1 Matrix Inverses

An *inverse* of a matrix A is a matrix that $AA^{-1} = A^{-1}A = I$.

1.1 Examples

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{8} \end{bmatrix} \text{ are inverses}$$

Since $AB = BA = I$, we can write $B = A^{-1}$.

1.2 Properties

Not all matrices have inverses. For example:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ has no inverse}$$

While $AB = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = B$ does.

2 Connection to Linear Transformations

If T is a linear transformation with associated matrix A (as $T(x) = Ax$), then the inverse transformation has associated matrix A^{-1}.

3 2x2 Inverse Formula

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then if $ad - bc \neq 0$:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This is a special case of a more general formula (involving the determinant and adjugate) that works for any size matrix. For 2x2, it's easy to verify that $AA^{-1} = I$.

4 Using Linear Systems to Find the Inverse

Suppose A has inverse A^{-1}. Then $AA^{-1} = I$.

So $A \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

To find A^{-1}, we must solve the system $Ax = e_i$ for each column.

Instead of solving $[Ax_1][Ax_2]$ individually, we can combine these into one augmented matrix:

$$[A|e_1, e_2, \ldots] = [A|I]$$

If A doesn't have n pivots (one in each row), then A^{-1} does not exist.

If A has n pivots, then the algorithm produces $[I|A^{-1}]$.

5 Summary of Inverse Existence

- If A doesn't have n pivots, then A^{-1} doesn't exist
- If A has n pivots, then A is invertible and the RREF is $[I|A^{-1}]$

6 Example: Finding A^{-1}

Find A^{-1} if it exists:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

We form the augmented matrix $[A|I]$:

$$\begin{bmatrix} 2 & 1 & 3 & | & 1 & 0 & 0 \\ 1 & 0 & 1 & | & 0 & 1 & 0 \\ 3 & 1 & 4 & | & 0 & 0 & 1 \end{bmatrix}$$

After row operations to get RREF:

$$\begin{bmatrix} 1 & 0 & 0 & | & -1 & 3 & -1 \\ 0 & 1 & 0 & | & 3 & -8 & 3 \\ 0 & 0 & 1 & | & 1 & -2 & 1 \end{bmatrix}$$

Therefore:

$$A^{-1} = \begin{bmatrix} -1 & 3 & -1 \\ 3 & -8 & 3 \\ 1 & -2 & 1 \end{bmatrix}$$

7 Using Inverses to Solve Systems

If A exists, we can solve $Ax = b$ by multiplying both sides by A^{-1}:

$$\begin{aligned} A^{-1}Ax &= A^{-1}b \\ Ix &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

Note: This is not the solution $A \to R$. However, given A has n pivots (is invertible), this solution is unique.