MTH565: Matrix Groups

End Semester Examination

Monsoon Semester 2025

Date:	**Time:**
13th December 2025	9:30 AM – 1:30 PM
Duration:	**Total Marks:**
4 Hours 00 Minutes	40

OPEN BOOK EXAMINATION

INSTRUCTIONS TO CANDIDATES

(i) This is an **open book examination**. You may refer to notes, and other printed materials.

(ii) Electronic devices (phones, tablets, laptops) are **not permitted**.

(iii) If you are using some result, state it clearly. You can use only those results which were discussed in the lectures.

Good Luck!

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EXAMINATION QUESTIONS

Question 1. Prove or disprove the following statements. [10 marks]

(i) Let $A \in O(n)$ such that $\det A = -1$. Then

$$O(n) = SO(n) \cup \{A \cdot B : B \in SO(n)\}$$

[2]

(ii) For $n \geq 2$, $O(n)$ is isomorphic to $SO(n) \times \mathbb{Z}_2$. [2]

(iii) $SL(n, \mathbb{K})$ is **not** compact. [2]

(iv) Let

$$G = \{A \in GL_n(\mathbb{R}) : \det A = 2^k \text{ for some } \; k \in \mathbb{Z}\}.$$

Then G is a matrix group. [2]

(v) Consider a set

$$A = \{a + \imath b \in \mathbb{H} : a^2 + b^2 = 1\}.$$

Then A is a normal subgroup of $Sp(1)$.

(You may assume that it is a subgroup of $Sp(1)$). [2]

Question 2. In the following problem, we will prove that $Sp(1) \times Sp(1)$ is a double cover of $SO(4)$. You may assume that $Sp(1)$ is a matrix group. [11 marks]

(i) If G_1, G_2 are two matrix groups, show that $G_1 \times G_2$ is a matrix group. So, $Sp(1) \times Sp(1)$ is a matrix group. [1]

(ii) For any $(q_1, q_2) \in Sp(1) \times Sp(1)$, show that the map $\varphi(q) : \mathbb{H}(\cong \mathbb{R}^4) \to \mathbb{H}$, defined as $\varphi(q)(\boldsymbol{v}) = q_1 \boldsymbol{v} \bar{q}_2$, is an orthogonal linear transformation. (You do not need to show that it is a linear transformation). [1]

(iii) From part (ii), with respect to the basis $\{1, i, j, k\}$ of $\mathbb{H}$, $\varphi(q)$ can be regarded as an element of $O(4)$. Show that for any $\boldsymbol{v} \in \mathbb{H}$, the map

$$\varphi : Sp(1) \times Sp(1) \to O(4), \quad (q_1, q_2) \mapsto q_1 \boldsymbol{v} \bar{q}_2.$$

is a homomorphism. Also, show that the image will lie in $SO(4)$. [1 + 1]

(iv) By finding the kernel of the map, show that it is 2-to-1 map. [2]

(v) Show that it is a local diffeomorphism. [4]

(vi) Show that the map is surjective and hence conclude that $Sp(1) \times Sp(1)$ is a double cover of $SO(4)$. [1]

Question 3. Let us define the *Affine group* as

$$\mathrm{Aff}_n(\mathbb{R}) := \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} : A \in GL_n(\mathbb{R}) \ \text{ and } \ v \in \mathbb{R}^n \right\}.$$

[8 marks]

(i) Show that $\mathrm{Aff}_n(\mathbb{R})$ is a matrix group. [2]

(ii) Is $\mathrm{Aff}_n(\mathbb{R})$ is compact? [1]

(iii) Find the Lie algebra of $\mathrm{Aff}_n(\mathbb{R})$ and hence find the dimension of $\mathrm{Aff}_n(\mathbb{R})$. [3]

(iv) Let $\mathfrak{aff}_n(\mathbb{R})$ denotes the Lie algebra of $\mathrm{Aff}_n(\mathbb{R})$. For any $X, Y \in \mathfrak{aff}_n(\mathbb{R})$, find the Lie bracket $[X, Y]$. [2]

Question 4. Let G be a matrix group and $\mathcal{L}(G)$ be its Lie algebra. Prove that the tangent space to G at $g \in G$ is

$$T_g G = \{Xg : X \in \mathcal{L}(G)\} = \{gX : X \in \mathcal{L}(G)\}.$$

[3 marks]

Question 5. Let $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Calculate e^A. [4 marks]

Question 6. Consider a basis $\{E_1, E_2, E_3\}$ for $\mathfrak{so}_3$ such that

$$[E_1, E_2] = E_3, \quad [E_2, E_3] = E_1, \quad [E_3, E_1] = E_2.$$

[9 marks]

(i) Show that $\mathfrak{so}_3$ does not have a non trivial proper ideal, that is, only proper ideal of $\mathfrak{so}_3$ is 0. [3]

(ii) Let $\mathrm{Isom}(\mathbb{R}^2)$ denotes the set of all isometries of $\mathbb{R}^2$. Compute its Lie algebra. [2]

(iii) Is the Lie algebra $\mathcal{L}(\mathrm{Isom}(\mathbb{R}^2))$ isomorphic to $\mathfrak{so}_3$ as a Lie algebra? (Hint: See if $\mathcal{L}(\mathrm{Isom}(\mathbb{R}^2))$ has any proper ideal). [4]

— End of Examination —

Some of the results that can be used

- $\mathfrak{so}(n)$ = set of all skew symmetric matrices.
- If M_1, M_2 are two manifolds, then $T_{(p,q)}(M_1 \times M_2) \cong T_pM_1 \oplus T_qM_2$.
- If $qr = rq$ for any $r \in \mathbb{H}$, then $q \in \mathbb{R}$.
- Let G and H be any matrix groups and $f : G \to H$ be a smooth group homomorphism. If f is a local diffeomorphism at $I \in G$, then f is a local diffeomorphism at any point of G.