Lecture Notes: Sets, Operations, and Maps

1 Sets

$$\begin{aligned}
\mathbb{N} &= \{0, 1, 2, \ldots\} && \text{Natural numbers} \\
\mathbb{Z} &= \{\ldots, -2, -1, 0, 1, 2, \ldots\} && \text{Integers} \\
\mathbb{Q} &= \left\{\frac{k}{n} \,|\, k, n \in \mathbb{N}, n \neq 0\right\} && \text{Rational numbers} \\
\mathbb{R} &= && \text{Real numbers} \\
\mathbb{C} &= \{a + bi \,|\, a, b \in \mathbb{R}\} && \text{Complex numbers} \\
& && \text{where } i \text{ is the \textit{imaginary unit} characterized by } i^2 = -1. \\
\mathbb{R}^d &= \{(a_1, a_2, \ldots, a_d) \,|\, a_i \in \mathbb{R}\} && \text{d-dimensional space} \\
\mathbb{R}^\infty &= \{(a_0, a_1, a_2, \ldots) \,|\, a_i \in \mathbb{R}\} && \text{Space of infinite sequences of real numbers}
\end{aligned}$$

2 Set Operations

Let A, B be *sets*.

$$\begin{aligned}
A \cup B &= \{c \,|\, c \in A \text{ or } c \in B\} && \text{Union} \\
A \cap B &= \{c \,|\, c \in A \text{ and } c \in B\} && \text{Intersection} \\
A \backslash B &= \{c \,|\, c \in A \text{ and } c \notin B\} && \text{Difference} \\
A \Delta B &= A \cup B \backslash (A \cap B) && \text{Symmetric difference}
\end{aligned}$$

3 Indexed Families of Sets

Let $(A_\alpha)_{\alpha \in I}$ be an *indexed family of sets* (index set I).

$$\begin{aligned}
\bigcup_{\alpha \in I} A_\alpha &= \{a \,|\, a \in A_\alpha \text{ for some } \alpha \in I\} && \text{Union over the index set} \\
\bigcap_{\alpha \in I} A_\alpha &= \{a \,|\, a \in A_\alpha \text{ for every } \alpha \in I\} && \text{Intersection over the index set}
\end{aligned}$$

4 Maps (Functions)

A *map* or *function* is a rule that assigns to each element of a set A a unique element of a set B.

Let A, B be sets. A map f is an assignment: to each $a \in A$, f assigns a unique $b \in B$, which is called the *value of f at a* and is denoted by $f(a)$.

Note: f assigns to each $a \in A$ a value, but there may exist $b \in B$ that is not $f(a)$ for any $a \in A$.

- A is called the *domain* (of definition) of f. - B is called the *target space* of f.

If $A' \subset A$, then the map f' (0.15

$$f' : A' \to B$$
$$a \mapsto f'(a) = f(a)$$

is called the *restriction of f to A'* (denoted by $f' = f|_{A'}$).

5 Examples of Maps

1. $f : \mathbb{R} \to \mathbb{R} \qquad x \mapsto sin(x) \qquad f = sin$
2. $f : \mathbb{R}\backslash\{0\} \to \mathbb{R} \qquad x \mapsto \frac{1}{x}$
3. $g : \mathbb{R} \to \mathbb{R}$

$$x \mapsto \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

4. $f : V \to$ set of subspaces of $V \qquad v \mapsto span(v)$
5. $D : P(\mathbb{R}) \to P(\mathbb{R}) \qquad p(x) \mapsto p'(x) = p' \qquad$ *derivative*