Lecture Notes on Sets and Maps

Sets

Sets of Numbers

$$\begin{aligned}
\mathbb{N} &= \{0, 1, 2, ...\} && \textit{natural numbers} \\
\mathbb{Z} &= \{..., -2, -1, 0, 1, 2, ...\} && \textit{integers} \\
\mathbb{Q} &= \left\{\frac{k}{n} | k, n \in \mathbb{N}, n \neq 0\right\} && \textit{rational numbers} \\
\mathbb{R} &= && \textit{real numbers} \\
\mathbb{C} &= \{a + ib | a, b \in \mathbb{R}\} && \textit{complex numbers} \\
&\text{where } i \text{ is the } \textit{imaginary unit} \text{ characterized by } i^2 = -1. \\
\mathbb{R}^d &= \{(a_1, ..., a_d) | a_i \in \mathbb{R}\} && \textit{d-dimensional space} \\
\mathbb{R}^\infty &= \{(a_0, a_1, a_2, ...) | a_i \in \mathbb{R}\} && \textit{space of infinite sequences of reals}
\end{aligned}$$

Set Operations

Let A, B be sets. We can perform operations on these sets:

- **Union ($A \cup B$)**: $\{c | c \in A \text{ or } c \in B\}$
- **Intersection ($A \cap B$)**: $\{c | c \in A \text{ and } c \in B\}$
- **Difference ($A \backslash B$)**: $\{c | c \in A \text{ and } c \notin B\}$
- **Symmetric Difference ($A \Delta B$)**: $(A \cup B) \backslash (A \cap B)$

Indexed Families of Sets

Let $(A_\alpha)_{\alpha \in I}$ be an indexed family of sets (with the index set being I).

Examples

1. $I = \mathbb{N}$, $A_\alpha = \{\alpha, \alpha + 1\}$ 2. $I = \mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$, $A_\alpha = [\alpha, \infty)$

Union and Intersection of Indexed Families

$$\bigcup_{\alpha \in I} A_\alpha = \{a | a \in A_\alpha \text{ for some } \alpha \in I\}$$

$$\bigcap_{\alpha \in I} A_\alpha = \{a | a \in A_\alpha \text{ for every } \alpha \in I\}$$

Maps

* **Map** = **function** = **mapping** = **transformation**.

Let A, B be sets. A map $f : A \to B$ is an assignment that assigns to each element $a \in A$ a unique element $b \in B$, called the value of f at a, denoted by $f(a)$.

The set A is called the **domain** of f, and the set B is called the **target space** of f.

Restriction of a Map

If $A' \subseteq A$, then the map $f' : A' \to B$ defined by $f'(a) = f(a)$ for all $a \in A'$ is called the **restriction** of f to A' (denoted by $f' = f|_{A'}$).

Examples of Maps

- **$f : \mathbb{R} \to \mathbb{R}, x \mapsto sin(x)$** ($f$ is the sine function)
- **$f : \mathbb{R}^+ \to \mathbb{R}, x \mapsto \sqrt{x}$** ($f$ is the square root function)
- **$g : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} 1/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$** ($g$ is the reciprocal function with a special case at $x = 0$)
- **$f : V \to$ set of subspaces of V$, v \mapsto span(v)$** (f is the span function)
- **$D : P(\mathbb{R}) \to P(\mathbb{R}), p(x) \mapsto p'(x)$** ($D$ is the derivative operator for polynomials)