# Introduction to Symplectic Geometry

Sachchidanand Prasad

## Contents



**Course information:**

- **Course name** : Introduction to Symplectic Geometry
- **Instructor** : Sachchidanand Prasad
- **Time** : Wednesday 17:30 - 19:00
- **Course webpage** : Link to the course website
- **References** :


[1] *Lectures on Symplectic Geometry*, by Ana Cannas da Silva.

[2] *Introduction to Symplectic Topology*, by Dusa McDuff and Dietmar Salamon.

[3] *Lectures on Symplectic Manifolds*, by Alan Weinstein.

[4] *Symplectic Techniques in Physics*, by Victor Guillemin and Shlomo Sternberg.

# 1 Introduction

*The word symplectic was invented by Hermann Weyl in 1939. He replaced the Latin roots in the word complex, com-plexus, by the corresponding Greek roots sym-plektikos.*

## 1.1 An overview of geometry

- **Geometry**: Background Space (smooth manifold) + extra structure (tensor)
  - Riemannian geometry: smooth manifold + metric structure
    - metric structure = positive-definite symmetric 2-tensor
  - Complex geometry: smooth manifold + complex structure
    - complex structure = involutive endomorphism ($(1,1)$-tensor)
  - Symplectic geometry: smooth manifold + symplectic structure
    - symplectic structure = closed non-degenerate 2-form
  - Contact geometry: smooth manifold + contact structure
    - contact structure = "local contact 1-form"

In both symplectic and Riemannian geometry the main object of study is a smooth manifold equipped with a bilinear form on each tangent space. In the Riemannian manifold, this form is a symmetric, nondegenerate, positive definite form, turning each tangent space into normed vector space. On the other hand, in symplectic geometry, we instead require a skew-symmetric bilinear form on each tangent space, again varying smoothly. We still require that at each point $p$ in our manifold $M$, a skew-symmetric 2-form $\omega_p$ should be nondegenerate, that is,

$$\omega_p(X,Y) = 0 \ \ \forall \ Y \in T_pM, \text{ then } X \equiv 0.$$

Finally, note that because $\omega$ is a skew-symmetric 2-form, it must be closed, that is, $\mathrm{d}\omega = 0$.
We will now compare both geometry and from next lecture onwards we will discuss in more details. We will use the following notations:

- $M$ : real finite dimensional smooth manifold without boundary.
- $C^\infty(M) = \{f : M \to \mathbb{R} : f \text{ is smooth}\}$.
- $\chi(M) = \{X : M \to TM : X \text{ is a vector field}\}$.
- $\Omega^k(M) = \{\omega : TM \times TM \times \cdots \times TM \to \mathbb{R}\}$.

We now start some comparison between Riemannian geometry and symplectic geometry:

(1a) Riemannian manifold is a pair $(M, \langle\cdot,\cdot\rangle)$, where

- $\langle\cdot,\cdot\rangle : \chi(M) \times \chi(M) \to C^\infty(M)$ satisfies $\langle X,Y\rangle = \langle Y,X\rangle$ and $\langle fX + gY, Z\rangle = f\langle X,Z\rangle + g\langle Y,Z\rangle$.
- $\langle\cdot,\cdot\rangle$ is positive definite.

(2a) Symplectic manifold is a pair $(M,\omega)$ where

- $\omega \in \Omega^2(M)$ is bilinear.
- $\omega$ is nondegenerate.
- $\omega$ is closed, that is $d\omega = 0$.

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(1b) Every smooth manifold is a Riemannian manifold.

(2b) Not all manifolds are Symplectic. The necessary conditions are:

- $\dim M = \text{even}$.
- $M$ is oriented.
- If $M$ is compact, then $H^2_{dR}(M, \mathbb{R}) \neq 0$.

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(1c) Isometry: Two Riemannian manifolds $(M_1, {}_1\langle\cdot,\cdot\rangle)$ and $(M_2, {}_2\langle\cdot,\cdot\rangle)$ are isometric if there exists a $C^1$ map $\varphi : M_1 \to M_2$ such that

$$ {}_2\Big\langle d\varphi_p(X), d\varphi_p(Y)\Big\rangle_{\varphi(p)} = {}_1\langle X, Y\rangle_p $$

(2c) Similarly, we have symplectomorphism between two symplectic manifolds.

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(1d) Curvature is a local invariant in Riemannian manifolds.

(2d) There are no local invariants (apart from dimension) in symplectic manifolds. According to the Darboux-Weinstein theorem, *given any two symplectic manifolds of the same finite dimension, they look alike locally.*

# 2 Symplectic algebra

In this lecture, we will mostly recall the linear algebra preliminaries for our course. More precisely, we will deal with linear symplectic algebra which we will be using through out the course.

## 2.1 Some basic definitions

**Definition 2.1** (Vector sapce). *A set $(V, +, \cdot)$ is said to be a vector space over a field $\mathbb{F}$ if the operations*

$$ + : V \times V \to V \text{ and } \cdot : \mathbb{F} \times V \to V, $$

*satisfies the following properties. For any $v, v_1, v_2, v_3$ and $\alpha, \beta \in \mathbb{F}$ we have the following.*

1. *(Commutativity) $v_1 + v_2 = v_2 + v_1$.*
2. *(Associativity) $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$.*
3. *(Existence of additive identity) There exists $0 \in V$ such that for any $v \in V$ $0 + v = v = v + 0$.*

4. *(Existence of additive inverse) For any $v \in V$, there exists $w$ such that $v + w = 0 = w + v$. We will denote $w = -v$.*

5. *(Multiplicative identity) For any $v \in V$, $1 \cdot v = v$.*

6. *(Multiplication associativity) $(\alpha\beta) \cdot v = \alpha \cdot (\beta \cdot v)$.*

7. *(Distribution law)*

   - $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$.
   - $\alpha(v_1 + v_2) = \alpha \cdot v_1 + \alpha \cdot v_2$.

Our field will always be either $\mathbb{R}$ or $\mathbb{C}$.

**Definition 2.2** (Lienar map)**.** *Let $T : V \to W$ be a map between two vector spaces $V$ and $W$. Then $T$ is said to be linear if,*

$$T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2),$$

*for $v_1, v_2 \in V$ and $\alpha, \beta \in \mathbb{F}$.*

**Definition 2.3** (Dual space)**.** *If $V$ is a vector space over a field $\mathbb{F}$. Then the dual space of $V$, denoted by $V^*$, is defined by*

$$V^* := \{\varphi : V \to \mathbb{F} : \varphi \text{ is linear}\}.$$

**Definition 2.4** (Bilinear map)**.** *Let $V, W, S$ be vector spaces over a field $\mathbb{F}$. The a bilinear map $B$ is a map*

$$B : V \times W \to S$$

*such that $B$ is linear in each argument. That is, $B(\cdot, w) : V \to S$ and $B(v, \cdot) : W \to S$ is linear for any $v \in V$ and $w \in W$.*

**Definition 2.5.** *A bilinear form $\omega$ on a vector space $V$ is a bilinear map $B : V \times V \to \mathbb{F}$. The bilinear form $\omega$ is said to be nondegenerate if the kernel*

$$\ker \omega := \{v \in V : \omega(v, w) = 0 \text{ for all } w \in V\}$$

*is trivial.*

We identify a bilinear form $\omega$ on $E$ with the linear mapping $u \mapsto (v \mapsto \omega(u, v))$ for $u, v \in E$.

**Definition 2.6.** *Let $\omega$ be a bilinear form on a vector space $V$.*

1. *$\omega$ is said to be symmetric if $\omega(v, w) = \omega(w, v)$ for any $v, w \in V$.*

2. *$\omega$ is said to be skew-symmetric if $\omega(v, w) = -\omega(w, v)$ for any $v, w \in V$.*

## 2.2 Symplectic vector space

The first important notions that we introduce are the symplectic form and the symplectic vector space. We also define the concept of canonical form of a symplectic form and the symplectic basis of a symplectic vector space. Throughout this notes, we will assume $V$ to be a vector space of finite dimension.

**Definition 2.7.** *The pair $(V, \omega)$ is said to be symplectic vector space if $\omega : V \times V \to \mathbb{R}$ is skew-symmetric, nondegenerate bilinear form. We call $\omega$ a symplectic form on $E$.*

*Remark.* It follows from the definition that $\omega(v, v) = 0$ for any $v \in V$.

*Example* 2.8. On $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ we define $\omega$ by

$$\omega((\mathbf{x},\mathbf{y}),(\mathbf{x}',\mathbf{y}')) := \sum_{i=1}^{n}(x_i y_i' - x_i' y_i) = \langle \mathbf{x},\mathbf{y}'\rangle - \langle \mathbf{x}',\mathbf{y}\rangle.$$

We claim that $\omega$ is a symplectic form on $\mathbb{R}^{2n}$. It is clear that $\omega$ is a bilinear form. Further, we need to check two things:

(i) $\omega$ is skew-symmetric.

For any $(\mathbf{a},\mathbf{b}),(\mathbf{c},\mathbf{d}) \in \mathbb{R}^n \times \mathbb{R}^n$, we have

$$\begin{aligned}\omega((\mathbf{c},\mathbf{d}),(\mathbf{a},\mathbf{b})) &= \langle \mathbf{b},\mathbf{c}\rangle - \langle \mathbf{a},\mathbf{d}\rangle, \text{ and}\\ \omega((\mathbf{a},\mathbf{b}),(\mathbf{c},\mathbf{d})) &= \langle \mathbf{a},\mathbf{d}\rangle - \langle \mathbf{b},\mathbf{c}\rangle = -\omega((\mathbf{c},\mathbf{d}),(\mathbf{a},\mathbf{b})).\end{aligned}$$

(ii) $\omega$ is nondegenerate.

Let $\omega((\mathbf{x},\mathbf{y}),(\mathbf{a},\mathbf{b})) = 0$ for any $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n$. We need to show that $(\mathbf{x},\mathbf{y}) = (\mathbf{0},\mathbf{0})$. Take $\mathbf{a} = \mathbf{0}$ and $\mathbf{b} = \mathbf{x}$, then

$$\omega((\mathbf{x},\mathbf{y}),(\mathbf{0},\mathbf{x})) = 0 \implies \langle \mathbf{x},\mathbf{x}\rangle - \langle \mathbf{y},\mathbf{0}\rangle = 0 \implies \mathbf{x} = 0.$$

Similarly, one can show that $\mathbf{y} = 0$ and hence, $\omega$ is nondegenerate.

This is called standard symplectic form on $\mathbb{R}^n \times \mathbb{R}^n$.

The above example can also be written in the following form:

*Example* 2.9. Let $V = \mathbb{R}^{2n}$ with a basis $\{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\}$ and define $\omega$ as

$$\omega(e_i, e_j) = 0, \quad \omega(f_i, f_j) = 0 \quad \text{and } \omega(e_i, f_j) = \delta_{ij}.$$

Then $\omega$ is standard symplectic form on $V$.

*Example* 2.10. Let $V$ be any vector space of dimension $n$ and $V^*$ denotes its dual. If $E = V \oplus V^*$ and define

$$\omega : E \times E \to \mathbb{R}, \quad \omega((v,\alpha),(v',\alpha')) = \alpha'(v) - \alpha(v'),$$

then $(E,\omega)$ is a symplectic vector space.

Since $\alpha$ and $\alpha'$ are linear maps, it is clear that $\omega$ is a bilinear form. Let us show it is skew-symmetric and nondegenerate.

(i) $\omega$ is skew-symmetric.

For any $v, v' \in V$ and $\alpha, \alpha' \in V^*$, we have

$$\begin{aligned}\omega((v,\alpha),(v',\alpha')) &= \alpha'(v) - \alpha(v')\\ &= -(\alpha(v') - \alpha'(v))\\ &= \omega((v',\alpha'),(v,\alpha)).\end{aligned}$$

(ii) $\omega$ is nondegenerate.

Let $\omega((v,\alpha),(w,\beta)) = 0$ for any $(w,\beta) \in E$. We need to show that $v = 0$ and $\alpha \equiv 0$. Observe that for any $\beta \in V^*$

$$\omega((v,\alpha),(0,\beta)) = \beta(v) = 0 \implies v = 0.$$

Similarly, for any $w \in V$,

$$\omega((v,\alpha),(w,0)) = \alpha(w) = 0 \implies \alpha = 0.$$

Thus $(E,\omega)$ is a symplectic vector space.

**Definition 2.11.** *Let $(V,\omega)$ is a symplectic vector space, then for any subspace $W \subseteq V$, we define the* $\omega$-orthogonal space

$$W^{\omega} := \{v \in V : \omega(v,w) = 0,\ \forall\ w \in W\}.$$

**Proposition 2.12.** *Let $V$ be a $k$-dimensional vector space over $\mathbb{R}$ and $\omega$ be a bilinear form.*

1. *If $\omega$ is symmetric with rank $r$, then there exists a basis $\mathcal{B}$ of $V$ such that with respect to $\mathcal{B}$,*

$$[\omega]_{\mathcal{B}} = \begin{bmatrix} \epsilon_1 & & & & & \\ & \ddots & & & & \\ & & \epsilon_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}, \text{ where } \epsilon_i = \pm 1, \quad i = 1,2,\ldots,r.$$

2. *If $\omega$ is skew-symmetric with rank $r$, then $r = 2n$ and there is a basis $\mathcal{B}$ of $V$ relative to which*

$$[\omega]_{\mathcal{B}} = \begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ where } I_n \text{ is the identity matrix of size } n.$$

*Proof.* 1. Proof is left.

2. Since $\omega \neq 0$, we can choose $e_1, f_1 \in V$ such that $\omega(e_1,f_1) \neq 0$ (this must implies that both the vectors are linearly independent). By rescaling $e_1$, we can further assume that $\omega(e_1,f_1) = 1$. Define $W_1 := \operatorname{span}\{e_1,f_1\}$. Since, $\omega$ is skew-symmetric, we have $\omega(e_1,e_1) = 0 = \omega(f_1,f_1)$. Thus, the restriction of $\omega$ on $W_1$ is

$$[\omega]_{\{e_1,f_1\}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let $W_2$ be the $\omega$-orthogonal complement of $W_1$, that is, $W_2 = W_1^{\omega}$. It is clear that $W_1 \cap W_2 = \{0\}$. We claim that $V = W_1 \oplus W_2$. Note that for any $v \in V$, we have

$$\omega\left(e_1, v - \omega(v,f_1)e_1 + \omega(v,e_1)f_1\right) = 0 \text{ and}$$
$$\omega\left(f_1, v - \omega(v,f_1)e_1 + \omega(v,e_1)f_1\right) = 0.$$

Thus, $v - \omega(v,f_1)e_1 + \omega(v,e_1)f_1 \in W_2^{\omega}$ and hence $V = W_1 \oplus W_2$. We can repeat the process on $W_2$ and find $e_2$ and $f_2$ such that $\omega(e_2,f_2) = 1$. Now the matrix will be

$$[\omega]_{\{e_1,e_2,f_1,f_2\}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

Inductively, we get a basis

$$\mathcal{B} = \{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\}$$

such that $[\omega]_{\mathcal{B}}$ will be in the given form.

□

*Remark.* Since we focus on non-degenerate skew-symmetric bilinear form, that is, rank $= 2n$, we may consider only the case with matrix representation $\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ and $V$ must be of even dimension.

We just showed the following corollary.

**Corollary 2.13.** *Every finite dimensional symplectic vector space $(V, \omega)$ has even dimension.*

*Exercise* 2.14. Show that the space of skew-symmetric bilinear form is isomorphic to the space $\wedge^2 V^*$ of the second exterior product of $V^*$.

So if $\mathcal{B} = \{e_1, \ldots, e_{2n}\}$ is a basis for $V$, and $\mathcal{B}^*$ is its dual, then for any $\omega \in \wedge^* V$ with the matrix $(\omega_{ij})$, relative to $\mathcal{B}$, can also be written as

$$[\omega]_{\mathcal{B}} = \sum_{i<j} \omega_{ij} e_i^* \wedge e_j^*.$$

*Remark.* Since elements of $\wedge^2 V^*$ are represented by anti-symmetric matrices and with all the entries of the main diagonal equal to 0, for a vector space $V$ of dimension $2n$ we have $\dim \wedge^2 V^* = \frac{2n(2n-1)}{2} = n(2n-1)$.

**Corollary 2.15** (Canonical form of $\omega$)**.** *For every skew-symmetric bilinear form $\omega$, there exists a basis $\mathcal{B} = \{e_1, \ldots, e_{2n}\}$ of $V$ such that*

$$[\omega]_{\mathcal{B}} = \sum_{i<j} e_i^* \wedge e_j^*.$$

This representation is called a canonical form of $\omega$ and we call $\mathcal{B}$ a symplectic basis of $V$.

## 2.3 Symplectomorphism

**Definition 2.16.** *Two symplectic vector spaces $(V_1, \omega_1)$ and $(V_2, \omega_2)$ are called* symplectomorphic *if there exists an isomorphism $\varphi : V_1 \to V_2$ of vector space such that $\omega_2(\varphi(x), \varphi(y)) = \omega_1(x, y)$. In other words, $\varphi^*\omega_2 = \omega_1$. We call $\varphi$ a* symplectomorphism*. We will write $V_1 \cong V_2$.*

*Exercise* 2.17. What can you conclude if $\dim V_1 = \dim V_2$ and $\varphi : V_1 \to V_2$ satisfies $\varphi^*\omega_2 = \omega_1$?

We claim that $\varphi$ is injective. If $v \in \ker\varphi$, then for any $v' \in V_1$, we have

$$\omega_1(v, v') = \omega_2(\varphi(v), \varphi(v')) = 0.$$

Since, $\omega_1$ is nondegenerate, $v = 0$. Since the dimension matches, $V_1 \cong V_2$.

*Exercise* 2.18. Show that the set of all symplectomorphisms of a symplectic vector space $(V, \omega)$ forms a group under the composition.

**Definition 2.19.** *The group of symplectomorphism of a symplectic vector space $(V, \omega)$ is called* symplectic group *and we will denote this by $\mathrm{Sp}(V)$.*

*Example* 2.20. Some examples on symplectomorphism:

1. $V = \mathbb{R}^{2n}$ and $\omega((\mathbf{a}, \mathbf{b}), (\mathbf{c}, \mathbf{d})) = \langle \mathbf{a}, \mathbf{d} \rangle - \langle \mathbf{c}, \mathbf{b} \rangle$.

   - $\varphi(e_i) = f_i$ and $\varphi(f_i) = -e_i$. If we change $\varphi(f_i) = e_i$, will it work?

     It is clear that $\varphi$ is an isomorphism. We just need to show that $\varphi^*\omega = \omega$. Note that

     $$\omega\left(e_i, e_j\right) = 0 = \omega\left(f_i, f_j\right).$$
     $$\delta_{ij} = \omega\left(\varphi(e_i), \varphi(f_j)\right) = \omega\left(f_i, -e_j\right) = -\omega\left(f_i, e_j\right) = \omega\left(e_j, f_i\right) = \delta_{ij}.$$

   - $\varphi(e_i) = e_i + f_i$ and $\varphi(f_i) = f_i$.

   - For any invertible matrix $X$,

     $$\varphi(e_i) = \sum_j X_{ij} e_j \text{ and } \varphi(f_i) = \sum_j \left(X^{-1}\right)_{ji} f_j.$$

2. We show in Example 2.10 $E = V \oplus V^*$ is a symplectic vector space. We can give a symplectomorphism on $E$ as follows. Let $T : V \to V$ be an isomorphism and $T^* : V^* \to V^*$ be the dual map. Then
   $$T \oplus T^* : E \to E,$$
   is a symplectomorphism.

3. Let $V$ be a complex vector space of complex dimension $n$, with complex, positive definite inner product (=Hermitian metric) $h : V \times V \to \mathbb{C}$.Then $V$, viewed as a real vector space, with bilinear form the imaginary part $\omega = \mathrm{Im}(h)$ is a symplectic vector space. Every unitary map $V \to V$ preserves $h$, hence also $\omega$ and is therefore symplectic.

*Exercise* 2.21. Show that $\mathbb{R}^{2n}$, $E$ and the third example are symplectomorphic.

**Proposition 2.22.** *Every symplectic vector space $(V, \omega)$ of dimension $2n$ is symplectomorphic to $\mathbb{R}^{2n}$ with the canonical symplectic form.*

As a consequence of the above proposition, we have the following theorem, which we call *Linear Darboux theorem*.

**Theorem 2.23** (Linear Darboux Theorem)**.** *For any symplectic vector space* $(V,\omega)$ *there exists a basis* $\mathcal{B} = \{e_i, f_i\}_{i=1}^n$ *of* $V$ *such that*

$$\omega(e_i, e_j) = 0 = \omega(f_i, f_j) \text{ and } \omega(e_i, f_j) = \delta_{ij} \quad \forall\, i, j.$$

*This basis is called a* Darboux basis *of* $V$*.*

The above theorem is equivalent to following statements:

(i) Any symplectic vector space is even-dimensional.

(ii) Any even dimensional vector space admits a linear symplectic form.

(iii) Up to linear symplectomorphisms, there is a unique linear symplectic form on each even dimensional vector space.

## 2.4 Subspaces of a symplectic vector space

Recall the Definition 2.11 of $\omega$-perpendicular space. Note that with our assumption that $V$ is finite dimensional, $\omega$ is nondegenerate if and only if the map

$$\omega^\flat : V \to V^*, \quad \omega^\flat(v)(w) = \omega(v,w)\ \forall\, v, w \in V$$

is an isomorphism.

*Note.* For any subspace $W \subset V$, we have

$$W^\omega = \left(w^\flat\right)^{-1}(\mathrm{ann}(W)),$$

where $\mathrm{ann}(W)$ is the annihilator of $W$, that is, the set of all $f \in V^*$ such that $f(w) = 0$ for $w \in W$.

We have

$$v \in \mathrm{ann}(W) \iff \text{for any } w \in W,\ \left(\omega^\flat(v)\right)(w) = 0 \iff \omega(v,w) = 0 \iff v \in W^\omega.$$

**Definition 2.24.** *A subspace* $W \subseteq V$ *of a symplectic vector space is called*

*(i)* isotropic *if* $W \subseteq W^\omega$*, that is,* $\omega\big|_{W\times W} = 0$*;*

*(ii)* co-isotropic *if* $W^\omega \subseteq W$*, that is,* $W^\omega$ *is isotropic;*

*(iii)* Lagrangian *if* $W^\omega = W$*, that is,* $W$ *is isotropic and co-isotropic;*

*(iv)* symplectic *if* $\omega_{W\times W}$ *is nondegenerate.*

*The set of Lagrangian subspaces of* $V$ *is called the* Lagrangian Grassmannian *and denoted* $\mathrm{Lag}(V)$*.*

*Exercise* 2.25. Show that $W$ is symplectic if and only if $W \cap W^\omega = \{0\}$ .

*Exercise* 2.26. Let $(V,\omega)$ is a symplectic vector space and $W$ be any subspace of $V$. Consider the map $\varphi : V \to W^*$, defined by $\varphi(v) = \omega(v)(w)$ for any $v \in V$ and $w \in W$. Show that $\varphi$ is surjective. Deduce that $\dim W^\omega = \dim V - \dim W$. Also, show that $(W^\omega)^\omega = W$.

*Remark.* 1. From Exercise 2.26, we conclude that if $\dim V = 2n$, then all the isotropic subspaces have dimension smaller or equal $n$, all the co-isotropic have dimension bigger or equal $n$ and all the Lagrangian subspace have dimension $n$.

2. If $W \subseteq V$ is symplectic subspace, then it follows from the definition that $W \cap W^\omega = \{0\}$ and therefore, from the dimension sum restriction (Exercise 2.26), we must have $V = W \oplus W^\omega$.

*Example* 2.27. Every 1-dimensional subspace of $V$ is isotropic and every subspace with codimension 1 is co-isotropic.

*Example* 2.28. Consider $V = \mathbb{R}^{2n}$ with canonical symplectic form $\omega$. Define

$$\begin{aligned} W_1 &= \operatorname{span}\{e_1, e_2\}. \quad \text{Isotropic} \\ W_2 &= \operatorname{span}\{e_1, e_2, \dots, e_n, f_3, f_4, \dots, f_n\}. \quad \text{Co-isotropic} \\ W_3 &= \operatorname{span}\{e_1, e_2, \dots, e_n\}. \quad \text{Lagrangian} \\ W_4 &= \operatorname{span}\{e_1, f_1\}. \quad \text{Symplectic} \end{aligned}$$

*Exercise* 2.29. Let $(V, \omega)$ be a symplectic vector space and $W$ be any subspace of $V$.

1. Show that if $W$ is isotropic, then $\dim W \leq \frac{1}{2} \dim V$.
2. Show that if $W$ is Lagrangian, then $\dim W = \frac{1}{2} \dim V$.
3. Show that if $W$ is Lagrangian, then any basis $\mathcal{B}_W = \{e_1, e_2, \dots, e_n\}$ of $W$ can be extended to a symplectic basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ of $V$.

**Proposition 2.30.** *For any symplectic vector space $(V, \omega)$, there exists a Lagrangian subspace $L$.*

*Proof.* Since for every $v \in V$ we have $\omega(v, v) = 0$, $V$ has an isotropic subspace. Let $L \subsetneq V$ be a maximal isotropic subspace of $V$, that is, it is not contained in any isotropic subspace of strictly larger dimension. Then we claim that $L$ is Lagrangian, that is, $L^\omega = L$. We only need to show that $L$ is co-isotropic. Suppose not, take $v \in L^\omega \setminus L$, then $L' = L \oplus \operatorname{span}\{v\}$ is isotropic and larger than $L$. ☐

An immediate consequence is that any symplectic vector space $V$ has even dimension: For if $L$ is a Lagrangian subspace

$$\dim V = \dim L + \dim L^\omega = 2 \dim L.$$

From this proof we also conclude that a maximal isotropic subspace is a Lagrangian subspace. Therefore we have the following corollary.

**Corollary 2.31.** *Every isotropic subspace is contained in a Lagrangian subspace.*

**Some properties:** Let $W, W_1, W_2$ be a subspaces of a symplectic vector space $(V, \omega)$.

(1) $\dim W + \dim W^\omega = \dim V$.

> As per the hint given in Exercise 2.26, consider the map
>
> $$\varphi : V \to W^*, \quad \varphi(v) = \omega(v, \cdot).$$
>
> Note that
>
> $$\ker \varphi = \{v \in V : \varphi(v) = 0\} = \{v \in V : \omega(v,w) = 0 \forall\, w \in W\} = W^\omega.$$
>
> Now, we claim that $\varphi$ is surjective. Let $f \in W^*$, that is, $f : W \to \mathbb{R}$ is linear. As $W \subsetneq V$, we can extend the map $f$ to $V$, say $\tilde{f}$. Since $\tilde{\varphi} : V \to V^*$, $v \mapsto \omega(v, \cdot)$ is an isomorphism, there exists $v \in V$ such that $\tilde{\varphi}(v) = \tilde{f}$. Thus, $\varphi(v) = f$. So, Image$(\varphi) = W^*$. Thus, using rank-nullity theorem, we have
>
> $$\dim V = \dim W^* + \dim W^\omega = \dim W + \dim W^\omega.$$

(2) $(W^\omega)^\omega = W$.

> Note that for any $w \in W$,
>
> $$\omega(w,v) = 0, \forall\, v \in W^\omega \implies w \in (W^\omega)^\omega \implies W \subseteq (W^\omega)^\omega.$$
>
> Using the dimension formula, we have
>
> $$\begin{aligned} \dim W^\omega + \dim (W^\omega)^\omega = \dim V = \dim W + \dim W^\omega \\ \implies \dim W = \dim (W^\omega)^\omega \implies W = (W^\omega)^\omega. \end{aligned}$$

(3) If $W_1 \subseteq W_2$, then $W_2^\omega \subseteq W_1^\omega$.

> Let $w_2' \in W_2^\omega$. For any $w_2 \in W_2$, $\omega(w_2, w_2') = 0$. For any $w_1 \in W_1 \subseteq W_2$, we must have
>
> $$\omega(w_2', w_1) = 0 \implies w_2 \in W_1^\omega.$$

(4) If $W$ is symplectic, then $W \oplus W^\omega = V$.

> We have
>
> $$\dim (W + W^\omega) = \dim W + \dim W^\omega - \dim (W \cap W^\omega) \implies W + W^\omega = V.$$

(5) Every 1-dimensional subspace is isotropic.

(6) Every codimensional 1 subspace is co-isotropic.

(7) If $W$ is Lagrangian, then $\dim W = \frac{1}{2} \dim V$.

(8) If $W$ is Lagrangian, then any basis $\{e_1, e_2, \dots, e_n\}$ of $W$ can be extended to a symplectic basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$.

(9) If $W$ is Lagrangian, then $(V, \omega)$ is symplectomorphic to the space $(W \oplus W^*, \Omega)$, where $\Omega(x \oplus \alpha, y \oplus \beta) = \beta(x) - \alpha(y)$.

*Example* 2.32 (Quotient Space). Let $(V, \omega)$ is a symplectic vector space and $W$ be any isotropic subspace of $V$. We define $V/W := \{[v] = v + W : v \in V\}$. Then there is a natural symplectic form on $W^\omega / W$.

Define
$$\Omega : W^\omega/W \times W^\omega/W \to \mathbb{R}, \quad \Omega([v],[v']) := \omega(v,v').$$
We need to verify that $\Omega$ is well-defined and is a symplectic form on $W^\omega/W$.

(i) **Well-defined:**

For any $v,v' \in W^\omega$ and $w,w' \in W$ we have

$$\omega(v+w,v'+w') = \omega(v,v') + \underbrace{\omega(v,w')}_{0} + \underbrace{\omega(w,v')}_{0} + \underbrace{\omega(w,w')}_{0} = \omega(v,v').$$

Here the middle terms vanish because of orthogonality and last term vanishes because $W$ is isotropic.

(ii) **Bilinear:** Exercise.

(iii) **Skew-symmetric:** Exercise.

(iv) **Non-degenerate**

For any $v,v' \in V$, let

$$\Omega([v],[v']) = 0 \implies \omega(v,v') = 0 \implies v = 0.$$

**Proposition 2.33.** *Given any finite collection of Lagrangian subspaces $L_1, L_2, \ldots, L_k$, of a symplectic there exists a Lagrangian subspace $L$ with $L \cap L_i = \{0\}$ for all $i = 1,2,\ldots,k$.*

*Proof.* We will write the proofs in steps:

1. **Step 1 :** Choose a maximal isotropic subspace $L$ of $V$ such that $L \cap L_i = \{0\}$.

   We can choose such an isotropic subspace because finite union of proper subspace can not be the full space. Since $L_i$'s are Lagrangian, so they are proper subspace of $V$ and hence $\bigcup_i L_i \subsetneq V$. Choose a $v \in V$ which is not in any $L_i$. Then span$\{v\}$ is an isotropic subspace of $V$ such that intersection with any $L_i$ is trivial.

2. **Step 2 :** We will show that $L$ is Lagrangian. Suppose it is false, that is, $L^\omega \subsetneq L$. From Example 2.32, we know that $L^\omega/L$ has a symplectic form. Let $\pi : L^\omega \to L^\omega/L$ be the quotient map. Then we have the following claims.

   (a) For each $i$, the space $\pi(L_i \cap L^\omega)$ is isotropic.

   Let $\Omega$ as defined in Example 2.32. Then we need to show that $\Omega\big|_{\pi(L_i \cap L^\omega)} = 0$. This is easy as for any $[v],[v'] \in \pi(L^\omega \cap L_i)$

   $$\Omega([v],[v]') = \omega(v,v') = 0.$$

   The last equality is because $v,v' \in L_i$ which is Lagrangian, in particular it is isotropic.

   (b) There exists a one dimensional space $F \subseteq L^\omega/L$ such that $F$ is transversal to each of $\pi(L^\omega \cap L_i)$.

   Similar to the step 1, we can choose an element $[v] \in L^\omega/L$ away from each of $\pi(L^\omega \cap L_i)$. Define $F = \text{span}\{[v]\}$. It is clear that $F \cap (L^\omega \cap L_i) = \{0\}$, and hence they are transversal.

   Now we note that

- $L' := \pi^{-1}(F)$ is isotropic subspace of $V$;
- $L \subsetneq L'$ and
- $L' \cap L_i = \{0\}$ for each $i$.

Combing all this, we get a contradiction to the choice of $L$. Thus, $L$ is Lagrangian.

□

*Remark.* As a consequence of Proposition 2.33, one can give an alternative proof of the Linear Darboux Theorem (Theorem 2.23).

**Theorem 2.34.** *Every symplectic vector space $(V, \omega)$ of dimension $2n$ is symplectomorphic to $\mathbb{R}^{2n}$ with the canonical symplectic form.*

*Proof.* Use Proposition 2.30, let $L_1$ be a Lagrangian subspace of $V$. Now, using Proposition 2.33, choose a Lagrangian subspace $L_2$ which is transversal to $L_1$. Then, the map

$$L_1 \times L_2 \to \mathbb{R}, \quad (l_1, l_2) \mapsto \omega(l_1, l_2).$$

is nondegenerate. This gives an isomorphism between $L_1$ and $L_2^*$ as shown by the composition

$$\psi : L_1 \overset{i}{\hookrightarrow} V \xrightarrow{\omega^\flat} V^* \xrightarrow{i^*} L_2^*.$$

Note that

$$\begin{aligned}\ker(\psi) &= \left\{l_1 \in L_1 : \psi(l_1) = i^*\left(\omega^\flat\left(i(l_1)\right)\right) = 0\right\} \\ &= \{l_1 \in L_1 : i^*\left(\omega(l_1, v)\right) = 0, \ \ v \in V\} \\ &= \{0\}.\end{aligned}$$

Now, let $\{e_1, e_2, \ldots, e_n\}$ be a basis for $L_1$ and $f_1, f_2, \ldots, f_n$ be the dual basis for $L_2^*$. Since $L_1$ and $L_2$ are transversal to each other, $\{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\}$ is a symplectic basis for $V$. □

**Corollary 2.35.** *Let $(V_1, \omega_1)$ and $(V_2, \omega_2)$ be two symplectic vector spaces of same dimension. Let $L_1, L_1' \subseteq V_1, L_2, L_2' \subseteq V_2$ be Lagrangian subspace such that $L_1 \cap L_1'$ and $L_2 \cap L_2'$ are trivial. Then there exists a symplectomorphism $\varphi : V_1 \to V_2$ such that $\varphi(L_1) = L_2$ and $\varphi(L_1') = L_2'$.*

## 2.5 Compatible Complex Structures

**Definition 2.36.** *Let $V$ be a real vector space. Then an automorphism $J : V \to V$ is said to be a* complex structure *on $V$ if $J^2 = -Id$, that is, $J(J(v)) = -v$ for $v \in V$. We will write this as $(V, \mathbb{R}, J)$.*

*Example* 2.37. Take $V = \mathbb{R}$. If $V$ has a complex structure $J$, then $J^2 = -Id$. Let $J(1) = \alpha$.

$$J(J(1)) = -1 \implies \alpha^2 = -1,$$

a contradiction as $\alpha \in \mathbb{R}$. So, on $\mathbb{R}$, there is no complex structure.

*Example* 2.38. Tak $V = \mathbb{R}^2$. Define $J$ as

$$J(e_1) = e_2 \text{ and } J(e_2) = -e_1.$$

Then it is easy to see that $J$ is a complex structure on $\mathbb{R}^2$.

*Example* 2.39. Note that if $n$ is odd, then $\mathbb{R}^n$ does not admit a complex structure. As if there is a complex structure $J$, then we must have

$$J^2 = -Id \implies \det\left(J^2\right) = (-1)^n = -1 \implies (\det J)^2 = -1,$$

not possible.

The real vector space $\mathbb{R}^n$ admits a complex structure if and only if $n$ is even. Let $n = 2k$. If $\{e_1, \ldots, e_k, f_1, \ldots, f_k\}$ is a basis of $\mathbb{R}^n$, then define $J : \mathbb{R}^{2k} \to \mathbb{R}^{2k}$, as

$$J(e_I) = f_i \text{ and } J(f_i) = -e_i.$$

Then $J$ is a complex structure on $\mathbb{R}^n$. On the other hand, if $J$ exists, then

$$\det(J^2) = (-1)^n \implies (\det J)^2 = (-1)^n \implies n \text{ is even.}$$

**Theorem 2.40.** *Every real vector space $(V, \mathbb{R}, J)$ with a complex structure $J$ is even dimensional.*

**Corollary 2.41.** *Every symplectic vector space admits a complex structure.*

**Definition 2.42.** *A complex structure $J$ on a symplectic vector space $(V, \omega)$ is said to be* $\omega$-compatible*, if*

$$G_J(v, w) = \omega(v, J(w))$$

*is an inner product.*

*Remark.* We note that $J$ is $\omega$-compatible implies

$$\omega(Jv_1, Jv_2) = \omega(v_1, v_2).$$

By using the properties of $g$, we have

$$\omega(Jv_1, Jv_2) = g(Jv_1, v_2) = g(v_2, Jv_1) = \omega\left(v_2, J^2 v_1\right) = \omega(v_2, -v_1) = \omega(v_1, v_2).$$

Indeed,

$$J \text{ is } \omega\text{-compatible} \iff \begin{cases} \omega(Jv_1, Jv_2) = \omega(v_1, v_2) \\ \omega(v, Jv) > 0, \quad \forall\, v \neq 0. \end{cases}$$

Thus, following tha above remark, we get if $J$ is $\omega$-compatible, then $J \in \mathrm{Sp}(V)$.

The next result say that compatible complex structures always exist on symplectic vector space.

**Theorem 2.43.** *Let $(V, \omega)$ be a symplectic vector space. Then there is a compatible complex structure $J$ on $V$.*

*Proof.* Since $V$ is a vector space, we choose an inner product $G$ on $V$. Since $\omega$ and $G$ are nondegenerate, the maps

$$\left.\begin{array}{lcl} u \in V & \xmapsto{\omega^*(u)} & \omega(u, \cdot) \in V^* \\ \\ v \in V & \xmapsto{G^*(v)} & G(v, \cdot) \in V^* \end{array}\right\} \text{are isomorphisms between } V \text{ and } V^*.$$

Hence, we can find a linear map $A : V \to V$ such that $\omega(u, v) = G(u, Av)$.

Note that for any $u, v \in V$,

$$\begin{aligned}
&\omega^*(u)(v) = \omega(u,v) = G(u,Av) = G^*(u)(Av) \\
\implies &G^*(u) \circ A = \omega^*(u) \ \forall \ u \in V \implies G^* \circ A = \omega^* \\
\implies &A = G^{*^{-1}} \circ \omega^*.
\end{aligned}$$

If $A^2 = -Id$, then $A$ is a compatible complex structure on $V$. Let us suppose that $A^2 \neq -Id$. Note that,

$$G\left(A^T u, v\right) = G(u, Av) = \omega(u,v) = -\omega(v,u) = -G(v,Au) = G(-Au, v),$$

which implies $A^T = -A$, that is, $A$ is skew-symmetric. We also note that

- $\left(AA^T\right)^T = AA^T$, that is, $AA^T$ is symmetric and
- for any $u \neq 0$, $G\left(AA^T u, u\right) = G\left(A^T u, A^T u\right) > 0$, that is, $AA^T$ is positive definite.

This implies that $AA^T$ is diagonalizable with positive eigenvalues $\lambda_i$. So there exists $P \in GL(n, \mathbb{R})$ such that

$$AA^T = P\mathrm{diag}\left(\lambda_1, \lambda_2, \ldots, \lambda_{2n}\right).$$

So, we may take any real power of $AA^T$. In particular,

$$\sqrt{AA^T} := P\mathrm{diag}\left(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_{2n}}\right)P.$$

Then $\sqrt{AA^T}$ is symmetric and positive-definite. Let

$$J = \left(\sqrt{AA^T}\right)^{-1} A.$$

This factorization $A = \sqrt{AA^T}J$ is called *polar decompostion* of $A$. Since $A$ commutes with $\sqrt{AA^T}$, $J$ must commute with $\sqrt{AA^T}$. Now we have

- $J^T = A^T \sqrt{AA^T}^{-1} = \sqrt{AA^T}(-A) = -J$ and
- Since $A^T = -A$, we have $JJ^T = Id$. Thus, $J^2 = -Id$.

So, $J$ is a complex structure on $V$. Now it remains to show that it is compatible with $\omega$, that is $G_j(u,v) = \omega(u, Jv)$ is an inner product. Equivalently,

(i) $\omega(Ju, Jv) = \omega(u,v)$.

$$\omega(Ju, Jv) = G(Ju, AJv) = G\left(u, J^t AJv\right) = G\left(u, AJ^t Jv\right) = G(u, Av) = \omega(u,v).$$

(ii) For any $v \neq 0$, $\omega(v, Jv) > 0$.

$$\omega(v, Jv) = G(v, AJv) = G\left(v, \sqrt{AA^T} v\right) > 0.$$

□

The set of all symmetric bilinear forms is a vector space and hence we can topologize it. Let $\mathrm{Met}(V)$ is the set of metrics (inner products) on $V$. The following theorem follows from Theorem 2.43.

**Theorem 2.44.** *Let $(V,\omega)$ be a symplectic vector space. There is a canonical continuous surjective map*

$$\mathcal{F} : \mathrm{Met}(V) \to \mathcal{J}(V,\omega), \quad g \mapsto J_g,.$$

*where $J_g$ satisfies $g(v,w) = \omega\big(v, J_g w\big)$. Moreover, the other way map $\mathcal{G} : \mathcal{J}(V,\omega) \to \mathrm{Met}(V)$, $J \mapsto g_J$ associating to each compatible complex structure the corresponding Riemannian structure is a section, that is, $\mathcal{F} \circ \mathcal{G}(J) = J$.*

**Proposition 2.45.** *Let $(V,\omega)$ be a symplectic vector space of dimension $2n$ and $J$ be any complex structure on $V$. Then the following are equivalent.*

*(i)* *$J$ is $\omega$-compatible.*

*(ii)* *$(V,\omega)$ has a symplectic basis of the form*

$$\{v_1, v_2, \ldots, v_n, J_1 v_1, J_2 v_2, \ldots, J_n v_n\}.$$

*(iii)* *There exists an isomorphism $\varphi : \mathbb{R}^{2n} \to V$ such that*

$$\varphi^* \omega = \omega_0 \text{ and } \varphi^* J = J_0,$$

*where $\omega_0$ and $J_0$ are standard symplectic form and complex structure on $\mathbb{R}^{2n}$, respectively.*

*(iv)* *$J$ satisfies $\omega(v, Jv) > 0$ for $v \neq 0$ and for any Lagrangian subspace $L$ of $V$, the subspace $JL$ must be Lagrangian.*

*Proof.* (i) $\implies$ (ii):

By Proposition 2.30, choose a Lagrangian subspace $L$ of $V$. Choose an orthogonal basis $\{v_1, \ldots, v_n\}$ of $L$ with respect to the inner product induced by $J$ and $\omega$. Then it is easy to see that $\{v_1, \ldots, v_n, Jv_1, \ldots, Jv_n\}$ is a symplectic basis.

(ii) $\implies$ (iii):

Define

$$\Phi : \mathbb{R}^{2n} \to V, \quad z \mapsto \sum_{i=1}^{n} (x_i v_i + y_i J v_i),$$

where $z = (x_1, \ldots, x_n, y_1, \ldots, y_n)$.

(iii) $\implies$ (i): Exercise.

(i) $\implies$ (iv):

If $J$ is $\omega$-compatible the for any $v \neq 0$, $\omega(v, Jv) > 0$. Moreover any $u, v \in L$ we have $\omega(Ju, Jv) = \omega(u, v) = 0$, so $\omega|_{JL} = 0$. This together with the fact that $J$ is an isomorphism proves that $JL$ is Lagrangian.

(iv) $\implies$ (i):

We prove that $g(v,w) = \omega(v, Jw)$ defines an inner product. It is positive definite and bilinear by assumption. Let $g$ is not symmetric. That is, there exists $u, v \in V$ such that

$$g(u,v) \neq g(v,u) \implies \omega(u, Jv) \neq \omega(v, Ju).$$

Since $v \neq 0$, $\omega(v, Jv) > 0$. Now define

$$w := u - \frac{\omega(v,Ju)}{\omega(v,Jv)} v,$$

so we get

$$\omega(w,Jv) = \omega(u,Jv) - \omega(v,Ju) \neq 0$$

and hence we have that $w, Jv$ and $v, Jw$ are linearly independent. Moreover, since

$$\omega(v,Jw) = \omega(v,Ju) - \frac{\omega(v,Ju)}{\omega(v,Jv)}\omega(v,Jv) = 0,$$

there is a Lagrangian $L$ containing $v$ and $Jw$. So it follows that $Jv, w \in JL$, but since $\omega(w,Jv) \neq 0$, $JL$ can not be Lagrangian, a contradiction.

$\square$

**Corollary 2.46.** *The space $\mathcal{J}(V,\omega)$ is contractible.*

*Proof.* Since $\mathrm{Met}(V)$ is convex, Theorem 2.44 implies that $\mathcal{J}(V,\omega)$ is contractible. $\square$

## 2.6 Symplectic Groups

We recall that if $(V,\omega)$ is a symplectic vector space of dimension $2n$, then $\mathrm{Sp}(V)$ denotes the space of all symplectomorphisms of $V$. We can write,

$$\mathrm{Sp}(V) = \{\varphi : V \to V : \varphi^*\omega = \omega\}$$

By Theorem 2.23, $V$ is symplectomorphic to $\mathbb{R}^{2n}$ with the canonical symplectic form. Thus, $\mathrm{Sp}(V) \cong \mathrm{Sp}(\mathbb{R}^n) =: \mathrm{Sp}(2n)$. Thus, we can see $\mathrm{Sp}(V) \cong \mathrm{Sp}(2n) \subseteq \mathrm{GL}(2n,\mathbb{R})$. If $A \in \mathrm{Sp}(2n)$, then for any $v, w \in V$

$$\begin{aligned}
\omega(Av,Aw) = \omega(v,w) &\implies (Av)^T \begin{bmatrix} 0 & Id_n \\ -Id_n & 0 \end{bmatrix} (Aw) = v^T \begin{bmatrix} 0 & Id_n \\ -Id_n & 0 \end{bmatrix} w \\
&\implies v^T A^T J A w = v^T J w \\
&\implies A^T J A = J.
\end{aligned}$$

Thus,

$$\mathrm{Sp}(2n) := \left\{ A \in \mathrm{GL}(2n,\mathbb{R}) : A^T J A = J \right\}.$$

**Properties of symplectic group**

- If $A \in \mathrm{Sp}(2n)$, then $\det A = \pm 1$. We will later show that $\det A = 1$.

$$\begin{aligned}
A^T J A = J &\implies \det A^T \cdot \det J \cdot \det A = \det J \\
&\implies \det A \det A^T = 1 \quad (\text{since } \det J = 1) \\
&\implies (\det A)^2 = 1 \quad (\text{since } \det A = \det A^T) \\
&\implies \det A = \pm 1.
\end{aligned}$$

- If $A \in \mathrm{Sp}(2n)$, then $A^T \in \mathrm{Sp}(2n)$.

$$
\begin{aligned}
A^T J A = J &\implies A^{-1} J \left(A^T\right)^{-1} = J^{-1} \\
&\implies A^{-1}(-J)\left(A^{-1}\right)^T = -J \\
&\implies A^{-1} J \left((A^{-1})\right)^T = J \\
&\implies J = AJA^T \implies A^T \in \mathrm{Sp}(2n).
\end{aligned} \tag{$\star$}
$$

- For any $A \in \mathrm{Sp}(2n)$, $A^{-1} \in \mathrm{Sp}(2n)$. (Follows from ($\star$)).
- $\dim \mathrm{Sp}(2n) = 2n^2 + n$.

*Proof.* The general linear group $\mathrm{GL}(2n, \mathbb{R})$ acts on $\wedge^2 V^*$ transitively as

$$
\varphi : \mathrm{GL}(2n, \mathbb{R}) \times \wedge^2 V^* \to \wedge^2 V^*, \quad (A, \omega) \mapsto A^* \omega.
$$

This makes $\wedge^2 V^*$ a homogeneous space. The stabilizer at $\omega$ is exactly $\mathrm{Sp}(2n)$ which a closed subgroup of $\mathrm{GL}(2n, \mathbb{R})$. So, the coset space $\mathrm{GL}(2n, \mathbb{R})/\mathrm{Sp}(2n)$ is homeomorphic to $\wedge^2 V^*$ and hence

$$
\begin{aligned}
\dim \mathrm{Sp}(2n) &= \dim \mathrm{GL}(2n, \mathbb{R}) - \dim \wedge^2 V^* \\
&= (2n)^2 - \frac{2n(2n-1)}{2} = 2n^2 + n.
\end{aligned}
$$

□

- The Lie algebra of $\mathrm{Sp}(2n)$ is

$$
\mathfrak{sp}(2n) = \left\{ X \in M(2n, \mathbb{R}) : X^T J + JX = 0 \right\}.
$$

*Proof.* Let $\gamma : [0,1] \to \mathrm{Sp}(2n)$ passing through $Id_n$. Then $\gamma$ must satisfy $\gamma^T J \gamma = J$. Taking the derivative at $t = 0$, we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma(t)^T J \gamma(t)\right)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(J)\Big|_{t=0} = 0 &\implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma(t)^T\right) J \gamma(t)\Big|_{t=0} + \gamma(t)^T J \frac{\mathrm{d}}{\mathrm{d}t}(\gamma(t))\Big|_{t=0} = 0 \\
&\implies \dot{\gamma}^T(0) J + J \dot{\gamma}(0) = 0.
\end{aligned}
$$

Thus $\mathfrak{sp}(2n) \subseteq \left\{ X \in M(2n, \mathbb{R}) : X^T J + JX = 0 \right\}$. For the other equality, we have

$$
\begin{aligned}
X^T J + JX = 0 &\implies X^T J = -JX \\
&\implies X^T = J(-X)J^{-1} \\
&\implies \exp(tX^T) = \exp\left(J(-tX)J^{-1}\right) = J \exp(tX)^{-1} J^{-1} \\
&\implies \exp(tX)^T J = J \exp(tX)^{-1} \\
&\implies \exp(tX)^T J \exp(tX) = J.
\end{aligned}
$$

Thus, $X \in \mathfrak{sp}(2n)$. □

In other way, we see that $\xi \in \mathfrak{sp}(2n)$ implies $\xi \in \mathfrak{gl}(2n)$ such that $\omega(\xi v, w) + \omega(v, \xi w) = 0$.

- For any vector space $V$, $\mathrm{Sp}(V)$ is not compact.
- Looking at the symplectic group as a as an inverse of a regular value.

Let $\mathfrak{so}(2n)$ denotes the space of all skew-symmetric symmetric matrices of size $2n$. We recall that

- Let $M$ and $N$ be smooth manifolds of dimensions $m$ and $n$, respectively, and $f : M \to N$ be a smooth function. A point $p \in N$ is said to be *regular value* of $f$ if for any $x \in f^{-1}\{p\}$, the differential map $df_x : T_xM \to T_pN$ is surjective.
- If $p$ is a regular value of $f$, then $f^{-1}\{p\}$ is a smooth manifold of dimension $\dim M - \dim N$.
- Tangent space of a vector space is itself.

Look at the map

$$f : M(2n, \mathbb{R}) \to \mathfrak{so}(2n), \quad X \mapsto X^T JX.$$

Then the differential of $f$ at $X$ will be

$$df_X : M(2n, \mathbb{R}) \to \mathfrak{so}(2n), \quad df_X(A) = A^T JX + X^T JA.$$

Note that $f^{-1}(J) = \left\{X : X^T JX = J\right\} = \mathrm{Sp}(2n)$. We claim that $J$ is a regular value of $f$. Let $X \in f^{-1}(J) = \mathrm{Sp}(2n)$. Take any $B \in \mathfrak{so}(2n)$. Define $A == -\frac{1}{2}XJB$. Then observe that

$$\begin{aligned}
df_X(A) &= -\frac{1}{2}\left((XJB)^T JX + X^T J(XJB)\right) \\
&= -\frac{1}{2}\left(B^T J^T (X^T JX) + (X^T JX)JB\right) \\
&= -\frac{1}{2}\left(B^T (-J)J + JJB\right) \\
&= -\frac{1}{2}\left(-B(Id) - B(Id)\right) = B.
\end{aligned}$$

Thus, $J$ is a regular value and hence $\mathrm{Sp}(2n)$ is a smooth manifold of dimension equals to

$$\dim \mathrm{Sp}(2n) = \dim M(2n, \mathbb{R}) - \dim(\mathfrak{so}(2n)) = (2n)^2 - \frac{2n(2n-1)}{2n} = 2n^2 + n.$$

Since it a submanifold of $\mathrm{GL}(2n, \mathbb{R})$, it is a Lie group and its Lie algebra is

$$\mathfrak{sp}(2n) = T_{Id}\mathrm{Sp}(2n) = \ker df_{Id} = \left\{A \in M(2n, \mathbb{R}) : A^T J + JA = 0\right\}.$$

Let $U(V) \subseteq \mathrm{Sp}(V)$ be the collection of all automorphisms which preserves the Hermitian structure for a given compatible complex structure $J \in \mathcal{J}(V, \omega)$. We call $U(V)$ a unitary group. Fix $J \in J(V, \omega)$. Let $G$ be the inner product induced from $J$ and $\omega$. If $J'$ is another compatible complex structure, then the map $A : V \to V$ taking an orthonormal basis with respect to $(J, G)$ into one for $(J', G')$ and satisfies $A^* J' = J$. This shows:

**Proposition 2.47.** *The action of the symplectic group $\mathrm{Sp}(V)$ on the space $\mathcal{J}(V, \omega)$ of compatible complex structures is transitive with stabilizer at $J$ equal to the unitary group $U(V)$. That is, $\mathcal{J}(V, \omega)$ may be viewed as a homogeneous space*

$$\mathcal{J}(V, \omega) = \mathrm{Sp}(V)/U(V).$$

*This shows, in particular, that $\mathrm{Sp}(V)$ is connected. (If a Lie group $G$ acts transitively on a connected smooth manifold $X$, and there exists a point $x \in X$ such that the stabilizer $G_x$ is connected, then so is $G$.) This implies that $A \in \mathrm{Sp}(V)$, then the determinant of $A$ is 1.*

*Exercise* 2.48. Consider the matrix

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A, B, C$ and $D$ are real $n \times n$ matrices. Prove that $X$ is symplectic if and only if $A^T C, B^T D$ are symmetric and $A^T D - B^T C = I$. In particular for $n = 1$, we have $\mathrm{Sp}(2) = \mathrm{SL}(2, \mathbb{R})$. Also,

$$X^{-1} = \begin{bmatrix} D^T & -B^T \\ -C^T & A^T \end{bmatrix}.$$

### Eigenvalues of Symplectic matrices

**Theorem 2.49.** *Let $A \in \mathrm{Sp}(2n)$. Then*

$$\lambda \in \sigma(A) \iff \lambda^{-1} \in \sigma(A),$$

*and the multiplicities of $\lambda$ and $\lambda^{-1}$ agree. If $\pm 1$ are eigenvalues of $A$, then it occurs with even multiplicity.*

*Proof.* Note that the determinant of $A$ is 1. We also know that for any $A \in \mathrm{GL}(2n, \mathbb{R})$ the eigenvalues appear in complex conjugate pairs of equal multiplicity. For $A \in \mathrm{Sp}(2n)$, the eigenvalues $\lambda, \lambda^{-1}$ have equal multiplicity. Also, note that $A \in \mathrm{Sp}(2n)$ implies

$$A^T J A = J \implies A^T = J A^{-1} J^{-1},$$

which means $A$ and $A^{-1}$ are similar matrices. Thus, the multiplicity of $\lambda$ and $\lambda^{-1}$ has to be even. □

Now we wil talk about the eigenspaces of a symplectic matrix.

**Lemma 2.50.** *Let $A \in \mathrm{Sp}(2n)$ be diagonalizable with all real eigenvalues. We denote $E_\lambda$ the eigenspace for the eigenvalue $\lambda$. Then we have*

$$E_\lambda^\omega = \oplus_{\lambda\mu \neq 1} E_\mu$$

*Proof.* Let $v \in E_\lambda$ and $w \in E_\mu$, then we have

$$\omega(v, w) = \omega(Av, Aw) = \lambda\mu\omega(Av, Aw),$$

so for $\lambda\mu \neq 1$, and $v \in E_\lambda$ it holds $\omega(v, w) = 0$ for all $w \in E_\mu$. Hence $E_\mu \subseteq E_\lambda^\omega$. Now we apply the formula $\dim V = \dim U + \dim U^\omega$ together with the hypothesis that $A$ is diagonalizable establishes the result. □

**Lemma 2.51.** *Let $A \in \mathrm{Sp}(2n)$ be diagonalizable with all real eigenvalues. The for any $\alpha \geq 0$, $A^\alpha \in \mathrm{Sp}(2n)$.*

*Proof.* Since $A$ is diagonalizable with real eigenvalues, there exists a orthonormal basis of eigenvectors. Hence, if we denote $E_{\lambda_i}$, the eigenspace for the eigenvalue $\lambda_i$, then

$$\mathbb{R}^{2n} = \bigoplus_{i=1}^{k} E_{\lambda_i}.$$

Then for any two vectors $u, v \in \mathbb{R}^{2n}$, we have

$$u = \sum_{i=1}^{k} u_i \quad \text{and} \quad v = \sum_{i=1}^{k} v_i,$$

where $u_i, v_i \in E_{\lambda_i}$. As $A$ is symplectic, we have

$$\omega(u_i, v_j) = \omega(Au_i, Av_j) = \lambda_i \lambda_j \omega(u_i, v_j).$$

Hence, either $\lambda_i \lambda_j = 1$ or $\omega(u_i, v_j) = 0$. Therefore, for any $\alpha \geq 0$,

$$\begin{aligned}
\omega\left(A^\alpha u, A^\alpha v\right) &= \sum_{i,j=1}^{k} \omega\left(A^\alpha u_i, A^\alpha v_j\right) \\
&= \sum_{i,j=1}^{k} \left(\lambda_i \lambda_j\right)^\alpha \omega\left(u_i, v_j\right) \\
&= \sum_{i,j=1}^{k} \omega(u_i, v_j) = \omega(u, v).
\end{aligned}$$

□

**Proposition 2.52.** *1. The unitary group $U(n)$ is a maximal compact subgroup of the symplectic linear group $\mathrm{Sp}(2n)$.*

*2. The inclusion of $U(n)$ into $\mathrm{Sp}(2n)$ is a homotopy equivalence. In particular, $\mathrm{Sp}(2n)$ is connected.*

*Proof.* We will first prove (2). Define a map

$$f : \mathrm{Sp}(2n) \times [0,1] \to \mathrm{Sp}(2n), \quad f_t(A) := f(A, t) = A\left(A^T A\right)^{-\frac{t}{2}}.$$

We made the following claims:

- For any $t \in [0,1]$ and $A \in \mathrm{Sp}(2n)$, $f(A, t) \in \mathrm{Sp}(2n)$.

  Since $A^T A$ is symmetric positive definite symplectic matrix and hence its inverse will also be symmetric, positive definite and symplectic matrix. Thus, using Lemma 2.51, $\left(A^T A\right)^{-\frac{t}{2}} \in \mathrm{Sp}(2n)$.

- $f$ is a deformation retraction.

  It is clear that $f$ is continuous function. Note that $f_0(A) = A$, thus $f_0 = id$. Also, for any $A \in U(n)$, $f_t = id$. Finally, for any $A \in \mathrm{Sp}(2n)$, $f_t(A) \in U(n)$.

It proves (2).

To prove (1), let $G$ be a compact subgroup of $\mathrm{Sp}(2n)$ such that $U(n) \subsetneq G$. Let $A \in G \setminus U(n)$. In the previous proof we saw that $f_1(A) = A\left(A^T A\right)^{-\frac{1}{2}} \in U(n) \subset G$ and hence symmetric, positive definite symplectic matrix $B = (A^T A)^{\frac{1}{2}} \in G \setminus U(n)$. This implies $B$ has an eigenvalue $\lambda > 1$. Thus, the sequence $(B^n)$ do not have a convergent subsequence $G$, which is a contradiction to our assumption that $G$ is compact. □

Recall that given an invertible matrix $A$, we have $A = QS$, where $Q$ is an orthogonal matrix and $S$ is a positive definite symmetric matrix. More precisely, $S = (A^T A)^{-1}$ and $Q = A(A^T A)^{-1}$. If $A \in \mathrm{Sp}(2n)$, then

$$\begin{aligned}
A^T J A = J \implies A &= J^{-1}(A^T)^{-1} J = J^{-1}\left((QS)^T\right)^{-1} J = J^{-1}\left(SQ^T\right)^{-1} J \\
&J^{-1}\left((Q^T)^{-1} S^{-1}\right) J = \left(J^{-1}(Q^T)^{-1} J\right)\left(J^{-1} S^{-1} J\right).
\end{aligned}$$

Note that $J^{-1}(Q^T)^{-1}J$ is an orthogonal matrix and $J^{-1}S^{-1}J$ is a positive definite symmetric matrix and thus, this is another polar decomposition of $A$. We know that polar decomposition of any invertible matrix is unique. Thus,

$$J^{-1}(Q^T)^{-1}J = Q \implies Q^TJQ = J \text{ and } J^{-1}S^{-1}J = S \implies S^TJS = J.$$

Note that Proposition 2.52 implies that the fundamental group of $\mathrm{Sp}(2n)$ is same as that of $U(n)$. In the next proposition, we will prove that the fundamental group of $U(n)$ is $\mathbb{Z}$.

**Proposition 2.53.** *The fundamental group of $U(n)$ is isomorphic to $\mathbb{Z}$. The determinant map $\det_{\mathbb{C}} : U(n) \to U(1) \cong \mathbb{S}^1$ induces an isomorphism of fundamental group.*

*Proof.* The determinant map

$$\det : U(n) \to U(1) \cong \mathbb{S}^1$$

is a fibre bundle with fibre $\mathrm{SU}(n)$.

- Recall that if $E, B$ are topological space and $\pi : E \to B$ is a continuous surjective map, then $(E, B, \pi, F)$ is called *fibre bundle* if local triviality holds, that is, for any $x \in B$ there exists a neighborhood $U_x \subseteq B$ of $x$ such that $\pi^{-1}(U) \cong U \times F$ and the following diagram is commutative.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\cong \text{(homeo)}"] \arrow[d] & U \times F \arrow[dl] \\ U & \end{tikzcd}$$

- If $(E, B, \pi, F)$ is a fibre bundle, then it induces a long exact sequence in the homotopy groups.

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$$

This implies, we have a long exact sequence of homotopy groups

$$\cdots \to \pi_1(\mathrm{SU}(n)) \to \pi_1(U(n)) \to \pi_1(\mathbb{S}^1) \to \pi_0(\mathrm{SU}(n)) \to \cdots.$$

Since $\mathrm{SU}(n)$ is path-connected, $\pi_0(\mathrm{SU}(n)) \cong 0$. Let us prove that $\mathrm{SU}(n)$ is simply connected. We will prove this by induction. For $n = 1$, $\mathrm{SU}(1) = 1$, so it is simply connected. Now we have the fibre bundle

$$\begin{tikzcd} \mathrm{SU}(n-1) \arrow[r] & \mathrm{SU}(n) \arrow[d] & A \in \mathrm{SU}(n) \arrow[d, mapsto] \\ & \mathbb{S}^{2n-1} & A(:,1) \end{tikzcd}$$

This in the long exact sequence will give

$$\cdots \to \pi_2\left(\mathbb{S}^{2n-1}\right) \to \pi_1(\mathrm{SU}(n-1)) \to \pi_1(\mathrm{SU}(n)) \to \pi_1\left(\mathbb{S}^{2n-1}\right) \to \cdots.$$

As $n > 1$, $\mathbb{S}^{2n-1}$ is simply connected and by the induction hypothesis, $\pi_1(\mathrm{SU}(n-1)) \cong 0$. Thus, $\pi_1(\mathrm{Su}(n)) \cong 0$. This gives that $\pi_1(U(n)) \cong \pi(\mathbb{S}^1) \cong \mathbb{Z}$. □

## 2.7 Maslov Index

From Proposition 2.52 and Proposition 2.53, it follows that the fundamental group of $\mathrm{Sp}(2n)$ is isomorphic to $\mathbb{Z}$. An explicit isomorphism from $\pi_1(\mathrm{Sp}(2n))$ to $\mathbb{Z}$ is given by the Maslov index.

**Theorem 2.54.** *There exists a unique function $\mu : \pi_1(\mathrm{Sp}(2n)) \to \mathbb{Z}$ called the* Maslov index *which assigns an integer to every loop $\alpha : \mathbb{S}^1 \to \mathrm{Sp}(2n)$ of symplectic matrices and satisfies the following axioms.*

*(1) (Homotopy) Two loops in $\mathrm{Sp}(2n)$ are homotopic if and only if they have the same Maslov index.*

*(2) (Product) For any two loops $\alpha, \beta \in \pi_1(\mathrm{Sp}(2n))$ we have*

$$\mu(\alpha \star \beta) = \mu(\alpha) + \mu(\beta).$$

*In particular, the constant loop has Maslov index $0$.*

*(3) (Direct sum) If $n_1 + n_2 = n$ identify $\mathrm{Sp}(2n_1) \oplus \mathrm{Sp}(2n_2)$ in the obvious way with a subgroup of $\mathrm{Sp}(2n)$, then*

$$\mu(\alpha \oplus \beta) = \mu(\alpha) + \mu(\beta).$$

*(4) (Normalization) The loop $\alpha$ defined by*

$$\alpha(t) = \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix},$$

*has Maslov index $1$.*

*Proof.* Define the map

$$\rho : \mathrm{Sp}(2n) \to \mathbb{S}^1, \rho(X) := \det(Q),$$

where $Q$ is the orthogonal matrix in the polar decomposition of the matrix $A$. We define the Maslov index as

$$\mu : \pi_1(\mathrm{Sp}(2n)) \to \mathbb{Z}, \quad \gamma \mapsto \deg(\rho \circ \gamma).$$

Equivalently,

$$\mu(\gamma) := \widetilde{\rho \circ \gamma}(1) - \widetilde{\rho \circ \gamma}(0),$$

where $\widetilde{\rho \circ \gamma} : \mathbb{R} \to \mathbb{R}$ is a continuous lift of the loop $\rho \circ \gamma$. Now it is easy to check it satisfies all the axioms. Also, the uniqueness is easy to follow. $\square$