Sets and Maps

Sets

$$
\begin{aligned}
\mathbb{N} &= \{0, 1, 2, \dots\} \qquad \text{*natural numbers*} \\
\mathbb{Z} &= \{\dots, -2, -1, 0, 1, 2, \dots\} \qquad \text{*integers*} \\
\mathbb{Q} &= \left\{ \frac{k}{n} \mid k, n \in \mathbb{N}, n \neq 0 \right\} \qquad \text{*rationals*} \\
\mathbb{R} &= \qquad \text{*real numbers*} \\
\mathbb{C} &= \{a + ib \mid a, b \in \mathbb{R}\} \qquad \text{*complex numbers*} \\
& \quad i = \ \text{*imaginary unit*} \\
& \quad \ \text{characterized by } i^2 = -1. \\
\mathbb{R}^d &= \{(a_1, \dots, a_d) \mid a_i \in \mathbb{R}\} \qquad \text{*d-dimensional space*} \\
\mathbb{R}^\infty &= \{(a_0, a_1, a_2, \dots) \mid a_i \in \mathbb{R}\} \qquad \text{*space of infinite sequences of reals*}
\end{aligned}
$$

Operations on Sets

$$
\begin{aligned}
A \cup B &= \{c \mid c \in A \text{ or } c \in B\} \qquad \text{*union*} \\
A \cap B &= \{c \mid c \in A \text{ and } c \in B\} \qquad \text{*intersection*} \\
A \backslash B &= \{c \in A \mid c \notin B\} \qquad \text{*difference*} \\
A \triangle B &= A \backslash B \cup B \backslash A \\
&= (A \cup B) \cap (A \cap B) \qquad \text{*symmetric difference*}
\end{aligned}
$$

Indexed Family of Sets

Let $(A_\alpha)_{\alpha\in I}$ be an *indexed family of sets* (index set I).

Examples:

1. $I = \mathbb{N}$,
   $A_\alpha := \{\alpha, \alpha+1\}$
2. $I = \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$,
   $A_\alpha := [\alpha, \infty)$

Union:

$$\begin{aligned}\bigcup_{\alpha\in I} A_\alpha &= \{a \mid a \in A_\alpha \text{ for some } \alpha \in I\} \\ &= \{a \mid \exists \alpha \in I \text{ such that } a \in A_\alpha\}\end{aligned}$$

Intersection:

$$\begin{aligned}\bigcap_{\alpha\in I} A_\alpha &= \{a \mid a \in A_\alpha \text{ for every } \alpha \in I\} \\ &= \{a \mid \forall \alpha \in I, a \in A_\alpha\}\end{aligned}$$

Maps

Maps are also called *functions*, *mappings*, or *transformations*.

Let A, B be sets. A *map* f

$$f : A \to B$$

$$a \mapsto f(a)$$

is an *assignment*: to $\forall a \in A$, f assigns a unique $b \in B$ which is called the *value of f at a* and is denoted by $f(a)$.

Note: f assigns to $\forall a \in A$ a value, but there may exist $b \in B$ s.t. $f(a_1) = f(a_2)$ $\forall a_1, a_2 \in A$.

A is called the *domain* of f.

B is called the *target space* of f.

Restriction

If $A' \subset A$, then the map f'

$$f' : A' \to B$$

$$a \mapsto f'(a) = f(a)$$

is called the *restriction* of f to A' (denoted by $f' = f|_{A'}$).

Examples:

1. $f : \mathbb{R} \to \mathbb{R}$
$$x \mapsto \sin(x)$$
   $\to f = \sin$
2. $f : \mathbb{R} \backslash \{0\} \to \mathbb{R}$
$$x \mapsto \frac{1}{x}$$
3. $g : \mathbb{R} \to \mathbb{R}$
$$x \mapsto \begin{cases} \frac{1}{x} & x \neq 0 \\ 2 & x = 0 \end{cases}$$
4. $f : V \to$ set of subspaces of $V = \mathcal{S}$
$$v \mapsto \text{span}(v)$$
5. $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$
$$p(x) \mapsto p'(x) = p'$$
      derivative