Day 4 : 14th January, 2026

Exercise 4.1: (*Cone of a space*)

Given a space X, consider the quotient space (known as the *cone* of X)

$$CX := \frac{X \times [0,1]}{X \times \{0\}}.$$

Draw the cones for the following spaces.

i) $X = \{0, 1\}$.

ii) $X = \{0, 1, 2\}$.

iii) $X = [0, 1]$.

iv) $X = S^1$.

v) Show that for any sphere S^n, the cone is homeomorphic to $(n+1)$-disc, that is, $CS^n \cong D^{n+1}$.

Exercise 4.2: (*Suspension of sphere*)

Given a space X, consider the quotient space (known as the *(unreduced) suspension space* of X)

$$\Sigma X := \frac{X \times [0,1]}{X \times \{0,1\}}.$$

Show that for any $n \geq 0$, we have a homeomorphism $S^{n+1} \cong \Sigma S^n$, where $S^n = \left\{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\right\}$ is the n-sphere as a subspace in $\mathbb{R}^{n+1}$.

Exercise 4.3: (*Attaching map*) $5 + (3+2) + 5$ = 15

a) Suppose $A \subset X$ is a subspace and $f : A \to Y$ is a continuous map. On the disjoint union $X \sqcup Y$, consider the relation $u \sim v$ if and only if

i) $u = v$, or

ii) $u, v \in A,\ \ f(u) = f(v)$, or

iii) $u \in A,\ \ v = f(u)$, or

iv) $v \in A,\ \ u = f(v)$, or

Check that $\sim$ is an equivalence relation on $X \sqcup Y$.

b) Let us denote the quotient space under the equivalence relation as $X \cup_f Y$, which is called the *attaching space* obtained by the *attaching map* f. Check that the maps

$$\begin{aligned} p : X &\to X \cup_f Y \\ x &\mapsto [x], \end{aligned} \qquad \text{and} \qquad \begin{aligned} q : Y &\to X \cup_f Y \\ y &\mapsto [y] \end{aligned}$$

are continuous. Moreover, check that $p|_A = q \circ f$.

c) Suppose we have maps $\varphi : X \to Z$ and $\psi : Y \to Z$ such that the outer square in the diagram commutes (i.e, $\varphi\ |_A = \psi \circ f$): Then, show that there exists a unique continuous map $h : X \cup_f Y \to Z$ making the triangles commutative, i.e, $h \circ p = \varphi$ and $h \circ q = \psi$.

Definition 4.4: (*Group action*)

Let G be a group and X be a topological space. We say that *G acts on X* if there exists a map $\varphi : G \times X \to X$ such that

i) For any $x \in X$, $\varphi(e, x) = x$;
ii) For any $g, h \in G$ and $x \in X$, $\varphi(g, \varphi(h, x)) = \varphi(g \cdot h, x)$.

From now onward, we will write $\varphi(g, x) = g \cdot x$.

We say X is a G-space if an action of G on X is given. On any G-space, we have a natural equivalence relation defined as

$$x \sim y \Leftrightarrow \exists\ g \in G \text{ such that } \ y = g \cdot x.$$

The equivalence classes are called *orbits* of G. The corresponding quotient space is denoted by X/G.

Exercise 4.5: (*Based on quotient space via a group*)

i) Let $X = \mathbb{R}$ and $G = \mathbb{Z}$. Let G acts on X by $n \cdot x = x + n$, for $x \in X$ and $n \in G$. The quotient space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to S^1.
ii) Take $X = \mathbb{R}^2$ and $G = \mathbb{Z}$. The action is given by

$$n \cdot (x, y) = (x + n, y).$$

Show that X/G is homeomorphic to infinite cylinder $\{(x, y, z) : x^2 + y^2 = 1, z \in \mathbb{R}\}$.
iii) Take $X = S^n$ and $G = \mathbb{Z}_2 = \{\pm 1\}$. The action is defined as $-1 \cdot x = -x$ for $x \in S^n$. Then $X/G \cong \mathbb{RP}^n$.

Exercise 4.6: (*Lens Spaces*)

Consider $S^3 \subseteq \mathbb{C}^2$ as

$$S^3 := \left\{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\right\}.$$

Let p, q be co-prime numbers. Let a generator $g \in \mathbb{Z}_p$ acts on S^3 by

$$g \cdot (z_1, z_2) = \left(e^{(2\pi\iota)/p} z_1, e^{(2\pi\iota)/q} z_2\right).$$

The quotient space is denoted by $L(p, q)$ and is called a *Lens space*.

i) Show that $L(2, 1)$ is homeomorphic to $\mathbb{RP}^3$.
ii) If p divides $q - q'$, then $L(p, q) \cong L(p, q')$.

4.1. Hausdorff Quotient Spaces

Definition 4.7: (*Open equivalence relation*)

Let $\sim$ be an equivalence relation on a topological space X.For any set $A \subseteq X$, we define

$$[A] := \{x \in X : \exists\ a \in A \text{ such that } \ a \sim x\}.$$

The equivalence is called *open* if $[A]$ is open whenever A is open in X.

Exercise 4.8: (*Open quivalence relation and the quotient map*)

An equivalence relation $\sim$ is open if and only if the quotient map $\pi : X \to X/\sim$ is open.

Proposition 4.9: (*Quotient space is Hausdorff*)

Let $\sim$ be an open equivalence relation on a space X. Then

$$R = \{(x, y) : x \sim y\} \subseteq X \times X$$

is closed if and only if the quotient space $X/\sim$ is Hausdorff.

Proof : Let $X/\sim$ is Hausdorff. We will show that $R^c = (X \times X) \setminus R$ is open in $X \times X$. Let $(x, y) \in R^c$, that is, $x \nsim y$. This implies $[x] \neq [y]$ and hence there exists open neighborhoods U and V of $[x]$ and $[y]$, respectively, such that $U \cap V = \emptyset$. Let $\pi : X \to X/\sim$ be the quotient map. Then $\tilde{U} := \pi^{-1}(U)$ and $\tilde{V} := \pi^{-1}(V)$ are open neighborhoods of x and y, respectively. We will show that $\tilde{U} \times \tilde{V} \subseteq R^c$. Suppose $(a, b) \in \left(\tilde{U} \times \tilde{V}\right) \cap R$, then $(a, b) \in R$ implies $a \sim b$ which implies $U \ni [a] = [b] \in V$, a contradiction. Thus, $\tilde{U} \times \tilde{V} \subseteq R^c$ and hence R is closed.

On the other hand, let R be closed in $X \times X$. We need to show that $X/\sim$ is Hausdorff. Let $[x] \neq [y] \in X/\sim$. This implies $x \nsim y$ which implies $(x, y) \in R^c$. Since R^c is open, there exists basic open sets $U \times V$ such that $(x, y) \in U \times V \subseteq R^c$. Since the relation is open, $\tilde{U} := \pi(U)$ and $\tilde{V} := \pi(V)$ are open subsets of $X/\sim$. Since $U \times V \in R^c$, so $\tilde{U} \cap \tilde{V} = \emptyset$ and hence $X/\sim$ is Hausdorff. □

Exercise 4.10: (*Real projective space is Hausdorff*)

Show that $\mathbb{RP}^n$ is Hausdorff.