

Day 3 : 12th January, 2026

3.1. Quotient Space Let X be a set and \sim be an equivalence relation on X . Let X/\sim be the set of all equivalence classes of \sim . We have a natural map, called the *quotient map*

$$\pi : X \rightarrow X/\sim, \quad x \mapsto [x].$$

If we further assume that X is a topological space, we then want to introduce a topology on the quotient space X/\sim such that the quotient map π is continuous. One choice for the same is indiscrete topology on X/\sim . However, we would like to have the largest possible topology on X/\sim such that π is continuous. If \mathcal{T} is such a topology on X/\sim , then for any $V \in \mathcal{T}$, the set $\pi^{-1}(V)$ must be open in X . This suggests the following.

Definition 3.1: (Quotient topology)

With the notation above, we define the *quotient topology* on X/\sim as

$$\mathcal{T} = \{V \subseteq X/\sim : \pi^{-1}(V) \text{ is open in } X\}.$$

The map $\pi : X \rightarrow X/\sim$ is called a *quotient map*. The space $(X/\sim, \mathcal{T})$ is called the *quotient space*.

Proposition 3.2: (Quotient topology is a topology)

Let X be a topological space and \sim is an equivalence relation on X . Then the quotient topology on X/\sim is the largest topology for which the natural quotient map $\pi : X \rightarrow X/\sim$ is continuous.

Theorem 3.3: (Universal Mapping Property)

Let $\pi : X \rightarrow X/\sim$ be a quotient map. A map $f : X/\sim \rightarrow Y$ is continuous if and only if $f \circ \pi$ is continuous.

Proof : Since π is continuous, so continuity of f implies $f \circ \pi$ is continuous. For the other way, if $f \circ \pi$ is continuous, then we need to show that f is continuous. For that, if $V \subseteq Y$ is an open set, then we need to show that $f^{-1}(V)$ is open in X/\sim . But V is open in X/\sim implies $\pi^{-1}(V)$ is open in X and

$$\begin{aligned} (f \circ \pi)^{-1}(V) &= \pi^{-1}(f^{-1}(V)) \Rightarrow \pi^{-1}(f^{-1}(V)) \text{ is open} \\ &\Rightarrow f^{-1}(V) \text{ is open in } X/\sim. \end{aligned}$$

□

The next theorem tells us how to generate quotient space.

Theorem 3.4: (Generate a quotient space)

Let X, Y be two topological space and $f : X \rightarrow Y$ be a continuous function. Define an equivalence relation \sim on X as $x_1 \sim x_2 \Leftrightarrow f(x_1) = f(x_2)$. Then there exists a continuous function $g : X/\sim \rightarrow Y$ such that $f = g \circ \pi$.

Exercise 3.5:

Prove the above theorem.

Caution 3.6: (A different description)

Many a times, we define a quotient topology directly on a set.

Definition 3.7: (Quotient map)

Given a space (X, \mathcal{T}) and a function $f : X \rightarrow Y$ to a set Y , the *quotient topology* on Y is defined as

$$\mathcal{T}_f := \{U \mid f^{-1}(U) \in \mathcal{T}\}.$$

The map $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}_f)$ is called a *quotient map*. In other words, f is a quotient map if $U \subset Y$ is open if and only if $f^{-1}(U) \subset X$ is open.

- The quotient topology \mathcal{T}_f is indeed a topology on Y , and $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}_f)$ is continuous.

Combining all, we have the following.

Theorem 3.8: (Universal property of quotient topology)

Suppose (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) are given. Then, for any set function, $q : X \rightarrow Y$, the following are equivalent.

1. \mathcal{T}_Y is the quotient topology induced by q (in other words, q is a quotient map).
2. \mathcal{T}_Y is the finest (i.e., largest) topology for which q is continuous.
3. \mathcal{T}_Y is the unique topology having the following property : for any space (Z, \mathcal{T}_Z) and any set map $f : Y \rightarrow Z$, we have f is continuous if and only if $f \circ q$ is continuous

Remark 3.9: (Quotient map and surjectivity)

Suppose $f : X \rightarrow Y$ is a quotient map. Assume that f is *not* surjective. Then, for any $y \in Y \setminus f(X)$ we have $f^{-1}(y) = \emptyset \subset X$ open, and hence, $\{y\}$ is open in Y . In other words, $Y \setminus f(X)$ has the discrete topology. Also, $f(X) \subset Y$ is both an open and closed set. Hence, the open and closed sets of $f(X)$ in the subspace topology are precisely the same in the actual (quotient) topology on Y . For these reasons, we can (and usually we do) assume that a quotient map is surjective.

Remark 3.10: (Surjective map and equivalence relation)

Suppose $f : X \rightarrow Y$ is a surjective map. Then, the collection $\bigsqcup_{y \in Y} f^{-1}(y)$ is a partition on X , and hence, induces an equivalence relation. Indeed, we can define $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$. Conversely, given any equivalence relation \sim on X , we see that $q : X \rightarrow X/\sim$, is a surjective map, where X/\sim is the collection of all equivalence classes under the relation \sim .

Given a set map $f : X \rightarrow Y$, a subset $S \subset X$ is called *saturated* (or *f-saturated*) if $S = f^{-1}(f(S))$ holds.

Exercise 3.11: (Saturated open set)

Given a quotient map $q : X \rightarrow Y$, a set $U \subset X$ is q -saturated if and only if it is the union of the equivalence classes of its elements (i.e, $U = \bigcup_{x \in U} [x]$).

Example 3.12: ($[0, 1]/\{0, 1\}$ is S^1)

Consider the quotient space obtained from $[0, 1]$ got by identifying the end points 0 and 1. That is, $X := [0, 1]/\sim$, where

$$x \sim y \Leftrightarrow x = y \text{ or } x, y \in \{0, 1\}.$$

Show that X is homeomorphic to circle S^1 .

Proof : Consider the map $f : [0, 1] \rightarrow S^1$ given by $f(t) = e^{2\pi it} = (\cos 2\pi t, \sin 2\pi t)$. Clearly, f is continuous and surjective. Also, $f(0) = (1, 0) = f(1)$. Passing to the quotient $X = [0, 1]/\{0, 1\}$, we get a map $\tilde{f} : X \rightarrow S^1$ defined by $\tilde{f}([x]) = f(x)$. It is easy to see that \tilde{f} is well-defined, and hence, by the property of the quotient topology, \tilde{f} is continuous. Now, \tilde{f} is surjective (as f was), and moreover, it is injective.

In order to show \tilde{f} is open, we consider the two cases.

- i) Suppose $V \subset X$ is an open set, such that $[0] = [1] = \{0, 1\} \notin V$. Then, $q^{-1}(V) \subset [0, 1]$ is an open set, which is actually contained in $(0, 1)$. In particular, $q^{-1}(V)$ is a union of open intervals. Observe that (by drawing picture or otherwise) f maps such open intervals to open arcs of S^1 (which are open in S^1). Then, $\tilde{f}(V) = f(q^{-1}(V))$ is open.
- ii) Suppose $V \subset X$ is an open set, such that $[0] = [1] = \{0, 1\} \in V$. Then, $q^{-1}(V)$ is the union of open intervals of $(0, 1)$, as well as, $[0, \varepsilon_1) \cup (1 - \varepsilon_2, 1]$ for some $\varepsilon_1, \varepsilon_2 > 0$. We have already seen that any open intervals get mapped to open arcs. Also, $f([0, \varepsilon_1) \cup (1 - \varepsilon_2, 1])$ is another open arc in S^1 containing the point $(1, 0)$. Thus, $\tilde{f}(V) = f(q^{-1}(V))$ is open in S^1 .

Hence, $\tilde{f} : X \rightarrow S^1$ is a homeomorphism. □

The following result allows us to identify quotient spaces with other concrete spaces.

Theorem 3.13: (Identify quotient spaces with other space)

Let X and Y be compact topological spaces. Assume further that Y is Hausdorff. Let $f : X \rightarrow Y$ be surjective continuous map. Define an equivalence relation \sim by declaring $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$. Then X/\sim is homeomorphic to Y .

With the above theorem it is easy to show that $[0, 1]/\sim$ is S^1 .

Exercise 3.14:

- i) Show that the quotient space got by identifying two of the opposite sides of a rectangle is homeomorphic to a cylinder.
- ii) Let $X = [0, 1] \times [0, 1]$. Identify

$$\begin{aligned} &\{(0, t) : t \in [0, 1]\} \text{ with } \{(1, t) : t \in [0, 1]\} \\ &\{(s, 0) : s \in [0, 1]\} \text{ with } \{(s, 1) : s \in [0, 1]\}. \end{aligned}$$

Show that the quotient space obtained from the above identification is a torus, that is, $S^1 \times S^1$.

From any space X and a subset $A \subseteq X$, the space X/A stands for the quotient space of X with respect to the equivalence relation

$$x_1 \sim x_2 \Leftrightarrow x_1 = x_2 \text{ or } x_1, x_2 \in A.$$

Thus, X/A is the space obtained from X by collapsing A to a single point.

Exercise 3.15: (\mathbb{R}/\mathbb{Z} is S^1)

Consider the quotient space $X = \mathbb{R}/\mathbb{Z}$, where the equivalence relation is given as $a \sim b$ if and only $a - b \in \mathbb{Z}$. Show that X is homeomorphic to the circle S^1 .