Lecture Notes on Sets and Maps

Sets

$$
\begin{aligned}
\mathbb{N} &= \{0, 1, 2, \dots\} && \text{natural numbers} \\
\mathbb{Z} &= \{-2, -1, 0, 1, 2, \dots\} && \text{integers} \\
\mathbb{Q} &= \left\{\frac{k}{n} \mid k, n \in \mathbb{N}, n \neq 0\right\} && \text{rationals} \\
\mathbb{R} &= \mathbb{R} && \text{real numbers} \\
\mathbb{C} &= \{a + bi \mid a, b \in \mathbb{R}\} && \text{complex numbers} \\
& \quad \text{where } i \text{ is the imaginary unit, characterized by } i^2 = -1. \\
\mathbb{R}^d &= \{(a_1, \dots, a_d) \mid a_i \in \mathbb{R}\} && \text{d-dimensional space} \\
\mathbb{R}^\infty &= \{(a_0, a_1, a_2, \dots) \mid a_i \in \mathbb{R}\} && \text{space of infinite sequences of reals}
\end{aligned}
$$

Set Operations

Let A, B be sets.

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\begin{aligned}
A \cup B &= \{c \mid c \in A \text{ or } c \in B\} && \textit{union} \\
A \cap B &= \{c \mid c \in A \text{ and } c \in B\} && \textit{intersection} \\
A \setminus B &= \{c \in A \mid c \notin B\} && \textit{difference} \\
A \triangle B &= A \cup B \setminus (A \cap B) && \textit{symmetric difference}
\end{aligned}
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Indexed Families of Sets

Let $(A_\alpha)_{\alpha \in I}$ be an *indexed family of sets* (index set I).

1. Let $I = \mathbb{N}$ and $A_\alpha = \{\alpha, \alpha + 1\}$.
2. Let $I = \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ and $A_\alpha = [\alpha, \infty)$.

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\begin{aligned}
\bigcup_{\alpha \in I} A_\alpha &= \{a \mid a \in A_\alpha \text{ for some } \alpha \in I\} && \textit{union} \\
&= \{a \mid \exists \alpha \in I \text{ such that } a \in A_\alpha\} \\
\bigcap_{\alpha \in I} A_\alpha &= \{a \mid a \in A_\alpha \text{ for every } \alpha \in I\} && \textit{intersection} \\
&= \{a \mid \forall \alpha \in I, a \in A_\alpha\}
\end{aligned}
$$

Maps

Map, *function*, *mapping*, and *transformation* are all synonyms. Let A, B be sets (+). A *map* f:

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\begin{aligned}
f : A &\to B \\
a &\mapsto f(a)
\end{aligned}
$$

is an *assignment*: to $\forall a \in A$, f assigns a unique $b \in B$, which is called the *value* of f at a and is denoted by $f(a)$.
Note: f assigns to $\forall a \in A$ a value, but there may exist $b \in B$ st $f(a_1) \neq b$, $\forall a_1 \in A$.

A is called the *domain* (of definition) of f. B is the *target space* of f.

If $A' \subseteq A$, then the map f':

$$\begin{aligned} f' : A' &\to B \\ a &\mapsto f'(a) = f(a) \end{aligned}$$

is called the *restriction* of f to A' (denoted by $f' = f|_{A'}$).

Examples:

1. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \sin(x)$. Here, $f = \sin$ is a function.
2. $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $x \mapsto \frac{1}{x}$.
3. $g : \mathbb{R} \to \mathbb{R}$, $x \mapsto \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.
4. $f : V \to \{\text{set of subspaces of } V\}$, $v \mapsto \text{span}(v)$ where V is a vector space.
5. $D : P(\mathbb{R}) \to P(\mathbb{R})$, $p(x) \mapsto p'(x)$ where $p'(x)$ is the derivative of $p(x)$.