Set Theory and Maps

1 Sets

1.1 Basic Sets

* **Natural Numbers ($\mathbb{N}$):**

$$\mathbb{N} = \{0, 1, 2, ...\}$$

* **Integers ($\mathbb{Z}$):**

$$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$$

* **Rational Numbers ($\mathbb{Q}$):**

$$\mathbb{Q} = \left\{ \frac{k}{n} \mid k, n \in \mathbb{N}, n \neq 0 \right\}$$

* **Real Numbers ($\mathbb{R}$):** * **Complex Numbers ($\mathbb{C}$):**

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$$

- **Imaginary Unit (i):** characterized by $i^2 = -1$.

1.2 Spaces

* **d-dimensional Space ($\mathbb{R}^d$):**

$$\mathbb{R}^d = \{(a_1, ..., a_d) \mid a_i \in \mathbb{R}\}$$

* **Space of Infinite Sequences of Reals ($\mathbb{R}^\infty$):**

$$\mathbb{R}^\infty = \{(a_0, a_1, a_2, ...) \mid a_i \in \mathbb{R}\}$$

2 Set Operations

Let A, B be *sets*.
* **Union ($A \cup B$):**

$$A \cup B = \{c \mid c \in A \text{ or } c \in B\}$$

* **Intersection ($A \cap B$):**

$$A \cap B = \{c \mid c \in A \text{ and } c \in B\}$$

* **Difference ($A \backslash B$):**

$$A \backslash B = \{c \mid c \in A \text{ and } c \notin B\}$$

* **Symmetric Difference ($A \Delta B$):**

$$A \Delta B = (A \cup B) \backslash (A \cap B)$$

3 Indexed Families of Sets

Let $(A_\alpha)_{\alpha \in I}$ be an *indexed family of sets* (index set I).

- **Union:**
$$\bigcup_{\alpha \in I} A_\alpha = \{a \mid a \in A_\alpha \text{ for some } \alpha \in I\}$$
- **Intersection:**
$$\bigcap_{\alpha \in I} A_\alpha = \{a \mid a \in A_\alpha \text{ for every } \alpha \in I\}$$

4 Maps

A *map* (or *function* or *mapping* or *transformation*) is a rule that assigns to each element of a set A (the *domain*) a unique element of a set B (the *target space*).

- **Notation:** $f : A \to B$
- **Value:** $f(a)$ is the element in B assigned to $a \in A$.
- **Restriction:** If $A' \subseteq A$, then $f' : A' \to B$ where $f'(a) = f(a)$ is called the *restriction* of f to A', denoted $f' = f|_{A'}$.

4.1 Examples

1. $**f : \mathbb{R} \to \mathbb{R}, x \mapsto \sin(x)**$ (Sine function)
2. $**f : \mathbb{R}\backslash\{0\} \to \mathbb{R}, x \mapsto \frac{1}{x}**$ (Reciprocal function)
3. $**g : \mathbb{R} \to \mathbb{R}, x \mapsto 0$ if $x = 0$ and $x \mapsto \frac{1}{x}$ if $x \neq 0**$ (Piecewise function)
4. $**f : V \to$ set of subspaces of V, $v \mapsto span(v)**$ (Span of a vector)
5. $**D : P(\mathbb{R}) \to P(\mathbb{R}), p(x) \mapsto p'(x)**$ (Derivative)