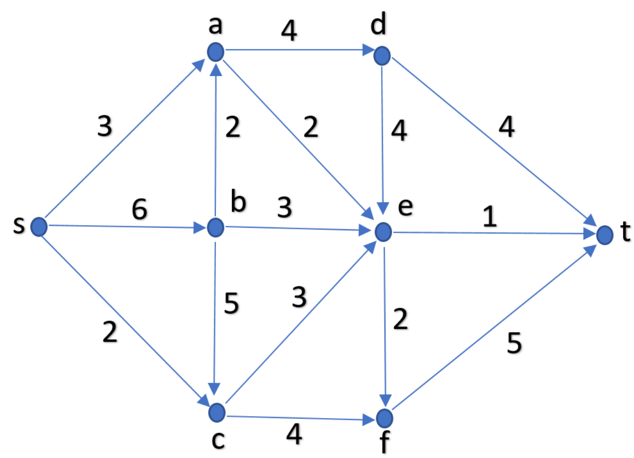
Algorithms - CS5800 - Summer 2, 2017 - Assignment 5

1. Apply the Ford-Fulkerson algorithm on the following flow network to find a maximal flow and a minimal cut.



Explain the steps of your work.
For the last iteration of the algorithm:

(a) Draw the network with the capacities of each edge and the values of the flow on each edge (before the last iteration is performed).

(b) Draw the residual network.

(c) Apply the BFS algorithm on the residual network to find a path from s to t.

(d) Explain how to define the "improved flow".

(e) Draw the residual network for the improved flow.

(f) Show a minimal cut of the (original) flow network.

Solution: in a separate file.

2. Let $G = (V, E)$ be a flow network such that the capacity of each edge is either 5 or 4.
Prove or disprove: The value of a maximal flow for G can't be 11.

Solution: This is true. For such a network the capacity of a minimal cut is:

$$\text{cap}(S, \overline{S}) = a \cdot 5 + b \cdot 4$$

for some $a, b \in \mathbb{N}$.
This clearly can't equal 11.

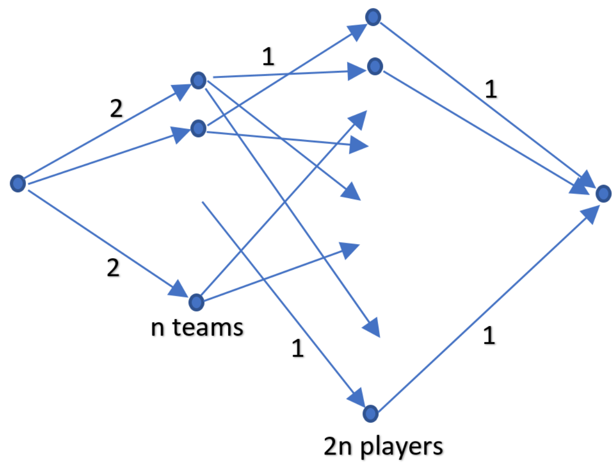
3. The teams in the NBA chooses new players each year in a process called "the draft". Each year there are n teams and $2n$ players.
The league wants to change the rules of the draft so that each team will give a list of players that it is willing to get and some algorithm will match 2 players

for each team, out of the list of players the team is willing to get.
Find a sufficient and necessary condition for it to be possible for the algorithm to give each team 2 players that the team wants.

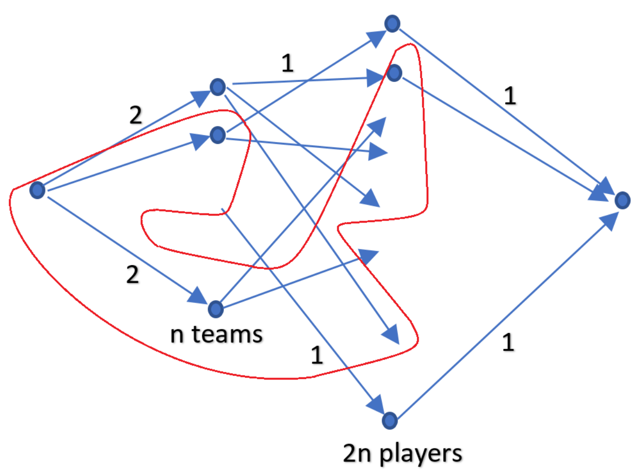
<u>Solution:</u> For a team t_i let us denote by A_{t_i} the set of players that the team is willing to get.
The condition is that for any subset of teams $I \subseteq \{1, ..., n\}$, we have:

$$\left| \bigcup_{i \in I} A_{t_i} \right| \geq 2|I|$$

Consider the following flow network:



Clearly there exists a flow of value $2n$ in this network, if and only if there exists a solution to the problem.
It is also clear that if there exists a solution then the condition holds since for any subset of teams I, the union of the sets of players they want, contains the $2|I|$ players that they get (in the solution).
To see the other direction, consider a general s-t cut of the graph:



We can write $S = \{s\} \cup T \cup P$, where T is a set of teams and P is a set of players.

The capacity of such a cut would be:

$$\operatorname{cap}(S,\overline{S}) = \sum_{t\notin T} 2 + \sum_{(t,p)\in((T\times\overline{P})\cap E)} 1 + \sum_{p\in P} 1$$

Now, each player that the teams in T want, is either in P or in $\overline{P}$. If it is in P, it contributes 1 to the above sums (in the third sum). If it is in $\overline{P}$ it also contributes 1 to the above sums (in the second sum). therefore, the above sum is bigger or equal to:

$$\geq \sum_{t\notin T} 2 + \left|\bigcup_{t\in T} A_t\right| \underbrace{\geq}_{\text{By our assumption}} \sum_{t\notin T} 2+ \geq 2|T| = 2n$$

4. Let $G=(V,E)$ be a flow network with capacity $c: E\to\mathbb{R}_{\geq 0}$.
For 2 s-t cuts $(S,\overline{S})$ and $(S',\overline{S'})$, we define their intersection to be $(S\cap S', \overline{S\cap S'})$. Show that if $(S,\overline{S})$ and $(S',\overline{S'})$ are both minimal cuts, then their intersection $(S\cap S', \overline{S\cap S'})$ is also a minimal cut.

<u>Solution:</u> Let $f_0: E\to\mathbb{R}$ be a maximal flow for G.
Since S is a minimal cut and f_0 is a maximal flow, we have:

$$\operatorname{val}(f_0) = \operatorname{cap}(S,\overline{S}) = \sum_{(u,v)\in((S\times\overline{S})\cap E)} c(u,v)$$

On the other hand we have:

$$\operatorname{val}(f_0) = f_0(S,\overline{S}) - f_0(\overline{S},S) \leq f_0(S,\overline{S}) \leq \sum_{(u,v)\in((S\times\overline{S})\cap E)} c(u,v)$$

It follows that:

$$f_0(S,\overline{S}) = \sum_{(u,v)\in((S\times\overline{S})\cap E)} c(u,v)$$

and:

$$f_0(\overline{S},S) = 0$$

We see that we must have $f_0(u,v) = \operatorname{cap}(u,v)$ for any edge of the form $(u,v)\in((S\times\overline{S})\cap E)$,
and $f_0(u,v)=0$ for any edge of the form $(u,v)\in((\overline{S}\times S)\cap E)$.
Let (u,v) be an edge from $S\cap S'$ to $\overline{S\cap S'}$ (that is $u\in S\cap S'$, $v\in\overline{S\cap S'}$). It follows that $v\notin S$ or $v\notin S'$. Assume $v\notin S$. The edge (u,v) goes from S to $\overline{S}$ and hence by what we saw $f_0(u,v)=c(u,v)$.
Let (v,u) be an edge from $\overline{S\cap S'}$ to $S\cap S'$. It follows that $v\notin S$ or $v\notin S'$. Assume $v\notin S$. The edge (v,u) goes from $\overline{S}$ to S and hence by what we saw $f_0(v,u)=0$.
We see that for any edge e from $S\cap S'$ to $\overline{S\cap S'}$ we have:

$$f_0(e) = c(e)$$

and for any edge e from $\overline{S \cap S'}$ to $S \cap S'$ we have:

$$f_0(e) = 0$$

It follows that:

$$\text{val}(f_0) = f(S \cap S', \overline{S \cap S'}) - f(\overline{S \cap S'}, S \cap S') = \text{cap}(S \cap S') - 0 = \text{cap}(S \cap S')$$

and thus by the min cut max flow theorem $S \cap S'$ is a minimal cut.

5. Optional (A bonus might be given for a good and elegant solution)
   Let $G = (V, E)$ be a flow network with integer capacities:

   $$c : E \to \mathbb{N}$$

   Find an algorithm that finds a minimal cut $(S, \overline{S})$ of G with minimal number of edges of the form (u, v) with $u, \in S$, and $v \in \overline{S}$.