1 Inverse Matrices

An *inverse matrix* is invertible if there is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$.

1.1 Examples

- $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -1 & 3/2 \end{bmatrix}$ are inverses
- Since $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we can write $B = A^{-1}$
- Not all matrices have inverses. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has no inverse

1.2 Connection to Linear Transformations

If T is a linear transformation with associated matrix A (so $T(x) = Ax$), then the inverse transformation exists when A^{-1} exists.

2 2x2 Inverse Formula

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then if $ad - bc \neq 0$:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This is a special case of Cramer's formula (which is complicated to compute but has the same form for nxn matrices).

2.1 Using Cross-Section to Find the Inverse

Suppose A has inverse A^{-1}. Then $AA^{-1} = [A_1, A_2] = I$

So $A_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

To find A^{-1}, we must solve the system $Ax = e_i$ for each column vector.

Instead of solving $[A|e_1], [A|e_2], \ldots, [A|e_n]$ individually, we can solve $[A|I]$ at once.

2.2 Gaussian Elimination Method

If A doesn't have n pivots (one in each row):

- A^{-1} does not exist
- A is not invertible

If A has n pivots, then the algorithm produces $[I|A^{-1}]$.

3 Example: Finding A^{-1}

Find A^{-1}, if it exists:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

Solution steps:

$$\begin{bmatrix} 2 & 1 & 3 & | & 1 & 0 & 0 \\ 1 & 0 & 1 & | & 0 & 1 & 0 \\ 3 & 1 & 4 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - \frac{1}{2}R_1} \begin{bmatrix} 2 & 1 & 3 & | & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & | & -\frac{1}{2} & 1 & 0 \\ 3 & 1 & 4 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 - \frac{3}{2}R_1} \begin{bmatrix} 2 & 1 & 3 & | & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & | & -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & | & -\frac{3}{2} & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2} \begin{bmatrix} 2 & 1 & 3 & | & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & | & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & | & -1 & -1 & 1 \end{bmatrix}$$

Since A doesn't have 3 pivots, A^{-1} does not exist.

4 Using Inverses to Solve Systems

If A exists, we can solve $Ax = b$ by multiplying both sides by A^{-1}:

$$\begin{aligned} A^{-1}Ax &= A^{-1}b \\ Ix &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

Note: A^{-1} has a solution if A is invertible (has n pivots). However, even if A has n pivots, this solution is expensive to compute compared to Gaussian elimination.