Matrix Groups: Assignment #1

Based on untill isometry

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Problem 1

If $\lambda \in \mathbb{H}$ commutes with every element of $\mathbb{H}$, prove that $\lambda \in \mathbb{R}$.

Problem 2

Let $q \in \mathbb{H}$, and define

$$\mathbb{C} \cdot q = \{\lambda \cdot q : \lambda \in \mathbb{C}\} \subset \mathbb{H} \quad \text{and} \quad q \cdot \mathbb{C} = \{q \cdot \lambda : \lambda \in \mathbb{C}\} \subset \mathbb{H}.$$

(i) Let $g_1 : \mathbb{H} \to \mathbb{C}^2$, $(a + b \cdot \boldsymbol{i} + c \cdot \boldsymbol{j} + d \cdot \boldsymbol{k} \mapsto (a + b \cdot \boldsymbol{i}, c + d \cdot \boldsymbol{i})$. Show that $g_1(\mathbb{C} \cdot q)$ is a one-dimensional subspace $\mathbb{C}$-subspace of $\mathbb{C}^2$.

(ii) Define an indetification $\tilde{g}_1 : \mathbb{H} \to \mathbb{C}^2$ such that $\tilde{g}_1(q \cdot \mathbb{C})$ is a one-dimensional $\mathbb{C}$-subspace of $\mathbb{C}^2$.

Problem 3

Recall that for any subsegt $X \subset \mathbb{R}^2$, the symmetry group

$$\mathrm{Symm}(X) := \{f \in \mathrm{Isom}(X) : f(X) = X\}.$$

(i) Consider $X \subset \mathbb{R}^2$. Show that if $\mathrm{Symm}(X)$ is a finite set, then its elements must share a common fixed point and hence isomorphic to a subgroup of $O(2)$.

(ii) The only finite subgroups of $O(2)$ are $\mathbb{Z}_m$ and D_m, where D_m is the dihedral group.

Problem 4

Think of $Sp(1)$ as the group of unit-length quaternions; that is,

$$Sp(1) = \{q \in \mathbb{H} : |q| = 1\}.$$

(i) For every $q \in Sp(1)$, show that the conjugation map $C_q : \mathbb{H} \to \mathbb{H}$, defined as $C_q(\boldsymbol{v}) = q \cdot \boldsymbol{v} \cdot \overline{\boldsymbol{v}}$, is an orthogonal linear transformation. Thus, with respect to the natural basis $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ of $\mathbb{H}$, C_q can be regarded as an element of $O(4)$.

(ii) For every $q \in Sp(1)$, verify that $C_{q(1)} = 1$ and therefore, that C_q sends $\mathrm{Im}(\mathbb{H}) = \mathrm{span}(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ to itself. Conclude that the restriction $C_q \mid_{\mathrm{Im}(\mathbb{H})}$ can be regarded as an element of $O(3)$.

(iii)Define $\varphi : Sp(1) \to O(3)$ as

$$\varphi(q) = C_q \mid_{\mathrm{Im}(\mathbb{H})}.$$

Verify that φ is a group homomorphism.

(iv) Verify that the kernel of φ is $\{1, -1\}$ and therefore, that φ is two-to-one map.

(v) [Bonus Problem] Show that the image of φ is $SO(3)$.

(vi) Also show that $Sp(1)$ is homomorphic to $SU(2)$.

(vii) Finally can you identify something from here.

From now onward a matrix group G means a closed (topological closed) subgroup of $GL_n(\mathbb{F})$. By closedness, we mean that if a sequence in G has a limit in $GL_n(\mathbb{F})$, then that limit must lie in G.

Problem 5

Let G be a matrix group and $H \subset G$ be closed subgroup of G. Prove that H is a matrix group.

Problem 6

Prove that $\mathrm{Aff}_n(\mathbb{F})$ is NOT closed in $M_{n+1}(\mathbb{F})$ but it is a matrix group. Is it compact?

Problem 7

A matrix $A \in M_n(\mathbb{F})$ is called *upper triangular matrix* if all entries below the diagonal are zero, that is, $a_{ij} = 0$ for $i > j$. Prove that

$$UT_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) : A \text{ is an upper triangular matrix}\}$$

is not closed in $M_n(\mathbb{F})$.

Note that if the diagonal elements are not zero, then $UT_n(\mathbb{R})$ is a subset of $GL_n(\mathbb{R})$. Is it a matrix group?

Problem 8

Prove that $\mathrm{Isom}(\mathbb{R}^n)$ is a matrix group. Is it compact?

Problem 9

Let$G \subset GL_n(\mathbb{R})$ be a compact subgroup.

(i) Prove that every element of G has determinant 1 or -1.

(ii) Is it true that $G \subset O(n)$?

Problem 10

There are two natural functions from $SU(n) \times U(1)$ to $U(n)$. The first is $f_1(A, \lambda) = \lambda \cdot A$. The second is $f_2(A, \lambda) =$ the result of multiplying each entry of the first row of A times λ.

(i) Prove that f_1 is an n-to-1 homomorphism.

(ii) Prove that f_2 is a homeomorphism but not a homomorphism when $n \geq 2$.