Algebraic Topology I: Homework #4

Based on category theory

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Problem 1

i) Let X be a set. Describe explicitly,
   a) the objects of the discrete category, $\mathbf{Disc}$,
   b) the morphisms,
   c) composition.

ii) Show that every function : $X \to Y$ defines a functor $\mathbf{Disc}(X) \to \mathbf{Disc}(Y)$, and that every such functor arises uniquely this way.

Problem 2

A binary relation $\preceq$ on a set X is called a preorder if it is reflexive and transitive. Define a category $\mathscr{P}$ by:

$$\begin{aligned}&\text{Obj : elements of } P\\&\text{a morphism } x \to y \text{ exists iff } x \preceq y.\end{aligned}$$

Show that $\mathscr{P}$ is a category.

Problem 3

i) Let X be a topological space and

$$C(X) := \{f : X \to X : f \text{ is continuous}\}.$$

Show that $C(X)$ is a commutative ring with unity under pointwise operations:

$$f + g := x \mapsto f(x) + g(x) \quad \text{and} \quad f \cdot g : x \mapsto f(x)g(x).$$

for all $x \in X$.

ii) Show that $X \mapsto C(X)$ gives a (contravariant) functor $\mathbf{Top} \to \mathbf{Rings}$.

Problem 4

Let p be a fixed prime in $\mathbb{Z}$. Let $\mathbf{Ab}$ be the category of abelain groups. Define a functor

$$F : \mathbf{Ab} \to \mathbf{Ab}, \quad G \mapsto G/pG$$
$$F(f) : x + pG \mapsto f(x) + pH,$$

where $f : G \to H$ is a group homomorphism.

i) Show that if f is surjection, then $F(f)$ is a surjection.

ii) Give an example of an injective homomorphism f for which $F(f)$ is not injective.

Problem 5

Let $\mathscr{C}$ and $\mathscr{D}$ be categories, and let $\sim$ be a congruence on $\mathscr{C}$. If $T : \mathscr{C} \to \mathscr{D}$ is a functor with $T(f) = T(g)$, whenever $f \sim g$, then T defines a functor $T' : \mathscr{C}' \to \mathscr{D}$ by

$$T'(X) = T(X), \text{for any object} \quad X \quad \text{and}$$
$$T([f]) = T(f) \text{ for any morphism } \quad f.$$

where $\mathscr{C}$ is the quotient cateogry

Problem 6

Let G be a group. Show that G gives rise to a cateogory $\mathscr{G}$ with single object, denoted $*$, and $\mathrm{Hom}_{\mathscr{G}}(*, *) = G$.