Day 2 : 7th January, 2026

2.1. Interior

Definition 2.1: (*Interior of a set*)

Given $A \subset X$, the *interior* of A, denoted $\mathring{A}$ (or $\operatorname{int}(A)$), is the largest open set contained in A. A point $x \in \mathring{A}$ is called an *interior point* of A.

Exercise 2.2: (*Interior of open sets*)

For any $A \subset X$ show that $\mathring{A}$ is the union of all open sets contained in A. In particular, show that $A \subset X$ is open if and only if $A = \mathring{A}$.

Exercise 2.3: (*Interior point*)

Given $A \subset X$, show that a point $x \in X$ is an interior point of A if and only if there exists some open set $U \subset X$ such that $x \in U \subset A$.

2.2. Boundary

Definition 2.4: (*Boudary of a set*)

Given $A \subset X$, the *boundary* of A, denoted ∂A (or $\operatorname{bd}(A)$), is defined as

$$\partial A = \overline{A} \cap \overline{(X \setminus A)}.$$

Clearly boundary of any set is always a closed set. Also, observe the following. Given any $A \subset X$, a point $x \in X$ can satisfy exactly one of the following.

a) There exists an open set U with $x \in U \subset A$ (whence x is an interior point of A).
b) There exists an open set U with $x \in U \subset X \setminus A$ (whence x is an interior point of $X \setminus A$).
c) For any open set U with $x \in U$, we have $U \cap A \neq \emptyset$ and $U \cap (X \setminus A) = \emptyset$ (whence x is a boundary point of A).

Exercise 2.5:

Given $A \subset X$, show that

$$\partial A = \{x \in X \mid \text{for any } U \subset X \text{ open, with } x \in U \text{ , we have } U \cap A \neq \emptyset \neq U \cap (X \setminus A)\}$$

Exercise 2.6:

Find out the boundaries of A, when

a) $A = \{(x, y) \mid x^2 + y^2 < 1\} \subset \mathbb{R}^2$, and
b) $A = \{(x, y, z) \mid x^2 + y^2 < 1,\ z = 0\} \subset \mathbb{R}^3$.

Exercise 2.7: (*Boundary properties*)

Let $A \subset X$ be given.

a) Show that $\overline{A} = \mathring{A} \sqcup \partial A$ (i.e., a disjoint union).
b) Show that A is open if and only if $\partial A = \overline{A} \setminus A$.

Exercise 2.8: (*Boundary computation*)

Compute the boundary of the following subsets $A \subset X$.

a) X is any space, and $A = X$.
b) X is any space, and $A = \emptyset$.
c) X is a discrete space, and $\emptyset \neq A \subsetneq X$.
d) X is an indiscrete space, and $\emptyset \neq A \subsetneq X$.
e) $X = \mathbb{R}$ and $A = \mathbb{Z}$.
f) $X = \mathbb{R}$ and $A = \mathbb{Q}$.
g) $X = \mathbb{R}$ and $A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$.

2.3. Subspaces

Definition 2.9: (*Subspace topology*)

Given a topological space $(X, \mathcal{T})$ and a subset $A \subset X$, the *subspace topology* on A is defined as the collection

$$\mathcal{T}_A := \{U \subset A \mid U = A \cap O \text{ for some } O \in \mathcal{T}\}.$$

We say $(A, \mathcal{T}_A)$ is a subspace of $(X, \mathcal{T})$.

Exercise 2.10:

Suppose $U \subset X$ is an open set. What are the open subsets of U in the subspace topology? What are the closed sets?

Proposition 2.11: (*Closure in subspace*)

Let $Y \subset X$ be a subspace. Then, a subset of Y is closed in Y if and only if it is the intersection of Y with a closed set of X. Consequently, for any $A \subset Y$, the closure of A in the subspace topology is given as $\overline{A}^Y = \overline{A} \cap Y$.

Exercise 2.12:

Prove the above proposition.

Exercise 2.13: (*Interior and subspace*)

Prove or disprove : Let $Y \subset X$ be a subspace, and $A \subset Y$. Then, the interior of A in Y (with respect the subspace topology) is $\mathring{A} \cap Y$.

2.4. Continuous function

Definition 2.14: (*Continuous function*)

Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a function $f : X \to Y$ is said to be *continuous* if $f^{-1}(U) \in \mathcal{T}_X$ for any $U \in \mathcal{T}_Y$ (i.e., pre-image of open sets are open).

Exercise 2.15: (*Pre-image of closed set*)

Show that $f : X \to Y$ is continuous if and only if preimage of closed sets of Y is closed in X.

Exercise 2.16: (*Continuity : closure and interior*) $5 + 5 + 5 = 15$

Given a map $f : X \to Y$, show that the following are equivalent.

a) f is continuous.
b) For any subset $A \subset X$, we have $f\left(\overline{A}\right) \subset \overline{f(A)}$.
c) For any subset $B \subset Y$, we have $f^{-1}(\operatorname{int}(B)) \subset \operatorname{int}\left(f^{-1}(B)\right)$.

Exercise 2.17: (*Continuity of the identity*)

Suppose X is equipped given topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. Show that $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if and only if $\mathrm{Id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous.

Exercise 2.18: (*Maps from discrete topology*) $5 + 5 + 5 = 15$

Suppose $(X, \mathcal{T})$ is a topological space. Show that the following are equivalent.

a) X has the discrete topology, i.e., $\mathcal{T} = \mathcal{P}(X)$.
b) Given any space Y, any function $f : X \to Y$ is continuous.
c) The map $\mathrm{Id} : (X, \mathcal{T}) \to (X, \mathcal{P}(X))$ is continuous.

Exercise 2.19: (*Maps into indiscrete topology*) 5

Suppose X is a topological space. Show that the topology is indiscrete if and only if given any space Y, any function $f : Y \to X$ is continuous.

Hint : Consider $Y = X$ equipped with the indiscrete topology, and $f = \mathrm{Id}$.

Exercise 2.20: (*Subspace and inclusion*) $5 + 5 = 10$

Suppose $(X, \mathcal{T})$ is a space, and some $A \subset X$ is equipped with the subspace topology $\mathcal{T}_A$.

a) Show that the inclusion map $\iota : A \hookrightarrow X$ is continuous.

b) Suppose $\mathcal{S}$ is some topology on A such that the inclusion map $\iota : (A, \mathcal{S}) \hookrightarrow (X, \mathcal{T})$ is continuous. Show that $\mathcal{S}$ is finer than $\mathcal{T}_A$.

Definition 2.21: (*Open map*)

Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a function $f : X \to Y$ is said to be *open* if $f(U) \in \mathcal{T}_Y$ for any $U \in \mathcal{T}_X$ (i.e, image of opens sets are open).

Exercise 2.22: (*Openness of the identity*)

Suppose X is equipped given topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. Show that $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ if and only if $\mathrm{Id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is open.

Exercise 2.23: (*Openness of bijection*)

Suppose $f : X \to Y$ is a bijection. Show that f is open if and only if f^{-1} is continuous.

Definition 2.24: (*Homeomorphism*)

Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a function $f : X \to Y$ is said to be a *homeomorphism* if the following holds.

a) f is bijective, with inverse $f^{-1} : Y \to X$.

b) f is continuous.

c) f is open (or equivalently, f^{-1} is continuous).

Exercise 2.25: (*Continuous bijective map*)

For $0 \le t < 1$, consider $f(t) = (\cos 2\pi t, \sin 2\pi t)$. Check that $f : [0, 1) \to \mathbb{R}^2$ is a continuous, injective map. Draw the image. Is it a homeomorphism onto the image (with the corresponding subspace topologies)?

Fact 2.26: (*Invariance of domain*)

In general, a continuous bijection need not be a homeomorphism. However, there is a special situation known as the *Invariance of domain*. Suppose $U \subset \mathbb{R}^n$ is an open set. Consider a continuous injective map $f : U \to \mathbb{R}^n$. Denote $V := f(U)$. Clearly, $f : U \to V$ is a continuous bijection.

It is a very important theorem in topology that states : V is open and $f : U \to V$ is a homeomorphism.

Definition 2.27: (*Closed map*)

Given two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a function $f : X \to Y$ is said to be *closed* if $f(C)$ is closed in Y for any closed set $C \subset X$.

Exercise 2.28: (*Open and closed map*)

Give examples of continuous maps which are :

a) open, but not closed,
b) closed, but not open,
c) neither open nor closed,
d) both open and closed.

Hint : Consider $f_1(x, y) = x$, $f_2(x) = \begin{cases} 0 & x<0 \\ x & x\geq 0 \end{cases}$, $f_3(x) = \sin(x)$, and $f_4(x) = x$.

Exercise 2.29: (*Continuity is local*)

Suppose $X = \bigcup U_\alpha$, for some open sets U_α. Show that $f : X \to Y$ is continuous if and only if $f|_{U_\alpha} \to Y$ is continuous for all α.

Theorem 2.30: (*Pasting lemma*)

Suppose $X = A \cup B$, for some closed sets $A, B \subset X$. Let $f : A \to Y, g : B \to Y$ be given continuous maps, such that $f(x) = g(x)$ for any $x \in A \cap B$. Then, there exists a (unique) continuous map $h : X \to Y$ such that $h(x) = \begin{cases} f(x) & x\in A \\ g(x) & x\in B. \end{cases}$

Proof : Clearly, h is a well-defined function, and it is uniquely defined. We show that h is continuous. Let $C \subset Y$ be a closed set. Then,

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C).$$

Now, $f^{-1}(C) \subset A$ and $g^{-1}(C) \subset B$ are closed sets (in the subspace topology). But then they are closed in X, since A, B are closed. Then, $h^{-1}(C)$ is closed. Since C was arbitrary, we have h is continuous. $\square$

Definition 2.31: (*Hausdorff space*)

A space X is called *Hausdorff* (or a T_2*-space*) if for any $x, y \in X$ with $x \neq y$, there exists open neighborhoods $x \in U_x \subset X, y \in U_y \subset X$, such that $U_x \cap U_y = \emptyset$. In other words, any two points of a Hausdorff space can be separated by open sets.

Exercise 2.32: (*Metric spaces are Hausdorff*)

If (X, d) is a metric space, then show that the metric topology is Hausdorff.