Complex Variables: Solution to Homework #1

Based on algebra of complex numbers

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Problem 1

Let $\mathbb{C}$ denotes the set of all complex numbers. Then show the following.

1. Addition and product operations on $\mathbb{C}$ are commutative. That is, for any $z_1, z_2 \in \mathbb{C}$, we have
$$z_1 + z_2 = z_2 + z_1 \quad \text{and} \quad z_1 \cdot z_2 = z_2 \cdot z_1$$
2. Addition and product operations on $\mathbb{C}$ are associative. That is, for any $z_1, z_2, z_3 \in \mathbb{C}$, we have
$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \quad \text{and} \quad (z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$$

Solution

1. Addition on $\mathbb{C}$ is commutative. Let $z_1 = x_1 + \iota y_1$ and $z_2 = x_2 + \iota y_2$. Then
$$\begin{aligned} z_1 + z_2 &= (x_1 + \iota y_1) + (x_2 + \iota y_2) \\ &= (x_1 + x_2) + \iota(y_1 + y_2) \\ &= (x_2 + x_1) + \iota(y_2 + y_1) \quad \text{commutativity of } \mathbb{R} \\ &= (x_2 + \iota y_2) + \iota(x_1 + \iota y_1) \\ &= z_2 + z_1. \end{aligned}$$
Similarly, one can show that the product is commutative (by using the commutativity of product in $\mathbb{R}$.)
2. Again, we will only discuss the associativity of addition and the product will be shown similarly. Let $z_k = x_k + \iota y_k$, for $k = 1, 2, 3$.
$$\begin{aligned} (z_1 + z_2) + z_3 &= (x_1 + \iota y_1 + x_2 + \iota y_2) + x_3 + \iota y_3 \\ &= (x_1 + x_2) + \iota(y_1 + y_2) + x_3 + \iota y_3 \\ &= (x_1 + x_2) + x_3 + \iota(y_1 + y_2) + \iota y_3 \\ &= x_1 + (x_2 + x_3) + \iota y_1 + \iota(y_2 + y_3) \\ &= x_1 + \iota y_1 + (x_2 + \iota y_2 + x_3 + \iota y_3) \\ &= z_1 + (z_2 + z_3). \end{aligned}$$

Problem 2

Represent the following complex numbers in the form of $a + \iota b$, where a and b are real numbers.

1. $\frac{1}{3+4\iota}$
2. $\frac{3+5\iota}{2-7\iota}$
3. $\frac{1}{\iota}$
4. $\frac{1}{x+\iota y}$, where $x^2 + y^2 = 7$.
5. $(1+\iota)^5$.

Solution

1. $\frac{1}{3+4\iota}$. We can write,

$$\frac{1}{3+4\iota} \times \frac{3-4\iota}{3-4\iota} = \frac{3-4\iota}{3^2+4^2} = \frac{3}{25} - \iota\frac{4}{25}.$$

   Thus, $a = \frac{3}{25}$ and $b = -\frac{4}{25}$.

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2. $\frac{3+5\iota}{2-7\iota}$. Consider

$$\frac{3+5\iota}{2-7\iota} \times \frac{2+7\iota}{2+7\iota} = \frac{6+21\iota+10\iota-35}{2^2+7^2} = \frac{-29}{53} + \iota\frac{31}{53}.$$

   Thus, $a = -\frac{29}{53}$ and $b = \frac{31}{53}$.

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3. $\frac{1}{\iota}$.

$$\frac{1}{\iota} = \frac{1}{\iota} \times \frac{\iota}{\iota} = \frac{\iota}{-1} = -\iota.$$

   Thus, $a = 0$ and $b = -1$.

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4. $\frac{1}{x+\iota y}$.

$$\frac{1}{x+\iota y} \times \frac{x-\iota y}{x-\iota y} = \frac{x}{x^2+y^2} + \iota\frac{y}{x^2+y^2} = \frac{x}{7} + \iota\frac{y}{7}.$$

   Thus, $a = \frac{x}{7}$ and $b = \frac{y}{7}$.

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5. $(1+\iota)^5$. Note that

$$(1+\iota)^2 = 1 + \iota^2 + 2\iota = 1 - 1 + 2\iota = 2\iota.$$

   Thus,

$$\begin{aligned}(1+\iota)^5 &= (1+\iota)^2 \cdot (1+\iota)^2 \cdot (1+\iota) \\ &= 2\iota \cdot 2\iota \cdot (1+\iota) \\ &= -4(1+\iota) \\ &= -4 - 4\iota.\end{aligned}$$

   Thus, $a = -4$ and $b = -4$.

Problem 3

Let

$$z_1 = 2+3\iota, \quad z_2 = 3\iota, \quad z_3 = 3-4\iota \text{ and } z_4 = 1-\iota.$$

Simplify the following.

1. $\frac{z_1+z_2\cdot z_3}{z_4}$.
2. $z_1 \cdot z_2 \cdot z_3 \cdot z_4$.
3. $\frac{\frac{z_1}{z_2}}{z_3} - z_4$.

Solution

1. We want to find $\frac{z_1+z_2\cdot z_3}{z_4}$. Note that

$$z_2 \cdot z_3 = 3\iota \cdot (3-4\iota) = 9\iota - 12\iota^2 = 9\iota + 12.$$

Also,

$$\frac{1}{z_4} = \frac{1}{1-\iota} = \frac{1}{1-\iota} \times \frac{1+\iota}{1+\iota} = \frac{1+\iota}{2}.$$

Thus,

$$\begin{aligned}\frac{z_1+z_2\cdot z_3}{z_4} &= (z_1+z_2\cdot z_3)\cdot\frac{1}{z_4} \\ &= ((2+3\iota)+(12+9\iota))\cdot\frac{1+\iota}{2} \\ &= (14+12\iota)\cdot\frac{1+\iota}{2} \\ &= (7+6\iota)(1+\iota) \\ &= 1+13\iota\end{aligned}$$

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2. We want to find $z_1 \cdot z_2 \cdot z_3 \cdot z_4$. In the previous probelm we have already found $z_3 \cdot z_3$. So,

$$\begin{aligned} z_1\cdot(z_2\cdot z_3)\cdot z_4 &= (2+3\iota)\cdot(12+9\iota)\cdot(1-\iota) \\ &= (2+3\iota)\cdot(21-3\iota) \\ &= 51+57\iota.\end{aligned}$$

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3. We want to simplify $\frac{\frac{z_1}{z_2}}{z_3} - z_4$. At first consider

$$\frac{z_1}{z_2} = \frac{2+3\iota}{3\iota} = \frac{2}{3\iota}+1 = -\frac{2}{3}\iota+1.$$

Similarly,

$$\frac{\frac{z_1}{z_2}}{z_3} = \frac{-\frac{2}{3}\iota+1}{3-4\iota} = \frac{1-\frac{2}{3}\iota}{3-4\iota}\times\frac{3+4\iota}{3+4\iota} = \frac{1}{25}\left(\frac{17}{3}+2\iota\right).$$

Problem 4

Look at the following figure and write the corresponding complex number. Each grid shows one unit. For example, the complex number corresponding to the point $(2, 2)$ will be $2 + 2\iota$.

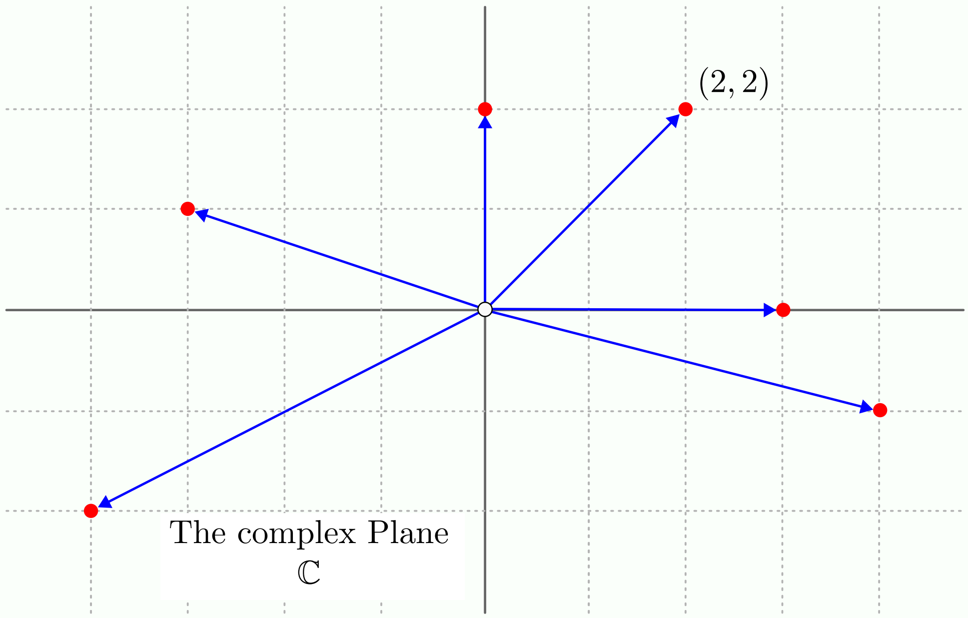
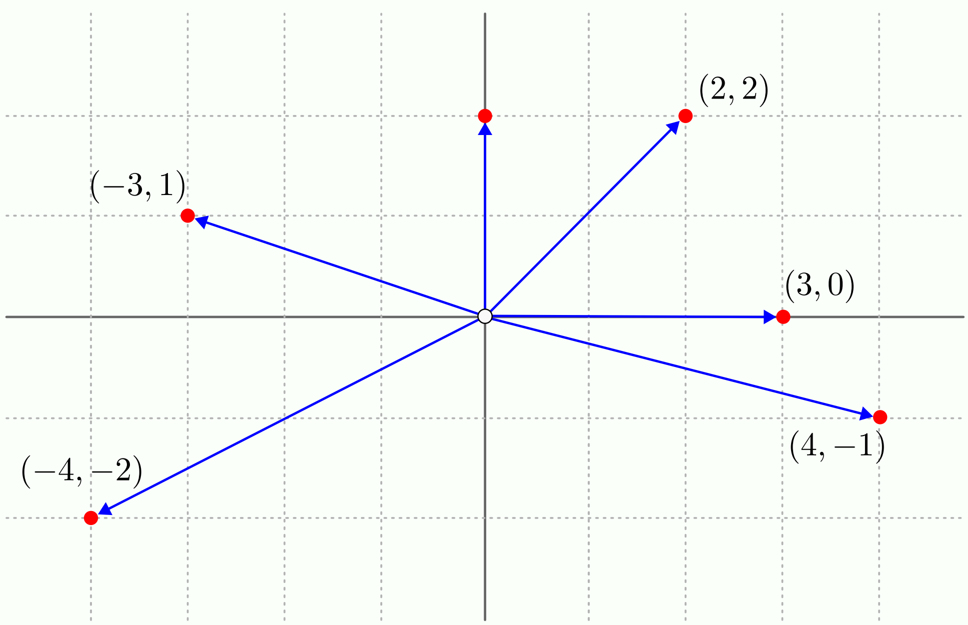


Figure 1: The complex plane

Solution



Problem 5

Geometrically demonstrate the following.

- Sum of two complex numbers.
- Product of complex numbers.

Solution

- Sum of two complex numbers z_1 and z_2.

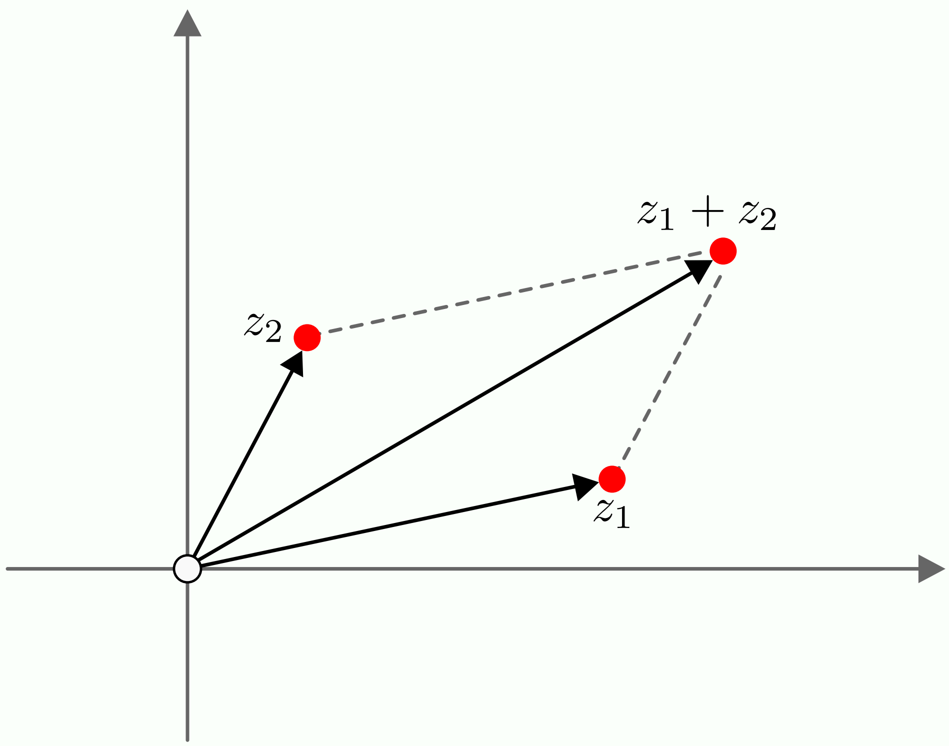


Figure 3: Sum of z_1 and z_2

Problem 6

We want to understand the geometric meaning of difference of two complex numbers. Answer the following steps to understand the geometric meaning of difference of two complex numbers, say $z_1 - z_2$.

- Draw the complex number z_1 and z_2. It is an arbitrary choice, your drawing maybe different from your friends' drawing.
- Draw the complex number $-z_2$.
- Use the previous problem to draw the complex number $z_1 + (-z_2)$.