1 Matrix Inverses

An *inverse* of a matrix A is a matrix that $AA^{-1} = A^{-1}A = I$.

1.1 Examples

- $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$ are inverses
- Since $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we can write $B = A^{-1}$
- Not all matrices have inverses: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has no inverse

1.2 Connection to Linear Transformations

If T is a linear transformation with standard matrix A (so $T(x) = A\vec{x}$), then the inverse transformation exists if and only if A^{-1} exists.

2 2x2 Inverse Formula

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then if $ad - bc \neq 0$:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This is a special case of a more general formula (involving the determinant and adjugate) that works for any size matrix.

2.1 Using Gauss-Jordan to Find the Inverse

Suppose A has inverse A^{-1}. Then:

$$AA^{-1} = I$$

$$A[A^{-1}] = [I]$$

So to find A^{-1}, we need to solve the system $Ax = e_i$ for each standard basis vector e_i.

Instead of solving $[A|e_1], [A|e_2], \ldots, [A|e_n]$ individually, we can streamline the process by solving $[A|I]$ in one go.

2.2 Invertibility Conditions

If A is $n \times n$, then the following are equivalent:

- A has an inverse
- $Ax = 0$ has only the trivial solution
- The RREF of A is I
- A has n pivots
- $\det A \neq 0$

3 Example: Finding A^{-1}

Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Find A^{-1}, if it exists.

$$\begin{aligned} [A|I] &= \begin{bmatrix} 2 & 1 & 0 & | & 1 & 0 & 0 \\ 1 & 2 & 1 & | & 0 & 1 & 0 \\ 0 & 1 & 2 & | & 0 & 0 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 0 & | & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{8} \\ 0 & 1 & 0 & | & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & | & \frac{1}{8} & -\frac{1}{4} & \frac{5}{8} \end{bmatrix} \end{aligned}$$

Therefore,

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{8} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{8} & -\frac{1}{4} & \frac{5}{8} \end{bmatrix}$$

3.1 Using Inverses to Solve Systems

If A is invertible, we can solve $Ax = b$ by multiplying both sides by A^{-1}:

$$\begin{aligned} A^{-1}Ax &= A^{-1}b \\ Ix &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

However, this is not an efficient method for solving systems in practice.