1 Inverse Matrices

An inverse matrix A is invertible if there is a matrix B such that AB = BA = I.

1.1 Examples

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \text{ are inverses}$$

Since AB = BA = I, we can write B = A^{-1}
Not all matrices have inverses:

$$A = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \text{ has no inverse}$$

1.2 Connection to Linear Transformations

Let T be a linear transformation with standard matrix A (in $\mathbb{R}^n \to \mathbb{R}^n$). The inverse of T is a linear transformation with standard matrix A^{-1}.

2 2x2 Inverse Formula

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then if $ad - bc \neq 0$:

$$A^{-1} = \tfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This is a special case of a more general formula (involving the determinant and adjugate matrix) which is more complicated to compute (but we'll see later).

2.1 Using Cross-Section to Find the Inverse

Suppose A has inverse A'. Then AA' = I.
So $A \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
To find A', we must solve the system Ax = e1 and Ay = e2, where e1, e2 are the standard basis vectors.

Instead of solving [Ax], [Ay], [...] individually, we can combine them into one augmented matrix:

$$[A|e_1, e_2, ...] = [A|I]$$

If A is invertible, this reduces to $[I|A^{-1}]$

3 Summary: Finding the Inverse

- If A doesn't have n pivots, then A doesn't exist
- If A has n pivots, then the algorithm works and we get A^{-1}

4 Example

Find A^{-1}, if it exists:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

$$[A|I] = \begin{bmatrix} 2 & 1 & 3 & | & 1 & 0 & 0 \\ 1 & 0 & 1 & | & 0 & 1 & 0 \\ 3 & 1 & 4 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & | & 0 & 1 & 0 \\ 0 & 1 & 1 & | & 1 & -2 & 0 \\ 0 & 0 & 1 & | & -1 & 1 & 1 \end{bmatrix}$$

Therefore:

$$A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \text{ (Check: } AA^{-1} = I\text{)}$$

5 Using Inverses to Solve Systems

If A exists, we can solve Ax = b by multiplying by A^{-1}:

$$A^{-1}Ax = A^{-1}b$$

$(A^{-1}A)x = A^{-1}b$
$Ix = A^{-1}b$
$x = A^{-1}b$

Hence $A^{-1}b$ is the solution to Ax = b. However, given A has n pivots (is invertible), this solution is unique.