# Algebraic Topology I: Homework #2

Based on review of point set topology

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**Theory :**

i) Given a space $X$ and a subset $A \subset X$, a point $x \in X$ is called a *limit point* (or *cluster point*, or *point of accumulation*) of $A$ if for any open set $U \subset X$, with $x \in U$, we have $A \cap U$ contains a point other than $x$.

ii) Given $A \subset X$, the *closure* of $A$, denoted $\overline{A}$ (or $\mathrm{cl}(A)$), is the smallest closed set of $X$ that contains $A$.

iii) Given $A \subset X$, the *interior* of $A$, denoted $\mathring{A}$ (or $\mathrm{int}(A)$), is the largest open set contained in $A$. A point $x \in \mathring{A}$ is called an *interior point* of $A$.

iv) Given $A \subset X$, the *boundary* of $A$, denoted $\partial A$ (or $\mathrm{bd}(A)$), is defined as

$$\partial A = \overline{A} \cap \overline{(X \setminus A)}.$$

v) Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a function $f : X \to Y$ is said to be a *homeomorphism* if the following holds.

   a) $f$ is bijective, with inverse $f^{-1} : Y \to X$.

   b) $f$ is continuous.

   c) $f$ is open (or equivalently, $f^{-1}$ is continuous).

**Problem 1**

Show that if $A$ is a closed set of $X$, then $A$ contains all of its limit points. Give an example of a space $X$ and a subset $A \subset X$, such that

i) there is a limit point $x$ of $A$ which is not an element of $A$, and
ii) there is an element $a \in A$ which is not a limit point of $A$.

**Problem 2**

Show the following.

i) For any $A \subset X$, $\mathring{A}$ is the union of all open sets contained in $A$. In particular, show that $A \subset X$ is open if and only if $A = \mathring{A}$.
ii) Given $A \subset X$, a point $x \in X$ is an interior point of $A$ if and only if there exists some open set $U \subset X$ such that $x \in U \subset A$.

**Problem 3**

Compute the boundary of the following subsets $A \subset X$.

a) $X$ is any space, and $A = X$.
b) $X$ is any space, and $A = \emptyset$.
c) $X$ is a discrete space, and $\emptyset \neq A \subsetneq X$.
d) $X$ is an indiscrete space, and $\emptyset \neq A \subsetneq X$.
e) $X = \mathbb{R}$ and $A = \mathbb{Z}$.
f) $X = \mathbb{R}$ and $A = \mathbb{Q}$.
g) $X = \mathbb{R}$ and $A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$.

**Problem 4**

Let $A \subset X$ and $B \subset Y$ be two topological spaces. Let $X \times Y$ be equipped with the product topology and $A \times B$ be equipped with the subspace topology.

i) Show that $A \times B$ is homeomorphic to $B \times A$.
ii) Prove that if $A, B$ are closed, then $A \times B$ is closed in $X \times Y$.
iii) Prove that $\overline{A \times B} = \overline{A} \times \overline{B}$. Is it true for interiors? What about the boundaries?

**Problem 5**

i) Prove that a space $X$ is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$.
ii) Prove that if $Y$ is a Hausdorff space and $f : X \to Y$ is a continuous map, then the graph

$$\Gamma_f := \{(x, f(x)) : x \in X\} \subseteq X \times Y$$

is closed in $X \times Y$.

iii) Let $Y$ be a compact space. Prove that if a map $f : X \to Y$ has closed graph $\Gamma_f$, then $f$ is continuous.

**Problem 6**

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that its graph is

i) closed;

ii) connected;

iii) path connected.

**Problem 7**

*Pasting lemma:* Suppose $X = A \cup B$, for some closed sets $A, B \subset X$. Let $f : A \to Y, g : B \to Y$ be given continuous maps, such that $f(x) = g(x)$ for any $x \in A \cap B$. Then, there exists a (unique) continuous map $h : X \to Y$ such that

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B. \end{cases}$$

**Problem 8**

i) Prove that $\mathbb{R}^n \setminus \mathbb{R}^m$ is homeomorphic to $\mathbb{S}^{n-m-1} \times \mathbb{R}^{m+1}$.

ii) Prove that

$$\mathbb{S}^n \cap \left\{x \in \mathbb{R}^{n+1} : x_1^2 + \ldots + x_k^2 \leq x_{k+1}^2 + \ldots + x_{n_1}^2\right\}$$

is homeomorphic to $\mathbb{S}^{k-1} \times \mathbb{D}^{n-k+1}$.

iii) Prove that $GL_n(\mathbb{R})$ is homeomorphic to $SL_n(\mathbb{R}) \times GL_1(\mathbb{R})$.