Algebraic Topology I: Homework #8

Based on fundamental group

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Problem 1

Recall that for any based topoligcal space (X, x_0), $\pi_1(X, x_0)$ is the collection of homotopy classes of loops in X based at x_0. We have seen that with the usual concatenation, it forms a group. In this exercise, we will learn that the parameter of the concatenation does not change the result.

Let $\alpha, \beta : [0,1] \to X$ be two loops based at x_0. Define

$$(\alpha \star \beta)(s) = \begin{cases} \alpha(3s), & 0 \leq s \leq \frac{1}{3} \\ \beta\left(\frac{3s-1}{2}\right), & \frac{1}{3} \leq s \leq 1. \end{cases}$$

For any $[\alpha], [\beta] \in \pi_1(X, x_0)$, define

$$[\alpha] \star [\beta] := [\alpha \star \beta].$$

i) Show that $\alpha \star \beta$ is a loop based at x_0. Show that $\star$ is well defined.
ii) Show that the operation $\star$ induces a well-defined operation on homotopy classes of loops.
iii) Let $*$ denote the usual concatenation

$$(\alpha * \beta)(s) = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Show that $\alpha * \beta \simeq \alpha \star \beta$.
iv) Deduce that the operation $\star$ defines the same group structure on $\pi_1(X, x_0)$ as the usual concatenation.

Generalization: For $0 < a < 1$, define

$$(\alpha *_a \beta)(s) := \begin{cases} \alpha\left(\frac{s}{a}\right), & 0 \leq s \leq a \\ \beta\left(\frac{s-a}{1-a}\right), & a \leq t \leq 1 \end{cases}$$

Show that for every $a \in (0,1)$, the induced group structure on $\pi_1(X, x_0)$ is the same.

Problem 2

Consider the assignment

$$\pi_1 : \mathbf{Top}_* \to \mathbf{Groups}$$

a) $(X, x_0) \mapsto \pi_1(X, x_0)$, and
b) for any morphism $f : (X, x_0) \to (Y, y_0)$, continuous,

$$\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\alpha] \mapsto [f \circ \alpha].$$

Chech π_1 is a functor. That is, you need to check:
i) If $[\alpha] = [\beta]$, then $[f \circ \alpha] = [f \circ \beta]$.

ii) It is a homomorphism. That is,

$$\pi_1(f)([\alpha] * [\beta]) = \pi_1(f)([\alpha]) * \pi_1(f)([\beta]),$$
$$f \circ (\alpha * \beta) = (f \circ \alpha) * (f \circ \beta).$$

iii) For identity map

$$\pi_1(\mathrm{id}_X) = \mathrm{Id}_{\pi_1(X)}.$$

iv) If $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $\pi_1(g \circ f) = \pi_1(f) \circ \pi_1(g)$.

v) If $f : (X, x_0) \to (Y, y_0)$ is an isomorphism (that is, a homeomorphism in this category), then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism (that is, a group isomorphism in this category).

Note: Usually $\pi_1(f)$ is denoted by f_* and is called as *homomorphism induced by f*.

Problem 3

Let x_0 and x_1 be points of a path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$, where

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1), \quad [\gamma] \mapsto [\overline{\alpha}] * [f] * [\alpha].$$

Problem 4

Let G be a topological group with operation $\cdot$ and identity element x_0. Let $\Omega(G, x_0)$ denote the set of all loops in G based at x_0. For $f, g \in \Omega(G, x_0)$, define

$$(f \otimes g)(s) := f(s) \cdot g(s).$$

i) Show that this operation makes the set $\Omega(G, x_0)$ into a group.

ii) Show that this operation induces a group operation $\otimes$ on $\pi_1(G, x_0)$.

iii) Show that the two group operations $*$ and $\otimes$ on $\pi_1(G, x_0)$ re the same.

   Hint: Compute $\left(f * e_{x_0}\right) \otimes \left(e_{x_0} * g\right)$.

iv) Show that $\pi_1(G, x_0)$ is abelain.