Sets

$\mathbb{N} = \{0, 1, 2, \dots\}$ natural numbers

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ integers

$\mathbb{Q} = \{ \frac{k}{n} \mid k, n \in \mathbb{N},\ n \neq 0 \}$

rationals

$\mathbb{R}$ = real numbers

$\mathbb{C} = \{ a + ib \mid a, b \in \mathbb{R} \}$ = complex

i = "imaginary unit"
characterized by $i^2 = -1$.

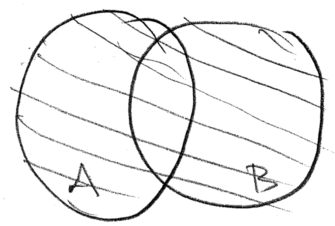
$\mathbb{R}^d = \{ (a_1, \dots, a_d) \mid a_i \in \mathbb{R} \}$ d-dimensional space

$\mathbb{R}^\infty = \{ (a_0, a_1, a_2, \dots) \mid a_i \in \mathbb{R} \}$
space of <u>infinite sequences</u> of reals

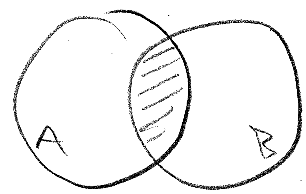
A, B sets. <u>Operations</u>:

$$A \cup B = \{ c \mid c \in A \text{ or } c \in B \}$$ <u>union</u>

$$A \cap B = \{ c \mid c \in A \text{ and } c \in B \}$$ <u>intersection</u>

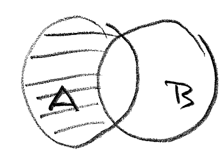


$A \cup B$



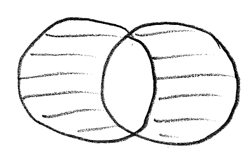
$A \cap B$

$$A \setminus B = \{ c \in A \mid c \notin B \} :$$
"<u>difference</u>"



$A \setminus B$

$$A \triangle B = A \setminus B \cup B \setminus A :$$



$$= (A \cup B) \setminus A \cap B \quad \text{etc.}$$

"<u>symmetric difference</u>"

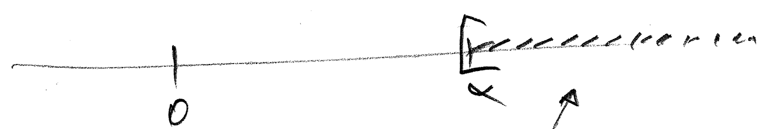
Let $(A_\alpha)_{\alpha \in I}$ be an indexed family of sets (index set I).

<u>EX</u> ① $I = \mathbb{N}$,

$$A_\alpha \underset{\substack{\uparrow \\ \text{definition}}}{:=} \{\alpha, \alpha+1\}$$

② $I = \mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$

$$A_\alpha := [\alpha, \infty)$$



infinite interval

union: $$\bigcup_{\alpha \in I} A_\alpha := \{a \mid a \in A_\alpha \text{ for some } \alpha \in I\}$$
$$= \{a \mid \exists\, \alpha \in I \text{ such that } a \in A_\alpha\}$$

intersection $$\bigcap_{\alpha \in I} A_\alpha = \{a \mid a \in A_\alpha \underbrace{\text{fore every } \alpha \in I}_{\forall \alpha \in I}\}.$$

Maps in general.

Map = function = mapping (= transformation)

Let A, B be sets ($\neq \emptyset$). A map f

$$f : A \longrightarrow B$$
$$\quad a \longmapsto f(a)$$

is an assignment: to $\forall a \in A$ f assign a unique $b \in B$ which is called the value of f at a and is denoted by $f(a)$.

Note: f assigns to $\forall\, a \in A$ a value, but there may exists $b \in B$ st $f(a) \neq b$ $\forall a \in A$.

A is called the domain (of definition) of f

B is — " — the target space of f.

If $A' \subseteq A$, then the <u>map f'</u>

$$\begin{aligned} f' : A' &\longrightarrow B \\ \overset{\Psi}{a} &\longmapsto f'(a) := f(a) \end{aligned}$$

is called the <u>restriction of f to A'</u>

(denoted by $f' = f\restriction_{A'}$)

<u>EX</u>: ① $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \sin(x)$ $\rightarrow$ $f = \sin$ fct.

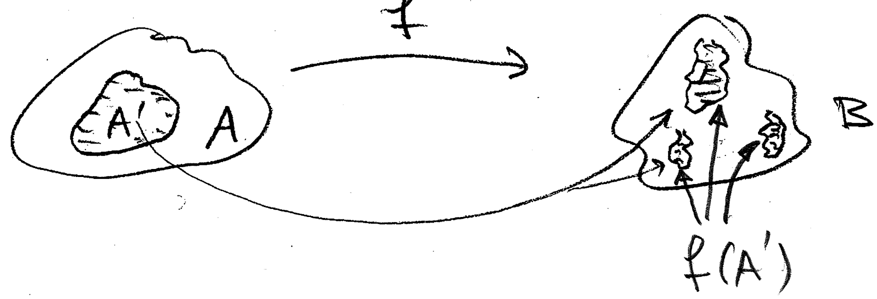
② $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $x \longmapsto 1/x$

③ $g : \mathbb{R} \to \mathbb{R}$, $x \longmapsto \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$

④ $f : V \longrightarrow$ set of subspaces of $V = \mathcal{S}$
$\overset{\Psi}{\vec{v}} \longmapsto \operatorname{span}(\vec{v})$

⑤ $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$, $p(x) \longmapsto \frac{d}{dx} p(x) = p'$ <u>derivative</u>

Let



$$f(A') := \{ b \in B \mid \exists\, a \in A : f(a) = b \} =$$

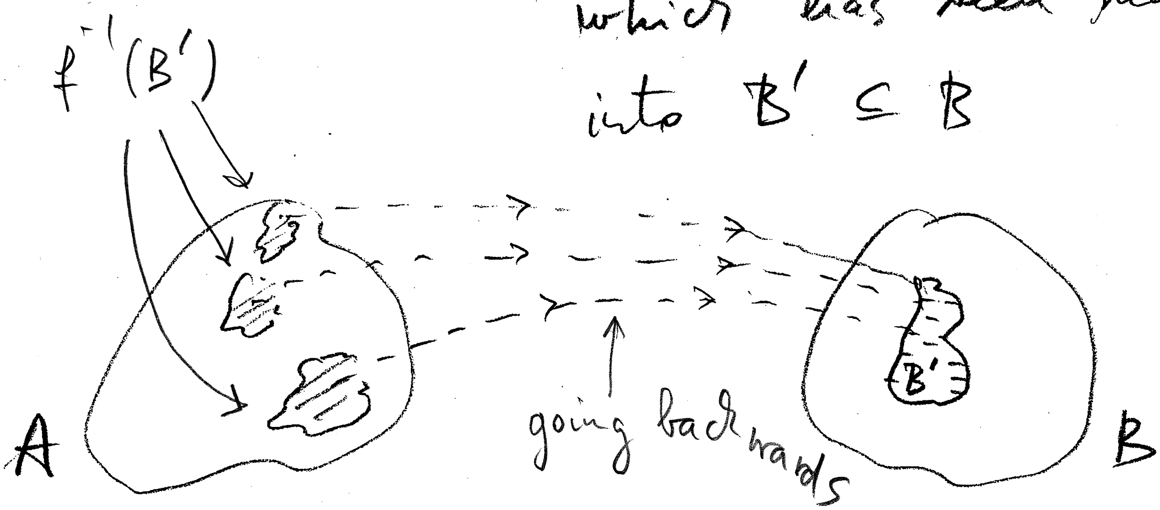
[the image of the subset $A' \subseteq A$] $= \{ f(a) \mid a \in A' \}$

image = "forward image"

Similarly we define the inverse image (backwards image) of $B' \subseteq B$

$$f^{-1}(B') := \{ a \in A \mid f(a) \in B' \}$$

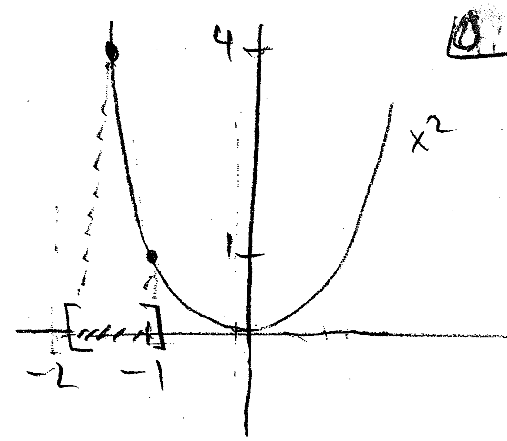
all the elements a in A which has been mapped into $B' \subseteq B$



⚠ f^{-1} here is NOT the inverse map which may not even exist!

Ex $\quad f : \mathbb{R} \to \mathbb{R}$

$\qquad x \mapsto x^2$



$A' = [-2, -1]$

$f(A') = [1, 4]$

let $B' = [1, 4]$

$f^{-1}(B') = \{ x \in \mathbb{R} \mid x^2 \in [1, 4] \}$

$= [-2, -1] \cup [1, 2]$

($\rightarrow$ in general $f^{-1}(f(A')) \supseteq A'$)

etc.

<u>Lemma</u> : $(B_\alpha)_{\alpha \in I}$, $B_\alpha \subseteq B$.

then

1) $f^{-1}\left(\bigcup_\alpha B_\alpha\right) = \bigcup_\alpha f^{-1} B_\alpha$

2) $f^{-1}\left(\bigcap_\alpha B_\alpha\right) = \bigcap_\alpha f^{-1} B_\alpha$

3) $f^{-1}(B^c) = (f^{-1}B)^c$

$\ulcorner$ HW. $\lrcorner$

Special case : $I = \{1,2\}$.

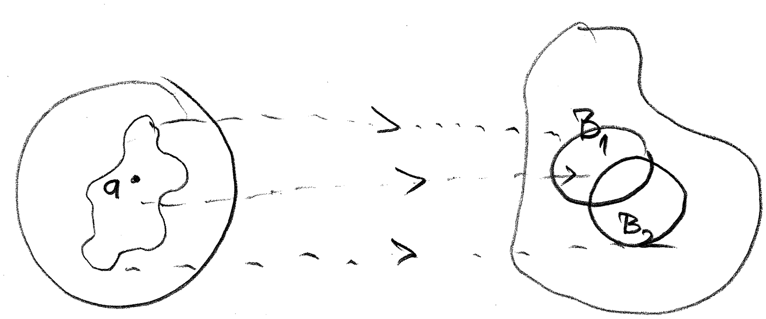
1) $f^{-1}(B_1 \cup B_2) = f^{-1}B_1 \cup f^{-1}B_2$

$\ulcorner$ Let $a \in f^{-1}(B_1 \cup B_2) \Longleftrightarrow$

$f(a) \in B_1 \cup B_2 \Leftrightarrow \underbrace{f(a) \in B_1}_{\Leftrightarrow\, a \in f^{-1}B_1}$ OR $\underbrace{f(a) \in B_2}_{a \in f^{-1}B_2}$

$\Longleftrightarrow a \in f^{-1}B_1 \cup f^{-1}B_2$ $\lrcorner$

etc.



Def $f : A \to B$ is called

(1) <u>injective</u> if $f(a) = f(a') \Rightarrow a = a'$
(one to one) (distinct elements have distinct values)

(2) <u>surjective</u>: if $f(A) = B$, $\iff$
(onto)
$\forall\, b \in B \;\exists\, a$ s.t. $f(a) = b$.

(3) <u>bijective</u> if <u>injective</u> + <u>surjective</u>.

In this case each $a \in A$ corresponds to exactly one $b \in B$ (and vice versa) and we can define the <u>inverse</u> f^{-1}

$$f^{-1} : B \longrightarrow A$$
$$b \longmapsto \text{the unique } a \in A \text{ with } f(a) = b$$

Then $f^{-1} f(a) = a \quad \forall a \in A.$

$f f^{-1}(b) = b \quad \forall b \in B$

EX

① $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ is <u>not injective</u>

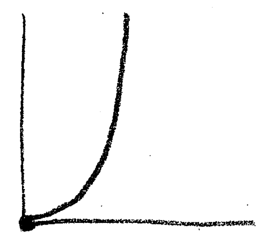
(since $f(-1) = f(1) = 1$ and $-1 \neq 1$)

is <u>not surjective</u> since

$\nexists\, x \in \mathbb{R}$ st $f(x) = x^2 = -1$,

(and $-1 \in \mathbb{R}$).

② $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $x \longmapsto x^2$ is <u>bijective</u>



and the inverse map is given

$f^{-1} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $x \longmapsto \sqrt{x}$.

③ $D : \mathcal{P}(\mathbb{R}) \longrightarrow \mathcal{P}(\mathbb{R})$ ← polynomials

$p(x) \longmapsto p'(x)$ is <u>surjective</u>

⌈ Let $p \in \mathcal{P}(\mathbb{R})$. Must find $q \in \mathcal{P}(\mathbb{R})$ with $Dq = p$

if $p = \sum_{k=0..n} a_k x^k$ and we set $q = \sum_{k=1}^{n+1} \frac{a_{k-1}}{k} x^k$

$\Rightarrow Dq = \sum_{k=1..n+1} a_{k-1} x^{k-1} = \sum_{k=0..n} a_k x^k = p$ qed ⌋

D is <u>not injective</u> since

$$D(p_1) = 0 = D(p_2)$$

where $p_1(x) \equiv 1$, $p_2(x) \equiv 2$ (constant polynomials)

but $p_1 \neq p_2$.

<u>Composition of maps</u>

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, dashed, "h = g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

A, B, C sets, f, g maps. Then

$$h : A \to C$$
$$a \mapsto g(f(a)) =: h(a)$$

is called the <u>composition</u> of f, g and is denoted by $\boxed{g \circ f.}$

Size (cardinality) of sets.

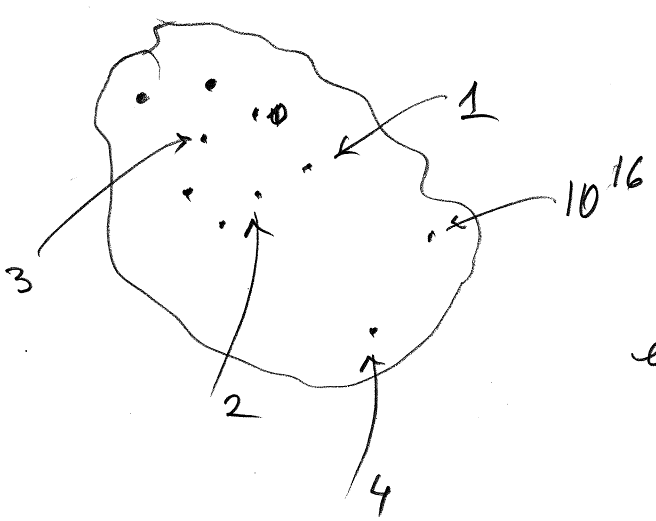
$|A| :=$ "number of elements of A"

- well-defined for finite sets

if A infinite, $|A| = \infty$.

Def A is called countably infinite if we can enumerate all its elements

i.e. if $\exists\ \varphi : \mathbb{N} \to A$ bijective.

$A =$ 

etc.

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Def $|A| = |B|$ (A, B have the same size)

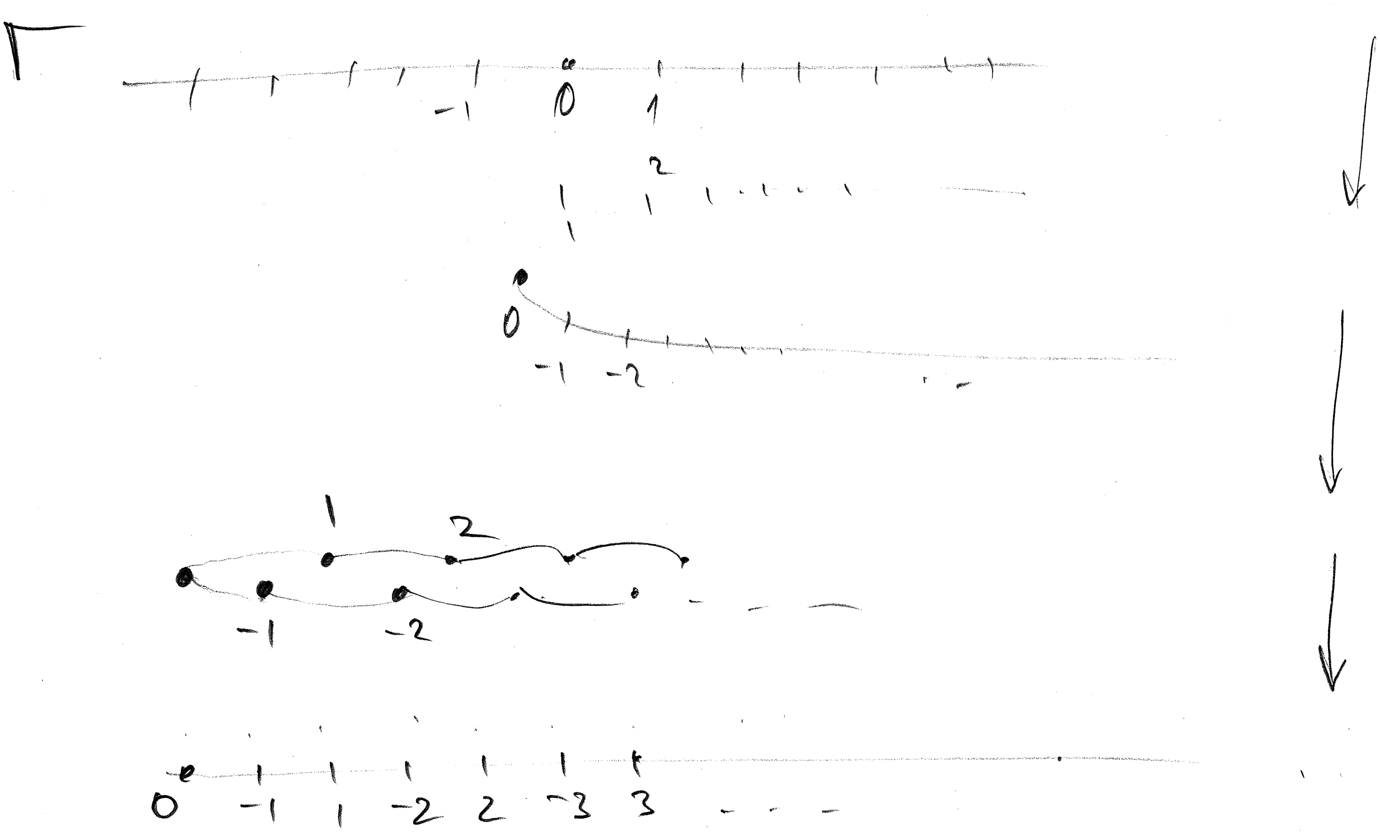
$\iff \exists\ \varphi : A \to B$ bijective.

So A is countably infinite

$\iff$ has the cardinality of $\mathbb{N}$.

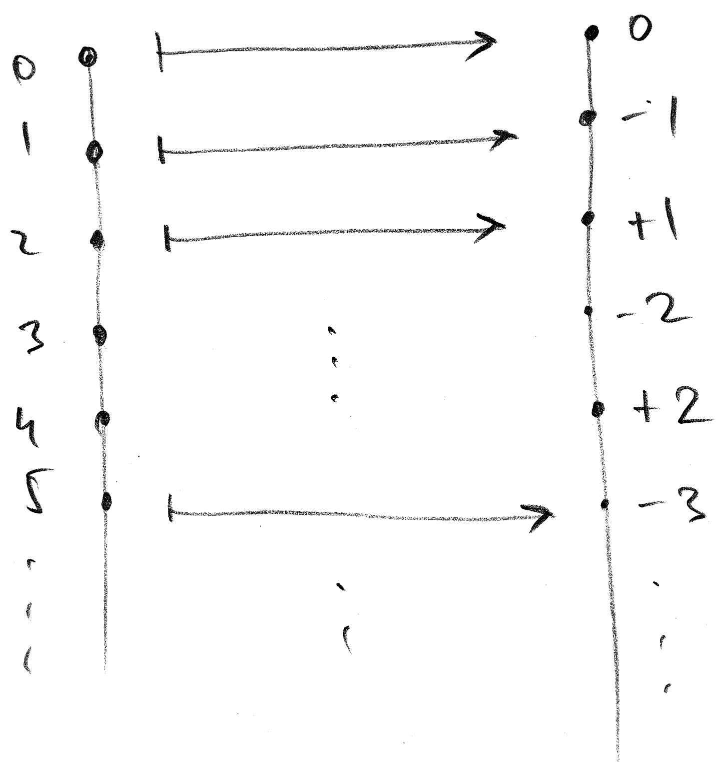
EX is $|\mathbb{N}| \overset{?}{=} |\mathbb{Z}|$?

A: yes



and the corresponding bijection φ (called the "enumeration") is

$$\varphi : \mathbb{N} \longrightarrow \mathbb{Z}$$



etc.

an explicite formula:

$$\varphi(k) = (-1)^k \left[\frac{k+1}{2} \right]$$

integer part of $\frac{k+1}{2}$

EX. $|\mathbb{Q}| = ?$

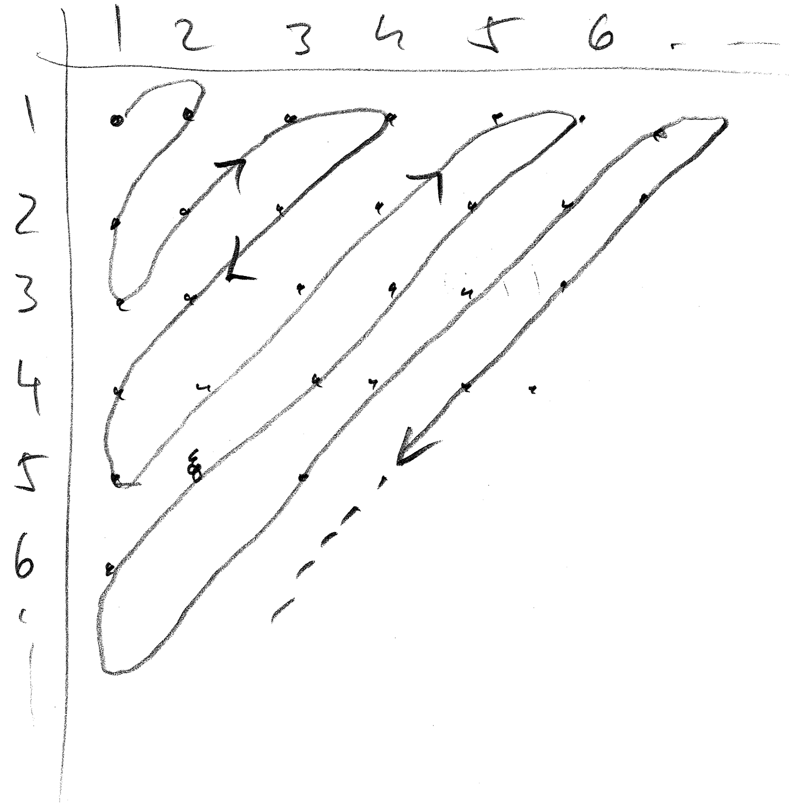
Claim $|\mathbb{Q}| = |\mathbb{N}|$.

$q \in \mathbb{Q} \Rightarrow q = \frac{k}{n}$ ← relative prime (fraction is simplified)

$n \downarrow$ \ $k \rightarrow$	1	2	3	4	5	
1	1	2	3	4	5	6
2	$\frac{1}{2}$	$\cancel{\frac{2}{2}}$	$\frac{3}{2}$	$\cancel{4/2}$	$\frac{5}{2}$	
3	$\frac{1}{3}$	$\frac{2}{3}$	$\cancel{3/3}$	$4/3$		
4	$\frac{1}{4}$	$\cancel{\frac{2}{4}}$	$\frac{3}{4}$			
5	$\frac{1}{5}$	$\frac{2}{5}$				

$\otimes$ means not rel. prime.

How to enumerate them?



!

so the enumeration is:

$$1,\ 2,\ \tfrac{1}{2},\ \tfrac{1}{3},\ 3,\ 4,\ \tfrac{3}{2},\ \tfrac{2}{3},\ \tfrac{1}{4},\ \tfrac{1}{5},\ 5,\ 6 \ldots$$

first second third ... etc.

Q: Is every infinite set countable?

A: No.

Thm (Cantor) Let A be a set and define the **power set** $\mathcal{P}(A)$ as

$$\mathcal{P}(A) := \{ B \mid B \subseteq A \}$$

the set of all subsets of A.

Then $|\mathcal{P}(A)| \gneq |A|$.

⌜no proof⌟ (easy...).

In particular $|\mathcal{P}(\mathbb{N})| > |\mathbb{N}|$

"uncountable"

Fact: $\mathbb{R}$ is uncountable.

if $|A| = n \in \mathbb{N}$ i.e. A is finite

then $|\mathcal{P}(A)| = 2^n$.

⌜

a_1	a_2	a_3	$\dots$	a_n
0	1	1	0 1 $\dots$	1

this sequence corresponds to the subset $\{a_2, a_3, a_5 \dots, a_n\}$

Indeed there are as many subsets as 0-1 sequences of length n

$\Rightarrow 2^n$

⌟