Poisson Distribution and Queues

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What is a Poisson Distribution?

Definition and Context

The **Poisson distribution** is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time or space, provided that these events occur with a **known constant mean rate** and **independently** of the time since the last event.

Key Characteristics

1. **Occurrence Over an Interval:** The events (such as phone calls, accidents, or defects) happen over a continuous interval (time, area, volume, etc.).
2. **Constant Average Rate:** The average number of occurrences, denoted by λ (the rate parameter), is constant over the interval.
3. **Independence:** The events occur independently; the occurrence of one event does not affect the probability of another event occurring.
4. **Low Probability of Simultaneous Events:** In any very small sub-interval, the probability of more than one event occurring is negligible.

PMF of a Poisson Process

Probability Mass Function (PMF)

For a Poisson random variable X (which counts the number of events in the interval), the probability of observing exactly k events is given by:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad \text{for } k = 0, 1, 2, \ \dots$$

where:

- λ is the average (expected) number of events in the interval,
- e is the base of the natural logarithm (approximately 2.71828),
- $k!$ is the factorial of k.

Mean and Variance

- **Mean:** $E[X] = \lambda$
- **Variance:** $\text{Var}(X) = \lambda$

Because both the mean and variance are equal to λ, the Poisson distribution is uniquely characterized by a single parameter.

Word Problem #1

Word Problem: A call center receives an average of 10 calls per hour. What is the probability that exactly 15 calls will be received in a particular hour?

Step-by-Step Solution:

1. **Identify the Rate Parameter (λ):**
   - Given: $\lambda = 10$ calls per hour.
2. **Determine the Event of Interest:**
   - $k = 15$ calls.
3. **Apply the PMF Formula:**

$$P(X = 15) = \frac{10^{15} e^{-10}}{15!}$$

4. **Calculation:**
   - Calculate 10^{15} and $15!$ (15 factorial).
   - e^{-10} is computed using a calculator.

   Although you may use a calculator for exact values, the formula itself is:

$$\frac{10^{15} \cdot e^{-10}}{1307674368000} \approx 0.0347.$$

This means there is about a 3.47% chance, which is the probability $P(X = 15)$ for a Poisson process with $\lambda = 10$.

Word Problem #2

Word Problem: On average, 2 traffic accidents occur per day in a small town. What is the probability that exactly 5 accidents occur in a day?

Step-by-Step Solution:

1. **Rate Parameter (λ):**
   - $\lambda = 2$ accidents per day.
2. **Event of Interest:**
   - $k = 5$ accidents.
3. **Apply the PMF Formula:**

$$P(X = 5) = \frac{2^5 e^{-2}}{5!}$$

4. **Calculations:**
   - $2^5 = 32$,
   - $5! = 120$,
   - $e^{-2} \approx 0.1353$.
5. **Plug in the Values:**

$$P(X = 5) = \frac{32 \times 0.1353}{120} \approx \frac{4.3296}{120} \approx 0.0361.$$

6. **Conclusion:** There is approximately a 3.61% chance of observing exactly 5 accidents in

Word Problem #3

Word Problem: A web server receives on average 50 requests per minute. What is the probability that in a given minute, the server receives exactly 45 requests?

Step-by-Step Solution:

1. **Rate Parameter (λ):**
   - $\lambda = 50$ requests per minute.
2. **Event of Interest:**
   - $k = 45$ requests.
3. **Apply the PMF Formula:**

$$P(X = 45) = \frac{50^{45}e^{-50}}{45!}$$

4. **Calculation Considerations:**
   - The numbers are large, and the computation is typically done using a calculator or computer software.
   - The expression 50^{45} represents a large number, as does 45!. Using logarithms or software helps manage these computations.

The numerical value is approximately

$$P(X = 45) \approx 0.046,$$

or about **4.6%**.

Word Problem #4

Word Problem: A factory produces a large batch of items, and on average, there are 3 defects per 1000 items. If you examine 1000 items, what is the probability of finding no defects?

Step-by-Step Solution:

1. **Rate Parameter (λ):**
   - $\lambda = 3$ defects per 1000 items.
2. **Event of Interest:**
   - $k = 0$ defects.
3. **Apply the PMF Formula:**

$$P(X = 0) = \frac{3^0 e^{-3}}{0!}$$

4. **Simplify the Expression:**
   - $3^0 = 1$,
   - $0! = 1$,
   - Thus,

$$P(X = 0) = e^{-3} \approx 0.0498.$$

5. **Conclusion:** There is about a 4.98% chance of finding no defects in a batch of 1000 items.

Word Problem #5

Word Problem: A radioactive sample emits particles at an average rate of 5 particles per second. What is the probability that exactly 7 particles are emitted in one second?

Step-by-Step Solution:

1. **Rate Parameter (λ):**
   - $\lambda = 5$ particles per second.

2. **Event of Interest:**
   - $k = 7$ particles.

3. **Apply the PMF Formula:**

$$P(X = 7) = \frac{5^7 e^{-5}}{7!}$$

4. **Calculate Step-by-Step:**
   - $5^7 = 78125$,
   - $7! = 5040$,
   - $e^{-5} \approx 0.0067379.$

5. **Plug in the Values:**

$$P(X = 7) = \frac{78125 \times 0.0067379}{5040} \approx \frac{526.18}{5040} \approx 0.1044.$$

6. **Conclusion:** There is approximately a 10.44% chance that exactly 7 particles will be

Defenition of Discrete Poisson dist.

Definition of the Poisson Distribution

A Poisson-distributed random variable X with parameter $\lambda > 0$ has the probability mass function (PMF):

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \ \ldots$$

where:

- λ is the expected number of occurrences in a fixed interval.
- $e^{-\lambda}$ ensures the probabilities sum to 1.

Mathematical Expectation (Discrete)

The expectation is given by:

$$E[X] = \sum_{k=0}^{\infty} kP(X = k)$$

Step 1: Substitute the PMF

$$E[X] = \sum_{k=0}^{\infty} k\frac{e^{-\lambda}\lambda^k}{k!}$$

Step 2: Factor k out

$$E[X] = e^{-\lambda} \sum_{k=1}^{\infty} \frac{k\lambda^k}{k!}$$

Since $k! = k \cdot (k-1)!$, rewrite:

Cancelling k in the numerator and denominator:

$$E[X] = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}$$

Step 3: Recognizing the series

The summation resembles the Taylor series expansion of e^{λ}:

$$\sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = \lambda e^{\lambda}$$

Thus: $E[X] = e^{-\lambda} \cdot \lambda e^{\lambda} = \lambda$

Conclusion: $E[X] = \lambda$

which means the expected number of occurrences in an interval is simply λ.

Variance (Discrete)

Variance is defined as: $Var(X) = E[X^2] - (E[X])^2$

Step 1: Compute $E[X^2]$

First, the second moment $E[X^2]$ is: $E[X^2] = \sum_{k=0}^{\infty} k^2 P(X = k)$

Using the identity: $k^2 = k(k-1) + k$

we rewrite: $E[X^2] = \sum_{k=0}^{\infty} (k(k-1) + k) P(X = k)$

Step 2: Compute Two Summations

Splitting into two summations: $E[X^2] = \sum_{k=0}^{\infty} k(k-1) P(X = k) + \sum_{k=0}^{\infty} k P(X = k$

From earlier, $E[X] = \lambda$. Now, let's evaluate the first term: $\sum_{k=0}^{\infty} k(k-1) P(X = k)$

By similar expansion techniques as before, this turns out to be:

$\lambda^2 + \lambda$ Thus: $E[X^2] = \lambda^2 + \lambda + \lambda = \lambda^2 + 2\lambda$

Step 3: Compute $Var(X)$ $Var(X) = E[X^2] - (E[X])^2$

$Var(X) = (\lambda^2 + 2\lambda) - \lambda^2$ $Var(X) = \lambda$

which means the variance of the Poisson distribution is equal to its mean.

Continuous Poisson Distribution

Below is an explanation of the "continuous" side of a Poisson process. Although the classic Poisson distribution is defined only for nonnegative integers (counting events), many applications of the Poisson process involve measuring time continuously. In these contexts, we study the waiting times until events occur. Two important continuous distributions arise naturally:

1. **The Exponential Distribution:** This models the waiting time until the first event of a Poisson process.
2. **The Erlang (or Gamma) Distribution:** This models the waiting time until the kth event in a Poisson process.

Continuous Poisson Distribution (Exponential)

Poisson Process Recap

- A Poisson process is used to model random events occurring over time (or space) at a constant average rate λ (events per unit time).
- The number of events in a fixed interval of length t is governed by the discrete Poisson distribution:

$$P(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, 2, \ \ldots$$

Exponential Distribution

- **Definition:** The waiting time T until the first event in a Poisson process is exponentially distributed.
- **Probability Density Function (PDF):** $$f_T(t) = \lambda e^{-\lambda t}, \quad t \geq 0.$$
- **Key Properties:**
  - **Mean:** $\frac{1}{\lambda}$
  - **Memoryless Property:** The probability of waiting an additional time does not depend on how much time has already elapsed.

Continuous Poisson Distribution (Erlang)

Erlang (Gamma) Distribution

- **Definition:** The waiting time T_k until the kth event is the sum of k independent exponential random variables (for a Poisson process with rate λ).
- **PDF (for integer k – sometimes called the Erlang distribution):**

$$f_{T_k}(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!}, \quad t \geq 0.$$

- **Mean:** $E[T_k] = \frac{k}{\lambda}$
- **Variance:** $\mathrm{Var}(T_k) = \frac{k}{\lambda^2}$

CDF and PDF of Distributions

Continuous Analogues in Poisson Processes:

- **Exponential Distribution:** Models the waiting time until the first event. PDF: $f_T(t) = \lambda e^{-\lambda t}$ and CDF: $F_T(t) = 1 - e^{-\lambda t}$.
- **Erlang (Gamma) Distribution:** Models the waiting time until the kth event. PDF for integer k:

$$f_{T_k}(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!}.$$

Word Problem #6

Word Problem: A call center receives calls at an average rate of 10 calls per hour ($\lambda = 10$ per hour). What is the probability that the waiting time until the first call is less than 3 minutes?

1. **Convert Units:**
   - The rate is 10 per hour.
   - Convert 3 minutes to hours:

$$3 \text{ minutes} = \frac{3}{60} = 0.05 \text{ hours}.$$

2. **Model the Waiting Time:**
   - The waiting time T until the first event is exponentially distributed:

$$f_T(t) = \lambda e^{-\lambda t}, \quad t \geq 0.$$

   - Here, $\lambda = 10$ per hour.

3. **Find the Cumulative Distribution Function (CDF):**
   - The CDF for an exponential distribution is: $F_T(t) = P(T \leq t) = 1 - e^{-\lambda t}.$

4. **Plug in the Values:** $F_T(0.05) = 1 - e^{-10 \times 0.05} = 1 - e^{-0.5}.$

5. **Calculate the Probability:**
   - Using $e^{-0.5} \approx 0.6065$, $F_T(0.05) = 1 - 0.6065 = 0.3935.$

6. **Conclusion:** The probability that the waiting time until the first call is less than 3 minutes is approximately **39.35%**.

Word Problem #7

Word Problem: In the same call center ($\lambda = 10$ calls/hour), what is the probability that the waiting time until the third call is more than 12 minutes?

1. **Convert Units:**
   - Convert 12 minutes to hours: $$12 \text{ minutes} = \frac{12}{60} = 0.2 \text{ hours.}$$
2. **Model the Waiting Time for the Third Call:**
   - The waiting time until the 3rd call, T_3, follows an Erlang (Gamma) distribution with parameters $k = 3$ and $\lambda = 10$.
3. **CDF of the Erlang Distribution:**
   - The CDF for T_3 is: $$F_{T_3}(t) = 1 - e^{-\lambda t} \sum_{i=0}^{2} \frac{(\lambda t)^i}{i!}.$$

- We want the probability that $T_3 > 0.2$ hours, which is: $$P(T_3 > 0.2) = 1 - F_{T_3}(0.2).$$

4. **Compute $F_{T_3}(0.2)$:**
   - First, $\lambda t = 10 \times 0.2 = 2$.
   - Sum the series for $i = 0$ to 2: $$\sum_{i=0}^{2} \frac{2^i}{i!} = \frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} = 1 + 2 + 2 = 5.$$
   - Then, $$F_{T_3}(0.2) = 1 - e^{-2} \cdot 5.$$

   With $e^{-2} \approx 0.1353$, we have: $$F_{T_3}(0.2) = 1 - 5 \times 0.1353 = 1 - 0.6765 = 0.3235.$$
5. **Find the Complement:** $$P(T_3 > 0.2) = 1 - 0.3235 = 0.6765.$$
6. **Conclusion:** The probability that the waiting time for the third call is more than 12 minutes is approximately **67.65%**.

Word Problem #8

Word Problem: The time between traffic accidents in a small town is exponentially distributed with a mean of 30 minutes. What is the probability that the waiting time for the next accident exceeds 45 minutes?

1. **Identify the Mean and Rate λ:**
   - Mean waiting time = 30 minutes.
   - The rate λ is the reciprocal of the mean:

$$\boxed{\lambda = \frac{1}{30} \text{ per minute.}}$$

2. **Write the Exponential CDF:**
   - For waiting time T:

$$P(T > t) = e^{-\lambda t}.$$

3. **Plug in $t = 45$ minutes:** $P(T > 45) = e^{-(1/30)\times 45} = e^{-1.5}.$
4. **Calculate:**
   - $e^{-1.5} \approx 0.2231.$
5. **Conclusion:** The probability that the waiting time between accidents exceeds 45 minutes is about **22.31%**.

Word Problem #9

Word Problem: Buses arrive at a bus stop following a Poisson process at an average rate of 6 buses per hour. What is the probability that you have to wait more than 15 minutes for the next bus?

1. **Convert 15 Minutes to Hours:** $$15 \text{ minutes} = \frac{15}{60} = 0.25 \text{ hours.}$$
2. **Determine λ:**
   - Here, $\lambda = 6$ buses per hour.
3. **Use the Exponential Distribution:**
   - The probability that the waiting time T is greater than t is: $P(T > t) = e^{-\lambda t}$.
4. **Plug in the Values:**

$$P(T > 0.25) = e^{-6 \times 0.25} = e^{-1.5}.$$

5. **Calculate:**
   - $e^{-1.5} \approx 0.2231.$
6. **Conclusion:** The probability of waiting more than 15 minutes for the next bus is approximately **22.31%**.

Word Problem #10

Word Problem: The lifetime (time until failure) of a certain machine is exponentially distributed with a mean of 100 hours. What is the probability that the machine lasts at least 150 hours?

1. **Determine the Mean and λ:**
   - Mean lifetime = 100 hours.
   - The rate λ is:

$$\boxed{\lambda = \frac{1}{100} \text{ per hour.}}$$

2. **Use the Exponential Distribution's Survival Function:**
   - The probability that the machine lasts at least t hours: $P(T \geq t) = e^{-\lambda t}.$

3. **Plug in $t = 150$ Hours:**

$$P(T \geq 150) = e^{-(1/100)\times 150} = e^{-1.5}.$$

4. **Calculate:**
   - $e^{-1.5} \approx 0.2231.$

5. **Conclusion:** The probability that the machine lasts at least 150 hours is approximately **22.31%**.

Definition of Continuous Probability

Poisson Process Definition

A **Poisson process** with rate $\lambda > 0$ is characterized by:

- The number of events in a time interval t follows a Poisson distribution:

$$P(N(t) = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}, \quad k = 0, 1, 2, \ \dots$$

- The **waiting times** between consecutive events follow an exponential distribution with parameter λ:

$$f_T(t) = \lambda e^{-\lambda t}, \quad t \geq 0$$

where T represents the time until the next event.

Expectation (Continuous)

Expectation $E[T]$

The expectation of the **waiting time** T is given by: $E[T] = \int_0^\infty t f_T(t) dt$

Step 1: Substitute $f_T(t)$ $E[T] = \int_0^\infty t\lambda e^{-\lambda t} dt$

Step 2: Use Integration by Parts

Let:

- $u = t$, so $du = dt$
- $dv = \lambda e^{-\lambda t} dt$, so $v = -e^{-\lambda t}$

Using integration by parts: $\int udv = uv - \int vdu$

we get: $E[T] = [-te^{-\lambda t}]_0^\infty + \int_0^\infty e^{-\lambda t} dt$

Since $te^{-\lambda t} \to 0$ as $t \to \infty$, and evaluating the integral: $E[T] = 0 + \frac{1}{\lambda}$

Conclusion: $E[T] = \frac{1}{\lambda}$

meaning the expected waiting time between events is **inversely** proportional to λ.