Day 7 : 2nd February, 2026

Definition 7.1: (*Functor*)

Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A *covariant functor* $F : \mathscr{C} \to \mathscr{D}$ consists of the following data.

i) For each object $A \in \mathscr{C}$, there is an object $F(A) \in \mathscr{D}$.
ii) For each morphism $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, there is a morphism $F(f) \in \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$ such that
   a) F preserves the identity, that is, for any object $A \in \mathscr{C}$,
$$F(1_A) = 1_{F(A)}.$$
   b) F preserves the composition, that is, if $f : A \to B$ and $g : B \to C$, then
$$F(g \circ f) = F(g) \circ F(f).$$

We say $F : \mathscr{C} \to \mathscr{D}$ is a *contravariant functor* if it reverses the morphism and the composition, that is, for $f : A \to B$ and $g : B \to C$ in $\mathscr{C}$

$$F(f) : F(B) \to F(A) \text{ in } \mathscr{D} \quad \text{and}$$
$$F(g \circ f) = F(f) \circ F(g).$$

Example 7.2: (*Some functors*)

1. Let $\mathscr{C}$ be any category. Then we always have the identity functor $\mathrm{Id} : \mathscr{C} \to \mathscr{C}$.
2. The easiest example of functors are *Forgetful functors*. For any mathematical object defined as a set with some additional structure, we can "forget" the extra structure and get a forgetful functor.
   i) $F : \mathbf{Top} \to \mathbf{Sets}$ defined as follows. For any topological space X, $F(X)$ is the underlying set and if $f : X \to Y$ is a continuous function, then $F(f)$ is the same function. So F forgets the topological structure of X and also forgets the continuous functions between topological spaces.
   ii) Similarly, there is a functor $\mathbf{Groups} \to \mathbf{Sets}$ forgetting the group structure on groups and a functor $\mathbf{Rings} \to \mathbf{Sets}$ forgetting the ring structure on rings.
   iii) We can also have forgetful functor which does not forget all the structures. For example, if $\mathbf{Ab}$ is the category of abelian groups, then a functor from $\mathbf{Ab} \to \mathbf{Groups}$ forgets the abelian structure in groups and remember just underlying group structure. Similarly, we can have functors $\mathbf{Rings} \to \mathbf{Ab}$ which forgets the multiplicative structure and remembers the underlying additive group.
3. Let X and Y be two sets and $f : X \to Y$ be a function. We have a functor between the two discrete categories and conversely. That is, given a function $f : X \to Y$ one can give a functor and given a functor $F : \mathbf{Disc(X)} \to \mathbf{Disc(Y)}$ one can define $f : X \to Y$.
4. Let M be a fixed topological space. Then
$$P_M : \mathbf{Top} \to \mathbf{Top}, \quad X \mapsto X \times M$$
is a functor.
5. Hom *functors.*
   i) Let $\mathscr{C}$ be any category and A be an object in $\mathscr{C}$. Then we have the following functor
$$\mathrm{Hom}(A, \_) : \mathscr{C} \to \mathbf{Sets}, \quad B \mapsto \mathrm{Hom}(A, B).$$

For any $f : X \to Y$ in $\mathscr{C}$,

$$f_* := \mathrm{Hom}(A, f) : \mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y), \quad g \mapsto f \circ g.$$

This is a covariant functor.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, dashed, "f \circ g"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

ii) We can also define a contravariant functor as follows.

$$\mathrm{Hom}(\_, A) : \mathscr{C} \to \mathbf{Sets}, \quad B \mapsto \mathrm{Hom}(B, A).$$

For any $f : X \to Y$ in $\mathscr{C}$,

$$f^* := \mathrm{Hom}(f, A) : \mathrm{Hom}(X, A) \to \mathrm{Hom}(Y, A), \quad g \mapsto g \circ f.$$

This is a contravariant functor.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dashed, "g \circ f"'] & Y \arrow[d, "g"] \\ & A \end{tikzcd}$$

6. Let $\mathbf{Vect}$ be the category of vector space over a field $\mathbb{F}$. Here morphisms are $\mathbb{F}$-linear maps. For any two vector spaces V and W, $\mathrm{Hom}(V, W)$ is a vector space over $\mathbb{F}$. We have two Hom functors,

$$\mathrm{Hom}(V, \_) : \mathbf{Vect} \to \mathbf{Vect}$$
$$\mathrm{Hom}(\_, V) : \mathbf{Vect} \to \mathbf{Vect}.$$

7.1. Homotopy Equivalence

Definition 7.3: (*Path*)

A *path* in a topological space X from x to y is a continuous map $\gamma : [a, b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$. We will write this as $\gamma_{[a,b]} : x \to y$, that is, γ is a path joining x to y.

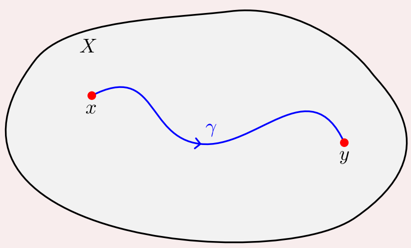

Figure 1: Path between x and y

We can *reparametrize* and use the unit interval $[0,1]$ as a source. That is,

$$\tilde{\gamma}:[0,1]\to X,\quad \tilde{\gamma}(t)=\gamma(a(1-t)+bt).$$

Think it like we have changed the speed of the particle from point x to the point y.

If $\gamma:[0,1]\to X$ is a path joining x to y, then the *inverse path* $\overline{\gamma}(t)=\gamma(1-t)$ is a path joining y to x. Also, if we have two paths $\gamma_{[0,1]}:x\to y$ and $\eta_{[0,1]}:y\to z$, then we can concatenate these two paths to get a new path $(\gamma*\eta)_{[0,1]}:x\to z$ defined as

$$(\gamma*\eta)(t)=\begin{cases}\gamma(2t) & \text{if } 0\le t\le \frac{1}{2}\\ \eta(2t-1) & \text{if } \frac{1}{2}\le t\le 1.\end{cases}$$

Note that by pasting lemma, $\gamma*\eta$ is continuous.

A *constant path* at $x_0\in X$, denoted as γ_{x_0}, is a constant function $\gamma_{x_0}(t)=x_0$.

Exercise 7.4: (*Equivalence relation on X*)

Let X be a topological space. Define a relation $\sim$ on X by

$$x\sim y \Leftrightarrow \exists \text{ a path } \gamma:x\to y.$$

Show that $\sim$ is an equivalence relation on X.

The set of equivalence classes will be denoted by

$$\pi_0(X):=X/\sim = \text{set of path components of } X.$$

Definition 7.5: (*Path connected*)

A space X is said to be *path connected* or *0-connected* if given any $x,y\in X$, there is a path γ in X joining x and y.

Exercise 7.6: (*Equivalent statements for path connectedness*)

Let X be a topological space. The following are equivalent:

i) X is path connected.
ii) $\pi_0(X)$ is singleton.
iii) Any continuous function $f:\{0,1\}\to X$ ha a continuous extension $F:[0,1]\to X$.

Exercise 7.7: (π_0 *as a functor*)

Consider the category of topological spaces. A continuous function, $f:X\to Y$, induces a map

$$\pi_0(f):\pi_0(X)\to\pi_0(Y),\quad [x]\mapsto[f(x_0)].$$

i) Show that $\pi_0:\mathbf{Top}\to\mathbf{Sets}$ is a (covariant) functor.
ii) If $f:X\to Y$ is a homeomorphism, then $\pi_0(f):\pi_0(X)\to\pi_0(Y)$ is a bijection.

iii) If $f : X \to Y$ is a homeomorphism, then for any $A \subset X$, we have an induced homeomorphism from $X \setminus A \to Y \setminus f(A)$.

Exercise 7.8:

Using the above exercise, show the following:

i) For $n > 1$, $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$.

ii) The space $\mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$ is not homeomorphic to $\mathbb{R}$ or $\mathbb{R}^2$.

Path Category

Note that path concatenation is not associative as a map-level operation. Indeed, a path is not just a geometric curve but a curve together with a parametrization, which we may think of as "time". When we form $(\alpha * \beta) * \gamma$, the path α is traversed during the first quarter of the time interval, β during the second quarter, and γ during the entire second half. On the other hand, in $\alpha * (\beta * \gamma)$, the path α is traversed during the entire first half of the time interval, while β and γ are compressed into the second half. Thus, although both composite paths trace the same route from x to w, they do so at different speeds and reach intermediate points at different times. Since these parametrization differ, the resulting paths are not equal as functions $[0, 1] \to X$, and concatenation is therefore not associative.

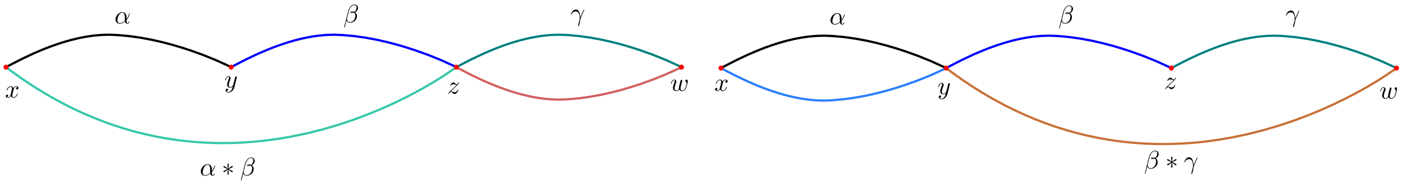

Figure 2: Concatenation is not associative

Exercise 7.9: (*Concatenation is not associative*)

Show that the concatenation of paths is not associative. That is, if $\alpha_{[0,1]} : x \to y, \beta_{[0,1]} : y \to z$ and $\gamma_{[0,1]} : z \to w$, then it is not necessary that

$$(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma).$$

We can rectify this defect by using parameter intervals of different length. So let us consider paths of the form $\alpha : [0, a] \to X, \beta : [0, b] \to X$ with $\alpha(a) = \beta(0)$ and $a, b \geq 0$. Their composition

$$\beta \circ \alpha = \gamma : [0, a + b] \to X, \quad \gamma(t) = \begin{cases} \alpha(t) & 0 \leq t \leq a \\ \beta(a - t) & a \leq t \leq a + b. \end{cases}$$

In this manner we get a category $\mathscr{P}(X)$ defined as

$$\mathrm{Ob}(\mathscr{P}(X)) = \text{points of } \ X$$
$$\mathrm{Hom}(x, y) = \{\gamma : [0, a] \to X : \gamma(0) = x \text{ and } \gamma(a) = x \text{ for some } a \geq 0\}.$$

Exercise 7.10: (*Category of paths*)

i) Show that for any topological space X $\mathscr{P}(X)$ is a category.

ii) For any continuous function $f : X \to Y$, we get a functor

$$\mathscr{P}(f) : \mathscr{P}(X) \to \mathscr{P}(Y), \quad \gamma \mapsto f \circ \gamma.$$