1 Inverse Matrices

An $n \times n$ matrix A is invertible if there is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$.

1.1 Examples

- $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ are inverses
- Since $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we can write $B = A^{-1}$
- Not all $n \times n$ matrices have inverses
- $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ has no inverse
- $AB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq I$

1.2 Connection to Linear Transformations

A^{-1} is the inverse transformation with respect to A (or T_A^{-1} w.r.t. T_A). The inverse of T is a linear transformation whose associated matrix is A^{-1}.

2 2x2 Inverse Formula

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then if $ad - bc \neq 0$:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This is a special case of a more general formula (involving the determinant and adjugate) which is more complicated to compute.

2.1 Using Gauss-Jordan to Find the Inverse

Suppose A has inverse A^{-1} Then $AA^{-1} = I$ So $A[A^{-1}] = [I]$

To find A^{-1}, we must solve the system $Ax = e_i$ for each i.

Instead of solving $[A][x_1] = [e_1]$, $[A][x_2] = [e_2]$, ..., $[A][x_n] = [e_n]$ individually, we can streamline the process by augmenting A with I:

$[A|I]$

If A doesn't have n pivots (one in each row), then A^{-1} does not exist. If A has n pivots, then the algorithm produces $[I|A^{-1}]$.

2.2 Summary

- If A doesn't have n pivots, then A^{-1} doesn't exist
- If A has n pivots, then we obtain A^{-1}

3 Example

Find A^{-1}, if it exists:

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & -2 & | & 1 & 0 & 0 \\ 4 & 9 & -3 & | & 0 & 1 & 0 \\ -2 & -3 & 7 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2-2R_1} \begin{bmatrix} 2 & 4 & -2 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & -2 & 1 & 0 \\ -2 & -3 & 7 & | & 0 & 0 & 1 \end{bmatrix}$$
$$\xrightarrow{R_3+R_1} \begin{bmatrix} 2 & 4 & -2 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & -2 & 1 & 0 \\ 0 & 1 & 5 & | & 1 & 0 & 1 \end{bmatrix}$$
$$\xrightarrow{R_3-R_2} \begin{bmatrix} 2 & 4 & -2 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & -2 & 1 & 0 \\ 0 & 0 & 4 & | & 3 & -1 & 1 \end{bmatrix}$$

Therefore:

$$A^{-1} = \begin{bmatrix} \frac{11}{8} & -\frac{1}{4} & \frac{1}{8} \\ -\frac{5}{8} & \frac{3}{4} & -\frac{1}{8} \\ \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix} \quad (\text{Check: } AA^{-1} = I)$$

3.1 Using Inverses to Solve Systems

If A exists, we can solve $Ax = b$ by multiplying both sides by A^{-1}:

$$\begin{aligned} A^{-1}Ax &= A^{-1}b \\ Ix &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

Note: $A^{-1}b$ has solution x if b (and A) have no pivots. However, even if A has n pivots (is invertible), this solution is unique.