# Sets and Maps

## Sets

$$
\begin{aligned}
\mathbb{N} &= \{0, 1, 2, \dots\} && \text{*natural numbers*} \\
\mathbb{Z} &= \{\dots, -2, -1, 0, 1, 2, \dots\} && \text{*integers*} \\
\mathbb{Q} &= \{\frac{k}{n} \mid k, n \in \mathbb{N}, n \neq 0\} && \text{*rationals*} \\
\mathbb{R} &= && \text{*real numbers*} \\
\mathbb{C} &= \{a + bi \mid a, b \in \mathbb{R}\} && \text{*complex numbers*} \\
& && \text{*imaginary unit*} \\
& && \text{*characterized by* } i^2 = -1. \\
\mathbb{R}^d &= \{(a_1, \dots, a_d) \mid a_i \in \mathbb{R}\} && \text{*d-dimensional space*} \\
\mathbb{R}^\infty &= \{(a_0, a_1, a_2, \dots) \mid a_i \in \mathbb{R}\} && \text{*space of infinite sequences of reals*}
\end{aligned}
$$

## Set Operations

Let $A$, $B$ be sets.

$$
\begin{aligned}
A \cup B &= \{c \mid c \in A \text{ or } c \in B\} && \text{*union*} \\
A \cap B &= \{c \mid c \in A \text{ and } c \in B\} && \text{*intersection*}
\end{aligned}
$$

$$
\begin{aligned}
A \setminus B &= \{c \in A \mid c \notin B\} && \text{*difference*} \\
A \Delta B &= A \cup B \setminus (A \cap B) && \text{*symmetric difference*}
\end{aligned}
$$

## Indexed Family of Sets

Let $(A_\alpha)_{\alpha \in I}$ be an *indexed family* of *sets* (*index set* $I$).

**Examples:**

1. $I = \mathbb{N}$ $A_\alpha := \{\alpha, \alpha + 1\}$
2. $I = \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ $A_\alpha := [\alpha, \infty)$

**Union:**

$$\bigcup_{\alpha \in I} A_\alpha = \{a \mid a \in A_\alpha \text{ for some } \alpha \in I\}$$
$$= \{a \mid \exists \alpha \in I \text{ such that } a \in A_\alpha\}$$

**Intersection:**

$$\bigcap_{\alpha \in I} A_\alpha = \{a \mid a \in A_\alpha \text{ for every } \alpha \in I\}$$
$$= \{a \mid \forall \alpha \in I, a \in A_\alpha\}$$

## Maps

**Definition**: Let $A$, $B$ be sets. A *map* $f$ is an *assignment* that assigns a unique element $b \in B$ to each element $a \in A$. This element $b$ is called the *value of $f$ at $a$* and is denoted by $f(a)$.

**Note:** $f$ assigns to each $a \in A$ a value, but there may exist $b \in B$ such that $f(a) \neq b$ for some $a \in A$.

$A$ is called the *domain* of $f$.
$B$ is called the *target space* of $f$.

If $A' \subseteq A$, then the *map* $f'$ (0.15

$$f' : A' \to B$$
$$a \mapsto f'(a) = f(a)$$

is called the *restriction* of $f$ to $A'$ (denoted by $f' = f|_{A'}$).
**Examples:**

1. $f : \mathbb{R} \to \mathbb{R}$ $x \mapsto \sin(x)$ $f = \sin$ *function*
2. $f : \mathbb{R}_{>0} \to \mathbb{R}$ $x \mapsto y^x$
3. $g : \mathbb{R} \to \mathbb{R}$ $x \mapsto \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$
4. $f : V \to$ set of subspaces of $V = S$ $v \mapsto \text{span}(v)$
5. $D : P(\mathbb{R}) \to P(\mathbb{R})$ $p(x) \mapsto p'(x) = p'$ *derivative*