Sets

Natural numbers are represented by the set:

$$\mathbb{N} = \{0, 1, 2, ...\}$$

Integers are represented by the set:

$$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$$

Rational numbers are represented by the set:

$$\mathbb{Q} = \left\{ \frac{k}{n} \mid k \in \mathbb{Z}, n \in \mathbb{N}, n \neq 0 \right\}$$

Real numbers are represented by the set $\mathbb{R}$.

Complex numbers are represented by the set:

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$$

where i is the *imaginary unit*, characterized by $i^2 = -1$.

d-dimensional space over $\mathbb{R}$ is:

$$\mathbb{R}^d = \{(a_1, ..., a_d) \mid a_i \in \mathbb{R}\}$$

The space of infinite sequences of real numbers is:

$$\mathbb{R}^\infty = \{(a_0, a_1, a_2, ...) \mid a_i \in \mathbb{R}\}$$

Set Operations

Given two sets A and B:

Union: The union of A and B, denoted $A \cup B$, is the set of all elements that are in A, in B, or in both.

$$A \cup B = \{c \mid c \in A \text{ or } c \in B\}$$

Intersection: The intersection of A and B, denoted $A \cap B$, is the set of all elements that are in both A and B.

$$A \cap B = \{c \mid c \in A \text{ and } c \in B\}$$

Difference: The difference of A and B, denoted $A \setminus B$, is the set of all elements that are in A but not in B.

$$A \setminus B = \{c \in A \mid c \notin B\}$$

Symmetric Difference: The symmetric difference of A and B, denoted $A \Delta B$, can be expressed in the following ways:

$$\begin{aligned} A \Delta B &= A \setminus B \cup B \setminus A \\ &= (A \cup B) \setminus (A \cap B) \end{aligned}$$

Indexed Families of Sets

Let $(A_\alpha)_{\alpha \in I}$ be an indexed family of sets, with index set I.

Union: The union of the family, denoted $\bigcup_{\alpha \in I} A_\alpha$, is:

$$\bigcup_{\alpha \in I} A_\alpha = \{\alpha \mid a \in A_\alpha \text{ for some } \alpha \in I\}$$

This can also be expressed as:

$$\bigcup_{\alpha \in I} A_\alpha = \{a \mid \exists \alpha \in I \text{ such that } a \in A_\alpha\}$$

Intersection: The intersection of the family, denoted $\bigcap_{\alpha \in I} A_\alpha$, is:

$$\bigcap_{\alpha \in I} A_\alpha = \{a \mid a \in A_\alpha \text{ for every } \alpha \in I\}$$

This can also be expressed as:

$$\bigcap_{\alpha \in I} A_\alpha = \{a \mid \forall \alpha \in I, a \in A_\alpha\}$$

Maps

A **map** (also called a *function*, *mapping*, or *transformation*) is a rule that assigns a unique output to each input.

Let A and B be sets. A map f from A to B, denoted $f : A \to B$, is an assignment such that for every element $a \in A$, there exists a unique element $b \in B$, called the *value of f at a* and denoted by $f(a)$.

Note: While f assigns a value to every element in A, there may exist elements in B that are not mapped to by any element in A.

Domain and Target Space: Given a map $f : A \to B$:

- A is called the *domain* of f.
- B is called the *target space* of f.

Restriction: If $A' \subseteq A$, then the *restriction* of f to A', denoted by $f' = f|_{A'}$, is the map $f' : A' \to B$ defined by $f'(a) = f(a)$ for all $a \in A'$.

Examples:

1. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \sin(x)$. This map is often written as $f = \sin$.
2. $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $x \mapsto \frac{1}{x}$.
3. $g : \mathbb{R} \to \mathbb{R}$, $x \mapsto \begin{cases} x^2 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$
4. $f : V \to \{\text{set of subspaces of } V\}$, $v \mapsto \text{span}(v)$, where V is a vector space.
5. $D : P(\mathbb{R}) \to P(\mathbb{R})$, $p(x) \mapsto p'(x)$, where $P(\mathbb{R})$ is the set of polynomials with real coefficients and $p'(x)$ is the derivative of $p(x)$.