1 Inverse Matrices

An *inverse matrix* A is invertible if there is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$.

1.1 Examples

$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$ are inverses

Since $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we can write $B = A^{-1}$

Not all matrices have inverses:

$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ has no inverse

Since $AB = \begin{bmatrix} 0 & 0 \\ a+b & a+b \end{bmatrix} \neq I$

1.2 Connection to Linear Transformations

A^{-1} is the *inverse transformation* with respect to A (or T_A^{-1} w.r.t. T_A).

The inverse of T is a linear transformation whose associated matrix is A^{-1}.

2 2x2 Inverse Formula

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then if $ad - bc \neq 0$:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This is a special case of a more general formula (which is complicated to compute but has the same idea for matrices of any size).

2.1 Using Gauss-Jordan to Find the Inverse

Suppose A has inverse A^{-1}. Then $AA^{-1} = I$ So $A[A^{-1}] = [I]$

To find A^{-1}, we need to solve the system $Ax = e_i$ for each i.

Instead of solving $[Ax][Ax][Ax]$ individually, we can combine these systems into one matrix equation:

$$[A|e_1, e_2, ..., e_n] \to [I|A^{-1}]$$

If A doesn't have n pivots (one in each row), then A^{-1} does not exist.

If A has n pivots, then the algorithm produces A^{-1}.

2.2 Summary

- If A doesn't have n pivots, then A^{-1} doesn't exist
- If A has n pivots, then A is invertible and the RREF is $[I|A^{-1}]$

3 Example

Find A^{-1}, if it exists:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 & | & 1 & 0 & 0 \\ 1 & 0 & 1 & | & 0 & 1 & 0 \\ 3 & 1 & 4 & | & 0 & 0 & 1 \end{bmatrix}$$

After row operations:

$$\begin{bmatrix} 1 & 0 & 0 & | & -1 & 3 & -1 \\ 0 & 1 & 0 & | & 3 & -8 & 3 \\ 0 & 0 & 1 & | & 1 & -2 & 1 \end{bmatrix}$$

Therefore:

$$A^{-1} = \begin{bmatrix} -1 & 3 & -1 \\ 3 & -8 & 3 \\ 1 & -2 & 1 \end{bmatrix} \quad \text{(Check: } AA^{-1} = I\text{)}$$

4 Using Inverses to Solve Systems

If A exists, we can solve $Ax = b$ by multiplying both sides by A^{-1}:

$$\begin{aligned} A^{-1}Ax &= A^{-1}b \\ Ix &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

However, this solution is $O(n^3)$ (as matrix multiplication is $O(n^3)$), which is inefficient.