Differential and Integral Calculus

KochiTechGroup

Differentiation through First Principles

The **first principle of differentiation**, also known as the **definition of a derivative**, is given by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Now, let's compute five simple derivatives using this principle:

1. $f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

Expanding,

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} (2x + h) = 2x$$

Differentiation Example #1

1. $f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

Expanding,

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} (2x + h) = 2x$$

Differentiation Example #2

2. $f(x) = x^3$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

Expanding,

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2$$

Example #3

3. $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Multiply by the conjugate:

$$= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Example #4

4. $f(x) = 1/x$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

Taking common denominator,

$$= \lim_{h \to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{-h}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)}$$

$$= -\frac{1}{x^2}$$

Example #5

$f(x) = \ln(x)$

$$f'(x) = \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h}$$

Using logarithm properties:

$$= \lim_{h \to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h}$$

Approximating for small h,

$$\ln(1 + h/x) \approx h/x$$

$$= \lim_{h \to 0} \frac{h/x}{h} = \frac{1}{x}$$

Example #6

$$f(x) = e^x$$

$$f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

Factor out e^x:

$$= \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h}$$

$$= \lim_{h \to 0} e^x \cdot \frac{e^h - 1}{h}$$

Since $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$,

$$= e^x$$

Example #7

$f(x) = \sin(x)$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

Using trigonometric identity:

$$\sin(x+h) - \sin(x) = 2\cos\left(x + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)$$

Approximating $\sin(h/2) \approx h/2$,

$$= \lim_{h \to 0} \frac{2\cos\left(x + \frac{h}{2}\right) \cdot \frac{h}{2}}{h}$$

$$= \lim_{h \to 0} \cos(x + h/2)$$

SUVAT equation in Physics

SUVAT equations describe motion under constant acceleration. They are derived using **differentiation** and **integration** from the basic kinematic definitions.

The five main SUVAT equations involve:

- s = displacement
- u = initial velocity
- v = final velocity
- a = acceleration
- t = time

The fundamental relationship between **velocity, acceleration, and displacement** is:

$$a = \frac{dv}{dt}, \quad v = \frac{ds}{dt}$$

Derivation of First SUVAT Equation

1st SUVAT Equation: $v = u + at$

Using acceleration definition:

$$a = \frac{dv}{dt}$$

Rearrange:

$$dv = a dt$$

Integrate both sides:

$$\int dv = \int a dt$$

$$v = at + C$$

At $t = 0$, velocity is u, so:

$$u = C$$

Thus, we get: $v = u + at$

Second SUVAT Equation

2nd SUVAT Equation: $s = ut + \frac{1}{2}at^2$

Using velocity definition:

$$v = \frac{ds}{dt}$$

Substituting $v = u + at$ from the first equation:

$$\frac{ds}{dt} = u + at$$

Rearrange:

$$ds = (u + at)dt$$

Integrate both sides:

$$\int ds = \int (u + at)dt$$

$$s = ut + \frac{1}{2}at^2 + C$$

At $t = 0$, displacement $s = 0$, so $C = 0$, giving:

$$s = ut + \frac{1}{2}at^2$$

Third SUVAT equation

3rd SUVAT Equation: $v^2 = u^2 + 2as$

From **1st SUVAT equation:** $v = u + at$

Multiply both sides by v: $vv = (u + at)v$

Since $v = \frac{ds}{dt}$, substitute:

$$v\frac{ds}{dt} = (u + at)v$$

Rearrange: $vds = (u + at)ds$

Integrate both sides: $\int vdv = \int (u + at)ds$

Using $ds = vdt$, replace: $\int vdv = \int (u + at)vdt$

Using $v = u + at$,

$$\int vdv = \int (u + at)(u + at)dt$$

Expanding and integrating:

$$\frac{1}{2}v^2 = \frac{1}{2}u^2 + uat + \frac{1}{2}a^2t^2$$

Rearrange: $v^2 = u^2 + 2as$

Maxima and Minima – Problem #1

Problem:

A company's revenue function is given by:

$$R(x) = -5x^2 + 100x$$

where x is the number of items sold. Find the number of items that maximizes revenue.

Solution:

1. First derivative: $R^{'}(x) = -10x + 100$
   - Set $R^{'}(x) = 0$:

$$-10x + 100 = 0 \Rightarrow x = 10$$

2. Second derivative: $R^{''}(x) = -10$
   - Since $R^{''}(x) < 0$, $x = 10$ is a **maximum**.
3. **Conclusion:** The company should produce **10 units** for maximum revenue.

Maxima and Minima – Problem #2

Problem:

The total cost $C(x)$ of producing x items is given by:

$$C(x) = x^2 - 12x + 60$$

Find the number of items that minimize production cost.

Solution:

1. First derivative: $C'(x) = 2x - 12$
   - Set $C'(x) = 0$:

$$2x - 12 = 0 \Rightarrow x = 6$$

2. Second derivative: $C''(x) = 2$
   - Since $C''(x) > 0$, $x = 6$ is a **minimum.**
3. **Conclusion:** Producing **6 units** minimizes cost.

Maxima and Minima – Example #3

Problem:

A farmer wants to build a rectangular pen using **100 meters** of fencing. If one side is along an existing wall, find the maximum area.

Solution:

1. **Let width =** x **meters,** then total fencing is:

$$\text{Length} = (100 - 2x)$$

$$A(x) = x(100 - 2x) = 100x - 2x^2$$

2. First derivative: $A'(x) = 100 - 4x$
   - Set $A'(x) = 0$:

$$100 - 4x = 0 \Rightarrow x = 25$$

3. Second derivative: $A''(x) = -4$
   - Since $A''(x) < 0$, $x = 25$ is a **maximum.**
4. **Conclusion:** The maximum area is when **width = 25 meters**.

Maxima and Minima – Problem #4

Problem:

The fuel consumption rate of a car in liters per kilometer is modeled by:

$$F(v) = v^2 - 20v + 200$$

where v is the speed in km/h. Find the speed that minimizes fuel consumption.

Solution:

1. First derivative: $F'(v) = 2v - 20$
   - Set $F'(v) = 0$:

$$2v - 20 = 0 \Rightarrow v = 10$$

2. Second derivative: $F''(v) = 2$
   - Since $F''(v) > 0$, $v = 10$ is a **minimum.**
3. **Conclusion:** The most fuel-efficient speed is **10 km/h**.

Numerical Differentiation

Calculus of Finite Differences – Method #1

Forward Difference Approximation

Formula

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

where h is a small step size.

Analytical Derivation

Using Taylor expansion:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + O(h^3)$$

Ignoring higher-order terms, we obtain:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

Numerical Example

Given $f(x) = x^2$, compute $f'(2)$ using $h = 0.01$:

$$f'(2) \approx \frac{(2.01)^2 - 2^2}{0.01} = \frac{4.0401 - 4}{0.01} = 4.01$$

Backward Difference Approximation – Part #2

Backward Difference Approximation

Formula

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$

Analytical Derivation

Taylor expansion at $x - h$:

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) + O(h^3)$$

Rearranging:

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$

Numerical Example

For $f(x) = x^3$, estimate $f'(2)$:

$$f'(2) \approx \frac{2^3 - (1.99)^3}{0.01} = \frac{8 - 7.880599}{0.01} = 11.94$$

Central Difference Approximation – Method #3

Central Difference Approximation

Formula $$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

This method is more accurate as it cancels error terms.

Analytical Derivation

Taylor expansions:
$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + O(h^3)$$
$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) + O(h^3)$$

Subtracting: $$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

Numerical Example

For $f(x) = \sin(x)$, estimate $f'(1)$:

$$f'(1) \approx \frac{\sin(1.01) - \sin(0.99)}{2 \times 0.01} = \frac{0.846 - 0.836}{0.02} = 0.5$$