1 Matrix Inverses

An *inverse* of a matrix A is denoted A^{-1}. A matrix A such that $AA^{-1} = I$.

1.1 Examples

- $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix}$ are inverses
- Since $AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we can write $B = A^{-1}$
- Not all matrices have inverses. $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ has no inverse

1.2 Connection to Linear Transformations

Let T be a linear transformation with associated matrix A (in FTLA or FTVS). The inverse of T is a linear transformation with associated matrix A^{-1}.

2 2x2 Inverse Formula

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then if $ad - bc \neq 0$:

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

There is a general $n \times n$ inverse formula (involving the determinant) but it's complicated to compute (we'll see this later). For now, we'll focus on the 2x2 case.

2.1 Using Cross-Section to Find the Inverse

Suppose A has inverse A^{-1}. Then $AA^{-1} = I$ and $A^{-1}A = I$.

So $A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Then $A^{-1} = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$

To find A^{-1}, we must solve the system $Ax = e_1$ and $Ay = e_2$. Instead of solving $[Ax], [Ay], [Az]$ individually, we can streamline this process by solving one system at once:

$$A[x_1\ y_1\ z_1] = [e_1\ e_2\ e_3]$$

If A doesn't have n pivots (one in each row), then A^{-1} does not exist.

If A has n pivots, then the algorithm $\frac{1}{n}[A\ I] \to [I\ A^{-1}]$ works.

2.2 Example

Find A^{-1}, if it exists:

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 3 & | & 1 & 0 & 0 \\ 1 & 0 & 1 & | & 0 & 1 & 0 \\ 0 & 1 & 2 & | & 0 & 0 & 1 \end{pmatrix}$$

(Reduced row echelon form steps omitted)

$$\begin{pmatrix} 1 & 0 & 0 & | & 2 & -1 & -1 \\ 0 & 1 & 0 & | & -4 & 3 & 1 \\ 0 & 0 & 1 & | & 1 & -1 & 0 \end{pmatrix}$$

Therefore, $A^{-1} = \begin{pmatrix} 2 & -1 & -1 \\ -4 & 3 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ (Check: $AA^{-1} = I$)

3 Using Inverses to Solve Systems

If A exists, we can solve $Ax = b$ by multiplying both sides by A^{-1}:

$$A^{-1}Ax = A^{-1}b$$
$$Ix = A^{-1}b$$
$$x = A^{-1}b$$

Here $A^{-1}b$ is a solution to $Ax = b$. However, even if A has n pivots (is invertible), this solution is unique.