# Introduction to Symplectic Geometry

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## Contents



**Course information:**

- **Course name** : Introduction to Symplectic Geometry
- **Instructor** : Sachchidanand Prasad
- **Time** : Wednesday 17:30 - 19:00
- **Course webpage** : Link to the course website
- **References** :

[1] *Lectures on Symplectic Geometry*, by Ana Cannas da Silva.

[2] *Introduction to Symplectic Topology*, by Dusa McDuff and Dietmar Salamon.

[3] *Lectures on Symplectic Manifolds*, by Alan Weinstein.

[4] *Symplectic Techniques in Physics*, by Victor Guillemin and Shlomo Sternberg.

# 1 Introduction

*The word symplectic was invented by Hermann Weyl in 1939. He replaced the Latin roots in the word complex, com-plexus, by the corresponding Greek roots sym-plektikos.*

## 1.1 An overview of geometry

- **Geometry**: Background Space (smooth manifold) + extra structure (tensor)
  - Riemannian geometry: smooth manifold + metric structure
    - metric structure = positive-definite symmetric 2-tensor
  - Complex geometry: smooth manifold + complex structure
    - complex structure = involutive endomorphism ($(1,1)$-tensor)
  - Symplectic geometry: smooth manifold + symplectic structure
    - symplectic structure = closed non-degenerate 2-form
  - Contact geometry: smooth manifold + contact structure
    - contact structure = "local contact 1-form"

In both symplectic and Riemannian geometry the main object of study is a smooth manifold equipped with a bilinear form on each tangent space. In the Riemannian manifold, this form is a symmetric, nondegenerate, positive definite form, turning each tangent space into normed vector space. On the other hand, in symplectic geometry, we instead require a skew-symmetric bilinear form on each tangent space, again varying smoothly. We still require that at each point $p$ in our manifold $M$, a skew-symmetric 2-form $\omega_p$ should be nondegenerate, that is,

$$\omega_p(X,Y) = 0 \ \ \forall \ Y \in T_pM, \text{ then } X \equiv 0.$$

Finally, note that because $\omega$ is a skew-symmetric 2-form, it must be closed, that is, $\mathrm{d}\omega = 0$.
We will now compare both geometry and from next lecture onwards we will discuss in more details. We will use the following notations:

- $M$ : real finite dimensional smooth manifold without boundary.
- $C^\infty(M) = \{f : M \to \mathbb{R} : f \text{ is smooth}\}$.
- $\chi(M) = \{X : M \to TM : X \text{ is a vector field}\}$.
- $\Omega^k(M) = \{\omega : TM \times TM \times \cdots \times TM \to \mathbb{R}\}$.

We now start some comparison between Riemannian geometry and symplectic geometry:

(1a) Riemannian manifold is a pair $(M, \langle\cdot,\cdot\rangle)$, where

- $\langle\cdot,\cdot\rangle : \chi(M) \times \chi(M) \to C^\infty(M)$ satisfies $\langle X,Y\rangle = \langle Y,X\rangle$ and $\langle fX + gY, Z\rangle = f\langle X,Z\rangle + g\langle Y,Z\rangle$.
- $\langle\cdot,\cdot\rangle$ is positive definite.

(2a) Symplectic manifold is a pair $(M,\omega)$ where

- $\omega \in \Omega^2(M)$ is bilinear.
- $\omega$ is nondegenerate.
- $\omega$ is closed, that is $\mathrm{d}\omega = 0$.

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(1b) Every smooth manifold is a Riemannian manifold.

(2b) Not all manifolds are Symplectic. The necessary conditions are:

- $\dim M = \text{even}$.
- $M$ is oriented.
- If $M$ is compact, then $H^2_{dR}(M, \mathbb{R}) \neq 0$.

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(1c) Isometry: Two Riemannian manifolds $(M_1, {}_1\langle\cdot,\cdot\rangle)$ and $(M_2, {}_2\langle\cdot,\cdot\rangle)$ are isometric if there exists a $C^1$ map $\varphi : M_1 \to M_2$ such that

$$_2\Big\langle d\varphi_p(X), d\varphi_p(Y)\Big\rangle_{\varphi(p)} = {}_1\langle X, Y\rangle_p$$

(2c) Similarly, we have symplectomorphism between two symplectic manifolds.

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(1d) Curvature is a local invariant in Riemannian manifolds.

(2d) There are no local invariants (apart from dimension) in symplectic manifolds. According to the Darboux-Weinstein theorem, *given any two symplectic manifolds of the same finite dimension, they look alike locally.*

# 2 Symplectic algebra

In this lecture, we will mostly recall the linear algebra preliminaries for our course. More precisely, we will deal with linear symplectic algebra which we will be using through out the course.

## 2.1 Some basic definitions

**Definition 2.1** (Vector sapce). *A set $(V, +, \cdot)$ is said to be a vector space over a field $\mathbb{F}$ if the operations*

$$+ : V \times V \to V \text{ and } \cdot : \mathbb{F} \times V \to V,$$

*satisfies the following properties. For any $v, v_1, v_2, v_3$ and $\alpha, \beta \in \mathbb{F}$ we have the following.*

1. *(Commutativity) $v_1 + v_2 = v_2 + v_1$.*
2. *(Associativity) $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$.*
3. *(Existence of additive identity) There exists $0 \in V$ such that for any $v \in V$ $0 + v = v = v + 0$.*

4. *(Existence of additive inverse) For any $v \in V$, there exists $w$ such that $v + w = 0 = w + v$. We will denote $w = -v$.*

5. *(Multiplicative identity) For any $v \in V$, $1 \cdot v = v$.*

6. *(Multiplication associativity) $(\alpha\beta) \cdot v = \alpha \cdot (\beta \cdot v)$.*

7. *(Distribution law)*

   - $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$.
   - $\alpha(v_1 + v_2) = \alpha \cdot v_1 + \alpha \cdot v_2$.

Our field will always be either $\mathbb{R}$ or $\mathbb{C}$.

**Definition 2.2** (Lienar map). *Let $T : V \to W$ be a map between two vector spaces $V$ and $W$. Then $T$ is said to be linear if,*

$$T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2),$$

*for $v_1, v_2 \in V$ and $\alpha, \beta \in \mathbb{F}$.*

**Definition 2.3** (Dual space). *If $V$ is a vector space over a field $\mathbb{F}$. Then the dual space of $V$, denoted by $V^*$, is defined by*

$$V^* := \{\varphi : V \to \mathbb{F} : \varphi \text{ is linear}\}.$$

**Definition 2.4** (Bilinear map). *Let $V, W, S$ be vector spaces over a field $\mathbb{F}$. The a bilinear map $B$ is a map*

$$B : V \times W \to S$$

*such that $B$ is linear in each argument. That is, $B(\cdot, w) : V \to S$ and $B(v, \cdot) : W \to S$ is linear for any $v \in V$ and $w \in W$.*

**Definition 2.5.** *A bilinear form $\omega$ on a vector space $V$ is a bilinear map $B : V \times V \to \mathbb{F}$. The bilinear form $\omega$ is said to be nondegenerate if the kernel*

$$\ker \omega := \{v \in V : \omega(v, w) = 0 \text{ for all } w \in V\}$$

*is trivial.*

We identify a bilinear form $\omega$ on $E$ with the linear mapping $u \mapsto (v \mapsto \omega(u, v))$ for $u, v \in E$.

**Definition 2.6.** *Let $\omega$ be a bilinear form on a vector space $V$.*

1. *$\omega$ is said to be symmetric if $\omega(v, w) = \omega(w, v)$ for any $v, w \in V$.*

2. *$\omega$ is said to be skew-symmetric if $\omega(v, w) = -\omega(w, v)$ for any $v, w \in V$.*

## 2.2 Symplectic vector space

The first important notions that we introduce are the symplectic form and the symplectic vector space. We also define the concept of canonical form of a symplectic form and the symplectic basis of a symplectic vector space. Throughout this notes, we will assume $V$ to be a vector space of finite dimension.

**Definition 2.7.** *The pair $(V, \omega)$ is said to be symplectic vector space if $\omega : V \times V \to \mathbb{R}$ is skew-symmetric, nondegenerate bilinear form. We call $\omega$ a symplectic form on $E$.*

*Remark.* It follows from the definition that $\omega(v, v) = 0$ for any $v \in V$.

*Example* 2.8. On $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ we define $\omega$ by

$$\omega((\mathbf{x},\mathbf{y}),(\mathbf{x}',\mathbf{y}')) := \sum_{i=1}^{n}(x_i y_i' - x_i' y_i) = \langle \mathbf{x},\mathbf{y}'\rangle - \langle \mathbf{x}',\mathbf{y}\rangle.$$

We claim that $\omega$ is a symplectic form on $\mathbb{R}^{2n}$. It is clear that $\omega$ is a bilinear form. Further, we need to check two things:

(i) $\omega$ is skew-symmetric.

For any $(\mathbf{a},\mathbf{b}),(\mathbf{c},\mathbf{d}) \in \mathbb{R}^n \times \mathbb{R}^n$, we have

$$\begin{aligned}\omega((\mathbf{c},\mathbf{d}),(\mathbf{a},\mathbf{b})) &= \langle \mathbf{b},\mathbf{c}\rangle - \langle \mathbf{a},\mathbf{d}\rangle, \text{ and}\\ \omega((\mathbf{a},\mathbf{b}),(\mathbf{c},\mathbf{d})) &= \langle \mathbf{a},\mathbf{d}\rangle - \langle \mathbf{b},\mathbf{c}\rangle = -\omega((\mathbf{c},\mathbf{d}),(\mathbf{a},\mathbf{b})).\end{aligned}$$

(ii) $\omega$ is nondegenerate.

Let $\omega((\mathbf{x},\mathbf{y}),(\mathbf{a},\mathbf{b})) = 0$ for any $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n$. We need to show that $(\mathbf{x},\mathbf{y}) = (\mathbf{0},\mathbf{0})$. Take $\mathbf{a} = \mathbf{0}$ and $\mathbf{b} = \mathbf{x}$, then

$$\omega((\mathbf{x},\mathbf{y}),(\mathbf{0},\mathbf{x})) = 0 \implies \langle \mathbf{x},\mathbf{x}\rangle - \langle \mathbf{y},\mathbf{0}\rangle = 0 \implies \mathbf{x} = 0.$$

Similarly, one can show that $\mathbf{y} = 0$ and hence, $\omega$ is nondegenerate.

This is called standard symplectic form on $\mathbb{R}^n \times \mathbb{R}^n$.

The above example can also be written in the following form:

*Example* 2.9. Let $V = \mathbb{R}^{2n}$ with a basis $\{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\}$ and define $\omega$ as

$$\omega(e_i, e_j) = 0, \quad \omega(f_i, f_j) = 0 \quad \text{and } \omega(e_i, f_j) = \delta_{ij}.$$

Then $\omega$ is standard symplectic form on $V$.

*Example* 2.10. Let $V$ be any vector space of dimension $n$ and $V^*$ denotes its dual. If $E = V \oplus V^*$ and define

$$\omega : E \times E \to \mathbb{R}, \quad \omega((v,\alpha),(v',\alpha')) = \alpha'(v) - \alpha(v'),$$

then $(E,\omega)$ is a symplectic vector space.

Since $\alpha$ and $\alpha'$ are linear maps, it is clear that $\omega$ is a bilinear form. Let us show it is skew-symmetric and nondegenerate.

(i) $\omega$ is skew-symmetric.

For any $v, v' \in V$ and $\alpha, \alpha' \in V^*$, we have

$$\begin{aligned}\omega((v,\alpha),(v',\alpha')) &= \alpha'(v) - \alpha(v')\\ &= -(\alpha(v') - \alpha'(v))\\ &= \omega((v',\alpha'),(v,\alpha)).\end{aligned}$$

(ii) $\omega$ is nondegenerate.

Let $\omega((v,\alpha),(w,\beta)) = 0$ for any $(w,\beta) \in E$. We need to show that $v = 0$ and $\alpha \equiv 0$. Observe that for any $\beta \in V^*$

$$\omega((v,\alpha),(0,\beta)) = \beta(v) = 0 \implies v = 0.$$

Similarly, for any $w \in V$,

$$\omega((v,\alpha),(w,0)) = \alpha(w) = 0 \implies \alpha = 0.$$

Thus $(E,\omega)$ is a symplectic vector space.

**Definition 2.11.** *Let $(V,\omega)$ is a symplectic vector space, then for any subspace $W \subseteq V$, we define the* $\omega$-orthogonal space

$$W^{\omega} := \{v \in V : \omega(v,w) = 0,\ \forall\ w \in W\}.$$

**Proposition 2.12.** *Let $V$ be a $k$-dimensional vector space over $\mathbb{R}$ and $\omega$ be a bilinear form.*

1. *If $\omega$ is symmetric with rank $r$, then there exists a basis $\mathcal{B}$ of $V$ such that with respect to $\mathcal{B}$,*

$$[\omega]_{\mathcal{B}} = \begin{bmatrix} \epsilon_1 & & & & & \\ & \ddots & & & & \\ & & \epsilon_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}, \text{ where } \epsilon_i = \pm 1, \quad i = 1,2,\ldots,r.$$

2. *If $\omega$ is skew-symmetric with rank $r$, then $r = 2n$ and there is a basis $\mathcal{B}$ of $V$ relative to which*

$$[\omega]_{\mathcal{B}} = \begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ where } I_n \text{ is the identity matrix of size } n.$$

*Proof.* 1. Proof is left.

2. Since $\omega \neq 0$, we can choose $e_1, f_1 \in V$ such that $\omega(e_1,f_1) \neq 0$ (this must implies that both the vectors are linearly independent). By rescaling $e_1$, we can further assume that $\omega(e_1,f_1) = 1$. Define $W_1 := \operatorname{span}\{e_1,f_1\}$. Since, $\omega$ is skew-symmetric, we have $\omega(e_1,e_1) = 0 = \omega(f_1,f_1)$. Thus, the restriction of $\omega$ on $W_1$ is

$$[\omega]_{\{e_1,f_1\}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let $W_2$ be the $\omega$-orthogonal complement of $W_1$, that is, $W_2 = W_1^{\omega}$. It is clear that $W_1 \cap W_2 = \{0\}$. We claim that $V = W_1 \oplus W_2$. Note that for any $v \in V$, we have

$$\omega\left(e_1, v - \omega(v,f_1)e_1 + \omega(v,e_1)f_1\right) = 0 \text{ and}$$
$$\omega\left(f_1, v - \omega(v,f_1)e_1 + \omega(v,e_1)f_1\right) = 0.$$

Thus, $v - \omega(v,f_1)e_1 + \omega(v,e_1)f_1 \in W_2^{\omega}$ and hence $V = W_1 \oplus W_2$. We can repeat the process on $W_2$ and find $e_2$ and $f_2$ such that $\omega(e_2,f_2) = 1$. Now the matrix will be

$$[\omega]_{\{e_1,e_2,f_1,f_2\}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

Inductively, we get a basis

$$\mathcal{B} = \{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\}$$

such that $[\omega]_{\mathcal{B}}$ will be in the given form.

□

*Remark.* Since we focus on non-degenerate skew-symmetric bilinear form, that is, rank $= 2n$, we may consider only the case with matrix representation $\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ and $V$ must be of even dimension.

We just showed the following corollary.

**Corollary 2.13.** *Every finite dimensional symplectic vector space $(V,\omega)$ has even dimension.*

*Exercise* 2.14. Show that the space of skew-symmetric bilinear form is isomorphic to the space $\wedge^2 V^*$ of the second exterior product of $V^*$.

So if $\mathcal{B} = \{e_1,\dots,e_{2n}\}$ is a basis for $V$, and $\mathcal{B}^*$ is its dual, then for any $\omega \in \wedge^* V$ with the matrix $(\omega_{ij})$, relative to $\mathcal{B}$, can also be written as

$$[\omega]_{\mathcal{B}} = \sum_{i<j} \omega_{ij} e_i^* \wedge e_j^*.$$

*Remark.* Since elements of $\wedge^2 V^*$ are represented by anti-symmetric matrices and with all the entries of the main diagonal equal to 0, for a vector space $V$ of dimension $2n$ we have $\dim \wedge^2 V^* = \frac{2n(2n-1)}{2} = n(2n-1)$.

**Corollary 2.15** (Canonical form of $\omega$)**.** *For every skew-symmetric bilinear form $\omega$, there exists a basis $\mathcal{B} = \{e_1,\dots,e_{2n}\}$ of $V$ such that*

$$[\omega]_{\mathcal{B}} = \sum_{i<j} e_i^* \wedge e_j^*.$$

This representation is called a canonical form of $\omega$ and we call $\mathcal{B}$ a symplectic basis of $V$.

## 2.3 Symplectomorphism

**Definition 2.16.** *Two symplectic vector spaces $(V_1, \omega_1)$ and $(V_2, \omega_2)$ are called* symplectomorphic *if there exists an isomorphism $\varphi : V_1 \to V_2$ of vector space such that $\omega_2(\varphi(x), \varphi(y)) = \omega_1(x, y)$. In other words, $\varphi^*\omega_2 = \omega_1$. We call $\varphi$ a* symplectomorphism. *We will write $V_1 \cong V_2$.*

*Exercise* 2.17. What can you conclude if $\dim V_1 = \dim V_2$ and $\varphi : V_1 \to V_2$ satisfies $\varphi^*\omega_2 = \omega_1$?

We claim that $\varphi$ is injective. If $v \in \ker \varphi$, then for any $v' \in V_1$, we have

$$\omega_1(v, v') = \omega_2(\varphi(v), \varphi(v')) = 0.$$

Since, $\omega_1$ is nondegenerate, $v = 0$. Since the dimension matches, $V_1 \cong V_2$.

*Exercise* 2.18. Show that the set of all symplectomorphisms of a symplectic vector space $(V, \omega)$ forms a group under the composition.

**Definition 2.19.** *The group of symplectomorphism of a symplectic vector space $(V, \omega)$ is called* symplectic group *and we will denote this by $\mathrm{Sp}(V)$.*

*Example* 2.20. Some examples on symplectomorphism:

1. $V = \mathbb{R}^{2n}$ and $\omega((\mathbf{a}, \mathbf{b}), (\mathbf{c}, \mathbf{d})) = \langle \mathbf{a}, \mathbf{d} \rangle - \langle \mathbf{c}, \mathbf{b} \rangle$.

   - $\varphi(e_i) = f_i$ and $\varphi(f_i) = -e_i$. If we change $\varphi(f_i) = e_i$, will it work?

     It is clear that $\varphi$ is an isomorphism. We just need to show that $\varphi^*\omega = \omega$. Note that

     $$\omega\left(e_i, e_j\right) = 0 = \omega\left(f_i, f_j\right).$$
     $$\delta_{ij} = \omega\left(\varphi(e_i), \varphi(f_j)\right) = \omega\left(f_i, -e_j\right) = -\omega\left(f_i, e_j\right) = \omega\left(e_j, f_i\right) = \delta_{ij}.$$

   - $\varphi(e_i) = e_i + f_i$ and $\varphi(f_i) = f_i$.

   - For any invertible matrix $X$,

     $$\varphi(e_i) = \sum_j X_{ij} e_j \text{ and } \varphi(f_i) = \sum_j \left(X^{-1}\right)_{ji} f_j.$$

2. We show in Example 2.10 $E = V \oplus V^*$ is a symplectic vector space. We can give a symplectomorphism on $E$ as follows. Let $T : V \to V$ be an isomorphism and $T^* : V^* \to V^*$ be the dual map. Then
   $$T \oplus T^* : E \to E,$$
   is a symplectomorphism.

3. Let $V$ be a complex vector space of complex dimension $n$, with complex, positive definite inner product (=Hermitian metric) $h : V \times V \to \mathbb{C}$.Then $V$, viewed as a real vector space, with bilinear form the imaginary part $\omega = \mathrm{Im}(h)$ is a symplectic vector space. Every unitary map $V \to V$ preserves $h$, hence also $\omega$ and is therefore symplectic.

*Exercise* 2.21. Show that $\mathbb{R}^{2n}$, $E$ and the third example are symplectomorphic.

**Proposition 2.22.** *Every symplectic vector space $(V, \omega)$ of dimension $2n$ is symplectomorphic to $\mathbb{R}^{2n}$ with the canonical symplectic form.*

As a consequence of the above proposition, we have the following theorem, which we call *Linear Darboux theorem*.

**Theorem 2.23** (Linear Darboux Theorem)**.** *For any symplectic vector space $(V,\omega)$ there exists a basis $\mathcal{B} = \{e_i, f_i\}_{i=1}^n$ of $V$ such that*

$$\omega(e_i, e_j) = 0 = \omega(f_i, f_j) \text{ and } \omega(e_i, f_j) = \delta_{ij} \quad \forall\, i,j.$$

*This basis is called a* Darboux basis *of $V$.*

The above theorem is equivalent to following statements:

(i) Any symplectic vector space is even-dimensional.

(ii) Any even dimensional vector space admits a linear symplectic form.

(iii) Up to linear symplectomorphisms, there is a unique linear symplectic form on each even dimensional vector space.

## 2.4 Subspaces of a symplectic vector space

Recall the Definition 2.11 of $\omega$-perpendicular space. Note that with our assumption that $V$ is finite dimensional, $\omega$ is nondegenerate if and only if the map

$$\omega^\flat : V \to V^*, \quad \omega^\flat(v)(w) = \omega(v,w)\ \forall\, v,w \in V$$

is an isomorphism.

*Note.* For any subspace $W \subset V$, we have

$$W^\omega = \left(w^\flat\right)^{-1}(\operatorname{ann}(W)),$$

where $\operatorname{ann}(W)$ is the annihilator of $W$, that is, the set of all $f \in V^*$ such that $f(w) = 0$ for $w \in W$.

We have

$$v \in \operatorname{ann}(W) \iff \text{for any } w \in W,\ \left(\omega^\flat(v)\right)(w) = 0 \iff \omega(v,w) = 0 \iff v \in W^\omega.$$

**Definition 2.24.** *A subspace $W \subseteq V$ of a symplectic vector space is called*

*(i)* isotropic *if $W \subseteq W^\omega$, that is, $\omega\big|_{W\times W} = 0$;*

*(ii)* co-isotropic *if $W^\omega \subseteq W$, that is, $W^\omega$ is isotropic;*

*(iii)* Lagrangian *if $W^\omega = W$, that is, $W$ is isotropic and co-isotropic;*

*(iv)* symplectic *if $\omega_{W\times W}$ is nondegenerate.*

*The set of Lagrangian subspaces of $V$ is called the* Lagrangian Grassmannian *and denoted* $\operatorname{Lag}(V)$.

*Exercise* 2.25. Show that $W$ is symplectic if and only if $W \cap W^\omega = \{0\}$ .

*Exercise* 2.26. Let $(V,\omega)$ is a symplectic vector space and $W$ be any subspace of $V$. Consider the map $\varphi : V \to W^*$, defined by $\varphi(v) = \omega(v)(w)$ for any $v \in V$ and $w \in W$. Show that $\varphi$ is surjective. Deduce that $\dim W^\omega = \dim V - \dim W$. Also, show that $(W^\omega)^\omega = W$.

*Remark.* 1. From Exercise 2.26, we conclude that if $\dim V = 2n$, then all the isotropic subspaces have dimension smaller or equal $n$, all the co-isotropic have dimension bigger or equal $n$ and all the Lagrangian subspace have dimension $n$.

2. If $W \subseteq V$ is symplectic subspace, then it follows from the definition that $W \cap W^\omega = \{0\}$ and therefore, from the dimension sum restriction (Exercise 2.26), we must have $V = W \oplus W^\omega$.

*Example* 2.27. Every 1-dimensional subspace of $V$ is isotropic and every subspace with codimension 1 is co-isotropic.

*Example* 2.28. Consider $V = \mathbb{R}^{2n}$ with canonical symplectic form $\omega$. Define

$$\begin{aligned} W_1 &= \operatorname{span}\{e_1, e_2\}. \quad \text{Isotropic} \\ W_2 &= \operatorname{span}\{e_1, e_2, \ldots, e_n, f_3, f_4, \ldots, f_n\}. \quad \text{Co-isotropic} \\ W_3 &= \operatorname{span}\{e_1, e_2, \ldots, e_n\}. \quad \text{Lagrangian} \\ W_4 &= \operatorname{span}\{e_1, f_1\}. \quad \text{Symplectic} \end{aligned}$$

*Exercise* 2.29. Let $(V, \omega)$ be a symplectic vector space and $W$ be any subspace of $V$.

1. Show that if $W$ is isotropic, then $\dim W \leq \frac{1}{2}\dim V$.
2. Show that if $W$ is Lagrangian, then $\dim W = \frac{1}{2}\dim V$.
3. Show that if $W$ is Lagrangian, then any basis $\mathcal{B}_W = \{e_1, e_2, \ldots, e_n\}$ of $W$ can be extended to a symplectic basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of $V$.

**Proposition 2.30.** *For any symplectic vector space $(V, \omega)$, there exists a Lagrangian subspace $L$.*

*Proof.* Since for every $v \in V$ we have $\omega(v, v) = 0$, $V$ has an isotropic subspace. Let $L \subsetneq V$ be a maximal isotropic subspace of $V$, that is, it is not contained in any isotropic subspace of strictly larger dimension. Then we claim that $L$ is Lagrangian, that is, $L^\omega = L$. We only need to show that $L$ is co-isotropic. Suppose not, take $v \in L^\omega \setminus L$, then $L' = L \oplus \operatorname{span}\{v\}$ is isotropic and larger than $L$. □

An immediate consequence is that any symplectic vector space $V$ has even dimension: For if $L$ is a Lagrangian subspace

$$\dim V = \dim L + \dim L^\omega = 2\dim L.$$

From this proof we also conclude that a maximal isotropic subspace is a Lagrangian subspace. Therefore we have the following corollary.

**Corollary 2.31.** *Every isotropic subspace is contained in a Lagrangian subspace.*