Matrix Groups: Homework #5

Based on matrices over other fields

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Problem 1

(i) Let p be a prime. Prove that if $p \mid ab$, then p divides either a or b.

(ii) In a field $\mathbb{F}$, prove that if $ab = 0$, then either $a = 0$ or $b = 0$.

Problem 2

Let

$$U(n) = \{A \in GL_n(\mathbb{C}) : AA^* = I_n = A^*A\},$$

where A^* is the conjugate transpose. For example,

$$A = \begin{pmatrix} 2+\iota & 1 \\ 1-\iota & 2\iota \end{pmatrix}, \text{then} \quad A^* = \begin{pmatrix} 2-\iota & 1+\iota \\ 1 & -2\iota \end{pmatrix}.$$

This is called *unitary group* (analogus to set of orthogonal group). Similarly, we have *special unitary group* which is

$$SU(n) = \{A \in U(n) : \det A = 1\}.$$

(i) Can you identify the groups $U(1)$ and $SU(1)$?

(ii) Prove that $SU(2) = \left\{ \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right) : a, b \in \mathbb{C} \text{ and } |a|^2 + |b|^2 = 1 \right\}$.

(iii) Show that $SU(2)/\{\pm I\} \cong SO(3)$.

Problem 3

Determine the groups $GL_1(\mathbb{C}), SL_1(\mathbb{C}), O_1(\mathbb{C})$ and $SO_1(\mathbb{C})$.

Problem 4

This problem involve calculations for matrix group over $\mathbb{Z}_p$.

(i) How many elements are there in the group $GL_2(\mathbb{Z}_3)$?

(ii) How many are there in $SL_2(\mathbb{Z}_3)$?

(iii) Find the inverse of the matrix $\left(\begin{smallmatrix} 2 & 2 \\ 1 & 2 \end{smallmatrix}\right)$ in $GL_2(\mathbb{Z}_7)$.

Problem 5

We want to define an injective homomorphism $\varphi_n : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R})$. Given any $A \in M_n(\mathbb{C})$, we have a corresponding linear map $L_A : \mathbb{C}^n \to \mathbb{C}^n$. Also, we have a canonical map $f_n : \mathbb{C}^n \to \mathbb{R}^{2n}$, $(a + \iota b_1, ..., a_n + \iota b_n) \mapsto (a_1, b_1, ..., a_n, b_n)$.

Given $A \in M_{n(\mathbb{C})}$, we need to determine $B = \varphi_n(A) \in M_{2n}(\mathbb{R})$, equivalently, we need to find a linear map $L_{\varphi_n(A)} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ so that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{C}^n \arrow[r, "f_n"] \arrow[d, "L_A"'] & \mathbb{R}^{2n} \arrow[d, "L_{\varphi_n(A)}"] \\
\mathbb{C}^n \arrow[r, "f_n"] & \mathbb{R}^{2n}
\end{tikzcd}$$

That is, $f_n \circ L_A = L_{\varphi_n(A)} \circ f_n$.

Consider

$$\varphi_1 : M_1(\mathbb{C}) \to M_2(\mathbb{R}), \quad a + \iota b \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Show that with this definition of φ_1 the above diagram is commutative.