MATRIX GROUPS

(MTH565)

Quiz 7

Thursday, 30th October 2025

Name: ______________________________

Roll Number: ______________________________

Obtained Marks: ______________________ **/10**

EXAMINATION INSTRUCTIONS

1. This is a **Closed Book Examination**.
2. Answer all questions in the space provided on subsequent pages.
3. Show all necessary working steps clearly and legibly.
4. State any theorems or results used. Only results discussed in lectures may be used without proof.
5. The total point for the problems is 12, but the maximum obtainable score is 10.
6. **Duration:** 60 minutes.

- -

Good Luck!

Problem Set

Problem 1

To prove $\mathbb{S}^2$ is a manifold, we will use the stereographic projection. For this, the function $f : \mathbb{S}^2 - \{(0,0,1)\} \to \mathbb{R}^2$ is defined so that $f(p)$ equals to xy-coordinates of the intersection of the plane $z = -1$ with the line containing $(0,0,1)$ and p, see Figure 1.

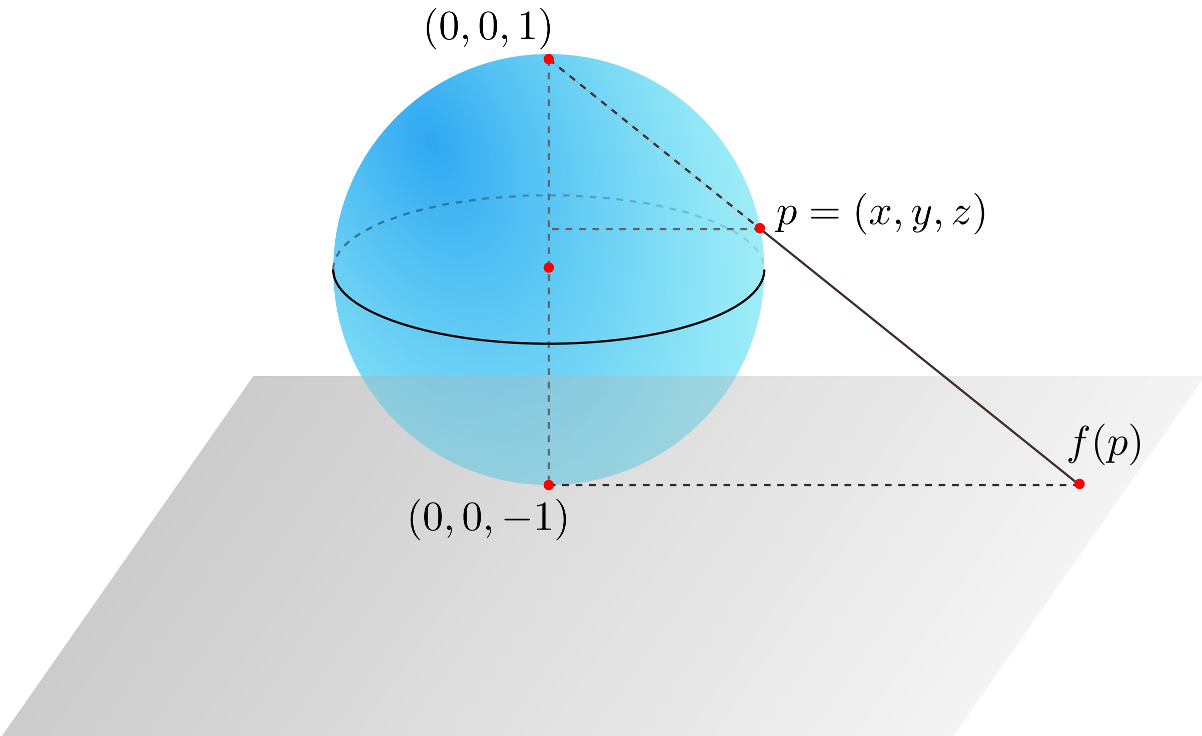

Figure 1: Stereographic projection

(i) Use similar triangles to show that

$$f(x,y,z) = \frac{2}{1-z}(x,y).$$

(ii) Find a formula for $f^{-1} : \mathbb{R}^2 \to \mathbb{S}^2 - \{(0,0,1)\}$.

(iii) Find a formula for the function $g : \mathbb{S}^2 - \{(0,0,-1)\} \to \mathbb{R}^2$, defined so that $g(p)$ equals the xy-coordinates of the intersection of the plane $z = 1$ with the line containing $(0,0,-1)$ and p. Also find a formula for $g^{-1} : \mathbb{R}^2 \to \mathbb{S}^2 - \{0,0,-1\}$.

(You can either write a formula and verify that is well defined or derived in a similar fashion.)

(iv) Find an explicit formula for the composition

$$g \circ f^{-1} : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}^2 - \{(0,0)\}.$$

$\boxed{2+2+2+1=7}$

Problem 2

Let G be a (not necessarily path-connected) matrix group (closed subgroup of $GL(n, \mathbb{K})$). Define the *identity component*, G_0, of G as

$$G_0 := \{g \in G : \exists \text{ continuous } \gamma : [0,1] \to G \text{ with } \gamma(0) = I \text{ and } \gamma(1) = g\}.$$

(i) Prove that G_0 is a matrix group.

(ii) Prove that G_0 is a normal subgroup of G.

$\boxed{3+2}$

SOLUTION SPACE

Solution (continued)

Solution (continued)