ROOT LOCUS

CONSTRUCTION RULES

1. K= 0 points

   Starting points

   Poles of G(s)H(s)

   Let n be the number of finite poles

2. K = ∞ points

   Terminating points

   Zeros of G(s)H(s)

   Let m be the number of finite zeros

3. Number of branches

Greater of m, n

4. Asymptotes : The branches of root locus which tends to infinity

$$\theta = \frac{(2q+1)\Pi}{n-m} \quad q = 0,1,2\text{-----}|n-m-1|$$

5. Centroid (Intersection of asymptotes)

$$\sigma = \frac{\Sigma(\text{finite poles}) - \Sigma(\text{finite zeros})}{\text{number of finite poles (n)} - \text{number of finite zeros (m)}}$$

6. Root locus on real axis

A point on the real axis lies on the locus if the number of open-loop poles and zeros on the real axis to the right of this point is odd.

6. Root locus on imaginary axis

The point where the root locus intersects the imaginary and the value of k can be determined from Routh criteria

8. Break Points

points at which multiple roots of the characteristic equation

$$\frac{dk}{ds} = 0$$

Solve for s?

9. Angle of Departure (Complex poles)

The angle at which the root locus leaves from a complex pole

$\emptyset_d = 180^o$ - (sum of angles of vectors to a complex pole in question from other poles- sum of angles of vectors to a complex pole in question from other zeros)

10. Angle of Arrival (Complex zeros)

The angle at which the root locus enters to a complex zero

$\emptyset_a$ = 180° - (sum of angles of vectors to a complex zero in question from other zeros- sum of angles of vectors to a complex zero in question from other poles)

11. The root locus is symmetrical about the real axis

12. The open loop gain k at any point on the locus

$$k = \frac{products\ of\ phasor\ lengths\ from\ th\ point\ to\ open\ loop\ poles}{}$$

Problem 1

The open loop transfer function of a unity negative feed back system is given by $\frac{k}{s(s+1)(s+2)}$

Draw the root locus as the value of k varies from zero to Infinity

Solution

Construction Rules:

1. K = 0 points 0, -1, -2 n = 3
2. K = ∞ points ∞,∞,∞ m = 0
3. Number of branches = 3
4. Asymptotes = $\frac{(2q+1)\Pi}{n-m}$ q = 0,1,2

   = 60^0, 180^0, 300^0
5. Centroid (Intersection of asymptotes)

$$\sigma = \frac{(0-1-2)-(0)}{(3-0)} = -1$$

6. Root locus on real axis lies between

   0 and -1

   -2 and $-\infty$

7. Root locus on imaginary axis

   Characteristic equation $1 + G(s)H(s) = 0$

   $s^3+3s^2+2s+k = 0$

   Draw the Routh table

s^3	1	2
s^2	3	k
s^1	$\frac{6-k}{3}$	0
s^0	k	0

For Stability k should lie between 0 and 6

When k = 6, the root locus intersects the imaginary axis at $s = \pm j\sqrt{2}$ rad/sec

8. Break Points

$$\frac{dk}{ds} = 0$$

$$s^3+3s^2+2s+k = 0$$

$$k = -(s^3+3s^2+2s)$$

$$\frac{dk}{ds} = -(3s^2+6s+2) = 0$$

$$s = -0.42, -1.58$$

$$k_{s=-0.42} = 0.38$$

$$k_{s=-1.58} = -0.38$$

Therefore s=-0.42 is a valid break away point.

9. Root locus is symmetrical with real axis

10. Angle of Departure

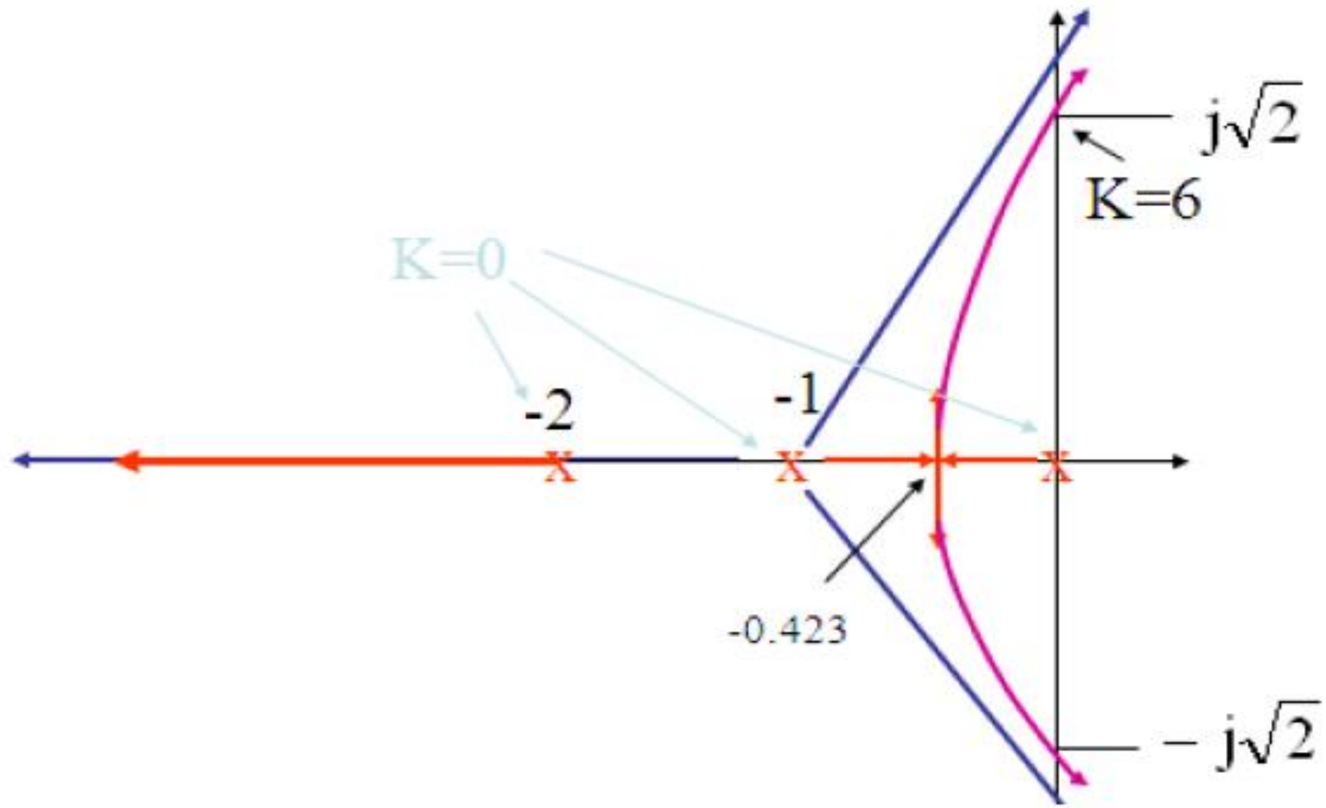
    No complex Poles

11. Angle of Arrival

    No complex zeros

12. The value of k at any point on the locus

    Graphical/Analytical



$j\sqrt{2}$
K=6
K=0
-2
-1
-0.423
$-j\sqrt{2}$

Problem 2

The open loop transfer function of a unity negative feed back system is given by $\frac{k(s+3)}{(s+1)(s+2)}$

Draw the root locus as the value of k varies from zero to Infinity

Solution

Construction Rules:

1. K = 0 points -1, -2 n = 2
2. K = ∞ points -3, ∞ m = 1
3. Number of branches = 2
4. Asymptotes = $\frac{(2q+1)\pi}{n-m}$

   Number of asymptotes = n-m = 1

   $\theta = \pi$
5. Centroid (Intersection of asymptotes)

   $\sigma = \frac{(-1-2)-(-3)}{2-1} = 0$

6. Root locus on real axis lies between

   -1 and -2

   -3 and -∞

7. Root locus on imaginary axis

   Characteristic equation $1 + G(s)H(s) = 0$

   $s^2 + (k+3)s + (2+3k) = 0$

   Draw the Routh table

s^2	1	2+3k
s^1	k +3	0
s^0	2+3k	0

   For Stability k must be greater than -0.66

- Root locus never intersects imaginary axis

8. Break Points

$$\frac{dk}{ds} = 0$$

$s^2 + 3s + 2 + k(s+3) = 0$

$k = -(s^2+3s+2)/(s+3)$

$$\frac{dk}{ds} = s^2+6s+7 = 0$$

$s = -1.59, -4.4$

$k_{s=-1.59} = 0.17$

$k_{s=-4.4} = 5.8$

Therefore s = - 1.59 is a valid break- away point and s = - 4.4 is a valid break- in point

9. Root locus is symmetrical with real axis

10. Angle of Departure

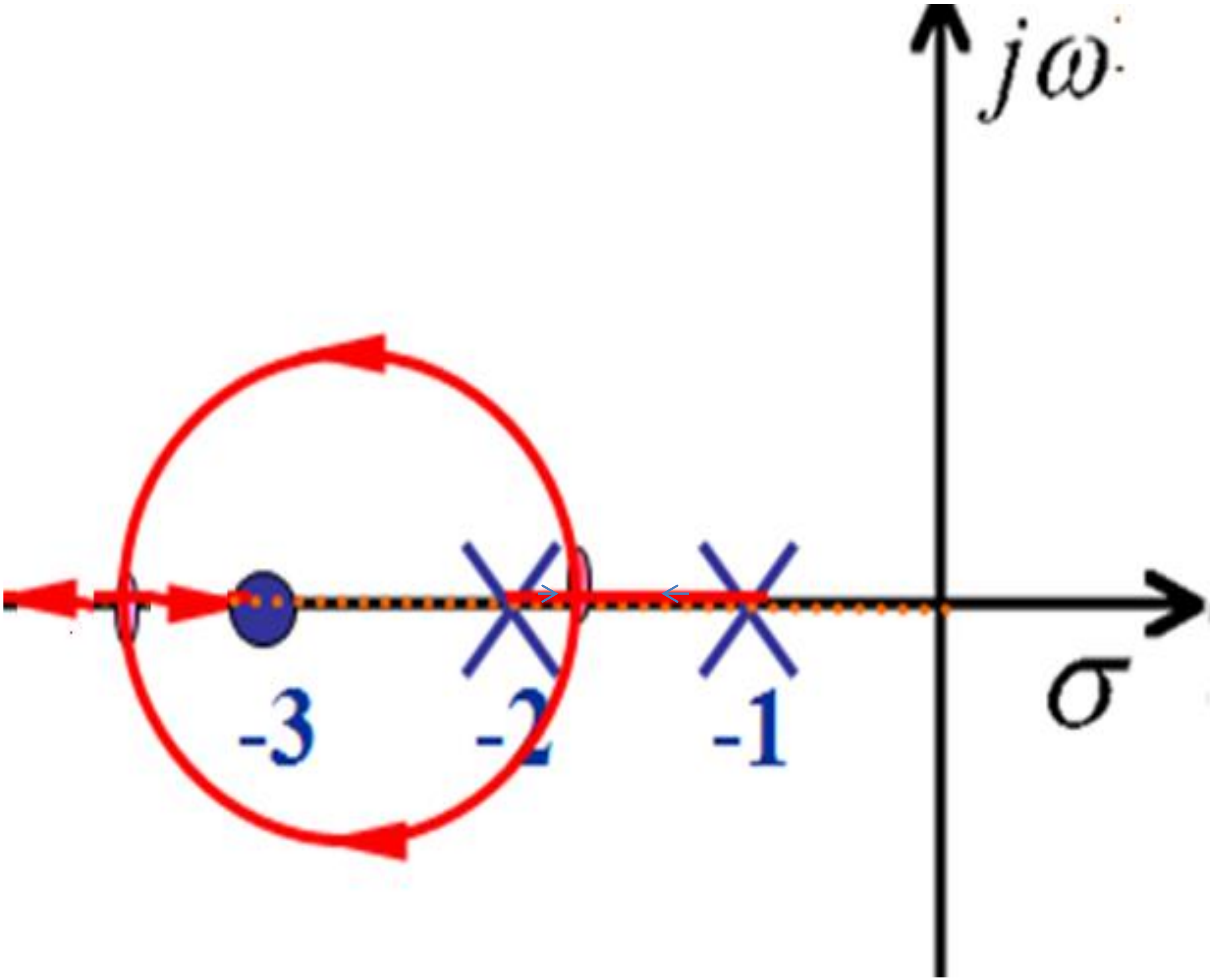
    No complex Poles

11. Angle of Arrival

    No complex zeros

12. The value of k at any point on the locus

    Graphical/Analytical



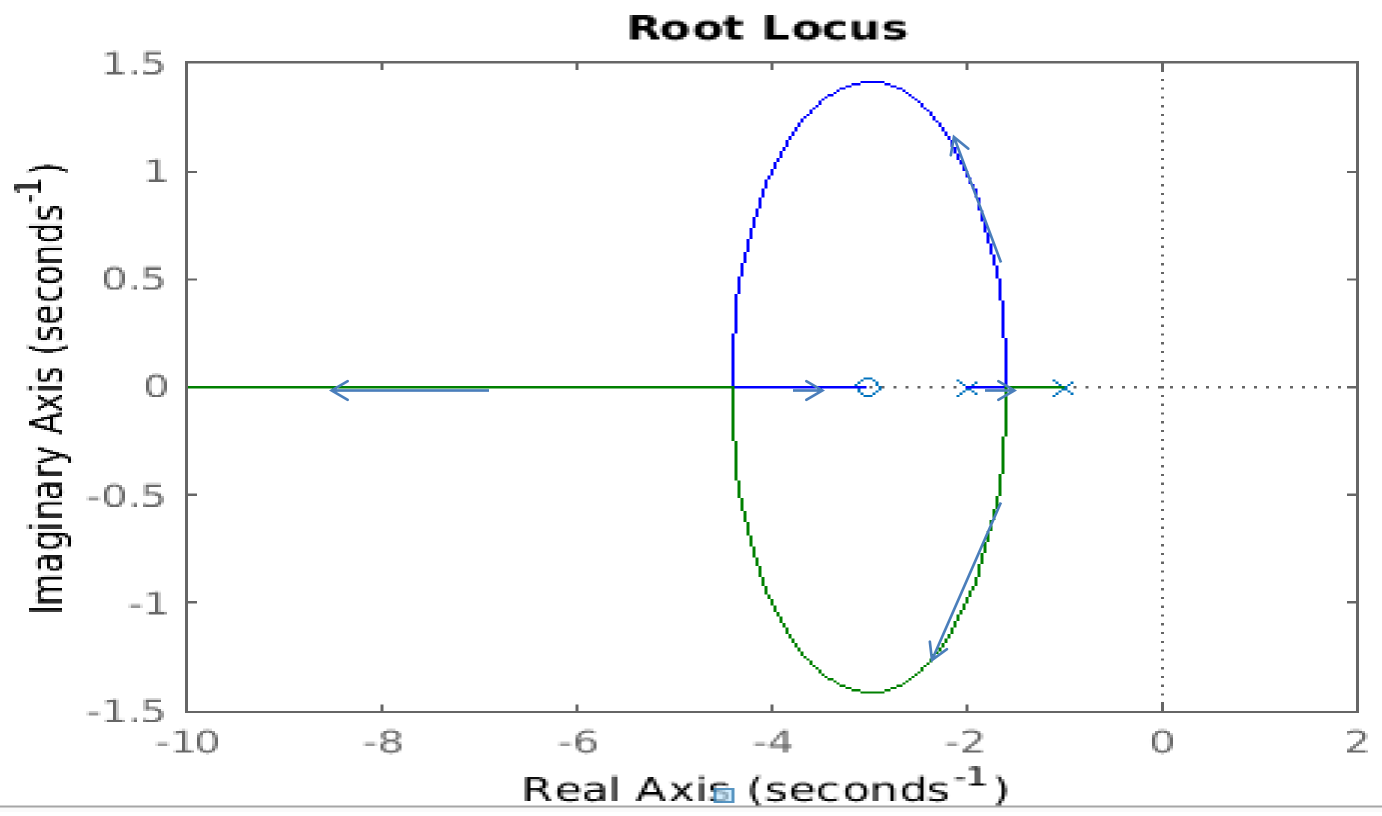
$j\omega$
σ
-3
-2
-1

- Prove that the root locus is in the form of Circle

  Centre = -3,0

  Radius = $\sqrt{2}$

```
>> num=[1 3];
>> den=[1 3 2];
>> rlocus(num,den)
```



Problem 3

The open loop transfer function of a unity negative feed back system is given by $\frac{k(s+1)(s+2)}{s(s+3)}$

Draw the root locus as the value of k varies from zero to Infinity

Solution

Construction Rules:

1. K = 0 points 0, -3 n = 2
2. K = ∞ points -1, -2 m = 2
3. Number of branches = 2
4. Asymptotes = $\frac{(2q+1)\Pi}{n-m}$

   Number of asymptotes = 0
5. Centroid (Intersection of asymptotes)

   No centriod

6. Root locus on real axis lies between

    0 and -1

    -2 and -3

7. Root locus on imaginary axis

   Characteristic equation 1 + G(s)H(s) = 0

   Never intersects imaginary axis

8. Break Points

    $\frac{dk}{ds} = 0$

    $s^3+3s^2+2s+k = 0$

    $k = -(s^2+3s)/(s^2+3s+2)$

    no break point

9. Root locus is symmetrical with real axis

10. Angle of Departure

    No complex Poles

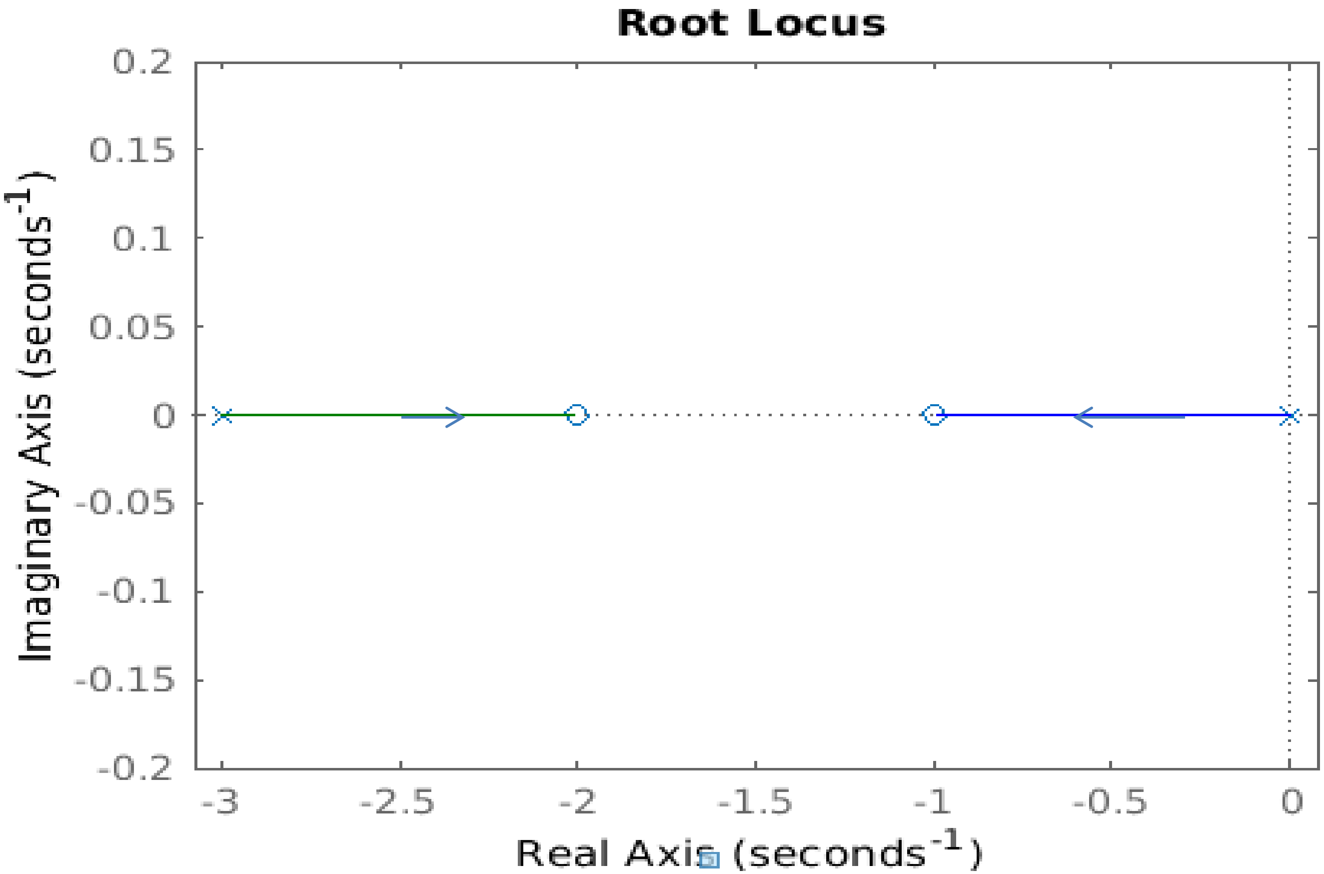
11. Angle of Arrival

    No complex zeros

12. The value of k at any point on the locus

    Graphical/Analytical

```
>> num=[1 3 2];
>> den=[1 3 0];
>> rlocus (num,den)
```



Problem 4

The open loop transfer function of a unity negative feed back system is given by $\frac{k(s+1)}{(s^2+4s+13)}$

Draw the root locus as the value of k varies from zero to Infinity

Solution

Construction Rules:

1. K = 0 points -2+3j, -2-3j n = 2
2. K = ∞ points -1, ∞ m = 1
3. Number of branches = 2
4. Asymptotes = $\frac{(2q+1)\pi}{n-m}$

   Number of asymptotes = n-m = 1

   $\theta = \pi$
5. Centroid (Intersection of asymptotes)

   $\sigma = \frac{(-2-2)-(-1)}{2-1}$ = -3

6. Root locus on real axis lies between

   -1 and $-\infty$

7. Root locus on imaginary axis

   Characteristic equation $1 + G(s)H(s) = 0$

   $s^2 + (k+4)s + (13+k) = 0$

   Draw the Routh table

s^2	1	13+k
s^1	k +4	0
s^0	13+k	0

- Root locus never intersects imaginary axis

8. Break Points

$$\frac{dk}{ds} = 0$$

$s^2 + 4s + 13 + k(s+1) = 0$

$k = -(s^2+4s+13)/(s+1)$

$$\frac{dk}{ds} = s^2+2s-9 = 0$$

$s = -4.16, 2.16$

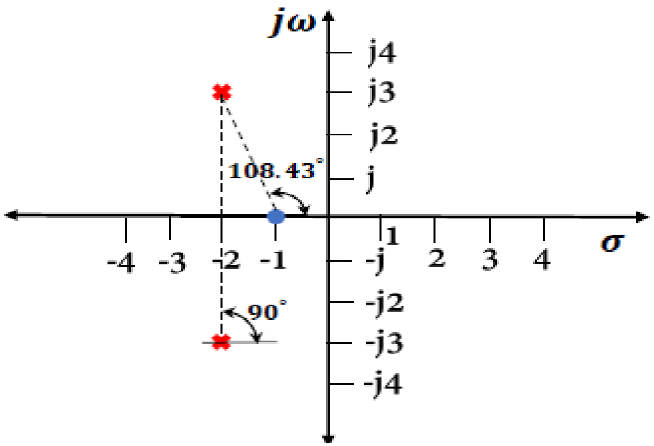
$k_{s=-4.16} = 4.3$

$k_{s=2.16} = -8.3$

Therefore $s = -4.16$ is a valid break- in point

9. Root locus is symmetrical with real axis

10. Angle of Departure at -2+j3 is



$\theta_d = 180 - [\theta_1 - \theta_2]$

$\theta_1 = 90$

$\theta 2 = 180 - \tan^{-1} 3/1 = 108.4$

$\theta_d = 198.4$

Similarly angle of departure at -2-3j is given by -198.4

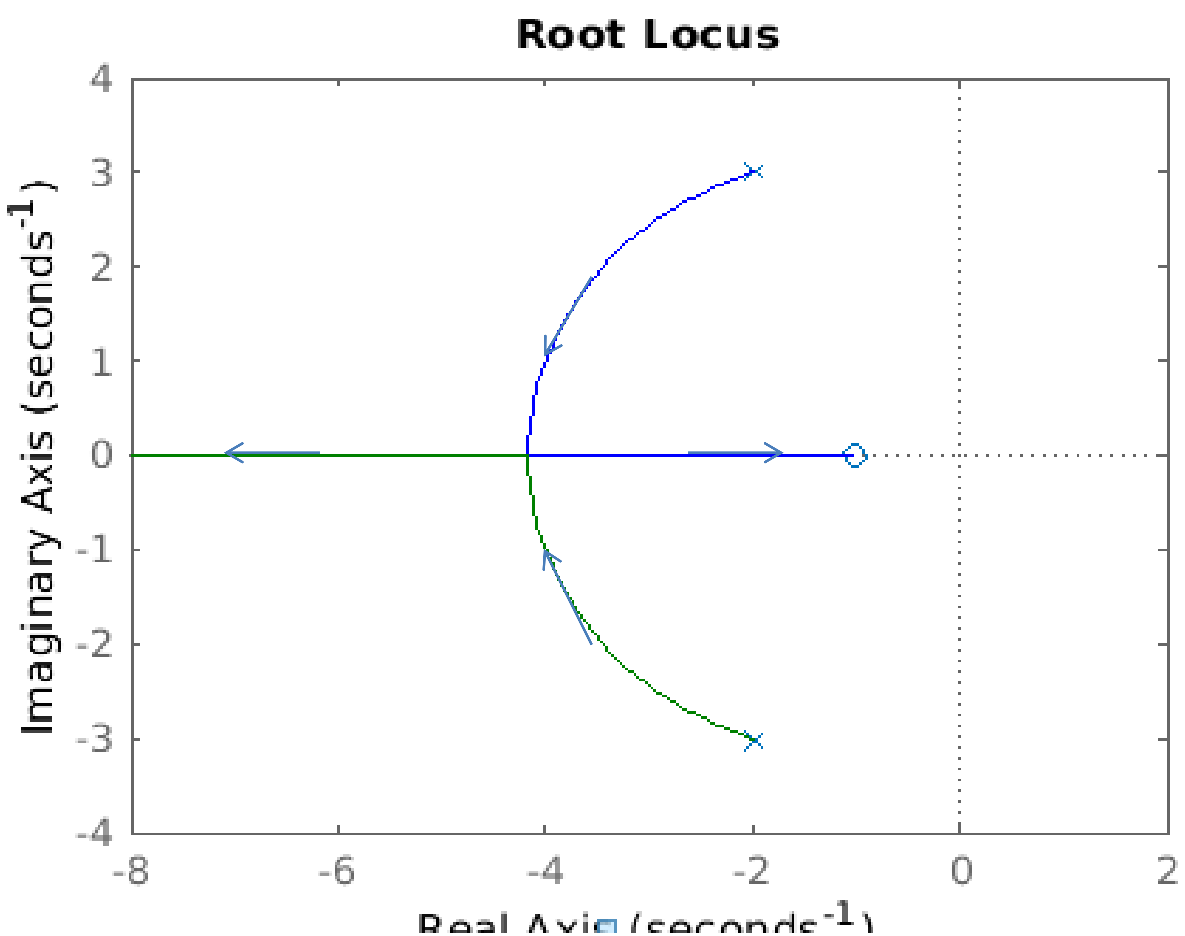
11. Angle of Arrival

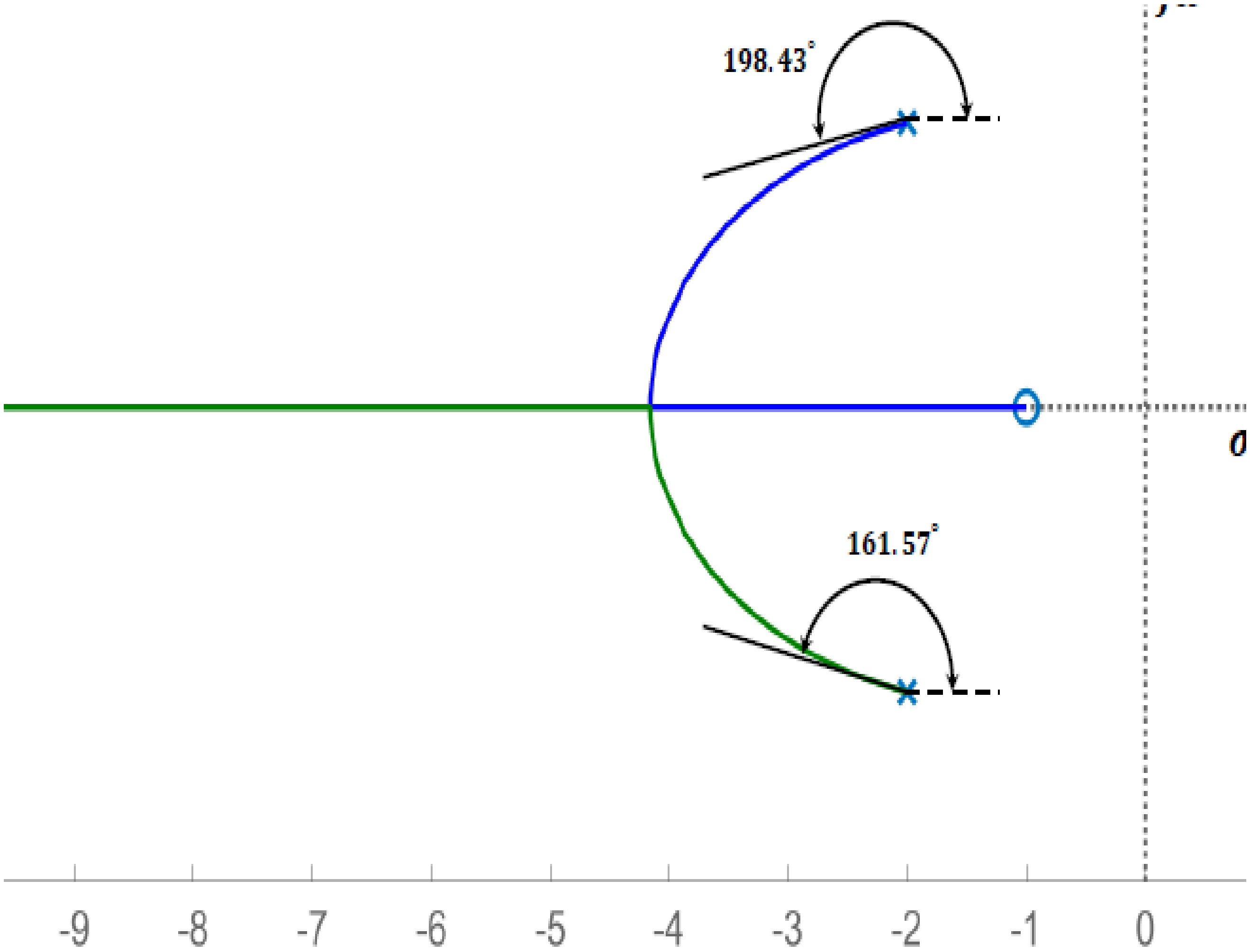
No complex zeros

12. The value of k at any point on the locus

Graphical/Analytical

```
>> num=[1 1];
>> den=[1 4 13];
>> rlocus (num,den)
```




198.43°
161.57°
σ
-9
-8
-7
-6
-5
-4
-3
-2
-1
0

Problem 5

The open loop transfer function of a unity negative feed back system is given by $\frac{k}{s(s^2+8s+32)}$

Draw the root locus as the value of k varies from zero to Infinity

Solution

Construction Rules:

1. K = 0 points 0, -4+4j, -4-4j n = 3
2. K = ∞ points ∞, ∞ , ∞ m = 0
3. Number of branches = 3
4. Asymptotes = $\frac{(2q+1)\pi}{n-m}$

   Number of asymptotes = n-m = 3

   θ = 60,180,300
5. Centroid (Intersection of asymptotes)

   $\sigma = \frac{(-4-4)-(0)}{3-0}$ = -2.67

6. Root locus on real axis lies between

   0 and -∞

7. Root locus on imaginary axis

   Characteristic equation $1 + G(s)H(s) = 0$

   $s^3 + 8s^2 + 32s + k = 0$

   Draw the Routh table

s^3	1	32
s^2	8	k
s^1	$\frac{256-k}{8}$	0
s^0	k	0

- Root locus intersects imaginary axis at $s = \pm 5.66$
- The value of at the point is 256

8. Break Points

$$\frac{dk}{ds} = 0$$

$$s^2 + 4s + 13 + k(s+1) = 0$$

$$k = -(s^3+8s^2+32s)$$

$$\frac{dk}{ds} = 3s^2+16s+32 = 0$$

$$s = -2.67 \pm 1.89j$$

Therefore there is no valid break points

9. Root locus is symmetrical with real axis

10. Angle of Departure at -4+j4 is

$\theta_d = 180 - [\theta_1 + \theta_2]$

$\theta_1 = 90$

$\theta 2 = 180 - \tan^{-1} 1 = 135$

$\theta_d = -45^0$

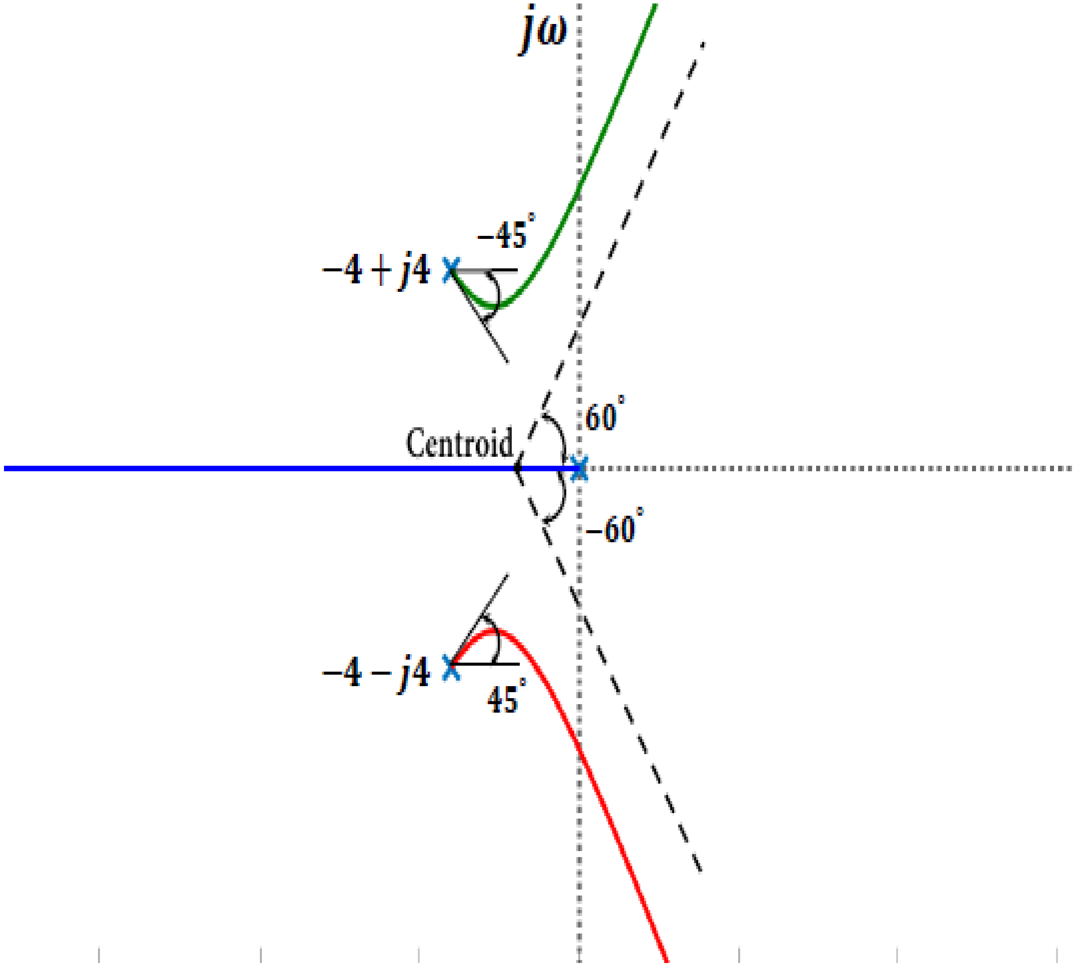
Similarly angle of departure at -4-4j is given by 45^0

11. Angle of Arrival

No complex zeros

12. The value of k at any point on the locus

Graphical/Analytical

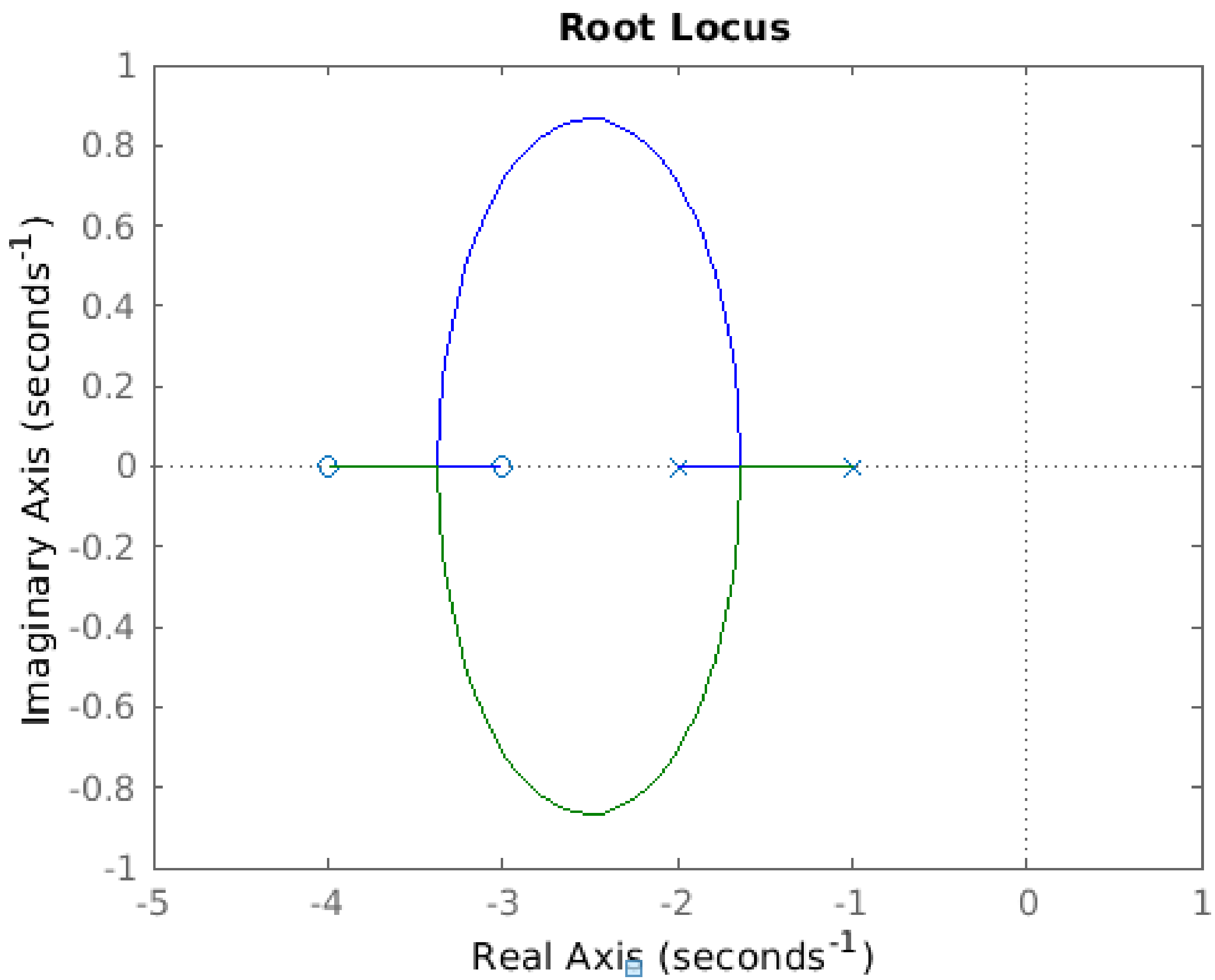


$j\omega$
-45°
$-4+j4$
60°
Centroid
-60°
$-4-j4$
45°

Problem 6

The open loop transfer function of a unity negative feed back system is given by $\frac{k(s+3)(s+4)}{(s+1)(s+2)}$

Draw the root locus as the value of k varies from zero to Infinity



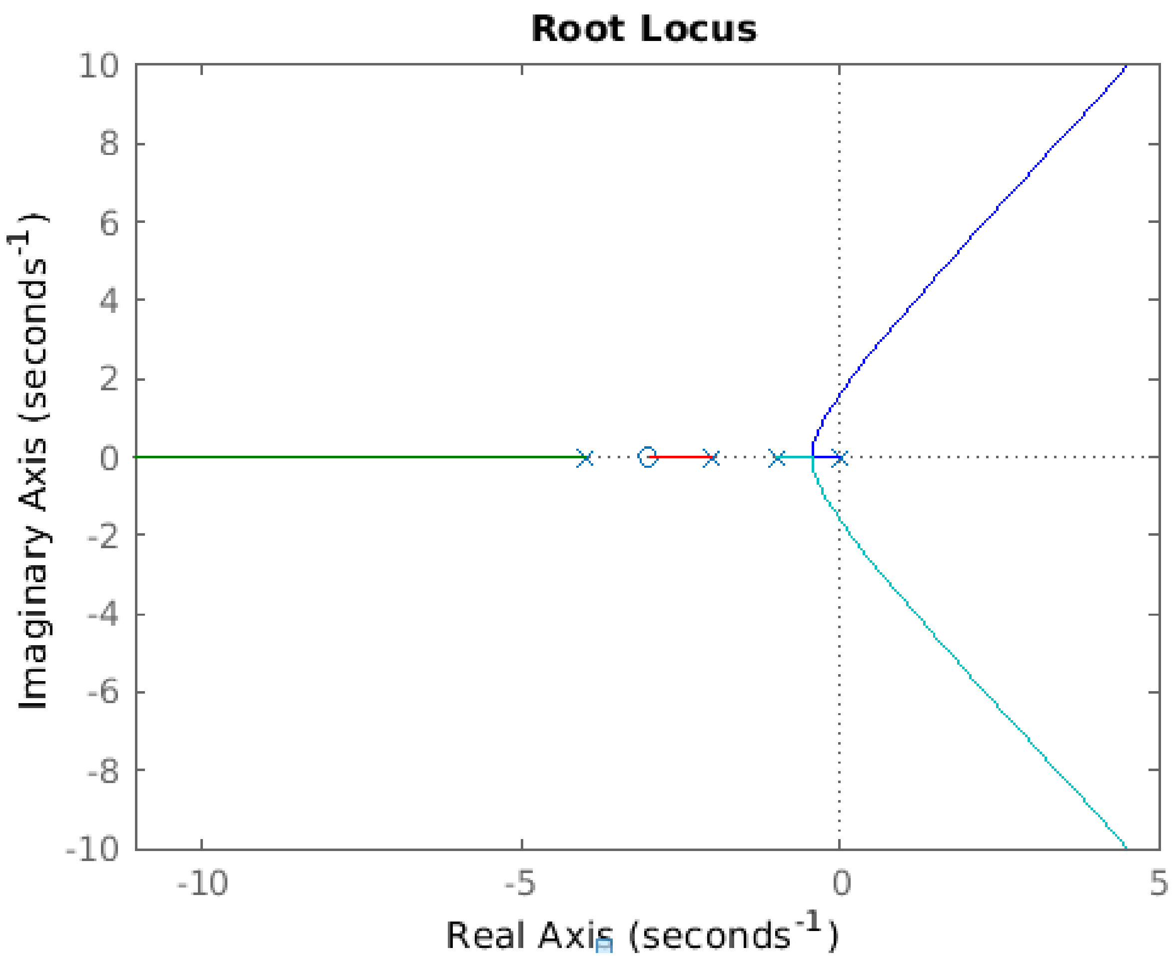
Root Locus
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-5
-4
-3
-2
-1
0
1
Imaginary Axis (seconds^-1)
Real Axis (seconds^-1)

Problem 7

The open loop transfer function of a unity negative feed back system is given by

$$\frac{k(s+3)}{s(s+1)(s+2)(s+4)}$$

Draw the root locus as the value of k varies from zero to Infinity

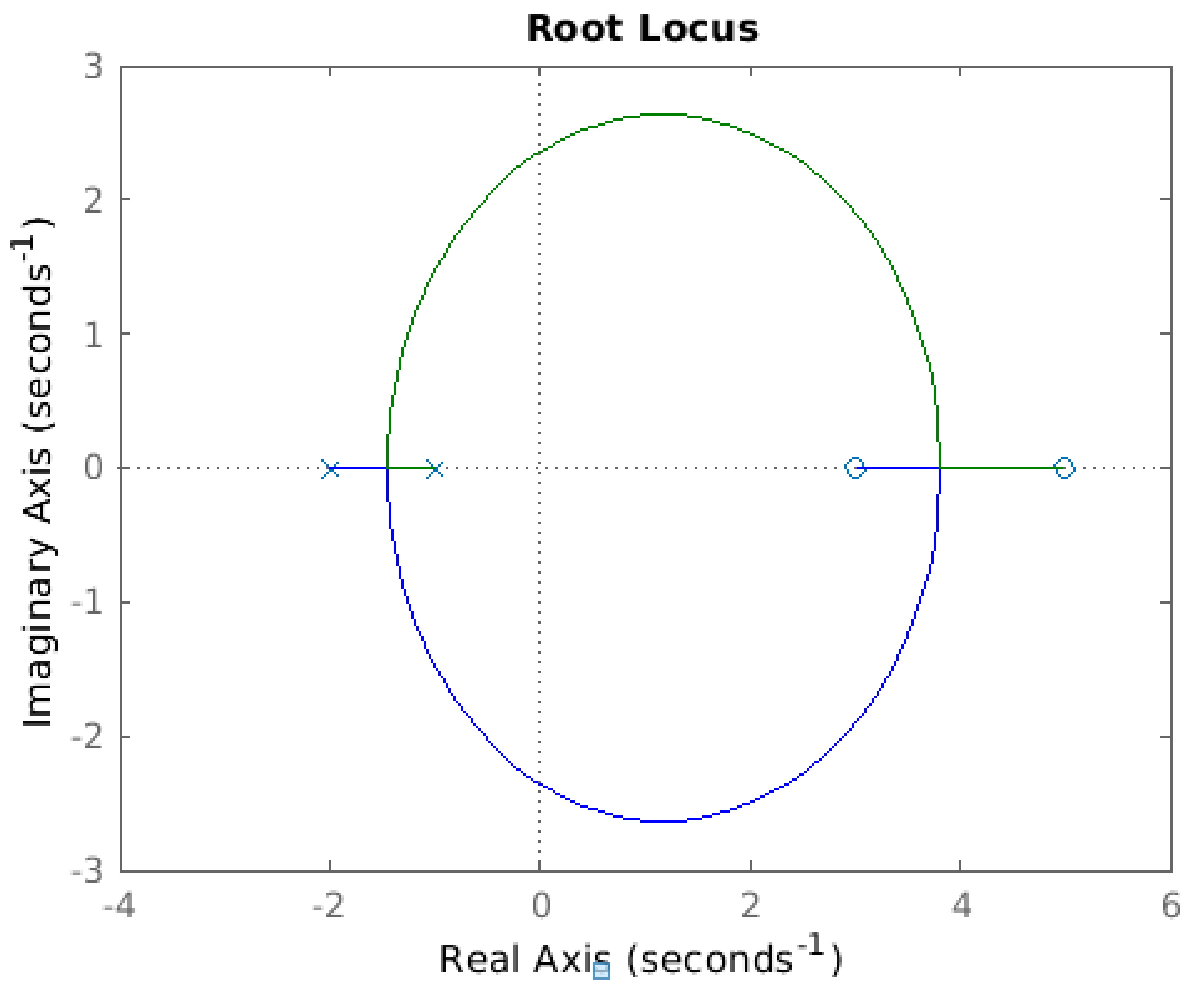


Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
10
8
6
4
2
0
-2
-4
-6
-8
-10
-10
-5
0
5

Problem 8

The open loop transfer function of a unity negative feed back system is given by $\frac{k(s-3)(s-5)}{(s+1)(s+2)}$

Draw the root locus as the value of k varies from zero to Infinity

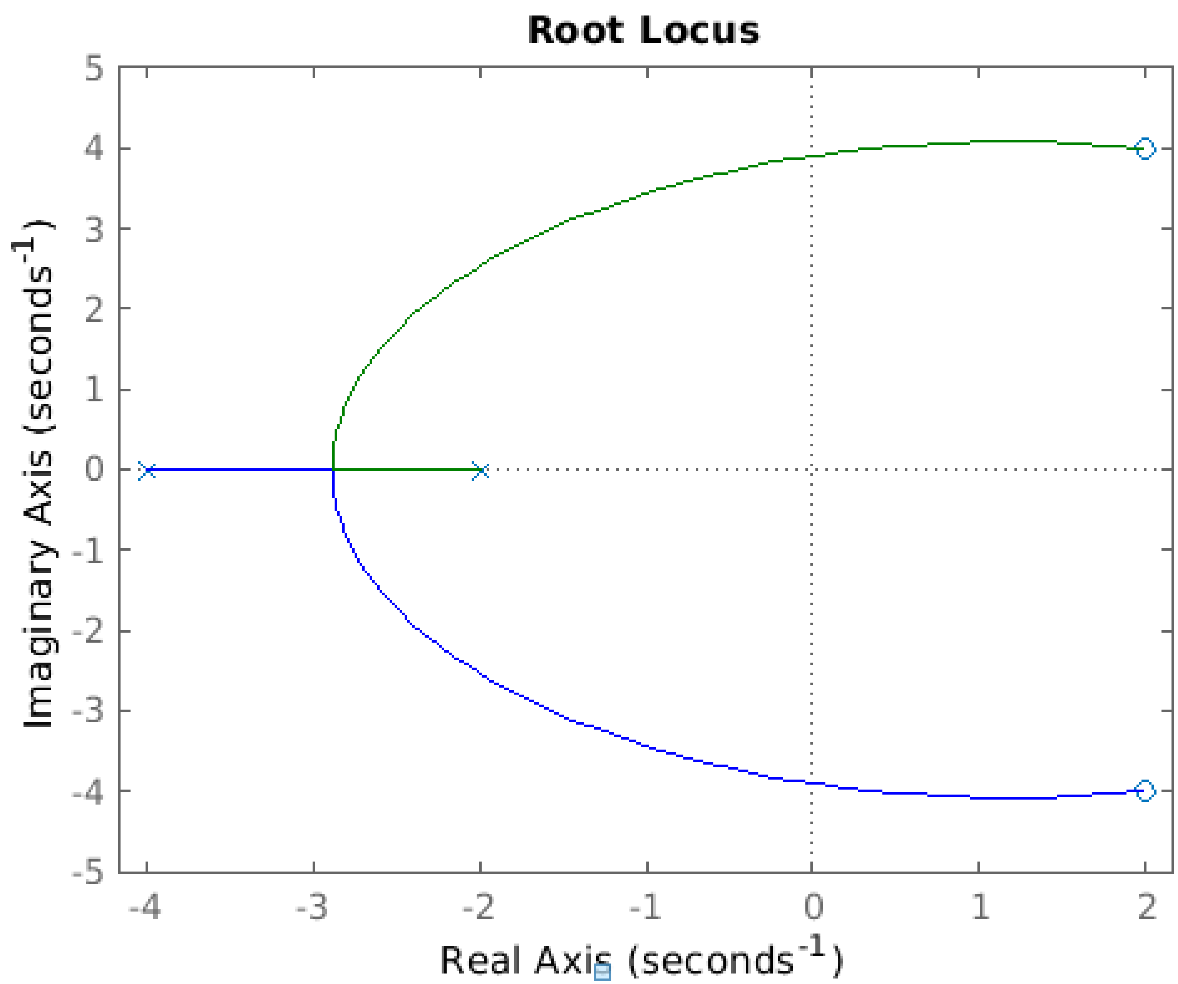


Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
3
2
1
0
-1
-2
-3
-4
-2
0
2
4
6

Problem 9

The open loop transfer function of a unity negative feed back system is given by $\frac{k(s^2-4s+20)}{(s+2)(s+4)}$

Draw the root locus as the value of k varies from zero to Infinity

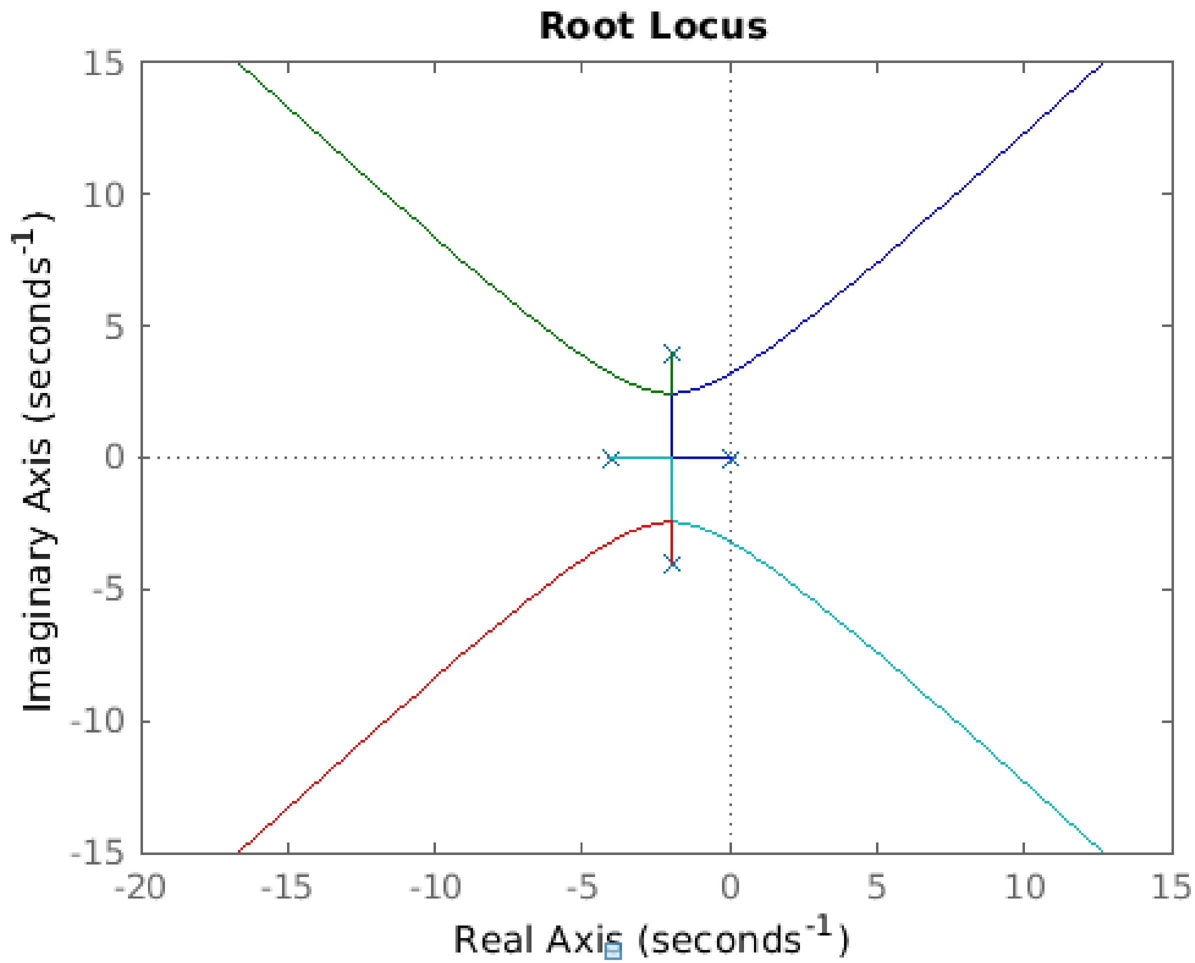

Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
5
4
3
2
1
0
-1
-2
-3
-4
-5
-4
-3
-2
-1
0
1
2

Problem 10

The open loop transfer function of a unity negative feed back system is given by

$$\frac{k}{s(s+4)(s^2+4s+20)}$$

Draw the root locus as the value of k varies from zero to Infinity



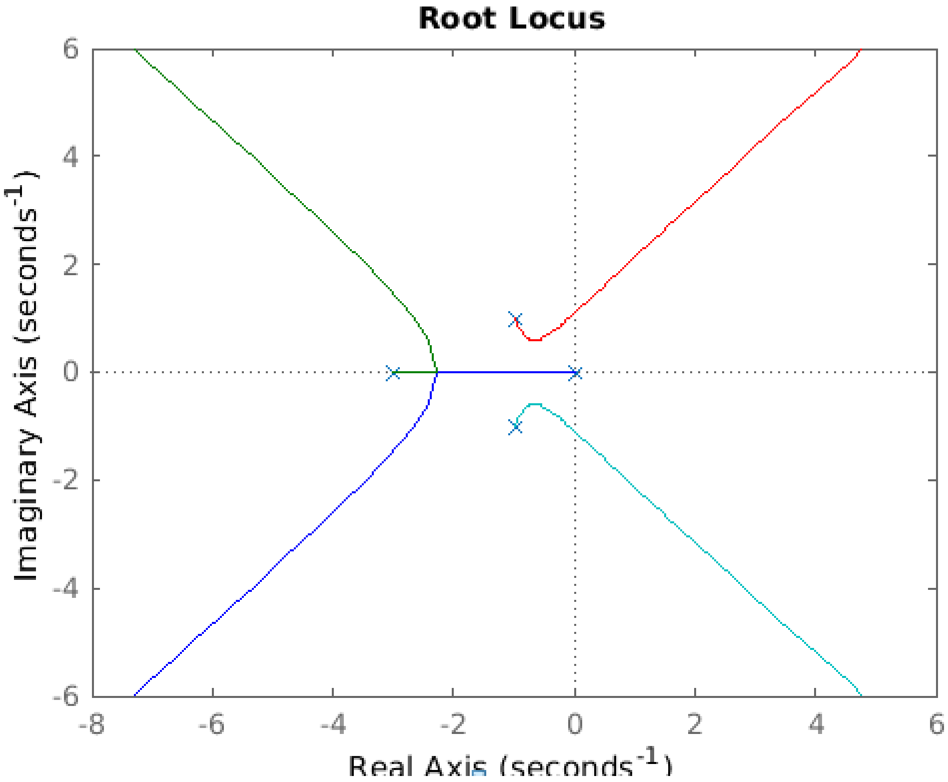
Root Locus
15
10
5
0
-5
-10
-15
Imaginary Axis (seconds^{-1})
-20
-15
-10
-5
0
5
10
15
Real Axis (seconds^{-1})

Problem 11

The open loop transfer function of a unity negative feed back system is given by

$$\frac{k}{s(s+3)(s^2+2s+2)}$$

Draw the root locus as the value of k varies from zero to Infinity



Root Locus
Imaginary Axis (seconds⁻¹)
Real Axis (seconds⁻¹)
6
4
2
0
-2
-4
-6
-8
-6
-4
-2
0
2
4
6

Problem 12

The open loop transfer function of a unity negative feed back system is given by $\frac{k(s^2+6s+10)}{(s^2+2s+10)}$

Draw the root locus as the value of k varies from zero to Infinity

Solution

Construction Rules:

1. K = 0 points -1+3j, -1-3j n = 2
2. K = ∞ points -3+j, -3-j m = 2
3. Number of branches = 2
4. Asymptotes = $\frac{(2q+1)\pi}{n-m}$

   Number of asymptotes = n-m = 0
5. Centroid (Intersection of asymptotes)

   No Centroid

6. Root locus on real axis
7. Root locus on imaginary axis
8. Break Points

$$\frac{dk}{ds} = 0$$

$k = -(s^2+2s+10)/(s^2+6s+10)$

$$\frac{dk}{ds} = 4s^2-40 = 0$$

$s = \pm 3.16$

$k_{\ s=3.16} = -0.66$

$k_{\ s=-3.16} = -13.3$

Therefore there is no valid break point

9. Root locus is symmetrical with real axis

10. Angle of Departure at -1+j3 is

$\theta_d = 180 - [\theta_1 - (\theta_2 + \theta_{3)}]$

$\theta_1 = 90$

$\theta_2 = \tan^{-1} 2/2 = 45$

$\theta_3 = \tan^{-1} 4/2 = 63.4$

$\theta_d = 180 - [90 - (45 + 63.4)] = 198.4$

Similarly angle of departure at -1-3j is given by -198.4

11. Angle of Arrival at -3+j is

$\theta_a = 180 - [\theta_1 - (\theta_2 + \theta_{3)}]$

$\theta_1 = 90$

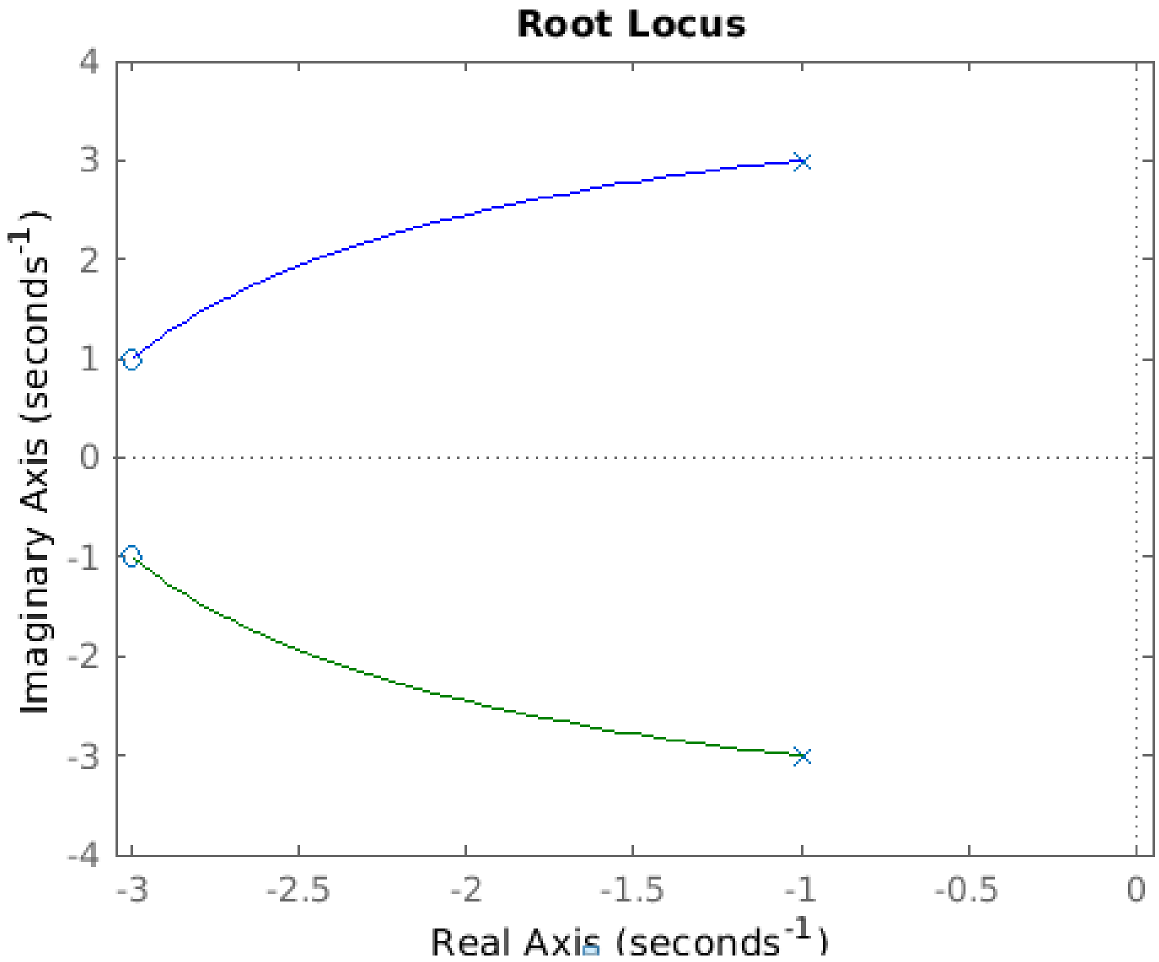
$\theta_2 = 270 - \tan^{-1} 2/2 = 225$

$\theta_3 = 180 - \tan^{-1} 4/2 = 117$

$\theta_a = 180 - [90 - (225 + 117)] = 432 \ (72)$

Similarly angle of arrival at -3-j is given by -432 (-72)

12. The value of k at any point on the locus: Graphical/Analytical


Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
4
3
2
1
0
-1
-2
-3
-4
-3
-2.5
-2
-1.5
-1
-0.5
0

Problem 13

The open loop transfer function of unity feedback system is given by $\dfrac{k(s+0.2)}{s^2(s+3.6)}$

Sketch the root locus of the system

Solution

Construction Rules:

1. K = 0 points 0, 0, -3.6 n = 3
2. K = ∞ points -0.2, ∞, ∞ m = 1
3. Number of branches = 3
4. Asymptotes = $\frac{(2q+1)\pi}{n-m}$

   Number of asymptotes = n-m = 2

   θ = 90,270
5. Centroid (Intersection of asymptotes)

   $\sigma = \frac{(-3.6)-(-0.2)}{3-1}$ = -1.7

6. Root locus on real axis lies between

   -0.2 and -3.6

7. Root locus on imaginary axis

   Characteristic equation $1 + G(s)H(s) = 0$

   $s^3 + 3.6s^2 + ks + 0.2k = 0$

   Draw the Routh table

s^3	1	k
s^2	3.6	0.2k
s^1	$0.94k$	0
s^0	0.2k	0

Root locus never intersects imaginary axis

8. Break Points $\frac{dk}{ds} = 0$

$s^3 + 3.6s^2 + k(s + 0.2) = 0$

$k = -(s^3 + 3.6s^2)/(s + 0.2)$

$\frac{dk}{ds} = 2s^3 + 4.2s^2 + 1.44s = 0$

$s = 0, -1.67, -0.43$

$K_{s=0} = 0$

$K_{s=\ -1.67} = 3.66$

$K_{s=\ -0.43} = 2.5$

Therefore s=-0.43, -1.67 are valid break points

9. Root locus is symmetrical with real axis

10. Angle of Departure

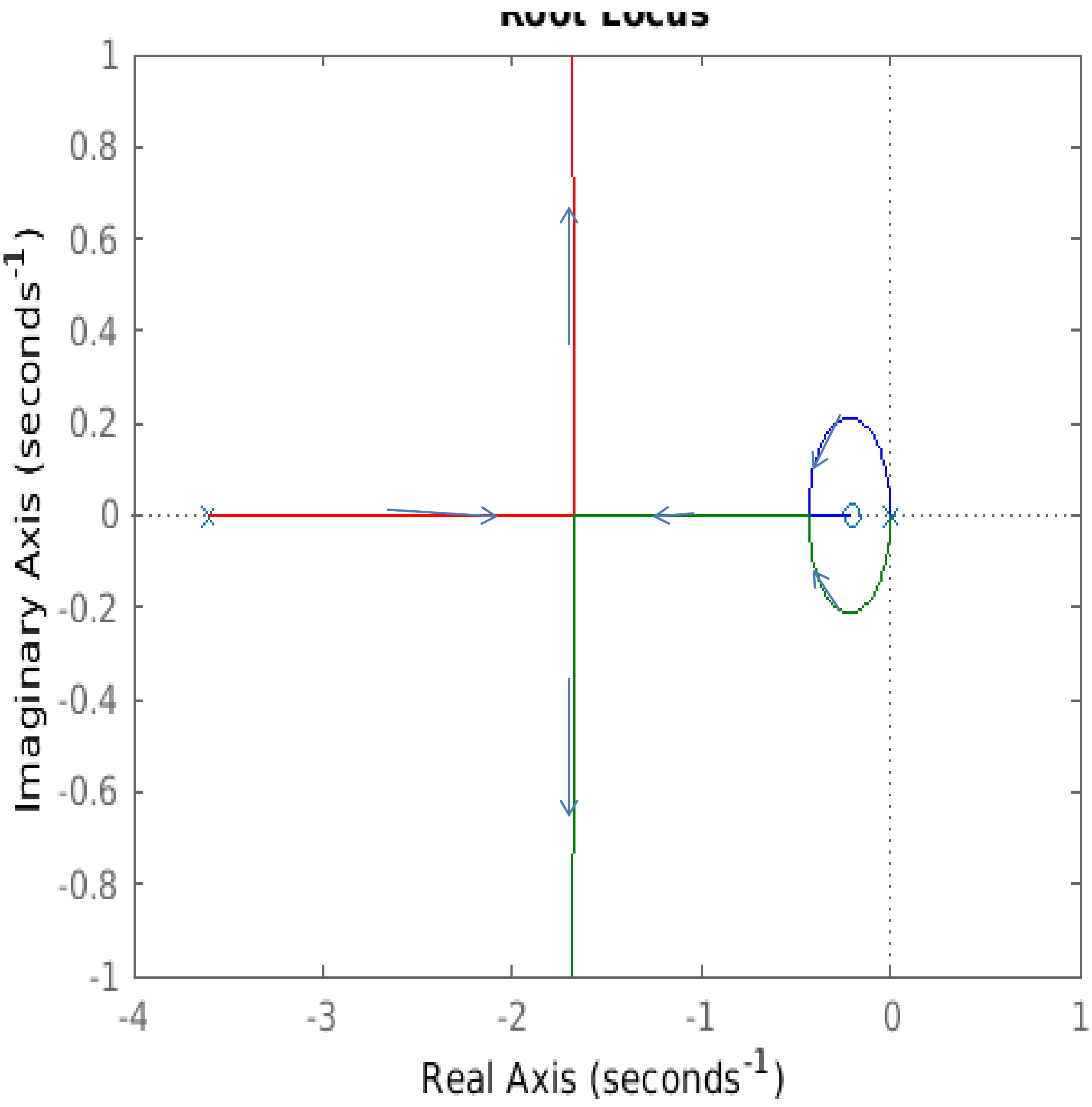
    No complex poles

11. Angle of Arrival

    No complex zeros

12. The value of k at any point on the locus

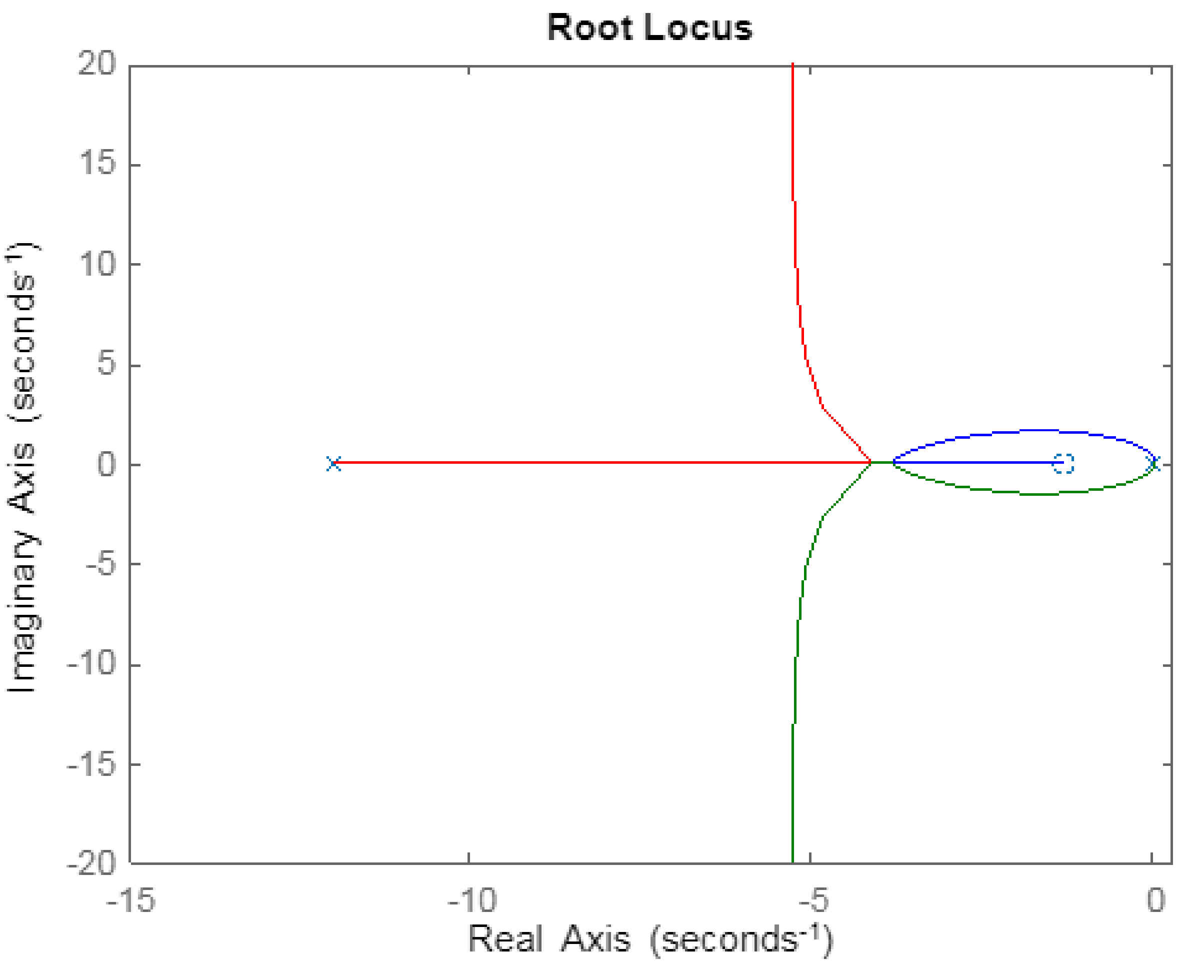
    Graphical/Analytical


Root Locus
Imaginary Axis (seconds⁻¹)
Real Axis (seconds⁻¹)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-4
-3
-2
-1
0
1

Problem 14

The open loop transfer function of unity feedback system is given by $\dfrac{k(s+1.33)}{s^2(s+12)}$

Sketch the root locus of the system

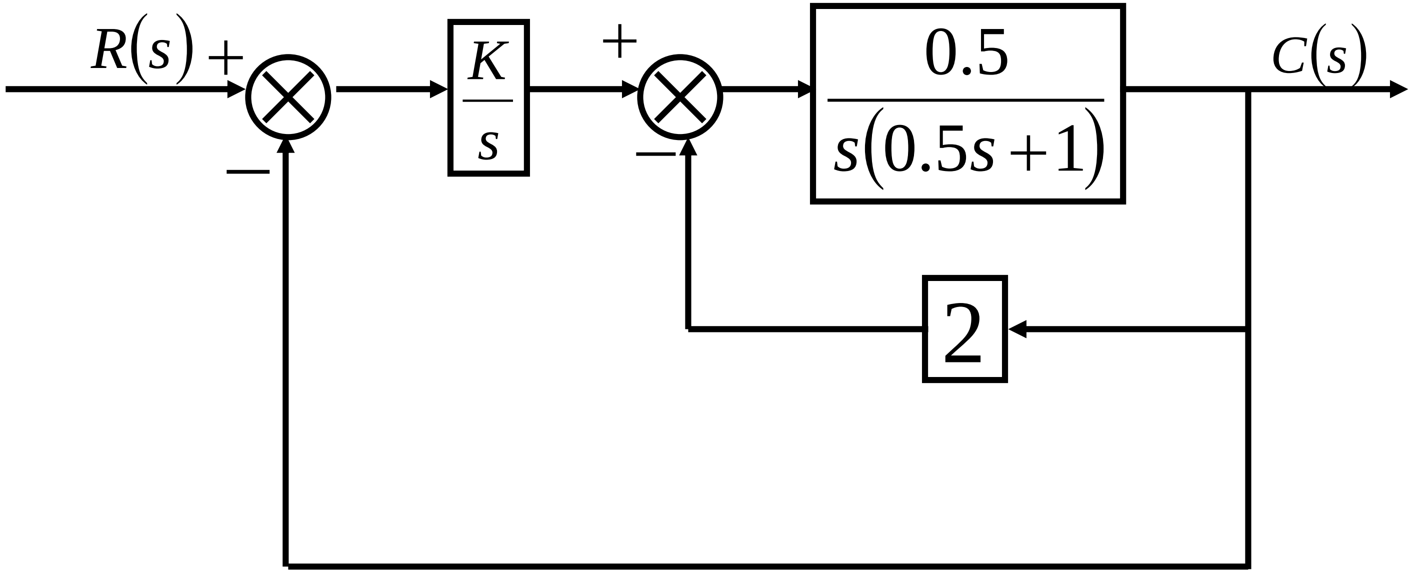

Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
20
15
10
5
0
-5
-10
-15
-20
-15
-10
-5
0

Problem 15

The open loop transfer function of a unity negative feed back system is given by $\frac{k}{s(s^2+4)}$

Draw the root locus as the value of k varies from zero to Infinity

Problem 16



Problem 17

The open loop transfer function of unity feedback system is given by $G(s) = \dfrac{2s + (p + 6)}{s^2 + 5s + 6}$
Sketch the root locus of the system with `p' as the parameter to be varied.

Solution

Given that $G(s) = \dfrac{2s + (p + 6)}{s^2 + 5s + 6}$ and $H(s) = 1$:

The closed loop transfer function is given by

$$\frac{G(s)}{1 + G(s)H(s)} = \frac{2s + p + 6}{s^2 + 7s + 12 + p}$$

The Characteristic equation of the system is

$$s2 + 7s + 12 + p = 0$$

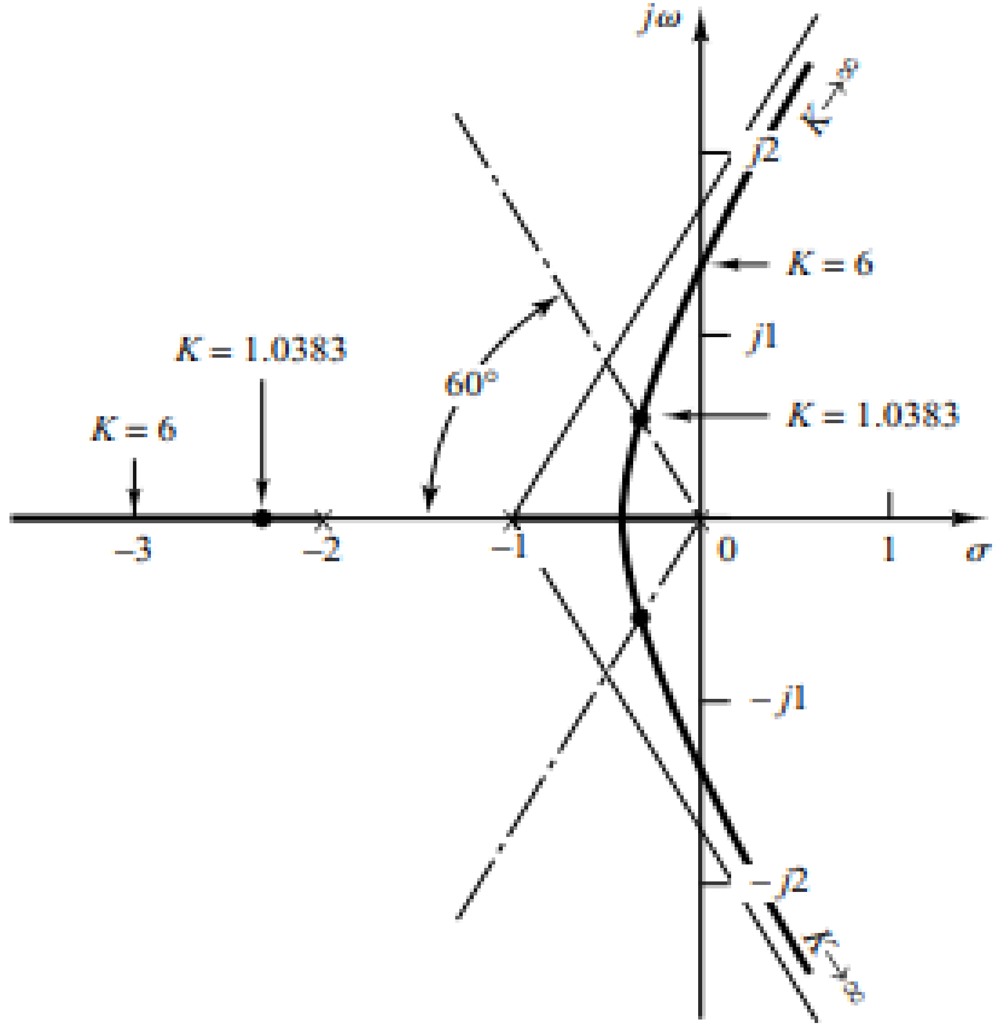
An equivalent system with the same characteristic equation and p as the gain is $1 + \dfrac{p}{s^2 + 7s + 12} = 0$

So, the root locus of an equivalent unity feedback system with open-loop transfer function with $\dfrac{p}{s^2 + 7s + 12}$ is considered

Problem

The open loop transfer function of a unity negative feed back system is given by $\frac{k}{s(s+1)(s+2)}$

Draw the root locus as the value of k varies from zero to Infinity. Determine the value of k such that the damping ratio is 0.5



$j\omega$
$K \to \infty$
$j2$
$K = 6$
$j1$
$K = 1.0383$
60°
$K = 1.0383$
$K = 6$
−3
−2
−1
0
1
σ
$-j1$
$-j2$
$K \to \infty$

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