

### Definition 6.1: (Subcategory)

Let  $\mathcal{C}$  be a category. A **subcategory**  $\mathcal{D}$  of  $\mathcal{C}$  is a category such that

- i)  $\text{Ob}(\mathcal{D}) \subseteq \text{Ob}(\mathcal{C})$ ;
- ii) for any  $A, B \in \text{Ob}(\mathcal{D})$ ,  $\text{Hom}_{\mathcal{D}}(A, B) \subseteq \text{Hom}_{\mathcal{C}}(A, B)$ ;
- iii) composition in  $\mathcal{D}$  is the same as composition in  $\mathcal{C}$ .

The third condition says that the function  $\text{Hom}_{\mathcal{D}}(A, B) \times \text{Hom}_{\mathcal{D}}(B, C) \rightarrow \text{Hom}_{\mathcal{D}}(A, C)$  is the restriction of the corresponding composition with subscripts  $\mathcal{C}$ .

### Example 6.2: (Examples of subcategory)

- i) Let  $\mathcal{C} = \mathbf{Top}$ . Then the following are subcategories of  $\mathcal{C}$ .
  - a) The objects are subspaces of  $(\mathbb{R}^n, \mathcal{T}_{\text{Euc}})$  and morphisms are continuous maps between spaces.
  - b) The objects are Hausdorff topological spaces.
  - c) The objects are compact topological spaces.
- ii) Let  $\mathcal{C} = \mathbf{Groups}$  be the category of groups. Then  $\mathbf{Ab}$ , the category of abelian groups is a subcategory. Category of rings  $\mathbf{Rings}$  is also a subcategory.
- iii) Consider the category  $\mathcal{C} = \mathbf{Top}^2$  with

$\text{Ob}(\mathcal{C}) =$  ordered pairs  $(X, A)$  where  $X$  is a topological space  
and  $A$  is a subspace of  $X$ .

$\text{Hom}((X, A), (Y, B)) =$  set of continuous functions  $f : X \rightarrow Y$  with  $f(A) \subseteq B$ .

Then  $\mathbf{Top}_*$  is a subcategory of  $\mathbf{Top}^2$ .

### Exercise 6.3:

- i) Let  $\mathcal{C}$  be a category and let  $A \in \text{Ob}(\mathcal{C})$ . Prove that  $\text{Hom}(A, A)$  has a unique identity  $1_A$ .
- ii) If  $\mathcal{D}$  is a subcategory of  $\mathcal{C}$ , and if  $A \in \text{Ob}(\mathcal{D})$ , then the identity of  $A$  in  $\text{Hom}_{\mathcal{D}}(A, A)$  is the identity  $1_A \in \text{Hom}_{\mathcal{C}}(A, A)$ .
- iii) Show that one may regard  $\mathbf{Top}$  as a subcategory of  $\mathbf{Top}^2$  if one identifies a space  $X$  with the pair  $(X, \emptyset)$ .

### Definition 6.4: (Commutative Diagram)

A **diagram** in a category  $\mathcal{C}$  is a directed graph whose vertices are labeled by objects of  $\mathcal{C}$  and whose directed edges are labeled by morphisms in  $\mathcal{C}$ . A **commutative diagram** in  $\mathcal{C}$  is a diagram in which for each pair of vertices, every two paths (composites) between them are equal as morphisms.

### Example 6.5: (A commutative diagram)

The following is a diagram

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
f' \downarrow & & \downarrow g \\
C & \xrightarrow{g'} & D
\end{array}$$

In addition, if  $g \circ f = g' \circ f'$ , then the diagram is commutative.

**Definition 6.6:** (Congruence)

A **congruence** on a category  $\mathcal{C}$  is an equivalence relation  $\sim$  on the class  $\bigcup_{(A,B)} \text{Hom}(A, B)$  of all morphisms in  $\mathcal{C}$  such that

- i)  $f \in \text{Hom}(A, B)$  and  $f \sim f'$  implies  $f' \in \text{Hom}(A, B)$ ;
- ii)  $f \sim f', g \sim g'$  and the composition  $g \circ f$  exists imply that  $g \circ f \sim g' \circ f'$ .

**Theorem 6.7:** (Quotient Category)

Let  $\mathcal{C}$  be a category with congruence  $\sim$ , and let  $[f]$  denote the equivalence class of a morphism  $f$ . Define  $\mathcal{C}'$  by

$$\begin{aligned}
\text{Ob}(\mathcal{C}') &= \text{Ob}(\mathcal{C}); \\
\text{Hom}_{\mathcal{C}'}(A, B) &= \{[f] : f \in \text{Hom}_{\mathcal{C}}(A, B)\}; \\
[g] \circ [f] &= [g \circ f].
\end{aligned}$$

Then  $\mathcal{C}'$  is a category. It is called a **quotient category**.

One usually denotes  $\text{Hom}_{\mathcal{C}'}(A, B)$  by  $[A, B]$ .

**Exercise 6.8:**

Show that  $\mathcal{C}'$  is a category.

**Exercise 6.9:** (Category of conjugacy classes)

Consider the category of groups **Groups**. Let  $G, H$  be two groups and  $f, g \in \text{Hom}(G, H)$ . Define

$$f \sim g \Leftrightarrow \text{there exists } a \in H \text{ such that } f(x) = ag(x)a^{-1}, \text{ for all } x \in G.$$

- i) Show that  $\sim$  is an equivalence relation on each  $\text{Hom}(G, H)$ .
- ii)

**Definition 6.10:** (Functor)

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. A **covariant functor**  $F : \mathcal{C} \rightarrow \mathcal{D}$  consists of the following data.

- i) For each object  $A \in \mathcal{C}$ , there is an object  $F(A) \in \mathcal{D}$ .
- ii) For each morphism  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ , there is a morphism  $F(f) \in \text{Hom}_{\mathcal{D}}(F(A), F(B))$  such that
  - a)  $F$  preserves the identity, that is, for any object  $A \in \mathcal{C}$ ,

$$F(1_A) = 1_{F(A)}.$$

- b)  $F$  preserves the composition, that is, if  $f : A \rightarrow B$  and  $g : B \rightarrow C$ , then

$$F(g \circ f) = F(g) \circ F(f).$$

We say  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a **contravariant functor** if it reverses the morphism and the composition, that is, for  $f : A \rightarrow B$  and  $g : B \rightarrow C$  in  $\mathcal{C}$

$$F(f) : F(B) \rightarrow F(A) \text{ in } \mathcal{D} \text{ and}$$

$$F(g \circ f) = F(f) \circ F(g).$$

### Example 6.11: (Some functors)

1. Let  $\mathcal{C}$  be any category. Then we always have the identity functor  $\text{Id} : \mathcal{C} \rightarrow \mathcal{C}$ .
2. The easiest example of functors are **Forgetful functors**. For any mathematical object defined as a set with some additional structure, we can “forget” the extra structure and get a forgetful functor.
  - i)  $F : \mathbf{Top} \rightarrow \mathbf{Sets}$  defined as follows. For any topological space  $X$ ,  $F(X)$  is the underlying set and if  $f : X \rightarrow Y$  is a continuous function, then  $F(f)$  is the same function. So  $F$  forgets the topological structure of  $X$  and also forgets the continuous functions between topological spaces.
  - ii) Similarly, there is a functor  $\mathbf{Groups} \rightarrow \mathbf{Sets}$  forgetting the group structure on groups and a functor  $\mathbf{Rings} \rightarrow \mathbf{Sets}$  forgetting the ring structure on rings.
  - iii) We can also have forgetful functor which does not forget all the structures. For example, if  $\mathbf{Ab}$  is the category of abelian groups, then a functor from  $\mathbf{Ab} \rightarrow \mathbf{Groups}$  forgets the abelian structure in groups and remember just underlying group structure. Similarly, we can have functors  $\mathbf{Rings} \rightarrow \mathbf{Ab}$  which forgets the multiplicative structure and remembers the underlying additive group.
3. Let  $X$  and  $Y$  be two sets and  $f : X \rightarrow Y$  be a function. We have a functor between the two discrete categories and conversely. That is, given a function  $f : X \rightarrow Y$  one can give a functor and given a functor  $F : \mathbf{Disc}(X) \rightarrow \mathbf{Disc}(Y)$  one can define  $f : X \rightarrow Y$ .
4. Let  $M$  be a fixed topological space. Then

$$P_M : \mathbf{Top} \rightarrow \mathbf{Top}, \quad X \mapsto X \times M$$

is a functor.

### 5. Hom functors.

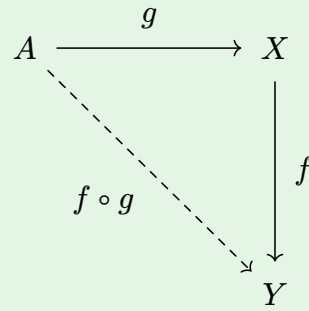
- i) Let  $\mathcal{C}$  be any category and  $A$  be an object in  $\mathcal{C}$ . Then we have the following functor

$$\text{Hom}(A, \_) : \mathcal{C} \rightarrow \mathbf{Sets}, \quad B \mapsto \text{Hom}(A, B).$$

For any  $f : X \rightarrow Y$  in  $\mathcal{C}$ ,

$$f_* := \text{Hom}(A, f) : \text{Hom}(A, X) \rightarrow \text{Hom}(A, Y), \quad g \mapsto f \circ g.$$

This is a covariant functor.



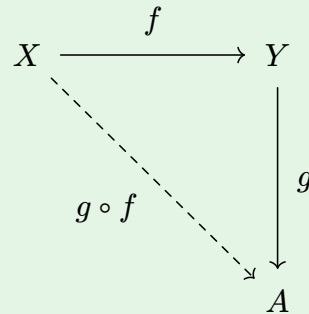
ii) We can also define a contravariant functor as follows.

$$\text{Hom}(\_, A) : \mathcal{C} \rightarrow \mathbf{Sets}, \quad B \mapsto \text{Hom}(B, A).$$

For any  $f : X \rightarrow Y$  in  $\mathcal{C}$ ,

$$f^* := \text{Hom}(f, A) : \text{Hom}(X, A) \rightarrow \text{Hom}(Y, A), \quad g \mapsto g \circ f.$$

This is a contravariant functor.



6. Let  $\mathbf{Vect}$  be the category of vector space over a field  $\mathbb{F}$ . Here morphisms are  $\mathbb{F}$ -linear maps. For any two vector spaces  $V$  and  $W$ ,  $\text{Hom}(V, W)$  is a vector space over  $\mathbb{F}$ . We have two  $\text{Hom}$  functors,

$$\text{Hom}(V, \_) : \mathbf{Vect} \rightarrow \mathbf{Vect}$$

$$\text{Hom}(\_, V) : \mathbf{Vect} \rightarrow \mathbf{Vect}.$$