Lecture Notes on Sets and Maps

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1 Sets

We will be working with the following fundamental sets:

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\begin{aligned}
&\mathbb{N} = \{0, 1, 2, ...\} && \text{*natural numbers*} \\
&\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\} && \text{*integers*} \\
&\mathbb{Q} = \left\{ \frac{k}{n} \mid k, n \in \mathbb{N}, n \neq 0 \right\} && \text{*rationals*} \\
&\mathbb{R} = && \text{*real numbers*} \\
&\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\} && \text{*complex numbers*}
\end{aligned}
$$

where i is the *imaginary unit* characterized by $i^2 = -1$.

1.1 Operations on Sets

Given two sets A and B, we can perform the following operations:

* **Union ($A \cup B$):** $\{c \mid c \in A \text{ or } c \in B\}$ - The set containing all elements that are in either A or B. * **Intersection ($A \cap B$):** $\{c \mid c \in A \text{ and } c \in B\}$ - The set containing only the elements that are in both A and B. * **Difference ($A \setminus B$):** $\{c \in A \mid c \notin B\}$ - The set containing all elements that are in A but not in B. * **Symmetric Difference ($A \Delta B$):** $(A \cup B) \setminus (A \cap B)$ - The set containing all elements that are in either A or B, but not in both.

1.2 Indexed Families of Sets

Let $(A_\alpha)_{\alpha \in I}$ be an *indexed family* of sets, where I is the *index set*. This refers to a collection of sets where each set is associated with an element from the index set.

Here are two examples:

1. $I = \mathbb{N}$: The index set is the set of natural numbers, and each set A_α is defined as $\{\alpha, \alpha + 1\}$.

2. $I = \mathbb{R}^+$: The index set is the set of positive real numbers, and each set A_α is defined as $[\alpha, \infty)$.

We can define the following:

* **Union of an indexed family:** $\bigcup_{\alpha \in I} A_\alpha = \{a \mid a \in A_\alpha \text{ for some } \alpha \in I\}$ * **Intersection of an indexed family:** $\bigcap_{\alpha \in I} A_\alpha = \{a \mid a \in A_\alpha \text{ for every } \alpha \in I\}$

2 Maps

A *map* (or *function*) f from a set A to a set B (denoted as $f : A \to B$) assigns a unique element in B to each element in A.

* The *domain* of f is the set A. * The *target space* of f is the set B.

If A' is a subset of A, the *restriction* of f to A' is a map $f' : A' \to B$ such that $f'(a) = f(a)$ for all $a \in A'$.

2.1 Examples of Maps

Here are some examples of maps:

1. $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto sin(x)$ - the sine function.
2. $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $x \mapsto \frac{1}{x}$.
3. $g : \mathbb{R} \to \mathbb{R}$, $x \mapsto \begin{cases} \frac{1}{x} & x \neq 0 \\ 2 & x = 0 \end{cases}$.
4. $f : V \to$ set of subspaces of V, $v \mapsto span(v)$.
5. $D : P(\mathbb{R}) \to P(\mathbb{R})$, $p(x) \mapsto p'(x)$ - the *derivative* operator.