MTH565: Matrix Groups

End Semester Examination Solution

Monsoon Semester 2025

Question 1. Prove or disprove the following statements. [10 marks]

(i) Let $A \in O(n)$ such that $\det A = -1$. Then

$$O(n) = SO(n) \cup \{A \cdot B : B \in SO(n)\}$$

[2]

(ii) For $n \geq 2$, $O(n)$ is isomorphic to $SO(n) \times \mathbb{Z}_2$. [2]

(iii) $SL(n, \mathbb{K})$ is **not** compact. [2]

(iv) Let

$$G = \{A \in GL_n(\mathbb{R}) : \det A = 2^k \text{ for some } \ k \in \mathbb{Z}\}.$$

Then G is a matrix group. [2]

(v) Consider a set

$$A = \{a + \imath b \in \mathbb{H} : a^2 + b^2 = 1\}.$$

Then A is a normal subgroup of $Sp(1)$.

(You may assume that it is a subgroup of $Sp(1)$). [2]

Solution

✓ TRUE

Since $A \in O(n)$ and $B \in SO(n)$, it is clear that $SO(n) \cup \{A \cdot B : B \in SO(n)\}$.

For the converse, let $X \in O(n)$ such that $\det X = -1$. If $\det X = 1$, then $X \in SO(n)$ and the other inclusion follows.

Take

$$B = A^{-1}X \in O(n) \ \text{ and } \ \det(B) = \det(A^{-1})\det(X) = 1 \Rightarrow B \in SO(n).$$

Thus, $O(n) \subset SO(n) \cup \{A \cdot B : B \in SO(n)\}$ and hence the equality.

× FALSE

The statement is true for odd positive integers n. (The proof can be seen in homework).

This is false for even numbers. In particular, we will show that $O(2)$ is not isomorphic to $SO(2) \times \mathbb{Z}_2$. The group $SO(2) \times \mathbb{Z}_2$ is abelian whereas $O(2)$ is not. For example, the matrices

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in O(2),$$

but they do not commute.

× FALSE

This is also false. Basically, this is true if $n \geq 2$. It is easy to see that $SL(1, \mathbb{K})$ is compact. For example, $SL(1, \mathbb{R}) = \{1\}$, which is compact.

✓ TRUE

We need to show that G is closed in $GL_n(\mathbb{R})$. Consider the map

$$f : GL_n(\mathbb{R}) \to \mathbb{R}, \quad A \mapsto \det(A).$$

Then,

$$G = f^{-1}\{2^k : k \in \mathbb{Z}\}.$$

Since f is continuous, it is enough to prove that the set $S = \{2^k : k \in \mathbb{Z}\}$ is closed in $\mathbb{R}$. Note that the set S is discrete and hence closed. Thus, G is a matrix group.

× FALSE

The subgroup A is not normal in $Sp(1)$. For that, take $g = \frac{1+j}{\sqrt{2}} \in Sp(1)$ and $h = \iota \in A$. Then,

$$\begin{aligned} ghg^{-1} &= \left(\frac{1+j}{\sqrt{2}}\right) \cdot \iota \cdot \left(\frac{1+j}{\sqrt{2}}\right)^{-1} \\ &= \frac{1}{2}(1+j) \cdot \iota \cdot (1-j) \\ &= \frac{1}{2}(\iota - k)(1-j) \\ &= \frac{1}{2}(\iota - k - k - \iota) = k \notin A. \end{aligned}$$

Thus, the subgroup A is not normal in $Sp(1)$.

Question 2. In the following problem, we will prove that $Sp(1) \times Sp(1)$ is a double cover of $SO(4)$. You can assume that $Sp(1)$ is a matrix group. [11 marks]

(i) If G_1, G_2 are two matrix groups, show that $G_1 \times G_2$ is a matrix group. So, $Sp(1) \times Sp(1)$ is a matrix group. [1]

(ii) For any $(q_1, q_2) \in Sp(1) \times Sp(1)$, show that the map $\varphi(q) : \mathbb{H}(\cong \mathbb{R}^4) \to \mathbb{H}$, defined as $\varphi(q)(\boldsymbol{v}) = q_1 \boldsymbol{v} \bar{q}_2$, is an orthogonal linear transformation. (You do not need to show that it is a linear transformation). [1]

(iii) From part (ii), with respect to the basis $\{1, i, j, k\}$ of $\mathbb{H}$, $\varphi(q)$ can be regarded as an element of $O(4)$. Show that for any $\boldsymbol{v} \in \mathbb{H}$, the map

$$\varphi : Sp(1) \times Sp(1) \to O(4), \quad (q_1, q_2) \mapsto q_1 \boldsymbol{v} \bar{q}_2.$$

is a homomorphism. Also, show that the image will lie in $SO(4)$. [1 + 1]

(iv) By finding the kernel of the map, show that it is 2-to-1 map. [2]

(v) Show that it is a local diffeomorphism at (I, I) and hence conclude that it is a local diffeomorphism at any point. [3 + 1]

(vi) Show that the map is surjective and hence conclude that $Sp(1) \times Sp(1)$ is a double cover of $SO(4)$. [1]

Solution

(i) Let $G_1 \subseteq GL_{n_1}(\mathbb{K})$ and $G_2 \subseteq GL_{n_2}(\mathbb{K})$. Then we can see the set

$$G_1 \times G_2 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in G_1, B \in G_2 \right\} \subseteq GL_{n_1+n_2}(\mathbb{K}).$$

Since G_1, G_2 are closed sets, the cartesian product $G_1 \times G_2$ is closed in $GL_{n_1+n_2}(\mathbb{K})$.

(ii) In order to show that $\varphi(q)$ is an orthogonal transformation, we will show that it preserves the norm (and hence it follows that it preserves the inner product). Since $\|q_1\| = 1 = \|\bar{q}_2\|$, we have

$$\|\varphi(q)(\boldsymbol{v})\| = \|q_1 \boldsymbol{v} \bar{q}_2\| = \|q_1\| \cdot \|\boldsymbol{v}\| \cdot \|\bar{q}_2\| = \|\boldsymbol{v}\|.$$

Therefore, $\varphi(q)$ is an orthogonal linear transformation.

(iii) φ is a group homomorphism.

- For any $q = (q_1, q_2)$, and $r = (r_1, r_2) \in Sp(1)$, we have

$$\begin{aligned}\varphi(q \cdot r)(\boldsymbol{v}) = \varphi(q_1 r_1, q_2 r_2) &= (q_1 r_1) \boldsymbol{v} (\overline{q_2 r_2}) \\ &= q_1 (r_1 \boldsymbol{v} \bar{r}_2) \bar{q}_2 = \varphi(q)(r_1 \boldsymbol{v} \bar{r}_2) \\ &= (\varphi(q) \circ \varphi(r))(\boldsymbol{v}).\end{aligned}$$

Now we will show that the image lie in $SO(4)$. Since $Sp(1) \times Sp(1)$ is connected, the image must be connected. As $\varphi(1,1)(\boldsymbol{v}) = \boldsymbol{v}$ corresponds to the identity transformation which is in $SO(4)$, hence the whole image will be in $SO(4)$.

(iv) If $q = (q_1, q_2) \in \ker(\varphi)$ then, $\varphi(q) = \mathrm{Id}$. That is, for any $\boldsymbol{v} \in \mathbb{H}$,

$$\varphi(q)(\boldsymbol{v}) = \boldsymbol{v} \Rightarrow q_1 \boldsymbol{v} \bar{q}_2 = \boldsymbol{v}.$$

Take $\boldsymbol{v} = 1$, then

$$q_1\bar{q}_2 = 1 \Rightarrow q_1 = q_2.$$

Thus, for any $\boldsymbol{v} \in \mathbb{H}$,

$$q_1\boldsymbol{v}\bar{q}_2 = v \Rightarrow q_1\boldsymbol{v}\bar{q}_1 = \boldsymbol{v} \Rightarrow q_1\boldsymbol{v} = \boldsymbol{v}q_1.$$

Since q_1 commutes with every element of $\boldsymbol{v}$, it must be real and hence $q_1 = \pm 1$. Therefore,

$$\ker(\varphi) = \{(I, I), (-I, -I)\}.$$

Hence φ is a two-to-one map.

(v) At first we will show that φ is a local diffeomorphism at (I, I). In order to show that φ is a local diffeomorphism at (I, I), we will show that

$$d\varphi_{(I,I)} : \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \to \mathfrak{so}(4)$$

is a vector space isomorphism. For that, we will show that $d\varphi_{(I,I)}$ maps the standard basis of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ to a basis of $\mathfrak{so}(4)$. For the point $(i, 0)$, we take a curve $\gamma(t) = (e^{it}, 1) \in Sp(1) \times Sp(1)$ and $\gamma'(0) = (i, 0)$. Thus, for any $\boldsymbol{v} \in \mathbb{H}$,

$$\begin{aligned} d\varphi_{(I,I)}(i, 0)(\boldsymbol{v}) = d\varphi_{(I,I)}(e^{it}, 1)(\boldsymbol{v}) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (e^{it}\boldsymbol{v}) \\ &= i\boldsymbol{v} = \begin{cases} i & \text{if } \boldsymbol{v} = 1 \\ -1 & \text{if } \boldsymbol{v} = i \\ k & \text{if } \boldsymbol{v} = j \\ -j & \text{if } \boldsymbol{v} = k. \end{cases} \end{aligned}$$

The matrix of the above is $\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Similarly, for $(j, 0)$, $(k, 0)$, $(0, i)$, $(0, j)$ and $(0, k)$ can be done and we conclude that it maps to a basis of $\mathfrak{so}(4)$. Just to show one more, note that for the same if we want

$$\begin{aligned} d\varphi_{(I,I)}(0, i)(\boldsymbol{v}) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \boldsymbol{v}e^{-it} = -\boldsymbol{v}i \\ &= \begin{cases} -i & \text{if } \boldsymbol{v} = 1 \\ 1 & \text{if } \boldsymbol{v} = i \\ k & \text{if } \boldsymbol{v} = j \\ -j & \text{if } \boldsymbol{v} = k. \end{cases} \end{aligned}$$

Thus, the matrix will be $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

Another way to show that $d\varphi_{(I,I)}$ is a vector space isomorphism, is by finding the kernel of the map. Note that for any $p, q \in \mathfrak{sp}(1)$, and $\boldsymbol{v} \in \mathbb{H}$, we have

$$\begin{aligned} d\varphi_{(I,I)}(p, q)(\boldsymbol{v}) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \varphi(e^{pt}, e^{tq})(\boldsymbol{v}) \\ &= p\boldsymbol{v} - \boldsymbol{v}q. \end{aligned}$$

If $(p, q) \in \ker\big(d\varphi_{(I,I)}\big)$, then

$$\begin{aligned} d\varphi_{(I,I)}(p,q) = 0 &\Rightarrow p\boldsymbol{v} - \boldsymbol{v}q = 0 \ \forall \ \boldsymbol{v} \in \mathbb{H} \\ &\Rightarrow p = q, \quad \text{take} \ \ \boldsymbol{v} = 1. \end{aligned}$$

Thus,

$$\begin{aligned} \ker\left(d\varphi_{(I,I)}\right) &= \{(p,p) \in \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) : p\boldsymbol{v} = \boldsymbol{v}p \ \text{ for any } \ \boldsymbol{v} \in \mathbb{H}\} \\ &= \{(0,0)\}. \end{aligned}$$

The last equality holds because, if $p \in \mathbb{H}$ commutes with every element of $\mathbb{H}$, then $p \in \mathbb{R}$. Here $p \in \mathfrak{sp}(1) = \text{span}\{i,j,k\}$. This implies $p = 0$. Therefore, kernel is trivial. Since $\dim(\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)) = 6 = \dim(\mathfrak{so}(4))$, and we have an injection, implies that $d\varphi_{(I,I)}$ is an isomorphism. This proves that φ is a local diffeomorphism at (I,I). By left translation, we can show that it is a local diffeomorphism at any point.

(vi) Since $Sp(1) \times Sp(1)$ is compact and the map φ is continuous, implies that $\text{im}(\varphi)$ is compact in $SO(4)$ and hence closed. Also, φ is a local diffeomorphism, so it is an open map and therefore, $\text{im}(\varphi)$ is open in $SO(4)$. Since $SO(4)$ is connected, $\text{im}(\varphi) = SO(4)$. Therefore, φ is surjective and hence $Sp(1) \times Sp(1)$ is a double cover of $SO(4)$.

Question 3. Let us define the *Affine group* as

$$\mathrm{Aff}_n(\mathbb{R}) := \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} : A \in GL_n(\mathbb{R}) \ \text{ and } \ v \in \mathbb{R}^n \right\}.$$

[8 marks]

(i) Show that $\mathrm{Aff}_n(\mathbb{R})$ is a matrix group. [2]

(ii) Is $\mathrm{Aff}_n(\mathbb{R})$ is compact? [1]

(iii) Find the Lie algebra of $\mathrm{Aff}_n(\mathbb{R})$ and hence find the dimension of $\mathrm{Aff}_n(\mathbb{R})$. [3]

(iv) Let $\mathfrak{aff}_n(\mathbb{R})$ denotes the Lie algebra of $\mathrm{Aff}_n(\mathbb{R})$. For any $X, Y \in \mathfrak{aff}_n(\mathbb{R})$, find the Lie bracket $[X, Y]$. [2]

Solution

(i) Since for any $X \in \mathrm{Aff}_n(\mathbb{R})$, write $X = \left(\begin{smallmatrix} A & v \\ 0 & 1 \end{smallmatrix}\right)$ for some $A \in GL_n(\mathbb{R})$ and $v \in \mathbb{R}^n$.This implies $X \in M_{n+1}(\mathbb{R})$. Also, $\det X = \det A \neq 0$ and hence $X \in GL_{n+1}(\mathbb{R})$. Now observe that

$$\mathrm{Aff}_n(\mathbb{R}) \cong GL_n(\mathbb{R}) \times \mathbb{R}^n, \quad \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \mapsto (A, v).$$

Since $GL_n(\mathbb{R}) \times \mathbb{R}^n$ is closed in $GL_{n+1}(\mathbb{R})$, the affine space $\mathrm{Aff}_n(\mathbb{R})$ is also closed in $GL_{n+1}(\mathbb{R})$ and hence is a matrix group.

(ii) Since $GL_n(\mathbb{R}) \times \mathbb{R}^n$ is not compact in for any $n \in \mathbb{N}$, the affine space $\mathrm{Aff}_n(\mathbb{R})$ is also not compact.

(iii) The Lie algebra will be

$$\begin{aligned} \mathfrak{aff}_n(\mathbb{R}) = T_{\left(\begin{smallmatrix} I & 0 \\ 0 & 1 \end{smallmatrix}\right)} \mathrm{Aff}_n(\mathbb{R}) &\cong T_{I,\mathbf{0}}(GL_n(\mathbb{R}) \times \mathbb{R}^n) \\ &\cong T_I GL_n(\mathbb{R}) \oplus T_{\mathbf{0}}(\mathbb{R}^n) \\ &\cong M_n(\mathbb{R}) \oplus \mathbb{R}^n. \end{aligned}$$

Thus,

$$\mathfrak{aff}_\mathfrak{n}(\mathbb{R}) \cong \left\{ \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} : A \in M_n(\mathbb{R}) \ \text{ and } \ v \in \mathbb{R}^n \right\}.$$

Hence, the dimension of $\mathrm{Aff}_n(\mathbb{R})$ is $n^2 + n$.

(iv) For any $X, Y \in \mathfrak{aff}_\mathfrak{n}(\mathbb{R})$, write $X = \left(\begin{smallmatrix} A & v \\ 0 & 0 \end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix} B & w \\ 0 & 0 \end{smallmatrix}\right)$. Then,

$$\begin{aligned} [X, Y] &= XY - YX \\ &= \begin{pmatrix} AB & Aw \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} BA & Bv \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} [A, B] & Aw - Bv \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Question 4. Let G be a matrix group and $\mathcal{L}(G)$ be its Lie algebra. Prove that the tangent space to G at $g \in G$ is

$$T_g G = \{Xg : X \in \mathcal{L}(G)\} = \{gX : X \in \mathcal{L}(G)\}.$$

[3 marks]

Solution

We will show that

$$T_g G = \{gX : X \in \mathcal{L}(G)\}.$$

The other one also follow the same method. First of all consider the map (the left translation by g)

$$L_g : G \to G, \quad h \mapsto gh.$$

This map is a smooth homomorphism with

$$\left(dL_g\right)_e : \mathcal{L}(G) \to T_g G, \quad X \mapsto gX.$$

Thus, we proved that $\{gX : X \in \mathcal{L}(G)\} \subseteq T_g G$.

For the converse, let $X \in T_g G$. Then there exists $\gamma : (-\varepsilon, \varepsilon) \to G$ such that $\gamma(0) = g$ and $\gamma'(0) = X$. Define a curve $\eta : (-\varepsilon, \varepsilon) \to G, \quad \eta(t) = g^{-1}\gamma(t)$. Then,

$$\eta(0) = e, \eta'(0) = g^{-1}X \in \mathcal{L}(G).$$

Thus, we can write $X = g(g^{-1}X) \in \{gX : X \in \mathcal{L}(G)\}$ and hence $T_g G \subseteq \{gX : X \in \mathcal{L}(G)\}$, and therefore, the equality.

Question 5. Let $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Calculate e^A. [4 marks]

Solution

Let us write

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} =: X + Y.$$

Note that

$$XY = YX \Rightarrow e^A = e^{X+Y} = e^X \cdot e^Y = \begin{pmatrix} e^a & 0 \\ 0 & e^a \end{pmatrix} \cdot e^Y.$$

Thus, we only need to find e^Y. We have seem in one of the quiz (quiz 5) that

$$e^Y = \begin{pmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{pmatrix}.$$

Therefore,

$$e^A = \begin{pmatrix} e^a & 0 \\ 0 & e^a \end{pmatrix} \cdot \begin{pmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{pmatrix} = \begin{pmatrix} e^a \cos b & e^a \sin b \\ -e^a \sin b & e^a \cos b \end{pmatrix}.$$

Question 6. Consider a basis $\{E_1, E_2, E_3\}$ for $\mathfrak{so}_3$ such that

$$[E_1, E_2] = E_3, \quad [E_2, E_3] = E_1, \quad [E_3, E_1] = E_2.$$

[9 marks]

(i) Show that $\mathfrak{so}_3$ does not have a non trivial proper ideal, that is, only proper ideal of $\mathfrak{so}_3$ is 0. [3]

(ii) Let $\mathrm{Isom}(\mathbb{R}^2)$ denotes the set of all isometries of $\mathbb{R}^2$. Compute its Lie algebra. [2]

(iii) Is the Lie algebra $\mathcal{L}(\mathrm{Isom}(\mathbb{R}^2))$ isomorphic to $\mathfrak{so}_3$ as a Lie algebra? (Hint: See if $\mathcal{L}(\mathrm{Isom}(\mathbb{R}^2))$ has any proper ideal). [4]

Solution

(i) Let $\mathfrak{h}$ be an ideal of $\mathfrak{so}(3)$. Then for any $X = a_1E_1 + a_2E_2 + a_3E_3 \in \mathfrak{h}$

$$\begin{aligned}[X, E_1] \in \mathfrak{h} &\Rightarrow a_2[E_2, E_1] + a_3[E_3, E_1] \in \mathfrak{h} \\ &\Rightarrow -a_2E_3 + a_3E_2 \in \mathfrak{h}.\end{aligned}$$

Again,

$$[-a_2E_3 + a_3E_2, E_3] \in \mathfrak{h} \Rightarrow a_3[E_2, E_3] \in \mathfrak{h} \Rightarrow a_3E_1 \in \mathfrak{h} \Rightarrow E_1 \in \mathfrak{h}.$$

Similarly, we can show that $E_2, E_3 \in \mathfrak{h}$ and hence $\mathfrak{h} = \mathfrak{so}(3)$. Therefore, $\mathfrak{so}(3)$ does not have a non-trivial proper ideal.

(ii) Recall that

$$\mathrm{Isom}(\mathbb{R}^2) = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} : A \in O(2), v \in \mathbb{R}^2 \right\}.$$

By the similar way (as we did for the $\mathrm{Aff}_n(\mathbb{R})$), $\mathcal{L}(\mathrm{Isom}(\mathbb{R}^2))$ is $\mathfrak{o}(2) \oplus \mathbb{R}^2$ as a vector space. More precisely,

$$\mathcal{L}(\mathrm{Isom}(\mathbb{R}^2)) = \left\{ \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} : A \in \mathfrak{o}(2), v \in \mathbb{R}^2 \right\}.$$

(iii) We claim that $\mathfrak{t} = \left\{ \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} : v \in \mathbb{R}^2 \right\}$ is an ideal of $\mathcal{L}(\mathrm{Isom}(\mathbb{R}^2))$. Since it is a proper ideal, it will conclude that the Lie algebra of $\mathrm{Isom}(\mathbb{R}^2)$ is not isomorphic to $\mathfrak{so}(3)$ as a Lie algebra. To see it is an ideal, let $X = \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathrm{Isom}(\mathbb{R}^2))$ and $Y = \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} \in \mathfrak{t}$. Then,

$$[X, Y] = \begin{pmatrix} 0 & Aw \\ 0 & 0 \end{pmatrix} \in \mathfrak{t}.$$

Thus, $\mathfrak{t}$ is a proper ideal of $\mathcal{L}(\mathrm{Isom}(\mathbb{R}^2))$.