1 Inverse Matrices

An *inverse matrix* A is invertible if there is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$.

1.1 Examples

- $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ are inverses
- Since $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we can write $B = A^{-1}$
- Not all matrices have inverses. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has no inverse.

1.2 Connection to Linear Transformations

A^{-1} is the inverse transformation with respect to matrix A (or T at $A\vec{x} = \vec{b}$). The inverse of T is a linear transformation whose associated matrix is A^{-1}.

2 2x2 Inverse Formula

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $T \det(A) \neq 0$:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This is a special case of a more general formula (involving the determinant and adjugate) that is complicated to compute (will see this later) for larger matrices.

2.1 Using Cross-Section to Find the Inverse

Suppose A has inverse A^{-1} Then $AA^{-1} = [A_1 A_2] = I$

So $A\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

To find A^{-1}, we must solve these systems:

$$A\vec{x}_1 = \vec{e}_1, \quad A\vec{x}_2 = \vec{e}_2$$

Instead of solving $[Ax_1][Ax_2] = [e_1 e_2]$ individually, we can combine them into one matrix equation:

$$A[x_1, x_2, \ldots] = [e_1, e_2, \ldots]$$

If A doesn't have n pivots (one in each row), then A doesn't exist. If A has n pivots, then the algorithm will produce A^{-1}.

2.2 Summary

- If A doesn't have n pivots, then A^{-1} doesn't exist.
- If A has n pivots, then A^{-1} exists and the RREF is $[I|A^{-1}]$.

3 Example: Finding A^{-1}

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$. Find A^{-1}, if it exists.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Row operations}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & -8 & 5 & 4 \\ 0 & 0 & 1 & 2 & -1 & -1 \end{array}\right]$$

Therefore,

$$A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ -8 & 5 & 4 \\ 2 & -1 & -1 \end{bmatrix}$$

(Check: Verify $AA^{-1} = I$)

3.1 Using Inverses to Solve Systems

If A exists, we can solve $A\vec{x} = \vec{b}$ by multiplying $A^{-1}\vec{b}$:

$$\begin{aligned} A\vec{x} &= \vec{b} \\ A^{-1}A\vec{x} &= A^{-1}\vec{b} \\ I\vec{x} &= A^{-1}\vec{b} \\ \vec{x} &= A^{-1}\vec{b} \end{aligned}$$

Note: A^{-1} has solutions if (and only if) A has n pivots (is invertible). When A^{-1} exists, this solution is unique.