Algebraic Topology I: Homework #7

Based on homotopy

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Problem 1

Show that if $f_0, f_1 : X \to Y$ are homotpic, then for any $g : Y \to Z$, the maps $g \circ f_0$ and $g \circ f_1$ are homotopic.

Problem 2

Let (X, x_0) be a pointed space. For a loop α based at x_0, let $[\alpha]$ denote the homotopy class of α, where homotopies are taken relative to the basepoint x_0. Define

$$\pi_1(X, x_0) = \{[\alpha] : \alpha \text{ is a loop based at } x_0\}.$$

In the lectures, we have seen pictorially the operation on $\pi_1(X, x_0)$ given by concatenation of loops. In this problem, you are asked to prove these properties formally and give explicit homotopies.

i) Show that for loops α and β based at x_0, the operation

$$[\alpha] * [\beta] = [\alpha * \beta]$$

is well defined.

ii) Show that there exists $[e] \in \pi_1(X, x_0)$ such that for any loop α based at x_0,

$$[\alpha] * [e] = [\alpha] = [e] * [\alpha].$$

iii) Show that for any loop α based at x_0, there exists a loop β based at x_0 such that

$$[\alpha] * [\beta] = [e] = [\beta] * [\alpha].$$

iv) Show that for any loops α, β, γ based at x_0,

$$([\alpha] * [\beta]) * [\gamma] = [\alpha] * ([\beta] * [\gamma]).$$

Problem 3

Recall that given space X and Y, the set $[X, Y]$ denote the set of all homotopy classes of maps of X into Y.

i) Let $\mathbb{I} = [0, 1]$, Show that for any space X, the set $[X, \mathbb{I}]$ is singleton.
ii) Show that if Y is path connected, then the set $[\mathbb{I}, Y]$ is singleton.
iii) Show that if Y is contractible, then for any X, the set $[X, Y]$ is singleton.
iv) Show that a contractible space is path connected.
v) Show that if X is contractible, then $\pi_1(X, x_0)$ is the trivial group.
vi) Show that if X is contractible and Y is path connected, then the set $[X, Y]$ is singleton.

Problem 4

If X is path connected, then show that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

Problem 5

Let α be a loop based at x_0. Show that if α is null-homotopic, then for any loop β based at x_0 we have

$$[\alpha] * [\beta] = [\beta] = [\beta] * [\alpha].$$

Problem 6

Use the standard homeomorphism

$$h : [a, b] \to [0, 1], \quad \mapsto \frac{s-a}{s-b}$$

to show that

$$f : [0,1] \to [0,1], \quad s \mapsto \begin{cases} 2s, & s \in \left[0, \frac{1}{4}\right] \\ s + \frac{1}{4}, & s \in \left[\frac{1}{4}, \frac{1}{2}\right] \\ \frac{s+1}{2}, & s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

is homotopic to the identity on $[0, 1]$.

Problem 7

Consider the identity map, $1_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$, as a closed curve on the torus $\mathbb{S}^1 \times \mathbb{S}^1$ in Figure 1 and find explicity two other closed curves on the torus such that all three belongs to different homotopy classes.

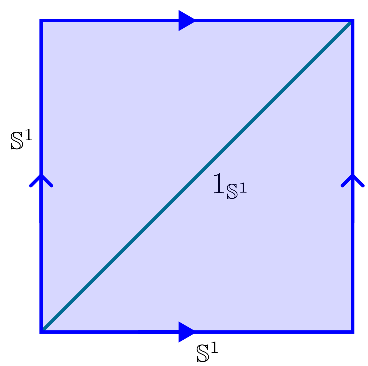

Figure 1: Loop on the torus

Problem 8

Show that the following X, Y are homotpically equivalent spaces. Are they homeomorphic?

i) $X = \mathbb{S}^n, Y = \mathbb{S}^n \times \mathbb{R}^m$;
ii) $X = \mathbb{R}^n, Y = \{0\}$;
iii) $X = \mathbb{S}^{n-1}, Y = \mathbb{R}^n \setminus \{0\}$;
iv) $X = \mathbb{S}^1 \vee \mathbb{S}^1, Y = \text{punctured torus}$;
v) $X = \mathbb{S}^1, Y = \text{punctured } \mathbb{RP}^2$.

Problem 9

Show that a circle $\mathbb{S}^1$, a cylinder $\mathbb{S}^1 \times [0, 1]$ and a solid torus $\mathbb{S}^1 \times \mathbb{D}^2$ are mutually homotopic.

Problem 10

Show that if $f : X \to \mathbb{S}^n$ is not surjective, then f is nullhomotopic. Give an example of a space X and a surjective map $f : X \to \mathbb{S}^n$ such that f is not null homotopic.