Time-of-Flight Rendering

Independent Study Exercises

Monte Carlo Rendering & Inverse Rendering

Contents

1 Introduction to Time-of-Flight Imaging

Background

Time-of-flight (ToF) imaging exploits the finite speed of light to recover depth and temporal information about a scene. Unlike conventional cameras that integrate all arriving light regardless of when it was emitted, ToF systems encode path length information into their measurements.

The key insight is that light traveling a distance d experiences a time delay $\tau = d/c$, where $c \approx 3 \times 10^8$ m/s is the speed of light. By modulating the light source and/or sensor in time, we can extract information about these delays and hence about scene geometry.

In this assignment, we develop the mathematical foundations for simulating ToF imaging systems, starting from the basic physics of light propagation and building up to amplitude-modulated continuous-wave (AMCW) and frequency-swept continuous-wave (FSCW) systems.

2 Time-Shifting and Radiance Propagation

Background

In standard rendering, we work with the steady-state radiance field $L(\mathbf{x}, \boldsymbol{\omega})$. For time-of-flight rendering, we must extend this to a *transient* radiance field $L(\mathbf{x}, \boldsymbol{\omega}, t)$ that captures how radiance varies with time.

The propagation of light in free space is governed by the fact that light travels at speed c along rays. This fundamental constraint relates the radiance at different points along a ray and leads to the key property we need for transient rendering: **time-shifting**.

2.1 The Method of Characteristics

To derive the time-shifting property, we'll use the *method of characteristics*, a standard technique for solving first-order PDEs. The idea is to find curves in space-time along which the PDE reduces to an ordinary differential equation (ODE).

Consider a ray parameterized by arc length s:

$$\mathbf{x}(s) = \mathbf{x}_0 + s\boldsymbol{\omega}$$

where $\mathbf{x}_0$ is the starting point and $\boldsymbol{\omega}$ is the unit direction. Light traveling along this ray at speed c relates position and time: if light is at $\mathbf{x}_0$ at time t_0, it reaches $\mathbf{x}(s)$ at time

$$t(s) = t_0 + \frac{s}{c}.$$

The curves $(s, t(s))$ are called *characteristic curves*. Along these curves, we can track how radiance evolves.

Exercise 1 (Transport Equation Along Characteristics)**.** Let $\tilde{L}(s) = L(\mathbf{x}(s), \boldsymbol{\omega}, t(s))$ denote the radiance evaluated along a characteristic curve.

(a) Using the chain rule, show that:

$$\frac{\mathrm{d}\tilde{L}}{\mathrm{d}s} = (\boldsymbol{\omega} \cdot \nabla_{\mathbf{x}})L + \frac{1}{c}\frac{\partial L}{\partial t}$$

Hint: Compute $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}$ and $\frac{\mathrm{d}t}{\mathrm{d}s}$ from the definitions above.

(b) In free space (no absorption, emission, or scattering), radiance is conserved along rays. What does this imply about $\frac{\mathrm{d}\tilde{L}}{\mathrm{d}s}$? Write the resulting PDE for $L(\mathbf{x}, \boldsymbol{\omega}, t)$.

2.2 From the Transport Equation to Time-Shifting

The transport equation tells us that $\tilde{L}(s) = L(\mathbf{x}(s), \boldsymbol{\omega}, t(s))$ is constant along characteristic curves. This has an immediate and powerful consequence: if we know the radiance at one point on a ray, we know it everywhere along that ray—we just need to account for the time delay.

Exercise 2 (Time-Shifting Property)**.** Let $\mathbf{y}$ and $\mathbf{x}$ be two points along a ray with $\mathbf{x} = \mathbf{y} + d\boldsymbol{\omega}$ for some distance $d > 0$. Consider the characteristic curve passing through $(\mathbf{x}, t)$, parameterized as $\mathbf{x}(s) = \mathbf{y} + s\boldsymbol{\omega}$ with $t(s) = t(0) + s/c$.

(a) Since $\mathbf{x}(d) = \mathbf{x}$ and $t(d) = t$, find $t(0)$. Then use the fact that $\tilde{L}$ is constant along characteristics (Exercise 1) to derive the **time-shifting property**:

$$L(\mathbf{x}, \boldsymbol{\omega}, t) = L\left(\mathbf{y}, \boldsymbol{\omega}, t - \frac{d}{c}\right)$$

(b) Interpret this result: what does it tell us about how radiance propagates?

Remark 1 (Application to Rendering)**.** The time-shifting property is fundamental to transient rendering. If a surface point $\mathbf{y}$ emits radiance $L_{\text{out}}(\mathbf{y}, \boldsymbol{\omega}, t')$, then a sensor at $\mathbf{x}$ (distance d away in direction $-\boldsymbol{\omega}$) observes this radiance at time $t = t' + d/c$:

$$L_{\text{sensor}}(t) = L_{\text{out}}\left(\mathbf{y}, \boldsymbol{\omega}, t - \frac{d}{c}\right)$$

In path tracing, we simply accumulate each path's contribution to a time bin determined by its total path length.

2.3 The Transient Rendering Equation

In steady-state rendering, the rendering equation relates outgoing radiance to emitted and reflected light:

$$L_o(\mathbf{x}, \boldsymbol{\omega}_o) = L_e(\mathbf{x}, \boldsymbol{\omega}_o) + \int_{\mathcal{H}^2} f_r(\mathbf{x}, \boldsymbol{\omega}_i, \boldsymbol{\omega}_o) L_i(\mathbf{x}, \boldsymbol{\omega}_i)(\boldsymbol{\omega}_i \cdot \mathbf{n})\, \mathrm{d}\boldsymbol{\omega}_i$$

For transient rendering, we extend this by adding time dependence. The key physical assumption is that **scattering is instantaneous**: light arriving at time t is immediately scattered at time t (no delay at the surface). This is valid for most materials at ToF timescales.

Exercise 3 (Transient Rendering Equation)**.** (a) Write the transient rendering equation for $L_o(\mathbf{x}, \boldsymbol{\omega}_o, t)$, incorporating time dependence in emission L_e and incident radiance L_i.

(b) Let $\mathbf{y}$ be the nearest visible surface point from $\mathbf{x}$ in direction $\boldsymbol{\omega}_i$, at distance d. Using the time-shifting property, express $L_i(\mathbf{x}, \boldsymbol{\omega}_i, t)$ in terms of the outgoing radiance at $\mathbf{y}$.

(c) Explain why this creates a recursive relationship where radiance at time t depends on radiance at *earlier* times.

3 AMCW Time-of-Flight Imaging

Background

Amplitude-Modulated Continuous-Wave (AMCW) ToF cameras are the most common consumer ToF technology (e.g., Microsoft Kinect, Intel RealSense). Instead of emitting short pulses and directly measuring time delays, AMCW systems:

1. Modulate the light source intensity sinusoidally at frequency f_m
2. Measure the phase shift between emitted and received signals
3. Convert phase shift to depth

The elegant insight is that phase measurement can be done with simple hardware (correlation with a reference signal), avoiding the need for picosecond-resolution timing circuits.

3.1 Phasor Representation

Working with sinusoidal signals is greatly simplified by the *phasor* representation.

Definition 1 (Phasor)**.** A sinusoidal signal $a(t) = A\cos(\omega t + \phi)$ can be represented as a complex number (phasor):

$$\tilde{A} = Ae^{\mathrm{i}\phi} \in \mathbb{C}$$

where $A = |\tilde{A}|$ is the amplitude and $\phi = \arg(\tilde{A})$ is the phase. The original signal is recovered as $a(t) = \mathrm{Re}[\tilde{A}e^{\mathrm{i}\omega t}]$.

The power of phasors is that **time delays become phase shifts**: if $a(t) = A\cos(\omega t)$ is delayed by time τ, then:

$$a(t-\tau) = A\cos(\omega(t-\tau)) = A\cos(\omega t - \omega\tau)$$

The delayed signal has phasor $\tilde{A}_{\text{delayed}} = Ae^{-\mathrm{i}\omega\tau} = \tilde{A}\cdot e^{-\mathrm{i}\omega\tau}$.

3.2 The AMCW Measurement Model

An AMCW system has three components, each described by a phasor:

1. **Light source**: Emits intensity $s(t) = s_0(1 + m_s\cos(\omega_m t))$
2. **Scene**: Transforms emitted light into received light via a linear response
3. **Sensor**: Correlates received light with a reference signal

Exercise 4 (Light Source and Depth Range)**.** Consider an AMCW light source with intensity:

$$s(t) = s_0\left(1 + m_s\cos(\omega_m t)\right)$$

where s_0 is the mean intensity, $m_s \in [0,1]$ is the modulation depth, and $\omega_m = 2\pi f_m$ is the angular modulation frequency.

(a) The modulated component of $s(t)$ is $s_0 m_s\cos(\omega_m t)$. What is its phasor $\tilde{S}$? (Use the convention that the source has zero phase at $t = 0$.)

(b) For a typical modulation frequency of $f_m = 20$ MHz, compute:

- The modulation wavelength $\lambda_m = c/f_m$
- The maximum unambiguous depth $d_{\max}$ (the depth at which the round-trip phase shift reaches 2π)

(c) Why is phase ambiguity a limitation of single-frequency AMCW systems?

3.3 Scene Response and Convolution

The scene acts as a linear time-invariant system. Light emitted at time t returns at various later times depending on path lengths in the scene. The scene's impulse response $h(\tau)$ gives the contribution of all paths with time delay τ:

$$r(t) = (s * h)(t) = \int_0^\infty s(t-\tau)h(\tau)\,\mathrm{d}\tau$$

For sinusoidal input, the Fourier transform of h fully characterizes the scene's effect.

Definition 2 (Scene Phasor)**.** The scene phasor at modulation frequency ω_m is:

$$\tilde{H}(\omega_m) = \int_0^\infty h(\tau)e^{-\mathrm{i}\omega_m \tau}\,\mathrm{d}\tau$$

This is the Fourier transform of $h(\tau)$ evaluated at ω_m.

Exercise 5 (Scene Response)**.** (a) For a single reflector at depth d with reflectivity ρ, the impulse response is $h(\tau) = \rho \cdot \delta(\tau - \tau_d)$ where $\tau_d = 2d/c$. Compute the scene phasor $\tilde{H}(\omega_m)$.

(b) Show that for sinusoidal source modulation, the received signal's phasor is $\tilde{R} = \tilde{S} \cdot \tilde{H}(\omega_m)$. What is the phase of $\tilde{R}$ for the single-reflector case?

(c) For a scene with multiple reflectors at depths $d_1, d_2, \ldots$ with reflectivities $\rho_1, \rho_2, \ldots$, write the scene phasor as a sum. What does this imply about separating contributions from different depths?

3.4 Sensor Demodulation

The sensor extracts the received signal's phasor by correlation with a reference signal. The sensor response is $g(t) = g_0(1 + m_g \cos(\omega_m t + \psi))$, where ψ is an adjustable phase offset. The correlation measurement is:

$$I(\psi) = \frac{1}{T}\int_0^T r(t)g(t)\,\mathrm{d}t$$

Exercise 6 (Four-Bucket Demodulation)**.** Let the received signal be $r(t) = r_0(1+m_r\cos(\omega_m t+\phi_r))$.

(a) Show that for long integration time $T \gg 2\pi/\omega_m$:

$$I(\psi) = r_0 g_0 \left(1 + \frac{m_r m_g}{2}\cos(\phi_r - \psi)\right)$$

Hint: Expand the product $r(t)g(t)$ and use the identity $\cos A \cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$. Terms oscillating at ω_m or $2\omega_m$ average to zero.

(b) The "four-bucket" algorithm samples at $\psi \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$. Write out the four measurements I_0, I_1, I_2, I_3 and show that:

$$\phi_r = \arctan\left(\frac{I_1 - I_3}{I_0 - I_2}\right)$$

(c) How is depth d computed from ϕ_r?

Remark 2 (The Phasor Imaging Equation)**.** The complete AMCW measurement can be elegantly expressed in phasor form. If we define the measurement phasor as $\tilde{I} = (I_0 - I_2) + \mathrm{i}(I_1 - I_3)$, then:

$$\tilde{I} = \tilde{S} \cdot \tilde{H}(\omega_m) \cdot \tilde{G}^*$$

where $\tilde{S}$, $\tilde{H}$, and $\tilde{G}$ are the source, scene, and sensor phasors. This factorization is the foundation of *phasor imaging* and enables elegant analysis of multi-frequency and coded ToF systems.

4 FSCW Time-of-Flight Imaging

Background

Frequency-Swept Continuous-Wave (FSCW) systems, also known as FMCW (frequency-modulated continuous-wave), use a different strategy. Instead of a fixed modulation frequency, the source frequency is swept over a range during the measurement period.
FSCW is widely used in radar and increasingly in optical systems. Its key advantage over AMCW is the ability to **resolve multiple returns at different depths** without phase ambiguity, because depth is encoded in frequency rather than phase.

4.1 Chirp Signals

Definition 3 (Linear Chirp)**.** A *linear chirp* signal has instantaneous frequency that increases linearly with time:

$$f(t) = f_0 + \frac{B}{T}t \quad \text{for } t \in [0, T]$$

where f_0 is the starting frequency, B is the bandwidth (total frequency sweep), and T is the sweep period.

Since instantaneous frequency is the time derivative of phase, $f(t) = \frac{1}{2\pi}\frac{\mathrm{d}\phi}{\mathrm{d}t}$, the phase is:

$$\phi(t) = 2\pi \int_0^t f(t')\,\mathrm{d}t' = 2\pi f_0 t + \frac{\pi B}{T}t^2$$

The chirp signal is therefore:

$$s(t) = A\cos\left(2\pi f_0 t + \frac{\pi B}{T}t^2\right)$$

4.2 Homodyne Detection and Phase Difference

The key to FSCW is *homodyne detection*: multiplying the received signal with the transmitted signal. To see why this extracts the phase difference, write two sinusoids using $\cos\phi = \mathrm{Re}[e^{\mathrm{i}\phi}] = \frac{1}{2}(e^{\mathrm{i}\phi} + e^{-\mathrm{i}\phi})$:

$$\begin{aligned} s(t) \cdot r(t) = \cos\phi_s \cdot \cos\phi_r &= \frac{1}{4}(e^{\mathrm{i}\phi_s} + e^{-\mathrm{i}\phi_s})(e^{\mathrm{i}\phi_r} + e^{-\mathrm{i}\phi_r}) \\ &= \frac{1}{4}(e^{\mathrm{i}(\phi_s+\phi_r)} + e^{\mathrm{i}(\phi_s-\phi_r)} + e^{-\mathrm{i}(\phi_s-\phi_r)} + e^{-\mathrm{i}(\phi_s+\phi_r)}) \\ &= \frac{1}{2}\cos(\phi_s + \phi_r) + \frac{1}{2}\cos(\phi_s - \phi_r) \end{aligned}$$

The sum-frequency term $\cos(\phi_s + \phi_r)$ oscillates rapidly and is removed by low-pass filtering. The difference-frequency term $\cos(\phi_s - \phi_r)$ oscillates slowly and encodes the **phase difference** $\phi_s(t) - \phi_r(t)$.

Exercise 7 (Beat Frequency and Depth)**.** In an FSCW system, the received signal from a reflector at depth d is a delayed copy of the transmitted signal: $r(t) = \rho \cdot s(t - \tau_d)$ where $\tau_d = 2d/c$.

The sensor performs homodyne detection: it multiplies received and transmitted signals and low-pass filters to extract the "beat" signal.

(a) Compute the phase difference $\phi_s(t) - \phi_r(t)$ between transmitted and received signals. Show that for $\tau_d \ll T$, this simplifies to approximately:

$$\phi_s(t) - \phi_r(t) \approx 2\pi f_0 \tau_d + \frac{2\pi B \tau_d}{T} t$$

(b) The beat frequency is the coefficient of t divided by 2π. Express the beat frequency f_b in terms of depth d.

4.3 Depth Resolution

The beat signal is analyzed using an FFT over the sweep period T, which has frequency resolution $\Delta f = 1/T$. Since depth and beat frequency are related by $d = f_b cT/(2B)$, the depth resolution is:

$$\Delta d = \frac{c}{2B}$$

Note that resolution depends *only* on bandwidth B, not on sweep time or center frequency. For $B = 1$ GHz, this gives $\Delta d = 15$ cm.

Exercise 8 (Multiple Reflectors in FSCW vs AMCW)**.** For a scene with two reflectors at depths d_1 and d_2, the beat signal is a sum of two sinusoids at frequencies $f_{b,1}$ and $f_{b,2}$.

(a) Describe how to process the beat signal to recover the two depths d_1 and d_2.

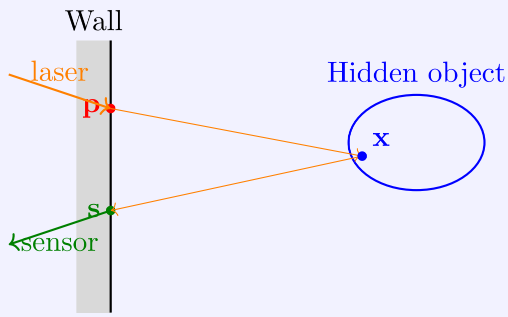
(b) Explain why single-frequency AMCW cannot separate these two returns.

5 Fermat Paths and Non-Line-of-Sight Imaging

Background

Non-line-of-sight (NLOS) imaging aims to reconstruct objects hidden from direct view by analyzing indirect reflections. A typical setup:

- A pulsed laser illuminates a point $\mathbf{p}$ on a visible "relay wall"
- Light bounces off hidden objects and returns to the wall
- A time-resolved sensor at point $\mathbf{s}$ measures the transient response
- By scanning $\mathbf{p}$ and/or $\mathbf{s}$, we gather data to reconstruct the hidden scene



The key mathematical insight is that at any measurement time τ, light is received from all hidden points $\mathbf{x}$ satisfying a constant path length constraint—and these points form an **ellipsoid**.

Exercise 9 (Ellipsoidal Constraints)**.** For the NLOS geometry above, let $\mathbf{x}$ be a point on the hidden object.

(a) Write the total path length $\ell(\mathbf{p}, \mathbf{x}, \mathbf{s})$ for light traveling from wall point $\mathbf{p}$ to hidden point $\mathbf{x}$ to sensor point $\mathbf{s}$.

(b) For fixed $\mathbf{p}$ and $\mathbf{s}$, what is the locus of points $\mathbf{x}$ that contribute to the measurement at time τ? (Express as an equation.)

(c) This locus is an ellipsoid with foci at $\mathbf{p}$ and $\mathbf{s}$. As τ increases, describe how this ellipsoid changes and what this means for the transient measurement.

5.1 Fermat Points and Discontinuities

Definition 4 (Fermat Point)**.** A *Fermat point* $\mathbf{x}^*$ on a hidden surface is a point where the path length $\ell(\mathbf{x}) = \|\mathbf{x}-\mathbf{p}\| + \|\mathbf{x}-\mathbf{s}\|$ is stationary with respect to motion along the surface. Equivalently, $\nabla_{\mathbf{x}}\ell$ is perpendicular to the surface (parallel to the surface normal).

Fermat points are significant because they create **discontinuities** or sharp features in the transient measurement. These occur when the expanding ellipsoid first touches (or last leaves) the hidden surface.

Exercise 10 (Specular Reflection at Fermat Points)**.** Let $\mathbf{n}$ be the surface normal at a point $\mathbf{x}$ on the hidden surface. Define unit vectors pointing toward the illumination and sensor:

$$\hat{\mathbf{d}}_p = \frac{\mathbf{p}-\mathbf{x}}{\|\mathbf{p}-\mathbf{x}\|}, \quad \hat{\mathbf{d}}_s = \frac{\mathbf{s}-\mathbf{x}}{\|\mathbf{s}-\mathbf{x}\|}$$

(a) Compute $\nabla_{\mathbf{x}}\ell$ and show that the Fermat condition (gradient parallel to $\mathbf{n}$) requires:

$$(\hat{\mathbf{d}}_p + \hat{\mathbf{d}}_s) \times \mathbf{n} = \mathbf{0}$$

(b) Show that this is equivalent to the law of specular reflection: the angle of incidence (from $\mathbf{p}$) equals the angle of reflection (toward $\mathbf{s}$).

(c) Explain physically why Fermat points correspond to specular reflections.

Exercise 11 (Reconstruction from Discontinuities)**.** Consider scanning the illumination point $\mathbf{p}$ while keeping the sensor at $\mathbf{s} = \mathbf{p}$ (confocal configuration). At each position $\mathbf{p}$, measure the time $\tau^*(\mathbf{p})$ of the first discontinuity (when the ellipsoid first touches the hidden surface).

(a) Each measurement $\tau^*(\mathbf{p})$ constrains the Fermat point $\mathbf{x}^*$ to lie on an ellipsoid (which becomes a sphere for $\mathbf{s} = \mathbf{p}$). Write this constraint equation.

(b) Explain how measurements from multiple wall positions $\mathbf{p}_1, \mathbf{p}_2, \ldots$ can be combined to localize $\mathbf{x}^*$.

(c) In the confocal case, the constraint simplifies to $\|\mathbf{x}^* - \mathbf{p}\| = c\tau^*/2$. This is a sphere of radius $c\tau^*/2$ centered at $\mathbf{p}$. How many measurements are needed to determine $\mathbf{x}^*$ in 3D?

6 Summary

ToF Technique	**Key Equation**	**Depth Encoding**
Transient rendering	$L(\mathbf{x}, \boldsymbol{\omega}, t) = L(\mathbf{y}, \boldsymbol{\omega}, t - d/c)$	Time bin from path length
AMCW	$\phi_r = -4\pi d/\lambda_m$	Phase of phasor
FSCW	$f_b = 2Bd/(cT)$	Beat frequency
NLOS	$\|\mathbf{x}-\mathbf{p}\| + \|\mathbf{x}-\mathbf{s}\| = c\tau$	Ellipsoidal constraint

Key Takeaways:

- **Time-shifting**: The transient transport equation implies radiance propagates along rays with time delay d/c. This is the foundation of all transient rendering.
- **AMCW**: Sinusoidal modulation encodes depth in phase. Four-bucket demodulation extracts the phasor. Limited by phase wrapping beyond $d_{\max} = c/(2f_m)$.
- **FSCW**: Frequency sweeps encode depth in beat frequency. FFT analysis separates multiple reflectors. Resolution $\Delta d = c/(2B)$ depends only on bandwidth.
- **NLOS**: Ellipsoidal geometry governs transient measurements. Fermat points (specular reflections) create discontinuities. Scanning and triangulation enable reconstruction of hidden scenes.