Derivative

Engineering Mathematics-I

Dr. (PhD) Sachchidanand Prasad

SPNREC, Araria

October 7, 2024

Today’s Goal

Today's Goal

- What is derivative?

Today's Goal

- What is derivative?
- Geometric meaning of derivative

Today's Goal

- What is derivative?
- Geometric meaning of derivative
- Some examples

Today's Goal

- What is derivative?
- Geometric meaning of derivative
- Some examples
- Rolle's theorem

Today's Goal

- What is derivative?
- Geometric meaning of derivative
- Some examples
- Rolle's theorem
- Lagrange's mean value theorem

What is derivative?

What is derivative?

Suppose we are given two points (x_1, y_1) and (x_2, y_2) on the line. Then the slope of the line is calculated by

What is derivative?

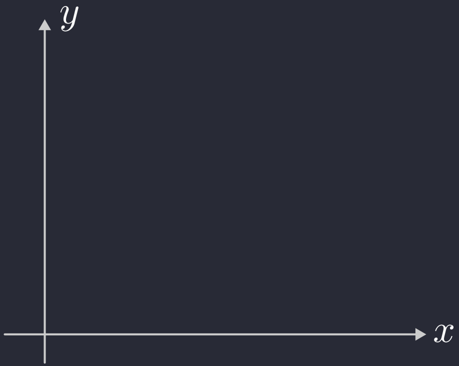
Suppose we are given two points (x_1, y_1) and (x_2, y_2) on the line. Then the slope of the line is calculated by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

What is derivative?

Suppose we are given two points (x_1, y_1) and (x_2, y_2) on the line. Then the slope of the line is calculated by

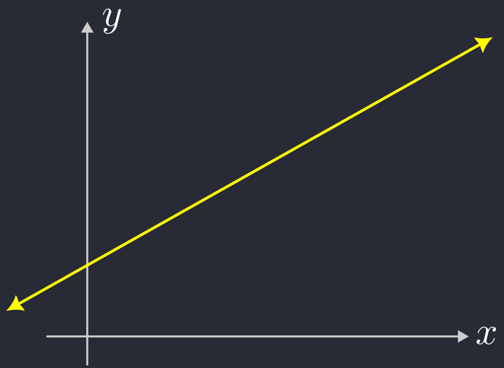
$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



What is derivative?

Suppose we are given two points (x_1, y_1) and (x_2, y_2) on the line. Then the slope of the line is calculated by

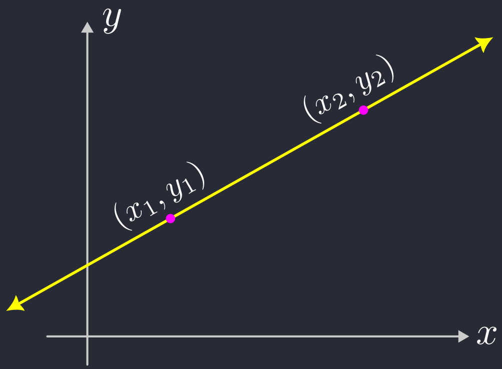
$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



What is derivative?

Suppose we are given two points (x_1, y_1) and (x_2, y_2) on the line. Then the slope of the line is calculated by

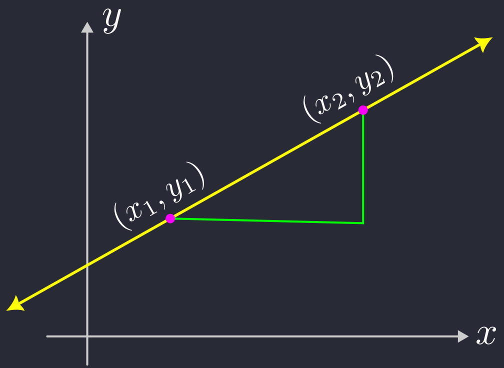
$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



What is derivative?

Suppose we are given two points (x_1, y_1) and (x_2, y_2) on the line. Then the slope of the line is calculated by

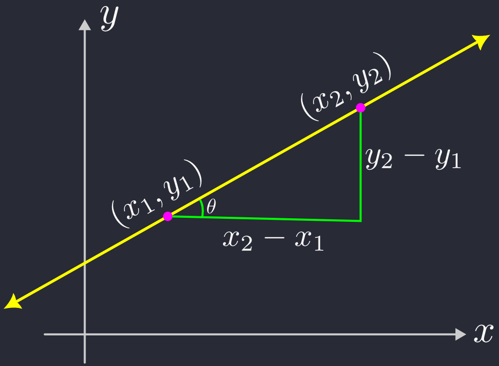
$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



What is derivative?

Suppose we are given two points (x_1, y_1) and (x_2, y_2) on the line. Then the slope of the line is calculated by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



Average Change

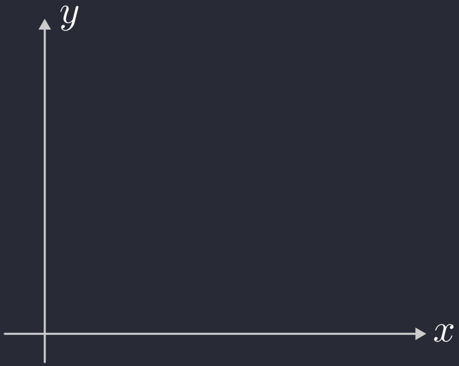
If we are given with a curve with two points, then the average rate of change is calculated by the slope of the secant line

Average Change

If we are given with a curve with two points, then the average rate of change is calculated by the slope of the secant line (*the line which passes through the two points on the curve*).

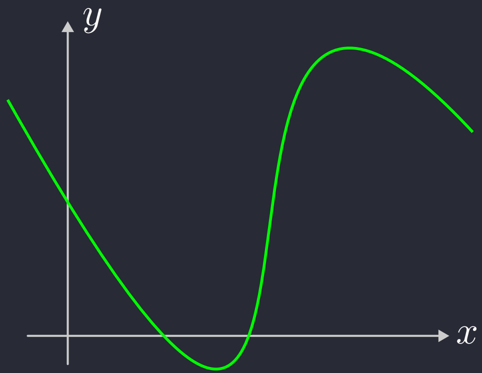
Average Change

If we are given with a curve with two points, then the average rate of change is calculated by the slope of the secant line (*the line which passes through the two points on the curve*).



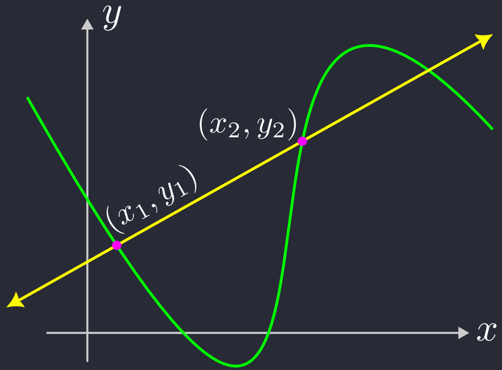
Average Change

If we are given with a curve with two points, then the average rate of change is calculated by the slope of the secant line (*the line which passes through the two points on the curve*).



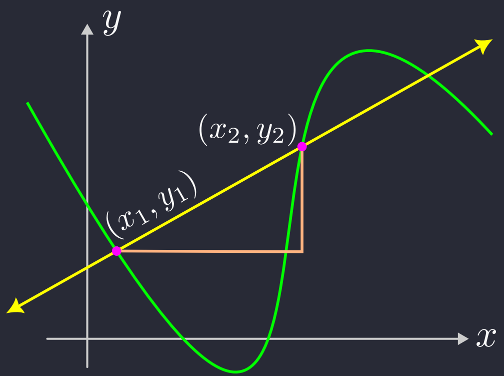
Average Change

If we are given with a curve with two points, then the average rate of change is calculated by the slope of the secant line (*the line which passes through the two points on the curve*).



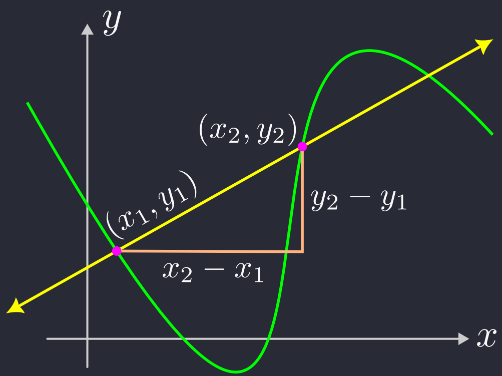
Average Change

If we are given with a curve with two points, then the average rate of change is calculated by the slope of the secant line (*the line which passes through the two points on the curve*).



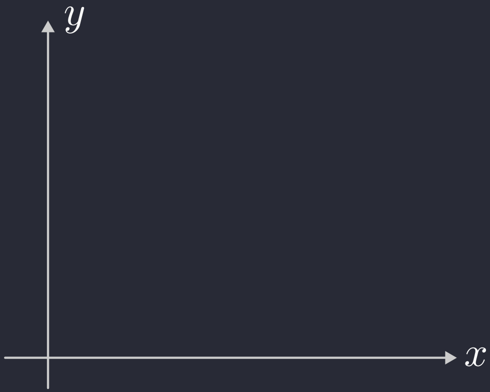
Average Change

If we are given with a curve with two points, then the average rate of change is calculated by the slope of the secant line (*the line which passes through the two points on the curve*).

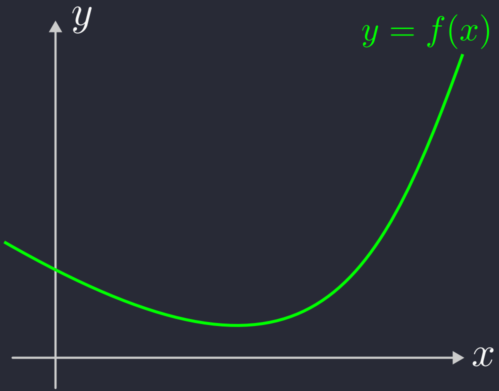


Average Change Cont...

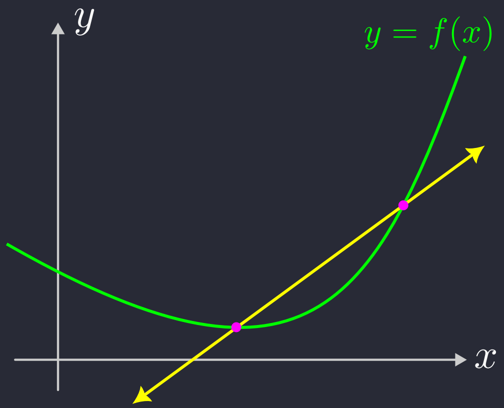
Average Change Cont...



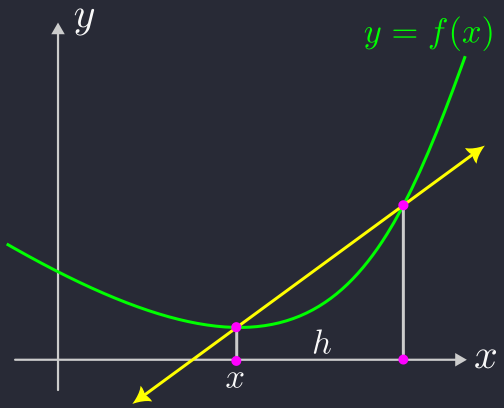
Average Change Cont...



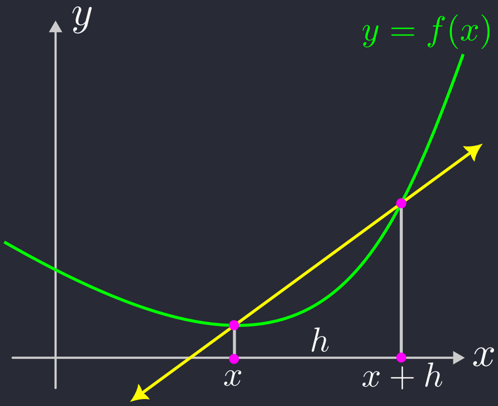
Average Change Cont...



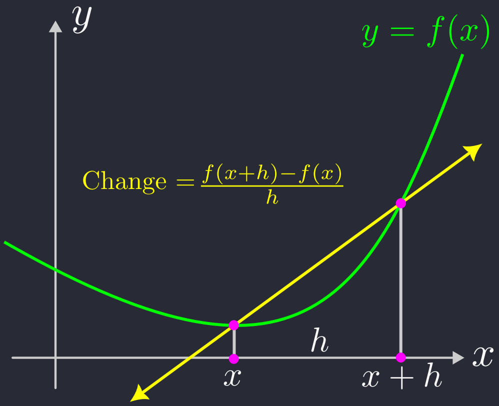
Average Change Cont...



Average Change Cont...



Average Change Cont...



Average Change Cont...

Average Change Cont...

- The slope of the secant line is also referred to as the *average rate of change* of f over the interval $[x, x+h]$.

Average Change Cont...

- The slope of the secant line is also referred to as the *average rate of change* of f over the interval $[x, x+h]$.
- Thus, average rate of change of f over the interval $[x, x+h]$ will be

Average Change Cont...

- The slope of the secant line is also referred to as the *average rate of change* of f over the interval $[x, x+h]$.
- Thus, average rate of change of f over the interval $[x, x+h]$ will be

$$\frac{f(x+h) - f(x)}{h}.$$

Average Change Cont...

- The slope of the secant line is also referred to as the *average rate of change* of f over the interval $[x, x+h]$.
- Thus, average rate of change of f over the interval $[x, x+h]$ will be

$$\frac{f(x+h) - f(x)}{h}.$$

- What will happen if h "tends" to 0?

Tangent Line

Tangent Line

As h tends to 0 the points x and $x+h$ coincide

Tangent Line

As h tends to 0 the points x and $x + h$ coincide and hence the secant line tends to the tangent line at x.

Tangent Line

As h tends to 0 the points x and $x + h$ coincide and hence the secant line tends to the tangent line at x. See the animation.

Tangent Line

As h tends to 0 the points x and $x + h$ coincide and hence the secant line tends to the tangent line at x. See the animation. Thus,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Slope of the tangent line is known as *Instantaneous Rate of Change*

Examples

Examples

Let us check the derivative of $f(x) = x^2$ is $2x$.

Examples

Let us check the derivative of $f(x) = x^2$ is $2x$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Examples

Let us check the derivative of $f(x) = x^2$ is $2x$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \end{aligned}$$

Examples

Let us check the derivative of $f(x) = x^2$ is $2x$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + h^2 + 2xh - x^2}{h} \end{aligned}$$

Examples

Let us check the derivative of $f(x) = x^2$ is $2x$.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + h^2 + 2xh - x^2}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 2hx}{h}
\end{aligned}$$

Examples

Let us check the derivative of $f(x) = x^2$ is $2x$.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + h^2 + 2xh - x^2}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 2hx}{h} \\
&= \lim_{h \to 0} (h + 2x)
\end{aligned}$$

Examples

Let us check the derivative of $f(x) = x^2$ is $2x$.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + h^2 + 2xh - x^2}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 2hx}{h} \\
&= \lim_{h \to 0} (h + 2x) \\
&= 2x.
\end{aligned}$$

Example Cont...

Example Cont...

Check the differentiability of the function $f(x) = |x|$.

Example Cont...

Check the differentiability of the function $f(x) = |x|$.

Note that for any $x \in \mathbb{R} \setminus \{0\}$, the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists.

Example Cont...

Check the differentiability of the function $f(x) = |x|$.

Note that for any $x \in \mathbb{R} \setminus \{0\}$, the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. For $x = 0$, we have

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

Example Cont...

Check the differentiability of the function $f(x) = |x|$.

Note that for any $x \in \mathbb{R} \setminus \{0\}$, the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. For $x = 0$, we have

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

Example Cont...

Check the differentiability of the function $f(x) = |x|$.

Note that for any $x \in \mathbb{R} \setminus \{0\}$, the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. For $x = 0$, we have

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

The left hand limit is -1

Example Cont...

Check the differentiability of the function $f(x) = |x|$.

Note that for any $x \in \mathbb{R} \setminus \{0\}$, the limit

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists. For $x = 0$, we have

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

The left hand limit is -1 and the right hand limit is 1,

Example Cont...

Check the differentiability of the function $f(x) = |x|$.

Note that for any $x \in \mathbb{R} \setminus \{0\}$, the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. For $x = 0$, we have

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

The left hand limit is -1 and the right hand limit is 1, hence the limit does not exist.

Example Cont...

Check the differentiability of the function $f(x) = |x|$.

Note that for any $x \in \mathbb{R} \setminus \{0\}$, the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. For $x = 0$, we have

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

The left hand limit is -1 and the right hand limit is 1, hence the limit does not exist. Thus, the function is not differentiable at $x = 0$.

Application of derivative

Application of derivative

We will discuss the following applications:

Application of derivative

We will discuss the following applications:

- Rolle's theorem

Application of derivative

We will discuss the following applications:

- Rolle's theorem
- Mean value theorem

Rolle's Theorem

Rolle's Theorem

Suppose f is defined on $[a, b]$ such that

Rolle's Theorem

Suppose f is defined on $[a, b]$ such that

- f is continuous on $[a, b]$

Rolle's Theorem

Suppose f is defined on $[a, b]$ such that

- f is continuous on $[a, b]$
- f is differentiable on (a, b)

Rolle's Theorem

Suppose f is defined on $[a, b]$ such that

- f is continuous on $[a, b]$
- f is differentiable on (a, b)
- $f(a) = f(b)$

Rolle's Theorem

Suppose f is defined on $[a, b]$ such that

- f is continuous on $[a, b]$
- f is differentiable on (a, b)
- $f(a) = f(b)$

Then, there exists $c \in (a, b)$ such that

$$f'(c) = 0.$$

Rolle's Theorem

Suppose f is defined on $[a, b]$ such that

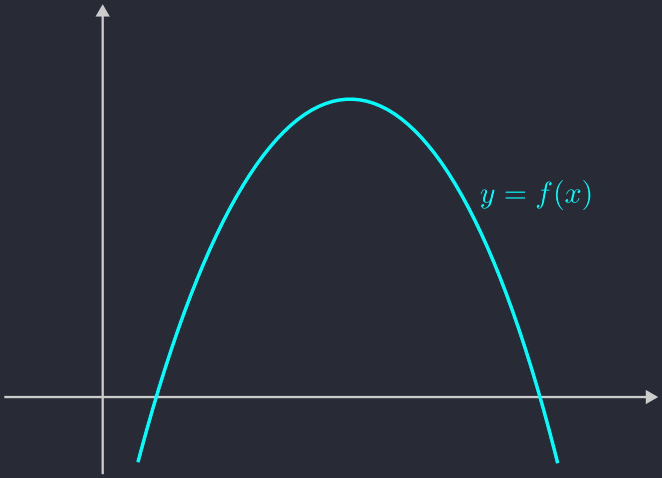
- f is continuous on $[a, b]$
- f is differentiable on (a, b)
- $f(a) = f(b)$

Then, there exists $c \in (a, b)$ such that

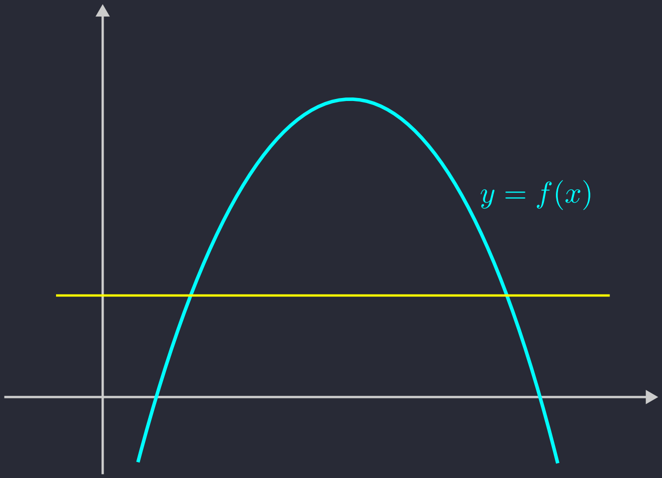
$$f'(c) = 0.$$

A pictorial view of Rolle's theorem

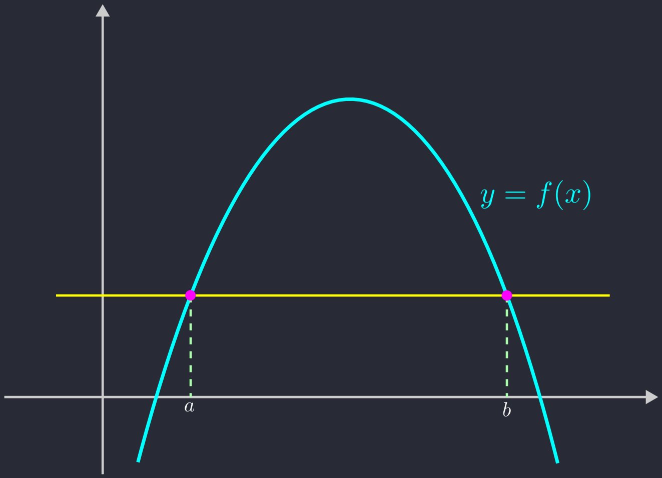
A pictorial view of Rolle's theorem



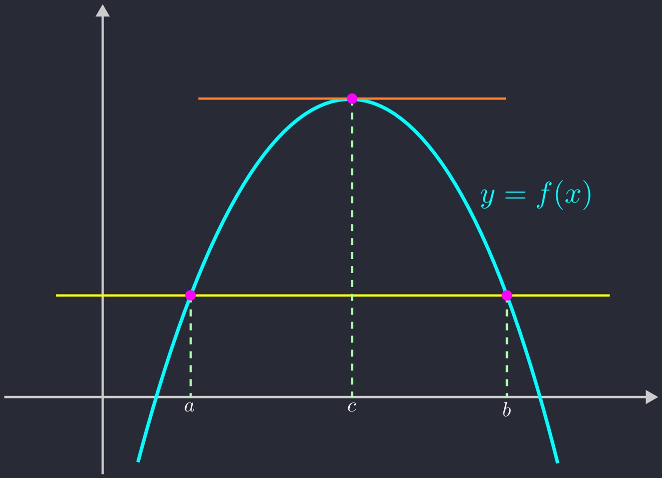
A pictorial view of Rolle's theorem



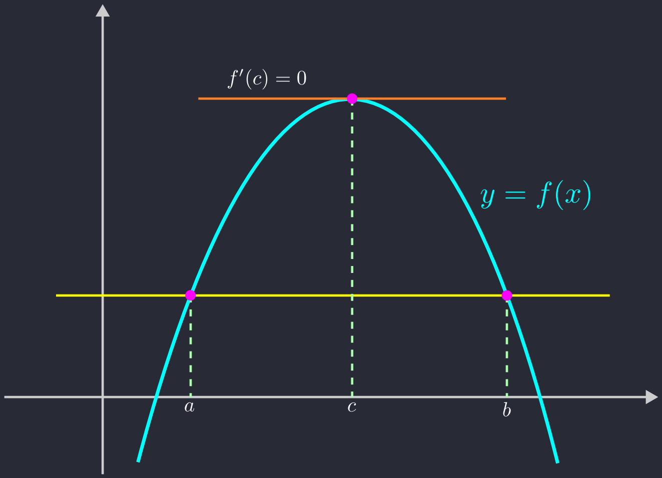
A pictorial view of Rolle's theorem



A pictorial view of Rolle's theorem



A pictorial view of Rolle's theorem



A pictorial view of Rolle's theorem

Show the animation

Problems on Rolle's Theorem

Problem

For the function $f(x) = x\left(x^2 - 1\right)$ *test for the applicability of Rolle's theorem in the interval* $[-1, 1]$ *and hence find* c *such that* $-1 < c < 1$.

Problem cont...

Solution

Given that the function is

$$f(x) = x\left(x^2 - 1\right).$$

Problem cont...

Solution

Given that the function is

$$f(x) = x\left(x^2 - 1\right).$$

We have

1. f *is continuous on* $[-1, 1]$,

Problem cont...

Solution

Given that the function is

$$f(x) = x\left(x^2 - 1\right).$$

We have

1. f *is continuous on* $[-1, 1]$,
2. f *is differentiable on* $(-1, 1)$

Problem cont...

Solution

Given that the function is

$$f(x) = x\left(x^2 - 1\right).$$

We have

1. f *is continuous on* $[-1, 1]$,
2. f *is differentiable on* $(-1, 1)$ *and*
3. $f(-1) = 0 = f(1)$.

Problem cont...

Solution

Given that the function is

$$f(x) = x\left(x^2 - 1\right).$$

We have

1. f *is continuous on* $[-1, 1]$,
2. f *is differentiable on* $(-1, 1)$ *and*
3. $f(-1) = 0 = f(1)$.

Since f satisfies the hypothesis of Rolle's theorem,

Problem cont...

Solution

Given that the function is

$$f(x) = x\left(x^2 - 1\right).$$

We have

1. f *is continuous on* $[-1, 1]$,
2. f *is differentiable on* $(-1, 1)$ *and*
3. $f(-1) = 0 = f(1)$.

Since f *satisfies the hypothesis of Rolle's theorem, there exists* $c \in (-1, 1)$ *such that* $f'(c) = 0$. *That is,*

Solution cont...

$$f'(c) = 0$$

Solution cont...

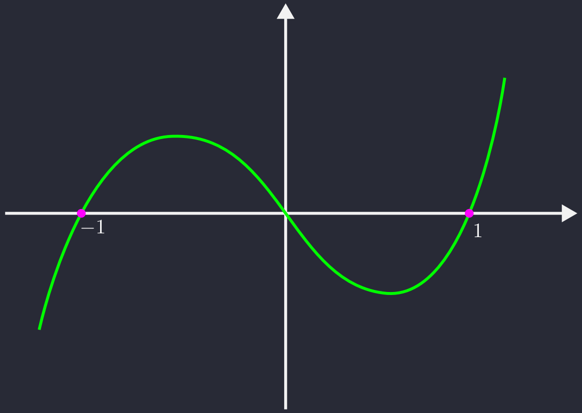
$$f'(c) = 0 \implies 3c^2 - 1 = 0$$

Solution cont...

$$f'(c) = 0 \implies 3c^2 - 1 = 0 \implies c = \pm\sqrt{\frac{1}{3}}.$$

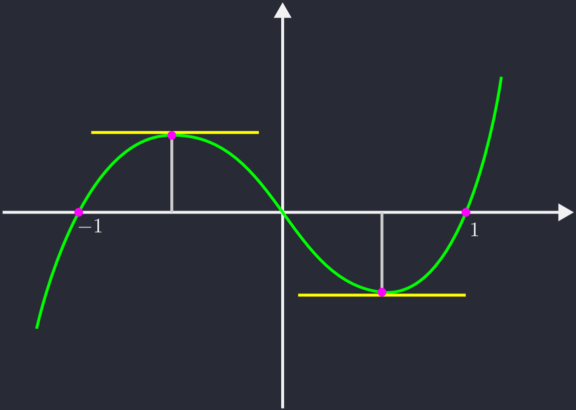
Solution cont...

$$f'(c) = 0 \implies 3c^2 - 1 = 0 \implies c = \pm\sqrt{\frac{1}{3}}.$$



Solution cont...

$$f'(c) = 0 \implies 3c^2 - 1 = 0 \implies c = \pm\sqrt{\frac{1}{3}}.$$



Problem 2

Problem

Verify the Rolle's theorem for

$$f(x) = \frac{\sin x}{e^x}, \quad in\ (0, \pi).$$

Problem 3

Problem

It is given that the Rolle's theorem holds for the function

$$f(x) = x^3 + bx^2 + cx, \quad 1 \leq x \leq 2$$

at the point $x = \frac{4}{3}$. *Find the value of* b *and* c.

Lagrange’s Mean Value Theorem

Lagrange's Mean Value Theorem

Suppose f is defined on $[a, b]$ such that

Lagrange's Mean Value Theorem

Suppose f is defined on $[a, b]$ such that

- f is continuous on $[a, b]$

Lagrange's Mean Value Theorem

Suppose f is defined on $[a, b]$ such that

- f is continuous on $[a, b]$
- f is differentiable on (a, b)

Lagrange's Mean Value Theorem

Suppose f is defined on $[a, b]$ such that

- f is continuous on $[a, b]$
- f is differentiable on (a, b)

Then, there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Lagrange's Mean Value Theorem

Suppose f is defined on $[a, b]$ such that

- f is continuous on $[a, b]$
- f is differentiable on (a, b)

Then, there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

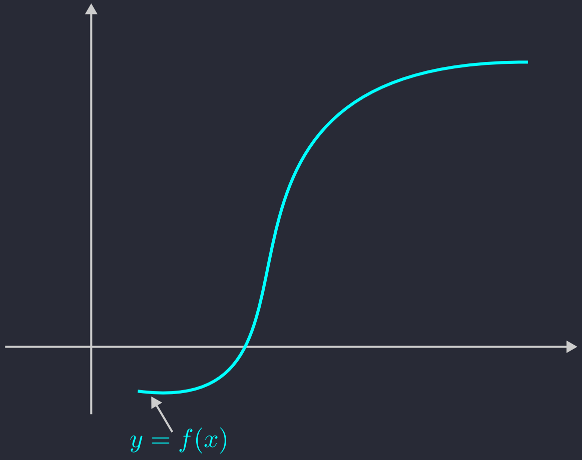
Proof.
Take

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

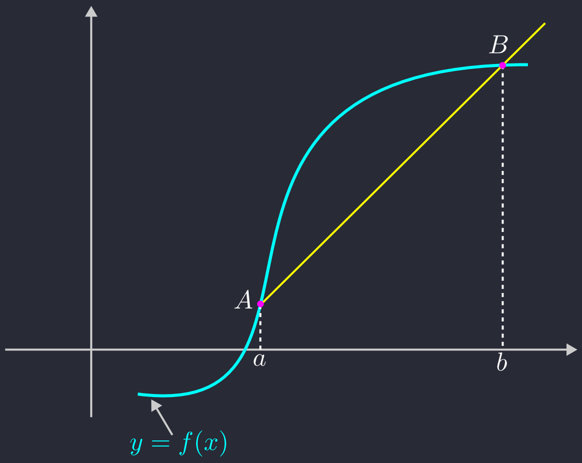
$\square$

A picture

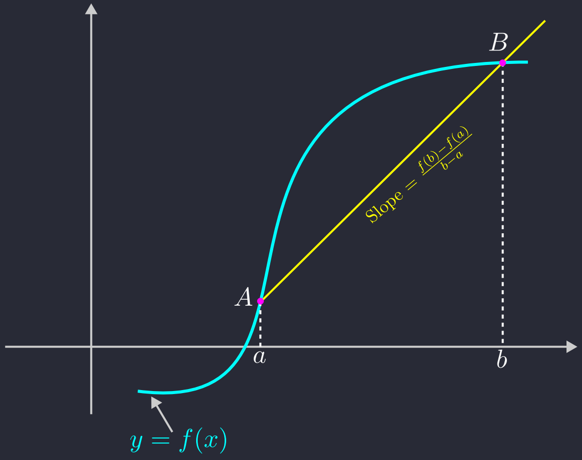
A picture



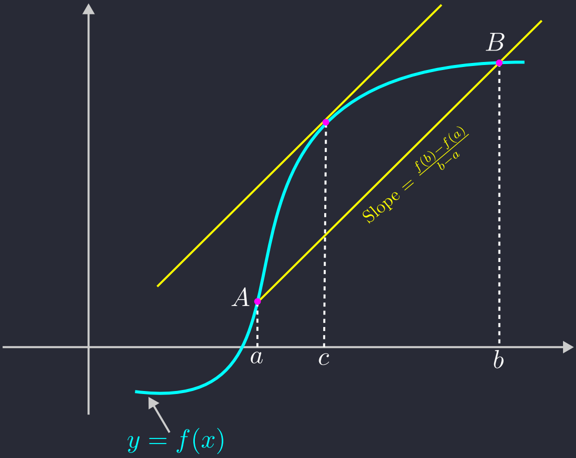
A picture



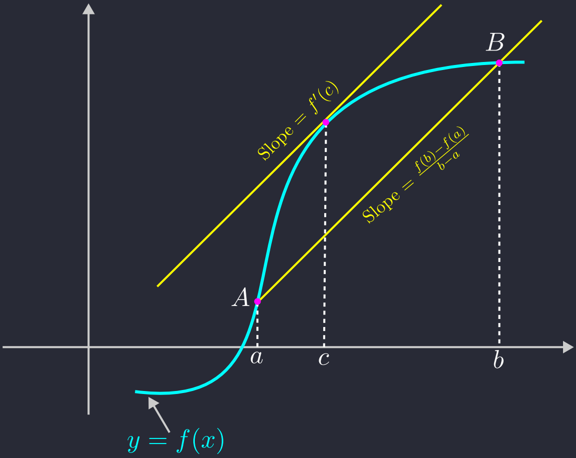
A picture



A picture



A picture



Problem-1

Problem

Verify the Lagrange's mean value theorem for the function

$$f(x) = x(x-1)(x-2), \quad a = 0 \text{ and } b = \frac{1}{2}.$$

Also find c.

Problem-2

Problem

Verify the Lagrange's mean value theorem for the function given below and find c

$$f(x) = \log x, \quad a = 1 \text{ and } b = e.$$

Cauchy's Mean Value Theorem

Let f and g be two functions defined on $[a, b]$ such that

Cauchy's Mean Value Theorem

Let f and g be two functions defined on $[a, b]$ such that

- f and g are continuous on $[a, b]$

Cauchy's Mean Value Theorem

Let f and g be two functions defined on $[a, b]$ such that

- f and g are continuous on $[a, b]$
- f and g are differentiable on (a, b)

Cauchy's Mean Value Theorem

Let f and g be two functions defined on $[a, b]$ such that

- f and g are continuous on $[a, b]$
- f and g are differentiable on (a, b) and
- $g'(x) \neq 0$ for any $x \in (a, b)$.

Cauchy's Mean Value Theorem

Let f and g be two functions defined on $[a, b]$ such that

- f and g are continuous on $[a, b]$
- f and g are differentiable on (a, b) and
- $g'(x) \neq 0$ for any $x \in (a, b)$.

Then there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$