Take-home quiz 5

15-468/668/868, Physics-based Rendering, Spring 2021
Carnegie Mellon University

Due Wednesday, Mar. 24, at 11:59 pm ET

Question 1

We discussed in class the *multi-sample* version of multiple importance sampling. It is worth exploring the mathematical properties of this procedure in more detail. Let's assume that we want to estimate an integral of the form:

$$F = \int_D f(x) \, \mathrm{d}\mu(x). \tag{1}$$

To this end, we are given T sampling distributions, each with a probability density function $p_t(x), x \in D, t = 1, \ldots, T$. We assume that each of these sampling distributions can be used to form an unbiased Monte Carlo estimate of the target integral of Equation (1). We also assume that we have a total sample budget of N.

Instead of using just one of these distributions, in multi-sample multiple importance sampling we proceed as follows: In pre-processing, we allocate n_t samples to each sampling distribution t, so that $\sum_{t=1}^{T} n_t = 1$. At runtime, first, we independently draw n_t samples $X_{t,n}, n = 1, \ldots, n_t$, from each sampling distribution t, for a total of N independent samples. Second, we form an estimate of the target integral as:

$$\langle F \rangle = \sum_{t=1}^{T} \frac{1}{n_t} \sum_{n=1}^{n_t} \frac{w_t(X_{t,n}) f(X_{t,n})}{p_t(X_{t,n})}, \tag{2}$$

where we get to select the weights $w_t(X)$ for each $t = 1, \ldots, T$ and $X \in D$.

1. Prove that the estimator of Equation (2) is unbiased if $\sum_{t=1}^{T} w_t(X) = 1$, for all $X \in D$.

2. Prove that the variance of the estimator of Equation (2) equals

$$\mathrm{Var}(\langle F \rangle) = \underbrace{\sum_{t=1}^{T} \frac{1}{n_t^2} \sum_{n=1}^{n_t} \mathbb{E}\left[F_{n,t}^2\right]}_{\equiv \mathcal{S}} - \sum_{t=1}^{T} \frac{1}{n_t^2} \sum_{n=1}^{n_t} \mathbb{E}\left[F_{n,t}\right]^2, \tag{3}$$

where $\mathbb{E}[\cdot]$ is expectation and we have defined

$$F_{n,t} \equiv \frac{w_t(X_{t,n}) f(X_{t,n})}{p_t(X_{t,n})}. \tag{4}$$

3. Prove that the term $\mathcal{S}$ in Equation (3) is minimized by setting the weights w_t for every $X \in D$ according to the *balance heuristic*:

$$w_t(X) = \frac{c_t p_t(X)}{\sum_{t=1}^{T} c_t p_t(X)}, \tag{5}$$

where we have defined $c_t \equiv \frac{n_t}{N}, t = 1, \ldots, T$. To prove this, you should first express $\mathcal{S}$ as an integral over D; then use the method of Lagrange multipliers to minimize the integrand this integral for each $X \in D$, subject to the constraint $\sum_{t=1}^{T} w_t(X) = 1$.

Deliverables

As described on the course website, solutions are submitted through Canvas.

- **Write-up:** Your write-up should be typeset in LaTeX, and show consist of your answers to the quiz questions. **Please note that we do not accept handwritten scans for your write-up.**
- **Submission:** Your submission should be a PDF file, `<andrew-id.pdf>`, with your write-up. **Please do not submit ZIP files.**

Credits

Inspiration for the questions in this quiz came from Veach and Guibas [1].

References

[1] E. Veach and L. J. Guibas. Optimally combining sampling techniques for monte carlo rendering. In *Proceedings of the 22nd annual conference on Computer graphics and interactive techniques*, pages 419–428, 1995.