Probability Axioms

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Basic Algebra of Probability (Kolmogrov)

Axioms and Definitions

A **probability space** is a triplet (Ω, F, P) where:

1. **Sample Space:** Ω is the set of all possible outcomes.
2. **Sigma-Algebra (F):** F is a collection of subsets of Ω (called events) satisfying:
   - **(A1) Non-emptiness:** $\Omega \in F$.
   - **(A2) Complements:** If $A \in F$ then $A^c = \Omega \setminus A \in F$.
   - **(A3) Countable Unions:** If $A_1, A_2, \cdots \in F$, then $\bigcup_{i=1}^{\infty} A_i \in F$.
3. **Probability Measure $P: F \to [0, 1]$:** This function satisfies:
   - **(P1) Non-negativity:** $P(A) \geq 0$ for all $A \in F$.
   - **(P2) Normalization:** $P(\Omega) = 1$.
   - **(P3) Countable Additivity:** For any sequence of mutually disjoint events $A_1, A_2, ..$ (i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$),

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

Basic Algebra of Events (aka Set Theory)

Basic Set Operations and Their Properties

For any two events A, B (subsets of Ω) we define:

- **Union:** $A \cup B = \{\omega \in \Omega : \omega \in A \text{ or } \omega \in B\}$.
- **Intersection:** $A \cap B = \{\omega \in \Omega : \omega \in A \text{ and } \omega \in B\}$.
- **Complement:** $A^c = \Omega \setminus A = \{\omega \in \Omega : \omega \notin A\}$.

1. Commutative Laws $\quad A \cup B = B \cup A \quad \text{and} \quad A \cap B = B \cap A.$

2. Associative Laws

$$(A \cup B) \cup C = A \cup (B \cup C) \quad \text{and} \quad (A \cap B) \cap C = A \cap (B \cap C).$$

3. Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

4. Identity Laws • $A \cup \emptyset = A.$ • $A \cap \Omega = A.$

5. Idempotent Laws $\quad A \cup A = A \quad \text{and} \quad A \cap A = A.$

6. Complementation Properties $\quad A \cup A^c = \Omega. \quad A \cap A^c = \emptyset. \quad (A^c)^c = A.$

7. De Morgan's Laws $\quad (A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c.$

Problem #1

Problem: In a school, let

- A be the event that a student passes Mathematics, and
- B be the event that the student passes English.

Suppose 70% of the students pass Mathematics, 65% pass English, and 50% pass both.

Questions: a) Find the probability that a student passes at least one of the two subjects (i.e. the union $A \cup B$). b) Find the probability that a student fails both subjects.

Solution:

a) **Algebra of Events (Inclusion–Exclusion Principle):**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

$$P(A \cup B) = 0.70 + 0.65 - 0.50 = 0.85.$$

b) **Complement:** The event of failing both subjects is the complement of passing at least one subject: $P((A \cup B)^c) = 1 - P(A \cup B) = 1 - 0.85 = 0.15.$

Problem #2

Problem: In a survey, let

- T be the event that a respondent likes tea, and
- C be the event that the respondent likes coffee.

It is found that 80% like tea, 75% like coffee, and 65% like both.

Questions: a) Determine the probability that a respondent likes either tea or coffee (i.e. $T \cup C$). b) What is the probability that a respondent likes neither beverage?

Solution:

a) Using the inclusion−exclusion principle:

$$P(T \cup C) = P(T) + P(C) - P(T \cap C) = 0.80 + 0.75 - 0.65 = 0.90.$$

b) The probability of liking neither is the complement of $T \cup C$:

$$P((T \cup C)^c) = 1 - 0.90 = 0.10.$$

Problem #3

3. Game Show Prizes

Problem: On a game show, let

- A be the event that a contestant wins a main prize, and
- B be the event that the contestant wins a bonus prize.

Suppose the probability of winning a main prize is 40%, the probability of winning the bonus prize is 30%, and 15% win both prizes.

Questions: a) Find the probability that a contestant wins at least one prize. b) Using De Morgan's law, determine the probability that the contestant wins no prize.

Solution:

a)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.40 + 0.30 - 0.15 = 0.55.$$

b) The probability of winning no prize is the complement:

$$P((A \cup B)^c) = 1 - 0.55 = 0.45.$$

Note: De Morgan's law tells us that $(A \cup B)^c = A^c \cap B^c$, which confirms that "no prize" means the contestant got neither A nor B.

Problem #4

5. Library Book Genres

Problem: In a library, let

- F be the event that a book is fiction, and
- I be the event that a book is illustrated.

Suppose 55% of the books are fiction, 35% are illustrated, and 20% are both.

Questions: Find the probability that a randomly selected book is either fiction or illustrated.

Solution:

Using the formula:

$$P(F \cup I) = P(F) + P(I) - P(F \cap I) = 0.55 + 0.35 - 0.20 = 0.70.$$

Problem #5

Problem: In a university, let

- E be the event that a student is an active member of the English Club, and
- F be the event that a student is an active member of the French Club.

Suppose 40% of the students are in the English Club, 30% are in the French Club, and 15% are in both clubs.

Questions: a) Use the algebra of events to find the probability that a randomly selected student is a member of at least one of these clubs. b) Find the probability that a student is in exactly one club (i.e. either English or French but not both).

Solution:

a)

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) = 0.40 + 0.30 - 0.15 = 0.55.$$

b) The event "exactly one club" is

$$(E \setminus F) \cup (F \setminus E),$$

and its probability is:

$$P(\text{exactly one}) = P(E \cup F) - P(E \cap F) = 0.55 - 0.15 = 0.40.$$

Problem #6

Problem: In a manufacturing plant, there are two machines. Let

- M_1 be the event that the first machine fails on a given day, and
- M_2 be the event that the second machine fails on a given day.

Suppose the probability of M_1 is 0.10, the probability of M_2 is 0.15, and the probability that they both fail is 0.05.

Questions: a) Calculate the probability that at least one machine fails. b) Use the complement to find the probability that neither machine fails.

Solution:

a)

$$P(M_1 \cup M_2) = P(M_1) + P(M_2) - P(M_1 \cap M_2) = 0.10 + 0.15 - 0.05 = 0.20.$$

b)

$$P((M_1 \cup M_2)^c) = 1 - 0.20 = 0.80.$$

BirthDay Problem

Problem: In a room of 23 people, what is the probability that at least two share the same birthday (assume 365 equally likely birthdays)?

Step-by-Step Analysis:

1. **Calculate the Complement:** The easier path is to calculate the probability that all 23 people have different birthdays, and then subtract that from 1.
2. **Probability that All Birthdays Are Different:**

$$P(\text{all different}) = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \cdots \times \frac{365-22}{365}.$$

Numerically, this product is approximately 0.4927.

3. **Compute the Desired Probability:**

$$P(\text{at least one shared}) = 1 - 0.4927 \approx 0.5073.$$

Birthday problem revisited

Problem: In a room of 30 people, what is the probability that at least two share the same birthday? (Assume 365 equally likely birthdays.)

Solution:

1. **Complement – All Different Birthdays:**

$$P(\text{all different}) = \frac{365}{365} \times \frac{364}{365} \times \cdots \times \frac{365-29}{365}.$$

2. **Calculation:** This product is approximately 0.2937.
3. **At Least One Shared Birthday:**

$$P(\text{shared birthday}) = 1 - 0.2937 \approx 0.7063.$$

Problem #9

Problem: A fair coin is flipped three times. What is the probability of getting exactly two heads?

Step-by-Step Analysis:

1. **Determine the Total Outcomes:** Each coin toss has 2 outcomes (Head or Tail). For three tosses, the total number of outcomes is:

$$2^3 = 8.$$

2. **Count the Favorable Outcomes:** To have exactly two heads, you must choose 2 tosses out of 3 to be heads. The number of ways to do this is given by the binomial coefficient:

$$\binom{3}{2} = 3.$$

The arrangements could be: HHT, HTH, THH.

3. **Calculate the Probability:**

$$P(\text{exactly 2 heads}) = \frac{3}{8} = 0.375.$$

Dice Problem

Problem: A six-sided die is rolled twice. What is the probability that the sum of the two rolls is 7?

Step-by-Step Analysis:

1. **Determine the Total Outcomes:** Each die has 6 outcomes, so for two dice:

$$6 \times 6 = 36.$$

2. **Identify the Favorable Outcomes:** The pairs of numbers that sum to 7 are: $(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$ This gives 6 favorable outcomes.
3. **Calculate the Probability:**

$$P(\text{sum} = 7) = \frac{6}{36} = \frac{1}{6} \approx 0.1667.$$

Uniform Distribution Problem

Problem: A random number is drawn uniformly from the interval [2, 8]. What is the probability that the number is between 3 and 5?

Solution:

1. **Uniform Density:** For X uniform on $[a, b]$, the density is

$$f(x) = \frac{1}{b - a}.$$

Here, $b - a = 8 - 2 = 6$.

2. **Probability Calculation:** The desired interval length is $5 - 3 = 2$. Therefore,

$$P(3 \leq X \leq 5) = \frac{2}{6} = \frac{1}{3} \approx 0.3333.$$

Answer: Approximately 33.33% probability.

Club Allocation

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- F be the event that a student is an active member of the French Club.

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Questions: a) Use the algebra of events to find the probability that a randomly selected student is a member of at least one of these clubs. b) Find the probability that a student is in exactly one club (i.e. either English or French but not both).

Solution:

a)

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) = 0.40 + 0.30 - 0.15 = 0.55.$$

b) The event "exactly one club" is

$$(E \setminus F) \cup (F \setminus E),$$

and its probability is:

$$P(\text{exactly one}) = P(E \cup F) - P(E \cap F) = 0.55 - 0.15 = 0.40.$$

Candidate Allocation

Problem: Consider a company where

- A is the event that a candidate's resume meets the minimum qualifications, and
- B is the event that the candidate passes the interview.

Suppose 80% of applicants meet qualifications, 70% of those meet the interview criteria, and overall, 60% of candidates satisfy both.

Questions: a) What is the probability a randomly chosen candidate qualifies in at least one phase? b) What is the probability a candidate does not satisfy either condition?
a) Here, "qualifies in at least one phase" means the candidate either meets the resume criteria or passes the interview (or both). Using inclusion−exclusion (if we assume interview is conducted only on those who qualify, then $P(B) = 0.70$ of the qualified group, but here it appears the figures are already given with overlap $P(A \cap B) = 0.60$). However, we treat A and B as separate events with:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

We need $P(B)$ overall. Notice that $P(B \mid A) = \frac{0.60}{0.80} = 0.75$. If the interview is only for qualified candidates, one might take $P(B) = 0.60$ overall. Assuming $P(B) = 0.60$ overall (combined outcome provided),

$$P(A \cup B) = 0.80 + 0.60 - 0.60 = 0.80.$$

$$P((A \cup B)^c) = 1 - 0.80 = 0.20.$$

Questions , if any