Axiomatic characterization of the interval function of unit interval graphs

1 Introduction

Transit functions on discrete structures are introduced by Mulder [15] mainly to generalize the concept of betweenness in an axiomatic way. A transit function is an abstract notion of an interval. Given a nonempty finite set V, a *transit function* R is defined as a function $R : V \times V \to 2^V$ satisfying the three axioms

(t1) $u \in R(u, v)$, for all $u, v \in V$.

(t2) $R(u, v) = R(v, u)$, for all $u, v \in V$.

(t3) $R(u, u) = \{u\}$, for all $u \in V$.

More detailed introductions and definitions related to transit function and their applications can be found in the following papers: with emphasis on betweenness [4, 6, 7, 13, 14]; on intervals [1, 3, 6, 14, 16, 17, 20, 22, 23] and on convexity [3, 5, 12, 13, 14].

The *underlying graph* G_R of a transit function R is the graph with vertex set V, where two distinct vertices u and v are joined by an edge if and only if $R(u, v) = \{u, v\}$. Note that in general, G and G_R need not be isomorphic graphs, see [15].

A u, v-*shortest path* in a connected graph G is a u, v-path in G containing the minimum number of edges. The length of a shortest u, v-path P (that is, the number of edges in P) is the standard distance in G.
The interval function of a connected graph G is defined as

$$I_G(u, v) = \{w \in V\colon\ w \text{ lies on some shortest } u, v\text{-path}\}.$$

Consider the transit function R defined on nonempty set V, Nebeský initiated a very interesting problem on the interval function I of a connected graph G with vertex set V. The problem is the following:“ Is it possible to give a characterization of I using a set of simple axioms (first order axioms) defined on R? ”

Nebeský [18, 17] proved that there exists such a characterization for the interval function $I(u, v)$ by using axioms on the transit function R. In further papers that followed [19, 20, 21, 22], Nebeský improved the formulation and proof of this characterization. Finally, Mulder and Nebeský [16] gave an optimal characterization of the interval function of a connected graph by using a minimal set of axioms. Recently, in [10] the axiomatic characterization of the interval function of a connected graph is extended to that of a disconnected graph.
But the first systematic study of the interval function is due to Mulder in [14]. The axiomatic characterization of the interval function of trees presented by Sholander in [25] with a partial proof. Chvátal et al. [11] obtained the completion of this proof. Recently, new characterizations of the interval function of trees using three different sets of axioms which are weaker than those presented in [11, 25] are discussed in [1]. Along with this, the axiomatic characterization of the interval function of block graphs is also obtained in [1]. Axiomatic characterization of the interval function of median graphs, modular graphs, geodetic graphs, (claw, paw)-free graphs and bipartite graphs are respectively described in

[14, 16, 19, 8, 9].

A graph $G = (V, E)$ with $V = v_1, v_2, ..., v_n$ is an *interval graph* if there exists a family of closed intervals $\mathcal{M} = \{I_1, I_2, ..., I_n\}$ (interval model) associated to the vertices such that $I_i \cap I_j \neq \emptyset \Leftrightarrow (v_i, v_j) \in E$. If an interval graph admits a model with all intervals of the same length, it will be called *unit interval graph*. In other words, a graph G is a *unit interval* graph when G is the intersection graph of a collection of equal-sized intervals on the real line. Let G be a unit interval graph. Then we know that there exists a linear ordering $v_1, v_2, ..., v_n$ of $V(G)$ such that if there exists an edge joining v_i to v_j, then $v_i, ..., v_j$ form a complete subgraph. Moreover, it will be *proper* if it admits an interval model where any interval is not properly contained in another. The following Theorem shows the forbidden induced subgraphs for the unit interval graphs.

Theorem 1 ([2, 24])**.** *Given a graph G. The following are equivalent:*

(a) G is unit interval graph,

(b) G is proper interval graph,

(c) G is claw, net, tent-free and chordal.

In this paper, we attempt to present axiomatic characterization of the interval function of unit interval graphs. For this purpose we introduce the following axioms:

$(u1)$ if $R(u, v, w) \neq \emptyset$, then $R(u, v, w) \cap \{u, v, w\} = \{u\}$ or $\{v\}$ or $\{w\}$, for all $u, v, w \in V$.

$(u2)$ if $R(u, v) \cap R(u, w) \setminus \{u\} \neq \emptyset$ and $R(v, w) \cap R(u, v) \setminus \{v\} \neq \emptyset$ and $R(u, w) \cap R(v, w) \setminus \{w\} \neq \emptyset$, then $R(u, v, w) \cap \{u, v, w\} = \{u\}$ or $\{v\}$ or $\{w\}$, for all $u, v, w \in V$.

2 Interval function of chordal graphs

Changat et al. in [7] characterized the graphs for which the interval function satisfies the axiom $(J0)$:

(J_0) For any pairwise distinct elements $u, v, x, y \in V$ such that $x \in R(u, y)$ and $y \in R(x, v)$, we have $x \in R(u, v)$.

and these graphs are precisely *Ptolemaic graphs* (Ptolemaic graphs are exactly chordal graphs that are 3-fan-free).

In the following, we present (J_0') axiom, which characterize the interval function of only chordal graphs.

(J_0') if $v \in R(u, w)$, $w \in R(v, y)$ and $|(R(u, w) \setminus \{v\}) \cap (R(v, y) \setminus \{w\})| \neq 1$ then $v \in R(u, y)$, for all distinct $u, v, w, y \in V$.

Theorem 2. *The interval function I of a connected graph G, satisfies (J_0') if and only if G is a chordal graph.*

Proof. Suppose that G is not a chordal graph. Therefore G contains an induced cycle C of length at least 4. Suppose the length of C is even. Hence we have $x_1, x_2, x_3, ..., x_{2n}$, $n \geq 2$ as the vertices of C. If $n = 2$, let $x_1 = u$, $x_2 = v$, $x_3 = w$ and $x_4 = y$ and if $n > 2$, let $x_1 = u$, $x_2 = v$, $x_4 = w$ and $x_{2+\frac{2n}{2}} = y$. (i.e. the antipodal vertex to $x_2 = v$ on C). It is easy to see that $v \in I(u,w)$, $w \in I(v,y)$ and $|(I(u,w) \setminus \{v\}) \cap (I(v,y) \setminus \{w\})| \neq 1$ but $v \notin R(u,y)$. Now suppose the length of C is odd. let $x_1 = u$, $x_2 = v$, $x_3 = w$ and $x_{2+\lfloor \frac{(2n)+1}{2} \rfloor} = y$. (i.e. the antipodal vertex to both $x_1 = u$ and $x_2 = v$ on C). It is easy to see that I does not fulfill (J_0').

Conversely, let G be a chordal graph. Suppose I dose not satisfy (J_0') on G. Then there exist vertices u, v, w, y in G such that $v \in I(u,w)$, $w \in I(v,y)$ and $|(I(u,w) \setminus \{v\}) \cap (I(v,y) \setminus \{w\})| \neq 1$ but $v \notin R(u,y)$. Let P and Q be a u,w-shortest path containing v and a v,y-shortest path containing w respectively and R be a u,y-shortest path that intersect P the most. Let u_1 be the last common vertex of P and R. u_1 lies on the u,v-subpath of P. (u_1 maybe equal to u but $u_1 \neq v$, since v does not lie on R.) Let v_1 be the first common vertex of R and Q. Since by assumption u_1 was the last common vertex of P and R, hence v_1 must lie on w,y-subpath of Q and v_1 maybe equal to v but $v_1 \neq y$. Now consider the cycle $C_1 : u_1 \to P \to v \to Q \to w \to Q \to v_1 \to R \to u_1$, the length of C_1 is at least 4. If the length of C_1 is 4, then C_1 is induced, a contradiction. Therefore, the length of C_1 ia at least 5. Note that C_1 can not be induced since G is a chordal graph. To avoid having an induced hole, there must exist some chords in C_1. There are no possibility of having chords between u_1v-subpath of P and $P \cap Q \setminus \{v\}$, since every such a chord is in contradiction with P being a shortest u,w-path containing x. Suppose there exist a chord between $u_1 \to P \to v$ and $w \to Q \to v_1$. Let u_1' be the last such vertex on $u_1 \to P \to v$ which is adjacent to $u_1'' \in w \to Q \to v_1$ such that u_1'' is the nearest vertex to w. Therefore the cycle $C_2 : u_1' \to P \to v \to Q \to w \to Q \to u_1''u_1'$ is of length at least 4, which is a contradiction. Hence there is no chord between $u_1 \to P \to v$ and $w \to Q \to v_1$. Therefore C_1 contains two induced paths $P' : u_1 \to P \to v \to Q \to w \to v_1$ of length at least 4 and $P'' : u_1 \to R \to v_1$ of length at least 2, (since otherwise C_1 is induced cycle of length 5, which is a contradiction.) Now let $P' : x_0x_1, ..., x_k$ and $P'' : z_0z_1, ..., z_l$, where $x_0 = u_1 = z_0$ and $x_k = v_1 = z_l$. Note that $l < k$, since P'' is a subpath of R but P' is not a part of any u,y-shortest path. $x_1z_1 \in E(G)$, since otherwise we have an induced cycle of length at least 4 on vertices $\{x_0, x_1, z_1, ...\}$. Moreover, $x_1z_2 \notin E(G)$, since otherwise we get a contradiction by the choice of $v \neq x_1$ or by the fact that v is not on a shortest u,v-path if $v = x_1$. Similarly, $x_1z_i \notin E(G)$, for every $i \in \{3, ..., l\}$. Now z_1x_2 is an edge, otherwise we have an induced cycle of length at least 4 on vertices $\{x_1, x_2, z_1, ...\}$. Now to avoid having an induced cycle on $\{x_2, z_1, x_3...\}$ there must be chord between z_1x_3 or z_2x_2. Suppose z_1x_3 is an edge in G, but now we get a contradiction with assumption that $v \in P$, if $x_3 = w$ and any $x_i = w$, for $i \geq 4$. Hence if z_1x_3 is an edge in G then $x_1 = v$ and $x_2 = w$ but in this case $|(I(u,w) \setminus \{v\}) \cap (I(v,y) \setminus \{w\})| = \{z_1\}$, which is contradiction with assumption. Therefore the only possibility is z_2x_2 is an edge in G. But now if $x_1 = v$ and $x_2 = w$ we have $|(I(u,w) \setminus \{v\}) \cap (I(v,y) \setminus \{w\})| = \{z_1\}$, which is a contradiction with assumption.

□

3 Interval function of claw-free graphs

$(u1)$ if $R(u,v,w) \neq \emptyset$, then $R(u,v,w) \cap \{u,v,w\} = \{u\}$ or $\{v\}$ or $\{w\}$, for all $u, v, w \in V$.

Theorem 3. *The interval function I of a connected graph G, satisfies $(u1)$ if and only if G is a claw-free graph.*

Proof. Suppose that G contains a claw as an induced subgraphs. It is easy to see that in a graph with an induced claw shown in Figure 1, there exist u, v, w such that $I(u,v,w) \neq \emptyset$, but $I(u,v,w) \cap \{u,v,w\} = \emptyset$ and I does not fulfill $(u1)$.

Conversely, let G be a claw-free graph. Suppose I does not satisfy $(u1)$. Hence there exist u,v,w in G such that $I(u,v,w) \neq \emptyset$, but $R(u,v,w) \cap \{u,v,w\} \neq \{u\}$ or $\{v\}$ or $\{w\}$. Consider u,v-shortest path. Note that w does not lie on u,v-shortest path, otherwise $I(u,v,w) = \{w\}$ and $I(u,v,w) \cap \{u,v,w\} = \{w\}$, a contradiction with assumption. Moreover, u,w-shortest path intersects u,v-shortest path, otherwise $I(u,v,w) \cap \{u,v,w\} = \{u\}$, a contradiction with assumption. Let x be the first common vertex between u,w-shortest path and u,v-shortest path. It is easy to see that we get a claw on x and neighbors of x, $N(x) = \{z,y,y'\}$, where y' lies on u,x subpath of P and y lies on x,v subpath of P and z lies on x,w subpath of Q. To avoid having claw as an induced subgraph there must be edges between z,y' or y,y' or z,y. But z,y' and y,y' are not an edge since we get a contradiction with P and Q being the shortest path containing x. Suppose z,y is an edge. If $z = w$ and $y = v$ we get an induced paw on $\{y',x,v,w\}$ and $I(u,v,w) = \emptyset$ or if $z \neq w$ and $y \neq v$ we have an induced net $\{y',x,y,z\}$ and neighborhood of z on x,w-subpath of Q and neighborhood of y on x,v-subpath of P. It is easy to see that $I(u,v,w) = \emptyset$, a contradiction with assumption. $\square$

4 Interval function of net-free graphs

$(u2)$ for any $u_1 \neq u_2 \neq u_3 \neq v_1 \neq v_2 \neq v_3 \in V$, $R(u_1,v_3) = \{u_1,v_2,v_3\}, R(u_2,v_1) = \{u_1,v_1,v_3\}, R(u_3,v_2) = \{u_3,v_1,v_2\}$, then exist $v,v' \in \{v_1,v_2,v_3\}$ such that $R(v,v') = \{v,v'\}$ or exists $i \in \{1,2,3\}$ such that $R(u_i,v_i) = \{u_i,v_i\}$.

Explanation: $R(u_1,v_3) = \{u_1,v_2,v_3\}, R(u_2,v_1) = \{u_1,v_1,v_3\}, R(u_3,v_2) = \{u_3,v_1,v_2\}$ imply the net formed by the edges $u_1v_2, u_2v_3, u_3v_1, v_1v_2, v_1v_3, v_2v_3$. And the edges u_1v_3, u_2v_1, u_3v_2 do not exist. Then, there is an edge between the vertices of $\{v_1,v_2,v_3\}$ or the edge u_iv_i for some $i \in \{1,2,3\}$.

5 Interval function of (claw, net)-free graphs

$(u2)$ if $R(u,v) \cap R(u,w) \setminus \{u\} \neq \emptyset$ and $R(v,w) \cap R(u,v) \setminus \{v\} \neq \emptyset$ and $R(u,w) \cap R(v,w) \setminus \{w\} \neq \emptyset$, then $R(u,v,w) \cap \{u,v,w\} = \{u\}$ or $\{v\}$ or $\{w\}$, for all $u,v,w \in V$.

Theorem 4. *The interval function I of a connected graph G, satisfies $(u2)$ if and only if G is a (claw, net)-free graph.*

Proof. Suppose that G contains claw or net as an induced subgraphs. It is easy to see that in a graph with an induced claw shown in Figure 1, there exist u,v,w such that $I(u,v) \cap I(u,w) \setminus \{u\} = z$ and $I(v,w) \cap I(u,v) \setminus \{v\} = z$ and $I(u,w) \cap I(v,w) \setminus \{w\} = z$, and $I(u,v,w) \cap \{u,v,w\} = z$ and $R(u,v,w) \cap \{u,v,w\} = \emptyset$. Hence the interval function I does not satisfy $(u2)$. Moreover, If G contains a net as an induced subgraph, we can find vertices u,v,w shown in Figure 1, such that $I(u,v) \cap I(u,w) \setminus \{u\} = x$ and $I(v,w) \cap I(u,v) \setminus \{v\} = y$ and $I(u,w) \cap I(v,w) \setminus \{w\} = z$, and $I(u,v,w) = \emptyset$. There I does not fulfill $(u2)$.

Conversely, let G be a (claw, net)-free graph. Assume that $I(u,v) \cap I(u,w) \setminus \{u\} \neq \emptyset$ and $I(v,w) \cap I(u,v) \setminus \{v\} \neq \emptyset$ and $I(u,w) \cap I(v,w) \setminus \{w\} \neq \emptyset$. But I does not satisfy $(u2)$ on G. Let P and Q be a u,v-shortest path and u,w-shortest path respectively. Note that w does not lie on P, since otherwise, Q is a subpath of P and therefore $I(v,w) \cap I(u,w) \setminus \{w\} =$

$\emptyset$, a contradiction with assumption. Furthermore, Q must interest P, since otherwise, $I(u,v) \cap I(u,w) \setminus \{u\} = \emptyset$, a contradiction with assumption. Let x be the first common vertex between P and Q. It is easy to see that we have a claw on x and neighbors of x, $N(x) = \{z, y, y'\}$, where y' lies on u, x subpath of P and y lies on x, v subpath of P and z lies on x, w subpath of Q. Not that $z \neq w$ otherwise $I(u,w) \cap I(v,w) \setminus \{w\} = \emptyset$, a contradiction with assumption and also $y \neq v$ otherwise $I(v,w) \cap I(v,u) \setminus \{v\} = \emptyset$. To avoid having a claw an induced subgraph there must be edges between z, y' or y, y' or z, y. But z, y' and y, y' are not an edge since we get a contradiction with P and Q being the shortest path containing x. Suppose z, y is an edge. Now we have an induced net on $\{x, z, y, y'\}$, neighborhood of z on x, w-subpath of Q and neighborhood of y on x, v-subpath of P. Which is a final contradiction.

$\square$

6 Interval function of tent-free graphs

($u3$) for any $a \neq b \neq c \neq x \neq y \neq z \in V$, $R(x,y) = \{a, x, y\}, R(y,z) = \{b, y, z\}, R(x,z) = \{c, x, z\}, R(a,b,c) = \{a, b, c\}$, then $R(a,z) = \{a, z\}$ or $R(b,x) = \{b, x\}$ or $R(c,y) = \{c, y\}$.

Explanation: $R(x,y) = \{a, x, y\}, R(y,z) = \{b, y, z\}, R(x,z) = \{c, x, z\}$ imply that a, y, b, z, c, x is a cycle an the edges xy, yz, xz do not exist. $R(a,b,c) = \{a, b, c\}$ together the above 3 conditions imply that $\{a, b, c\}$ is a triangle. Then, we have a tent and one of the three possible edges must exist.

($u3$) for any $a \neq b \neq c \neq x \neq y \neq z \in V$, $R(a,b) = \{a, b\}$, $R(a,c) = \{a, c\}$, $R(b,c) = \{b, c\}$, $a \in R(x,z)$, $b \in R(y,z)$, $c \in R(x,y)$, then $R(x,b) = \{x, b\}$ or $R(y,a) = \{y, a\}$ or $R(z,c) = \{z, c\}$.

So far we know that if the interval function I of a connected graph G satisfies axiom ($u3$), then G does not contain net, tent, X-house$+e$ and $3-fan+e$ as an induced subgraph. Note that an edge e is adjacent to some specific vertex in X-house and 3-fan. Furthermore, the family of forbidden induced subgraphs for the interval function satisfied by the axioms($u3$) might be bigger. I think axiom ($u3$) is not a good axiom, we need only tent-free graphs. I tried but have not found a new axiom for tent-free graphs yet.

7 Interval function of unit interval graphs

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