1 Matrix Inverses

An invertible matrix A is one that has an *inverse* A^{-1} such that $AA^{-1} = A^{-1}A = I$.

1.1 Examples

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

A is invertible, but B is not as we can write $B = A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Not all matrices have inverses. For example:

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ has no inverse}$$

1.2 Connection to Linear Transformations

A^{-1} is the inverse transformation with respect to A (as T is to V). The inverse of T is a linear transformation that "undoes" the original transformation.

2 2x2 Inverse Formula

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then if $ad - bc \neq 0$:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{1}$$

This is a special case of Cramer's formula (which is complicated to compute but has the same form for any size matrix).

2.1 Using Cross-Section to Find the Inverse

Suppose A has inverse A^{-1}:

$$AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = I$$
$$A^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

So to find A^{-1}, we must solve the system $Av = e_i$ where e_i are the standard basis vectors.

Instead of solving $[Ax][Ay] = [e_1][e_2]$ individually, we can combine them into one matrix equation:

$$[A|e_1, e_2, \ldots] = [A|I] \tag{2}$$

If A doesn't have an inverse (i.e., is singular), this process will fail at some point.

If A has an inverse, then the algorithm:

$$[A|I] \to [I|A^{-1}] \tag{3}$$

3 Example: Finding A^{-1}

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Find A^{-1}, if it exists.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - \frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 + \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{3}{4} & \frac{7}{4} & \frac{1}{4} & \frac{1}{2} & 1 \end{array}\right]$$

$$\xrightarrow{\text{various steps}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{2} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{array}\right]$$

Therefore:

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \tag{4}$$

3.1 Using Inverses to Solve Systems

If A exists, we can solve $Ax = b$ by multiplying both sides by A^{-1}:

$$\begin{aligned} A^{-1}Ax &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

However, if A has n pivots (i.e., is invertible), this solution is unique.