Lecture Notes on Sets and Maps

1 Sets

The following are common sets in mathematics.

$$\begin{aligned}
\mathbb{N} &= \{0, 1, 2, \dots\} && \text{*natural numbers*}\\
\mathbb{Z} &= \{\dots, -2, -1, 0, 1, 2, \dots\} && \text{*integers*}\\
\mathbb{Q} &= \left\{\frac{k}{n} \mid k, n \in \mathbb{N}, n \neq 0\right\} && \text{*rationals*}\\
\mathbb{R} &= && \text{*real numbers*}\\
\mathbb{C} &= \{a + ib \mid a, b \in \mathbb{R}\} && \text{*complex numbers*}\\
&\quad \text{where *i* is the *imaginary unit* characterized by } i^2 = -1.\\
\mathbb{R}^d &= \{(a_1, \dots, a_d) \mid a_i \in \mathbb{R}\} && \text{*d-dimensional space*}\\
\mathbb{R}^\infty &= \{(a_0, a_1, a_2, \dots) \mid a_i \in \mathbb{R}\} && \text{*space of infinite sequences of reals*}
\end{aligned}$$

2 Set Operations

Let A, B be *sets*. The following are the common set operations:

$$\begin{aligned}
A \cup B &= \{c \mid c \in A \text{ or } c \in B\} && \text{*union*}\\
A \cap B &= \{c \mid c \in A \text{ and } c \in B\} && \text{*intersection*}\\
A \backslash B &= \{c \in A \mid c \notin B\} && \text{*difference*}\\
A \triangle B &= A \backslash B \cup B \backslash A && \text{*symmetric difference*}
\end{aligned}$$

3 Indexed Families of Sets

Let $(A_\alpha)_{\alpha \in I}$ be an *indexed family of sets* (index set I).

- **Example 1:** Let $I = \mathbb{N}$ and $A_\alpha := \{\alpha, \alpha + 1\}$.
- **Example 2:** Let $I = \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ and $A_\alpha := [\alpha, \infty)$.

The following operations are useful when working with indexed families of sets:

$$\begin{aligned}
\bigcup_{\alpha \in I} A_\alpha &= \{a \mid a \in A_\alpha \text{ for some } \alpha \in I\}\\
&= \{a \mid \exists \alpha \in I \text{ such that } a \in A_\alpha\} && \text{*union*}\\
\bigcap_{\alpha \in I} A_\alpha &= \{a \mid a \in A_\alpha \text{ for every } \alpha \in I\} && \text{*intersection*}
\end{aligned}$$

4 Maps

A *map* (or *function* or *mapping* or *transformation*) is an assignment from one set to another.

- Let A, B be sets. A map f is an assignment:

$$\begin{aligned} f : A &\to B \\ a &\mapsto f(a). \end{aligned}$$

  For each $a \in A$, f assigns a unique $b \in B$ which is called the *value* of f at a and is denoted by $f(a)$.

- *Note:* f assigns to each $a \in A$ a value, but there may exist $b \in B$ that is not equal to $f(a)$ for any $a \in A$.

- A is called the *domain* (of definition) of f.

- B is called the *target space* of f.

- **Example 1:** Let $f : \mathbb{R} \to \mathbb{R}$ be the *sine function*:

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \sin(x) \end{aligned}$$

- **Example 2:** Let $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be the function

$$\begin{aligned} f : \mathbb{R}\backslash\{0\} &\to \mathbb{R} \\ x &\mapsto 1/x \end{aligned}$$

- **Example 3:** Let $g : \mathbb{R} \to \mathbb{R}$ be the function

$$\begin{aligned} g : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases} \end{aligned}$$

- **Example 4:** Let $f : V \to$ set of subspaces of $V = S$ be the function

$$\begin{aligned} f : V &\to S \\ v &\mapsto \operatorname{span}(v) \end{aligned}$$

  where V is a vector space.

- **Example 5:** Let $D : P(\mathbb{R}) \to P(\mathbb{R})$ be the *derivative function*

$$\begin{aligned} D : P(\mathbb{R}) &\to P(\mathbb{R}) \\ p(x) &\mapsto p'(x). \end{aligned}$$

5 Restriction

If $A' \subset A$, then the map f'

$$\begin{aligned} f' : A' &\to B \\ a &\mapsto f'(a) := f(a) \end{aligned}$$

is called the *restriction* of f to A' (denoted by $f' = f|_{A'}$).