Sets and Maps

Sets

$$\begin{aligned}
\mathbb{N} &= \{0, 1, 2, \dots\} \qquad \textit{natural numbers} \\
\mathbb{Z} &= \{-2, -1, 0, 1, 2, \dots\} \qquad \textit{integers} \\
\mathbb{Q} &= \left\{\frac{k}{n} \mid k, n \in \mathbb{N}, n \neq 0\right\} \qquad \textit{rationals} \\
\mathbb{R} &= \qquad \textit{real numbers} \\
\mathbb{C} &= \{a + ib \mid a, b \in \mathbb{R}\} \qquad \textit{complex numbers} \\
&\qquad i = \textit{imaginary unit, characterized by } i^2 = -1. \\
\mathbb{R}^d &= \{(a_1, a_2, \dots, a_d) \mid a_i \in \mathbb{R}\} \qquad \textit{d-dimensional space} \\
\mathbb{R}^\infty &= \{(a_0, a_1, a_2, \dots) \mid a_i \in \mathbb{R}\} \qquad \textit{space of infinite sequences of real numbers}
\end{aligned}$$

Set Operations

$$\begin{aligned}
A \cup B &= \{c \mid c \in A \text{ or } c \in B\} \qquad \textit{union} \\
A \cap B &= \{c \mid c \in A \text{ and } c \in B\} \qquad \textit{intersection} \\
A \setminus B &= \{c \in A \mid c \notin B\} \qquad \textit{difference} \\
A \Delta B &= (A \cup B) \setminus (A \cap B) \qquad \textit{symmetric difference}
\end{aligned}$$

Indexed Families of Sets

Let $(A_\alpha)_{\alpha \in I}$ be an *indexed family* of *sets* (*index set* I).

Examples:

1. $I = \mathbb{N}$, $A_\alpha := \{\alpha, \alpha + 1\}$
2. $I = \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$, $A_\alpha := [\alpha, \infty)$

Union of an Indexed Family:

$$\begin{aligned}
\bigcup_{\alpha \in I} A_\alpha &= \{a \mid a \in A_\alpha \text{ for some } \alpha \in I\} \\
&= \{a \mid \exists \alpha \in I \text{ such that } a \in A_\alpha\}
\end{aligned}$$

Intersection of an Indexed Family:

$$\begin{aligned}
\bigcap_{\alpha \in I} A_\alpha &= \{a \mid a \in A_\alpha \text{ for every } \alpha \in I\} \\
&= \{a \mid \forall \alpha \in I, a \in A_\alpha\}
\end{aligned}$$

Maps

Maps (also known as *functions* or *mappings*) are assignments that associate each element in a set A (*domain*) to a unique element in another set B (*target space*).

Notation: $f : A \to B$

$a \mapsto f(a)$ - indicates that element a from A is mapped to element $f(a)$ in B.

Value of a Map: $f(a)$ is called the *value* of the map f at element a.

Restriction of a Map: If A' is a subset of A, then the map f' from A' to B defined by $f'(a) = f(a)$ is called the *restriction* of f to A'.

Examples of Maps

1. *sin function*: $f : \mathbb{R} \to \mathbb{R}, x \mapsto \sin(x)$
2. *Power function*: $f : \mathbb{R}[2, 3] \to \mathbb{R}, x \mapsto x^y$
3. *Piecewise defined function*: $g : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$
4. *Span function (linear algebra)*: $f : V \to$ set of subspaces of $V, v \mapsto span(v)$
5. *Derivative operator (calculus)*: $D : P(\mathbb{R}) \to P(\mathbb{R}), p(x) \mapsto p'(x)$