Matrix Groups: Homework #2

Based on linear transformations

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Conventions

1. In all the problem below, unless specified, V, W denotes the vector spaces over $\mathbb{R}$.
2. Linear map and linear transformation will be used interchangeably.

Theory

Let $\{v_1, v_2, ..., v_m\}$ and $\{w_1, w_2, ..., w_n\}$ be bases of V and W respectively. Let $T : V \to W$ be a linear map. Since $T(v_i) \in W$ for each i, we can write

$$T(v_i) = \sum_{j=1}^{n} a_{ij} w_j, \quad 1 \le i \le m.$$

Then the matrix of T with respect to the above choice of bases is

$$[T]_W^V = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1i} & a_{2i} & \dots & a_{mi} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}_{m \times n}.$$

Problem 1

This problem we have already discussed in today's lecture. Try to go through the steps and think it through The portions which involve group theory, and if you are not comfortable, you can skip that.

Consider all rotations of the plane $\mathbb{R}^2$ about the origin. Work through the following steps to describe each rotation by an explicit linear map and then identify the full set.

1. **Rotation formula**

   Write the rotation by angle θ as a linear transformation:

   $$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

   acting on column vectors $\begin{pmatrix} x \\ y \end{pmatrix}$.

   Show that R_θ is linear and sends the unit circle to itself.

2. **Basic properties**

   a) Prove that $\det(R_\theta) = 1$, so $R_\theta \in \mathrm{GL}_2(\mathbb{R})$.

   b) Show $R_\theta^T R_\theta = I$, so $R_\theta \in \mathrm{SO}(2)$.

3. **Composition and inverses**

   Compute $R_\theta R_\varphi$ and prove:

$$R_\theta R_\varphi = R_{\theta+\varphi}.$$

Identify the identity R_0 and inverse R_θ^{-1}.

4. **Identify the set of all rotations**

   Let $\mathcal{R} = \{R_\theta : \theta \in \mathbb{R}\}$.

   a) Show $\mathcal{R}$ is closed under composition and inverses, hence a subgroup of $\mathrm{GL}_2(\mathbb{R})$ (the subgroup part can be left for now).

   b) Show $\mathcal{R} \subseteq \mathrm{SO}(2)$.

   c) Prove that any $A \in \mathrm{SO}(2)$ equals R_θ for some θ. Conclude $\mathcal{R} = \mathrm{SO}(2)$.

5. **Group isomorphisms**. This part will be discussed later.

   Show:

$$\mathcal{R} \cong S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$$

   via the maps $R_\theta \mapsto e^{i\theta}$ and $\theta \mapsto [\theta]$.

Problem 2

The following problem discusses the relation between composition of linear maps and corresponding matrices.

Let V, W, U be vector spaces. Let $\{v_1, \ldots, v_m\}$, $\{w_1, \ldots, w_n\}$ and $\{u_1, \ldots, u_k\}$ be bases for V, W and U, respectively. Let

$$T : V \to W, \quad S : W \to U$$

be linear maps. Let A and B be the matrices associated with T and S, respectively. Then the matrix associated with $S \circ T$ is AB.

Hint: Write

$$T(v_i) = \sum_{j=1}^{n} a_{ij} w_j \text{ and } S(w_j) = \sum_{r=1}^{k} b_{jr} u_r.$$

Then show that $S \circ T(v_i) = \sum_{r=1}^{k} c_{ir} u_r$, where $c_{ir} = \sum_{j=1}^{n} a_{ij} b_{jr}$. Finally conclude that the matrix is AB.

Problem 3

For a linear transformation $T : V \to W$, define

$$\begin{aligned} \ker T &:= \{v \in V : T(v) = 0\} \text{ and} \\ \operatorname{im} T &:= T(V) = \{w \in W : \exists v \in V \text{ such that } T(v) = w\}. \end{aligned}$$

Show that $\ker T$ is a subspace of V and $\operatorname{im} T$ is a subspace of W.

Hint: For a vector space V, to check W is a subspace of V it is enough to check the following properties:

- Closed under addition
- Closed under scalar multiplication
- $0 \in W$

Problem 4

Find the range (dimension of the image) and kernel of $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + z \\ x + y + 2z \\ 2x + y + 3z \end{pmatrix}.$$

Problem 5

Let $V = \mathbb{R}^2$. Let $\mathcal{B} = \{(9, 2), (4, -3)\}$; and $\mathcal{C} = \{(2, 1), (-3, 1)\}$ be ordered bases of V. Find the change of basis matrix $P_{C \leftarrow B}$ and $P_{B \leftarrow C}$ if $\mathcal{B}$ and $\mathcal{C}$ are ordered bases of $\mathbb{R}^2$.