure*) if every open neighborhood of x intersects A. An adherent point which is *not* a limit point is called an *isolated point* of A (which is then necessarily an element of A).

Definition 1.20: (*Closure of a set*)

Given $A \subset X$, the *closure* of A, denoted $\overline{A}$ (or $\mathrm{cl}(A)$), is the smallest closed set of X that contains A.

Exercise 1.21:

Show that $A \subset X$ is closed if and only if $A = \overline{A}$.

Exercise 1.22:

For any $A \subset X$, show that $\overline{A}$ is the intersection of all closed sets of X containing A. In particular, $A \subset \overline{A}$.