1 Inverse Matrices

An *inverse* to a matrix A is another matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$.

1.1 Examples

- $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$ are inverses
- Since $AB = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = 5I$, we can write $B = \frac{1}{5}A^{-1}$
- Not all matrices have inverses. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has no inverse.
- Since $AB = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix} = 5B$

1.2 Connection to Linear Transformations

Let T be a linear transformation with associated matrix A (in "standard" basis). Then the inverse transformation exists if and only if A has an inverse. The matrix of the inverse transformation is A^{-1}.

2 2x2 Inverse Formula

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then if $ad - bc \neq 0$:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{1}$$

This is a special case of a more general formula (involving the determinant and adjugate matrix, which are complicated to compute but follow this idea).

2.1 Using Cross-Section to Find the Inverse

Suppose A has inverse A^{-1} Then $AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

So $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Instead of solving $[Ax_1], [Ax_2], \ldots, [Ax_n]$ individually, we can combine these into one system and solve once:

$$[A|e_1, e_2, \ldots, e_n] \rightsquigarrow [I|A^{-1}] \tag{2}$$

If A doesn't have n pivots (one in each row), then A doesn't exist.
If A has n pivots, then the algorithm will produce A^{-1}.

2.2 Summary

- If A doesn't have n pivots, then A^{-1} doesn't exist
- If A has n pivots, then the RREF is $[I|A^{-1}]$

3 Example

Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Find A^{-1}, if it exists.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{array}\right]$$

Therefore, $A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$ (Check: Verify $AA^{-1} = I$)

4 Using Inverses to Solve Systems

If A exists, we can solve $Ax = b$ by multiplying by A^{-1}:

$$\begin{aligned} A^{-1}Ax &= A^{-1}b \\ Ix &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

However, this is not the solution $A^{-1}b$. Moreover, even if A has an inverse (i.e., is invertible), this solution is unique.