1 Inverse Matrices

An $n \times n$ matrix A is invertible if there is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$.

1.1 Examples

- $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ are inverses

- Since $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we can write $B = A^{-1}$

- Not all matrices have inverses. e.g., $A = \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix}$ has no inverse

1.2 Connection to Linear Transformations

Let T be a linear transformation with associated matrix A (in FTLA's V-W). The inverse of T is a linear transformation with associated matrix A^{-1}.

2 2x2 Inverse Formula

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then if $ad - bc \neq 0$:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note: $\det A = ad - bc$ is called the *determinant*

This is a special 2x2 inverse formula (extending to determinants which we will see later). For now, this formula is complicated to compute (until we learn how).

2.1 Using Gauss-Jordan to Find the Inverse

Suppose A has inverse A^{-1} Then $AA^{-1} = [A|I] \sim [I|A^{-1}]$

So to find A^{-1}, we must solve the system $Ax = e_i$ for each i.

Instead of solving $[A|e_1], [A|e_2], \ldots, [A|e_n]$ individually, we can solve one augmented matrix $[A|I]$

If A doesn't have n pivots (one in each row), then A^{-1} does not exist.

If A has n pivots, then the algorithm will produce $[I|A^{-1}]$

2.2 Summary: Finding Inverse Using Gauss-Jordan

- If A doesn't have n pivots, then A^{-1} doesn't exist

- If A has n pivots, then the algorithm will produce $[I|A^{-1}]$

3 Example: Finding A^{-1}

Find A^{-1}, if it exists:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

Solution:

$$[A|I] = \begin{bmatrix} 2 & 1 & 3 & | & 1 & 0 & 0 \\ 1 & 0 & 1 & | & 0 & 1 & 0 \\ 3 & 1 & 4 & | & 0 & 0 & 1 \end{bmatrix}$$

(Perform row operations...)

$$[I|A^{-1}] = \begin{bmatrix} 1 & 0 & 0 & | & -1 & 3 & -1 \\ 0 & 1 & 0 & | & -2 & -5 & 3 \\ 0 & 0 & 1 & | & 1 & -2 & 0 \end{bmatrix}$$

Therefore,

$$A^{-1} = \begin{bmatrix} -1 & 3 & -1 \\ -2 & -5 & 3 \\ 1 & -2 & 0 \end{bmatrix}$$

(Check: Verify $AA^{-1} = I$)

4 Using Inverses to Solve Systems

If A exists, we can solve $Ax = b$ by multiplying by A^{-1}:

$$\begin{aligned} A^{-1}Ax &= A^{-1}b \\ Ix &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

Note: $Ax = b$ has solution $x = A^{-1}b$. However, even if A has n pivots (so it is invertible), this solution is no more efficient than Gauss-Jordan elimination.