Stable matching

September 11, 2017

Overview

1. Stable matching

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- We will begin by looking at an algorithm that demonstrates many of the ideas we will encounter during our course.
- This algorithm is due to Gale and Shapley (David Gale and Lloyd Shapley $\sim$ 1962).
  Actually, it was in use before they published it.

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- We begin with a simple version of the problem.
- The problem: Given a set of n women $\{w_1, ..., w_n\}$ and a set of n men $\{m_1, ..., m_n\}$, we want to find a pairing so that each women and each men can get married. We want the matching we find to be stable.
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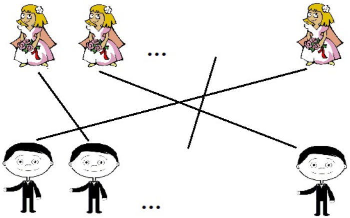

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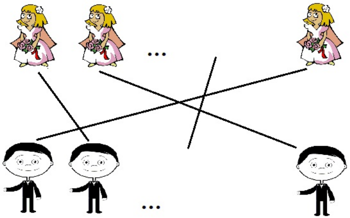

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- Each woman has a list of men ordered according to her preferences.
  Similarly, each man has a list of women ordered according to his preferences.
- Assume in our matching, some women w_1 is married to w_2 but she would prefer to be married to a men m_2 who's married to a women w_2.
  If m_2 also prefer w_1 over w_2, then the matching we have is not stable, since it is likely that w_1 and m_2 will leave their partners and get married.

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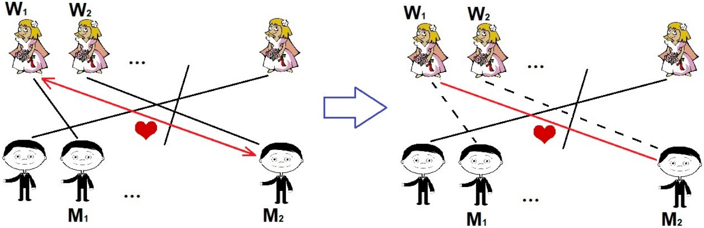

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- We see that we would like to find a matching such that for **every** women w we have:
  If m is a men that w would prefer to marry (over the man she is currently married to), then m prefers his current wife over w.
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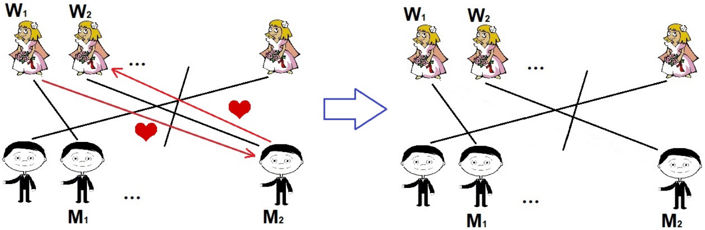

- More formally:
  For sets W and M of n women and n men, a perfect matching $S \subseteq W \times M$ is a set of n pairs of the form (w, m) with $w \in W$ and $m \in M$, such that each $w \in W$ belongs to exactly one pair and each $m \in M$ belongs to exactly one pair.
- If there are 2 pairs $(w, m), (w', m') \in S$ such that w prefers m' over m and m' prefers w over w' then we say that (m, w') is an instability with respect to S.

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  Given W, M and the lists of preferences, we want to find a perfect matching S such that there are **no** instabilities.
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Designing the Algorithm:

- The initial situation is that everyone are unmarried.
  At this point an unmarried woman w would chose the man m that is ranked highest in her list.
  w and m become engaged.
- In a situation where some women are free and some are engaged, a free woman w, proposes to the man m that she ranked highest among the men she haven't yet proposed to.
  If m is free they become engaged. If m is engaged they become engaged or not, according to m's preferences.
- The algorithm terminates when there are no free women.
  At this point all of the engagements become final (marriages).

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Preferences:

	Woman 1	Woman 2	Woman 3
Mike	Emma	Alice	Elizabeth
John	Alice	Elizabeth	Emma
Bob	Emma	Elizabeth	Alice

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Alice	Mike	John	Bob
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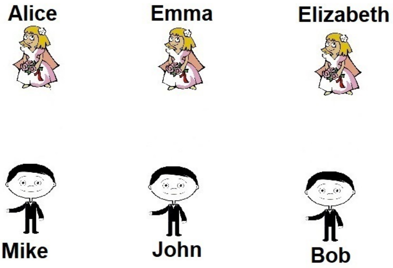

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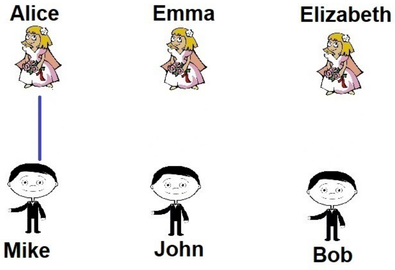

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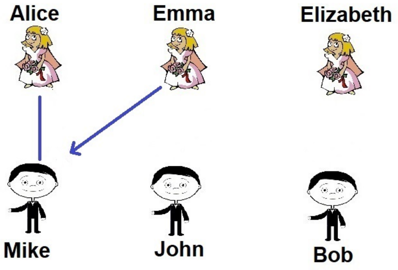

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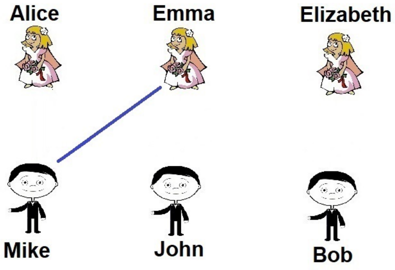

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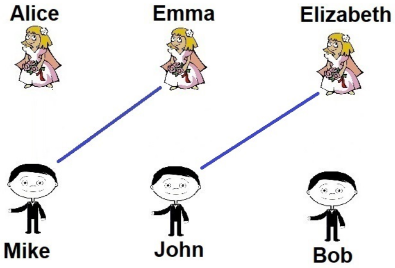

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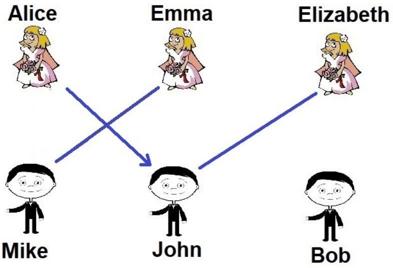

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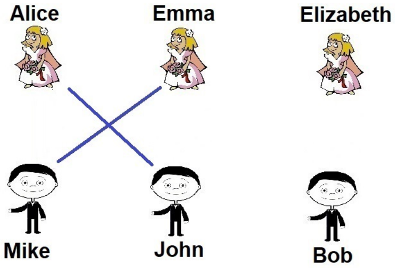

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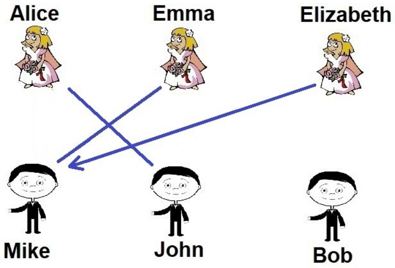

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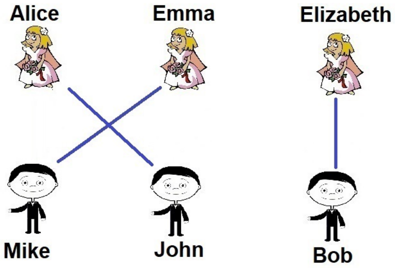

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  1. It is not clear that this algorithm terminates.
  2. It is not clear that this algorithm returns a perfect matching.
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- The facts that we mentioned suggest that the algorithm we describe has some "progress" in each step.
(You may think of it as if the algorithm has a direction in which it progresses.)
- Theorem: The Gale-Shapley algorithm terminates after at most n^2 steps.
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- Proof:
  In each step of the algorithm some woman proposes to some man to whom she never proposed before.
  We will denote the number of iterations by t.
  If we denote by $P(t)$, the set of pairs (w, m) such that w proposed to m by the end of iteration t, then it is clear that for all t we have:

$$P(t+1) > p(t)$$

  Since there are only n^2 possible pairs (w, m), the algorithm must terminate after at most n^2 iterations. ■

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- Next we want to show that the set of pairs that the algorithm returns is a perfect matching.
- What do we need to show?
- The only way a woman can remain unpaired at the end of the algorithm, is if she went over all of her list and none of the men agreed to get engaged to her.
  (Some might agree to get engaged but later reject her.)
- We will show:
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- Proof: Assume at some point during the execution of the algorithm, a woman w is free and had already proposed to all the men.
  It follows that all the men are engaged at this point. (This follows from the fact that after a man gets a proposal he remains engaged.)
  However, since the set of engaged pairs forms a matching, it follows that there are n engaged men and n engaged women. This is a contradiction. ■

- Corollary: The set of pairs that the algorithm returns is a perfect matching.
- Proof: Clearly, the set of engaged pairs is a matching at each point in the algorithm. The only way for the algorithm to terminate with a free woman is if this woman had proposed to all the men. However, by what we showed, for a free woman there must be a man to whom she did not propose. ■

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- It remains to show that the perfect matching that G-S algorithm produces is a stable matching.
- Theorem: If S is the perfect matching returned by the G-S algorithm, then S is a stable matching.

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- Proof: If S is not a stable matching then there are 2 pairs (w, m) and (w', m') which causes an instability with respect to S.
  For these 2 pairs we have:
  1. w prefers m' over m
  2. m' prefers w over w'

  From the fact that w prefers m' over m it follows that w proposed to m' before it proposed to m. Since w ends up with a different man them m', it follows that (at some point) w was rejected by m' for another woman w'' that m' prefers over w.
  By what we said, m' can only end up with a woman he prefers over w''. However he ends up with w' which he like less than w which he likes less than w''. This is a contradiction. ■

- As we mentioned, the algorithm can be executed in different ways. We have a choice of which free woman will propose.
- We may ask:
  Do we get the same perfect matching for any possible execution of the algorithm?
- We will show that the answer to this question is positive.
- The way we will prove that we always get the same perfect matching, is by showing that regardless of how we apply the algorithm, every woman ends up with the best possible match for her.

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- We say that a man m is valid for a woman w if there exists a stable matching that contains the pair (w, m).
- We say that a man m is the best valid partner for a woman w, if for any other valid man m', the woman w prefers m over m'.
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- We define the following set of pairs:

$$\hat{S} = \{(w, \text{best}(w)) | w \in W\}$$

- We will prove:
  Theorem: For any possible execution of the G-S algorithm, the resulted matching equals the set $\hat{S}$.
- This shows:
  1. $\hat{S}$ is a stable perfect matching.
  2. The result of the G-S algorithm is the best result for all the women simultaneously.
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- Proof: Assume the theorem is false. It follows that there exists an execution of the algorithm such that some woman is paired with a man that is not her best possible partner.
  Since the women propose to the men in decreasing order of their preferences, the fact that some woman w end up with a man which is not her best possible match, it means that at some point, w is rejected by best(w).
  Consider the first point in the execution of the algorithm where some woman w is rejected by a valid partner. It must be that w is rejected by $best(w)$. At this point, $best(w)$ is either engaged to a woman he prefers over w, or he is engaged to w and has been proposed to, by a woman he prefers over w. We denote the woman that best(w) prefers over w by w'.
  Since best(w) is a valid partner of w, it follows that there exists a stable matching S' that has the pair $(w, \text{best}(w))$. In S', the woman w' is paired with some $m \neq \text{best}(w)$. Since we are considering the first rejection by a valid partner in the execution of the algorithm, it follows that at this point w' had not been rejected by a valid partner. Since w' proposes in decreasing order and m is a valid partner of w', it follows that w' prefers best(w) over m.
  On the other hand, we know that best(w) prefers w' over w.
  It follows that the 2 pairs: $(w, \text{best}(w))$ and (w', m) are an instability in S'. This is a contradiction.
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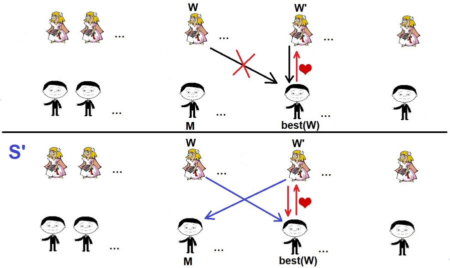

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  Theorem: The G-S algorithm pairs each man with his worst partner.

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- <u>Proof:</u> We know that the result of G-S algorithm is $\hat{S}$.
  If there is a pair (w, m) in $\hat{S}$ such that $w \neq \text{worst}(m)$, then there is a stable matching S' in which m is paired with $\text{worst}(m)$ and w is paired with some $m' \neq m$.
  In S': $(\text{worst}(m), m), (w, m')$.
  But m is the best for w and clearly m prefers w over $\text{worst}(m)$. It follows that the above is an instability in S'.
  This is a contradiction. $\blacksquare$

Stable matching

- Some questions to think of:
  1. Is it true that there always exists a stable matching such that someone gets her/his first choice ?
  2. Do we have a unique stable matching? How can we tell if the stable matching found by the G-S algorithm is unique?
  3. How do we generalize the algorithm?
     (For example if there are n applicants to m jobs.)

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