1 Inverse Matrices

An $n \times n$ matrix A is invertible if there is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$.

1.1 Examples

- $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$ are inverses
- Since $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we can write $B = A^{-1}$
- Not all $n \times n$ matrices have inverses
- $C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has no inverse
- Since $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$, $B = A^{-1}$

1.2 Connection to Linear Transformations

If T is a linear transformation with associated matrix A (in standard bases), then the inverse transformation exists if and only if A^{-1} exists. The matrix of the inverse transformation is A^{-1}.

2 2x2 Inverse Formula

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then if $ad - bc \neq 0$,

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This is a special case of a more general formula (involving the determinant and adjugate matrix) that is more complicated to compute (but we'll see later).

2.1 Using Cross-Section to Find the Inverse

Suppose A has inverse A^{-1} Then $AA^{-1} = [A_1 A_2] \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = I$

So $Ax_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $Ax_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

To find A^{-1}, we must solve the system $Ax = e_i$ for each standard basis vector e_i.

Instead of solving $[A|e_1], [A|e_2], \ldots, [A|e_n]$ individually, we can combine these into one augmented matrix $[A|I]$.

If A doesn't have n pivots (one in each row), then A^{-1} does not exist. If A has n pivots, then the algorithm produces $[I|A^{-1}]$.

2.2 Summary

- If A doesn't have n pivots, then A^{-1} doesn't exist
- If A has n pivots, then the RREF of $[A|I]$ is $[I|A^{-1}]$

3 Example: Finding A^{-1}

Find A^{-1} if it exists:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 1 & 4 & 0 & 0 & 1 \end{bmatrix} &\sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 & -3 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} \end{aligned}$$

Therefore, $A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -3 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ (Check: $AA^{-1} = I$)

4 Using Inverses to Solve Systems

If A exists, we can solve $Ax = b$ by multiplying by A^{-1}:

$$\begin{aligned} A^{-1}Ax &= A^{-1}b \\ Ix &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

However, A^{-1} has subtraction (in its entries), thus solution is unique.