1. A nonempty set $A \subseteq \mathbb{R}$ is said to be bounded above in $\mathbb{R}$ if there exists a real number M such that for all $a \in A$ we have $a \leq M$.

2. A nonempty set $A \subseteq \mathbb{R}$ is said to be bounded below in $\mathbb{R}$ if there exists a real number m such that for all $a \in A$ we have $a \geq M$.

3. A nonempty subset $A \subseteq \mathbb{R}$ is said to be bounded if it is bounded above and bounded below.

4. A sequence (x_n) is said to be convergent, if there exists a real number l such that for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have
$$|x_n - l| < \epsilon.$$

5. The real number l is said to be the limit of the sequence.

6. If a sequence (x_n) is convergent, then the limit is unique.

7. The space of real (or complex) convergent sequences forms a vector space over $\mathbb{R}$ (or $\mathbb{C}$).

8. Let $A \subseteq \mathbb{R}$ be a non-empty set. We say that $\alpha \in \mathbb{R}$ is a least upper bound of A if
   - α is an upper bound of A and
   - if β is an upper bound of A, then $\alpha \leq \beta$.

9. Let $A \subseteq \mathbb{R}$ be a non-empty set. We say that $\alpha \in \mathbb{R}$ is a greatest lower bound of A if
   - α is a lower bound of A and
   - if β is a lower bound of A, then $\alpha \geq \beta$.

10. Let A be a nonempty subset of $\mathbb{R}$. Let
$$\begin{aligned} M &= \text{an upper bound of } A \\ m &= \text{a lower bound of } A \\ \alpha &= \text{the least upper bound of } A \\ \beta &= \text{he greatest lower bound of } A. \end{aligned}$$
Then, we have
$$m \leq \beta \leq \alpha \leq M.$$

There are two equivalent Archimedean properties.

1. $\mathbb{N}$ is not bounded above in $\mathbb{R}$. That is give any $x \in \mathbb{N}$, there exists a natural number n such that $x > n$.

2. Given any $x, y \in \mathbb{R}$ there exists a natural number n such that $nx > y$.

1. If the sequences (x_n) and (y_n) are convergent, then the product will converge. That is,

$$\lim_{n\to\infty}(x_n \cdot y_n) = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n.$$

$$\lim_{n\to\infty} x_n = l$$

- A sequence (x_n) converges if and only if $(|x_n|)$ converges.
- Let (x_n) be a sequence such that $x_n \neq 0$ for all n. Let the sequence is convergent. Then the sequence $\left(\frac{1}{x_n}\right)$ converges and

$$\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{\lim_{n\to\infty} x_n}.$$

A sequence $(x_n) \subseteq \mathbb{R}$ is said to be Cauchy if for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$ we have $|x_m - x_n| < \epsilon$.

A sequence $(x_n) \subseteq \mathbb{R}$ is convergent if and only if it is Cauchy.

Space of all Cauchy sequences form a vector space over $\mathbb{R}$. A sequence (x_n) is said to be increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$ and it is said to be decreasing if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. We say a sequence (x_n) to be monotone if it is either increasing or decreasing.

Theorem 1. Let (x_n) is a sequence. Then we have the following results.

1. If (x_n) is increasing then it is convergent if and only if it is bounded above.
2. If (x_n) is decreasing then it is convergent if and only if it is bounded below.
3. If (x_n) is monotone then it is convergent if and only if it is bounded.

Definition 1. Let $f : \mathbb{N} \to \mathbb{R}$ be a sequence and S be any infinite subset of $\mathbb{N}$. Then a subsequence is the restriction of f to the set S.

Theorem 2. Given any real sequence (x_n) there exists a monotone subsequence.

Theorem 3 (Bolzano Weierstrass Theorem)**.** Every bounded sequence has a convergent subsequence.

Theorem 4. Let $(x_n), (y_n)$ and (z_n) be sequences such that $x_n \to \alpha$, $y_n \to \alpha$ and $x_n \leq z_n \leq y_n$ for all n. Then $z_n \to \alpha$.

Theorem 5.

1. Let $0 \leq r < 1$, then $r^n \to 0$.
2. $-1 < t < 1$, thn $t^n \to 0$.
3. Let $|r| < 1$, then $nr^n \to 0$.
4. Let $a > 0$, then $a^{\frac{1}{n}} \to 0$.

5. $n^{\frac{1}{n}} \to 1$.
6. For any $a \in \mathbb{R}$, we have $\frac{a^n}{n!} \to 0$.
7. Let $(x_n) \to 0$. Let (s_n) denotes the arithmetic mean defined by

$$s_n = \frac{x_1 + \cdots + x_n}{n}.$$

   Then $s_n \to 0$.

Definition 2. A series $\sum_{n=1}^{\infty} a_n$ is said to be convergent if the sequence of partial sum converges. That is, the sequence

$$S_N = \sum_{n=1}^{N} a_n$$

converges.

Definition 3. The series $\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ converges.

1. If $\sum a_n$ converges, then $an \to 0$. The converse need nit bve true. For example, $\sum \frac{1}{n}$ is not converges but $\frac{1}{n} \to 0$.
2. If a series is absolutely convergent, then it is convergent. Th converse need not be true. For example, $\sum \frac{(-1)^n}{n}$ is convergent, but $\sum \frac{1}{n}$ is not.
3. For a positive number p the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

   converges if and only if $p > 1$.
4. The space of all (absolutely) convergent series forms a vector space over $\mathbb{R}$.

1 Convergence Tests

1. Let $\sum a_n$ be a series such that $a_n \geq 0$ for all n.
   - If $\sum a_n$ converges and $|b_n| \leq a_n$ for all n, then $\sum b_n$ converges.
   - If $\sum a_n$ diverges and $a_n \geq b_n$, then $\sum b_n$ also diverges.
2. Suppose $\sum a_n$ and $\sum b_n$ are two series. Suppose that $r = \lim \left| \frac{a_n}{b_n} \right|$ exists, and $0 < r < \infty$. Then $\sum a_n$ converges absolutely if and only if $\sum b_n$ converges absolutely.

3. Suppose (a_n) is a decreasing sequence of positive terms. Then the series $\sum a_n$ converges if and only if the series $\sum 2^k a_{2^k}$ converges.

4. Let $\sum a_n$ be a series and let $r = \lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$. Then the series

   (a) converges absolutely if $r < 1$,

   (b) diverges if $r > 1$.

   (c) If $r = 1$, then the test is inconclusive.

5. Let $\sum a_n$ be a series and let $r = \lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}}$. Then the series

   (a) converges absolutely if $r < 1$,

   (b) diverges if $r > 1$.

   (c) If $r = 1$, then the test is inconclusive.

2 Limit

Definition 4. Let $f : U \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Let $a \in U$. We say that $\lim\limits_{x\to a} f(x) = l$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in (a-\delta, a+\delta)$ we have $|f(x) - l| < \epsilon$.

We say that $\lim\limits_{x\to a} f(x) = l$ if for every sequence $(x_n) \subseteq U$ converging to a the sequence $(f(x_n))$ should converge to l.

If $f, g : U \to \mathbb{R}$ be two functions, then we have

$$\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x),$$

provided both limit exist. We have

$$\lim_{x\to a}(cf(x)) = c\lim_{x\to a} f(x),$$

provided the limit exists.

3 Continuity of a function

Definition 5. Let $f : U \subseteq \mathbb{R} \to \mathbb{R}$ be a function. We say that f is continuous at $a \in U$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in (a-\delta, a+\delta)$ we have

$$|f(x) - f(a)| < \epsilon.$$

Definition 6. We say that $f : U \subseteq \mathbb{R}$ is continuous at $a \in U$ if for every sequence (x_n) converging to a we have $f(x_n) \to f(a)$.