Set Theory and Maps

Sets

$$\begin{aligned}
\mathbb{N} &= \{0, 1, 2, ...\} \qquad \textit{natural numbers} \\
\mathbb{Z} &= \{..., -2, -1, 0, 1, 2, ...\} \qquad \textit{integers} \\
\mathbb{Q} &= \left\{\frac{k}{n} \mid k, n \in \mathbb{N}, n \neq 0\right\} \qquad \textit{rational numbers} \\
\mathbb{R} &= \qquad \textit{real numbers} \\
\mathbb{C} &= \{a + ib \mid a, b \in \mathbb{R}\} \qquad \textit{complex numbers} \\
& i \text{ is the } \textit{imaginary unit}\text{, characterized by } i^2 = -1. \\
\mathbb{R}^d &= \{(a_1, ..., a_d) \mid a_i \in \mathbb{R}\} \qquad \textit{d-dimensional space} \\
\mathbb{R}^\infty &= \{(a_0, a_1, a_2, ...) \mid a_i \in \mathbb{R}\} \qquad \textit{space of infinite sequences of reals}
\end{aligned}$$

Set Operations

$$\begin{aligned}
A \cup B &= \{c \mid c \in A \text{ or } c \in B\} \qquad \textit{union} \\
A \cap B &= \{c \mid c \in A \text{ and } c \in B\} \qquad \textit{intersection} \\
A \setminus B &= \{c \in A \mid c \notin B\} \qquad \textit{difference} \\
A \Delta B &= (A \cup B) \setminus (A \cap B) \qquad \textit{symmetric difference}
\end{aligned}$$

Indexed Families of Sets

Let $(A_\alpha)_{\alpha \in I}$ be an *indexed family of sets* (index set I).

Examples:

1. $I = \mathbb{N}$, $A_\alpha = \{\alpha, \alpha + 1\}$ 2. $I = \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$, $A_\alpha = [\alpha, \infty)$

Operations on indexed families:

$$\begin{aligned}
\bigcup_{\alpha \in I} A_\alpha &= \{a \mid a \in A_\alpha \text{ for some } \alpha \in I\} \qquad \textit{union} \\
\bigcap_{\alpha \in I} A_\alpha &= \{a \mid a \in A_\alpha \text{ for every } \alpha \in I\} \qquad \textit{intersection}
\end{aligned}$$

Maps

Definition: A *map* $f : A \to B$ is an assignment that assigns to each element $a \in A$ a unique element $b \in B$, denoted by $f(a)$.

* *Domain*: A * *Target Space*: B

Restriction: If $A' \subseteq A$, then the map $f' : A' \to B$, $a' \to f'(a') = f(a')$ is called the *restriction of f to A'* (denoted by $f' = f|_{A'}$).

Examples:

1. $f : \mathbb{R} \to \mathbb{R}$, $x \to \sin(x)$ 2. $f : \mathbb{R}^+ \to \mathbb{R}$, $x \to \sqrt{x}$ 3. $g : \mathbb{R} \to \mathbb{R}$, $x \to \frac{1}{x}$ if $x \neq 0$, $x \to 0$ if $x = 0$ 4. $f : V \to$ set of subspaces of V, $v \to \text{span}(v)$ (where V is a vector space) 5. $D : P(\mathbb{R}) \to P(\mathbb{R})$, $p(x) \to p'(x)$ (the derivative)