

1 Inverse Matrices

An non-singular matrix A is invertible if there is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$.

1.1 Examples

$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$ are inverses

Since $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ we can write $B = A^{-1}$

Not all matrices have inverses:

$C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ has no inverse.

Since $AB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq I$

1.2 Connection to Linear Transformations

$A : T \rightarrow V$ is a linear transformation with associated matrix A (as T is A 's V basis)

The inverse of T is a linear transformation with associated matrix A^{-1}

2 2x2 Inverse Formula

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then if $ad - bc \neq 0$:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Proof: Show $AA^{-1} = I$ and $A^{-1}A = I$

There is a general nxn inverse formula (involving the determinant and adjugate) but it's complicated to compute (we'll see this later). For now, we'll use row reduction to find the inverse (if it exists).

3 Using Gauss-Jordan to Find the Inverse

Suppose A has inverse A^{-1}

Then $AA^{-1} = I$

So $[A|I] \rightarrow [I|A^{-1}]$

To find A^{-1} , we must solve the system $Ax = e_i$ for each i .

Instead of solving $[A|e_1], [A|e_2], \dots, [A|e_n]$ individually, we can solve $[A|I]$ once to get all solutions at the same time.

If A doesn't have n pivots (one in each row) then A^{-1} does not exist.

If A has n pivots, then the algorithm $\frac{1}{2}[I|A^{-1}] \rightarrow [A^{-1}|I]$ gives A^{-1} .

3.1 Summary

- If A doesn't have n pivots, then A^{-1} doesn't exist
- If A has n pivots, then the RREF is $[I \mid A^{-1}]$

4 Example

Find A^{-1} , if it exists:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 1 & 0 \\ -2 & 2 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/6 & 1/6 & 1/6 \\ 0 & 1 & 0 & 2/3 & -1/3 & -1/3 \\ 0 & 0 & 1 & -1/6 & -1/6 & 1/6 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} -1/6 & 1/6 & 1/6 \\ 2/3 & -1/3 & -1/3 \\ -1/6 & -1/6 & 1/6 \end{bmatrix}$$

(Check: Verify $AA^{-1} = I$)

5 Using Inverses to Solve Systems

If A is invertible, we can solve $Ax = b$ by multiplying both sides by A^{-1} :

$$A^{-1}Ax = A^{-1}b$$

$$x = A^{-1}b$$

Hence $x_i = \sum_j (A^{-1})_{ij} b_j$

However, solving A^{-1} is (in practice) more complicated than solving $Ax = b$ directly.