Chapter 1

Introduction

Distributed computing systems are characterized by their structure: a typical distributed computing system will consist of some large number of interacting devices that each run their own programs but that are affected by receiving messages or observing shared-memory updates from other devices. Examples of distributed computing systems range from simple systems in which a single client talks to a single server to huge amorphous networks like the Internet as a whole.

As distributed systems get larger, it becomes harder and harder to predict or even understand their behavior. Part of the reason for this is that we as programmers have not yet developed the kind of tools for managing complexity (like subroutines or objects with narrow interfaces, or even simple structured programming mechanisms like loops or if/then statements) that are standard in sequential programming. Part of the reason is that large distributed systems bring with them large amounts of inherent **nondeterminism**—unpredictable events like delays in message arrivals, the sudden failure of components, or in extreme cases the nefarious actions of faulty or malicious machines opposed to the goals of the system as a whole. Because of the unpredictability and scale of large distributed systems, it can often be difficult to test or simulate them adequately. Thus there is a need for theoretical tools that allow us to prove properties of these systems that will let us use them with confidence.

The first task of any theory of distributed systems is modeling: defining a mathematical structure that abstracts out all relevant properties of a large distributed system. There are many foundational models for distributed systems, but for this class we will follow [AW04] and use simple automaton-based models. Here we think of the system as a whole as passing from one

global state or **configuration** to another in response to *events*, e.g. local computation at some processor, an operation on shared memory, or the delivery of a message by the network. The details of the model will depend on what kind of system we are trying to represent:

- **Message passing** models (which we will cover in Part I) correspond to systems where processes communicate by sending messages through a network. In **synchronous message-passing**, every process sends out messages at time t that are delivered at time $t+1$, at which point more messages are sent out that are delivered at time $t+2$, and so on: the whole system runs in lockstep, marching forward in perfect synchrony. Such systems are difficult to build when the components become too numerous or too widely dispersed, but they are often easier to analyze than **asynchronous** systems, where messages are delivered eventually after some unknown delay. Variants on these models include **semi-synchronous** systems, where message delays are unpredictable but bounded, and various sorts of timed systems. Further variations come from restricting which processes can communicate with which others, by allowing various sorts of failures (**crash failures** that stop a process dead, **Byzantine failures** that turn a process evil, or **omission failures** that drop messages in transit), or—on the helpful side—by supplying additional tools like **failure detectors** (Chapter 11) or **randomization** (Chapter 23).

- **Shared-memory** models (Part II) correspond to systems where processes communicate by executing operations on shared objects that in the simplest case are typically simple memory cells supporting read and write operations (), but which could be more complex hardware primitives like **compare-and-swap** (§18.1.3), **load-linked/store-conditional** (§18.1.3), **atomic queues**, or more exotic objects from the seldom-visited theoretical depths. Practical shared-memory systems may be implemented as **distributed shared-memory** (Chapter 16) on top of a message-passing system in various ways.

  Like message-passing systems, shared-memory systems must also deal with issues of asynchrony and failures, both in the processes and in the shared objects.

- Other specialized models emphasize particular details of distributed systems, such as the labeled-graph models used for analyzing routing or the topological models used to represent some specialized agreement problems (see Chapter 28.

We'll see many of these at some point in this course, and examine which of them can simulate each other under various conditions.

Properties we might want to prove about a model include:

- **Safety** properties, of the form "nothing bad ever happens" or more precisely "there are no bad reachable states of the system." These include things like "at most one of the traffic lights at the intersection of Busy and Main is ever green." Such properties are typically proved using **invariants**, properties of the state of the system that are true initially and that are preserved by all transitions; this is essentially a disguised induction proof.

- **Liveness** properties, of the form "something good eventually happens." An example might be "my email is eventually either delivered or returned to me." These are not properties of particular states (I might unhappily await the eventual delivery of my email for decades without violating the liveness property just described), but of executions, where the property must hold starting at some finite time. Liveness properties are generally proved either from other liveness properties (e.g., "all messages in this message-passing system are eventually delivered") or from a combination of such properties and some sort of timer argument where some progress metric improves with every transition and guarantees the desirable state when it reaches some bound (also a disguised induction proof).

- **Fairness** properties are a strong kind of liveness property of the form "something good eventually happens to everybody." Such properties exclude **starvation**, a situation where most of the kids are happily chowing down at the orphanage ("some kid eventually eats something" is a liveness property) but poor Oliver Twist is dying for lack of gruel in the corner.

- **Simulations** show how to build one kind of system from another, such as a reliable message-passing system built on top of an unreliable system (TCP), a shared-memory system built on top of a message-passing system (distributed shared-memory), or a synchronous system build on top of an asynchronous system (**synchronizers**—see Chapter 13).

- **Impossibility results** describe things we can't do. For example, the classic **Two Generals** impossibility result (Chapter 3) says that it's impossible to guarantee agreement between two processes across an

unreliable message-passing channel if even a single message can be lost. Other results characterize what problems can be solved if various fractions of the processes are unreliable, or if asynchrony makes timing assumptions impossible. These results, and similar lower bounds that describe things we can't do quickly, include some of the most technically sophisticated results in distributed computing. They stand in contrast to the situation with sequential computing, where the reliability and predictability of the underlying hardware makes proving lower bounds extremely difficult.

There are some basic proof techniques that we will see over and over again in distributed computing.

For **lower bound** and **impossibility** proofs, the main tool is an **indistinguishability** argument. Here we construct two (or more) executions in which some process has the same input and thus behaves the same way, regardless of what algorithm it is running. This exploitation of process's ignorance is what makes impossibility results possible in distributed computing despite being notoriously difficult in most areas of computer science.[1]

For **safety properties**, statements that some bad outcome never occurs, the main proof technique is to construct an **invariant**. An invariant is essentially an induction hypothesis on reachable configurations of the system; an invariant proof shows that the invariant holds in all initial configurations, and that if it holds in some configuration, it holds in any configuration that is reachable in one step.

Induction is also useful for proving **termination** and **liveness** properties, statements that some good outcome occurs after a bounded amount of time. Here we typically structure the induction hypothesis as a **progress measure**, showing that some sort of partial progress holds by a particular time, with the full guarantee implied after the time bound is reached.

[1] An exception might be lower bounds for data structures, which also rely on a process's ignorance.

Part I

Message passing

Chapter 2

Model

See [AW04, Chapter 2] for details. We'll just give the basic overview here.

2.1 Basic message-passing model

We have a collection of n **processes** $p_1 \dots p_2$, each of which has a **state** consisting of a state from from state set Q_i, together with an inbuf and outbuf component representing messages available for delivery and messages posted to be sent, respectively. Messages are point-to-point, with a single sender and recipient: if you want broadcast, you have to pay for it. A **configuration** of the system consists of a vector of states, one for each process. The configuration of the system is updated by an **event**, which is either a **delivery event** (a message is moved from some process's outbuf to the appropriate process's inbuf) or a **computation event** (some process updates its state based on the current value of its inbuf and state components, possibly adding new messages to its outbuf). An **execution segment** is a sequence of alternating configurations and events $C_0, \phi_1, C_1, \phi_2, \dots$, in which each triple $C_i \phi_{i+1} C_{i+1}$ is consistent with the transition rules for the event ϕ_{i+1} (see [AW04, Chapter 2] or the discussion below for more details on this) and the last element of the sequence (if any) is a configuration. If the first configuration C_0 is an **initial configuration** of the system, we have an **execution**. A **schedule** is an execution with the configurations removed.

2.1.1 Formal details

Each process i has, in addition to its state state_i, a variable $\mathsf{inbuf}_i[j]$ for each process j it can receive messages from and $\mathsf{outbuf}_i[j]$ for each process j it

can send messages to. We assume each process has a **transition function** that maps tuples consisting of the inbuf values and the current state to a new state plus zero or one messages to be added to each outbuf (note that this means that the process's behavior can't depend on which of its previous messages have been delivered or not). A computation event comp(i) applies the transition function for i, emptying out all of i's inbuf variables, updating its state, and adding any outgoing messages to i's outbuf variables. A delivery event del(i, j, m) moves message m from $\mathsf{outbuf}_i[j]$ to $\mathsf{inbuf}_j[i]$.

Some implicit features in this definition:

- A process can't tell when its outgoing messages are delivered, because the outbuf_i variables aren't included in the **accessible state** used as input to the transition function.
- Processes are **deterministic**: The next action of each process depends only on its current state, and not on extrinsic variables like the phase of the moon, coin-flips, etc. We may wish to relax this condition later by allowing coin-flips; to do so, we will need to extend the model to incorporate probabilities.
- Processes must process all incoming messages at once. This is not as severe a restriction as one might think, because we can always have the first comp(i) event move all incoming messages to buffers in the state_i variable, and process messages sequentially during subsequent comp(i) events.
- It is possible to determine the accessible state of a process by looking only at events that involve that process. Specifically, given a schedule S, define the **restriction** $S|i$ to be the subsequence consisting of all comp(i) and del(j, i, m) events (ranging over all possible j and m). Since these are the only events that affect the accessible state of i, and only the accessible state of i is needed to apply the transition function, we can compute the accessible state of i looking only at $S|i$. In particular, this means that i will have the same accessible state after any two schedules S and S' where $S|i = S'|i$, and thus will take the same actions in both schedules. This is the basis for **indistinguishability proofs** (§3.2), a central technique in obtaining lower bounds and impossibility results.

A curious feature of this particular model is that communication channels are not modeled separately from processes, but instead are split across

processes (as the inbuf and outbuf variables). This leads to some oddities like having to distinguish the accessible state of a process (which excludes the outbufs) from the full state (which doesn't). A different approach (taken, for example, by [Lyn96]) would be to have separate automata representing processes and communication channels. But since the resulting model produces essentially the same executions, the exact details don't really matter.

2.1.2 Network structure

It may be the case that not all processes can communicate directly; if so, we impose a network structure in the form of a directed graph, where i can send a message to j if and only if there is an edge from i to j in the graph. Typically we assume that each process knows the identity of all its neighbors.

For some problems (e.g., in peer-to-peer systems or other **overlay networks**) it may be natural to assume that there is a fully-connected underlying network but that we have a dynamic network on top of it, where processes can only send to other processes that they have obtained the addresses of in some way.

2.2 Asynchronous systems

In an **asynchronous** model, only minimal restrictions are placed on when messages are delivered and when local computation occurs. A schedule is said to be **admissible** if (a) there are infinitely many computation steps for each process, and (b) every message is eventually delivered. (These are **fairness** conditions.) The first condition (a) assumes that processes do not explicitly terminate, which is the assumption used in [AW04]; an alternative, which we will use when convenient, is to assume that every process either has infinitely many computation steps or reaches an explicit halting state.

2.2.1 Example: client-server computing

Almost every distributed system in practical use is based on **client-server** interactions. Here one process, the **client**, sends a **request** to a second process, the **server**, which in turn sends back a **response**. We can model this interaction using our asynchronous message-passing model by describing what the transition functions for the client and the server look like: see Algorithms 2.1 and 2.2.

```
1 initially do
2 |  send request to server
```

Algorithm 2.1: Client-server computation: client code

```
1 upon receiving request do
2 |  send response to client
```

Algorithm 2.2: Client-server computation: server code

The interpretation of Algorithm 2.1 is that the client sends request (by adding it to its outbuf) in its very first computation event (after which it does nothing). The interpretation of Algorithm 2.2 is that in any computation event where the server observes request in its inbuf, it sends response.

We want to claim that the client eventually receives response in any admissible execution. To prove this, observe that:

1. After finitely many steps, the client carries out a computation event. This computation event puts request in its outbuf.
2. After finitely many more steps, a delivery event occurs that moves request to the server's inbuf.
3. After finitely many more steps, the server executes a computation event that causes it to send response.
4. After finitely many more steps, a delivery event occurs that moves response to the client's inbuf.
5. After finitely many more steps, the client executes a computation event that causes it to process response (and do nothing, given that we haven't include any code to handle this response).

Each step of the proof is justified by the constraints on admissible executions. If we could run for infinitely many steps without a particular process doing a computation event or a particular message being delivered, we'd violate those constraints.

Most of the time we will not attempt to prove the correctness of a protocol at quite this level of tedious detail. But if you are only interested in distributed algorithms that people actually use, you have now seen a proof of correctness for 99.9% of them, and do not need to read any further.

2.3 Synchronous systems

A **synchronous message-passing** system is exactly like an asynchronous system, except we insist that the schedule consists of alternating phases in which (a) every process executes a computation step, and (b) all messages are delivered. The combination of a computation phase and a delivery phase is called a **round**. Synchronous systems are effectively those in which all processes execute in lock-step, and there is no timing uncertainty. This makes protocols much easier to design, but makes them less resistant to real-world timing oddities. Sometimes this can be dealt with by applying a **synchronizer** (Chapter 13), which transforms synchronous protocols into asynchronous protocols at a small cost in complexity.

2.4 Complexity measures

There is no explicit notion of time in the asynchronous model, but we can define a time measure by adopting the rule that every message is delivered and processed at most 1 time unit after it is sent. Formally, we assign time 0 to the first event, and assign the largest time we can to each subsequent event, subject to the rule that if a message m from i to j is created at time t, then the time for the delivery of m from i to j and the time for the following computation step of j are both no greater than $j+1$. This is consistent with an assumption that message propagation takes at most 1 time unit and that local computation takes 0 time units. Another way to look at this is that it is a definition of a time unit in terms of maximum message delay together with an assumption that message delays dominate the cost of the computation. This last assumption is pretty much always true for real-world networks with any non-trivial physical separation between components, thanks to speed of light limitations.

The **time complexity** of a protocol (that terminates) is the time of the last event before all processes finish.

Note that looking at **step complexity**, the number of computation events involving either a particular process (**individual step complexity**) or all processes (**total step complexity**) is not useful in the asynchronous model, because a process may be scheduled to carry out arbitrarily many computation steps without any of its incoming or outgoing messages being delivered, which probably means that it won't be making any progress. These complexity measures will be more useful when we look at shared-memory models (Part II).

For a protocol that terminates, the **message complexity** is the total number of messages sent. We can also look at message length in bits, total bits sent, etc., if these are useful for distinguishing our new improved protocol from last year's model.

For synchronous systems, time complexity becomes just the number of rounds until a protocol finishes. Message complexity is still only loosely connected to time complexity; for example, there are synchronous **leader election** (Chapter 6) algorithms that, by virtue of grossly abusing the synchrony assumption, have unbounded time complexity but very low message complexity.

Chapter 3

Coordinated attack

(See also [Lyn96, §5.1].)

The **Two Generals** problem was the first widely-known distributed consensus problem, described in 1978 by Jim Gray [Gra78, §5.8.3.3.1], although the same problem previously appeared under a different name [AEH75].

The setup of the problem is that we have two generals on opposite sides of an enemy army, who must choose whether to attack the army or retreat. If only one general attacks, his troops will be slaughtered. So the generals need to reach agreement on their strategy.

To complicate matters, the generals can only communicate by sending messages by (unreliable) carrier pigeon. We also suppose that at some point each general must make an irrevocable decision to attack or retreat. The interesting property of the problem is that if carrier pigeons can become lost, there is no protocol that guarantees agreement in all cases unless the outcome is predetermined (e.g. the generals always attack no matter what happens). The essential idea of the proof is that any protocol that does guarantee agreement can be shortened by deleting the last message; iterating this process eventually leaves a protocol with no messages.

Adding more generals turns this into the **coordinated attack** problem, a variant of **consensus**; but it doesn't make things any easier.

3.1 Formal description

To formalize this intuition, suppose that we have $n \geq 2$ generals in a synchronous system with unreliable channels—the set of messages received in round $i + 1$ is always a subset of the set sent in round i, but it may be a proper subset (even the empty set). Each general starts with an input 0

(retreat) or 1 (attack) and must output 0 or 1 after some bounded number of rounds. The requirements for the protocol are that, in all executions:

Agreement All processes output the same decision (0 or 1).

Validity If all processes have the same input x, and no messages are lost, all processes produce output x. (If processes start with different inputs or one or more messages are lost, processes can output 0 or 1 as long as they all agree.)

Termination All processes terminate in a bounded number of rounds.[1]

Sadly, there is not protocol that satisfies all three conditions. We show this in the next section.

3.2 Impossibility proof

To show coordinated attack is impossible,[2] we use an **indistinguishability proof**.

The basic idea of an indistinguishability proof is this:

- Execution A is **indistinguishable** from execution B for some process p if p sees the same things (messages or operation results) in both executions.
- If A is indistinguishable from B for p, then p does the same thing in both executions.

So far, pretty dull. But now let's consider a chain of executions $A = A_0 A_1 \ldots A_k = B$, where A_i is indistinguishable from A_{i+1} for some process p_i. Suppose also that we are trying to solve an agreement task, where every process must output the same value. Then since p_i outputs the same value

[1] **Bounded** means that there is a fixed upper bound on the length of any execution. We could also demand merely that all processes terminate in a *finite* number of rounds. In general, finite is a weaker requirement than bounded, but if the number of possible outcomes at each step is finite (as they are in this case), they're equivalent. The reason is that if we build a tree of all configurations, each configuration has only finitely many successors, and the length of each path is finite, then **König's lemma** (see `http://en.wikipedia.org/wiki/Konig's_lemma`) says that there are only finitely many paths. So we can take the length of the longest of these paths as our fixed bound. [BG97, Lemma 3.1]

[2] Without making additional assumptions, always a caveat when discussing impossibility.

in A_i and A_{i+1}, every process outputs the same value in A_i and A_{i+1}. By induction on k, every process outputs the same value in A and B, even though A and B may be very different executions.

This gives us a tool for proving impossibility results for agreement: show that there is a path of indistinguishable executions between two executions that are supposed to produce different output. Another way to picture this: consider a graph whose nodes are all possible executions with an edge between any two indistinguishable executions; then the set of output-0 executions can't be adjacent to the set of output-1 executions. If we prove the graph is connected, we prove the output is the same for all executions.

For coordinated attack, we will show that no protocol satisfies all of agreement, validity, and termination using an indistinguishability argument. The key idea is to construct a path between the all-0-input and all-1-input executions with no message loss via intermediate executions that are indistinguishable to at least one process.

Let's start with $A = A_0$ being an execution in which all inputs are 1 and all messages are delivered. We'll build executions A_1, A_2, etc. by pruning messages. Consider A_i and let m be some message that is delivered in the last round in which any message is delivered. Construct A_{i+1} by not delivering m. Observe that while A_i is distinguishable from A_{i+1} by the recipient of m, on the assumption that $n \geq 2$ there is some other process that can't tell whether m was delivered or not (the recipient can't let that other process know, because no subsequent message it sends are delivered in either execution). Continue until we reach an execution A_k in which all inputs are 1 and no messages are sent. Next, let A_{k+1} through A_{k+n} be obtained by changing one input at a time from 1 to 0; each such execution is indistinguishable from its predecessor by any process whose input didn't change. Finally, construct A_{k+n} through A_{2k+n} by adding back messages in the reverse process used for A_0 through A_k. This gets us to an execution A_{k+n} in which all processes have input and no messages are lost. If agreement holds, then the indistinguishability of adjacent executions to some process means that the common output in A_0 is the same as in A_{2k+n}. But validity requires that A_0 outputs 1 and A_{2k+n} outputs 0: so validity is violated.

3.3 Randomized coordinated attack

So we now know that we can't solve the coordinated attack problem. But maybe we want to solve it anyway. The solution is to change the problem.

Randomized coordinated attack is like standard coordinated attack,

but with less coordination. Specifically, we'll allow the processes to flip coins to decide what to do, and assume that the communication pattern (which messages get delivered in each round) is fixed and independent of the coin-flips. This corresponds to assuming an **oblivious adversary** that can't see what is going on at all or perhaps a **content-oblivious adversary** that can only see where messages are being sent but not the contents of the messages. We'll also relax the agreement property to only hold with some high probability:

Randomized agreement For any adversary A, the probability that some process decides 0 and some other process decides 1 given A is at most ϵ.

Validity and termination are as before.

3.3.1 An algorithm

Here's an algorithm that gives $\epsilon = 1/r$. (See [Lyn96, §5.2.2] for details or [VL92] for the original version.) A simplifying assumption is that network is complete, although a strongly-connected network with r greater than or equal to the diameter also works.

- First part: tracking information levels
  - Each process tracks its "information level," initially 0. The state of a process consists of a vector of (input, information-level) pairs for all processes in the system. Initially this is (my-input, 0) for itself and $(\perp, -1)$ for everybody else.
  - Every process sends its entire state to every other process in every round.
  - Upon receiving a message m, process i stores any inputs carried in m and, for each process j, sets $\mathsf{level}_i[j]$ to $\max(\mathsf{level}_i[j], \mathsf{level}_m[j])$. It then sets its own information level to $\min_j(\mathsf{level}_i[j]) + 1$.
- Second part: deciding the output
  - Process 1 chooses a random key value uniformly in the range $[1, r]$.
  - This key is distributed along with $\mathsf{level}_i[1]$, so that every process with $\mathsf{level}_i[1] \geq 0$ knows the key.

– A process decides 1 at round r if and only if it knows the key, its information level is greater than or equal to the key, and all inputs are 1.

3.3.2 Why it works

Termination Immediate from the algorithm.

Validity

- If all inputs are 0, no process sees all 1 inputs (technically requires an invariant that processes' non-null views are consistent with the inputs, but that's not hard to prove.)
- If all inputs are 1 and no messages are lost, then the information level of each process after k rounds is k (prove by induction) and all processes learn the key and all inputs (immediate from first round). So all processes decide 1.

Randomized Agreement

- First prove a lemma: Define $\mathsf{level}_i^t[k]$ to be the value of $\mathsf{level}_i[k]$ after t rounds. Then for all i, j, k, t, (1) $\mathsf{level}_i[j]^t \geq \mathsf{level}_j[j]^{t-1}$ and (2) $|\mathsf{level}_i[k]^t - \mathsf{level}_j[k]^t| \leq 1$. As always, the proof is by induction on rounds. Part (1) is easy and boring so we'll skip it. For part (2), we have:
  - After 0 rounds, $\mathsf{level}_i^0[k] = \mathsf{level}_j^0[k] = -1$ if neither i nor j equals k; if one of them is k, we have $\mathsf{level}_k^0[k] = 0$, which is still close enough.
  - After t rounds, consider $\mathsf{level}_i^t[k] - \mathsf{level}_i^{t-1}[k]$ and similarly $\mathsf{level}_j^t[k] - \mathsf{level}_j^{t-1}[k]$. It's not hard to show that each can jump by at most 1. If both deltas are $+1$ or both are 0, there's no change in the difference in views and we win from the induction hypothesis. So the interesting case is when $\mathsf{level}_i[k]$ stays the same and $\mathsf{level}_j[k]$ increases or vice versa.
  - There are two ways for $\mathsf{level}_j[k]$ to increase:
    * If $j \neq k$, then j received a message from some j' with $\mathsf{level}_{j'}^{t-1}[k] > \mathsf{level}_j^{t-1}[k]$. From the induction hypothesis, $\mathsf{level}_{j'}^{t-1}[k] \leq \mathsf{level}_i^{t-1}[k] + 1 = \mathsf{level}_i^t[k]$. So we are happy.
    * If $j = k$, then j has $\mathsf{level}_j^t[j] = 1 + \min_{k \neq j} \mathsf{level}_j^t[k] \leq 1 + \mathsf{level}_j^t[i] \leq 1 + \mathsf{level}_i^t[i]$. Again we are happy.
- Note that in the preceding, the key value didn't figure in; so everybody's level at round r is independent of the key.

- So now we have that $\mathsf{level}_i^r[i]$ is in $\{\ell, \ell+1\}$, where ℓ is some fixed value uncorrelated with key. The only way to get some process to decide 1 while others decide 0 is if $\ell + 1 \geq \mathsf{key}$ but $\ell < \mathsf{key}$. (If $\ell = 0$, a process at this level doesn't know key, but it can still reason that $0 < \mathsf{key}$ since key is in $[1, r]$.) This can only occur if $\mathsf{key} = \ell + 1$, which occurs with probability at most $1/r$ since key was chosen uniformly.

3.3.3 Almost-matching lower bound

The bound on the probability of disagreement in the previous algorithm is almost tight. Varghese and Lynch show that no synchronous algorithm can get a probability of disagreement less than $\frac{1}{r+1}$, using a stronger validity condition that requires that the processes output 0 if any input is 0. This is a natural assumption for database commit, where we don't want to commit if any process wants to abort. We restate their result below:

Theorem 3.3.1. *For any synchronous algorithm for randomized coordinated attack that runs in r rounds that satisfies the additional condition that all non-faulty processes decide 0 if any input is 0,* $\Pr[\textit{disagreement}] \geq 1/(r+1)$.

Proof. Let ϵ be the bound on the probability of disagreement. Define $\mathsf{level}_i^t[k]$ as in the previous algorithm (whatever the real algorithm is doing). We'll show $\Pr[i \text{ decides } 1] \leq \epsilon \cdot (\mathsf{level}_i^r[i] + 1)$, by induction on $\mathsf{level}_i^r[i]$.

- If $\mathsf{level}_i^r[i] = 0$, the real execution is indistinguishable (to i) from an execution in which some other process j starts with 0 and receives no messages at all. In that execution, j must decide 0 or risk violating the strong validity assumption. So i decides 1 with probability at most ϵ (from the disagreement bound).

- If $\mathsf{level}_i^r[i] = k > 0$, the real execution is indistinguishable (to i) from an execution in which some other process j only reaches level $k-1$ and thereafter receives no messages. From the induction hypothesis, $\Pr[j \text{ decides } 1] \leq \epsilon k$ in that pruned execution, and so $\Pr[i \text{ decides } 1] \leq \epsilon(k+1)$ in the pruned execution. But by indistinguishability, we also have $\Pr[i \text{ decides } 1] \leq \epsilon(k+1)$ in the original execution.

Now observe that in the all-1 input execution with no messages lost, $\mathsf{level}_i^r[i] = r$ and $\Pr[i \text{ decides } 1] = 1$ (by validity). So $1 \leq \epsilon(r+1)$, which implies $\epsilon \geq 1/(r+1)$. □

Chapter 4

Broadcast and convergecast

Here we'll describe protocols for propagating information throughout a network from some central initiator and gathering information back to that same initiator. We do this both because the algorithms are actually useful and because they illustrate some of the issues that come up with keeping time complexity down in an asynchronous message-passing system.

4.1 Flooding

Flooding is about the simplest of all distributed algorithms. It's dumb and expensive, but easy to implement, and gives you both a broadcast mechanism and a way to build rooted spanning trees.

We'll give a fairly simple presentation of flooding roughly following Chapter 2 of [AW04].

4.1.1 Basic algorithm

The basic flooding algorithm is shown in Algorithm 4.1. The idea is that when a process receives a message M, it forwards it to all of its neighbors unless it has seen it before, which it tracks using a single bit seen-message.

Theorem 4.1.1. *Every process receives M after at most D time and at most $|E|$ messages, where D is the diameter of the network and E is the set of (directed) edges in the network.*

Proof. Message complexity: Each process only sends M to its neighbors once, so each edge carries at most one copy of M.

Time complexity: By induction on $d(\mathsf{root}, v)$, we'll show that each v receives M for the first time no later than time $d(\mathsf{root}, v) \leq D$. The base

```
1  initially do
2      if pid = root then
3          seen-message ← true
4          send M to all neighbors
5      else
6          seen-message ← false
7  upon receiving M do
8      if seen-message = false then
9          seen-message ← true
10         send M to all neighbors
```

Algorithm 4.1: Basic flooding algorithm

case is when $v = \mathsf{root}$, $d(\mathsf{root}, v) = 0$; here root receives message at time 0. For the induction step, Let $d(\mathsf{root}, v) = k > 0$. Then v has a neighbor u such that $d(\mathsf{root}, u) = k - 1$. By the induction hypothesis, u receives M for the first time no later than time $k - 1$. From the code, u then sends M to all of its neighbors, including v; M arrives at v no later than time $(k - 1) + 1 = k$. □

Note that the time complexity proof also demonstrates correctness: every process receives M at least once.

As written, this is a one-shot algorithm: you can't broadcast a second message even if you wanted to. The obvious fix is for each process to remember which messages it has seen and only forward the new ones (which costs memory) and/or to add a **time-to-live** (TTL) field on each message that drops by one each time it is forwarded (which may cost extra messages and possibly prevents complete broadcast if the initial TTL is too small). The latter method is what was used for searching in `http://en.wikipedia.org/wiki/Gnutella`, an early peer-to-peer system. An interesting property of Gnutella was that since the application of flooding was to search for huge (multiple MiB) files using tiny (100 byte) query messages, the actual bit complexity of the flooding algorithm was not especially large relative to the bit complexity of sending any file that was found.

We can optimize the algorithm slightly by not sending M back to the node it came from; this will slightly reduce the message complexity in many cases but makes the proof a sentence or two longer. (It's all a question of what you want to optimize.)

4.1.2 Adding parent pointers

To build a spanning tree, modify Algorithm 4.1 by having each process remember who it first received M from. The revised code is given as Algorithm 4.2

```
1  initially do
2      if pid = root then
3          parent ← root
4          send M to all neighbors
5      else
6          parent ← ⊥
7  upon receiving M from p do
8      if parent = ⊥ then
9          parent ← p
10
11         send M to all neighbors
```

Algorithm 4.2: Flooding with parent pointers

We can easily prove that Algorithm 4.2 has the same termination properties as Algorithm 4.1 by observing that if we map parent to seen-message by the rule $\perp \to$ **false**, anything else $\to$ **true**, then we have the same algorithm. We would like one additional property, which is that when the algorithm **quiesces** (has no outstanding messages), the set of parent pointers form a rooted spanning tree. For this we use induction on time:

Lemma 4.1.2. *At any time during the execution of Algorithm 4.2, the following invariant holds:*

1. *If u.parent $\neq \perp$, then u.parent.parent $\neq \perp$ and following parent pointers gives a path from u to* root.

2. *If there is a message M in transit from u to v, then u.parent $\neq \perp$.*

Proof. We have to show that any event preserves the invariant.

Delivery event M used to be in u.outbuf, now it's in v.inbuf, but it's still in transit and u.parent is still not $\perp$.[1]

[1] This sort of extraneous special case is why I personally don't like the split between outbuf and inbuf used in [AW04], even though it makes defining the synchronous model easier.

Computation event Let v receive M from u. There are two cases: if v.parent is already non-null, the only state change is that M is no longer in transit, so we don't care about u.parent any more. If v.parent is null, then

1. v.parent is set to u. This triggers the first case of the invariant. From the induction hypothesis we have that u.parent $\neq \perp$ and that there exists a path from u to the root. Then v.parent.parent $= u$.parent $\neq \perp$ and the path from $v \to u \to$ root gives the path from v.
2. Message M is sent to all of v's neighbors. Because M is now in transit from v, we need $v.parent \neq \perp$; but we just set it to u, so we are happy.

□

At the end of the algorithm, the invariant shows that every process has a path to the root, i.e., that the graph represented by the parent pointers is connected. Since this graph has exactly $|V| - 1$ edges (if we don't count the self-loop at the root), it's a tree.

Though we get a spanning tree at the end, we may not get a very good spanning tree. For example, suppose our friend the adversary picks some Hamiltonian path through the network and delivers messages along this path very quickly while delaying all other messages for the full allowed 1 time unit. Then the resulting spanning tree will have depth $|V| - 1$, which might be much worse than D. If we want the shallowest possible spanning tree, we need to do something more sophisticated: see the discussion of **distributed breadth-first search** in Chapter 5. However, we may be happy with the tree we get from simple flooding: if the message delay on each link is consistent, then it's not hard to prove that we in fact get a shortest-path tree. As a special case, flooding always produces a BFS tree in the synchronous model.

Note also that while the algorithm works in a directed graph, the parent pointers may not be very useful if links aren't two-way.

4.1.3 Termination

See [AW04, Chapter 2] for further modifications that allow the processes to detect termination. In a sense, each process can terminate as soon as it is done sending M to all of its neighbors, but this still requires some mechanism for clearing out the inbuf; by adding acknowledgments as described in

[AW04], we can terminate with the assurance that no further messages will be received.

4.2 Convergecast

A **convergecast** is the inverse of broadcast: instead of a message propagating down from a single root to all nodes, data is collected from outlying nodes to the root. Typically some function is applied to the incoming data at each node to summarize it, with the goal being that eventually the root obtains this function of all the data in the entire system. (Examples would be counting all the nodes or taking an average of input values at all the nodes.)

A basic convergecast algorithm is given in Algorithm 4.3; it propagates information up through a previously-computed spanning tree.

```
1  initially do
2      if I am a leaf then
3          send input to parent
4  upon receiving M from c do
5      append (c, M) to buffer
6      if buffer contains messages from all my children then
7          v ← f(buffer, input)
8          if pid = root then
9              return v
10         else
11             send v to parent
```

Algorithm 4.3: Convergecast

The details of what is being computed depend on the choice of f:

- If input $= 1$ for all nodes and f is sum, then we count the number of nodes in the system.
- If input is arbitrary and f is sum, then we get a total of all the input values.
- Combining the above lets us compute averages, by dividing the total of all the inputs by the node count.

- If f just concatenates its arguments, the root ends up with a vector of all the input values.

Running time is bounded by the depth of the tree: we can prove by induction that any node at height h (height is length of the longest path from this node to some leaf) sends a message by time h at the latest. Message complexity is exactly $n - 1$, where n is the number of nodes; this is easily shown by observing that each node except the root sends exactly one message.

Proving that convergecast returns the correct value is similarly done by induction on depth: if each child of some node computes a correct value, then that node will compute f applied to these values and its own input. What the result of this computation is will, of course, depend on f; it generally makes the most sense when f represents some associative operation (as in the examples above).

4.3 Flooding and convergecast together

A natural way to build the spanning tree used by convergecast is to run flooding first. This also provides a mechanism for letting the leaves know that they are leaves and initiating the protocol. The combined algorithm is shown as Algorithm 4.4.

However, this may lead to very bad time complexity for the convergecast stage. Consider a wheel-shaped network consisting of one central node p_0 connected to nodes $p_1, p_2, \ldots, p_{n-1}$, where each p_i is also connected to p_{i+1}. By carefully arranging for the $p_i p_{i+1}$ links to run much faster than the $p_0 p_i$ links, the adversary can make flooding build a tree that consists of a single path $p_0 p_1 p_2 \ldots p_{n-1}$, even though the diameter of the network is only 2. While it only takes 2 time units to build this tree (because every node is only one hop away from the initiator), when we run convergecast we suddenly find that the previously-speedy links are now running only at the guaranteed ≤ 1 time unit per hop rate, meaning that convergecast takes $n - 1$ time.

This may be less of an issue in real networks, where the latency of links may be more uniform over time, meaning that a deep tree of fast links is still likely to be fast when we reach the convergecast step. But in the worst case we will need to be more clever about building the tree. We show how to do this in Chapter 5.

```
1  initially do
2      children ← ∅
3      nonChildren ← ∅
4      if pid = root then
5          parent ← root
6          send init to all neighbors
7      else
8          parent ← ⊥
9  upon receiving init from p do
10     if parent = ⊥ then
11         parent ← p
12         send init to all neighbors
13     else
14         send nack to p
15 upon receiving nack from p do
16     nonChildren ← nonChildren ∪ {p}
17 as soon as children ∪ nonChildren includes all my neighbors do
18     v ← f(buffer, input)
19     if pid = root then
20         return v
21     else
22         send ack(v) to parent
23 upon receiving ack(v) from k do
24     add (k, v) to buffer
25     add k to children
```

Algorithm 4.4: Flooding and convergecast combined

Chapter 5

Distributed breadth-first search

Here we describe some algorithms for building a **breadth-first search** (**BFS**) tree in a network. All assume that there is a designated **initiator** node that starts the algorithm. At the end of the execution, each node except the initiator has a parent pointer and every node has a list of children. These are consistent and define a BFS tree: nodes at distance k from the initiator appear at level k of the tree.

In a synchronous network, **flooding** (§4.1) solves BFS; see [AW04, Lemma 2.8, page 21] or [Lyn96, §4.2]. So the interesting case is when the network is asynchronous.

In an asynchronous network, the complication is that we can no longer rely on synchronous communication to reach all nodes at distance d at the same time. So instead we need to keep track of distances explicitly, or possibly enforce some approximation to synchrony in the algorithm. (A general version of this last approach is to apply a synchronizer to one of the synchronous algorithms using a **synchronizer**; see Chapter 13.)

To keep things simple, we'll drop the requirement that a parent learn the IDs of its children, since this can be tacked on as a separate notification protocol, in which each child just sends one message to its parent once it figures out who its parent is.

5.1 Using explicit distances

This is a translation of the AsynchBFS automaton from [Lyn96, §15.4]. It's a very simple algorithm, closely related to Dijkstra's algorithm for shortest

paths, but there is otherwise no particular reason to use it; it is dominated by the $O(D)$ time and $O(DE)$ message complexity synchronizer-based algorithm described in §5.3. (Here D is the **diameter** of the network, the maximum distance between any two nodes.)

The idea is to run flooding with distances attached. Each node sets its distance to 1 plus the smallest distance sent by its neighbors and its parent to the neighbor supplying that smallest distance. A node notifies all its neighbors of its new distance whenever its distance changes.

Pseudocode is given in Algorithm 5.1

```
1  initially do
2      if pid = initiator then
3          distance ← 0
4          send distance to all neighbors
5      else
6          distance ← ∞
7  upon receiving d from p do
8      if d + 1 < distance then
9          distance ← d + 1
10         parent ← p
```

Algorithm 5.1: AsynchBFS algorithm (from [Lyn96])

(See [Lyn96] for a precondition-effect description, which also includes code for buffering outgoing messages.)

The claim is that after at most $O(VE)$ messages and $O(D)$ time, all distance values are equal to the length of the shortest path from the initiator to the appropriate node. The proof is by showing the following:

Lemma 5.1.1. *The variable* $\mathsf{distance}_p$ *is always the length of some path from initiator to* p*, and any message sent by* p *is also the length of some path from* initiator *to* p*.*

Proof. The second part follows from the first; any message sent equals p's current value of distance. For the first part, suppose p updates its distance; then it sets it to one more than the length of some path from initiator to p', which is the length of that same path extended by adding the pp' edge. □

We also need a liveness argument that says that $\mathsf{distance}_p = d(\mathsf{initiator}, p)$ no later than time $d(\mathsf{initiator}, p)$. Note that we can't detect when distance stabilizes to the correct value without a lot of additional work.

In [Lyn96], there's an extra $|V|$ term in the time complexity that comes from message pile-ups, since the model used there only allows one incoming message to be processed per time units (the model in [AW04] doesn't have this restriction). The trick to arranging this to happen often is to build a graph where node 1 is connected to nodes 2 and 3, node 2 to 3 and 4, node 3 to 4 and 5, etc. This allows us to quickly generate many paths of distinct lengths from node 1 to node k, which produces k outgoing messages from node k. It may be that a more clever analysis can avoid this blowup, by showing that it only happens in a few places.

5.2 Using layering

This approach is used in the *LayeredBFS* algorithm in [Lyn96], which is due to Gallager [Gal82].

Here we run a sequence of up to $|V|$ instances of the simple algorithm with a distance bound on each: instead of sending out just 0, the initiator sends out $(0, \mathsf{bound})$, where bound is initially 1 and increases at each phase. A process only sends out its improved distance if it is less than bound.

Each phase of the algorithm constructs a partial BFS tree that contains only those nodes within distance bound of the root. This tree is used to report back to the root when the phase is complete. For the following phase, notification of the increase in bound increase is distributed only through the partial BFS tree constructed so far. With some effort, it is possible to prove that in a bidirectional network that this approach guarantees that each edge is only probed once with a new distance (since distance-1 nodes are recruited before distance-2 nodes and so on), and the bound-update and acknowledgment messages contribute at most $|V|$ messages per phase. So we get $O(E + VD)$ total messages. But the time complexity is bad: $O(D^2)$ in the worst case.

5.3 Using local synchronization

The reason the layering algorithm takes so long is that at each phase we have to phone all the way back up the tree to the initiator to get permission to go on to the next phase. We need to do this to make sure that a node is only recruited into the tree once: otherwise we can get pile-ups on the channels as in the simple algorithm. But we don't necessarily need to do this globally. Instead, we'll require each node at distance d to delay sending out a recruiting message until it has confirmed that none of its neighbors

will be sending it a smaller distance. We do this by having two classes of messages:[1]

- exactly(d): "I know that my distance is d."
- more-than(d): "I know that my distance is $> d$."

The rules for sending these messages for a non-initiator are:

1. I can send exactly(d) as soon as I have received exactly($d-1$) from at least one neighbor and more-than($d-2$) from all neighbors.
2. I can send more-than(d) if $d = 0$ or as soon as I have received more-than($d-1$) from all neighbors.

The initiator sends exactly(0) to all neighbors at the start of the protocol (these are the only messages the initiator sends).

My distance will be the unique distance that I am allowed to send in an exactly(d) messages. Note that this algorithm terminates in the sense that every node learns its distance at some finite time.

If you read the discussion of synchronizers in Chapter 13, this algorithm essentially corresponds to building the **alpha synchronizer** into the synchronous BFS algorithm, just as the layered model builds in the **beta synchronizer**. See [AW04, §11.3.2] for a discussion of BFS using synchronizers. The original approach of applying synchronizers to get BFS is due to Awerbuch [Awe85].

We now show correctness. Under the assumption that local computation takes zero time and message delivery takes at most 1 time unit, we'll show that if $d(\text{initiator}, p) = d$, (a) p sends more-than(d') for any $d' < d$ by time d', (b) p sends exactly(d) by time d, (c) p never sends more-than(d') for any $d' \geq d$, and (d) p never sends exactly(d') for any $d' \neq d$. For parts (c) and (d) we use induction on d'; for (a) and (b), induction on time. This is not terribly surprising: (c) and (d) are safety properties, so we don't need to talk about time. But (a) and (b) are liveness properties so time comes in.

Let's start with (c) and (d). The base case is that the initiator never sends any more-than messages at all, and so never sends more-than(0), and any non-initiator never sends exactly(0). For larger d', observe that if a non-initiator p sends more-than(d') for $d' \geq d$, it must first have received

[1] In an earlier version of these notes, these messages where called distance(d) and not-distance(d); the more self-explanatory exactly and more-than terminology is taken from [BDLP08].

more-than$(d'-1)$ from all neighbors, including some neighbor p' at distance $d-1$. But the induction hypothesis tells us that p' can't send more-than$(d'-1)$ for $d'-1 \geq d-1$. Similarly, to send exactly(d') for $d' < d$, p must first have received exactly$(d'-1)$ from some neighbor p', but again p' must be at distance at least $d-1$ from the initiator and so can't send this message either. In the other direction, to send exactly(d') for $d' > d$, p must first receive more-than$(d'-2)$ from this closer neighbor p', but then $d'-2 > d-2 \geq d-1$ so more-than$(d'-2)$ is not sent by p'.

Now for (a) and (b). The base case is that the initiator sends exactly(0) to all nodes at time 0, giving (a), and there is no more-than(d') with $d' < 0$ for it to send, giving (b) vacuously; and any non-initiator sends more-than(0) immediately. At time $t+1$, we have that (a) more-than(t) was sent by any node at distance $t+1$ or greater by time t and (b) exactly(t) was sent by any node at distance t by time t; so for any node at distance $t+2$ we send more-than$(t+1)$ no later than time $t+1$ (because we already received more-than(t) from all our neighbors) and for any node at distance $t+1$ we send exactly$(t+1)$ no later than time $t+1$ (because we received all the preconditions for doing so by this time).

Message complexity: A node at distance d sends more-than(d') for all $0 < d' < d$ and exactly(d) and no other messages. So we have message complexity bounded by $|E| \cdot D$ in the worst case. Note that this is gives a bound of $O(DE)$, which is slightly worse than the $O(E + DV)$ bound for the layered algorithm.

Time complexity: It's immediate from (a) and (b) that all messages that are sent are sent by time D, and indeed that any node p learns its distance at time $d(\mathsf{initiator}, p)$. So we have optimal time complexity, at the cost of higher message complexity. I don't know if this trade-off is necessary, or if a more sophisticated algorithm could optimize both.

Our time proof assumes that messages don't pile up on edges, or that such pile-ups don't affect delivery time (this is the default assumption used in [AW04]). A more sophisticated proof could remove this assumption.

One downside of this algorithm is that it has to be started simultaneously at all nodes. Alternatively, we could trigger "time 0" at each node by a broadcast from the initiator, using the usual asynchronous broadcast algorithm; this would give us a BFS tree in $O(|E| \cdot D)$ messages (since the $O(|E|)$ messages of the broadcast disappear into the constant) and $2D$ time. The analysis of time goes through as before, except that the starting time 0 becomes the time at which the last node in the system is woken up by the broadcast. Further optimizations are possible; see, for example, the paper of Boulinier *et al.* [BDLP08], which shows how to run the same algorithm

with constant-size messages.

Chapter 6

Leader election

See [AW04, Chapter 3] or [Lyn96, Chapter 3] for details.

The basic idea of leader election is that we want a single process to declare itself leader and the others to declare themselves non-leaders. The non-leaders may or may not learn the identity of the leader as part of the protocol; if not, we can always add an extra phase where the leader broadcasts its identity to the others. Traditionally, leader election has been used as a way to study the effects of symmetry, and many leader election algorithms are designed for networks in the form of a ring.

A classic result of Angluin [Ang80] shows that leader election in a ring is impossible if the processes do not start with distinct identities. The proof is that if everybody is in the same state at every step, they all put on the crown at the same time. We discuss this result in more detail in §6.1.

With ordered identities, a simple algorithm due to Le Lann [LL77] and Chang and Roberts [CR79] solves the problem in $O(n)$ time with $O(n^2)$ messages: I send out my own id clockwise and forward any id bigger than mine. If I get my id back, I win. This works with a unidirectional ring, doesn't require synchrony, and never produces multiple leaders. See §6.2.1 for more details.

On a bidirectional ring we can get $O(n \log n)$ messages and $O(n)$ time with power-of-2 probing, using an algorithm of Hirschberg and Sinclair [HS80]. This is described in §6.2.2.

An evil trick: if we have synchronized starting, known n, and known id space, we can have process with id i wait until round $i \cdot n$ to start sending its id around, and have everybody else drop out when they receive it; this way only one process (the one with smallest id) ever starts a message and only n messages are sent [FL87]. But the running time can be pretty bad.

For general networks, we can apply the same basic strategy as in LeLann-Chang-Roberts by having each process initiate a broadcast/convergecast algorithm that succeeds only if the initiator has the smallest id. This is described in more detail in §6.3.

Some additional algorithms for the asynchronous ring are given in §§6.2.3 and 6.2.4. Lower bounds are shown in §6.4.

6.1 Symmetry

A system exhibits **symmetry** if we can permute the nodes without changing the behavior of the system. More formally, we can define a symmetry as an **equivalence relation** on processes, where we have the additional properties that all processes in the same equivalence class run the same code; and whenever p is equivalent to p', each neighbor q of p is equivalent to the corresponding neighbor q' of p'.

An example of a network with a lot of symmetries would be an **anonymous ring**, which is a network in the form of a cycle (the ring part) in which every process runs the same code (the anonymous part). In this case all nodes are equivalent. If we have a line, then we might or might not have any non-trivial symmetries: if each node has a **sense of direction** that tells it which neighbor is to the left and which is to the right, then we can identify each node uniquely by its distance from the left edge. But if the nodes don't have a sense of direction,[1] we can flip the line over and pair up nodes that map to each other.

Symmetries are convenient for proving impossibility results, as observed by Angluin [Ang80]. The underlying theme is that without some mechanism for **symmetry breaking**, a message-passing system escape from a symmetric initial configuration. The following lemma holds for **deterministic** systems, basically those in which processes can't flip coins:

Lemma 6.1.1. *A symmetric deterministic message-passing system that starts in an initial configuration in which equivalent processes have the same state has a synchronous execution in which equivalent processes continue to have the same state.*

Proof. Easy induction on rounds: if in some round p and p' are equivalent and have the same state, and all their neighbors are equivalent and have the same state, then p and p' receive the same messages from their neighbors

[1] Typically, this means that the nodes can tell their neighbors apart, but that their code behaves the same way if the left and right neighbors are flipped.

and can proceed to the same state (including outgoing messages) in the next round. □

An immediate corollary is that you can't do leader election in an anonymous system with a symmetry that puts each node in a non-trivial equivalence class, because as soon as I stick my hand up to declare I'm the leader, so do all my equivalence-class buddies.

With **randomization**, Lemma 6.1.1 doesn't directly apply, since we can break symmetry by having my coin-flips come up differently from yours. It does show that we can't guarantee convergence to a single leader in any fixed amount of time (because otherwise we could just fix all the coin flips to get a deterministic algorithm). Depending on what the processes know about the size of the system, it may still be possible to show that a randomized algorithm necessarily fails in some cases.

A more direct way to break symmetry is to assume that all processes have **identities**; now processes can break symmetry by just declaring that the one with the smaller or larger identity wins. This approach is taken in the algorithms in the following sections.

6.2 Leader election in rings

Here we'll describe some basic leader election algorithms for rings. Historically, rings were the first networks in which leader election was studied, because they are the simplest networks whose symmetry makes the problem difficult, and because of the connection to token-ring networks, a method for congestion control in local-area networks that is no longer used much.

6.2.1 The Le-Lann-Chang-Roberts algorithm

.

This is about the simplest leader election algorithm there is. It works in a **unidirectional ring**, where messages can only travel clockwise.[2] The algorithms works does not require synchrony, but we'll assume synchrony to make it easier to follow.

Formally, we'll let the state space for each process i consist of two variables: leader, initially 0, which is set to 1 if i decides it's a leader; and maxId, the largest id seen so far. We assume that i denotes i's position rather than

[2] We'll see later in §6.2.3 that the distinction between unidirectional rings and bidirectional rings is not a big deal, but for now let's imagine that having a unidirectional ring is a serious hardship.

its id, which we'll write as id_i. We will also treat all positions as values mod n, to simplify the arithmetic.

Code for the LCR algorithm is given in Algorithm 6.1.

```
1  initially do
2      leader ← 0
3      maxId ← id_i
4      send id_i to clockwise neighbor
5  upon receiving j do
6      if j = id_i then
7          leader ← 1
8      if j > maxId then
9          maxId ← j
10         send j to clockwise neighbor
```

Algorithm 6.1: LCR leader election

6.2.1.1 Proof of correctness for synchronous executions

By induction on the round number k. The induction hypothesis is that in round k, each process i's leader bit is 0, its maxId value is equal to the largest id in the range $(i-k)\ldots i$, and that it sends id_{i-k} if and only if id_{i-k} is the largest id in the range $(i-k)\ldots i$. The base case is that when $k=0$, $\mathsf{maxId} = \mathsf{id}_i$ is the largest id in $i\ldots i$, and i sends id_i. For the induction step, observe that in round $k-1$, $i-1$ sends $\mathsf{id}_{(i-1)-(k-1)} = \mathsf{id}_{i-k}$ if and only if it is the largest in the range $(i-k)\ldots(i-1)$, and that i adopts it as the new value of maxId and sends it just in case it is larger than the previous largest value in $(i-k+1)\ldots(i-1)$, i.e., if it is the largest value in $(i-k)\ldots i$.

Finally, in round $n-1$, $i-1$ sends $\mathsf{id}_{i-N} = \mathsf{id}_i$ if and only if i is the largest id in $(i-n+1)\ldots i$, the whole state space. So i receives id_i and sets $\mathsf{leader}_i = 1$ if and only if it has the maximum id.

6.2.1.2 Performance

It's immediate from the correctness proof that the protocols terminates after exactly n rounds.

To count message traffic, observe that each process sends at most 1 message per round, for a total of $O(n^2)$ messages. This is a tight bound

since if the ids are in decreasing order $n, n-1, n-2, \ldots 1$, then no messages get eaten until they hit n.

6.2.2 The Hirschberg-Sinclair algorithm

Basically the same as for LCR but both the protocol and the invariant get much messier. To specify the protocol, it may help to think of messages as mobile agents and the state of each process as being of the form (local-state, {agents I'm carrying}). Then the sending rule for a process becomes *ship any agents in whatever direction they want to go* and the transition rule is *accept any incoming agents and update their state in terms of their own internal transition rules.* An agent state for LCR will be something like (original-sender, direction, hop-count, max-seen) where direction is R or L depending on which way the agent is going, hop-count is initially 2^k when the agent is sent and drops by 1 each time the agent moves, and max-seen is the biggest id of any node the agent has visited. An agent turns around (switches direction) when hop-count reaches 0.

To prove this works, we can mostly ignore the early phases (though we have to show that the max-id node doesn't drop out early, which is not too hard). The last phase involves any surviving node probing all the way around the ring, so it will declare itself leader only when it receives its own agent from the left. That exactly one node does so is immediate from the same argument for LCR.

Complexity analysis is mildly painful but basically comes down to the fact that any node that sends a message 2^k hops had to be a winner at phase 2^{k-1}, which means that it is the largest of some group of 2^{k-1} ids. Thus the 2^k-hop senders are spaced at least 2^{k-1} away from each other and there are at most $n/2^{k-1}$ of them. Summing up over all $\lceil \lg n \rceil$ phases, we get $\sum_{k=0}^{\lceil \lg n \rceil} 2^k n/2^{k-1} = O(n \log n)$ messages and $\sum_{k=0}^{\lceil \lg n \rceil} 2^k = O(n)$ time.

6.2.3 Peterson's algorithm for the unidirectional ring

This algorithm is due to Peterson [Pet82] and assumes an asynchronous, unidirectional ring. It gets $O(n \log n)$ message complexity in all executions.

The basic idea (2-way communication version): Start with n candidate leaders. In each of at most $\lg n$ asynchronous phases, each candidate probes its nearest neighbors to the left and right; if its ID is larger than the IDs of both neighbors, it survives to the next phase. Non-candidates act as relays passing messages between candidates. As in Hirschberg and Sinclair (§6.2.2), the probing operations in each phase take $O(n)$ messages, and at least half

of the candidates drop out in each phase. The last surviving candidate wins when it finds that it's its own neighbor.

To make this work in a 1-way ring, we have to simulate 2-way communication by moving the candidates clockwise around the ring to catch up with their unsendable counterclockwise messages. Peterson's algorithm does this with a two-hop approach that is inspired by the 2-way case above; in each phase k, a candidate effectively moves two positions to the right, allowing it to look at the ids of three phase-k candidates before deciding to continue in phase $k+1$ or not. Here is a very high-level description; it assumes that we can buffer and ignore incoming messages from the later phases until we get to the right phase, and that we can execute sends immediately upon receiving messages. Doing this formally in terms of I/O automata or the model of §2.1 means that we have to build explicit internal buffers into our processes, which we can easily do but won't do here (see [Lyn96, pp. 483–484] for the right way to do this.)

We can use a similar trick to transform any bidirectional-ring algorithm into a unidirectional-ring algorithm: alternative between phases where we send a message right, then send a virtual process right to pick up any left-going messages deposited for us. The problem with this trick is that it requires two messages per process per phase, which gives us a total message complexity of $O(n^2)$ if we start with an $O(n)$-time algorithm. Peterson's algorithm avoids this by only propagating the surviving candidates.

Pseudocode for Peterson's algorithm is given in Algorithm 6.2.

Note: the phase arguments in the probe messages are useless if one has FIFO channels, which is why [Lyn96] doesn't use them. Note also that the algorithm does not elect the process with the highest ID, but the process that is carrying the sole surviving candidate in the last phase.

Proof of correctness is essentially the same as for the 2-way algorithm. For any pair of adjacent candidates, at most one of their current IDs survives to the next phase. So we get a sole survivor after $\lg n$ phases. Each process sends or relays at most 2 messages per phases, so we get at most $2n \lg n$ total messages.

6.2.4 A simple randomized $O(n \log n)$-message algorithm

An alternative to running a more sophisticated algorithm is to reduce the average cost of LCR using randomization. The presentation here follows the average-case analysis done by Chang and Roberts [CR79].

Run LCR where each id is constructed by prepending a long random bit-string to the real id. This gives uniqueness (since the real id's act as

```
1  procedure candidate()
2      phase ← 0
3      current ← pid
4      while true do
5          send probe(phase, current)
6          wait for probe(phase, x)
7          id_2 ← x
8          send probe(phase, current)
9          wait for probe(phase, x)
10         id_3 ← x
11         if id_2 = current then
12             I am the leader!
13             return
14         else if id_2 > current and id_2 > id_3 do
15             current ← id_2
16             phase ← phase + 1
17         else
18             switch to relay()
19 procedure relay()
20     upon receiving probe(p, i) do
21         send probe(p, i)
```

Algorithm 6.2: Peterson's leader-election algorithm

tie-breakers) and something very close to a random permutation on the constructed id's. When we have unique random id's, a simple argument shows that the i-th largest id only propagates an expected n/i hops, giving a total of $O(nH_n) = O(n \log n)$ hops.[3] Unique random id's occur with high probability provided the range of the random sequence is $\gg n^2$.

The downside of this algorithm compared to Peterson's is that knowledge of n is required to pick random id's from a large enough range. It also has higher bit complexity since Peterson's algorithm is sending only IDs (in the official version) without any random padding.

6.3 Leader election in general networks

For general networks, a simple approach is to have each node initiate a breadth-first-search and convergecast, with nodes refusing to participate in the protocol for any initiator with a lower id. It follows that only the node with the maximum id can finish its protocol; this node becomes the leader. If messages from parallel broadcasts are combined, it's possible to keep the message complexity of this algorithm down to $O(DE)$.

More sophisticated algorithms reduce the message complexity by coalescing local neighborhoods similar to what happens in the Hirschberg-Sinclair and Peterson algorithms. A noteworthy example is an $O(n \log n)$ message-complexity algorithm of Afek and Gafni [AG91], who also show an $\Omega(n \log n)$ lower bound on message complexity for any synchronous algorithm in a complete network.

6.4 Lower bounds

Here we present two classic $\Omega(\log n)$ lower bounds on message complexity for leader election in the ring. The first, due to Burns [Bur80], assumes that the system is asynchronous and that the algorithm is **uniform**: it does not depend on the size of the ring. The second, due to Frederickson and Lynch [FL87], allows a synchronous system and relaxes the uniformity assumption, but requires that the algorithm can't do anything to ids but copy and compare them.

[3] Alternatively, we could consider the **average-case complexity** of the algorithm when we assume all $n!$ orderings of the ids are equally likely; this also gives $O(n \log n)$ expected message complexity [CR79].

6.4.1 Lower bound on asynchronous message complexity

Here we describe a lower bound for uniform asynchronous leader election in the ring. The description here is based on [AW04, §3.3.3]; a slightly different presentation can also be found in [Lyn96, §15.1.4]. The original result is due to Burns [Bur80]. We assume the system is deterministic.

The basic idea is to construct a bad execution in which n processes send lots of messages recursively, by first constructing two bad $(n/2)$-process executions and pasting them together in a way that generates many extra messages. If the pasting step produces $\Theta(n)$ additional messages, we get a recurrence $T(n) \geq 2T(n/2) + \Theta(n)$ for the total message traffic, which has solution $T(n) = \Omega(n \log n)$.

We'll assume that all processes are trying to learn the identity of the process with the smallest id. This is a slightly stronger problem that mere leader election, but it can be solved with at most an additional $2n$ messages once we actually elect a leader. So if we get a lower bound of $f(n)$ messages on this problem, we immediately get a lower bound of $f(n) - 2n$ on leader election.

To construct the bad execution, we consider "open executions" on rings of size n where no message is delivered across some edge (these will be partial executions, because otherwise the guarantee of eventual delivery kicks in). Because no message is delivered across this edge, the processes can't tell if there is really a single edge there or some enormous unexplored fragment of a much larger ring. Our induction hypothesis will show that a line of $n/2$ processes can be made to send at least $T(n/2)$ messages in an open execution (before seeing any messages across the open edge); we'll then show that a linear number of additional messages can be generated by pasting two such executions together end-to-end, while still getting an open execution with n processes.

In the base case, we let $n = 2$. Somebody has to send a message eventually, giving $T(2) \geq 1$.

For larger n, suppose that we have two open executions on $n/2$ processes that each send at least $T(n/2)$ messages. Break the open edges in both executions and paste the resulting lines together to get a ring of size n; similarly paste the schedules σ_1 and σ_2 of the two executions together to get a combined schedule $\sigma_1\sigma_2$ with at least $2T(n/2)$ messages. Note that in the combined schedule no messages are passed between the two sides, so the processes continue to behave as they did in their separate executions.

Let e and e' be the edges we used to past together the two rings. Extend $\sigma_1\sigma_2$ by the longest possible suffix σ_3 in which no messages are delivered

across e and e'. Since σ_3 is as long as possible, after $\sigma_1\sigma_2\sigma_3$, there are no messages waiting to be delivered across any edge except e and e' and all processes are **quiescent**—they will send no additional messages until they receive one.

Now consider the processes in the half of the ring with the larger minimum id. Because each process must learn the minimum id in the other half of the ring, each of these processes must receive a message in some complete execution, giving an additional $n/2 - 2$ messages (since two of the processes might receive undelivered messages on e or e' that we've already counted). But to prove our induction hypothesis, we need to keep one of e or e' open. Consider some execution $\sigma_1\sigma_2\sigma_3\sigma_4$ in which all messages delayed on both e and e' are delivered, and partition the $n/2$ process on the losing side into two groups based on whether the first message they get is triggered by opening e or e'. One of these two groups must contain at least half the $n/2 - 2$ processes who receive new messages, meaning that there is an execution $\sigma_1\sigma_2\sigma_3\sigma_4'$ in which we open up only one edge and still get an additional $(n/2 - 2)/2 = \Theta(n)$ messages. This concludes the proof.

6.4.2 Lower bound for comparison-based algorithms

Here we give an $\Omega(n \log n)$ lower bound on messages for synchronous-start comparison-based algorithms in bidirectional synchronous rings. For full details see [Lyn96, §3.6], [AW04, §3.4.2], or the original JACM paper by Frederickson and Lynch [FL87].

Basic ideas:

- Two fragments $i \ldots i+k$ and $j \ldots j+k$ of a ring are **order-equivalent** provided $\mathsf{id}_{i+a} > \mathsf{id}_{i+b}$ if and only if $\mathsf{id}_{j+a} > \mathsf{id}_{j+b}$ for $b = 0 \ldots k$.
- An algorithm is **comparison-based** if it can't do anything to IDs but copy them and test for $<$. The state of such an algorithm is modeled by some non-ID state together with a big bag of IDs, messages have a pile of IDs attached to them, etc. Two states/messages are equivalent under some mapping of IDs if you can translate the first to the second by running all IDs through the mapping. Alternate version: Executions of p_1 and p_2 are **similar** if they send messages in the same direction(s) in the same rounds, declare themselves leader at the same round; an algorithm is comparison-based based if order-equivalent rings yield similar executions for corresponding processes. This can be turned into the explicit-copying-IDs model by replacing the original protocol with a **full-information protocol** in which each

message is replaced by the ID and a complete history of the sending process (including all messages it has every received).

- Define an **active round** as a round in which at least 1 message is sent. Claim: actions of i after k active rounds depends up to an order-equivalent mapping of ids only on the order-equivalence class of ids in $i-k \ldots i+k$ (the k**-neighborhood** of i). Proof: by induction on k. Suppose i and j have order-equivalent $(k-1)$-neighborhoods; then after $k-1$ active rounds they have equivalent states by the induction hypothesis. In inactive rounds, i and j both receive no messages and update their states in the same way. In active rounds, i and j receive order-equivalent messages and update their states in an order-equivalent way.

- If we have an order of ids with a lot of order-equivalent k-neighborhoods, then after k active rounds if one process sends a message, so do a lot of other ones.

Now we just need to build a ring with a lot of order-equivalent neighborhoods. For n a power of 2 we can use the bit-reversal ring, e.g., id sequence $000, 100, 010, 110, 001, 101, 011, 111$ (in binary) when $n = 8$. Figure 6.1 gives a picture of what this looks like for $n = 32$.

For n not a power of 2 we look up Frederickson and Lynch [FL87] or Attiya *et al.* [ASW88]. In either case we get $\Omega(n/k)$ order-equivalent members of each equivalence class after k active rounds, giving $\Omega(n/k)$ messages per active round, which sums to $\Omega(n \log n)$.

For non-comparison-based algorithms we can still prove $\Omega(n \log n)$ messages for time-bounded algorithms, but it requires techniques from **Ramsey theory**, the branch of combinatorics that studies when large enough structures inevitably contain substructures with certain properties.[4] Here "time-bounded" means that the running time can't depend on the size of the ID space. See [AW04, §3.4.2] or [Lyn96, §3.7] for the textbook version, or [FL87, §7] for the original result.

The intuition is that for any fixed protocol, if the ID space is large enough, then there exists a subset of the ID space where the protocol

[4] The classic example is **Ramsey's Theorem**, which says that if you color the edges of a complete graph red or blue, while trying to avoid having any subsets of k vertices with all edges between them the same color, you will no longer be able to once the graph is large enough (for any fixed k). See [GRS90] for much more on the subject of Ramsey theory.

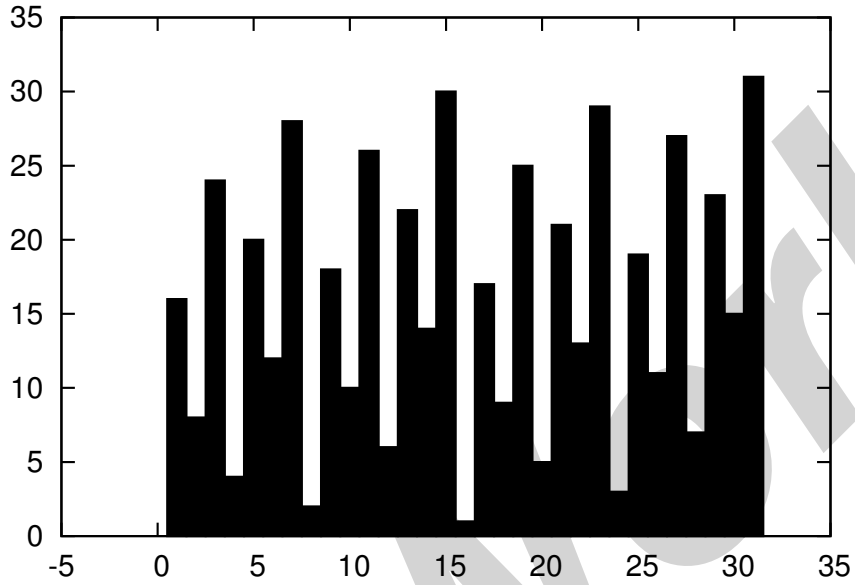


Figure 6.1: Labels in the bit-reversal ring with $n = 32$

acts like a comparison-based protocol. So the existence of an $O(f(n))$-message time-bounded protocol implies the existence of an $O(f(n))$-message comparison-based protocol, and from the previous lower bound we know $f(n)$ is $\Omega(n \log n)$. Note that time-boundedness is necessary: we can't prove the lower bound for non-time-bounded algorithms because of the $i \cdot n$ trick.

Chapter 7

Synchronous agreement

Here we'll consider synchronous agreement algorithm with stopping failures, where a process stops dead at some point, sending and receiving no further messages. We'll also consider Byzantine failures, where a process deviates from its programming by sending arbitrary messages, but mostly just to see how crash-failure algorithms hold up; for algorithms designed specifically for a Byzantine model, see Chapter 8.

If the model has communication failures instead, we have the coordinated attack problem from Chapter 3.

7.1 Problem definition

We use the usual synchronous model with n processes with binary inputs and binary outputs. Up to f processes may fail at some point; when a process fails, one or one or more of its outgoing messages are lost in the round of failure and all outgoing messages are lost thereafter.

There are two variants on the problem, depending on whether we want a useful algorithm (and so want strong conditions to make our algorithm more useful) or a lower bound (and so want weak conditions to make our lower bound more general). For algorithms, we will ask for these conditions to hold:

Agreement All non-faulty processes decide the same value.

Validity If all processes start with the same input, all non-faulty processes decide it.

Termination All non-faulty processes eventually decide.

For lower bounds, we'll replace validity with **non-triviality** (often called validity in the literature):

Non-triviality There exist failure-free executions A and B that produce different outputs.

Non-triviality follows from validity but doesn't imply validity; for example, a non-trivial algorithm might have the property that if all non-faulty processes start with the same input, they all decide something else. We'll start by using non-triviality, agreement, and termination to show a lower bound on the number of rounds needed to solve the problem.

7.2 Lower bound on rounds

Here we show that synchronous agreement requires at least $f + 1$ rounds if f processes can fail. This proof is modeled on the one in [Lyn96, §6.7] and works backwards from the final state; for a proof of the same result that works in the opposite direction, see [AW04, §5.1.4]. The original result (stated for Byzantine failures) is due to Dolev and Strong [DS83], based on a more complicated proof due to Fischer and Lynch [FL82]; see the chapter notes for Chapter 5 of [AW04] for more discussion of the history.

Like the similar proof for coordinated attack (§3.2), the proof uses an indistinguishability argument. But we have to construct a more complicated chain of intermediate executions.

A **crash failure** at process i means that (a) in some round r, some or all of the messages sent by i are not delivered, and (b) in subsequent rounds, no messages sent by i are delivered. The intuition is that i keels over dead in the middle of generating its outgoing messages for a round. Otherwise i behaves perfectly correctly. A process that crashes at some point during an execution is called **faulty**

We will show that if up to f processes can crash, and there are at least $f + 2$ processes, then at least $f + 1$ rounds are needed (in some execution) for any algorithm that satisfies agreement, termination, and non-triviality. In particular, we will show that if all executions run in f or fewer rounds, then the indistinguishability graph is connected; this implies non-triviality doesn't hold, because (as in §3.2), two adjacent states must decide the same value because of the agreement property.[1]

[1] The same argument works with even a weaker version of non-triviality that omits the requirement that A and B are failure-free, but we'll keep things simple.

Now for the proof. To simplify the argument, let's assume that all executions terminate in exactly f rounds (we can always have processes send pointless chitchat to pad out short executions) and that every processes sends a message to every other process in every round where it has not crashed (more pointless chitchat). Formally, this means we have a sequence of rounds $0, 1, 2, \ldots, f-1$ where each process sends a message to every other process (assuming no crashes), and a final round f where all processes decide on a value (without sending any additional messages).

We now want to take any two executions A and B and show that both produce the same output. To do this, we'll transform A's inputs into B's inputs one process at a time, crashing processes to hide the changes. The problem is that just crashing the process whose input changed might change the decision value—so we have to crash later witnesses carefully to maintain indistinguishability all the way across the chain.

Let's say that a process p **crashes fully** in round r if it crashes in round r and no round-r messages from p are delivered. The **communication pattern** of an execution describes which messages are delivered between processes without considering their contents—in particular, it tells us which processes crash and what other processes they manage to talk to in the round in which they crash.

With these definitions, we can state and prove a rather complicated induction hypothesis:

Lemma 7.2.1. *For any f-round protocol with $n \geq f+2$ process permitting up to f crash failures; any process p; and any execution A in which at most one processes crashes per round in rounds $0 \ldots r-1$, p crashes fully in round $r+1$, and no other processes crash; there is a sequence of executions $A = A_0 A_1 \ldots A_k$ such that each A_i is indistinguishable from A_{i+1} by some process, each A_i has at most one crash per round, and the communication pattern in A_k is identical to A except that p crashes fully in round r.*

Proof. By induction on $f - r$. If $r = f$, we just crash p in round r and nobody else notices. For $r < f$, first crash p in round r instead of $r+1$, but deliver all of its round-r messages anyway (this is needed to make space for some other process to crash in round $r+1$). Then choose some message m sent by p in round r, and let p' be the recipient of m. We will show that we can produce a chain of indistinguishable executions between any execution in which m is delivered and the corresponding execution in which it is not.

If $r = f - 1$, this is easy; only p' knows whether m has been delivered, and since $n \geq f+2$, there exists another non-faulty p'' that can't distinguish between these two executions, since p' sends no messages in round f or later.

If $r < f - 1$, we have to make sure p' doesn't tell anybody about the missing message.

By the induction hypothesis, there is a sequence of executions starting with A and ending with p' crashing fully in round $r + 1$, such that each execution is indistinguishable from its predecessor. Now construct the sequence

$$\begin{aligned} A &\to (A \text{ with } p' \text{ crashing fully in } r+1) \\ &\to (A \text{ with } p' \text{ crashing fully in } r+1 \text{ and } m \text{ lost}) \\ &\to (A \text{ with } m \text{ lost and } p' \text{ not crashing}). \end{aligned}$$

The first and last step apply the induction hypothesis; the middle one yields indistinguishable executions since only p' can tell the difference between m arriving or not and its lips are sealed.

We've shown that we can remove one message through a sequence of executions where each pair of adjacent executions is indistinguishable to some process. Now paste together $n - 1$ such sequences (one per message) to prove the lemma. □

The rest of the proof: Crash some process fully in round 0 and then change its input. Repeat until all inputs are changed.

7.3 Solutions

Here we give two solutions to synchronous agreement with crash failures. The first, due to Dolev and Strong [DS83], is more practical but does not generalize well to Byzantine failures. The second is a variant on the exponential information gathering algorithm of Pease, Shostak, and Lamport [PSL80], which propagates enough information that it can in principle simulate any other possible algorithm; it is mostly of interest because it can be used for the Byzantine case as well.

7.3.1 Flooding

We'll now show an algorithm that gets agreement, termination, and validity. Validity here is stronger than the non-triviality condition used in the lower bound, but the lower bound still applies: we can't do this in less than $f + 1$ rounds.

So let's do it in exactly $f + 1$ rounds. There are two standard algorithms, one of which generalizes to Byzantine processes under good conditions. We'll start with a simple approach based on flooding. This algorithm is described

in more detail in [AW04, §5.1.3] or [Lyn96, §6.2.1]; the original is due to Dolev and Strong [DS83].

Assumes very trustworthy processes. Each process keeps a set of (process, input) pairs, initially just $\{(\mathsf{myId}, \mathsf{myInput})\}$. At round r, I broadcast my set to everybody and take the union of my set and all sets I receive. At round $f + 1$, I decide on $f(S)$, where f is some fixed function from sets of process-input pairs to outputs that picks some input in S: for example, f might take the input with the smallest process-id attached to it, take the max of all known input values, or take the majority of all known input values.

Lemma 7.3.1. *After $f + 1$ rounds, all non-faulty processes have the same set.*

Proof. Let S_i^r be the set of process i after r rounds. What we'll really show is that if there are no failures in round k, then $S_i^r = S_j^r = S_i^{k+1}$ for all i, j, and $r > k$. To show this, observe that no faults in round k means that all processes that are still alive at the start of round k send their message to all other processes. Let L be the set of live processes in round k. At the end of round k, for i in L we have $S_i^{k+1} = \bigcup_{j \in L} S_j^k = S$. Now we'll consider some round $r = k + 1 + m$ and show by induction on m that $S_i^{k+m} = S$; we already did $m = 0$, so for larger m notice that all messages are equal to S and so S_i^{k+1+m} is the union of a whole bunch of S's. So in particular we have $S_i^{f+1} = S$ (since some failure-free round occurred in the preceding $f + 1$ rounds) and everybody decides the same value $f(S)$. □

Flooding depends on being able to trust second-hand descriptions of values; it may be that process 1 fails in round 0 so that only process 2 learns its input. If process 2 can suddenly tell 3 (but nobody else) about the input in round $f + 1$—or worse, tell a different value to 3 and 4—then we may get disagreement. This remains true even if Byzantine processes can't fake inputs (e.g., because an input value is really a triple $(i, v, \mathsf{signature}(v))$ using an unforgeable digital signature)—the reason is that a Byzantine process could horde some input $(i, v, \mathsf{signature}(v))$ until the very last round and then deliver it to only some of the non-faulty processes.

7.4 Exponential information gathering

The idea of **exponential information gathering** is that each process will do a lot of gossiping, but now its state is no longer just a flat set of

inputs, but a tree describing who it heard what from. We build this tree out of pairs of the form (id-sequence, input) where id-sequence is a sequence of intermediaries with no repetitions and input is some input. A process's state at each round is just a set of such pairs.

This is not really an improvement on flooding for crash failures, but it can be used as a basis for building an algorithm for Byzantine agreement (Chapter 8). Also useful as an example of a **full-information algorithm**, in which every process records all that it knows about the execution; in principle this allows the algorithm to simulate any other algorithm, which can sometimes be useful for proving lower bounds.

See [AW04, §5.2.4] or [Lyn96, §6.2.3] for more details than we provide here. The original exponential information-gathering algorithm (for Byzantine processes) is due to Pease, Shostak, and Lamport [PSL80].

Initial state is $(\langle\rangle, \mathsf{myInput})$.

At round r, process i broadcasts all pairs (w, v) where $|w| = r$ and i does not appear in w (these restrictions make the algorithm slightly less exponential). Upon receiving (w, v) from j, i adds (wj, v) to its list. If no message arrives from j in round r, i adds $(wj, \perp)$ to its list for all non-repeating w with $|w| = r$ (this step can also be omitted).

A tree structure is obtained by letting w be the parent of wj for each j.

At round $f + 1$, apply some fixed decision rule to the set of all values that appear in the tree (e.g. take the max, or decide on a default value v_0 if there is more than one value in the tree). That this works follows pretty much immediately from the fact that the set of node labels propagates just as in the flooding algorithm (which is why EIG isn't really an improvement). But there are some complications from the messages that aren't sent due to the i-not-in-w restriction on sending. So we have to write out a real proof. Below is basically just following the presentation in [Lyn96].

Let $\mathsf{val}(w, i)$ be the value v such that (w, v) appears in i's list at the end of round $f + 1$. We don't worry about what earlier round it appears in because we can compute that round as $|w| + 1$.

7.4.1 Basic invariants

- $\mathsf{val}(\langle\rangle, i) = i$'s input.
- Either $\mathsf{val}(wj, i)$ equals $\mathsf{val}(w, j)$ or $\mathsf{val}(wj, i) = \perp$ and j didn't send a message in round $|w| + 1$.

These are trivial.

7.4.2 Stronger facts

- If $\mathsf{val}(xjy, i) \neq \perp$ then $\mathsf{val}(x, j) = \mathsf{val}(xjy, i)$. Apply the invariant inductively.
- If $\mathsf{val}(w, i) \neq \perp$ then it equals $\mathsf{val}(\langle\rangle, j)$ for some j. Either $w = \langle\rangle$ and we win or we can apply the previous fact to $w = jy$.
- If $\mathsf{val}(w, i) \neq \perp$ then there is some w' that doesn't contain i such that $\mathsf{val}(w', i) = \mathsf{val}(w, i)$. Let $\mathsf{val}(w, i) = v$. If w doesn't contain i we are done, otherwise $w = w'iy$ for some w' and y, and thus $\mathsf{val}(w', i) = v$.

7.4.3 The payoff

Let S_i^r be the set of values in i's list after r rounds. We'll show that $S_i^{f+1} = S_j^{f+1}$ for all non-faulty i and j. Let v be in S_i^{f+1}. Then $v = \mathsf{val}(w, i)$ for some w that doesn't contain i (w here is really w' from before). If $|w| \leq f$, then i sends (w, v) to j at round $|wi|$ and so $\mathsf{val}(wi, j) = v$. Otherwise if $|w| = f + 1$, $w = xky$ for some non-faulty k, and from the first stronger fact we have $\mathsf{val}(x, k) = v$. Since k is non-faulty, it sends (x, v) to both i and j in round $|x|$ and we get $\mathsf{val}(xk, j) = v$. We've just shown v in S_i^{f+1} implies v in S_j^{f+1}, and by symmetry the converse holds, so the sets are equal.

This is a lot of work to avoid sending messages that contain my own id! However, by tacking on digital signatures, we can solve Byzantine agreement in the case where $f < n/2$: see [Lyn96, §6.2.4] for details.

7.4.4 The real payoff

Run the same algorithm in a Byzantine system with $n > 3f$ processes (treating bogus-looking messages as nulls), but compute the decision value by taking recursive majorities of non-null values down the tree. Details are in §8.2.1.

7.5 Variants

So far we have described **binary consensus**, since all inputs are 0 or 1. We can also allow larger input sets. With crash failures, this allows a stronger validity condition: the output must be equal to some input. Note that this stronger condition doesn't work if we have Byzantine failures. (Exercise: why not?)

Chapter 8

Byzantine agreement

Like synchronous agreement (as in Chapter 7) except that we replace crash failures with **Byzantine failures**, where a faulty process can ignore its programming and send any messages it likes. Since we are operating under a universal quantifier, this includes the case where the Byzantine processes appear to be colluding with each other under the control of a centralized adversary.

8.1 Lower bounds

We'll start by looking at lower bounds.

8.1.1 Minimum number of rounds

We've already seen an $f + 1$ lower bound on rounds for crash failures (see §7.2). This lower bound applies *a fortiori* to Byzantine failures, since Byzantine failures can simulate crash failures.

8.1.2 Minimum number of processes

We can also show that we need $n > 3f$ processes. For $n = 3$ and $f = 1$ the intuition is that Byzantine B can play non-faulty A and C off against each other, telling A that C is Byzantine and C that A is Byzantine. Since A is telling C the same thing about B that B is saying about A, C can't tell the difference and doesn't know who to believe. Unfortunately, this tragic soap opera is not a real proof, since we haven't actually shown that B can say exactly the right thing to keep A and C from guessing that B is evil.

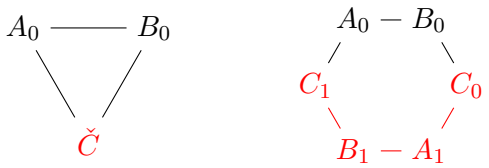


Figure 8.1: Three-process vs. six-process execution in Byzantine agreement lower bound. Processes A_0 and B_0 in right-hand execution receive same messages as in left-hand three-process execution with Byzantine $\check{C}$ simulation C_0 through C_1. So validity forces them to decide 0. A similar argument using Byzantine $\check{A}$ shows the same for C_0.

The real proof:[1] Consider an artificial execution where (non-Byzantine) A, B, and C are duplicated and then placed in a ring $A_0B_0C_0A_1B_1C_1$, where the digits indicate inputs. We'll still keep the same code for $n = 3$ on A_0, B_0, etc., but when A_0 tries to send a message to what it thinks of as just C we'll send it to C_1 while messages from B_0 will instead go to C_0. For any adjacent pair of processes (e.g. A_0 and B_0), the behavior of the rest of the ring could be simulated by a single Byzantine process (e.g. C), so each process in the 6-process ring behaves just as it does in some 3-process execution with 1 Byzantine process. It follows that all of the processes terminate and decide in the unholy 6-process Frankenexecution[2] the same value that they would in the corresponding 3-process Byzantine execution. So what do they decide?

Given two processes with the same input, say, A_0 and B_0, the giant execution is indistinguishable from an $A_0B_0\check{C}$ execution where $\check{C}$ is Byzantine (see Figure 8.1. Validity says A_0 and B_0 must both decide 0. Since this works for any pair of processes with the same input, we have each process deciding its input. But now consider the execution of $C_0A_1\check{B}$, where $\check{B}$ is Byzantine. In the big execution, we just proved that C_0 decides 0 and A_1 decides 1, but since the C_0A_1B execution is indistinguishable from the big execution to C_0 and A_1, they do the same thing here and violate agreement.

This shows that with $n = 3$ and $f = 1$, we can't win. We can generalize this to $n = 3f$. Suppose that there were an algorithm that solved Byzantine agreement with $n = 3f$ processes. Group the processes into groups of size f,

[1] The presentation here is based on [AW04, §5.2.3]. The original impossibility result is due to Pease, Shostak, and Lamport [PSL80]. This particular proof is due to Fischer, Lynch, and Merritt [FLM86].

[2] Not a real word.

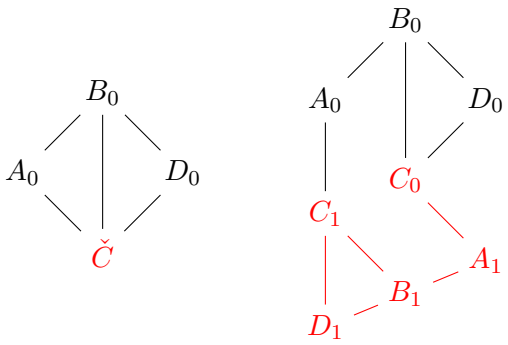


Figure 8.2: Four-process vs. eight-process execution in Byzantine agreement connectivity lower bound. Because Byzantine $\check{C}$ can simulate C_0, D_1, B_1, A_1, and C_1, A_0, B_0 and D_0 must all decide 0 or risk violating validity.

and let each of the $n = 3$ processes simulate one group, with everybody in the group getting the same input, which can only make things easier. Then we get a protocol for $n = 3$ and $f = 1$, an impossibility.

8.1.3 Minimum connectivity

So far, we've been assuming a complete communication graph. If the graph is not complete, we may not be able to tolerate as many failures. In particular, we need the connectivity of the graph (minimum number of nodes that must be removed to split it into two components) to be at least $2f+1$. See [Lyn96, §6.5] for the full proof. The essential idea is that if we have an arbitrary graph with a vertex cut of size $k < 2f + 1$, we can simulate it on a 4-process graph where A is connected to B and C (but not D), B and C are connected to each other, and D is connected only to B and C. Here B and C each simulate half the processes in the size-k cut, A simulates all the processes on one side of the cut and D all the processes on the other side. We then construct an 8-process artificial execution with two non-faulty copies of each of A, B, C, and D and argue that if one of B or C can be Byzantine then the 8-process execution is indistinguishable to the remaining processes from a normal 4-process execution. (See Figure 8.1.)

An argument similar to the $n > 3f$ proof then shows we violate one of validity or agreement: if we replacing C_0, C_1, and all the nodes on one side of the $C_0 + C_1$ cut with a single Byzantine $\check{C}$, we force the remaining non-faulty nodes to decide their inputs or violate validity. But then doing

the same thing with B_0 and B_1 yields an execution that violates agreement.

Conversely, if we have connectivity $2f+1$, then the processes can simulate a general graph by sending each other messages along $2f+1$ predetermined vertex-disjoint paths and taking the majority value as the correct message. Since the f Byzantine processes can only corrupt one path each (assuming the non-faulty processes are careful about who they forward messages from), we get at least $f+1$ good copies overwhelming the f bad copies. This reduces the problem on a general graph with sufficiently high connectivity to the problem on a complete graph, allowing Byzantine agreement to be solved if the other lower bounds are met.

8.1.4 Weak Byzantine agreement

(Here we are following [Lyn96, §6.6]. The original result is due to Lamport [Lam83].)

Weak Byzantine agreement is like regular Byzantine agreement, but validity is only required to hold if there are no faulty processes at all.[3] If there is a single faulty process, the non-faulty processes can output any value regardless of their inputs (as long as they agree on it). Sadly, this weakening doesn't improve things much: even weak Byzantine agreement can be solved only if $n \geq 3f+1$.

Proof: As in the strong Byzantine agreement case, we'll construct a many-process Frankenexecution to figure out a strategy for a single Byzantine process in a 3-process execution. The difference is that now the number of processes in our synthetic execution is much larger, since we want to build an execution where at least some of our test subjects think they are in a non-Byzantine environment. The trick is to build a very big, highly-symmetric ring so that at least some of the processes are so far away from the few points of asymmetry that might clue them in to their odd condition that the protocol terminates before they notice.

Fix some protocol that allegedly solves weak Byzantine agreement, and let r be the number of rounds for the protocol. Construct a ring of $6r$ processes $A_{01}B_{01}C_{01}A_{02}B_{02}C_{02}\ldots A_{0r}B_{0r}C_{0r}A_{10}B_{10}C_{10}\ldots A_{1r}B_{1r}C_{1r}$, where each X_{ij} runs the code for process X in the 3-process protocol with input i. For each adjacent pair of processes, there is a 3-process Byzantine execution

[3] An alternative might be to weaken agreement or termination to apply only if there are no non-faulty processes, but this makes the problem trivial. If we weaken agreement, we can just have each process decide whatever process 1 tells it to, and if we weaken termination, we can do more or less the same thing except that we only terminate if all the other processes tell us they heard the same value from process 1.

which is indistinguishable from the $6r$-process execution for that pair: since agreement holds in all Byzantine executions, each adjacent pair decides the same value in the big execution and so either everybody decides 0 or everybody decides 1 in the big execution.

Now we'll show that means that validity is violated in some no-failures 3-process execution. We'll extract this execution by looking at the execution of processes $A0_{r/2}B0_{r/2}C0_{r/2}$. The argument is that up to round r, any input-0 process that is at least r steps in the ring away from the nearest 1-input process acts like the corresponding process in the all-0 no-failures 3-process execution. Since $A_{0,r/2}$ is $3r/2 > r$ hops away from A_{1r} and similarly for $C_{0,r/2}$, our 3 stooges all decide 0 by validity. But now repeat the same argument for $A_{1,r/2}B_{1,r/2}C_{1,r/2}$ and get 3 new stooges that all decide 1. This means that somewhere in between we have two adjacent processes where one decides 0 and one decides 1, violating agreement in the corresponding 3-process execution where the rest of the ring is replaced by a single Byzantine process. This concludes the proof.

This result is a little surprising: we might expect that weak Byzantine agreement could be solved by allowing a process to return a default value if it notices anything that might hint at a fault somewhere. But this would allow a Byzantine process to create disagreement revealing its bad behavior to just one other process in the very last round of an execution otherwise headed for agreement on the non-default value. The chosen victim decides the default value, but since it's the last round, nobody else finds out. Even if the algorithm is doing something more sophisticated, examining the $6r$-process execution will tell the Byzantine process exactly when and how to start acting badly.

8.2 Upper bounds

Here we describe two upper bounds for Byzantine agreement, one of which gets an optimal number of rounds at the cost of many large messages, and the other of which gets smaller messages at the cost of more rounds. (We are following §§5.2.4–5.2.5 of [AW04] in choosing these algorithms.) Neither of these algorithms is state-of-the-art, but they demonstrate some of the issues in solving Byzantine agreement without the sometimes-complicated optimizations needed to get all the parameters of the algorithm down simultaneously.

8.2.1 Exponential information gathering gets $n = 3f + 1$

We'll show that a variant of Exponential Information Gathering as defined in §7.4 works with $n \geq 3f+1$ in $f+1$ rounds. This is the same technique used by Pease, Shostak, and Lamport [PSL80] to show that their impossibility result is tight.

Recall EIG gives us at each node a set of pairs $(\mathsf{path}, \mathsf{value})$ where path spans all sequences of 0 to n distinct ids and value is the input value forwarded along that path. We write $\mathsf{val}(w, i)$ for the value stored in i's list at the end of the protocol that is associated with path w. Because we can't trust these $\mathsf{val}(w, i)$ values to be an accurate description of any process's input if there is a Byzantine process in w, each process computes for itself replacement values $\mathsf{val}'(w, i)$ that use majority voting to try to get a more trustworthy picture of the original inputs.

Formally, we think of the set of paths as a tree where w is the parent of wj for each path w and each id j not in w. To apply EIG in the Byzantine model, ill-formed messages received from j are treated as missing messages, but otherwise the data-collecting part of EIG proceeds as in the crash failure model. However, we compute the decision value from the last-round values recursively as follows. First replace any missing pair involving a path w with $|w| = f + 1$ with $(w, 0)$. Then for each path w, define $\mathsf{val}'(w, i)$ to be the majority value among $\mathsf{val}'(wj, i)$ for all j, or $\mathsf{val}(w, i)$ if $|w| = f + 1$. Finally, we have process i decide $\mathsf{val}'(\langle\rangle, i)$ (which it can compute locally from its own stored values $\mathsf{val}(w, i)$).

The val' is a reconstruction of old values from later ones: as we move up the tree from wj to w we are moving backwards in time, until in the end we get the decision value $\mathsf{val}'(\langle\rangle, i)$ as a majority of reconstructed inputs $\mathsf{val}'(j, i)$. One way to think about this is that I don't trust j to give me the right value for wj—even when $w = \langle\rangle$ and j is just reporting its own input—so instead a take a majority of values of wj that j allegedly reported to other people. But since I don't trust those other people either, I use the same process recursively to construct those reports.

8.2.1.1 Proof of correctness

This is just a sketch of the proof from [Lyn96, §6.3.2]; essentially the same argument appears in [AW04, §5.2.4].

We start with a basic observation that good processes send and record values correctly:

Lemma 8.2.1. *If i, j, and k are all non-faulty then for all w, $\mathsf{val}(wk, i) = \mathsf{val}(wk, j) = \mathsf{val}(w, k)$.*

Proof. Trivial: k announces the same value $\mathsf{val}(w, k)$ to both i and j. □

More involved is this lemma, which says that when we reconstruct a value for a trustworthy process at some level, we get the same value that it sent us. In particular this will be used to show that the reconstructed inputs $\mathsf{val}'(j, i)$ are all equal to the real inputs for good processes.

Lemma 8.2.2. *If j is non-faulty then $\mathsf{val}'(wj, i) = \mathsf{val}(wj, i)$ for all non-faulty i and all w.*

Proof. By induction on $f + 1 - |wj|$. If $|wj| = f + 1$, then $\mathsf{val}'(wj, i) = \mathsf{val}(wj, i) = \mathsf{val}(w, j) = \mathsf{val}(wj, i)$. If $|wj| < f+1$, then $\mathsf{val}(wj, k) = \mathsf{val}(w, j)$ for all non-faulty k. It follows that $\mathsf{val}(wjk, i) = \mathsf{val}(w, j)$ for all non-faulty i and k (that do no appear in w). The bad guys report at most f bad values $\mathsf{val}(wj, k')$, but the good guys report at least $n - f - |wj|$ good values $\mathsf{val}(wj, k)$. Since $n \geq 3f + 1$ and $|wj| \leq f$, we have $n - f - |wj| \geq 3f + 1 - f - f \geq f + 1$ good values, which are a majority. □

We call a node w **common** $\mathsf{val}'(w, i) = \mathsf{val}'(w, j)$ for all non-faulty i, j. Lemma 8.2.2 says that wk is common if k is non-faulty. We can also show that any node whose children are all common is also common, whether or not the last process in its label is faulty.

Lemma 8.2.3. *Let wk be common for all k. Then w is common.*

Proof. Recall that, for $|w| < f + 1$, $\mathsf{val}'(w, i)$ is the majority value among all $\mathsf{val}'(wk, i)$. If all wk are common, then $\mathsf{val}'(wk, i) = \mathsf{val}'(wk, j)$ for all non-faulty i and j, so i and j compute the same majority values and get $\mathsf{val}'(w, i) = \mathsf{val}'(w, j)$. □

We can now prove the full result.

Theorem 8.2.4. *Exponential information gathering using $f + 1$ rounds in a synchronous Byzantine system with at most f faulty processes satisfies validity and agreement, provided $n \geq 3f + 1$.*

Proof. Validity: Immediate application of Lemmas 8.2.1 and 8.2.2 when $w = \langle\rangle$. We have $\mathsf{val}'(j, i) = \mathsf{val}(j, i) = \mathsf{val}(\langle\rangle, j)$ for all non-faulty j and i, which means that a majority of the $\mathsf{val}'(j, i)$ values equal the common input and thus so does $\mathsf{val}'(\langle\rangle, i)$.

Agreement: Observe that every path has a common node on it, since a path travels through $f+1$ nodes and one of them is good. If we then suppose that the root is not common: by Lemma 8.2.3, it must have a not-common child, that node must have a not-common child, etc. But this constructs a path from the root to a leaf with no not-common nodes, which we just proved can't happen. □

8.2.2 Phase king gets constant-size messages

The following algorithm, based on work of Berman, Garay, and Perry[BGP89], achieves Byzantine agreement in $2(f+1)$ rounds using constant-size messages, provided $n \geq 4f+1$. The description here is drawn from [AW04, §5.2.5]. The original Berman-Garay-Perry paper gives somewhat better bounds, but they're more complicated.

8.2.2.1 The algorithm

The basic idea of the algorithm is that we avoid the recursive majority voting of EIG by running a vote in each of $f+1$ *phases* through a **phase king**, some process chosen in advance to run the phase. Since the number of phases exceeds the number of faults, we eventually get a non-faulty phase king. The algorithm is structured so that one non-faulty phase king is enough to generate agreement and subsequent faulty phase kings can't undo the agreement.

Pseudocode appears in Algorithm 8.1. Each processes i maintains an array $\mathsf{pref}_i[j]$, where j ranges over all process ids. There are also utility values majority, kingMajority and multiplicity for each process that are used to keep track of what it hears from the other processes. Initially, $\mathsf{pref}_i[i]$ is just i's input and $\mathsf{pref}_i[j] = 0$ for $j \neq i$.

The idea of the algorithm is that in each phase, everybody announces their current preference (initially the inputs). If the majority of these preferences is large enough (e.g. all inputs are the same), everybody adopts the majority preference. Otherwise everybody adopts the preference of the phase king. The majority rule means that once the processes agree, they continue to agree despite bad phase kings. The phase king rule allows a good phase king to end disagreement. By choosing a different king in each phase, after $f+1$ phases, some king must be good. This intuitive description is justified below.

```
1  pref_i[i] = input
2  for j ≠ i do pref_i[j] = 0
3  for k ← 1 to f + 1 do
       // First round of phase k
4      send pref_i[i] to all processes (including myself)
5      pref_i[j] ← v_j, where v_j is the value received from process j
6      majority ← majority value in pref_i
7      multiplicity ← number of times majority appears in pref_i
       // Second round of phase k
8      if i = k then
           // I am the phase king
9          send majority to all processes
10     receive kingMajority from phase king
11     if multiplicity > n/2 + f then
12         pref_i[i] = majority
13     else
14         pref_i[i] = kingMajority
15 return pref_i[i]
```

Algorithm 8.1: Byzantine agreement: phase king

8.2.2.2 Proof of correctness

Termination is immediate from the algorithm.

For validity, suppose all inputs are v. We'll show that all non-faulty i have $\mathsf{pref}_i[i] = v$ after every phase. In the first round of each phase, process i receives at least $n - f$ messages containing v; since $n \geq 4f + 1$, we have $n - f \geq 3f + 1$ and $n/2 + f \leq (4f + 1)/2 + f = 3f + 1/2$, and thus these $n - f$ messages exceed the $n/2 + f$ threshold for adopting them as the new preference. So all non-faulty processes ignore the phase king and stick with v, eventually deciding v after round $2(f + 1)$.

For agreement, we'll ignore all phases up to the first phase with a non-faulty phase king. Let k be the first such phase, and assume that the pref values are set arbitrarily at the start of this phase. We want to argue that at the end of the phase, all non-faulty processes have the same preference. There are two ways that a process can set its new preference in the second round of the phase:

1. The process i observes a majority of more than $n/2+f$ identical values v and ignores the phase king. Of these values, more than $n/2$ of them were sent by non-faulty processes. So the phase king also receives these values (even if the faulty processes change their stories) and chooses v as its majority value. Similarly, if any other process j observes a majority of $n/2 + f$ identical values, the two $> n/2$ non-faulty parts of the majorities overlap, and so j also chooses v.

2. The process i takes its value from the phase king. We've already shown that i then agrees with any j that sees a big majority; but since the phase king is non-faulty, process i will agree with any process j that also takes its new preference from the phase king.

This shows that after any phase with a non-faulty king, all processes agree. The proof that the non-faulty processes continue to agree is the same as for validity.

8.2.2.3 Performance of phase king

It's not hard to see that this algorithm sends exactly $(f+1)(n^2+n)$ messages of 1 bit each (assuming 1-bit inputs). The cost is doubling the minimum number of rounds and reducing the tolerance for Byzantine processes. As mentioned earlier, a variant of phase-king with 3-round phases gets optimal fault-tolerance with $3(f + 1)$ rounds (but 2-bit messages). Still better is

a rather complicated descendant of the EIG algorithm due to Garay and Moses [GM98], which gets $f + 1$ rounds with $n \geq 3f + 1$ while still having polynomial message traffic.

Chapter 9

Impossibility of asynchronous agreement

The Fischer-Lynch-Paterson (FLP) result [FLP85] says that you can't do agreement in an asynchronous message-passing system if even one crash failure is allowed, unless you augment the basic model in some way, e.g. by adding randomization or failure detectors. After its initial publication, it was quickly generalized to other models including asynchronous shared memory [LAA87], and indeed the presentation of the result in [Lyn96, §12.2] is given for shared-memory first, with the original result appearing in [Lyn96, §17.2.3] as a corollary of the ability of message passing to simulate shared memory. In these notes, I'll present the original result; the dependence on the model is surprisingly limited, and so most of the proof is the same for both shared memory (even strong versions of shared memory that support e.g. atomic snapshots[1]) and message passing.

Section 5.3 of [AW04] gives a very different version of the proof, where it is shown first for two processes in shared memory, then generalized to n processes in shared memory by adapting the classic Borowsky-Gafni simulation [BG93] to show that two processes with one failure can simulate n processes with one failure. This is worth looking at (it's an excellent example of the power of simulation arguments, and BG simulation is useful in many other contexts) but we will stick with the original argument, which is simpler. We will look at this again when we consider BG simulation in Chapter 27.

[1] Chapter 19.

9.1 Agreement

Usual rules: **agreement** (all non-faulty processes decide the same value), **termination** (all non-faulty processes eventually decide some value), **validity** (for each possible decision value, there an execution in which that value is chosen). Validity can be tinkered with without affecting the proof much.

To keep things simple, we assume the only two decision values are 0 and 1.

9.2 Failures

A failure is an internal action after which all send operations are disabled. The adversary is allowed one failure per execution. Effectively, this means that any group of $n - 1$ processes must eventually decide without waiting for the n-th, because it might have failed.

9.3 Steps

The FLP paper uses a notion of *steps* that is slightly different from the send and receive actions of the asynchronous message-passing model we've been using. Essentially a step consists of receiving zero or more messages followed by doing a finite number of sends. To fit it into the model we've been using, we'll define a step as either a pair (p, m), where p receives message m and performs zero or more sends in response, or $(p, \perp)$, where p receives nothing and performs zero or more sends. We assume that the processes are deterministic, so the messages sent (if any) are determined by p's previous state and the message received. Note that these steps do not correspond precisely to delivery and send events or even pairs of delivery and send events, because what message gets sent in response to a particular delivery may change as the result of delivering some other message; but this won't affect the proof.

The fairness condition essentially says that if (p, m) or $(p, \perp)$ is continuously enabled it eventually happens. Since messages are not lost, once (p, m) is enabled in some configuration C, it is enabled in all successor configurations until it occurs; similarly $(p, \perp)$ is always enabled. So to ensure fairness, we have to ensure that any non-faulty process eventually performs any enabled step.

Comment on notation: I like writing the new configuration reached by applying a step e to C like this: Ce. The FLP paper uses $e(C)$.

9.4 Bivalence and univalence

The core of the FLP argument is a strategy allowing the adversary (who controls scheduling) to steer the execution away from any configuration in which the processes reach agreement. The guidepost for this strategy is the notion of **bivalence**, where a configuration C is **bivalent** if there exist traces T_0 and T_1 starting from C that lead to configurations CT_0 and CT_1 where all processes decide 0 and 1 respectively. A configuration that is not bivalent is **univalent**, or more specifically **0-valent** or **1-valent** depending on whether all executions starting in the configuration produce 0 or 1 as the decision value. (Note that bivalence or univalence are the only possibilities because of termination.) The important fact we will use about univalent configurations is that any successor to an x-valent configuration is also x-valent.

It's clear that any configuration where some process has decided is not bivalent, so if the adversary can keep the protocol in a bivalent configuration forever, it can prevent the processes from ever deciding. The adversary's strategy is to start in an initial bivalent configuration C_0 (which we must prove exists) and then choose only bivalent successor configurations (which we must prove is possible). A complication is that if the adversary is only allowed one failure, it must eventually allow any message in transit to a non-faulty process to be received and any non-faulty process to send its outgoing messages, so we have to show that the policy of avoiding univalent configurations doesn't cause problems here.

9.5 Existence of an initial bivalent configuration

We can specify an initial configuration by specifying the inputs to all processes. If one of these initial configurations is bivalent, we are done. Otherwise, let C and C' be two initial configurations that differ only in the input of one process p; by assumption, both C and C' are univalent. Consider two executions starting with C and C' in which process p is faulty; we can arrange for these executions to be indistinguishable to all the other processes, so both decide the same value x. It follows that both C and C' are x-valent. But since any two initial configurations can be connected by some chain of

such indistinguishable configurations, we have that all initial configurations are x-valent, which violations validity.

9.6 Staying in a bivalent configuration

Now start in a failure-free bivalent configuration C with some step $e = (p, m)$ or $e = (p, \perp)$ enabled in C. Let S be the set of configurations reachable from C without doing e or failing any processes, and let $e(S)$ be the set of configurations of the form $C'e$ where C' is in S. (Note that e is always enabled in S, since once enabled the only way to get rid of it is to deliver the message.) We want to show that $e(S)$ contains a failure-free bivalent configuration.

The proof is by contradiction: suppose that $C'e$ is univalent for all C' in S. We will show first that there are C_0 and C_1 in S such that each C_ie is i-valent. To do so, consider any pair of i-valent A_i reachable from C; if A_i is in S, let $C_i = A_i$. If A_i is not in S, let C_i be the last configuration before executing e on the path from C to A_i (C_ie is univalent in this case by assumption).

So now we have C_0e and C_1e with C_ie i-valent in each case. We'll now go hunting for some configuration D in S and step e' such that De is 0-valent but $De'e$ is 1-valent (or vice versa); such a pair exists because S is connected and so some step e' crosses the boundary between the $C'e = 0$-valent and the $C'e = 1$-valent regions.

By a case analysis on e and e' we derive a contradiction:

1. Suppose e and e' are steps of different processes p and p'. Let both steps go through in either order. Then $Dee' = De'e$, since in an asynchronous system we can't tell which process received its message first. But De is 0-valent, which implies Dee' is also 0-valent, which contradicts $De'e$ being 1-valent.

2. Now suppose e and e' are steps of the same process p. Again we let both go through in either order. It is not the case now that $Dee' = De'e$, since p knows which step happened first (and may have sent messages telling the other processes). But now we consider some finite sequence of steps $e_1e_2 \ldots e_k$ in which no message sent by p is delivered and some process decides in $Dee_1 \ldots e_k$ (this occurs since the other processes can't distinguish Dee' from the configuration in which p died in D, and so have to decide without waiting for messages from p). This execution fragment is indistinguishable to all processes except

p from $De'ee_1 \dots e_k$, so the deciding process decides the same value i in both executions. But Dee' is 0-valent and $De'e$ is 1-valent, giving a contradiction.

It follows that our assumption was false, and there is some reachable bivalent configuration $C'e$.

Now to construct a fair execution that never decides, we start with a bivalent configuration, choose the oldest enabled action and use the above to make it happen while staying in a bivalent configuration, and repeat.

9.7 Generalization to other models

To apply the argument to another model, the main thing is to replace the definition of a step and the resulting case analysis of 0-valent $De'e$ vs 1-valent Dee' to whatever steps are available in the other model. For example, in asynchronous shared memory, if e and e' are operations on different memory locations, they commute (just like steps of different processes), and if they are operations on the same location, either they commute (e.g. two reads) or only one process can tell whether both happened (e.g. with a write and a read, only the reader knows, and with two writes, only the first writer knows). Killing the witness yields two indistinguishable configurations with different valencies, a contradiction.

We are omitting a lot of details here. See [Lyn96, §12.2] for the real proof, or Loui and Abu-Amara [LAA87] for the generalization to shared memory, or Herlihy [Her91b] for similar arguments for a wide variety of shared-memory primitives. We will see many of these latter arguments in Chapter 18.

Chapter 10

Paxos

The **Paxos** algorithm for consensus in a message-passing system was first described by Lamport in 1990 in a tech report that was widely considered to be a joke (see `http://research.microsoft.com/users/lamport/pubs/pubs.html#lamport-paxos` for Lamport's description of the history). The algorithm was finally published in 1998 [Lam98], and after the algorithm continued to be ignored, Lamport finally gave up and translated the results into readable English [Lam01]. It is now understood to be one of the most efficient practical algorithms for achieving consensus in a message-passing system with failure detectors, mechanisms that allow processes to give up on other stalled processes after some amount of time (which can't be done in a normal asynchronous system because giving up can be made to happen immediately by the adversary).

We will describe only the basic Paxos algorithm. The WikiPedia article on Paxos (`http://en.wikipedia.org/wiki/Paxos`) gives a remarkably good survey of subsequent developments and applications.

10.1 Motivation: replicated state machines

A **replicated state machine** is an object that is replicated across multiple machines, with some mechanism for propagating operations on the object to all replicas. Formally, we think of an object as having a set of **states** Q, together with a transition relation δ that maps pairs of **operations** and states to pairs of **responses** and states. Applying an operation to the object means that we change the state to the output of δ and return the given response. If we have failures, we will need some sort of consensus protocol to coordinate which operations are applied and in what order. Paxos is a

common choice, although other choices of consensus protocols will work too.

10.2 The Paxos algorithm

The algorithm runs in a message-passing model with asynchrony and less than $n/2$ crash failures (but not Byzantine failures, at least in the original algorithm). As always, we want to get agreement, validity, and termination. The Paxos algorithm itself is mostly concerned with guaranteeing agreement and validity while allowing for the possibility of termination if there is a long enough interval in which no process restarts the protocol.

Processes are classified as **proposers**, **accepters**, and **learners** (a single process may have all three roles). The idea is that a proposer attempts to ratify a proposed decision value (from an arbitrary input set) by collecting acceptances from a majority of the accepters, and this ratification is observed by the learners. Agreement is enforced by guaranteeing that only one proposal can get the votes of a majority of accepters, and validity follows from only allowing input values to be proposed. The tricky part is ensuring that we don't get deadlock when there are more than two proposals or when some of the processes fail. The intuition behind how this works is that any proposer can effectively restart the protocol by issuing a new proposal (thus dealing with lockups), and there is a procedure to release accepters from their old votes if we can prove that the old votes were for a value that won't be getting a majority any time soon.

To organize this vote-release process, we attach a distinct proposal number to each proposal. The safety properties of the algorithm don't depend on anything but the proposal numbers being distinct, but since higher numbers override lower numbers, to make progress we'll need them to increase over time. The simplest way to do this in practice is to make the proposal number be a timestamp with the proposer's id appended to break ties. We could also have the proposer poll the other processes for the most recent proposal number they've seen and add 1 to it.

The revoting mechanism now works like this: before taking a vote, a proposer tests the waters by sending a prepare(n) message to all accepters, where n is the proposal number. An accepter responds to this with a promise never to accept any proposal with a number less than n (so that old proposals don't suddenly get ratified) together with the highest-numbered proposal that the accepter has accepted (so that the proposer can substitute this value for its own, in case the previous value was in fact ratified). If the proposer receives a response from a majority of the accepters, the proposer

then does a second phase of voting where it sends accept(n, v) to all accepters and wins if receives a majority of votes.

So for each proposal, the algorithm proceeds as follows:

1. The proposer sends a message prepare(n) to all accepters. (Sending to only a majority of the accepters is enough, assuming they will all respond.)

2. Each accepter compares n to the highest-numbered proposal for which it has responded to a prepare message. If n is greater, it responds with ack(n, v, n_v), where v is the highest-numbered proposal it has accepted and n_v is the number of that proposal (or $\perp$ and 0 if there is no such proposal). (An optimization at this point is to allow the accepter to send back nack(n') where n' is some higher number to let the proposer know that it's doomed and should back off and try again—this keeps a confused proposer who thinks it's the future from locking up the protocol until 2037.)

3. The proposer waits (possibly forever) to receive ack from a majority of accepters. If any ack contained a value, it sets v to the most recent (in proposal number ordering) value that it received. It then sends accept(n, v) to all accepters (or just a majority). You should think of accept as a demand ("Accept!") rather than acquiescence ("I accept")—the accepters still need to choose whether to accept or not.

4. Upon receiving accept(n, v), an accepter accepts v unless it has already received prepare(n') for some $n' > n$. If a majority of acceptors accept the value of a given proposal, that value becomes the decision value of the protocol.

Note that acceptance is a purely local phenomenon; additional messages are needed to detect which if any proposals have been accepted by a majority of accepters. Typically this involves a fourth round, where accepters send accepted(n, v) to all learners (often just the original proposer).

There is no requirement that only a single proposal is sent out (indeed, if proposers can fail we will need to send out more to jump-start the protocol). The protocol guarantees agreement and validity no matter how many proposers there are and no matter how often they start.

10.3 Informal analysis: how information flows between rounds

Call a **round** the collection of all messages labeled with some particular proposal n. The structure of the algorithm simulates a sequential execution in which higher-numbered rounds follow lower-numbered ones, even though there is no guarantee that this is actually the case in a real execution.

When an acceptor sends $\mathsf{ack}(n, v, n_v)$, it is telling the round-n proposer the last value preceding round n that it accepted. The rule that an acceptor only acknowledges a proposal higher than any proposal it has previously acknowledged prevents it from sending information "back in time"—the round n_v in an acknowledgment is always less than n. The rule that an acceptor doesn't accept any proposal earlier than a round it has acknowledged means that the value v in an $\mathsf{ack}(n, v, n_v)$ message never goes out of date—there is no possibility that an acceptor might retroactively accept some later value in round n' with $n_v < n' < n$. So the ack message values tell a consistent story about the history of the protocol, even if the rounds execute out of order.

The second trick is to use overlapping majorities to make sure that any value that is accepted is not lost. If the only way to decide on a value in round n is to get a majority of acceptors to accept it, and the only way to make progress in round n' is to get acknowledgments from a majority of acceptors, these two majorities overlap. So in particular the overlapping process reports the round-n proposal value to the proposer in round n', and we can show by induction on n' that this round-n proposal value becomes the proposal value in all subsequent rounds that proceed past the acknowledgment stage. So even though it may not be possible to detect that a decision has been reached in round n (say, because some of the acceptors in the accepting majority die without telling anybody what they did), no later round will be able to choose a different value. This ultimately guarantees agreement.

10.4 Safety properties

We now present a more formal analysis of the Paxos protocol. We consider only the safety properties of the protocol, corresponding to validity and agreement; without additional assumptions, Paxos does *not* guarantee termination.

Call a value *chosen* if it is accepted by a majority of accepters. The safety properties of Paxos are:

- No value is chosen unless it is first proposed. (This gives validity.)
- No two distinct values are both chosen. (This gives agreement.)

The first property is immediate from examination of the algorithm.

For the second property, we need some invariants. The intuition is that if some value is chosen, then a majority of accepters have accepted it for some proposal number n. Any proposal sent in an accept message with a higher number n' must be sent by a proposer that has seen an overlapping majority respond to its $\mathsf{prepare}(n')$ message. If we consider the process that overlaps, this process must have accepted v before it received $\mathsf{prepare}(n')$, since it can't accept afterwards, and unless it has accepted some other proposal since, it responds with $\mathsf{ack}(n', v, n)$. If these are the only values that the proposer receives with number n or greater, it chooses v as its new value.

Worrying about what happens in rounds between n and n' is messy, so we'll use two formal invariants (taken more or less directly from Lamport's paper):

Invariant 1 An accepter accepts a proposal numbered n if and only if it has not responded to a prepare message with a number $n' > n$.

Invariant 2 For any v and n, if a proposal with value v and number n has been issued (by sending accept messages), then there is a majority of accepters S such that either (a) no accepter in S has accepted any proposal numbered less than n, or (b) v is the value of the highest-numbered proposal among all proposals numbered less than n accepted by at least one accepter in S.

The proof of the first invariant is immediate from the rule for issuing acks.

The proof of the second invariant follows from the first invariant and the proposer's rule for issuing proposals: it can only do so after receiving ack from a majority of accepters—call this set S—and the value it issues is either the proposal's initial value if all responses are $\mathsf{ack}(n, \perp, 0)$, or the maximum value sent in by accepters in S if some responses are $\mathsf{ack}(n, v, n_v)$. In the first case we have case (a) of the invariant: nobody accepted any proposals numbered less than n before responding, and they can't afterwards. In the second case we have case (b): the maximum response value is the maximum-numbered accepted value within S at the time of each response, and again no new values numbered less than n will be accepted afterwards. Amazingly, none of this depends on the temporal ordering of different proposals or

messages: the accepters enforce that their acks are good for all time by refusing to change their mind about earlier rounds later.

So now we suppose that some value v is eventually accepted by a majority T with number n. Then we can show by induction on proposal number that all proposals issued with higher numbers have the same value (even if they were issued earlier). For any proposal accept(v', n') with $n' > n$, there is a majority S (which must overlap with T) for which either case (a) holds (a contradiction—once the overlapping accepter finally accepts, it violates the requirement that no proposal less than n' has been accepted) or case (b) holds (in which case by the induction hypothesis v' is the value of some earlier proposal with number $n' \geq n$, implying $v' = v$).

10.5 Learning the results

Somebody has to find out that a majority accepted a proposal in order to get a decision value out. The usual way to do this is to have a fourth round of messages where the accepters send chose(v, n) to some designated learner (usually just the original proposer), which can then notify everybody else if it doesn't fail first. If the designated learner does fail first, we can restart by issuing a new proposal (which will get replaced by the previous successful proposal because of the safety properties).

10.6 Liveness properties

We'd like the protocol to terminate eventually. Suppose there is a single proposer, and that it survives long enough to collect a majority of acks and to send out accepts to a majority of the accepters. If everybody else cooperates, we get termination in 3 message delays.

If there are multiple proposers, then they can step on each other. For example, it's enough to have two carefully-synchronized proposers alternate sending out prepare messages to prevent any accepter from every accepting (since an accepter promises not to accept accept(n, v) once it has responded to prepare$(n + 1)$). The solution is to ensure that there is eventually some interval during which there is exactly one proposer who doesn't fail. One way to do this is to use exponential random backoff (as popularized by Ethernet): when a proposer decides it's not going to win a round (e.g. by receiving a nack or by waiting long enough to realize it won't be getting any more acks soon), it picks some increasingly large random delay before

starting a new round; thus two or more will eventually start far enough apart in time that one will get done without interference.

A more abstract solution is to assume some sort of weak leader election mechanism, which tells each accepter who the "legitimate" proposer is at each time. The accepters then discard messages from illegitimate proposers, which prevents conflict at the cost of possibly preventing progress. Progress is however obtained if the mechanism eventually reaches a state where a majority of the accepters bow to the same non-faulty proposer long enough for the proposal to go through.

Such a weak leader election method is an example of a more general class of mechanisms known as **failure detectors**, in which each process gets hints about what other processes are faulty that eventually converge to reality. The particular failure detector in this case is known as the Ω failure detector; there are other still weaker ones that we will talk about later that can also be used to solve consensus. We will discuss failure detectors in detail in Chapter 11.

Chapter 11

Failure detectors

Failure detectors were proposed by Chandra and Toueg [CT96] as a mechanism for solving consensus in an asynchronous message-passing system with crash failures by distinguishing between slow processes and dead processes. The basic idea is that each process has attached to it a failure detector module that continuously outputs an estimate of which processes in the system have failed. The output need not be correct; indeed, the main contribution of Chandra and Toueg's paper (and a companion paper by Chandra, Hadzilacos, and Toueg [CHT96]) is characterizing just how bogus the output of a failure detector can be and still be useful.

We will mostly follow Chandra and Toueg in these notes; see the paper for the full technical details.

To emphasize that the output of a failure detector is merely a hint at the actual state of the world, a failure detector (or the process it's attached to) is said to **suspect** a process at time t if it outputs failed at that time. Failure detectors can then be classified based on when their suspicions are correct.

We use the usual asynchronous message-passing model, and in particular assume that non-faulty processes execute infinitely often, get all their messages delivered, etc. From time to time we will need to talk about time, and unless we are clearly talking about real time this just means any steadily increasing count (e.g., of total events), and will be used only to describe the ordering of events.

11.1 How to build a failure detector

Failure detectors are only interesting if you can actually build them. In a fully asynchronous system, you can't (this follows from the FLP result and the existence of failure-detector-based consensus protocols). But with timeouts, it's not hard: have each process ping each other process from time to time, and suspect the other process if it doesn't respond to the ping within twice the maximum round-trip time for any previous ping. Assuming that ping packets are never lost and there is an (unknown) upper bound on message delay, this gives what is known as an **eventually perfect failure detector**: once the max round-trip times rise enough and enough time has elapsed for the live processes to give up on the dead ones, all and only dead processes are suspected.

11.2 Classification of failure detectors

Chandra and Toueg define eight classes of failure detectors, based on when they suspect faulty processes and non-faulty processes. Suspicion of faulty processes comes under the heading of **completeness**; of non-faulty processes, **accuracy**.

11.2.1 Degrees of completeness

Strong completeness Every faulty process is eventually permanently suspected by every non-faulty process.

Weak completeness Every faulty process is eventually permanently suspected by some non-faulty process.

There are two temporal logic operators embedded in these statements: "eventually permanently" means that there is some time t_0 such that for all times $t \geq t_0$, the process is suspected. Note that completeness says nothing about suspecting non-faulty processes: a paranoid failure detector that permanently suspects everybody has strong completeness.

11.2.2 Degrees of accuracy

These describe what happens with non-faulty processes, and with faulty processes that haven't crashed yet.

Strong accuracy No process is suspected (by anybody) before it crashes.

Weak accuracy Some non-faulty process is never suspected.

Eventual strong accuracy After some initial period of confusion, no process is suspected before it crashes. This can be simplified to say that no non-faulty process is suspected after some time, since we can take end of the initial period of chaos as the time at which the last crash occurs.

Eventual weak accuracy After some initial period of confusion, some non-faulty process is never suspected.

Note that "strong" and "weak" mean different things for accuracy vs completeness: for accuracy, we are quantifying over suspects, and for completeness, we are quantifying over suspectors. Even a weakly-accurate failure detector guarantees that all processes trust the one visibly good process.

11.2.3 Boosting completeness

It turns out that any weakly-complete failure detector can be boosted to give strong completeness. Recall that the difference between weak completeness and strong completeness is that with weak completeness, somebody suspects a dead process, while with strong completeness, everybody suspects it. So to boost completeness we need to spread the suspicion around a bit. On the other hand, we don't want to break accuracy in the process, so there needs to be some way to undo a premature rumor of somebody's death. The simplest way to do this is to let the alleged corpse speak for itself: I will suspect you from the moment somebody else reports you dead until the moment you tell me otherwise.

Pseudocode is given in Algorithm 11.1.

```
1 initially do
2     suspects ← ∅
3 while true do
4     Let S be the set of all processes my weak detector suspects.
5     Send S to all processes.
6 upon receiving S from q do
7     suspects ← (suspects ∪ p) \ {q}
```

Algorithm 11.1: Boosting completeness

It's not hard to see that this boosts completeness: if p crashes, somebody's weak detector eventually suspects it, this process tells everybody else, and p never contradicts it. So eventually everybody suspects p.

What is slightly trickier is showing that it preserves accuracy. The essential idea is this: if there is some good-guy process p that everybody trusts forever (as in weak accuracy), then nobody ever reports p as suspect—this also covers strong accuracy since the only difference is that now every non-faulty process falls into this category. For eventual weak accuracy, wait for everybody to stop suspecting p, wait for every message ratting out p to be delivered, and then wait for p to send a message to everybody. Now everybody trusts p, and nobody every suspects p again. Eventual strong accuracy is again similar.

This will justify ignoring the weakly-complete classes.

11.2.4 Failure detector classes

Two degrees of completeness times four degrees of accuracy gives eight classes of failure detectors, each of which gets its own name. But since we can boost weak completeness to strong completeness, we can use this as an excuse to consider only the strongly-complete classes.

P **(perfect)** Strongly complete and strongly accurate: non-faulty processes are never suspected; faulty processes are eventually suspected by everybody. Easily achieved in synchronous systems.

S **(strong)** Strongly complete and weakly accurate. The name is misleading if we've already forgotten about weak completeness, but the corresponding W (weak) class is only weakly complete and weakly accurate, so it's the strong completeness that the S is referring to.

$\Diamond P$ **(eventually perfect)** Strongly complete and eventually strongly accurate.

$\Diamond S$ **(eventually strong)** Strongly complete and eventually weakly accurate.

Jumping to the punch line: P can simulate any of the others, S and $\Diamond P$ can both simulate $\Diamond S$ but can't simulate P or each other, and $\Diamond S$ can't simulate any of the others (See Figure 11.1—we'll prove all of this later.) Thus $\Diamond S$ is the weakest class of failure detectors in this list. However, $\Diamond S$ *is* strong enough to solve consensus, and in fact any failure detector (whatever

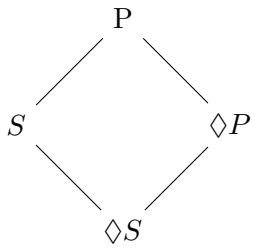


Figure 11.1: Partial order of failure detector classes. Higher classes can simulate lower classes.

its properties) that can solve consensus is strong enough to simulate $\Diamond S$ (this is the result in the Chandra-Hadzilacos-Toueg paper [CHT96])—this makes $\Diamond S$ the "weakest failure detector for solving consensus" as advertised. Continuing our tour through Chandra and Toueg [CT96], we'll show the simulation results and that $\Diamond S$ can solve consensus, but we'll skip the rather involved proof of $\Diamond S$'s special role from Chandra-Hadzilacos-Toueg.

11.3 Consensus with S

With the strong failure detector S, we can solve consensus for any number of failures.

In this model, the failure detectors as applied to most processes are completely useless. However, there is some non-faulty process c that nobody every suspects, and this is enough to solve consensus with as many as $n-1$ failures.

The basic idea of the protocol: There are three phases. In the first phase, the processes gossip about input values for $n-1$ asynchronous rounds. In the second, they exchange all the values they've seen and prune out any that are not universally known. In the third, each process decides on the lowest-id input that hasn't been pruned (minimum input also works since at this point everybody has the same view of the inputs).

Pseudocode is given in Algorithm 11.2

In phase 1, each process p maintains two partial functions V_p and δ_p, where V_p lists all the input values $\langle q, v_q \rangle$ that p has ever seen and δ_p lists only those input values seen in the most recent of $n-1$ asynchronous rounds. Both V_p and δ_p are initialized to $\{\langle p, v_p \rangle\}$. In round i, p sends (i, δ_p) to all processes. It then collects $\langle i, \delta_q \rangle$ from each q that it doesn't suspect and sets δ_p to $\bigcup_q \delta_q \setminus V_p$ (where q ranges over the processes from which p received a

```
1  V_p ← {⟨p, v_p⟩}
2  δ_p ← {⟨p, v_p⟩}
   // Phase 1
3  for i ← 1 to n − 1 do
4  |  Send ⟨i, δ_p⟩ to all processes.
5  |  Wait to receive ⟨i, δ_q⟩ from all q I do not suspect.
6  |  δ_p ← (⋃_q δ_q) \ V_p
7  |  V_p ← (⋃_q δ_q) ∪ V_p
   // Phase 2
8  Send ⟨n, δ_p⟩ to all processes.
9  Wait to receive ⟨n, δ_q⟩ from all q I do not suspect.
10 V_p ← (⋂_q V_q) ∩ V_p
   // Phase 3
11 return some input from V_p chosen via a consistent rule.
```

Algorithm 11.2: Consensus with a strong failure detector

message in round i) and sets V_p to $V_p \cup \delta_p$. In the next round, it repeats the process. Note that each pair $\langle q, v_q \rangle$ is only sent by a particular process p the first round after p learns it: so any value that is still kicking around in round $n-1$ had to go through $n-1$ processes.

In phase 2, each process p sends $\langle n, V_p \rangle$, waits to receive $\langle n, V_q \rangle$ from every process it does not suspect, and sets V_p to the intersection of V_p and all received V_q. At the end of this phase all V_p values will in fact be equal, as we will show.

In phase 3, everybody picks some input from their V_p vector according to a consistent rule.

11.3.1 Proof of correctness

Let c be a non-faulty process that nobody every suspects.

The first observation is that the protocol satisfies validity, since every V_p contains v_c after round 1 and each V_p can only contain input values by examination of the protocol. Whatever it may do to the other values, taking intersections in phase 2 still leaves v_c, so all processes pick some input value from a nonempty list in phase 3.

To get termination we have to prove that nobody ever waits forever for a message it wants; this basically comes down to showing that the first non-

faulty process that gets stuck eventually is informed by the S-detector that the process it is waiting for is dead.

For agreement, we must show that in phase 3, every V_p is equal; in particular, we'll show that every $V_p = V_c$. First it is necessary to show that at the end of phase 1, $V_c \subseteq V_p$ for all p. This is done by considering two cases:

1. If $\langle q, v_q \rangle \in V_c$ and c learns $\langle q, v_q \rangle$ before round $n-1$, then c sends $\langle q, v_q \rangle$ to p no later than round $n-1$, p waits for it (since nobody ever suspects c), and adds it to V_p.

2. If $\langle q, v_q \rangle \in V_c$ and c learns $\langle q, v_q \rangle$ only in round $n-1$, then $\langle q, v_q \rangle$ was previously sent through $n-1$ other processes, i.e., all of them. Each process $p \neq c$ thus added $\langle q, v_q \rangle$ to V_p before sending it and again $\langle q, v_q \rangle$ is in V_p.

(The missing case where $\langle q, v_q \rangle$ isn't in V_c we don't care about.)

But now phase 2 knocks out any extra elements in V_p, since V_p gets set to $V_p \cap V_c \cap$ (some other V_q's that are supersets of V_c). It follows that, at the end of phase 2, $V_p = V_c$ for all p. Finally, in phase 3, everybody applies the same selection rule to these identical sets and we get agreement.

11.4 Consensus with $\Diamond S$ and $f < n/2$

The consensus protocol for S depends on some process c never being suspected; if c is suspected during the entire (finite) execution of the protocol—as can happen with $\Diamond S$—then it is possible that no process will wait to hear from c (or anybody else) and the processes will all decide their own inputs. So to solve consensus with $\Diamond S$ we will need to assume fewer than $n/2$ failures, allowing any process to wait to hear from a majority no matter what lies its failure detector is telling it.

The resulting protocol, known as the **Chandra-Toueg consensus protocol**, is structurally similar to the consensus protocol in Paxos.[1] The difference is that instead of proposers blindly showing up, the protocol is divided into rounds with a rotating **coordinator** p_i in each round r with $r = i \pmod{n}$. The termination proof is based on showing that in any round where the coordinator is not faulty and nobody suspects it, the protocol finishes.

[1] See Chapter 10.

The consensus protocol uses as a subroutine a protocol for **reliable broadcast**, which guarantees that any message that is sent is either received by no non-faulty processes or exactly once by all non-faulty processes. Pseudocode for reliable broadcast is given as Algorithm 11.3. It's easy to see that if a process p is non-faulty and receives m, then the fact that p is non-faulty means that is successfully sends m to everybody else, and that the other non-faulty processes also receive the message at least once and deliver it.

```
1 procedure broadcast(m)
2     send m to all processes.
3 upon receiving m do
4     if I haven't seen m before then
5         send m to all processes
6         deliver m to myself
```

Algorithm 11.3: Reliable broadcast

Here's a sketch of the actual consensus protocol:

- Each process keeps track of a preference (initially its own input) and a timestamp, the round number in which it last updated its preference.
- The processes go through a sequence of asynchronous rounds, each divided into four phases:
  1. All processes send (round, preference, timestamp) to the coordinator for the round.
  2. The coordinator waits to hear from a majority of the processes (possibly including itself). The coordinator sets its own preference to some preference with the largest timestamp of those it receives and sends (round, preference) to all processes.
  3. Each process waits for the new proposal from the coordinator *or* for the failure detector to suspect the coordinator. If it receives a new preference, it adopts it as its own, sets timestamp to the current round, and sends (round, ack) to the coordinator. Otherwise, it sends (round, nack) to the coordinator.
  4. The coordinator waits to receive ack or nack from a majority of processes. If it receives ack from a majority, it announces the

current preference as the protocol decision value using reliable broadcast.

- Any process that receives a value in a reliable broadcast decides on it immediately.

Pseudocode is in Algorithm 19.

```
1  preference ← input
2  timestamp ← 0
3  for round ← 1 . . . ∞ do
4      Send ⟨round, preference, timestamp⟩ to coordinator
5      if I am the coordinator then
6          Wait to receive ⟨round, preference, timestamp⟩ from majority of
           processes.
7          Set preference to value with largest timestamp.
8          Send ⟨round, preference⟩ to all processes.
9      Wait to receive ⟨round, preference′⟩ from coordinator or to suspect
       coordinator.
10     if I received ⟨round, preference′⟩ then
11         preference ← preference′
12         timestamp ← round
13         Send ack(round) to coordinator.
14     else
15         Send nack(round) to coordinator.
16     if I am the coordinator then
17         Wait to receive ack(round) or nack(round) from a majority of
           processes.
18         if I received no nack(round) messages then
19             Broadcast preference using reliable broadcast.
```

11.4.1 Proof of correctness

For validity, observe that the decision value is an estimate and all estimates start out as inputs.

For termination, observe that no process gets stuck in phase 1, 2, or 4, because either it isn't waiting or it is waiting for a majority of non-faulty processes who all sent messages unless they have already decided (this is

why we need the nacks in phase 3). The loophole here is that processes that decide stop participating in the protocol; but because any non-faulty process retransmits the decision value in the reliable broadcast, if a process is waiting for a response from a non-faulty process that already terminated, eventually it will get the reliable broadcast instead and terminate itself. In phase 3, a process might get stuck waiting for a dead coordinator, but the strong completeness of $\Diamond S$ means that it suspects the dead coordinator eventually and escapes. So at worst we do finitely many rounds.

Now suppose that after some time t there is a process c that is never suspected by any process. Then in the next round in which c is the coordinator, in phase 3 all surviving processes wait for c and respond with ack, c decides on the current estimate, and triggers the reliable broadcast protocol to ensure everybody else decides on the same value. Since reliable broadcast guarantees that everybody receives the message, everybody decides this value *or some value previously broadcast*—but in either case everybody decides.

Agreement is the tricky part. It's possible that two coordinators both initiate a reliable broadcast and some processes choose the value from the first and some the value from the second. But in this case the first coordinator collected acks from a majority of processes in some round r, and all subsequent coordinators collected estimates from an overlapping majority of processes in some round $r' > r$. By applying the same induction argument as for Paxos, we get that all subsequent coordinators choose the same estimate as the first coordinator, and so we get agreement.

11.5 $f < n/2$ is still required even with $\Diamond P$

We can show that with a majority of failures, we're in trouble with just $\Diamond P$ (and thus with $\Diamond S$, which is trivially simulated by $\Diamond P$). The reason is that $\Diamond P$ can lie to us for some long initial interval of the protocol, and consensus is required to terminate eventually despite these lies. So the usual partition argument works: start half of the processes with input 0, half with 1, and run both halves independently with $\Diamond P$ suspecting the other half until the processes in both halves decide on their common inputs. We can now make $\Diamond P$ happy by letting it stop suspecting the processes, but it's too late.

11.6 Relationships among the classes

It's easy to see that P simulates S and $\Diamond P$ simulates $\Diamond S$ without modification. It's also immediate that P simulates $\Diamond P$ and S simulates $\Diamond S$ (make "eventually" be "now"), which gives a diamond-shaped lattice structure between the classes. What is trickier is to show that this structure doesn't collapse: $\Diamond P$ can't simulate S, S can't simulate $\Diamond P$, and $\Diamond S$ can't simulate any of the other classes.

First let's observe that $\Diamond P$ can't simulate S: if it could, we would get a consensus protocol for $f \geq n/2$ failures, which we can't do. It follows that $\Diamond P$ also can't simulate P (because P can simulate S).

To show that S can't simulate $\Diamond P$, choose some non-faulty victim process v and consider an execution in which S periodically suspects v (which it is allowed to do as long as there is some other non-faulty process it never suspects). If the $\Diamond P$-simulator ever responds to this by refusing to suspect v, there is an execution in which v really is dead, and the simulator violates strong completeness. But if not, we violate eventual strong accuracy. Note that this also implies S can't simulate P, since P can simulate $\Diamond P$. It also shows that $\Diamond S$ can't simulate either of $\Diamond P$ or P.

We are left with showing $\Diamond S$ can't simulate S. Consider a system where p's $\Diamond S$ detector suspects q but not r from the start of the execution, and similarly r's $\Diamond S$ detector also suspects q but not p. Run p and r in isolation until they give up and decide that q is in fact dead (which they must do eventually by strong completeness, since this run is indistinguishable from one in which q is faulty). Then wake up q and crash p and r. Since q is the only non-faulty process, we've violated weak accuracy.

Chandra and Toueg [CT96] give as an example of a natural problem that can be solved only with P the problem of **terminating reliable broadcast**, in which a single leader process attempts to send a message and all other processes eventually agree on the message if the leader is non-faulty but must terminate after finite time with a default *no message* return value if the leader is faulty.[2] The process is solvable using P by just having each process either wait for the message or for P to suspect the leader, which can only occur if the leader does in fact crash. If the leader is dead, the processes must eventually decide on no message; this separates P from $\Diamond S$ and $\Diamond P$ since we can then wake up the leader and let it send its message. But it also separates P from S, since we can have the S-detector only be

[2] This is a slight weakening of the problem, which however still separates P from the other classes. For the real problem see Chandra and Toueg [CT96].

accurate for non-leaders. For other similar problems see the paper.

Chapter 12

Logical clocks

Logical clocks assign a timestamp to all events in an asynchronous message-passing system that simulates real time, thereby allowing timing-based algorithms to run despite asynchrony. In general, they don't have anything to do with clock synchronization or wall-clock time; instead, they provide numerical values that increase over time and are consistent with the observable behavior of the system. In particular, messages are never delivered before they are sent, when time is measured using the logical clock.

12.1 Causal ordering

The underlying notion of a logical clock is **causal ordering**, a partial order on events that describes when one event e provably occurs before some other event e'.

For the purpose of defining casual ordering and logical clocks, we will assume that a schedule consists of **send events** and **receive events**, which correspond to some process sending a single message or receiving a single message, respectively.

Given two schedules S and S', call S and S' **similar** if $S|p = S'|p$ for all processes p; in other words, S and S' are similar if they are indistinguishable by all participants. We can define a causal ordering on the events of some schedule S implicitly by considering all schedules S' similar to S, and declare that $e < e'$ if e precedes e' in all such S. But it is usually more useful to make this ordering explicit.

Following [AW04, §6.1.1] (and ultimately [Lam78]), define the **happens-before** relation $\Rightarrow_S$ on a schedule S to consist of:

1. All pairs (e, e') where e precedes e' in S and e and e' are events of the

same process.

2. All pairs (e, e') where e is a send event and e' is the receive event for the same message.

3. All pairs (e, e') where there exists a third event e'' such that $e \Rightarrow_S e''$ and $e'' \Rightarrow_S e'$. (In other words, we take the **transitive closure** of the relation defined by the previous two cases.)

It is not terribly hard to show that this gives a partial order; the main observation is that if $e \Rightarrow_S e'$, then e precedes e' in S. So $\Rightarrow_S$ is a subset of the total order $<_S$ given by the order of events in S.

A **causal shuffle** S' of a schedule S is a permutation of S that is consistent with the happens-before relation on S; that is, if e happens-before e' in S, then e precedes e' in S'. The importance of the happens-before relation follows from this lemma:

Lemma 12.1.1. *Let S' be a permutation of the events in S. Then the following two statements are equivalent:*

1. *S' is a causal shuffle of S.*

2. *S' is the schedule of an execution fragment of a message-passing system with $S|p = S'|p$ for all S'.*

Proof. $(1 \Rightarrow 2)$. We need to show both similarity and that S' corresponds to some execution fragment. We'll show similarity first. Pick some p; then every event at p in S also occurs in S', and they must occur in the same order by the first case of the definition of the happens-before relation. This gets us halfway to showing S' is the schedule of some execution fragment, since it says that any events initiated by p are consistent with p's programming. To get the rest of the way, observe that any other events are receive events. For each receive event e' in S, there must be some matching send event e also in S; thus e and e' are both in S' and occur in the right order by the second case of the definition of happens-before.

$(2 \Rightarrow 1)$. First observe that since every event e in S' occurs at some process p, if $S'|p = S|p$ for all p, then there is a one-to-one correspondence between events in S' and S, and thus S' is a permutation of S. Now we need to show that S' is consistent with $\Rightarrow_S$. Let $e \Rightarrow_S e'$. There are three cases.

1. e and e' are events of the same process p and $e <_S e'$. But then $e <_{S'} e'$ because $S|p = S'|p$.

2. e is a send event and e' is the corresponding receive event. Then $e <_{S'} e'$ because S' is the schedule of an execution fragment.

3. $e \Rightarrow_S e'$ by transitivity. Then each step in the chain connecting e to e' uses one of the previous cases, and $e <_{S'} e'$ by transitivity of $<_{S'}$.

□

What this means: if I tell you $\Rightarrow_S$, then you know everything there is to know about the order of events in S that you can deduce from reports from each process together with the fact that messages don't travel back in time. But $\Rightarrow_S$ is a pretty big relation ($\Theta(|S|^2)$ bits with a naive encoding), and seems to require global knowledge of $<_S$ to compute. So we can ask if there is some simpler, easily computable description that works almost as well. This is where logical clocks come in.

12.2 Implementations

12.2.1 Lamport clock

Lamport's **logical clock** [Lam78] runs on top of any other message-passing protocol, adding additional state at each process and additional content to the messages (which is invisible to the underlying protocol). Every process maintains a local variable clock. When a process sends a message or executes an internal step, it sets $\mathsf{clock} \leftarrow \mathsf{clock} + 1$ and assigns the resulting value as the clock value of the event. If it sends a message, it piggybacks the resulting clock value on the message. When a process receives a message with timestamp t, it sets $\mathsf{clock} \leftarrow \max(\mathsf{clock}, t) + 1$; the resulting clock value is taken as the time of receipt of the message. (To make life easier, we assume messages are received one at a time.)

Theorem 12.2.1. *If we order all events by clock value, we get an execution of the underlying protocol that is locally indistinguishable from the original execution.*

Proof. Let $e <_L e'$ if e has a lower clock value than e'. If e and e' are two events of the same process, then $e <_L e'$. If e and e' are send and receive events of the same message, then again $e <_L e'$. So for *any* events e, e', if $e \Rightarrow_S e'$, then $e <_L e'$. Now apply Lemma 12.1.1. □

12.2.2 Neiger-Toueg-Welch clock

Lamport's clock has the advantage of requiring no changes in the behavior of the underlying protocol, but has the disadvantage that clocks are entirely under the control of the logical-clock protocol and may as a result make huge jumps when a message is received. If this is unacceptable—perhaps the protocol needs to do some unskippable maintenance task every 1000 clock ticks—then an alternative approach due to Neiger and Toueg [NT87] and Welch [Wel87] can be used.

Method: Each process maintains its own variable clock, which it increments whenever it feels like it. To break ties, the process extends the clock value to $\langle \mathsf{clock}, \mathsf{id}, \mathsf{eventCount} \rangle$ where eventCount is a count of send and receive events (and possibly local computation steps). As in Lamport's clock, each message in the underlying protocol is timestamped with the current extended clock value. Because the protocol can't change the clock values on its own, when a message is received with a timestamp later than the current extended clock value, its delivery is delayed until clock exceeds the message timestamp, at which point the receive event is assigned the extended clock value of the time of delivery.

Theorem 12.2.2. *If we order all events by clock value, we get an execution of the underlying protocol that is locally indistinguishable from the original execution.*

Proof. Again, we have that (a) all events at the same process occur in increasing order (since the event count rises even if the clock value doesn't, and we assume that the clock value doesn't drop) and (b) all receive events occur later than the corresponding send event (since we force them to). So Lemma 12.1.1 applies. □

The advantage of the Neiger-Toueg-Welch clock is that it doesn't impose any assumptions on the clock values, so it is possible to make clock be a real-time clock at each process and nonetheless have a causally-consistent ordering of timestamps even if the local clocks are not perfectly synchronized. If some process's clock is too far off, it will have trouble getting its messages delivered quickly (if its clock is ahead) or receiving messages (if its clock is behind)—the net effect is to add a round-trip delay to that process equal to the difference between its clock and the clock of its correspondent. But the protocol works well when the processes' clocks are closely synchronized, which has become a plausible assumption in the last 10-15

years thanks to the Network Time Protocol, cheap GPS receivers, and clock synchronization mechanisms built into most cellular phone networks.[1]

12.2.3 Vector clocks

Logical clocks give a *superset* of the happens-before relation: if $e \Rightarrow_S e'$, then $e <_L e'$ (or conversely, if $e \not<_L e'$, then it is not the case that $e \Rightarrow_S e'$). This is good enough for most applications, but what if we want to compute $\Rightarrow_S$ exactly?

Here we can use a **vector clock**, invented independently by Fidge [Fid91] and Mattern [Mat93]. Instead of a single clock value, each event is stamped with a vector of values, one for each process. When a process executes a local event or a send event, it increments only its own component x_p of the vector. When it receives a message, it increments x_p and sets each x_q to the max of its previous value and the value of x_q piggybacked on the message. We define $\mathsf{VC}(e) \leq \mathsf{VC}(e')$, where $\mathsf{VC}(e)$ is the value of the vector clock for e, if $\mathsf{VC}(e)_i \leq \mathsf{VC}(e')_i$ for all i.

Theorem 12.2.3. *Fix a schedule S; then for any e, e', $VC(e) < VC(e')$ if and only if $e \Rightarrow_S e'$.*

Proof. The if part follows immediately from the update rules for the vector clock. For the only if part, suppose e does not happen-before e'. Then e and e' are events of distinct processes p and p'. For $\mathsf{VC}(e) < \mathsf{VC}(e')$ to hold, we must have $\mathsf{VC}(e)_p < \mathsf{VC}(e')_p$; but this can occur only if the value of $\mathsf{VC}(e)_p$ is propagated to p' by some sequence of messages starting at p and ending at p' at or before e' occurs. In this case we have $e \Rightarrow_S e'$. □

12.3 Applications

12.3.1 Consistent snapshots

A **consistent snapshot** of a message-passing computation is a description of the states of the processes (and possibly messages in transit, but we can reduce this down to just states by keeping logs of messages sent and received) that gives the global configuration at some instant of a schedule that is a consistent reordering of the real schedule (a **consistent cut** in

[1] As I write this, my computer reports that its clock is an estimated 289 microseconds off from the timeserver it is synchronized to, which is less than a tenth of the round-trip delay to machines on the same local-area network and a tiny fraction of the round-trip delay to machines elsewhere, including the timeserver machine.

the terminology of [AW04, §6.1.2]. Without shutting down the protocol before taking a snapshot this is the about the best we can hope for in a message-passing system.

Logical time can be used to obtain consistent snapshots: pick some logical time and have each process record its state at this time (i.e. immediately after its last step before the time or immediately before its first step after the time). We have already argued that logical time gives a consistent reordering of the original schedule, so the set of values recorded is just the configuration at the end of an appropriate prefix of this reordering. In other words, it's a consistent snapshot.

If we aren't building logical clocks anyway, there is a simpler consistent snapshot algorithm due to Chandy and Lamport [CL85]. Here some central initiator broadcasts a snap message, and each process records its state and immediately forwards the snap message to all neighbors when it first receives a snap message. To show that the resulting configuration is a configuration of some consistent reordering, observe that (with FIFO channels) no process receives a message before receiving snap that was sent after the sender sent snap: thus causality is not violated by lining up all the pre-snap operations before all the post-snap ones.

The full Chandy-Lamport algorithm adds a second marker message that is used to sweep messages in transit out of the communications channels, which avoids the need to keep logs if we want to reconstruct what messages are in transit (this can also be done with the logical clock version). The idea is that when a process records its state after receiving the snap message, it issues a marker message on each outgoing channel. For incoming channels, the process all records all messages received between the snapshot and receiving a marker message on that channel (or nothing if it receives marker before receiving snap). A process only reports its value when it has received a marker on each channel. The marker and snap messages can also be combined if the broadcast algorithm for snap resends it on all channels anyway, and a further optimization is often to piggyback both on messages of the underlying protocol if the underlying protocol is chatty enough.

Note that Chandy-Lamport is equivalent to the logical-time snapshot using Lamport clocks, if the snap message is treated as a message with a very large timestamp. For Neiger-Toueg-Welch clocks, we get an algorithm where processes spontaneously decide to take snapshots (since Neiger-Toueg-Welch clocks aren't under the control of the snapshot algorithm) and delay post-snapshot messages until the local snapshot has been taken. This can be implemented as in Chandy-Lamport by separating pre-snapshot messages from post-snapshot messages with a marker message, and essentially turns

into Chandy-Lamport if we insist that a process advance its clock to the snapshot time when it receives a marker.

12.3.1.1 Property testing

Consistent snapshots are in principle useful for debugging (since one can gather a consistent state of the system without being able to talk to every process simultaneously), and in practice are mostly used for detecting **stable properties** of the system. Here a stable property is some predicate on global configurations that remains true in any successor to a configuration in which it is true, or (bending the notion of properties a bit) functions on configurations whose values don't change as the protocol runs. Typical examples are quiescence and its evil twin, deadlock. More exotic examples include total money supply in a banking system that cannot create or destroy money, or the fact that every process has cast an irrevocable vote in favor of some proposal or advanced its Neiger-Toueg-Welch-style clock past some threshold.

The reason we can test such properties using consistent snapshot is that when the snapshot terminates with value C in some configuration C', even though C may never have occurred during the actual execution of the protocol, there *is* an execution which leads from C to C'. So if P holds in C, stability means that it holds in C'.

Naturally, if P doesn't hold in C, we can't say much. So in this case we re-run the snapshot protocol and hope we win next time. If P eventually holds, we will eventually start the snapshot protocol after it holds and obtain a configuration (which again may not correspond to any global configuration that actually occurs) in which P holds.

12.3.2 Replicated state machines

The main application for suggested by Lamport in his logical-clocks paper [Lam78] was building a **replicated state machine**. In his construction, any process can at any time issue an operation on the object by broadcasting it with an attached timestamp. When a process receives an operation, it buffers it in a priority queue ordered by increasing timestamp. It can apply the first operation in the queue only when it can detect that no earlier operation will arrive, which it can do if it sees a message from every other process with a later timestamp (or after a timeout, if we have some sort of clock synchronization guarantee). It is not terribly hard to show that this guarantees that every replica gets the same sequence of operations applied

to it, and that these operations are applied in an order consistent with the processes' ability to determine the actual order in which they were proposed. Furthermore, if the processes spam each other regularly with their current clock values, each operation will take effect after at most two message delays (with Lamport clocks) if the clocks are not very well synchronized and after approximately one message delay (with Lamport or Neiger-Toueg-Welch clocks) if they are. A process can also execute read operations on its own copy immediately without notifying other processes (if it is willing to give up linearizability for sequential consistency).

However, this particular construction assumes no failures, so for poorly-synchronized clocks or systems in which sequentially-consistent reads are not good enough, replicated state machines are no better than simply keeping one copy of the object on a single process and having all operations go through that process: 2 message delays + 2 messages per operation for the single copy beats 2 message delays + many messages for full replication. But replicated state machines take less time under good conditions, and when augmented with more powerful tools like consensus or atomic broadcast are the basis of most fault-tolerant implementations of general shared-memory objects.

Chapter 13

Synchronizers

Synchronizers simulate an execution of a failure-free synchronous system in a failure-free asynchronous system. See [AW04, Chapter 11] or [Lyn96, Chapter 16] for a detailed (and rigorous) presentation.

13.1 Definitions

Formally, a synchronizer sits between the underlying network and the processes and does one of two things:

- A **global synchronizer** guarantees that no process receives a message for round r until *all processes* have sent their messages for round r.
- A **local synchronizer** guarantees that no process receives a message for round r until *all of that process's neighbors* have sent their messages for round r.

In both cases the synchronizer packages all the incoming round r messages m for a single process together and delivers them as a single action `recv`(p, m, r). Similarly, a process is required to hand over all of its outgoing round-r messages to the synchronizer as a single action `send`(p, m, r)—this prevents a process from changing its mind and sending an extra round-r message or two. It is easy to see that the global synchronizer produces executions that are effectively indistinguishable from synchronous executions, assuming that a synchronous execution is allowed to have some variability in exactly when within a given round each process does its thing. The local synchronizer only guarantees an execution that is locally indistinguishable from an execution of the global synchronizer: an individual process can't

tell the difference, but comparing actions at different (especially widely separated) processes may reveal some process finishing round $r+1$ while others are still stuck in round r or earlier. Whether this is good enough depends on what you want: it's bad for coordinating simultaneous missile launches, but may be just fine for adapting a synchronous message-passing algorithm (e.g. for distributed breadth-first search as described in Chapter 5) to an asynchronous system, if we only care about the final states of the processes and not when precisely those states are reached.

Formally, the relation between global and local synchronization is described by the following lemma:

Lemma 13.1.1. *For any schedule S of a locally synchronous execution, there is a schedule S' of a globally synchronous execution such that $S|p = S'|p$ for all processes p.*

Proof. Essentially, we use the same **happens-before** relation as in Chapter 12, and the fact that if a schedule S' is a causal shuffle of another schedule S (i.e., a permutation of T that preserves causality), then $S'|p = S|p$ for all p (Lemma 12.1.1).

Given a schedule S, consider a schedule S' in which the events are ordered first by increasing round and then by putting all sends before receives. This ordering is consistent with $\Rightarrow_S$, so it's a causal shuffle of S and $S'|p = S|p$. But it's globally synchronized, because no round-r operations at all happen before a round-$(r-1)$ operation. □

13.2 Implementations

These all implement at least a local synchronizer (the beta synchronizer is global). The names were chosen by their inventor, Baruch Awerbuch [Awe85].

The main difference between them is the mechanism used to determine when round-r messages have been delivered.

In the **alpha synchronizer**, every node sends a message to every neighbor in every round (possibly a dummy message if the underlying protocol doesn't send a message); this allows the receiver to detect when it's gotten all its round-r messages (because it expects to get a message from every neighbor) but may produce huge blow-ups in message complexity in a dense graph.

In the beta synchronizer, messages are acknowledged by their receivers (doubling the message complexity), so the senders can detect when all of their messages are delivered. But now we need a centralized mechanism to

collect this information from the senders and distribute it to the receivers, since any particular receiver doesn't know which potential senders to wait for. This blows up time complexity, as we essentially end up building a global synchronizer with a central leader.

The gamma synchronizer combines the two approaches at different levels to obtain a trade-off between messages and time that depends on the structure of the graph and how the protocol is organized.

Details of each synchronizer are given below.

13.2.1 The alpha synchronizer

The alpha synchronizer uses local information to construct a local synchronizer. In round r, the synchronizer at p sends p's message (tagged with the round number) to each neighbor p' or $\mathsf{noMsg}(r)$ if it has no messages. When it collects a message or noMsg from each neighbor for round r, it delivers all the messages. It's easy to see that this satisfies the local synchronization specification.

This produces no change in time but may drastically increase message complexity because of all the extra noMsg messages flying around. For a synchronous protocol that runs in T rounds with M messages, the same protocol running with the alpha synchronizer will run in T time units, but the message complexity may go up to $M + T \cdot |E|$ messages.

13.2.2 The beta synchronizer

The beta synchronizer centralizes detection of message delivery using a rooted directed spanning tree (previously constructed). When p' receives a round-r message from p, it responds with $\mathsf{ack}(r)$. When p collects an ack for all the messages it sent plus an OK from all of its children, it sends OK to its parent. When the root has all the ack and OK messages it is expecting, it broadcasts go. Receiving go makes p deliver the queued round-r messages.

This works because in order for the root to issue go, every round-r message has to have gotten an acknowledgment, which means that all round-r messages are waiting in the receivers' buffers to be delivered. For the beta synchronizer, message complexity increases slightly from M to $2M+2(n-1)$, but time complexity goes up by a factor proportional to the depth of the tree.

13.2.3 The gamma synchronizer

The gamma synchronizer combines the alpha and beta synchronizers to try to get low blowups on both time complexity and message complexity. The essential idea is to cover the graph with a spanning forest and run beta within each tree and alpha between trees. Specifically:

- Every message in the underlying protocol gets acked (including messages that pass between trees).
- When a process has collected all of its outstanding round-r acks, it sends OK up its tree.
- When the root of a tree gets all acks and OK, it sends ready to the roots of all adjacent trees (and itself). Two trees are adjacent if any of their members are adjacent.
- When the root collects ready from itself and all adjacent roots, it broadcasts go through its own tree.

As in the alpha synchronizer, we can show that no root issues go unless it and all its neighbors issue ready, which happens only after both all nodes in the root's tree and all their neighbors (some of whom might be in adjacent trees) have received acks for all messages. This means that when a node receives go it can safely deliver its bucket of messages.

Message complexity is comparable to the beta synchronizer assuming there aren't too many adjacent trees: $2M$ messages for sends and acks, plus $O(n)$ messages for in-tree communication, plus $O(E_{\text{roots}})$ messages for root-to-root communication. Time complexity per synchronous round is proportional to the depth of the trees: this includes both the time for in-tree communication, and the time for root-to-root communication, which might need to be routed through leaves.

In a particularly nice graph, the gamma synchronizer can give costs comparable to the costs of the original synchronous algorithm. An example in [Lyn96] is a ring of k-cliques, where we build a tree in each clique and get $O(1)$ time blowup and $O(n)$ added messages. This is compared to $O(n/k)$ time blowup for beta and $O(k)$ message blowup (or worse) for alpha. Other graphs may favor tuning the size of the trees in the forest toward the alpha or beta ends of the spectrum, e.g., if the whole graph is a clique (and we didn't worry about contention issues), we might as well just use beta and get $O(1)$ time blowup and $O(n)$ added messages.

13.3 Applications

See [AW04, §11.3.2] or [Lyn96, §16.5]. The one we have seen is distributed breadth-first search, where the two asynchronous algorithms we described in Chapter 5 were essentially the synchronous algorithms with the beta and alpha synchronizers embedded in them. But what synchronizers give us in general is the ability to forget about problems resulting from asynchrony provided we can assume no failures (which may be a very strong assumption) and are willing to accept a bit of overhead.

13.4 Limitations of synchronizers

Here we show some lower bounds on synchronizers, justifying our previous claim that failures are trouble and showing that global synchronizers are necessarily slow in a high-diameter network.

13.4.1 Impossibility with crash failures

The synchronizers above all fail badly if some process crashes. In the α synchronizer, the system slowly shuts down as a wave of waiting propagates out from the dead process. In the β synchronizer, the root never gives the green light for the next round. The γ synchronizer, true to its hybrid nature, fails in a way that is a hybrid of these two disasters.

This is unavoidable in the basic asynchronous model. Suppose that we had a synchronizer that could tolerate crash failures (here, the process that crashed in the asynchronous model would also appear to crash in the simulated synchronous model, but everybody else would keep going). Then we could use this fault-tolerant synchronizer to turn either of the synchronous agreement protocols from Chapter 7 into an asynchronous protocol tolerating arbitrarily many crash failures. But this contradicts the FLP impossibility result from Chapter 9.

We'll see more examples of this trick of showing that a particular simulation is impossible because it would allow us to violate impossibility results later, especially when we start looking at the strength of shared-memory objects in Chapter 18.

13.4.2 Unavoidable slowdown with global synchronization

The **session problem** gives a lower bound on the speed of a global synchronizer, or more generally on any protocol that tries to approximate synchrony

in a certain sense. Recall that in a global synchronizer, our goal is to produce a simulation that looks synchronous "from the outside"; that is, that looks synchronous to an observer that can see the entire schedule. In contrast, a local synchronizer produces a simulation that looks synchronous "from the inside"—the resulting execution is indistinguishable from a synchronous execution to any of the processes, but an outside observer can see that different processes execute different rounds at different times. The global synchronizer we've seen takes more time than a local synchronizer; the session problem shows that this is necessary.

In our description, we will mostly follow [AW04, §6.2.2].

A solution to the session problem is an asynchronous protocol in which each process repeatedly executes some **special action**. Our goal is to guarantee that these special actions group into s **sessions**, where a session is an interval of time in which every process executes at least one special action. We also want the protocol to terminate: this means that in every execution, every process executes a finite number of special actions.

A synchronous system can solve this problem trivially in s rounds: each process executes one special action per round. For an asynchronous system, a lower bound of Attiya and Mavronicolas [AM94] (based on an earlier bound of Arjomandi, Fischer, and Lynch [AFL83], who defined the problem in a slightly different communication model), shows that if the diameter of the network is D, there is no solution to the s-session problem that takes $(s-1)D$ time or less in the worst case. The argument is based on reordering events in any such execution to produce fewer than s sessions, using the happens-before relation from Chapter 12.

13.5 Outline of the proof

(See [AW04, §6.2.2] for the real proof.)

Fix some algorithm A for solving the s-session problem, and suppose that its worst-case time complexity is $(s-1)D$ or less. Consider some synchronous execution of A (that is, one where the adversary scheduler happens to arrange the schedule to be synchronous) that takes $(s-1)D$ rounds or less. Divide this execution into two segments: an initial segment β that includes all rounds with special actions, and a suffix δ that includes any extra rounds where the algorithm is still floundering around. We will mostly ignore δ, but we have to leave it in to allow for the possibility that whatever is happening there is important for the algorithm to work (e.g. to detect termination).

We now want to perform a causal shuffle on β that leaves it with only $s-1$ sessions. The first step is to chop β into at most $s-1$ segments $\beta_1, \beta_2, \ldots$ of at most D rounds each. Because the diameter of the network is D, there exist processes p_0 and p_1 such that no chain of messages starting at p_0 within some segment reaches p_1 before the end of the segment. It follows that for any events e_0 of p_0 and e_1 of p_1 in the same segment β_i, it is not the case that $e_0 \Rightarrow_{\beta\delta} e_1$. So there exists a causal shuffle of β_i that puts all events of p_0 after all events of p_1. By a symmetrical argument, we can similarly put all events of p_1 after all events of p_0. In both cases the resulting schedule is indistinguishable by all processes from the original.

So now we apply these shuffles to each of the segments β_i in alternating order: p_0 goes first in the even-numbered segments and p_1 goes first in the odd-numbered segments, yielding a sequence of shuffled segments β'_i. This has the effect of putting the p_0 events together, as in this example with $(s-1) = 4$:

$$\begin{aligned}\beta\delta|(p_0,p_1) &= \beta_1\beta_2\beta_3\beta_4\delta|(p_0,p_1)\\ &= \beta'_1\beta'_2\beta'_3\beta'_4\delta|(p_0,p_1)\\ &= (p_1p_0)(p_0p_1)(p_1p_0)(p_0p_1)\delta\\ &= p_1(p_0p_0)(p_1p_1)(p_0p_0)p_1\delta\end{aligned}$$

(here each p_0, p_1 stands in for a sequence of events of each process).

Now let's count sessions. We can't end a session until we reach a point where both processes have taken at least one step since the end of the last session. If we mark with a slash the earliest places where this can happen, we get a picture like this:

$$p_1p_0/p_0p_1/p_1p_0/p_0p_1/p_1\delta.$$

We have at most $s-1$ sessions! This concludes the proof.

Chapter 14

Quorum systems

14.1 Basics

In the past few chapters, we've seen many protocols that depend on the fact that if I talk to more than $n/2$ processes and you talk to more than $n/2$ processes, the two groups overlap. This is a special case of a **quorum system**, a family of subsets of the set of processes with the property that any two subsets in the family overlap. By choosing an appropriate family, we may be able to achieve lower load on each system member, higher availability, defense against Byzantine faults, etc.

The exciting thing from a theoretical perspective is that these turn a systems problem into a combinatorial problem: this means we can ask combinatorialists how to solve it.

14.2 Simple quorum systems

- Majority and weighted majorities
- Specialized read/write systems where write quorum is a column and read quorum a row of some grid.
- Dynamic quorum systems: get more than half of the most recent copy.
- Crumbling walls [PW97b, PW97a]: optimal small-quorum system for good choice of wall sizes.

14.3 Goals

- Minimize **quorum size**.
- Minimize **load**, defined as the minimum over all access strategies (probability distributions on quorums) of the maximum over all servers of probability it gets hit.
- Maximize **capacity**, defined as the maximum number of quorum accesses per time unit in the limit if each quorum access ties up a quorum member for 1 time unit (but we are allowed to stagger a quorum access over multiple time units).
- Maximize **fault-tolerance**: minimum number of server failures that blocks all quorums. Note that for standard quorum systems this is directly opposed to minimizing quorum size, since killing the smallest quorum stops us dead.
- Minimize **failure probability** = probability that every quorum contains at least one bad server, assuming each server fails with independent probability.

Naor and Wool [NW98] describe trade-offs between these goals (some of these were previously known, see the paper for citations):

- capacity = 1/load; this is obtained by selecting the quorums independently at random according to the load-minimizing distribution. In particular this means we can forget about capacity and just concentrate on minimizing load.
- load $\geq \max(c/n, 1/c)$ where c is the minimum quorum size. The first case is obvious: if every access hits c nodes, spreading them out as evenly as possible still hits each node c/n of the time. The second is trickier: Naor and Wool prove it using LP duality, but the argument essentially says that if we have some quorum Q of size c, then since every other quorum Q' intersects Q in at least one place, we can show that every Q' adds at least 1 unit of load in total to the c members of Q. So if we pick a random quorum Q', the average load added to all of Q is at least 1, so the average load added to some particular element of Q is at least $1/|Q| = 1/c$. Combining the two cases, we can't hope to get load better than $1/\sqrt{n}$, and to get this load we need quorums of size at least $\sqrt{n}$.

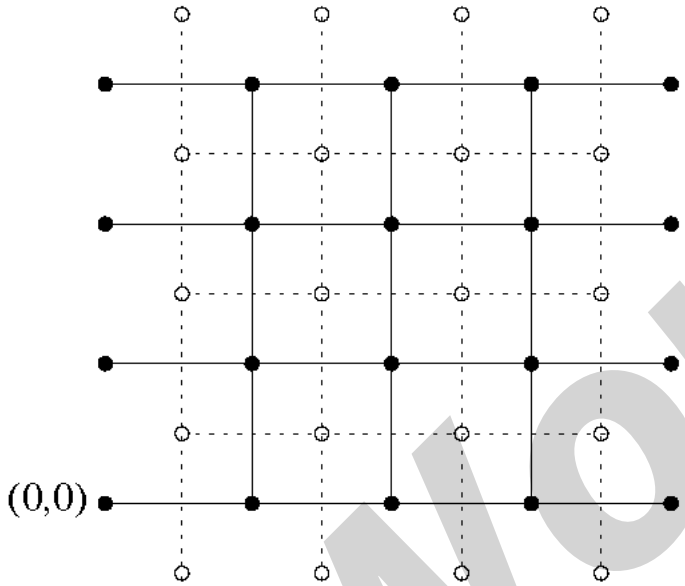


Figure 14.1: Figure 2 from [NW98]. Solid lines are $G(3)$; dashed lines are $G^*(3)$.

- failure probability is at least p when $p > 1/2$ (and optimal system is to just pick a single leader in this case), failure probability can be made exponentially small in size of smallest quorum when $p < 1/2$ (with many quorums). These results are due to Peleg and Wool [PW95].

14.4 Paths system

This is an optimal-load system from Naor and Wool [NW98] with exponentially low failure probability, based on percolation theory.

The basic idea is to build a $d \times d$ mesh-like graph where a quorum consists of the union of a top-to-bottom path (TB path) and a left-to-right path (LR path); this gives quorum size $O(\sqrt{n})$ and load $O(1/\sqrt{n})$. Note that the TB and LR paths are not necessarily direct: they may wander around for a while in order to get where they are going, especially if there are a lot of failures to avoid. But the smallest quorums will have size $2d + 1 = O(\sqrt{n})$.

The actual mesh is a little more complicated. Figure 14.1 reproduces the picture of the $d = 3$ case from the Naor and Wool paper.

Each server corresponds to a *pair* of intersecting edges, one from the

$G(d)$ grid and one from the $G^*(d)$ grid (the star indicates that $G^*(d)$ is the **dual graph**[1] of $G(d)$. A quorum consists of a set of servers that produce an LR path in $G(d)$ and a TB path in $G^*(d)$. Quorums intersect, because any LR path in $G(d)$ must cross some TB path in $G^*(d)$ at some server (in fact, each pair of quorums intersects in at least two places). The total number of elements n is $(d+1)^2$ and the minimum size of a quorum is $2d+1 = \Theta(\sqrt{n})$.

The symmetry of the mesh gives that there exists a LR path in the mesh if and only if there does not exist a TB path in its **complement**, the graph that has an edge only if the mesh doesn't. For a mesh with failure probability $p < 1/2$, the complement is a mesh with failure probability $q = 1 - p > 1/2$. Using results in percolation theory, it can be shown that for failure probability $q > 1/2$, the probability that there exists a left-to-right path is exponentially small in d (formally, for each p there is a constant $\phi(p)$ such that $\Pr[\exists \text{LR path}] \leq \exp(-\phi(p)d)$). We then have

$$\begin{aligned}
\Pr[\exists(\text{live quorum})] &= \Pr[\exists(\text{TB path}) \wedge \exists(\text{LR path})] \\
&= \Pr[\neg\exists(\text{LR path in complement}) \vee \neg\exists(\text{TB path in complement})] \\
&\leq \Pr[\neg\exists(\text{LR path in complement})] + \Pr[\neg\exists(\text{TB path in complement})] \\
&\leq 2\exp(-\phi(1-p)d) \\
&= 2\exp(-\Theta(\sqrt{n})).
\end{aligned}$$

So the failure probability of this system is exponentially small for any fixed $p < 1/2$.

See the paper [NW98] for more details.

14.5 Byzantine quorum systems

Standard quorum systems are great when you only have crash failures, but with Byzantine failures you have to worry about finding a quorum that includes a Byzantine serve who lies about the data. For this purpose you need something stronger. Following Malkhi and Reiter [MR98] and Malkhi *et al.* [MRWW01], one can define:

- A **b-disseminating quorum system** guarantees $|Q_1 \cap Q_2| \geq b + 1$ for all quorums Q_1 and Q_2. This guarantees that if I update a

[1] See `http://en.wikipedia.org/wiki/Dual_graph`; the basic idea is that the dual of a graph G embedded in the plane has a vertex for each region of G, and an edge connecting each pair of vertices corresponding to adjacent regions, where a region is a subset of the plane that is bounded by edges of G.

quorum Q_1 and you update a quorum Q_2, and there are at most b Byzantine processes, then there is some non-Byzantine process in both our quorums. Mostly useful if data is "self-verifying," e.g. signed with digital signatures that the Byzantine processes can't forge. Otherwise, I can't tell which of the allegedly most recent data values is the right one since the Byzantine processes lie.

- A **b-masking quorum system** guarantees $|Q_1 \cap Q_2| \geq 2b + 1$ for all quorums Q_1 and Q_2. (In other words, it's the same as a $2b$-disseminating quorum system.) This allows me to defeat the Byzantine processes through voting: given $2b + 1$ overlapping servers, if I want the most recent value of the data I take the one with the most recent timestamp that appears on at least $b + 1$ servers, which the Byzantine guys can't fake.

An additional requirement in both cases is that for any set of servers B with $|B| \leq b$, there is some quorum Q such that $Q \cap B = \emptyset$. This prevents the Byzantine processes from stopping the system by simply refusing to participate.

Note: these definitions are based on the assumption that there is some fixed bound on the number of Byzantine processes. Malkhi and Reiter [MR98] give more complicated definitions for the case where one has an arbitrary family $\{\mathcal{B}\}$ of potential Byzantine sets. The definitions above are actually simplified versions from [MRWW01].

The simplest way to build a b-disseminating quorum system is to use supermajorities of size at least $(n + b + 1)/2$; the overlap between any two such supermajorities is at least $(n + b + 1) - n = b + 1$. This gives a load of substantially more than $\frac{1}{2}$. There are better constructions that knock the load down to $\Theta(\sqrt{b/n})$; see [MRWW01].

For more on this topic in general, see the survey by by Merideth and Reiter [MR10].

14.6 Probabilistic quorum systems

The problem with all standard (or **strict**) quorum systems is that we need big quorums to get high fault tolerance, since the adversary can always stop us by knocking out our smallest quorum. A **probabilistic quorum system** or more specifically an **ϵ-intersecting quorum system** [MRWW01] improves the fault-tolerance by relaxing the requirements. For such a system we have not only a set system Q, but also a probability distribu-

tion w supplied by the quorum system designer, with the property that $\Pr[Q_1 \cap Q_2 = \emptyset] \leq \epsilon$ when Q_1 and Q_2 are chosen independently according to their weights.

14.6.1 Example

Let a quorum be any set of size $k\sqrt{n}$ for some k and let all quorums be chosen uniformly at random. Pick some quorum Q_1; what is the probability that a random Q_2 does not intersect Q_1? Imagine we choose the elements of Q_2 one at a time. The chance that the first element x_1 of Q_2 misses Q_1 is exactly $(n - k\sqrt{n})/n = 1 - k/\sqrt{n}$, and conditioning on x_1 through x_{i-1} missing Q_1 the probability that x_i also misses it is $(n - k\sqrt{n} - i + 1)/(n - i + 1) \leq (n - k\sqrt{n})/n = 1 - k/\sqrt{n}$. So taking the product over all i gives $\Pr[\text{all miss } Q_1] \leq (1 - k/\sqrt{n})^{k\sqrt{n}} \leq \exp(-k\sqrt{n})^{k/\sqrt{n)} = \exp(-k^2)$. So by setting $k = \Theta(\ln 1/\epsilon)$, we can get our desired ϵ-intersecting system.

14.6.2 Performance

Failure probabilities, if naively defined, can be made arbitrarily small: add low-probability singleton quorums that are hardly ever picked unless massive failures occur. But the resulting system is still ϵ-intersecting.

One way to look at this is that it points out a flaw in the ϵ-intersecting definition: ϵ-intersecting quorums may cease to be ϵ-intersecting conditioned on a particular failure pattern (e.g. when all the non-singleton quorums are knocked out by massive failures). But Malkhi *et al.* [MRWW01] address the problem in a different way, by considering only survival of **high quality** quorums, where a particular quorum Q is **δ-high-quality** if $\Pr[Q_1 \cap Q_2 = \emptyset | Q_1 = Q] \leq \delta$ and high quality if it's $\sqrt{\epsilon}$-high-quality. It's not hard to show that a random quorum is δ-high-quality with probability at least ϵ/δ, so a high quality quorum is one that fails to intersect a random quorum with probability at most $\sqrt{\epsilon}$ and a high quality quorum is picked with probability at least $1 - \sqrt{\epsilon}$.

We can also consider load; Malkhi *et al.* [MRWW01] show that essentially the same bounds on load for strict quorum systems also hold for ϵ-intersecting quorum systems: $\text{load}(S) \geq \max((\text{E}(|Q|)/n, (1 - \sqrt{\epsilon})^2/\text{E}(|Q|))$, where $\text{E}(|Q|)$ is the expected size of a quorum. The left-hand branch of the max is just the average load applied to a uniformly-chosen server. For the right-hand side, pick some high quality quorum Q' with size less than or equal to $(1 - \sqrt{\epsilon})\,\text{E}(|Q|)$ and consider the load applied to its most loaded member by its nonempty intersection (which occurs with probability at least

$1 - \sqrt{\epsilon}$) with a random quorum.

14.7 Signed quorum systems

A further generalization of probabilistic quorum systems gives **signed quorum systems** [Yu06]. In these systems, a quorum consists of some set of positive members (servers you reached) and negative members (servers you tried to reach but couldn't). These allow $O(1)$-sized quorums while tolerating $n - O(1)$ failures, under certain natural probabilistic assumptions. Because the quorums are small, the load on some servers may be very high: so these are most useful for fault-tolerance rather than load-balancing. See the paper for more details.

Part II

Shared memory

Chapter 15

Model

Basic shared-memory model. See also [AW04, §4.1].

The idea of shared memory is that instead of sending messages to each other, processes communicate through a pool of shared **objects**. These are typically **registers** supporting read and write operations, but fancier objects corresponding to more sophisticated data structures or synchronization primitives may also be included in the model.

It is usually assumed that the shared objects do not experience faults. This means that the shared memory can be used as a tool to prevent partitions and other problems that can arise in message passing if the number of faults get too high. As a result, for large numbers of processor failures, shared memory is a more powerful model than message passing, although we will see in Chapter 16 that both models can simulate each other provided a majority of processes are non-faulty.

15.1 Atomic registers

An **atomic register** supports read and write operations; we think of these as happening instantaneously, and think of operations of different processes as interleaved in some sequence. Each read operation on a particular register returns the value written by the last previous write operation. Write operations return nothing.

A process is defined by giving, for each state, the operation that it would like to do next, together with a transition function that specifies how the state will be updated in response to the return value of that operation. A configuration of the system consists of a vector of states for the processes and a vector of value for the registers. A sequential execution consists of a

sequence of alternating configurations and operations $C_0, \pi_1, C_1, \pi_2, C_2 \ldots$, where in each triple C_i, π_{i+1}, C_{i+1}, the configuration C_{i+1} is the result of applying π_{i+1} to configuration C_i. For read operations, this means that the state of the reading process is updated according to its transition function. For write operations, the state of the writing process is updated, and the state of the written register is also updated.

Pseudocode for shared-memory protocols is usually written using standard pseudocode conventions, with the register operations appearing either as explicit subroutine calls or implicitly as references to shared variables. Sometimes this can lead to ambiguity; for example, in the code fragment

$$\mathsf{done} \leftarrow \mathsf{leftDone} \wedge \mathsf{rightDone},$$

it is clear that the operation `write`(done, −) happens after `read`(leftDone) and `read`(rightDone), but it is not clear which of `read`(leftDone and `read`(rightDone) happens first. When the order is important, we'll write the sequence out explicitly:

```
1 leftIsDone ← read(leftDone)
2 rightIsDone ← read(rightDone)
3 write(done, leftIsDone ∧ rightIsDone)
```

Here leftIsDone and rightIsDone are internal variables of the process, so using them does not require read or write operations to the shared memory.

15.2 Single-writer versus multi-writer registers

One variation that does come up even with atomic registers is what processes are allowed to read or write a particular register. A typical assumption is that registers are **single-writer multi-reader**—there is only one process that can write to the register (which simplifies implementation since we don't have to arbitrate which of two near-simultaneous writes gets in last and thus leaves the long-term value), although it's also common to assume **multi-writer multi-reader** registers, which if not otherwise available can be built from single-writer multi-reader registers using atomic snapshot (see Chapter 19). Less common are **single-reader single-writer** registers, which act much like message-passing channels except that the receiver has to make an explicit effort to pick up its mail.

15.3 Fairness and crashes

From the perspective of a schedule, the fairness condition says that every processes gets to perform an operation infinitely often, unless it enters either a crashed or halting state where it invokes no further operations. (Note that unlike in asynchronous message-passing, there is no way to wake up a process once it stops doing operations, since the only way to detect that any activity is happening is to read a register and notice it changed.) Because the registers (at least in in multi-reader models) provide a permanent fault-free record of past history, shared-memory systems are much less vulnerable to crash failures than message-passing systems (though FLP[1] still applies); so in extreme cases, we may assume as many as $n - 1$ crash failures, which makes the fairness condition very weak. The $n - 1$ crash failures case is called the **wait-free** case—since no process can wait for any other process to do anything—and has been extensively studied in the literature.

For historical reasons, work on shared-memory systems has tended to assume crash failures rather than Byzantine failures—possibly because Byzantine failures are easier to prevent when you have several processes sitting in the same machine than when they are spread across the network, or possibly because in multi-writer situations a Byzantine process can do much more damage. But the model by itself doesn't put any constraints on the kinds of process failures that might occur.

15.4 Concurrent executions

Often, the operations on our shared objects will be implemented using lower-level operations. When this happens, it no longer makes sense to assume that the high-level operations occur one at a time—although an implementation may try to give that impression to its users. To model to possibility of concurrency between operations, we split an operation into an **invocation** and **response**, corresponding roughly to a procedure call and its return. The user is responsible for invoking the object; the object's implementation (or the shared memory system, if the object is taken as a primitive) is responsible for responding. Typically we will imagine that an operation is invoked at the moment it becomes pending, but there may be executions in which that does not occur. The time between the invocation and the response for an operation is the **interval** of the operation.

A **concurrent execution** is a sequence of invocations and responses,

[1] See Chapter 9.

where after any prefix of the execution, every response corresponds to some preceding invocation, and there is at most one invocation for each process—always the last—that does not have a corresponding response. How a concurrent execution may or may not relate to a sequential execution depends on the consistency properties of the implementation, as described below.

15.5 Consistency properties

Different shared-memory systems may provide various **consistency properties**, which describe how views of an object by different processes mesh with each other. The strongest consistency property generally used is **linearizability** [HW90], where an implementation of an object is **linearizable** if, for any concurrent execution of the object, there is a sequential execution of the object with the same operations and return values, where the (total) order of operations in the sequential execution is a linearization of the (partial) order of operations in the concurrent execution.

Less formally, this means that if operation a finishes before operation b starts in the concurrent execution, then a must come before b in the sequential execution. An equivalent definition is that we can assign each operation a **linearization point** somewhere between when its invocation and response, and the sequential execution obtained by assuming that all operations occur atomically at their linearization points is consistent with the specification of the object. Using either definition, we are given a fair bit of flexibility in how to order overlapping operations, which can sometimes be exploited by clever implementations (or lower bounds).

A weaker condition is **sequential consistency** [Lam79]. This says that for any concurrent execution of the object, there exists some sequential execution that is indistinguishable to all processes; however, this sequential execution might include operations that occur out of order from a global perspective. For example, we could have an execution of an atomic register where you write to it, then I read from it, but I get the initial value that precedes your write. This is sequentially consistent but not linearizable.

Mostly we will ask any implementations we consider to be linearizable. However, both linearizability and sequential consistency are much stronger than the consistency conditions provided by real multiprocessors. For some examples of weaker memory consistency rules, a good place to start might be the dissertation of Jalal Y. Kawash [Kaw00].

15.6 Complexity measures

There are several complexity measures for shared-memory systems.

Time Assume that no process takes more than 1 time unit between operations (but some fast processes may take less). Assign the first operation in the schedule time 1 and each subsequent operation the largest time consistent with the bound. The time of the last operation is the **time complexity**. This is also known as the **big-step** or **round** measure because the time increases by 1 precisely when every non-faulty process has taken at least one step, and a minimum interval during which this occurs counts as a big step or a round.

Total work The **total work** or **total step complexity** is just the length of the schedule, i.e. the number of operations. This doesn't consider how the work is divided among the processes, e.g. an $O(n^2)$ total work protocol might dump all $O(n^2)$ operations on a single process and leave the rest with almost nothing to do. There is usually not much of a direct correspondence between total work and time. For example, any algorithm that involves **busy-waiting**—where a process repeatedly reads a register until it changes—may have unbounded total work (because the busy-waiter might spin very fast) even though it runs in bounded time (because the register gets written to as soon as some slower process gets around to it). However, it is trivially the case that the time complexity is never greater than the total work.

Per-process work The **per-process work**, **individual work**, **per-process step complexity**, or **individual step complexity** measures the maximum number of operations performed by any single process. Optimizing for per-process work produces more equitably distributed workloads (or reveals inequitably distributed workloads). Like total work, per-process work gives an upper bound on time, since each time unit includes at least one operation from the longest-running process, but time complexity might be much less than per-process work (e.g. in the busy-waiting case above).

Remote memory references As we've seen, step complexity doesn't make much sense for processes that busy-wait. An alternative measure is **remote memory reference** complexity or **RMR** complexity. This measure charges one unit for write operations and the first read operation by each process following a write, but charges nothing for subsequent read operations if there are no intervening writes (see §17.5

for details). In this measure, a busy-waiting operation is only charged one unit. RMR complexity can be justified to a certain extent by the cost structure of multi-processor caching [MCS91, And90].

Contention In multi-writer or multi-reader situations, it may be bad to have too many processes pounding on the same register at once. The **contention** measures the maximum number of pending operations on any single register during the schedule (this is the simplest of several definitions out there). A single-reader single-writer algorithm always has contention at most 2, but achieving such low contention may be harder for multi-reader multi-writer algorithms. Of course, the contention is never worse that n, since we assume each process has at most one pending operation at a time.

Space Just how big are those registers anyway? Much of the work in this area assumes they are *very* big. But we can ask for the maximum number of bits in any one register (**width**) or the total size (**bit complexity**) or number (**space complexity**) of all registers, and will try to minimize these quantities when possible. We can also look at the size of the internal states of the processes for another measure of space complexity.

15.7 Fancier registers

In addition to stock read-write registers, one can also imagine more tricked-out registers that provide additional operations. These usually go by the name of **read-modify-write** (**RMW**) registers, since the additional operations consist of reading the state, applying some function to it, and writing the state back, all as a single atomic action. Examples of RMW registers that have appeared in real machines at various times in the past include:

Test-and-set bits A **test-and-set** operation sets the bit to 1 and returns the old value.

Fetch-and-add registers A **fetch-and-add** operation adds some increment (typically -1 or 1) to the register and returns the old value.

Compare-and-swap registers A **compare-and-swap** operation writes a new value only if the previous value is equal to a supplied test value.

These are all designed to solve various forms of **mutual exclusion** or locking, where we want at most one process at a time to work on some shared data structure.

Some more exotic read-modify-write registers that have appeared in the literature are

Fetch-and-cons Here the contents of the register is a linked list; a **fetch-and-cons** adds a new head and returns the old list.

Sticky bits (or sticky registers) With a **sticky bit** or **sticky register** [Plo89], once the initial empty value is overwritten, all further writes fail. The writer is not notified that the write fails, but may be able to detect this fact by reading the register in a subsequent operation.

Bank accounts Replace the write operation with *deposit*, which adds a non-negative amount to the state, and *withdraw*, which subtracts a non-negative amount from the state provided the result would not go below 0; otherwise, it has no effect.

These solve problems that are hard for ordinary read/write registers under bad conditions. Note that they all have to return something in response to an invocation.

There are also blocking objects like locks or semaphores, but these don't fit into the RMW framework.

We can also consider generic read-modify-write registers that can compute arbitrary functions (passed as an argument to the read-modify-write operation) in the modify step. Here we typically assume that the read-modify-write operation returns the old value of the register. Generic read-modify-write registers are not commonly found in hardware but can be easily simulated (in the absence of failures) using mutual exclusion.[2]

[2]See Chapter 17.

Chapter 16

Distributed shared memory

In **distributed shared memory**, our goal is to simulate a collection of memory locations or **registers**, each of which supports a **read** operation that returns the current state of the register and a **write** operation that updates the state. Our implementation should be **linearizable** [HW90], meaning that read and write operations appear to occur instantaneously (**atomically**) at some point in between when the operation starts and the operation finishes; equivalently, there should be some way to order all the operations on the registers to obtain a **sequential execution** consistent with the behavior of a real register (each read returns the value of the most recent write) while preserving the observable partial order on operations (where π_1 precedes π_2 if π_1 finishes before π_2 starts). Implicit in this definition is the assumption that implemented operations take place over some interval, between an **invocation** that starts the operation and a **response** that ends the operation and returns its value.[1]

In the absence of process failures, we can just assign each register to some process, and implement both read and write operations by remote procedure calls to the process (in fact, this works for arbitrary shared-memory objects). With process failures, we need to make enough copies of the register that failures can't destroy all of them. This creates an asymmetry between simulations of message-passing from shared-memory and vice versa; in the former case (discussed briefly in §16.1 below), a process that fails in the underlying shared-memory system only means that the same process fails in the simulated message-passing system. But in the other direction, not only does the failure of a process in the underlying message-passing system mean that the same process fails in the simulated shared-memory system, but the

[1] More details on the shared-memory model are given in Chapter 15.

simulation collapses completely if a majority of processes fail.

16.1 Message passing from shared memory

We'll start with the easy direction. We can build a reliable FIFO channel from single-writer single-reader registers using polling. The naive approach is that for each edge uv in the message-passing system, we create a (very big) register r_{uv}, and u writes the entire sequence of every message it has ever sent to v to r_{uv} every time it wants to do a new send. To receive messages, v polls all of its incoming registers periodically and delivers any messages in the histories that it hasn't processed yet.[2]

The ludicrous register width can be reduced by adding in an acknowledgment mechanism in a separate register ack_{vu}; the idea is that u will only write one message at a time to r_{uv}, and will queue subsequent messages until v writes in ack_{vu} that the message in r_{uv} has been received. With some tinkering, it is possible to knock r_{uv} down to only three possible states (sending 0, sending 1, and reset) and ack_{vu} down to a single bit (value-received, reset-received), but that's probably overkill for most applications.

Process failures don't affect any of these protocols, except that a dead process stops sending and receiving.

16.2 The Attiya-Bar-Noy-Dolev algorithm

Here we show how to implement shared memory from message passing. We'll assume that our system is asynchronous, that the network is complete, and that we are only dealing with $f < n/2$ crash failures. We'll also assume we only want to build single-writer registers, just to keep things simple; we can extend to multi-writer registers later.

Here's the algorithm, which is due to Attiya, Bar-Noy, and Dolev [ABND95]; see also [Lyn96, §17.1.3]. (Section 9.3 of [AW04] gives an equivalent algorithm, but the details are buried in an implementation of totally-ordered broadcast). We'll make n copies of the register, one on each process. Each process's copy will hold a pair $(\mathsf{value}, \mathsf{timestamp})$ where timestamps are (unbounded) integer values. Initially, everybody starts with $(\perp, 0)$. A process updates its copy with new values (v, t) upon receiving $\texttt{write}(v, t)$ from any other process p, provided t is greater than the process's current timestamp.

[2] If we are really cheap about using registers, and are willing to accept even more absurdity in the register size, we can just have u write every message it ever sends to r_u, and have each v poll all the r_u and filter out any messages intended for other processes.

It then responds to p with $\mathsf{ack}(v, t)$, whether or not it updated its local copy. A process will also respond to a message $\texttt{read}(u)$ with a response $\mathsf{ack}(\mathsf{value}, \mathsf{timestamp}, u)$; here u is a **nonce**[3] used to distinguish between different read operations so that a process can't be confused by out-of-date acknowledgments.

To write a value, the writer increments its timestamp, updates its value and sends $\texttt{write}(\mathsf{value}, \mathsf{timestamp})$ to all other processes. The write operation terminates when the writer has received acknowledgments containing the new timestamp value from a majority of processes.

To read a value, a reader does two steps:

1. It sends $\texttt{read}(u)$ to all processes (where u is any value it hasn't used before) and waits to receive acknowledgments from a majority of the processes. It takes the value v associated with the maximum timestamp t as its return value (no matter how many processes sent it).

2. It then sends $\texttt{write}(v, t)$ to all processes, and waits for a response $\mathsf{ack}(v, t)$ from a majority of the processes. Only then does it return.

(Any extra messages, messages with the wrong nonce, etc. are discarded.)

Both reads and writes cost $\Theta(n)$ messages ($\Theta(1)$ per process).

Intuition: Nobody can return from a write or a read until they are sure that subsequent reads will return the same (or a later) value. A process can only be sure of this if it knows that the values collected by a read will include at least one copy of the value written or read. But since majorities overlap, if a majority of the processes have a current copy of v, then the majority read quorum will include it. Sending $\texttt{write}(v, t)$ to all processes and waiting for acknowledgments from a majority is just a way of ensuring that a majority do in fact have timestamps that are at least t.

If we omit the `write` stage of a `read` operation, we may violate linearizability. An example would be a situation where two values (1 and 2, say), have been written to exactly one process each, with the rest still holding the initial value $\perp$. A reader that observes 1 and $(n-1)/2$ copies of $\perp$ will return 1, while a reader that observes 2 and $(n-1)/2$ copies of $\perp$ will return 2. In the absence of the `write` stage, we could have an arbitrarily long sequence of readers return 1, 2, 1, 2, ..., all with no concurrency. This

[3] A **nonce** is any value that is guaranteed to be used at most once (the term originally comes from cryptography, which in turn got it from linguistics). In practice, a reader will most likely generate a nonce by combining its process id with a local timestamp.

would not be consistent with any sequential execution in which 1 and 2 are only written once.

16.3 Proof of linearizability

Our intuition may be strong, but we still need a proof the algorithm works. In particular, we want to show that for any trace T of the ABD protocol, there is an trace of an atomic register object that gives the same sequence of invoke and response events. The usual way to do this is to find a **linearization** of the read and write operations: a total order that extends the observed order in T where $\pi_1 < \pi_2$ in T if and only if π_1 ends before π_2 starts. Sometimes it's hard to construct such an order, but in this case it's easy: we can just use the timestamps associated with the values written or read in each operation. Specifically, we define the timestamp of a write or read operation as the timestamp used in the `write`(v, t) messages sent out during the implementation of that operation, and we put π_1 before π_2 if:

1. π_1 has a lower timestamp than π_2, or
2. π_1 has the same timestamp as π_2, π_1 is a write, and π_2 is a read, or
3. π_1 has the same timestamp as π_2 and $\pi_1 <_T \pi_2$, or
4. none of the other cases applies, and we feel like putting π_1 first.

The intent is that we pick some total ordering that is consistent with both $<_T$ and the timestamp ordering (with writes before reads when timestamps are equal). To make this work we have to show (a) that these two orderings are in fact consistent, and (b) that the resulting ordering produces values consistent with an atomic register: in particular, that each read returns the value of the last preceding write.

Part (b) is easy: since timestamps only increase in response to writes, each write is followed by precisely those reads with the same timestamp, which are precisely those that returned the value written.

For part (a), suppose that $\pi_1 <_T \pi_2$. The first case is when π_2 is a read. Then before the end of π_1, a set S of more than $n/2$ processes send the π_1 process an $\mathsf{ack}(v1, t_1)$ message. Since local timestamps only increase, from this point on any $\mathsf{ack}(v_2, t_2, u)$ message sent by a process in S has $t_2 \geq t_1$. Let S' be the set of processes sending $\mathsf{ack}(v_2, t_2, u)$ messages processed by π_2. Since $|S| > n/2$ and $|S'| > n/2$, we have $S \cap S'$ is nonempty and so S' includes a process that sent $\mathsf{ack}(v_2, t_2)$ with $t_2 \geq t_1$. So π_2 is serialized after

π_1. The second case is when π_2 is a write; but then π_1 returns a timestamp that precedes the writer's increment in π_2, and so again is serialized first.

16.4 Proof that $f < n/2$ is necessary

This is pretty much the standard partition argument that $f < n/2$ is necessary to do anything useful in a message-passing system. Split the processes into two sets S and S' of size $n/2$ each. Suppose the writer is in S. Consider an execution where the writer does a write operation, but all messages between S and S' are delayed. Since the writer can't tell if the S' processes are slow or dead, it eventually returns. Now let some reader in S' attempt to read the simulated register, again delaying all messages between S and S'; now the reader is forced to return some value without knowing whether the S processes are slow or dead. If the reader doesn't return the value written, we lose. If by some miracle it does, then we lose in the execution where the write didn't happen and all the processes in S really were dead.

16.5 Multiple writers

So far we have assumed a single writer. The main advantage of this approach is that we don't have to do much to manage timestamps: the single writer can just keep track of its own. With multiple writers we can use essentially the same algorithm, but each write needs to perform an initial round of gathering timestamps so that it can pick a new timestamp bigger than those that have come before. We also extend the timestamps to be of the form $\langle \mathsf{count}, \mathsf{id} \rangle$, lexicographically ordered, so that two timestamps with the same count field are ordered by process id. The modified write algorithm is:

1. Send $\texttt{read}(u)$ to all processes and wait to receive acknowledgments from a majority of the processes.

2. Set my timestamp to $t = (\max_q \mathsf{count}_q + 1, \mathsf{id})$ where the max is taken over all processes q that sent me an acknowledgment. Note that this is a two-field timestamp that is compared lexicographically, with the id field used only to prevent duplicate timestamps.

3. Send $\texttt{write}(v, t)$ to all processes, and wait for a response $\mathsf{ack}(v, t)$ from a majority of processes.

This increases the cost of a write by a constant factor, but in the end we still have only a linear number of messages. The proof of linearizability

is essentially the same as for the single-writer algorithm, except now we must consider the case of two write operations by different processes. Here we have that if $\pi_1 <_T \pi_2$, then π_1 gets acknowledgments of its write with timestamp t_1 from a majority of processes before π_2 starts its initial phase to compute count. Since π_2 waits for acknowledgments from a majority of processes as well, these majorities overlap, so π_2's timestamp t_2 must exceed $t1$. So the linearization ordering previously defined still works.

16.6 Other operations

The basic ABD framework can be extended to support other operations.

One such operation is a **collect** [SSW91], where we read n registers in parallel with no guarantee that they are read at the same time. This can trivially be implemented by running n copies of ABD in parallel, and can be implemented with the same time and message complexity as ABD for a single register by combining the messages from the parallel executions into single (possibly very large) messages.

Chapter 17

Mutual exclusion

For full details see [AW04, Chapter 4] or [Lyn96, Chapter 10].

17.1 The problem

The goal is to share some critical resource between processes without more than one using it at a time—this is *the* fundamental problem in time-sharing systems.

The solution is to only allow access while in a specially-marked block of code called a **critical section**, and only allow one process at a time to be in a critical section.

A **mutual exclusion protocol** guarantees this, usually in an asynchronous shared-memory model.

Formally: We want a process to cycle between states **trying** (trying to get into critical section), **critical** (in critical section), **exiting** (cleaning up so that other processes can enter their critical sections), and **remainder** (everything else—essentially just going about its non-critical business). Only in the trying and exiting states does the process run the mutual exclusion protocol to decide when to switch to the next state; in the critical or remainder states it switches to the next state on its own.

17.2 Goals

(See also [AW04, §4.2], [Lyn96, §10.2].)

Core mutual exclusion requirements:

Mutual exclusion At most one process is in the critical state at a time.

No deadlock (progress) If there is at least one process in a trying state, then eventually some process enters a critical state; similarly for exiting and remainder states.

Note that the protocol is not required to guarantee that processes leave the critical or remainder state, but we generally have to insist that the processes at least leave the critical state on their own to make progress.

Additional useful properties (not satisfied by all mutual exclusion protocols; see [Lyn96, §10.4)]:

No lockout (**lockout-freedom**): If there is a particular process in a trying or exiting state, that process eventually leaves that state. This means that I don't starve because somebody else keeps jumping past me and seizing the critical resource before I can.

Stronger versions of lockout-freedom include explicit time bounds (how many rounds can go by before I get in) or **bounded bypass** (nobody gets in more than k times before I do).

17.3 Mutual exclusion using strong primitives

See [AW04, §4.3] or [Lyn96, 10.9]. The idea is that we will use some sort of **read-modify-write** register, where the RMW operation computes a new value based on the old value of the register and writes it back as a single atomic operation, usually returning the old value to the caller as well.

17.3.1 Test and set

A **test-and-set** operation does the following sequence of actions atomically:

```
1 oldValue ← read(bit)
2 write(bit, 1)
3 return oldValue
```

Typically there is also a second **reset** operation for setting the bit back to zero. For some implementations, this reset operation may only be used safely by the last process to get 0 from the test-and-set bit.

Because a test-and-set operation is atomic, if two processes both try to perform test-and-set on the same bit, only one of them will see a return value

of 0. This is not true if each process simply executes the above code on a stock atomic register: there is an execution in which both processes read 0, then both write 1, then both return 0 to whatever called the non-atomic test-and-set subroutine.

Test-and-set provides a trivial implementation of mutual exclusion, shown in Algorithm 17.1.

```
1 while true do
      // trying
2     while testAndSet(lock) = 1 do nothing
      // critical
3     (do critical section stuff)
      // exiting
4     reset(lock)
      // remainder
5     (do remainder stuff)
```

Algorithm 17.1: Mutual exclusion using test-and-set

It is easy to see that this code provides mutual exclusion, as once one process gets a 0 out of lock, no other can escape the inner while loop until that process calls the `reset` operation in its exiting state. It also provides progress (assuming the lock is initially set to 0); the only part of the code that is not straight-line code (which gets executed eventually by the fairness condition) is the inner loop, and if lock is 0, some process escapes it, while if lock is 1, some process is in the region between the `testAndSet` call and the `reset` call, and so it eventually gets to `reset` and lets the next process in (or itself, if it is very fast).

The algorithm does *not* provide lockout-freedom: nothing prevents a single fast process from scooping up the lock bit every time it goes through the outer loop, while the other processes ineffectually grab at it just after it is taken away. Lockout-freedom requires a more sophisticated turn-taking strategy.

17.3.2 A lockout-free algorithm using an atomic queue

Basic idea: In the trying phase, each process enqueues itself on the end of a shared queue (assumed to be an atomic operation). When a process comes to the head of the queue, it enters the critical section, and when exiting it dequeues itself. So the code would look something like Algorithm 17.2.

Note that this requires a queue that supports a `head` operation. Not all implementations of queues have this property.

```
1 while true do
      // trying
2     enq(Q, myId)
3     while head(Q) ≠ myId do nothing
      // critical
4     (do critical section stuff)
      // exiting
5     deq(Q)
      // remainder
6     (do remainder stuff)
```

Algorithm 17.2: Mutual exclusion using a queue

Here the proof of mutual exclusion is that only the process whose id is at the head of the queue can enter its critical section. Formally, we maintain an invariant that any process whose program counter is between the inner while loop and the call to `deq`(Q) must be at the head of the queue; this invariant is easy to show because a process can't leave the while loop unless the test fails (i.e., it is already at the head of the queue), no `enq` operation changes the head value (if the queue is nonempty), and the `deq` operation (which does change the head value) can only be executed by a process already at the head (from the invariant).

Deadlock-freedom follows from proving a similar invariant that every element of the queue is the id of some process in the trying, critical, or exiting states, so eventually the process at the head of the queue passes the inner loop, executes its critical section, and dequeues its id.

Lockout-freedom follows from the fact that once a process is at position k in the queue, every execution of a critical section reduces its position by 1; when it reaches the front of the queue (after some finite number of critical sections), it gets the critical section itself.

17.3.2.1 Reducing space complexity

Following [AW04, §4.3.2], we can give an implementation of this algorithm using a single read-modify-write (RMW) register instead of a queue; this drastically reduces the (shared) space needed by the algorithm. The reason this works is because we don't really need to keep track of the position of

each process in the queue itself; instead, we can hand out numerical tickets to each process and have the process take responsibility for remembering where its place in line is.

The RMW register has two fields, first and last, both initially 0. Incrementing last simulates an enqueue, while incrementing first simulates a dequeue. The trick is that instead of testing if it is at the head of the queue, a process simply remembers the value of the last field when it "enqueued" itself, and waits for the first field to equal it.

Algorithm 17.3 shows the code from Algorithm 17.2 rewritten to use this technique. The way to read the RMW operations is that the first argument specifies the variable to update and the second specifies an expression for computing the new value. Each RMW operation returns the old state of the object, before the update.

```
1 while true do
      // trying
2     position ← RMW(V, ⟨V.first, V.last + 1⟩)
      // enqueue
3     while RMW(V, V).first ≠ position.last do
4         nothing
      // critical
5     (do critical section stuff)
      // exiting
6     RMW(V, ⟨V.first + 1, V.last⟩)
      // dequeue
      // remainder
7     (do remainder stuff)
```

Algorithm 17.3: Mutual exclusion using read-modify-write

17.4 Mutual exclusion using only atomic registers

While mutual exclusion is easier using powerful primitives, we can also solve the problem using only registers.

17.4.1 Peterson's tournament algorithm

Algorithm 17.4 shows Peterson's lockout-free mutual exclusion protocol for two processes p_0 and p_1 [Pet81] (see also [AW04, §4.4.2] or [Lyn96, §10.5.1]).

It uses only atomic registers.

```
   shared data:
1  waiting, initially arbitrary
2  present[i] for i ∈ {0, 1}, initially 0
3  Code for process i:
4  while true do
       // trying
5      present[i] ← 1
6      waiting ← i
7      while true do
8          if present[¬i] = 0 then break
9
10         if waiting ≠ i then break
11
       // critical
12     (do critical section stuff)
       // exiting
13     present[i] = 0
       // remainder
14     (do remainder stuff)
```

Algorithm 17.4: Peterson's mutual exclusion algorithm for two processes

This uses three bits to communicate: present[0] and present[1] indicate which of p_0 and p_1 are participating, and waiting enforces turn-taking. The protocol requires that waiting be multi-writer, but it's OK for present[0] and present[1] to be single-writer.

In the description of the protocol, we write Lines 8 and 10 as two separate lines because they include two separate read operations, and the order of these reads is important.

17.4.1.1 Correctness of Peterson's protocol

Intuitively, let's consider all the different ways that the entry code of the two processes could interact. There are basically two things that each process does: it sets its own present in Line 5 and grabs the waiting variable in Line 6. Here's a typical case where one process gets in first:

1. p_0 sets present[0] $\leftarrow 1$

2. p_0 sets waiting $\leftarrow 0$
3. p_0 reads present$[1] = 0$ and enters critical section
4. p_1 sets present$[1] \leftarrow 1$
5. p_1 sets waiting $\leftarrow 1$
6. p_1 reads present$[0] = 1$ and waiting $= 1$ and loops
7. p_0 sets present$[0] \leftarrow 0$
8. p_1 reads present$[0] = 0$ and enters critical section

The idea is that if I see a 0 in your present variable, I know that you aren't playing, and can just go in.

Here's a more interleaved execution where the waiting variable decides the winner:

1. p_0 sets present$[0] \leftarrow 1$
2. p_0 sets waiting $\leftarrow 0$
3. p_1 sets present$[1] \leftarrow 1$
4. p_1 sets waiting $\leftarrow 1$
5. p_0 reads present$[1] = 1$
6. p_1 reads present$[0] = 1$
7. p_0 reads waiting $= 1$ and enters critical section
8. p_1 reads present$[0] = 1$ and waiting $= 1$ and loops
9. p_0 sets present$[0] \leftarrow 0$
10. p_1 reads present$[0] = 0$ and enters critical section

Note that it's the process that set the waiting variable last (and thus sees its own value) that stalls. This is necessary because the earlier process might long since have entered the critical section.

Sadly, examples are not proofs, so to show that this works in general, we need to formally verify each of mutual exclusion and lockout-freedom. Mutual exclusion is a safety property, so we expect to prove it using invariants. The proof in [Lyn96] is based on translating the pseudocode directly into

automata (including explicit program counter variables); we'll do essentially the same proof but without doing the full translation to automata. Below, we write that p_i is at line k if it the operation in line k is enabled but has not occurred yet.

Lemma 17.4.1. *If* present$[i] = 0$*, then* p_i *is at Line 5 or 14.*

Proof. Immediate from the code. □

Lemma 17.4.2. *If* p_i *is at Line 12, and* $p_{\neg i}$ *is at Line 8, 10, or 12, then* waiting $= \neg i$*.*

Proof. We'll do the case $i = 0$; the other case is symmetric. The proof is by induction on the schedule. We need to check that any event that makes the left-hand side of the invariant true or the right-hand side false also makes the whole invariant true. The relevent events are:

- Transitions by p_0 from Line 8 to Line 12. These occur only if present$[1] = 0$, implying p_1 is at Line Line 5 or 14 by Lemma 17.4.1. In this case the second part of the left-hand side is false.
- Transitions by p_0 from Line 10 to Line 12. These occur only if waiting $\neq 0$, so the right-hand side is true.
- Transitions by p_1 from Line 6 to Line 8. These set waiting to 1, making the right-hand side true.
- Transitions that set waiting to 0. These are transitions by p_0 from Line 6 to Line 10, making the left-hand side false.

□

We can now read mutual exclusion directly off of Lemma 17.4.2: if both p_0 and p_1 are at Line 12, then we get waiting $= 1$ and waiting $= 0$, a contradiction.

To show progress, observe that the only place where both processes can get stuck forever is in the loop at Lines 8 and 10. But then waiting isn't changing, and so some process i reads waiting $= \neg i$ and leaves. To show lockout-freedom, observe that if p_0 is stuck in the loop while p_1 enters the critical section, then after p_1 leaves it sets present$[1]$ to 0 in Line 13 (which lets p_0 in if p_0 reads present$[1]$ in time), but even if it then sets present$[1]$ back to 1 in Line 5, it still sets waiting to 1 in Line 6, which lets p_0 into the critical section. With some more tinkering this argument shows that p_1

enters the critical section at most twice while p_0 is in the trying state, giving 2-bounded bypass; see [Lyn96, Lemma 10.12]. With even more tinkering we get a constant time bound on the waiting time for process i to enter the critical section, assuming the other process never spends more than $O(1)$ time inside the critical section.

17.4.1.2 Generalization to n processes

(See also [AW04, §4.4.3].)

The easiest way to generalize Peterson's two-process algorithm to n processes is to organize a tournament in the form of log-depth binary tree; this method was invented by Peterson and Fischer [PF77]. At each node of the tree, the roles of the two processes are taken by the winners of the subtrees, i.e., the processes who have entered their critical sections in the two-process algorithms corresponding to the child nodes. The winner of the tournament as a whole enters the real critical section, and afterwards walks back down the tree unlocking all the nodes it won in reverse order. It's easy to see that this satisfies mutual exclusion, and not much harder to show that it satisfies lockout-freedom—in the latter case, the essential idea is that if the winner at some node reaches the root infinitely often then lockout-freedom at that node means that the winner of each child node reaches the root infinitely often.

The most natural way to implement the nodes is to have present[0] and present[1] at each node be multi-writer variables that can be written to by any process in the appropriate subtree. Because the present variables don't do much, we can also implement them as the OR of many single-writer variables (this is what is done in [Lyn96, §10.5.3]), but there is no immediate payoff to doing this since the waiting variables are still multi-writer.

Nice properties of this algorithm are that it uses only bits and that it's very fast: $O(\log n)$ time in the absence of contention.

17.4.2 Fast mutual exclusion

With a bit of extra work, we can reduce the no-contention cost of mutual exclusion to $O(1)$, while keeping whatever performance we previously had in the high-contention case. The trick (due to Lamport [Lam87]) is to put an object at the entrance to the protocol that diverts a solo process onto a "fast path" that lets it bypass the n-process mutex that everybody else ends up on.

Our presentation mostly follows [AW04][§4.4.5], which uses the **splitter**

abstraction of Moir and Anderson [MA95] to separate out the mechanism for diverting a lone process.[1] Code for a splitter is given in Algorithm 17.5.

```
   shared data:
1  atomic register race, big enough to hold an id, initially ⊥
2  atomic register door, big enough to hold a bit, initially open
3  procedure splitter(id)
4      race ← id
5      if door = closed then
6          return right
7      door ← closed
8      if race = id then
9          return stop
10     else
11         return down
```

Algorithm 17.5: Implementation of a splitter

A splitter assigns to each processes that arrives at it the value right, down, or stop. The useful properties of splitters are that if at least one process arrives at a splitter, then (a) at least one process returns right or stop; and (b) at least one process returns down or stop; (c) at most one process returns stop; and (d) any process that runs by itself returns stop. The first two properties will be useful when we consider the problem of **renaming** in Chapter 24; we will prove them there. The last two properties are what we want for mutual exclusion.

The names of the variables race and door follow the presentation in [AW04, §4.4.5]; Moir and Anderson [MA95], following Lamport [Lam87], call these X and Y. As in [MA95], we separate out the right and down outcomes—even though they are equivalent for mutex—because we will need them later for other applications.

The intuition behind Algorithm 17.5 is that setting door to closed closes the door to new entrants, and the last entrant to write its id to race wins (it's a slow race), assuming nobody else writes race and messes things up. The added cost of the splitter is always $O(1)$, since there are no loops.

To reset the splitter, write open to door. This allows new processes to enter the splitter and possibly return stop.

[1] Moir and Anderson call these things **one-time building blocks**, but the name **splitter** has become standard in subsequent work.

Lemma 17.4.3. *After each time that* door *is set to* open*, at most one process running Algorithm 17.5 returns* stop*.*

Proof. To simplify the argument, we assume that each process calls `splitter` at most once.

Let t be some time at which door is set to open ($-\infty$ in the case of the initial value). Let S_t be the set of processes that read open from door after time t and before the next time at which some process writes closed to door, and that later return stop by reaching Line 9.

Then every process in S_t reads door before any process in S_t writes door. It follows that every process in S_t writes race before any process in S_t reads race. If some process p is not the *last* process in S_t to write race, it will not see its own id, and will not return stop. But only one process can be the last process in S_t to write race.[2] □

Lemma 17.4.4. *If a process runs Algorithm 17.5 by itself starting from a configuration in which* door = open*, it returns* stop*.*

Proof. Follows from examining a solo execution: the process sets race to id, reads open from door, then reads id from race. This causes it to return stop as claimed. □

To turn this into an n-process mutex algorithm, we use the splitter to separate out at most one process (the one that gets stop) onto a **fast path** that bypasses the **slow path** taken by the rest of the processes. The slow-path process first fight among themselves to get through an n-process mutex; the winner then fights in a 2-process mutex with the process (if any) on the fast path.

Releasing the mutex is the reverse of acquiring it. If I followed the fast path, I release the 2-process mutex first then reset the splitter. If I followed the slow path, I release the 2-process mutex first then the n-process mutex. This gives mutual exclusion with $O(1)$ cost for any process that arrives before there is any contention ($O(1)$ for the splitter plus $O(1)$ for the 2-process mutex).

A complication is that if nobody wins the splitter, there is no fast-path process to reset it. If we don't want to accept that the fast path just breaks forever in this case, we have to include a mechanism for a slow-path process to reset the splitter if it can be assured that there is no fast-path process

[2] It's worth noting that this last process still might not return stop, because some later process—not in S_t—might overwrite race. This can happen even if nobody ever resets the splitter.

left in the system. The simplest way to do this is to have each process mark a bit in an array to show it is present, and have each slow-path process, while still holding all the mutexes, check on its way out if the door bit is set and no processes claim to be present. If it sees all zeros (except for itself) after seeing door = closed, it can safely conclude that there is no fast-path process and reset the splitter itself. The argument then is that the last slow-path process to leave will do this, re-enabling the fast path once there is no contention again. This approach is taken implicitly in Lamport's original algorithm, which combines the splitter and the mutex algorithms into a single miraculous blob.

17.4.3 Lamport's Bakery algorithm

See [AW04, §4.4.1] or [Lyn96, §10.7].

This is a lockout-free mutual exclusion algorithm that uses only single-writer registers (although some of the registers may end up holding arbitrarily large values). Code for the Bakery algorithm is given as Algorithm 17.6.

```
   shared data:
1  choosing[i], an atomic bit for each i, initially 0
2  number[i], an unbounded atomic register, initially 0
3  Code for process i:
4  while true do
       // trying
5      choosing[i] ← 1
6      number[i] ← 1 + max_{j≠i} number[j]
7      choosing[i] ← 0
8      for j ≠ i do
9          loop until choosing[j] = 0
10         loop until number[j] = 0 or ⟨number[i], i⟩ < ⟨number[j], j⟩
       // critical
11     (do critical section stuff)
       // exiting
12     number[i] ← 0
       // remainder
13     (do remainder stuff)
```

Algorithm 17.6: Lamport's Bakery algorithm

Note that several of these lines are actually loops; this is obvious for

Lines 9 and 10, but is also true for Line 6, which includes an implicit loop to read all $n - 1$ values of number$[j]$.

Intuition for mutual exclusion is that if you have a lower number than I do, then I block waiting for you; for lockout-freedom, eventually I have the smallest number. (There are some additional complications involving the choosing bits that we are sweeping under the rug here.) For a real proof see [AW04, §4.4.1] or [Lyn96, §10.7].

Selling point is a strong near-FIFO guarantee and the use of only single-writer registers (which need not even be atomic—it's enough that they return correct values when no write is in progress). Weak point is unbounded registers.

17.4.4 Lower bound on the number of registers

There is a famous result due to Burns and Lynch [BL93] that any mutual exclusion protocol using only read/write registers requires at least n of them. Details are in [Lyn96, §10.8]. A slightly different version of the argument is given in [AW04, 4.4.4]. The proof is another nice example of an indistinguishability proof, where we use the fact that if a group of processes can't tell the difference between two executions, they behave the same in both.

Assumptions: We have a protocol that guarantees mutual exclusion and progress. Our base objects are all atomic registers.

Key idea: In order for some process p to enter the critical section, it has to do at least one write to let the other processes know it is doing so. If not, they can't tell if p ever showed up at all, so eventually either some p' will enter the critical section and violate mutual exclusion or (in the no-p execution) nobody enters the critical section and we violate progress. Now suppose we can park a process p_i on each register r_i with a pending write to i; in this case we say that p_i **covers** r_i. If every register is so covered, we can let p go ahead and do whatever writes it likes and then deliver all the covering writes at once, wiping out anything p did. Now the other processes again don't know if p exists or not. So we can say something stronger: before some process p can enter a critical section, it has to write to an uncovered register.

The hard part is showing that we can cover all the registers without letting p know that there are other processes waiting—if p can see that other processes are waiting, it can just sit back and wait for them to go through the critical section and make progress that way. So our goal is to produce states in which (a) processes $p_1 \ldots, p_k$ (for some k) between them cover k registers, and (b) the resulting configuration is indistinguishable

from an idle configuration to $p_{k+1} \dots p_n$.

Lemma 17.4.5. *Starting from any idle configuration C, there exists an execution in which only processes $p_1 \dots p_k$ take steps that leads to a configuration C' such that (a) C' is indistinguishable by any of $p_{k+1} \dots p_n$ from some idle configuration C'' and (b) k registers are covered by $p_1 \dots p_k$ in C'.*

Proof. The proof is by induction on k. For $k = 1$, just run p_1 until it is about to do a write, let C' be the resulting configuration and let $C'' = C$.

For larger k, the essential idea is that starting from C, we first run to a configuration C_1 where $p_1 \dots p_{k-1}$ cover $k-1$ registers and C_1 is indistinguishable from an idle configuration by the remaining processes, and then run p_k until it covers one more register. If we let $p_1 \dots p_{k-1}$ go, they overwrite anything p_k wrote. Unfortunately, they may not come back to covering the same registers as before if we rerun the induction hypothesis (and in particular might cover the same register that p_k does). So we have to look for a particular configuration C_1 that not only covers $k-1$ registers but also has an extension that covers the same $k-1$ registers.

Here's how we find it: Start in C. Run the induction hypothesis to get C_1; here there is a set W_1 of $k-1$ registers covered in C_1. Now let processes p_1 through p_{k-1} do their pending writes, then each enter the critical section, leave it, and finish, and rerun the induction hypothesis to get to a state C_2, indistinguishable from an idle configuration by p_k and up, in which $k-1$ registers in W_2 are covered. Repeat to get sets W_3, W_4, etc. Since this sequence is unbounded, and there are only $\binom{r}{k-1}$ distinct sets of registers to cover (where r is the number of registers), eventually we have $W_i = W_j$ for some $i \neq j$. The configurations C_i and C_j are now our desired configurations covering the same $k-1$ registers.

Now that we have C_i and C_j, we run until we get to C_i. We now run p_k until it is about to write some register not covered by C_i (it must do so, or otherwise we can wipe out all of its writes while it's in the critical section and then go on to violate mutual exclusion). Then we let the rest of p_1 through p_{k-1} do all their writes (which immediately destroys any evidence that p_k ran at all) and run the execution that gets them to C_j. We now have $k-1$ registers covered by p_1 through p_{k-1} and a k-th register covered by p_k, in a configuration that is indistinguishable from idle: this proves the induction step. □

The final result follows by the fact that when $k = n$ we cover n registers; this implies that there are n registers to cover.

17.5 RMR complexity

It's not hard to see that we can't build a shared-memory mutex without busy-waiting: any process that is waiting can't detect that the critical section is safe to enter without reading a register, but if that register tells it that it should keep waiting, it is back where it started and has to read it again. This makes our standard step-counting complexity measures useless for describe the worst-case complexity of a mutual exclusion algorithm.

However, the same argument that suggests we can ignore local computation in a message-passing model suggests that we can ignore local operations on registers in a shared-memory model. Real multiprocessors have memory hierarchies where memory that is close to the CPU (or one of the CPUs) is generally much faster than memory that is more distant. This suggests charging only for **remote memory references**, or RMRs, where each register is local to one of the processes and only operations on non-local are expensive. This has the advantage of more accurately modeling real costs [MCS91, And90], and allowing us to build busy-waiting mutual exclusion algorithms with costs we can actually analyze.

As usual, there is a bit of a divergence here between theory and practice. Practically, we are interested in algorithms with good real-time performance, and RMR complexity becomes a heuristic for choosing how to assign memory locations. This gives rise to very efficient mutual exclusion algorithms for real machines, of which the most widely used is the beautiful MCS algorithm of Mellor-Crummey and Scott [MCS91]. Theoretically, we are interested in the question of how efficiently we can solve mutual exclusion in our formal model, and RMR complexity becomes just another complexity measure, one that happens to allow busy-waiting on local variables.

17.5.1 Cache-coherence vs. distributed shared memory

The basic idea of RMR complexity is that a process doesn't pay for operations on local registers. But what determines which operations are local?

In the **cache-coherent** model (CC for short), once a process reads a register it retains a local copy as long as nobody updates it. So if I do a sequence of read operations with no intervening operations by other processes, I may pay an RMR for the first one (if my cache is out of date), but the rest are free. The assumption is that each process can cache registers, and there is some cache-coherence protocol that guarantees that all the caches stay up to date. We may or may not pay RMRs for write operations or other read operations, depending on the details of the cache-coherence protocol,

but for upper bounds it is safest to assume that we do.

In the **distributed shared memory** model (DSM), each register is assigned permanently to a single process. Other processes can read or write the register, but only the owner gets to do so without paying an RMR. Here memory locations are nailed down to specific processes.

In general, we expect the cache-coherent model to be cheaper than the distributed shared-memory model, if we ignore constant factors. The reason is that if we run a DSM algorithm in a CC model, then the process p to which a register r is assigned incurs an RMR only if some other process q accesses p since p's last access. But then we can amortize p's RMR by charging q double. Since q incurs an RMR in the CC model, this tells us that we pay at most twice as many RMRs in DSM as in CC for any algorithm.

The converse is not true: there are (mildly exotic) problems for which it is known that CC algorithms are asymptotically more efficient than DSM algorithms [Gol11, DH04].

17.5.2 RMR complexity of Peterson's algorithm

As a warm-up, let's look at the RMR complexity of Peterson's two-process mutual exclusion algorithm (Algorithm 17.4). Acquiring the mutex requires going through mostly straight-line code, except for the loop that tests $\mathsf{present}[\neg i]$ and $\mathsf{waiting}$.

In the DSM model, spinning on $\mathsf{present}[\neg i]$ is not a problem (we can make it a local variable of process i). But $\mathsf{waiting}$ is trouble. Whichever process we don't assign it to will pay an RMR every time it looks at it. So Peterson's algorithm behaves badly by the RMR measure in this model.

Things are better in the CC model. Now process i may pay RMRs for its first reads of $\mathsf{present}[\neg i]$ and $\mathsf{waiting}$, but any subsequent reads are free unless process $\neg i$ changes one of them. But any change to either of the variables causes process i to leave the loop. It follows that process i pays at most 3 RMRs to get through the busy-waiting loop, giving an RMR complexity of $O(1)$.

RMR complexities for parts of a protocol that access different registers add just like step complexities, so the Peterson-Fischer tree construction described in §17.4.1.2 works here too. The result is $O(\log n)$ RMRs per critical section access, but only in the CC model.

17.5.3 Mutual exclusion in the DSM model

Yang and Anderson [YA95] give a mutual exclusion algorithm for the DSM model that requires $\Theta(\log n)$ RMRs to reach the critical section. This is now known to be optimal for deterministic algorithms [AHW08]. The core of the algorithm is a 2-process mutex similar to Peterson's, with some tweaks so that each process spins only on its own registers. Pseudocode is given in Algorithm 17.7; this is adapted from [YA95, Figure 1].

```
1  C[side(i)] ← i
2  T ← i
3  P[i] ← 0
4  rival ← C[¬side(i)]
5  if rival ≠ ⊥ and T = i then
6      if P[rival] = 0 then
7          P[rival] = 1
8      while P[i] = 0 do spin
9      if T = i then
10         while P[i] ≤ 1 do spin
   // critical section goes here
11 C[side(i)] ← ⊥
12 rival ← T
13 if rival ≠ i then
14     P[rival] ← 2
```

Algorithm 17.7: Yang-Anderson mutex for two processes

The algorithm is designed to be used in a tree construction where a process with id in the range $\{1 \ldots n/2\}$ first fights with all other processes in this range, and similarly for processes in the range $\{n/2 + 1 \ldots n\}$. The function side(i) is 0 for the first group of processes and 1 for the second. The variables $C[0]$ and $C[1]$ are used to record which process is the winner for each side, and also take the place of the present variables in Peterson's algorithm. Each process has its own variable $P[i]$ that it spins on when blocked; this variable is initially 0 and ranges over $\{0, 1, 2\}$; this is used to signal a process that it is safe to proceed, and tests on P substitute for tests on the non-local variables in Peterson's algorithm. Finally, the variable T is used (like waiting in Peterson's algorithm) to break ties: when $T = i$, it's i's turn to wait.

Initially, $C[0] = C[1] = \perp$ and $P[i] = 0$ for all i.

When I want to enter my critical section, I first set $C[\texttt{side}(i)]$ so you can find me; this also has the same effect as setting $\mathsf{present}[\texttt{side}(i)]$ in Peterson's algorithm. I then point T to myself and look for you. I'll block if I see $C[\neg\texttt{side}(i)] = 1$ and $T = i$. This can occur in two ways: one is that I really write T after you did, but the other is that you only wrote $C[\neg\texttt{side}(i)]$ but haven't written T yet. In the latter case, you will signal to me that T may have changed by setting $P[i]$ to 1. I have to check T again (because maybe I really did write T later), and if it is still i, then I know that you are ahead of me and will succeed in entering your critical section. In this case I can safely spin on $P[i]$ waiting for it to become 2, which signals that you have left.

There is a proof that this actually works in [YA95], but it's 27 pages of very meticulously-demonstrated invariants (in fairness, this includes the entire algorithm, including the tree parts that we omitted here). For intuition, this is not much more helpful than having a program mechanically check all the transitions, since the algorithm for two processes is effectively finite-state if we ignore the issue with different processes i jumping into the role of $\texttt{side}(i)$.

A slightly less rigorous proof but more human-accessible proof would be analogous to the proof of Peterson's algorithm. We need to show two things: first, that no two processes ever both enter the critical section, and second, that no process gets stuck.

For the first part, consider two processes i and j, where $\texttt{side}(i) = 0$ and $\texttt{side}(j) = 1$. We can't have both i and j skip the loops, because whichever one writes T last sees itself in T. Suppose that this is process i and that j skips the loops. Then $T = i$ and $P[i] = 0$ as long as j is in the critical section, so i blocks. Alternatively, suppose i writes T last but does so after j first reads T. Now i and j both enter the loops. But again i sees $T = i$ on its second test and blocks on the second loop until j sets $P[i]$ to 2, which doesn't happen until after j finishes its critical section.

Now let us show that i doesn't get stuck. Again we'll assume that i wrote T second.

If j skips the loops, then j sets $P[i] = 2$ on its way out as long as $T = i$; this falsifies both loop tests. If this happens after i first sets $P[i]$ to 0, only i can set $P[i]$ back to 0, so i escapes its first loop, and any j' that enters from the 1 side will see $P[i] = 2$ before attempting to set $P[i]$ to 1, so $P[i]$ remains at 2 until i comes back around again. If j sets $P[i]$ to 2 before i sets $P[i]$ to 0 (or doesn't set it at all because $T = j$, then $C[\texttt{side}(j)]$ is set to $\perp$ before i reads it, so i skips the loops.

If j doesn't skip the loops, then $P[i]$ and $P[j]$ are both set to 1 after i

and j enter the loopy part. Because j waits for $P[j] \neq 0$, when it looks at T the second time it will see $T = i \neq j$ and will skip the second loop. This causes it to eventually set $P[i]$ to 2 or set $C[\texttt{side}(j)]$ to $\perp$ before i reads it as in the previous case, so again i eventually reaches its critical section.

Since the only operations inside a loop are on local variables, the algorithm has $O(1)$ RMR complexity. For the full tree this becomes $O(\log n)$.

17.5.4 Lower bounds

For deterministic algorithms, there is a lower bound due to Attiya, Hendler, and Woelfel [AHW08] that shows that any one-shot mutual exclusion algorithm for n processes incurs $\Omega(n \log n)$ total RMRs in either the CC or DSM models (which implies that some single process incurs $\Omega(\log n)$ RMRs). This is based on an earlier breakthrough lower bound of Fan and Lynch [FL06] that proved the same lower bound for the number of times a register changes state. Both bounds are information-theoretic: a family of $n!$ executions is constructed containing all possible orders in which the processes enter the critical section, and it is shown that each RMR or state change only contributes $O(1)$ bits to choosing between them.

For randomized algorithms, Hendler and Woelfel [HW11] have an algorithm that uses $O(\log n / \log \log n)$ expected RMRs against an adaptive adversary and Bender and Gilbert [BG11] can do $O(\log^2 \log n)$ amortized expected RMRs against an oblivious adversary. Both bounds beat the deterministic lower bound. The adaptive-adversary bound is tight, due to a matching lower bound of Giakkoupis and Woelfel [GW12b] that holds even for systems that provide compare and swap objects. No non-trivial lower bound is currently known for an oblivious adversary.

Chapter 18

The wait-free hierarchy

In a shared memory model, it may be possible to solve some problems using **wait-free** protocols, in which any process can finish the protocol in a bounded number of steps, no matter what the other processes are doing (see Chapter 26 for more on this and some variants).

The **wait-free hierarchy** h_m^r classifies asynchronous shared-memory object types T by **consensus number**, where a type T has consensus number n if with objects of type T and atomic registers (all initialized to appropriate values[1]) it is possible to solve wait-free consensus (i.e., agreement, validity, wait-free termination) for n processes but not for $n+1$ processes. The consensus number of any type is at least 1, since 1-process consensus requires no interaction, and may range up to ∞ for particularly powerful objects.

The wait-free hierarchy was suggested by work by Maurice Herlihy [Her91b] that classified many common (and some uncommon) shared-memory objects by consensus number, and showed that an unbounded collection of objects with consensus number n together with atomic registers gives a wait-free implementation of any object in an n-process system. Various subsequent authors noticed that this did not give a **robust hierarchy** in the sense that combining two types of objects with consensus number n could solve wait-free consensus for larger n, and the hierarchy h_m^r was proposed by Prasad

[1]The justification for assuming that the objects can be initialized to an arbitrary state is a little tricky. The idea is that if we are trying to implement consensus from objects of type T that are themselves implemented in terms of objects of type S, then it's natural to assume that we initialize our simulated type-T objects to whatever states are convenient. Conversely, if we are using the ability of type-T objects to solve n-process consensus to show that they can't be implemented from type-S objects (which can't solve n-process consensus), then for both the type-T and type-S objects we want these claims to hold no matter how they are initialized.

Jayanti [Jay97] as a way of classifying objects that might be robust: an object is at level n of the h_m^r hierarchy if having unboundedly many objects plus unboundedly many registers solves n-process wait-free consensus but not $(n+1)$-process wait-free consensus.[2]

Whether or not the resulting hierarchy is in fact robust for arbitrary deterministic objects is still open, but Eric Ruppert [Rup00] subsequently showed that it is robust for RMW registers and objects with a read operation that returns the current state, and there is a paper by Borowsky, Gafni, and Afek [BGA94] that sketches a proof based on a topological characterization of computability[3] that h_m^r is robust for deterministic objects that don't discriminate between processes (unlike, say, single-writer registers). So for well-behaved shared-memory objects (deterministic, symmetrically accessible, with read operations, etc.), consensus number appears to give a real classification that allows us to say for example that any collection of read-write registers (consensus number 1), fetch-and-increments (2), test-and-set bits (2), and queues (2) is not enough to build a compare-and-swap (∞).

We won't attempt to do the full robustness proofs of Borowsky *et al.* [BGA94] or Ruppert [Rup00] that let us get away with this. Instead, we'll concentrate on Herlihy's original results and show that specific objects have specific consensus numbers when used in isolation. The procedure in each case will be to show an upper bound on the consensus number using a variant of Fischer-Lynch-Paterson (made easier because we are wait-free and don't have to worry about fairness) and then show a matching lower bound (for non-trivial upper bounds) by exhibiting an n-process consensus protocol for some n. Essentially everything below is taken from Herlihy's paper [Her91b], so reading that may make more sense than reading these notes.

18.1 Classification by consensus number

Here we show the position of various types in the wait-free hierarchy. The quick description is shown in Table 18.1; more details (mostly adapted from [Her91b]) are given below.

[2] The r in h_m^r stands for the registers, the m for having many objects of the given type. Jayanti [Jay97] also defines a hierarchy h_1^r where you only get finitely many objects. The h stands for "hierarchy," or, more specifically, $h(T)$ stands for the level of the hierarchy at which T appears [Jay11].

[3] See Chapter 28.

Consensus number	Defining characteristic	Examples
1	Read with interfering no-return RMW.	Registers, counters, generalized counters, max registers, atomic snapshots.
2	Interfering RMW; queue-like structures.	Test-and-set, fetch-and-write, fetch-and-add, queues, process-to-memory swap.
m		m-process consensus objects.
$2m-2$		Atomic m-register write.
∞	First write-like operation wins.	Queue with peek, fetch-and-cons, sticky bits, compare-and-swap, memory-to-memory swap, memory-to-memory copy.

Table 18.1: Position of various types in the wait-free hierarchy

18.1.1 Level 1: atomic registers, counters, other interfering RMW registers that don't return the old value

First observe that any type has consensus number at least 1, since 1-process consensus is trivial.

We'll argue that a large class of particularly weak objects has consensus number exactly 1, by running FLP with 2 processes. Recall from Chapter 9 that in the Fischer-Lynch-Paterson [FLP85] proof we classify states as bivalent or univalent depending on whether both decision values are still possible, and that with at least one failure we can always start in a bivalent state (this doesn't depend on what objects we are using, since it depends only on having invisible inputs). Since the system is wait-free there is no constraint on adversary scheduling, and so if any bivalent state has a bivalent successor we can just do it. So to solve consensus we have to reach a bivalent configuration C that has only univalent successors, and in particular has a 0-valent and a 1-valent successor produced by applying operations x and y of processes p_x and p_y.

Assuming objects don't interact with each other behind the scenes, x and y must be operations of the same object. Otherwise $Cxy = Cyx$ and we get a contradiction.

Now let's suppose we are looking at atomic registers, and consider cases:

- x and y are both reads, Then x and y commute: $Cxy = Cyx$, and we get a contradiction.
- x is a read and y is a write. Then p_y can't tell the difference between Cyx and Cxy, so running p_y to completion gives the same decision value from both Cyx and Cxy, another contradiction.
- x and y are both writes. Now p_y can't tell the difference between Cxy and Cy, so we get the same decision value for both, again contradicting that Cx is 0-valent and Cy is 1-valent.

There's a pattern to these cases that generalizes to other objects. Suppose that an object has a read operation that returns its state and one or more read-modify-write operations that don't return anything (perhaps we could call them "modify-write" operations). We'll say that the MW operations are **interfering** if, for any two operations x and y, either:

- x and y **commute**: $Cxy = Cyx$.
- One of x and y **overwrites** the other: $Cxy = Cy$ or $Cyx = Cx$.

Then no pair of read or modify-write operations can get us out of a bivalent state, because (a) reads commute; (b) for a read and MW, the non-reader can't tell which operation happened first; (c) and for any two MW operations, either they commute or the overwriter can't detect that the first operation happened. So any MW object with uninformative, interfering MW operations has consensus number 1.

For example, consider a counter that supports operations read, increment, decrement, and write: a write overwrites any other operation, and increments and decrements commute with each other, so the counter has consensus number 1. The same applies to a generalized counter that supports an atomic $x \leftarrow x + a$ operation; as long as this operation doesn't return the old value, it still commutes with other atomic increments. Max registers (reads on which return the largest value previously written) also have commutative updates, so they also have consensus number 1.

18.1.2 Level 2: interfering RMW objects that return the old value, queues (without peek)

Suppose now that we have a RMW object that returns the old value, and suppose that it is *non-trivial* in the sense that it has at least one RMW operation where the embedded function f that determines the new value is

not the identity (otherwise RMW is just read). Then there is some value v such that $f(v) \neq v$. To solve two-process consensus, have each process p_i first write its preferred value to a register r_i, then execute the non-trivial RMW operation on the RMW object initialized to v. The first process to execute its operation sees v and decides its own value. The second process sees $f(v)$ and decides the first process's value (which it reads from the register). It follows that non-trivial RMW object has consensus number *at least* 2.

In many cases, this is all we get. Suppose that the operations of some RMW type T are non-interfering in a way analogous to the previous definition, where now we say that x and y commute if they leave the object in the same state (regardless of what values are returned) and that y overwrites x if the object is always in the same state after both x and xy (again regardless of what is returned). The two processes p_x and p_y that carry out x and y know what happenened, but a third process p_z doesn't. So if we run p_z to completion we get the same decision value after both Cx and Cy, which means that Cx and Cy can't be 0-valent and 1-valent. It follows that no collection of RMW registers with interfering operations can solve 3-process consensus, and thus all such objects have consensus number 2. Examples of these objects include **test-and-set** bits, **fetch-and-add** registers, and **swap** registers that support an operation `swap` that writes a new value and return the previous value.

There are some other objects with consensus number 2 that don't fit this pattern. Define a **wait-free queue** as an object with enqueue and dequeue operations (like normal queues), where dequeue returns $\perp$ if the queue is empty (instead of blocking). To solve 2-process consensus with a wait-free queue, initialize the queue with a single value (it doesn't matter what the value is). We can then treat the queue as a non-trivial RMW register where a process wins if it successfully dequeues the initial value and loses if it gets empty.

However, enqueue operations are non-interfering: if p_x enqueues v_x and p_y enqueues v_y, then any third process can detect which happened first; similarly we can distinguish $\texttt{enq}(x)\texttt{deq}()$ from $\texttt{deq}()\texttt{enq}(x)$. So to show we can't do three process consensus we do something sneakier: given a bivalent state C with allegedly 0- and 1-valent successors $C\texttt{enq}(x)$ and $C\texttt{enq}(y)$, consider both $C\texttt{enq}(x)\texttt{enq}(y)$ and $C\texttt{enq}(y)\texttt{enq}(x)$ and run p_x until it does a $\texttt{deq}()$ (which it must, because otherwise it can't tell what to decide) and then stop it. Now run p_y until it also does a $\texttt{deq}()$ and then stop it. We've now destroyed the evidence of the split and poor hapless p_z is stuck. In the case of $C\texttt{deq}()\texttt{enq}(x)$ and $C\texttt{enq}(x)\texttt{deq}()$ on a non-empty queue we can kill the initial dequeuer immediately and then kill whoever dequeues x or the

value it replaced, and if the queue is empty only the dequeuer knows. In either case we reach indistinguishable states after killing only 2 witnesses, and the queue has consensus number at most 2.

Similar arguments work on stacks, deques, and so forth—these all have consensus number exactly 2.

18.1.3 Level ∞: objects where first write wins

These are objects that can solve consensus for any number of processes. Here are a bunch of level-∞ objects:

Queue with peek Has operations `enq`(x) and `peek`(), which returns the first value enqueued. (Maybe also `deq`(), but we don't need it for consensus). Protocol is to enqueue my input and then peek and return the first value in the queue.

Fetch-and-cons Returns old cdr and adds new car on to the head of a list. Use preceding protocol where `peek`() = `tail`(car :: cdr).

Sticky bit Has a `write` operation that has no effect unless register is in the initial $\perp$ state. Whether the `write` succeeds or fails, it returns nothing. The consensus protocol is to write my input and then return result of a read.

Compare-and-swap Has `CAS`(old, new) operation that writes new only if previous value is old. Use it to build a sticky bit.

Load-linked/store-conditional Like compare-and-swap split into two operations. The **o**peration reads a memory location and marks it. The **o**peration succeeds only if the location has not been changed since the preceding load-linked by the same process. Can be used to build a sticky bit.

Memory-to-memory swap Has `swap`(r_i, r_j) operation that atomically swaps contents of r_i with r_j, as well as the usual read and write operations for all registers. Use to implement fetch-and-cons. Alternatively, use two registers input[i] and victory[i] for each process i, where victory[i] is initialized to 0, and a single central register prize, initialized to 1. To execute consensus, write your input to input[i], then swap victory[i] with prize. The winning value is obtained by scanning all the victory registers for the one that contains a 1, then returning the corresponding input value.)

Memory-to-memory copy Has a copy(r_i, r_j) operation that copies r_i to r_j atomically. Use the same trick as for memory-to-memory swap, where a process copies prize to victory$[i]$. But now we have a process follow up by writing 0 to prize. As soon as this happens, the victory values are now fixed; take the leftmost 1 as the winner.[4]

Herlihy [Her91b] gives a slightly more complicated version of this procedure, where there is a separate prize$[i]$ register for each i, and after doing its copy a process writes 0 to all of the prize registers. This shows that memory-to-memory copy solves consensus for arbitrarily many processes even if we insist that copy operations can never overlap. The same trick also works for memory-to-memory swap, since we can treat a memory-to-memory swap as a memory-to-memory copy given that we don't care what value it puts in the prize$[i]$ register.

18.1.4 Level $2m - 2$: simultaneous m-register write

Here we have a (large) collection of atomic registers augmented by an m-register write operation that performs all the writes simultaneously. The intuition for why this is helpful is that if p_1 writes r_1 and r_{shared} while p_2 writes r_2 and r_{shared} then any process can look at the state of r_1, r_2 and r_{shared} and tell which write happened first. Code for this procedure is given in Algorithm 18.1; note that up to 4 reads may be necessary to determine the winner because of timing issues.

The workings of Algorithm 18.1 are straightforward:

- If the process reads $r_1 = r_2 = \perp$, then we don't care which went first, because the reader (or somebody else) already won.
- If the process reads $r_1 = 1$ and then $r_2 = \perp$, then p_1 went first.
- If the process reads $r_2 = 2$ and then $r_1 = \perp$, then p_2 went first. (This requires at least one more read after checking the first case.)
- Otherwise the process saw $r_1 = 1$ and $r_2 = 2$. Now read r_{shared}: if it's 1, p_2 went first; if it's 2, p_1 went first.

Algorithm 18.1 requires 2-register writes, and will give us a protocol for 2 processes (since the reader above has to participate somewhere to make the first case work). For m processes, we can do the same thing with m-register writes. We have a register $r_{pq} = r_{qp}$ for each pair of distinct processes p

[4] Or use any other rule that all processes apply consistently.

```
1  v_1 ← r_1
2  v_2 ← r_2
3  if v_1 = v_2 = ⊥ then
4      return no winner
5  if v_1 = 1 and v_2 = ⊥ then
       // p_1 went first
6      return 1
   // read r_1 again
7  v'_1 ← r_1
8  if v_2 = 2 and v'_1 = ⊥ then
       // p_2 went first
9      return 2
   // both p_1 and p_2 wrote
10 if r_shared = 1 then
11     return 2
12 else
13     return 1
```

Algorithm 18.1: Determining the winner of a race between 2-register writes. The assumption is that p_1 and p_2 each wrote their own ids to r_i and r_{shared} simultaneously. This code can be executed by any process (including but not limited to p_1 or p_2) to determine which of these 2-register writes happened first.

and q, plus a register r_{pp} for each p; this gives a total of $\binom{m}{2} + m = O(m^2)$ registers. All registers are initialized to $\perp$. Process p then writes its initial preference to some single-writer register pref_p and then simultaneously writes p to r_{pq} for all q (including r_{pp}). It then attempts to figure out the first writer by applying the above test for each q to r_{pq} (standing in for r_{shared}), r_{pp} (r_1) and r_{qq} (r_2). If it won against all the other processes, it decides its own value. If not, it repeats the test recursively for some p' that beat it until it finds a process that beat everybody, and returns its value. So m-register writes solve m-process wait-free consensus.

A further tweak gets $2m-2$: run two copies of an $(m-1)$–process protocol using separate arrays of registers to decide a winner for each group. Then add a second phase where each process has one register s_p, in which each process p from group 1 writes the winning id for its group simultaneously into s_p and s_q for each q in the other group. To figure out who won in the end, build a graph of all victories, where there is an edge from p to q if and only if p beat q in phase 1 or p's id was written before q's id in phase 2. The winner is the (unique) process with at least one outgoing edge and no incoming edges, which will be the process that won its own group (by writing first) and whose value was written first in phase 2.

18.1.4.1 Matching impossibility result

It might seem that the technique used to boost from m-process consensus to $(2m-2)$-process consensus could be repeated to get up to at least $\Theta(m^2)$, but this turns out not to be the case. The essential idea is to show that in order to escape bivalence, we have to get to a configuration C where *every* process is about to do an m-register write leading to a univalent configuration (since reads don't help for the usual reasons, and normal writes can be simulated by m-register writes with an extra $m-1$ dummy registers), and then argue that these writes can't overlap too much. So suppose we are in such a configuration, and suppose that Cx is 0-valent and Cy is 1-valent, and we also have many other operations $z_1 \dots z_k$ that lead to univalent states. Following Herlihy [Her91b], we argue in two steps:

1. There is some register that is written to by x alone out of all the pending operations. Proof: Suppose not. Then the 0-valent configuration $Cxyz_1 \dots z_k$ is indistinguishable from the 1-valent configuration $Cyz_1 \dots z_k$ by any process except p_x, and we're in trouble.

2. There is some register that is written to by x and y but not by any of the z_i. Proof:: Suppose not. The each register is written by at most

one of x and y, making it useless for telling which went first; or it is overwritten by some z_i, hiding the value that tells which went first. So $Cxyz_1 \dots z_k$ is indistinguishable from $Cyxz_1 \dots z_k$ for any process other than p_x and p_y, and we're still in trouble.

Now suppose we have $2m-1$ processes. The first part says that each of the pending operations (x, y, all of the z_i) writes to 1 single-writer register and at least k two-writer registers where k is the number of processes leading to a different univalent value. This gives $k+1$ total registers simultaneously written by this operation. Now observe that with $2m-1$ process, there is some set of m processes whose operations all lead to a b-valent state; so for any process to get to a $(\neg b)$-valent state, it must write $m+1$ registers simultaneously. It follows that with only m simultaneous writes we can only do $(2m-2)$-consensus.

18.1.5 Level m: m-process consensus objects

An m-process **consensus object** has a single `consensus` operation that, the first m times it is called, returns the input value in the first operation, and thereafter returns only $\perp$. Clearly this solves m-process consensus. To show that it doesn't solve $(m+1)$-process consensus even when augmented with registers, run a bivalent initial configuration to a configuration C where any further operation yields a univalent state. By an argument similar to the m-register write case, we can show that the pending operations in C must all be consensus operations on the same consensus object (anything else commutes or overwrites). Now run $Cxyz_1 \dots z_k$ and $Cyxz_1 \dots z_k$, where x and y lead to 0-valent and 1-valent states, and observe that p_k can't distinguish the resulting configurations because all it got was $\perp$. (Note: this works even if the consensus object isn't in its initial state, since we know that before x or y the configuration is still bivalent.)

So the m-process consensus object has consensus number m. This shows that h_m^r is nonempty at each level.

A natural question at this point is whether the inability of m-process consensus objects to solve $(m+1)$-process consensus implies robustness of the hierarchy. One might consider the following argument: given any object at level m, we can simulate it with an m-process consensus object, and since we can't combine m-process consensus objects to boost the consensus number, we can't combine any objects they can simulate either. The problem here is that while m-process consensus objects can simulate any object in a system with m processes (see below), it may be that some objects can do

more in a system with $m+1$ objects while still not solving $(m+1)$-process consensus. A simple way to see this would be to imagine a variant of the m-process consensus object that doesn't fail completely after m operations; for example, it might return one of the first two inputs given to it instead of $\perp$. This doesn't help with solving consensus, but it might (or might not) make it too powerful to implement using standard m-process consensus objects.

18.2 Universality of consensus

Universality of consensus says that any type that can implement n-process consensus can, together with atomic registers, give a wait-free implementation of any object in a system with n processes. That consensus is universal was shown by Herlihy [Her91b] and Plotkin [Plo89]. Both of these papers spend a lot of effort on making sure that both the cost of each operation and the amount of space used is bounded. But if we ignore these constraints, the same result can be shown using a mechanism similar to the replicated state machines of §10.1.

Here the processes repeatedly use consensus to decide between candidate histories of the simulated object, and a process successfully completes an operation when its operation (tagged to distinguish it from other similar operations) appears in a winning history. A round structure avoids too much confusion.

Details are given in Algorithm 18.2.

There are some subtleties to this algorithm. The first time that a process calls consensus (on $c[r]$), it may supply a dummy input; the idea is that it is only using the consensus object to obtain the agreed-upon history from a round it missed. It's safe to do this, because no process writes r to its round register until $c[r]$ is complete, so the dummy input can't be accidentally chosen as the correct value.

It's not hard to see that whatever h_{r+1} is chosen in $c[r+1]$ is an extension of h_r (it is constructed by appending operations to h_r), and that all processes agree on it (by the agreement property of the consensus object $c[r+1]$. So this gives us an increasing sequence of consistent histories. We also need to show that these histories are linearizable. The obvious linearization is just the most recent version of h_r. Suppose some call to $\texttt{apply}(\pi_1)$ finishes before a call to $\texttt{apply}(\pi_2)$ starts. Then π_1 is contained in some h_r when $\texttt{apply}(\pi_1)$ finishes, and since π_2 can only enter h by being appended at the end, we get π_1 linearized before π_2.

Finally, we need to show termination. The algorithm is written with a

```
1  procedure apply(π)
       // announce my intended operation
2      op[i] ← π
3      while true do
           // find a recent round
4          r ← max_j round[j]
           // obtain the history as of that round
5          if h_r = ⊥ then
6              h_r ← consensus(c[r], ⊥)
7          if π ∈ h_r then
8              return value π returns in h_r
           // else attempt to advance
9          h' ← h_r
10         for each j do
11             if op[j] ∉ h' then
12                 append op[j] to h'
13         h_{r+1} ← consensus(c[r + 1], h')
14         round[i] ← r + 1
```

Algorithm 18.2: A universal construction based on consensus

loop, so in principle it could run forever. But we can argue that no process after executes the loop more than twice. The reason is that a process p puts its operation in $\mathsf{op}[p]$ before it calculates r; so any process that writes $r' > r$ to round sees p's operation before the next round. It follows that p's value gets included in the history no later than round $r + 2$. (We'll see this sort of thing again when we do atomic snapshots in Chapter 19.)

Building a consistent shared history is easier with some particular objects that solve consensus. For example, a **fetch-and-cons** object that supplies an operation that pushes a new head onto a linked list and returns the old head trivially implements the common history above without the need for helping. One way to implement fetch-and-cons is with a swap object; to add a new element to the list, create a cell with its next pointer pointing to itself, then swap the next field with the head pointer for the entire list.

The solutions we've described here have a number of deficiencies that make them impractical in a real system (even more so than many of the algorithms we've described). If we store entire histories in a register, the register will need to be very, very wide. If we store entire histories as a linked list, it will take an unbounded amount of time to read the list. For solutions to these problems, see [AW04, 15.3] or the papers of Herlihy [Her91b] and Plotkin [Plo89].

Chapter 19

Atomic snapshots

We've seen in the previous chapter that there are a lot of things we can't make wait-free with just registers. But there are a lot of things we can. Atomic snapshots are a tool that let us do a lot of these things easily.

An **atomic snapshot object** acts like a collection of n single-writer multi-reader atomic registers with a special **snapshot** operation that returns (what appears to be) the state of all n registers at the same time. This is easy without failures: we simply lock the whole register file, read them all, and unlock them to let all the starving writers in. But it gets harder if we want a protocol that is wait-free, where any process can finish its own snapshot or write even if all the others lock up.

We'll give the usual sketchy description of a couple of snapshot algorithms. More details on early snapshot results can be found in [AW04, §10.3] or [Lyn96, §13.3]. There is also a reasonably recent survey by Fich on upper and lower bounds for the problem [Fic05].

19.1 The basic trick: two identical collects equals a snapshot

Let's tag any value written with a sequence number, so that each value written has a seqno field attached to it that increases over time. We can now detect if a new write has occurred between two reads of the same variable. Suppose now that we repeatedly perform **collects**—reads of all n registers—until two successive collects return exactly the same vector of values and sequence numbers. We can then conclude that precisely these values were present in the registers at some time in between the two collects. This gives us a very simple algorithm for snapshot. Unfortunately, it doesn't

terminate if there are a lot of writers around.[1] So we need some way to slow the writers down, or at least get them to do snapshots for us.

19.2 The Gang of Six algorithm

This is the approach taken by Afek and his five illustrious co-authors [AAD+93] (see also [AW04, §10.3] or [Lyn96, §13.3.2]): before a process can write to its register, it first has to complete a snapshot and leave the results behind with its write.[2] This means that if some slow process (including a slow writer, since now writers need to do snapshots too) is prevented from doing the two-collect snapshot because too much writing is going on, eventually it can just grab and return some pre-packaged snapshot gathered by one of the many successful writers.

Specifically, if a process executing a single snapshot operation σ sees values written by a single process i with three different sequence numbers s_1, s_2 and s_3, then it can be assured that the snapshot σ_3 gathered with sequence number s_3 started no earlier than s_2 was written (and thus no earlier than σ started, since σ read s_1 after it started) and ended no later than σ ended (because σ saw it). It follows that the snapshot can safely return σ_3, since that represents the value of the registers at some time inside σ_3's interval, which is contained completely within σ's interval.

So a snapshot repeatedly does collects until either (a) it gets two identical collects, in which case it can return the results (a **direct scan**, or (b) it sees three different values from the same process, in which case it can take the snapshot collected by the second write (an **indirect scan**). See pseudocode in Algorithm 19.1.

Amazingly, despite the fact that updates keep coming and everybody is trying to do snapshots all the time, a snapshot operation of a single process is guaranteed to terminate after at most $n + 1$ collects. The reason is that

[1] This isn't always a problem, since there may be external factors that keep the writers from showing up too much. Maurice Herlihy and I got away with using exactly this snapshot algorithm in an ancient, pre-snapshot paper on randomized consensus [AH90a].

[2] The algorithm is usually called the AADGMS algorithm by people who can remember all the names—or at least the initials—of the team of superheroes who came up with it (Afek, Attiya, Dolev, Gafni, Merritt, and Shavit) and Gang of Six by people who can't. Historically, this was one of three independent solutions to the problem that appeared at about the same time; a similar algorithm for **composite registers** was given by James Anderson [And94] and a somewhat different algorithm for **consistent scan** was given by Aspnes and Herlihy [AH90b]. The Afek *et al.* algorithm had the advantage of using bounded registers (in its full version), and so it and its name for atomic snapshot prevailed over its competitors.

in order to prevent case (a) from holding, the adversary has to supply at least one new value in each collect after the first. But it can only supply one new value for each of the $n-1$ processes that aren't doing collects before case (b) is triggered (it's triggered by the first process that shows up with a second new value). Adding up all the collects gives $1+(n-1)+1 = n+1$ collects before one of the cases holds. Since each collect takes $n-1$ read operations (assuming the process is smart enough not to read its own register), a snapshot operation terminates after at most n^2-1 reads.

```
1  procedure update_i(A, v)
2      s ← scan(A)
3      A[i] ← ⟨A[i].count + 1, v, s⟩
4  procedure scan(A)
5      initial ← collect(A)
6      previous ← initial while true do
7          s ← scan(A)
8          if s = previous then
               // Two identical collects
9              return s
10         else if ∃j : s[j].count ≥ initial[j].count + 2 do
               // Three different counts from j
11             return s[j].snapshot
12         else
               // Nothing useful, try again
13             previous ← s
```

Algorithm 19.1: Snapshot of [AAD+93] using unbounded registers

For a write operation, a process first performs a snapshot, then writes the new value, the new sequence number, and the result of the snapshot to its register (these are very wide registers). The total cost is n^2-1 read operations for the snapshot plus 1 write operation.

19.2.1 Linearizability

We now need to argue that the snapshot vectors returned by the Afek *et al.* algorithm really work, that is, that between each matching `invoke-snapshot` and `respond-snapshot` there was some actual time where the registers in the array contained precisely the values returned in the respond-snapshot

action. We do so by assigning a **linearization point** to each snapshot vector, a time at which it appears in the registers (which for correctness of the protocol had better lie within the interval between the snapshot invocation and response). For snapshots obtained through case (a), take any time between the two collects. For snapshots obtained through case (b), take the serialization point already assigned to the snapshot vector provided by the third write. In the latter case we argue by induction on termination times that the linearization point lies inside the snapshot's interval.

Note that this means that all snapshots were ultimately collected by two successive collects returning identical values, since any case-(b) snapshot sits on top of a finite regression of case-(b) snapshots that must end with a case-(a) snapshot. In practice what this means is that if there are many writers, eventually all of them will stall waiting for a case-(a) snapshot to complete, which happens because all the writers are stuck. So effectively the process of requiring writers to do snapshots first almost gives us a form of locking, but without the vulnerability to failures of a real lock. (In fact, crash failures just mean that there are fewer writers to screw things up, allowing snapshots to finish faster.)

19.2.2 Using bounded registers

The simple version of the Afek *et al.* algorithm requires unbounded registers (since sequence numbers may grow forever). One of the reasons why this algorithm required so many smart people was to get rid of this assumption: the paper describes a (rather elaborate) mechanism for recycling sequence numbers that prevents unbounded growth (see also [Lyn96, 13.3.3]). In practice, unbounded registers are probably not really an issue once one has accepted very large registers, but getting rid of them is an interesting theoretical problem.

It turns out that with a little cleverness we can drop the sequence numbers entirely. The idea is that we just need a mechanism to detect when somebody has done a lot of writes while a snapshot is in progress. A naive approach would be to have sequence numbers wrap around mod m for some small constant modulus m; this fails because if enough snapshots happen between two of my collects, I may notice none of them because all the sequence numbers wrapped around all the way. But we can augment mod-m sequence numbers with a second handshaking mechanism that detects when a large enough number of snapshots have occurred; this acts like the guard bit on an automobile odometer, than signals when the odometer has overflowed to prevent odometer fraud by just running the odometer forward an

extra million miles or so.

The result is the full version of Afek *et al.* [AAD+93]. (Our presentation here follows [AW04, 10.3].) The key mechanism for detecting odometer fraud is a **handshake**, a pair of single-writer bits used by two processes to signal each other that they have done something. Call the processes S (for *same*) and D (for *different*), and supposed we have handshake bits h_S and h_D. We then provide operations `tryHandshake` (signal that something is happening) and `checkHandshake` (check if something happened) for each process; these operations are asymmetric. The code is:

`tryHandshake`(S)**:** $h_S \leftarrow h_D$ (make the two bits the same)

`tryHandshake`(D)**:** $h_D \leftarrow \neg h_S$ (make the two bits different)

`checkHandshake`(S)**:** return $h_S \neq h_D$ (return true if D changed its bit)

`checkHandshake`(D)**:** return $h_S = h_D$ (return true if S changed its bit)

The intent is that `checkHandshake` returns true if the other process called `tryHandshake` after I did. The situation is a bit messy, however, since `tryHandshake` involves two register operations (reading the other bit and then writing my own). So in fact we have to look at the ordering of these read and write events. Let's assume that `checkHandshake` is called by S (so it returns true if and only if it sees different values). Then we have two cases:

1. `checkHandshake`(S) returns true. Then S reads a different value in h_D from the value it read during its previous call to `tryHandshake`(S). It follows that D executed a write as part of a `tryHandshake`(D) operation in between S's previous read and its current read.

2. `checkHandshake`(S) returns false. Then S reads the same value in h_D as it read previously. This does not necessarily mean that D didn't write h_D during this interval—it is possible that D is just very out of date, and did a write that didn't change the register value—but it does mean that D didn't perform both a read and a write since S's previous read.

How do we use this in a snapshot algorithm? The idea is that before performing my two collects, I will execute `tryHandshake` on my end of a pair of handshake bits for every other process. After performing my two collects, I'll execute `checkHandshake`. I will also assume each update (after

performing a snapshot) toggles a mod-2 sequence number bit on the value stored in its segment of the snapshot array. The hope is that between the toggle and the handshake, I detect any changes. (See [AW04, Algorithm 30] for the actual code.)

Does this work? Let's look at cases:

1. The toggle bit for some process q is unchanged between the two snapshots taken by p. Since the bit is toggled with each update, this means that an even number of updates to $q's$ segment occurred during the interval between p's writes. If this even number is 0, we are happy: no updates means no call to `tryHandshake` by q, which means we don't see any change in q's segment, which is good, because there wasn't any. If this even number is 2 or more, then we observe that each of these events precedes the following one:

   - p's call to `tryHandshake`.
   - p's first read.
   - q's first write.
   - q's call to `tryHandshake` at the start of its second scan.
   - q's second write.
   - p's second read.
   - p's call to `checkHandshake`.

   It follows that q both reads and writes the handshake bits in between p's calls to `tryHandshake` and `checkHandshake`, so p correctly sees that q has updated its segment.

2. The toggle bit for q has changed. Then q did an odd number of updates (i.e., at least one), and p correctly detects this fact.

What does p do with this information? Each time it sees that q has done a scan, it updates a count for q. If the count reaches 3, then p can determine that q's last scanned value is from a scan that is contained completely within the time interval of p's scan. Either this is a **direct scan**, where q actually performs two collects with no changes between them, or it's an **indirect scan**, where q got its value from some other scan completely contained within q's scan. In the first case p is immediately happy; in the second, we observe that this other scan is also contained within the interval of p's scan, and so (after chasing down a chain of at most $n-1$ indirect scans) we eventually reach a direct scan contained within it that provided the actual

value. In either case p returns the value of pair of adjacent collects with no changes between them that occurred during the execution of its scan operation, which gives us linearizability.

19.3 Faster snapshots using lattice agreement

The Afek *et al.* algorithm and its contemporaries all require $O(n^2)$ operations for each snapshot. It is possible to get this bound down to $O(n)$ using a more clever algorithm, [IMCT94] which is the best we can reasonably hope for in the worst case given that (a) even a collect (which doesn't guarantee anything about linearizability) requires $\Theta(n)$ operations when implemented in the obvious way, and (b) there is a linear lower bound, due to Jayanti, Tan, and Toueg [JTT00], on a large class of wait-free objects that includes snapshot.[3]

The first step, due to Attiya, Herlihy, and Rachman [AHR95], is a reduction to a related problem called **lattice agreement**.

19.3.1 Lattice agreement

A **lattice** is a partial order in which every pair of elements x, y has a least upper bound $x \vee y$ called the **join** of x and y and a greatest lower bound $x \wedge y$ called the **meet** of x and y. For example, we can make a lattice out of sets by letting join be union and meet be intersection; or we can make a lattice out of integers by making join be max and meet be min.

In the lattice agreement problem, each process starts with an input x_i and produces an output y_i, where both are elements of some lattice. The requirements of the problem are:

Comparability For all i, j, $y_i \leq y_j$ or $y_j \leq y_i$.

Downward validity For all i, $x_i \leq y_i$.

Upward validity For all i, $y_i \leq x_1 \vee x_2 \vee x_3 \vee \ldots \vee x_n$.

These requirements are analogous to the requirements for consensus. Comparability acts like agreement: the views returned by the lattice-agreement protocol are totally ordered. Downward validity says that each process will include its own input in its output. Upward validity acts like validity: an output can't include anything that didn't show up in some input.

[3] But see §21.5 for a faster alternative if we allow either randomization or limits on the number of times the array is updated.

For the snapshot algorithm, we also demand **wait-freedom**: each process terminates after a bounded number of its own steps, even if other processes fail.

Note that if we are really picky, we can observe that we don't actually need meets; a **semi-lattice** that provides only joins is enough. In practice we almost always end up with a full-blown lattice, because (a) we are working with finite sets, and (b) we generally want to include a bottom element $\perp$ that is less than all the other elements, to represent the "empty" state of our data structure. But any finite join-semi-lattice with a bottom element turns out to be a lattice, since we can define $x \wedge y$ as the join of all elements z such that $z \leq x$ and $z \leq y$. We don't *use* the fact that we are in a lattice anywhere, but it does save us two syllables not to have to say "semi-lattice agreement."

19.3.2 Connection to vector clocks

The first step in reducing snapshot to lattice agreement is to have each writer generates a sequence of increasing timestamps $r_1, r_2, \ldots$, and a snapshot corresponds to some vector of timestamps $[t_1, t_2 \ldots t_n]$, where t_i indicates the most recent write by p_i that is included in the snapshot (in other words, we are using vector clocks again; see §12.2.3). Now define $v \leq v'$ if $v_i \leq v'_i$ for all i; the resulting partial order is a lattice, and in particular we can compute $x \vee y$ by the rule $(x \vee y)_i = x_i \vee y_i$.

Suppose now that we have a bunch of snapshots that satisfy the comparability requirement; i.e., they are totally ordered. Then we can construct a sequential execution by ordering the snapshots in increasing order with each update operation placed before the first snapshot that includes it. This sequential execution is not necessarily a linearization of the original execution, and a single lattice agreement object won't support more than one operation for each process, but the idea is that we can nonetheless use lattice agreement objects to enforce comparability between concurrent executions of snapshot, while doing some other tricks (exploiting, among other things, the validity properties of the lattice agreement objects) to get linearizability over the full execution.

19.3.3 The full reduction

The Attiya-Herlihy-Rachman algorithm is given as Algorithm 19.2. It uses an array of registers R_i to hold round numbers (timestamps); an array S_i to hold values to scan; an unboundedly humongous array V_{ir} to hold views

obtained by each process in some round; and a collection of lattice-agreement objects LA_r, one for each round.

```
1  procedure scan()
       // First attempt
2      R_i ← r ← max(R_1 ... R_n, R_i + 1)
3      collect ← read(S_1 ... S_n)
4      view ← LA_r(collect)
5      if max(R_1 ... R_n) > R_i then
           // Fall through to second attempt
6      else
7          V_ir ← view
8          return V_ir
       // Second attempt
9      R_i ← r ← max(R_1 ... R_n, R_i + 1)
10     collect ← read(S_1 ... S_n)
11     view ← LA_r(collect)
12     if max(R_1 ... R_n) > R_i then
13         V_ir ← some nonempty V_jr
14         return V_ir
15     else
16         V_ir ← view
17         returnV_ir
```

Algorithm 19.2: Lattice agreement snapshot

The algorithm makes two attempts to obtain a snapshot. In both cases, the algorithm advances to the most recent round it sees (or its previous round plus one, if nobody else has reached this round yet), attempts a collect, and then runs lattice-agreement to try to get a consistent view. If after getting its first view it finds that some other process has already advanced to a later round, it makes a second attempt at a new, higher round r' and uses some view that it obtains in this second round, either directly from lattice agreement, or (if it discovers that it has again fallen behind), it uses an indirect view from some speedier process.

The reason why I throw away my view if I find out you have advanced to a later round is not because the view is bad for me but because it's bad for you: I might have included some late values in my view that you didn't see, breaking consistency between rounds. But I don't have to do this more

than once; if the same thing happens on my second attempt, I can use an indirect view as in [AAD+93], knowing that it is safe to do so because any collect that went into this indirect view started after I did.

The update operation is the usual update-and-scan procedure; for completeness this is given as Algorithm 19.3. To make it easier to reason about the algorithm, we assume that an update returns the result of the embedded scan.

```
1 procedure update_i(v)
2 |   S_i ← (S_i.seqno + 1, v)
3 |   return scan()
```

Algorithm 19.3: Update for lattice agreement snapshot

19.3.4 Why this works

We need to show three facts:

1. All views returned by the scan operation are comparable; that is, there exists a total order on the set of views (which can be extended to a total order on scan operations by breaking ties using the execution order).

2. The view returned by an update operation includes the update (this implies that future views will also include the update, giving the correct behavior for snapshot).

3. The total order on views respects the execution order: if π_1 and π_2 are scan operations that return v_1 and v_2, then $scan_1 <_S scan_2$ implies $view_1 \leq view_2$. (This gives us linearization.)

Let's start with comparability. First observe that any view returned is either a direct view (obtained from LA_r) or an indirect view (obtained from V_{jr} for some other process j). In the latter case, following the chain of indirect views eventually reaches some direct view. So all views returned for a given round are ultimately outputs of LA_r and thus satisfy comparability.

But what happens with views from different rounds? The lattice-agreement objects only operate within each round, so we need to ensure that any view returned in round r is included in any subsequent rounds. This is where checking round numbers after calling LA_r comes in.

Suppose some process i returns a direct view; that is, it sees no higher round number in either its first attempt or its second attempt. Then at the time it starts checking the round number in Line 5 or 12, no process has yet written a round number higher than the round number of i's view (otherwise i would have seen it). So no process with a higher round number has yet executed the corresponding collect operation. When such a process does so, it obtains values that are at least as current as those fed into LA_r, and i's round-r view is less than or equal to the vector of these values by upward validity of LA_r and thus less than or equal to the vector of values returned by $LA_{r'}$ for $r' > r$ by upward validity. So we have comparability of all direct views, which implies comparability of all indirect views as well.

To show that each view returned by scan includes the preceding update, we observe that either a process returns its first-try scan (which includes the update by downward validity) or it returns the results of a scan in the second-try round (which includes the update by downward validity in the later round, since any collect in the second-try round starts after the update occurs). So no updates are missed.

Now let's consider two scan operations π_1 and π_2 where π_1 precedes π_2 in the execution. We want to show that, for the views v_1 and v_2 that these scans return, $v_1 \leq v_2$. From the comparability property, the only way this can fail is if $v_2 < v_1$; that is, there is some update included in v_2 that is not included in v_1. But this can't happen; if π_2 starts after π_1 finishes, it starts after any update π_1 sees is already in one of the S_j registers, and so π_2 will include this update in its initial collect. (Slightly more formally, if s is the contents of the registers at some time between when π_1 finishes and π_2 starts, then $v_1 \leq s$ by upward validity and $s \leq v_2$ by downward validity of the appropriate LA objects.)

19.3.5 Implementing lattice agreement

There are several known algorithms for implementing lattice agreement, including the original algorithm of Attiya, Herlihy, and Rachman [AHR95] and an adaptive algorithm of Attiya and Fouren [AF01]. The best of them (assuming multi-writer registers) is Inoue *et al.*'s linear-time lattice agreement protocol [IMCT94].

The intuition behind this protocol is to implement lattice agreement using divide-and-conquer. The processes are organized into a tree, with each leaf in the tree corresponding to some process's input. Internal nodes of the tree hold data structures that will report increasingly large subsets of the inputs under them as they become available. At each internal node,

a double-collect snapshot is used to ensure that the value stored at that node is always the union of two values that appear in its children at the same time. This is used to guarantee that, so long as each child stores an increasing sequence of sets of inputs, the parent does so also.

Each process ascends the tree updating nodes as it goes to ensure that its value is included in the final result. A rather clever data structure is used to ensure that out-of-date smaller sets don't overwrite larger ones at any node, and the cost of using this data structure and carrying out the double-collect snapshot at a node with m leaves below it is shown to be $O(m)$. So the total cost of a snapshot is $O(n + n/2 + n/4 + \ldots 1) = O(n)$, giving the linear time bound.

Let's now look at the details of this protocol. There are two main components: the `Union` algorithm used to compute a new value for each node of the tree, and the `ReadSet` and `WriteSet` operations used to store the data in the node. These are both rather specialized algorithms and depend on the details of the other, so it is not trivial to describe them in isolation from each other; but with a little effort we can describe exactly what each component demands from the other, and show that it gets it.

The `Union` algorithm does the usual two-collects-without change trick to get the values of the children and then stores the result. In slightly more detail:

1. Perform `ReadSet` on both children. This returns a set of leaf values.
2. Perform `ReadSet` on both children again.
3. If the values obtained are the same in both collects, call `WriteSet` on the current node to store the union of the two sets and proceed to the parent node. Otherwise repeat the preceding step.

The requirement of the `Union` algorithm is that calling `ReadSet` on a given node returns a non-decreasing sequence of sets of values; that is, if `ReadSet` returns some set S at a particular time and later returns S', then $S \subseteq S'$. We also require that the set returned by `ReadSet` is a superset of any set written by a `WriteSet` that precedes it, and that it is equal to some such set. This last property only works if we guarantee that the values stored by `WriteSet` are all comparable (which is shown by induction on the behavior of `Union` at lower levels of the tree).

Suppose that all these conditions hold; we want to show that the values written by successive calls to `Union` are all comparable, that is, for any values S, S' written by union we have $S \subseteq S'$ or $S' \subseteq S$. Observe that

$S = L \cup R$ and $S' = L' \cup R'$ where L, R and L', R' are sets read from the children. Suppose that the `Union` operation producing S completes its snapshot before the operation producing S'. Then $L \subseteq L'$ (by the induction hypothesis) and $R \subseteq R'$, giving $S \subseteq S'$.

We now show how to implement the `ReadSet` and `WriteSet` operations. The main thing we want to avoid is the possibility that some large set gets overwritten by a smaller, older one. The solution is to have m registers $a[1 \dots m]$, and write a set of size s to every register in $a[1 \dots s]$ (each register gets a copy of the entire set). Because register $a[s]$ gets only sets of size s or larger, there is no possibility that our set is overwritten by a smaller one. If we are clever about how we organize this, we can guarantee that the total cost of all calls to `ReadSet` by a particular process is $O(m)$, as is the cost of the single call to `WriteSet` in `Union`.

Pseudocode for both is given as Algorithm 19.4. This is a simplified version of the original algorithm from [IMCT94], which does the writes in increasing order and thus forces readers to finish incomplete writes that they observe, as in Attiya-Bar-Noy-Dolev [ABND95] (see also Chapter 16).

```
   shared data: array a[1 ... m] of sets, initially ∅
   local data: index p, initially 0

 1 procedure WriteSet(S)
 2     for i ← |S| down to 1 do
 3         a[i] ← S

 4 procedure ReadSet()
       // update p to last nonempty position
 5     while true do
 6         s ← a[p]
 7         if p = m or a[p + 1] = ∅ then
 8             break
 9         else
10             p ← p + 1
11     return s
```

Algorithm 19.4: Increasing set data structure

Naively, one might think that we could just write directly to $a[|S|]$ and skip the previous ones, but this makes it harder for a reader to detect that

$a[|S|]$ is occupied. By writing all the previous registers, we make it easy to tell if there is a set of size $|S|$ or bigger in the sequence, and so a reader can start at the beginning and scan forward until it reaches an empty register, secure in the knowledge that no larger value has been written.[4] Since we want to guarantee that no reader every spends more that $O(m)$ operations on an array of m registers (even if it does multiple calls to `ReadSet`), we also have it remember the last location read in each call to `ReadSet` and start there again on its next call. For `WriteSet`, because we only call it once, we don't have to be so clever, and can just have it write all $|S| \leq m$ registers.

We need to show linearizability. We'll do so by assigning a specific linearization point to each high-level operation. Linearize each call to `ReadSet` at the last time that it reads $a[p]$. Linearize each call to `WriteSet`(S) at the first time at which $a[|S|] = S$ and $a[i] \neq \emptyset$ for every $i < |S|$ (in other words, at the first time that some reader might be able to find and return S); if there is no such time, linearize the call at the time at which it returns. Since every linearization point is inside its call's interval, this gives a linearization that is consistent with the actual execution. But we have to argue that it is also consistent with a sequential execution, which means that we need to show that every `ReadSet` operation returns the largest set among those whose corresponding `WriteSet` operations are linearized earlier.

Let R be a call to `ReadSet` and W a call to `WriteSet`(S). If R returns S, then at the time that R reads S from $a[|S|]$, we have that (a) every register $a[i]$ with $i < |S|$ is non-empty (otherwise R would have stopped earlier), and (b) $|S| = m$ or $a[|S| + 1] = \emptyset$ (as otherwise R would have kept going after later reading $a[|S| + 1]$. From the rule for when `WriteSet` calls are linearized, we see that the linearization point of W precedes this time and that the linearization point of any call to `WriteSet` with a larger set follows it. So the return value of R is consistent.

The payoff: unless we do more updates than snapshots, don't want to assume multi-writer registers, are worried about unbounded space, have a beef with huge registers, or care about constant factors, it costs no more time to do a snapshot than a collect. So in theory we can get away with assuming snapshots pretty much wherever we need them.

[4] This trick of reading in one direction and writing in another dates back to a paper by Lamport from 1977 [Lam77].

19.4 Practical snapshots using LL/SC

Though atomic registers are enough for snapshots, it is possible to get a much more efficient snapshot algorithm using stronger synchronization primitives. An algorithm of Riany, Shavit, and Touitou [RST01] uses **load-linked/store-conditional** objects to build an atomic snapshot protocol with linear-time snapshots and constant-time updates using small registers. We'll give a sketch of this algorithm here.

The RST algorithm involves two basic ideas: the first is a snapshot algorithm for a single scanner (i.e., only one process can do snapshots) in which each updater maintains two copies of its segment, a *high* copy (that may be more recent than the current scan) and a *low* copy (that is guaranteed to be no more recent than the current scan). The idea is that when a scan is in progress, updaters ensure that the values in memory at the start of the scan are not overwritten before the scan is completed, by copying them to the low registers, while the high registers allow new values to be written without waiting for the scan to complete. Unbounded sequence numbers, generated by the scanner, are used to tell which values are recent or not.

As long as there is only one scanner, nothing needs to be done to ensure that all scans are consistent. But extending the algorithm to multiple scanners is tricky. A simple approach would be to keep a separate low register for each concurrent scan—however, this would require up to n low registers and greatly increase the cost of an update. Instead, the authors devise a mechanism, called a **coordinated collect**, that allows the scanners collectively to implement a sequence of *virtual scans* that do not overlap. Each virtual scan is implemented using the single-scanner algorithm, with its output written to a common *view* array that is protected from inconsistent updates using LL/SC operations. A scanner participates in virtual scans until it obtains a virtual scan that is useful to it (this means that the virtual scan has to take place entirely within the interval of the process's actual scan operation); the simplest way to arrange this is to have each scanner perform two virtual scans and return the value obtained by the second one.

The paper puts a fair bit of work into ensuring that only $O(n)$ view arrays are needed, which requires handling some extra special cases where particularly slow processes don't manage to grab a view before it is reallocated for a later virtual scan. We avoid this complication by simply assuming an unbounded collection of view arrays; see the paper for how to do this right.

A more recent paper by Fatourou and Kallimanis [FK07] gives improved time and space complexity using the same basic technique.

19.4.1 Details of the single-scanner snapshot

The single-scanner snapshot is implemented using a shared currSeq variable (incremented by the scanner but used by all processes) and an array memory of n snapshot segments, each of which is divided into a high and low component consisting of a value and a timestamp. Initially, currSeq is 0, and all memory locations are initialized to $(\perp, 0)$. This part of the algorithm does not require LL/SC.

A call to `scan` copies the first of memory$[j]$.high or memory$[j]$.low that has a sequence number less than the current sequence number. Pseudocode is given as Algorithm 19.5.

```
1 procedure scan()
2     currSeq ← currSeq + 1
3     for j ← 0 to n − 1 do
4         h ← memory[j].high
5         if h.seq < currSeq then
6             view[j] ← h.value
7         else
8             view[j] ← memory[j].low.value
```

Algorithm 19.5: Single-scanner snapshot: `scan`

The `update` operation for process i cooperates by copying memory$[i]$.high to memory$[i]$.low if it's old.

The `update` operation always writes its value to memory$[i]$.high, but preserves the previous value in memory$[i]$.low if its sequence number indicates that it may have been present at the start of the most recent call to `scan`. This means that `scan` can get the old value if the new value is too recent. Pseudocode is given in Algorithm 19.6.

```
1 procedure update()
2     seq ← currSeq
3     h ← memory[i].high
4     if h.seq ≠ seq then
5         memory[i].low ← h
6     memory[i].high ← (value, seq)
```

Algorithm 19.6: Single-scanner snapshot: `update`

To show this actually works, we need to show that there is a linearization of the scans and updates that has each scan return precisely those values whose corresponding updates are linearized before it. The ordering is based on when each `scan` operation S increments currSeq and when each `update` operation U reads it; specifically:

- If U reads currSeq after S increments it, then $S < U$.
- If U reads currSeq before S increments it and S reads memory$[i]$.high (where i is the process carrying out U) before U writes it, then $S < U$.
- If U reads currSeq before S increments it, but S reads memory$[i]$.high after U writes it, then $U < S$.

Updates are ordered based on intervening scans (i.e., $U_1 < U_2$ if $U_1 < S$ and $S < U_2$ by the above rules), or by the order in which they read currSeq if there is no intervening scan.

To show this is a linearization, we need first to show that it extends the ordering between operations in the original schedule. Each of the above rules has $\pi_1 < \pi_2$ only if some low-level operation of π_1 precedes some low-level operation of π_2, with the exception of the transitive ordering of two update events with an intervening scan. But in this last case we observe that if $U_1 < S$, then U_1 writes memory$[i]$.high before S reads it, so if U_1 precedes U_2 in the actual execution, U_2 must write memory$[i]$.high after S reads it, implying $S < U_2$.

Now we show that the values returned by `scan` are consistent with the linearization ordering; that, is, for each i, `scan` copies to view$[i]$ the value in the last `update` by process i in the linearization. Examining the code for `scan`, we see that a `scan` operation S takes memory$[i]$.high if its sequence number is less than currSeq, i.e. if the `update` operation U that wrote it read currSeq before S incremented it and wrote memory$[i]$.high before S read it; this gives $U < S$. Alternatively, if `scan` takes memory$[i]$.low, then memory$[i]$.low was copied by some update operation U' from the value written to memory$[i]$.high by some update U that read currSeq before S incremented it. Here U' must have written memory$[i]$.high before S read it (otherwise S would have taken the old value left by U) and since U precedes U' (being an operation of the same process) it must therefor also have written memory$[i]$.high before S read it. So again we get the first case of the linearization ordering and $U < S$.

So far we have shown only that S obtains values that were linearized before it, but not that it ignores values that were linearized after it. So now let's consider some U with $S < U$. Then one of two cases holds:

- U reads currSeq after S increments it. Then U writes a sequence number in memory$[i]$.high that is greater than or equal to the currSeq value used by S; so S returns memory$[i]$.low instead, which can't have a sequence number equal to currSeq and thus can't be U's value either.
- U reads currSeq before S increments it but writes memory$[i]$.high after S reads it. Now S won't return U's value from memory$[i]$.high (it didn't read it), and won't get it from memory$[i]$.low either (because the value that *is* in memory$[i]$.high will have seq $<$ currSeq, and so S will take that instead).

So in either case, if $S < U$, then S doesn't return U's value. This concludes the proof of correctness.

19.4.2 Extension to multiple scanners

See the paper for details.

The essential idea: view now represents a *virtual scan* view$_r$ generated cooperatively by all the scanners working together in some asynchronous round r. To avoid conflicts, we update view$_r$ using LL/SC or compare-and-swap (so that only the first scanner to write wins), and pretend that reads of memory$[i]$ by losers didn't happen. When view$_r$ is full, start a new virtual scan and advance to the next round (and thus the next view$_{r+1}$).

19.5 Applications

Here we describe a few things we can do with snapshots.

19.5.1 Multi-writer registers from single-writer registers

One application of atomic snapshot is building multi-writer registers from single-writer registers. The idea is straightforward: to perform a write, a process does a snapshot to obtain the maximum sequence number, tags its own value with this sequence number plus one, and then writes it. A read consists of a snapshot followed by returning the value associated with the largest sequence number (breaking ties by process id). (See [Lyn96, §13.5] for a proof that this actually works.) This requires using a snapshot that doesn't use multi-writer registers, and turns out to be overkill in practice; there are simpler algorithms that give $O(n)$ cost for reads and writes based on timestamps (see [AW04, 10.2.3]).

With additional work, it is even possible to eliminate the requirement of multi-reader registers, and get a simulation of multi-writer multi-reader registers that goes all the way down to single-writer single-read registers, or even single-writer single-reader bits. See [AW04, §§10.2.1–10.2.2] or [Lyn96, §13.4] for details.

19.5.2 Counters and accumulators

Given atomic snapshots, it's easy to build a counter (supporting increment, decrement, and read operations); or, in more generality, an accumulator (supporting increments by arbitrary amounts); or, in even more generality, an object supporting any collection of commutative update operations (as long as these operations don't return anything). The idea is that each process stores in its segment the total of all operations it has performed so far, and a read operation is implemented using a snapshot followed by summing the results. This is a case where it is reasonable to consider multi-writer registers in building the snapshot implementation, because there is not necessarily any circularity in doing so.

19.5.3 Resilient snapshot objects

The previous examples can be generalized to objects with operations that either read the current state of the object but don't update it or update the state but return nothing, provided the update operations either overwrite each other (so that $Cxy = Cy$ or $Cyx = Cx$) or commute (so that $Cxy = Cyx$).

This was shown by Aspnes and Herlihy [AH90b] and improved on by Anderson and Moir [AM93] by eliminating unbounded space usage (this paper also defined the terms **snapshot objects** for those with separate read and update operations and **resilience** for the property that all operations commute or overwrite). The basic idea underneath both of these papers is to use the multi-writer register construction given above, but break ties among operations with the same sequence numbers by first placing overwritten operations before overwriting operations and only then using process ids.

This *almost* shows that snapshots can implement any object with consensus number 1 where update operations return nothing, because an object that violates the commute-or-overwrite condition in some configuration has consensus number at least 2 (see §18.1.2). It doesn't quite work (as observed in the Anderson-Moir paper), because the tie-breaking procedure assumes a static ordering on which operations overwrite each other, so that given

operations x and y where y overwrites x, y overwrites x in any configuration. But there may be objects with *dynamic* ordering, where y overwrites x in some configuration, x overwrites y in another, and perhaps even the two operations commute in yet another. This prevents us from achieving consensus, but also breaks the tie-breaking technique. So it may be possible that there are objects with consensus number 1 and no-return updates that we still can't implement using only registers.

Chapter 20

Lower bounds on perturbable objects

Being able to do snapshots in linear time means that we can build linearizable counters, generalized counters, max registers, etc. in linear time, by having each reader take a snapshot and combine the contributions of each updater using the appropriate commutative and associative operation. A natural question is whether we can do better by exploiting the particular features of these objects.

Unfortunately, the Jayanti-Tan-Toueg [JTT00] lower bound for **perturbable** objects says each of these objects requires $n-1$ space and $n-1$ steps for a read operation in the worst case, for any solo-terminating implementation from historyless objects.[1]

Here **perturbable** means that the object has a particular property that makes the proof work, essentially that the outcome of certain special executions can be changed by stuffing lots of extra update operations in the middle (see below for details). **Solo-terminating** means that a process finishes its current operation in a finite number of steps if no other process takes steps in between; it is a much weaker condition, for example, than wait-freedom. **Historyless objects** are those for which any operation that changes the state overwrites all previous operations (i.e., those for which covering arguments work, as long as the covering processes never report back what they say). Atomic registers are the typical example, while **swap objects** (with a swap operation that writes a new state while returning the old state) are the canonical example since they can implement any other

[1] A caveat is that we may be able to make almost all read operations cheaper, although we won't be able to do anything about the space bound. See Chapter 21.

historyless object (and even have consensus number 2, showing that even extra consensus power doesn't necessarily help here).

Below is a sketch of the proof. See the original paper [JTT00] for more details.

- Build executions of the form $\Lambda_k\Sigma_k\Pi$, where Λ_k is a preamble consisting of various complete update operations and k incomplete update operations, Σ_k delivers k delayed writes from the incomplete operations in Λ_k, and Π is a read operation whose first k reads are from registers written in Σ_k.
  - Induction hypothesis is that such an execution exists for each $k \leq n-1$.
  - Base case is $\Lambda_0\Sigma_0 = \langle\rangle$, covering 0 reads by Π.
- Now we look for a sequence of operations γ that change what Π returns in $\Lambda_k\gamma\Sigma_k\Pi$ (the object is **perturbable** if such a sequence always exists).
  - For a max register, let γ include a bigger write than all the others.
  - For a counter, let γ include at least n increments. The same works for a mod-m counter if m is at least $2n$.
    - Why n increments? With fewer increments, we can make Π return the same value by being sneaky about when the partial increments represented in Σ_k are linearized.
  - In contrast, historyless objects (including atomic registers) are not perturbable: if Σ_k includes a write that sets the value of the object, no set of operations inserted before it will change this value. (This is good, because we know that it only takes one atomic register to implement an atomic register.)
- Such a γ must write to some register not covered in Σ_k.
- Find a γ' that writes to the first uncovered register that Π looks at (if none exists, the reader is wasting a step), truncate before that write, and prepend the write to Σ_k.
  - In more detail: let $\gamma' = \alpha\beta\delta$, where β is the first write by γ' to the first register read by Π that is not covered by Σ_k. Let $\Lambda_{k+1} = \Lambda_k\alpha$ and $\Sigma_{k+1} = \beta\Sigma_k$. So now $\Lambda_{k+1}\Sigma_{k+1}\Pi = \Lambda_k\alpha\beta\Sigma_k\Pi$ and in particular Σ_{k+1} covers the first $k+1$ registers read by Π.

  - Note: γ' might be *much longer* than γ (this will be important later, when we want to get around the JTT lower bound).

- Repeat until we've covered $n-1$ registers. This implies that there *are* at least $n-1$ registers, and in the worst case a reader reads all of them.

Chapter 21

Restricted-use objects

Here we are describing work by Aspnes, Attiya, and Censor [AAC09], plus some extensions by Aspnes *et al.* [AACHE12] and Aspnes and Censor-Hillel [ACH13]. The idea is to place restrictions on the size of objects that would otherwise be subject to the Jayanti-Tan-Toueg bound [JTT00] (see Chapter 20), in order to get cheap implementations.

The central object that is considered in this work is a **max register**, for which read operation returns the largest value previously written, as opposed to the last value previously written. So after writes of 0, 3, 5, 2, 6, 11, 7, 1, 9, a read operation will return 11.

These are perturbable objects in the sense of the Jayanti-Tan-Toueg bound, so in the worst case a max-register read will have to read at least $n-1$ distinct atomic registers, giving an $n-1$ lower bound on both individual work and space. But we can get around this by considering bounded max registers (which only hold values in some range $0 \dots m-1$); these are not perturbable because once one hits its upper bound we can no longer insert new operations to change the value returned by a read.

21.1 Implementing bounded max registers

For $m = 1$, the implementation is trivial: write does nothing and read always returns 0.

For larger m, we'll show how to paste together two max registers left and right with m_0 and m_1 values together to get a max register r with $m_0 + m_1$ values. We'll think of each value stored in the max register as a bit-vector, with bit-vectors ordered lexicographically. In addition to left and right, we will need a 1-bit atomic register switch used to choose between them. The

read procedure is straightforward and is shown in Algorithm 21.1; essentially we just look at switch, read the appropriate register, and prepend the value of switch to what we get.

```
1 procedure read(r)
2     if switch = 0 then
3         return 0(read(left))
4     else
5         return 1(read(right))
```

Algorithm 21.1: Max register read operation

For write operations, we have two somewhat asymmetrical cases depending on whether the value we are writing starts with a 0 bit or a 1 bit. These are shown in Algorithm 21.2.

```
1 procedure write(r, 0x)
2     if switch = 0 then
3         write(left, x)
4 procedure write(r, 1x)
5     write(right, x)
6     switch ← 1
```

Algorithm 21.2: Max register write operations

The intuition is that the max register is really a big tree of switch variables, and we store a particular bit-vector in the max register by setting to 1 the switches needed to make `read` follow the path corresponding to that bit-vector. The procedure for writing $0x$ tests switch first, because once switch gets set to 1, any $0x$ values are smaller than the largest value, and we don't want them getting written to left where they might confuse particularly slow readers into returning a value we can't linearize. The procedure for writing $1x$ sets switch second, because (a) it doesn't need to test switch, since $1x$ always beats $0x$, and (b) it's not safe to send a reader down into right until some value has actually been written there.

It's easy to see that `read` and `write` operations both require exactly one operation per bit of the value read or written. To show that we get linearizability, we give an explicit linearization ordering (see the paper for a full proof that this works):

1. All operations that read 0 from switch go in the first pile.

   (a) Within this pile, we sort operations using the linearization ordering for left.

2. All operations that read 1 from switch or write 1 to switch go in the second pile, which is ordered after the first pile.

   (a) Within this pile, operations that touch right are ordered using the linearization ordering for right. Operations that don't (which are the "do nothing" writes for $0x$ values) are placed consistently with the actual execution order.

To show that this gives a valid linearization, we have to argue first that any `read` operation returns the largest earlier `write` argument and that we don't put any non-concurrent operations out of order.

For the first part, any `read` in the 0 pile returns 0`read`(left), and `read`(left) returns (assuming left is a linearizable max register) the largest value previously written to left, which will be the largest value linearized before the `read`, or the all-0 vector if there is no such value. In either case we are happy. Any `read` in the 1 pile returns 1`read`(right). Here we have to guard against the possibility of getting an all-0 vector if no `write` operations linearize before the `read`. But any `write` operation that writes $1x$ doesn't set switch to 1 until after it writes to right, so no `read` operation ever starts `read`(right) until after at least one `write` to right has completed, implying that that `write` to right linearizes before the `read` from right. So in this case as well all the second-pile operations linearize.

21.2 Encoding the set of values

If we structure our max register as a balanced tree of depth k, we are essentially encoding the values $0 \ldots 2^k - 1$ in binary, and the cost of performing a read or write operation on an m-valued register is exactly $k = \lceil \lg m \rceil$. But if we are willing to build an unbalanced tree, any **prefix code** will work.

The paper describes a method of building a max register where the cost of each operation that writes or reads a value v is $O(\log v)$. The essential idea is to build a tree consisting of a rightward path with increasingly large left subtrees hanging off of it, where each of these left subtrees is twice as big as the previous. This means that after following a path encoded as 1^k0, we hit a 2^k-valued max register. The value returned after reading some v' from this max register is $v' + (2^k - 1)$, where the $2^k - 1$ term takes into account

all the values represented by earlier max registers in the chain. Formally, this is equivalent to encoding values using an **Elias gamma code**, tweaked slightly by changing the prefixes from 0^k1 to 1^k0 to get the ordering right.

21.3 Unbounded max registers

While the unbalanced-tree construction could be used to get an unbounded max register, it is possible that read operations might not terminate (if enough writes keep setting 1 bits on the right path before the read gets to them) and for very large values the cost even of terminating reads becomes higher than what we can get out of a snapshot.

Here is the snapshot-based method: if each process writes its own contribution to the max register to a single-writer register, then we can read the max register by taking a snapshot and returning the maximum value. (It is not hard to show that this is linearizable.) This gives an unbounded max register with read and write cost $O(n)$. So by choosing this in preference to the balanced tree when m is large, the cost of either operation on a max register is $\min(\lceil \lg m \rceil, O(n))$.

We can combine this with the unbalanced tree by terminating the right path with a snapshot-based max register. This gives a cost for reads and writes of values v of $O(\min(\log v, n))$.

21.4 Lower bound

The $\min(\lceil \lg m \rceil, n-1)$ cost of a max register read turns out to be exactly optimal. Intuitively, we can show by a covering argument that once some process attempts to write to a particular atomic register, then any subsequent writes convey no additional information (because they can be overwritten by the first delayed write)—so in effect, no algorithm can use get more than one bit of information out of each atomic register.

For the lower bound proof, we consider solo-terminating executions in which $n-1$ writers do any number of max-register writes in some initial prefix Λ, followed by a single max-register read Π by process p_n. Let $T(m,n)$ be the optimal reader cost for executions with this structure with m values, and let r be the first register read by process p_n, assuming it is running an algorithm optimized for this class of executions (we do not even require it to be correct for other executions).

We are now going split up our set of values based on which will cause a write to write to r. Let S_k be the set of all sequences of writes that only

write values $\leq k$. Let t be the smallest value such that some execution in S_t writes to r (there must be some such t, or our reader can omit reading r, which contradicts the assumption that it is optimal).

Case 1 Since t is smallest, no execution in S_{t-1} writes to r. If we restrict writes to values $\leq t-1$, we can omit reading r, giving $T(t,n) \leq T(m,n)-1$ or $T(m,n) \geq T(t,n)+1$.

Case 2 Let α be some execution in S_t that writes to r.

- Split α as $\alpha'\delta\beta$ where δ is the first write to r by some process p_i.
- Construct a new execution $\alpha'\eta$ by letting all the max-register writes except the one performing δ finish.
- Now consider any execution $\alpha'\eta\gamma\delta$, where γ is any sequence of max-register writes with values $\geq t$ that excludes p_i and p_n. Then p_n always sees the same value in r following these executions, but otherwise (starting after $\alpha'\eta$) we have an $(n-1)$-process max-register with values t through $m-1$.
- Omit the read of r again to get $T(m,n) \geq T(m-t,n-1)+1$.

We've shown the recurrence $T(m,n) \geq \min_t(\max(T(t,n),T(m-t,n)))+1$, with base cases $T(1,n)=0$ and $T(m,1)=0$. The solution to this recurrence is exactly $\min(\lceil \lg m \rceil, n-1)$, with is the same, except for a constant factor on n, as the upper bound we got by choosing between a balanced tree for small m and a snapshot for $m \geq 2^{n-1}$. For small m, the recursive split we get is also the same as in the tree-based algorithm: call the r register switch and you can extract a tree from whatever algorithm somebody gives you. So this says that the tree-based algorithm is (up to choice of the tree) essentially the unique optimal bounded max register implementation for $m \leq 2^{n-1}$.

It is also possible to show lower bounds on randomized implementations of max registers and other restricted-use objects. See [AAC09, AACHH12] for examples.

21.5 Max-register snapshots

With some tinkering, it's possible to extend the max-register construction to get an array of max registers that supports snapshots. The description in this section follows [AACHE12].

Formally, a **max array** is an object a that supports an operation $\texttt{write}(a, i, v)$ that sets $a[i] \leftarrow \max(v, a[i])$, and an operation $\texttt{read}(a)$ that returns a snapshot of all components of the array. The first step in building this beast is to do it for two components. The resulting 2-component max array can then be used as a building block for larger max arrays and for fast restricted-used snapshots in general.

A $k \times \ell$ max array a is one that permits values in the range $0 \ldots k-1$ in $a[0]$ and $0 \ldots \ell - 1$ in $a[1]$. We think of $a[0]$ as the head of the max array and $a[1]$ as the tail. We'll show how to construct such an object recursively from smaller objects of the same type, analogous to the construction of an m-valued max register (which we can think of as a $m \times 1$ max array). The idea is to split head into two pieces left and right as before, while representing tail as a master copy stored in a max register at the top of the tree plus cached copies at every internal node. These cached copies are updated by readers at times carefully chosen to ensure linearizability.

The base of the construction is an ℓ-valued max register r, used directly as a $1 \times \ell$ max array; this is the case where the head component is trivial and we only need to store $a.\mathsf{tail} = r$. Here calling $\texttt{write}(a, 0, v)$ does nothing, while $\texttt{write}(a, 1, v)$ maps to $\texttt{write}(r, v)$, and $\texttt{read}(a)$ returns $\langle 0, \texttt{read}(r)\rangle$.

For larger values of k, paste a $k_1 \times \ell$ max array left and a $k_2 \times \ell$ max array right together to get a $(k_1 + k_2) \times \ell$ max array. This construction uses a switch variable as in the basic construction, along with an ℓ-valued max register tail that is used to store the value of $a[1]$. A call to $\texttt{write}(a, 1, v)$ operation writes tail directly, while $\texttt{write}(a, 0, v)$ and $\texttt{read}(a)$ follow the structure of the corresponding operations for a simple max register, with some extra work in read to make sure that the value in tail propagates into left and right as needed to ensure the correct value is returned.

Pseudocode is given in Algorithm 21.3.

The individual step complexity of each operation is easily computed. Assuming a balanced tree, $\texttt{write}(a, 0, v)$ takes exactly $\lg k$ steps, while $\texttt{write}(a, 1, v)$ costs exactly $\lg \ell$ steps; in both cases the cost is identical to that of a max-register write. Read operations are more complicated. In the worst case, we have two reads of $a.\mathsf{tail}$ and a write to $a.\mathsf{right}[1]$ at each level, plus up to two operations on $a.\mathsf{switch}$, for a total cost of at most $(3 \lg k - 1)(\lg \ell + 2) = O(\log k \log \ell)$ steps.

In the special case where $k = \ell$, we get that writes cost the same number of steps as in a single-component k-valued max register while the cost of reads is squared.

```
1  procedure write(a, i, v)
2      if i = 0 then
3          if v < k_1 then
4              if a.switch = 0 then
5                  write(a.left, 0, v)
6          else
7              write(a.right, 0, v − k_1)
8              a.switch ← 1
9      else
10         write(a.tail, v)

11 procedure read(a)
12     x ← read(a.tail)
13     if a.switch = 0 then
14         write(a.left, 1, x)
15         return read(a.left)
16     else
17         x ← read(a.tail)
18         write(a.right, 1, x)
19         return ⟨k_1, 0⟩ + read(a.right)
```

Algorithm 21.3: Recursive construction of a 2-component max array

21.5.1 Linearizability

In broad outline, the proof of linearizability follows the proof for a simple max register. But as with snapshots, we have to show that the ordering of the head and tail components are consistent.

The key observation is the following lemma.

Lemma 21.5.1. *Fix some execution of a max array a implemented as in Algorithm 21.3. Suppose this execution contains a* `read`(a) *operation* π_{left} *that returns* v_{left} *from* $a.\mathsf{left}$ *and a* `read`(a) *operation* π_{right} *that returns* v_{right} *from* $a.\mathsf{right}$*. Then* $v_{\mathsf{left}}[1] \leq v_{\mathsf{right}}[1]$.

Proof. Both $v_{\mathsf{left}}[1]$ and $v_{\mathsf{right}}[1]$ are values that were previously written to their respective max arrays by `read`(a) operations (such writes necessarily exist because any process that reads $a.\mathsf{left}$ or $a.\mathsf{right}$ writes $a.\mathsf{left}[1]$ or $a.\mathsf{right}[1]$ first). From examining the code, we have that any value written to $a.\mathsf{left}[1]$ was read from $a.\mathsf{tail}$ before $a.\mathsf{switch}$ was set to 1, while any value written to $a.\mathsf{right}[1]$ was read from $a.\mathsf{tail}$ after $a.\mathsf{switch}$ was set to 1. Since max-register reads are non-decreasing, we have than any value written to $a.\mathsf{left}[1]$ is less than or equal to any value written to $a.\mathsf{right}[1]$, proving the claim. □

The rest of the proof is tedious but straightforward: we linearize the `read`(a) and `write`$(a[0])$ operations as in the max-register proof, then fit the `write`$(a[1])$ operations in based on the tail values of the reads. The full result is:

Theorem 21.5.2. *If $a.\mathsf{left}$ and $a.\mathsf{right}$ are linearizable max arrays, and $a.\mathsf{tail}$ is a linearizable max register, then Algorithm 21.3 implements a linearizable max array.*

It's worth noting that the same unbalanced-tree construction used in §§21.2 and 21.3 can be used here as well; this gives a cost of $O(\min(\log v, n))$ for writes and $O(\min(\log v[0], n) \cdot \min(\log v[1], n))$ for reads, where v is the value written or read.

21.5.2 Application to standard snapshots

To build an ordinary snapshot object from 2-component max arrays, we construct a balanced binary tree in which each leaves holds a pointer to an individual snapshot element and each internal node holds a pointer to a partial snapshot containing all of the elements in the subtree of which

it is the root. The pointers themselves are non-decreasing indices into arrays of values that consist of ordinary (although possibly very wide) atomic registers.

When a process writes a new value to its component of the snapshot object, it increases the pointer value in its leaf and then propagates the new value up the tree by combining together partial snapshots at each step, using 2-component max arrays to ensure linearizability. The resulting algorithm is similar in many ways to the lattice agreement procedure of Inoue *et al.* [IMCT94] (see §19.3.5), except that it uses a more contention-tolerant snapshot algorithm than double collects and we allow processes to update their values more than once. It is also similar to some constructions of Jayanti [Jay02] for efficient computation of array aggregates (sum, min, max, etc.) using LL/SC, the main difference being that because the index values are non-decreasing, max arrays can substitute for LL/SC.

Each node in the tree except the root is represented by one component of a 2-component max array that we can think of as being owned by its parent, with the other component being the node's sibling in the tree. To propagate a value up the tree, at each level the process takes a snapshot of the two children of the node and writes the sum of the indices to the node's component in its parent's max array (or to an ordinary max register if we are at the root). Before doing this last write, a process will combine the partial snapshots from the two child nodes and write the result into a separate array indexed by the sum. In this way any process that reads the node's component can obtain the corresponding partial snapshot in a single register operation. At the root this means that the cost of obtaining a complete snapshot is dominated by the cost of the max-register read, at $O(\log v)$, where v is the number of updates ever performed.

A picture of this structure, adapted from [AACHE12], appears in Figure 21.1. The figure depicts an update in progress, with red values being the new values written as part of the update. Only some of the tables associated with the nodes are shown.

The cost of an update is dominated by the $O(\log n)$ max-array operations needed to propagate the new value to the root. This takes $O(\log^2 v \log n)$ steps.

The linearizability proof is trivial: linearize each update by the time at which a snapshot containing its value is written to the root (which necessarily occurs within the interval of the update, since we don't let an update finish until it has propagated its value to the top), and linearize reads by when they read the root. This immediately gives us an $O(\log^3 n)$ implementation—as long as we only want to use it polynomially many times—of anything

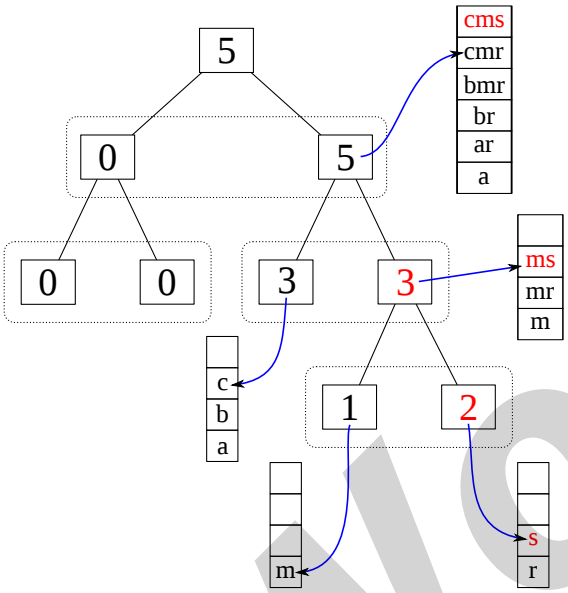


Figure 21.1: Snapshot from max arrays [AACHE12]

we can build from snapshot, including counters, generalized counters, and (by [AH90b, AM93]) any other object whose operations all commute with or overwrite each other in a static pattern.

Randomization can eliminate the need to limit the number of times the snapshot is used. The JTT bound still applies, so there will be occasional expensive operations, but we can spread these out randomly so that any particular operation has low expected cost. This gives a cost of $O(\log^3 n)$ expected steps for an unrestricted snapshot. See [ACH13] for details.

In some cases, further improvements are possible. The original max-registers paper [AAC09] gives an implementation of counters using a similar tree construction with only max registers that costs only $O(\log^2 n)$ for increments; here the trick is to observe that counters that can only be incremented by one are much easier to make linearizable, because there is no possibility of seeing an intermediate value that couldn't be present in a sequential execution.

Chapter 22

Common2

The **common2** class, defined by Afek, Weisberger, and Weisman [AWW93] consists of all read-modify-write objects where the modify functions either (a) all commute with each other or (b) all overwrite each other. We can think of it as the union of two simpler classes, the set of read-modify-write objects where all update operations commute, called **commuting objects** [AW99]; and the set of read-modify-write objects where all updates produce a value that doesn't depend on the previous state, called **historyless objects** [FHS98]).

From §18.1.2, we know that both commuting objects and historyless objects have consensus number at most 2, and that these objects have consensus number exactly 2 provided they supply at least one non-trivial update operation. The main result of Afek *et al.* [AWW93] is that commuting and historyless objects can all be implemented from any object with consensus number 2, even in systems with more than 2 processes. This gives a **completeness** result analogous to completeness results in complexity theory: any non-trivial common2 object can be used to implement any other common2 object.

The main result in the paper has two parts, reflecting the two parts of the common2 class: a proof that 2-process consensus plus registers is enough to implement all commuting objects (which essentially comes down to build a generalized fetch-and-add that returns an unordered list of all preceding operations); and a proof that 2-process consensus plus registers is enough to implement all overwriting objects (which is equivalent to showing that we can implement swap objects). The construction of the generalized fetch-and-add is pretty nasty, so we'll concentrate on the implementation of swap objects, limiting ourselves specifically to construction of single-use swap. For

the remaining results, you'll have to go to the paper itself [AWW93].

22.1 Test-and-set and swap for two processes

The first step is to get test-and-set.

Algorithm 22.1 shows how to turn 2-process consensus into 2-process test-and-set. The idea is that whoever wins the consensus protocol wins the test-and-set. This is linearizable, because if I run `TAS2` before you do, I win the consensus protocol by validity.

```
1 procedure TAS2()
2     if Consensus2(myId) = myId then
3         return 0
4     else
5         return 1
```

Algorithm 22.1: Building 2-process TAS from 2-process consensus

Once we have test-and-set for two processes, we can easily get one-shot swap for two processes. The trick is that a one-shot swap object always returns $\perp$ to the first process to access it and returns the other process's value to the second process. We can distinguish these two roles using test-and-set and add a register to send the value across. Pseudocode is in Algorithm 22.2.

```
1 procedure swap(v)
2     a[myId] = v
3     if TAS2() = 0 then
4         return ⊥
5     else
6         return a[¬myId]
```

Algorithm 22.2: Two-process one-shot swap from TAS

22.2 Building n-process TAS from 2-process TAS

To turn the `TAS2` into full-blown n-process `TAS`, start by staging a tournament along the lines of [PF77] (§17.4.1.2). Each process walks up a tree of nodes, and at each node it attempts to beat every process from the other

subtree using a $\texttt{TAS}_2$ object (we can't just have it fight one process, because we don't know which other process will have won the other subtree). A process drops out if it ever sees a 1. We can easily show that at most one process leaves each subtree with all zeros, including the whole tree itself.

Unfortunately, this process does not give a *linearizable* test-and-set object. It is possible that p_1 loses early to p_2, but then p_3 starts (elsewhere in the tree) after p_1 finishes, and races to the top, beating out p_2. To avoid this, we can follow [AWW93] and add a gate bit that locks out latecomers.[1]

The resulting construction looks something like Algorithm 22.3. This gives a slightly different interface that straight TAS; instead of returning 0 for winning and 1 for losing, the algorithm returns $\perp$ for winning and the id of some process that beats you for losing. It's not hard to see that this gives a linearizable test-and-set after translating the values back to 0 and 1 (the trick for linearizability is that any process that wins saw an empty gate, and so started before any other process finished). It also sorts the processes into a rooted tree, with each process linearizing after its parent (this latter claim is a little trickier, but basically comes down to a loser linearizing after the process that defeated it either on gate or on one of the TAS2 objects).

```
1  procedure compete(i)
       // check the gate
2      if gate ≠ ⊥ then
3          return gate
4      gate ← i
       // Do tournament, returning id of whoever I lose to
5      node ← leaf for i
6      while node ≠ root do
7          for each j whose leaf is below sibling of node do
8              if TAS2(t[i, j]) = 1 then
9                  return j
10         node ← node.parent
       // I win!
11     return ⊥
```

Algorithm 22.3: Tournament algorithm with gate

[1] The original version of this trick is from an earlier paper [AGTV92], where the gate bit is implemented as an array of single-writer registers.

22.3 Single-use swap objects

Here we'll show how to implement a **single-use swap object**, where each process is only allowed to execute a single `swap` operation. The essential idea is to explicitly string the processes into a sequence, where each process learns the identity of the process ahead of it. This sequence gives the linearization order and allows processes to compute their return values by reading the input stored by their predecessor.

The algorithm proceeds in asynchronous rounds, with the participants of each round organized into a tree using the `compete` procedure from Algorithm 22.3. The winner at round k will attempt to thread itself behind some process at round $k-1$, starting with the process it lost to at that round (or nobody if $k = 1$). In order for this to work, the round $k-1$ process must be locked down to round $k-1$ (and thread itself behind some other process at round $k-1$); this is done using a "trap" object implemented with a 2-process swap. If the target process escapes by calling the trap object first, it will leave behind the id of the process it lost to at round $k-1$, allowing the round-k winner to try again. If the round-k winner fails to trap anybody, it will eventually thread itself behind the round-$(k-1)$ winner, who is stuck at round $k-1$.

Only those processes that are ancestors of the process that beat the round-k winner may get trapped in round $k-1$; everybody else will escape and try again in a later round.

Pseudocode for the trap object is given in Algorithm 22.4. There are two operations. The `pass` operation is called by the process trying to escape; if it executes first, this process successfully escapes, but leaves behind the identity of somebody else to try. The `block` operation locks the target down so that `pass` fails. The shared data for a trap t consists of a two-process swap object $t[i, j]$ for each process i trying to block a process j. A utility procedure `passAll` is included that calls `pass` on all potential blockers until it fails.

It is not hard to see from the code that Algorithm 22.4 has the desired properties: if the passer reaches the swap object first, it is not blocked but leaves behind its value v for the passer; while if the blocker reaches the object first, it obtains no value but successfully blocks the passer.

The full swap construction is given in Algorithm 22.5.

This just implements the blocking strategy described before, with the main `swap` procedure implementing the round structure and the `findValue` helper procedure implementing the walk up the tree.

It's not hard to see that when all processes finish the protocol, they will

```
1  procedure block(t, j)
2      return swap(t[j, i], blocked)

3  procedure pass(t, j, v)
4      if swap(t[i, j], v) = blocked then
5          return false
6      else
7          return true

8  procedure passAll(t, v)
9      for j ← 1 to n do
10         if ¬pass(t, j, v) then return false
11
12     return true
```

Algorithm 22.4: Trap implementation from [AWW93]

be neatly arranged in a chain with each process except the first obtaining the value of its predecessor. Slightly less easy to see is that this ordering will be consistent with the observed execution order, which is necessary for linearizability. For a proof of linearizability, see the paper.

```
1  procedure swap (v)
2      input[i] ← v
3      for k ← 1 to n do
           // find our first target
4          t ← compete(tournament[k])
5          if t = ⊥ then
               // I am the round-k winner
6              return findValue(k − 1, t′)
7          else if ¬passAll(trap[k], t) do
               // I did not escape
8              return findValue(k, t)
9          else
               // I escaped, remember who I lost to
10             t′ ← t

11 procedure findValue(k, t)
12     if k = 0 then
13         return ⊥
14     else
15         repeat
16             x ← block(trap[k], t)
17             if x ≠ ⊥ then t ← x
18         until x = ⊥
19         return input[t]
```

Algorithm 22.5: Single-use swap from [AWW93]

Chapter 23

Randomized consensus and test-and-set

We've seen that we can't solve **consensus** in an asynchronous system with one crash failure [FLP85, LAA87], but that the problem becomes solvable using failure detectors [CT96]. An alternative that also allows us to solve consensus is to allow the processes to use randomization, by providing each process with a **local coin** that can generate random values that are immediately visible only to that process. The resulting **randomized consensus** problem replaces the **termination** requirement with **probabilistic termination**: all processes terminate with probability 1. The agreement and validity requirements remain the same.

In this chapter, we will describe how randomization interacts with the adversary, give a bit of history of randomized consensus, and then concentrate on recent algorithms for randomized consensus and the closely-related problem of randomized test-and-set. Much of the material in this chapter is adapted from notes for a previous course on randomized algorithms [Asp11] and my own recent papers [Asp12b, AE11, Asp12a].

23.1 Role of the adversary in randomized algorithms

Because randomized processes are unpredictable, we need to become a little more sophisticated in our handling of the adversary. As in previous asynchronous protocols, we assume that the adversary has control over timing, which we model by allowing the adversary to choose at each step which process performs the next operation. But now the adversary may do so

based on knowledge of the state of the protocol and its past evolution. How much knowledge we give the adversary affects its power. Several classes of adversaries have been considered in the literature; ranging from strongest to weakest, we have:

1. An **adaptive adversary**. This adversary is a function from the state of the system to the set of processes; it can see everything that has happened so far (including coin-flips internal to processes that have not yet been revealed to anybody else), but can't predict the future. It's known that an adaptive adversary can force any randomized consensus protocol to take $\Theta(n^2)$ total steps [AC08]. The adaptive adversary is also called a **strong adversary** following a foundational paper of Abrahamson [Abr88].

2. An **intermediate adversary** or **weak adversary** [Abr88] is one that limits the adversary's ability to observe or control the system in some way, without completely eliminating it. For example, a **content-oblivious adversary** [Cha96] or **value-oblivious adversary** [Aum97] is restricted from seeing the values contained in registers or pending write operations and from observing the internal states of processes directly. A **location-oblivious adversary** [Asp12b] can distinguish between values and the types of pending operations, but can't discriminate between pending operations based one which register they are operating on. These classes of adversaries are modeled by imposing an equivalence relation on partial executions and insisting that the adversary make the same choice of processes to go next in equivalent situations. Typically they arise because somebody invented a consensus protocol for the oblivious adversary below, and then looked for the next most powerful adversary that still let the protocol work.

   Weak adversaries often allow much faster consensus protocols than adaptive adversaries. Each of the above adversaries permits consensus to be achieved in $O(\log n)$ expected individual work using an appropriate algorithm. But from a mathematical standpoint, weak adversaries are a bit messy, and once you start combining algorithms designed for different weak adversaries, it's natural to move all the way down to the weakest reasonable adversary, the oblivious adversary described below.

3. A **oblivious adversary** has no ability to observe the system at all; instead, it fixes a sequence of process ids in advance, and at each step the next process in the sequence runs.

We will describe below a protocol that guarantees $O(\log \log n)$ expected individual work for an oblivious adversary. It is not known whether this is optimal; in fact, is is consistent with the best known lower bound (due to Attiya and Censor [AC08]) that consensus can be solved in $O(1)$ expected individual steps against an oblivious adversary.

23.2 History

The use of randomization to solve consensus in an asynchronous system with crash failures was proposed by Ben-Or *et al.*Ben-Or1983 for a message-passing model. Chor, Israeli, and Li [CIL94] gave the first wait-free consensus protocol for a shared-memory system, which assumed a particular kind of weak adversary. Abrahamson [Abr88] defined strong and weak adversaries and gave the first wait-free consensus protocol for a strong adversary; its expected step complexity was $\Theta\left(2^{n^2}\right)$. After failing to show that exponential time was necessary, Aspnes and Herlihy [AH90a] showed how to do consensus in $O(n^4)$ total work, a value that was soon reduced to $O(n^2 \log n)$ by Bracha and Rachman [BR91]. This remained the best known bound for the strong-adversary model until Attiya and Censor [AC08] showed matching $\Theta(n^2)$ upper and lower bounds for the problem; subsequent work [AC09] showed that it was also possible to get an $O(n)$ bound on individual work.

For weak adversaries, the best known upper bound on individual step complexity was $O(\log n)$ for a long time [Cha96, Aum97, Asp12b], with an $O(n)$ bound on total step complexity for some models [Asp12b]. More recent work has lowered the bound to $O(\log \log n)$, under the assumption of an oblivious adversary [Asp12a]. No non-trivial lower bound on expected individual step complexity is known, although there is a known lower bound on the distribution of of the individual step complexity [ACH10].

23.3 Reduction to simpler primitives

To show how to solve consensus using randomization, it helps to split the problem in two: we will first see how to detect *when* we've achieved agreement, and then look at *how* to achive agreement.

23.3.1 Adopt-commit objects

Most known randomized consensus protocols have a round-based structure where we alternative between generating and detecting agreement. Gafni [Gaf98] proposed **adopt-commit protocols** as a tool for detecting agreement, and these protocols were later abstracted as **adopt-commit objects** [MRRT08, AGGT09]. The version described here is largely taken from [AE11], which shows bounds on the complexity of adopt-commit objects.

An adopt-commit object supports a single operation, `AdoptCommit` (u), where u is an input from a set of m **values**. The result of this operation is an output of the form (commit, v) or (adopt, v), where the second component is a value from this set and the first component is a **decision bit** that indicates whether the process should decide value v immediately or adopt it as its preferred value in later rounds of the protocol.

The requirements for an adopt-commit object are the usual requirements of validity and termination, plus:

1. **Coherence.** If the output of some operation is (commit, v), then every output is either (adopt, v) or (commit, v).

2. **Convergence.** If all inputs are v, all outputs are (commit, v).

These last two requirement replace the agreement property of consensus. They are also strictly weaker than consensus, which means that a consensus object (with all its output labeled commit) is also an adopt-commit object.

The reason we like adopt-commit objects is that they allow the simple consensus protocol shown in Algorithm 23.1.

```
1 preference ← input
2 for r ← 1 ... ∞ do
3     (b, preference) ← AdoptCommit(AC[r], preference)
4     if b = commit then
5         return preference
6     else
7         do something to generate a new preference
```

Algorithm 23.1: Consensus using adopt-commit

The idea is that the adopt-commit takes care of ensuring that once somebody returns a value (after receiving commit), everybody else who doesn't

return adopts the same value (follows from coherence). Conversely, if everybody already has the same value, everybody returns it (follows from convergence). The only missing piece is the part where we try to shake all the processes into agreement. For this we need a separate object called a *conciliator*.

23.3.2 Conciliators

Conciliators are a weakened version of randomized consensus that replace agreement with **probabilistic agreement**: it's OK if the processes disagree sometimes as long as they agree with constant probability despite interference by the adversary. An algorithm that satisfies termination, validity, and probabilistic agreement is called a **conciliator**.[1]

The important feature of conciliators is that if we plug a conciliator that guarantees agreement with probability at least δ into Algorithm 23.1, then on average we only have to execute the loop $1/\delta$ times before every process agrees. This gives an expected cost equal to $1/\delta$ times the total cost of `AdoptCommit` and the conciliator. Typically we will aim for constant δ.

23.4 Implementing an adopt-commit object

What's nice about adopt-commit objects is that they can be implemented deterministically. Here we'll give a simple adopt-commit object for two values, 0 and 1. Optimal (under certain assumptions) constructions of m-valued adopt-commits can be found in [AE11].

Pseudocode is given in Algorithm 23.2.

Structurally, this is pretty similar to a splitter (see §17.4.2, except that we use values instead of process ids.

We now show correctness. Termination and validity are trivial. For coherence, observe that if I return (commit, v) I must have read $a[\neg v] = \textbf{false}$ before any process with $\neg v$ writes $a[\neg v]$; it follows that all such processes will see $\mathit{proposal} \neq \perp$ and return (adopt, v). For convergence, observe that if all processes have the same input v, they all write it to *proposal* and all observe $a[\neg v] = \textbf{false}$, causing them all to return (commit, v).

[1] Warning: This name has not really caught on in the general theory-of-distributed-computing community, and so far only appears in papers that have a particular researcher as a co-author [Asp12a, AE11, Asp12b]. Unfortunately, there doesn't seem to be a better name for the same object that has caught on. So we are stuck with it for now.

```
   shared data: a[0], a[1], initially false; proposal, initially ⊥
 1 procedure AdoptCommit(v)
 2     a[v] ← 1
 3     if proposal = ⊥ then
 4         proposal ← v
 5     else
 6         v ← proposal
 7     if a[¬v] = false then
 8         return (commit, v)
 9     else
10         return (adopt, v)
```

Algorithm 23.2: A 2-valued adopt-commit object

23.5 A one-register conciliator for an oblivious adversary

```
   shared data: register r, initially ⊥
 1 k ← 0
 2 while r = ⊥ do
 3     with probability 2^k/2n do
 4         write v to r
 5     else
 6         do a dummy operation
 7     k ← k + 1
 8 return r
```

Algorithm 23.3: Impatient first-mover conciliator from [Asp12b]

Algorithm 23.3 implements a conciliator using a single register; it works against an oblivious adversary.[2] This particular construction is taken from [Asp12b], and is based on an earlier algorithm of Chor, Israeli, and Li [CIL94]. The cost of this algorithm is expected $O(n)$ total work and $O(\log n)$ individual work. It's not known whether it is possible to improve on this bound.

The basic idea is that processes alternate between reading a register r

[2]Or any adversary dumb enough not to be able to block the write based on how the coin-flip turned out.

and (maybe) writing to the register; if a process reads a non-null value from the register, it returns it. Any other process that reads the same non-null value will agree with the first process; the only way that this can't happen is if some process writes a different value to the register before it notices the first write.

The random choice of whether to write the register or not avoids this problem. The idea is that even though the adversary can schedule a write at a particular time, because it's oblivious, it won't be able to tell if the process wrote (or was about to write) or did a no-op instead.

The basic version of this algorithm, due to Chor, Israeli, and Li [CIL94], uses a fixed $\frac{1}{2n}$ probability of writing to the register. So once some process writes to the register, the chance that any of the remaining $n-1$ processes write to it before noticing that it's non-null is at most $\frac{n-1}{2n} < 1/2$. It's also not hard to see that this algorithm uses $O(n)$ total operations, although it may be that one single process running by itself has to go through the loop $2n$ times before it finally writes the register and escapes.

Using increasing probabilities avoids this problem, because any process that executes the main loop $\lceil \lg n \rceil + 1$ times will write the register. This establishes the $O(\log n)$ per-process bound on operations. At the same time, an $O(n)$ bound on total operations still holds, since each write has at least a $\frac{1}{2n}$ chance of succeeding. The price we pay for the improvement is that we increase the chance that an initial value written to the register gets overwritten by some high-probability write. But the intuition is that the probabilities can't grow too much, because the probability that I write on my next write is close to the sum of the probabilities that I wrote on my previous writes—suggesting that if I have a high probability of writing next time, I should have done a write already.

Formalizing this intuition requires a little bit of work. Fix the schedule, and let p_i be the probability that the i-th write operation in this schedule succeeds. Let t be the least value for which $\sum_{i=1}^{t} p_i \geq 1/4$. We're going to argue that with constant probability one of the first t writes succeeds, and that the next $n-1$ writes by different processes all fail.

The probability that none of the first t writes succeed is

$$\begin{aligned}\prod_{i=1}^{t}(1-p_i) &\leq \prod_{i=1}^{t} e^{-p_i} \\ &= \exp\left(\sum_{i=1}^{t} p_i\right) \\ &\leq e^{-1/4}.\end{aligned}$$

Now observe that if some process q writes at or before the t-th write, then any process with a pending write either did no writes previously, or its last write was among the first $t-1$ writes, whose probabilities sum to less than $1/4$. In the first case, the process has a $\frac{1}{2n}$ chance of writing on its next attempt. In the second, it has a $\sum_{i \in S_q} p_i + \frac{1}{2n}$ chance of writing on its next attempt, where S_q is the set of indices in $1 \dots t-1$ where q attempts to write.

Summing up these probabilities over all processes gives a total of $\frac{n-1}{2n} + \sum_q \sum_{i \in S_q} p_i \leq 1/2 + 1/4 = 3/4$. So with probabililty at least $e^{-1/4}(1-3/4) = e^{-1/4}/4$, we get agreement.

23.6 Sifters

A faster conciliator can be obtained using a **sifter**, which is a mechanism for rapidly discarding processes using randomization [AA11] while keeping at least one process around. The idea of a sifter is to have each process either write a register (with low probability) or read it (with high probability); all writers and all readers that see $\perp$ continue to the next stage of the protocol, while all readers who see a non-null value drop out. An appropriately-tuned sifter will reduce n processes to at most $2\sqrt{n}$ processes on average; by iterating this mechanism, the expected number of remaining processes can be reduced to $1 + \epsilon$ after $O(\log \log n + \log(1/\epsilon))$ phases.

As with previous implementations of test-and-set (see Algorithm 22.3), it's often helpful to have a sifter return not only that a process lost but which process it lost to. This gives the implementation shown in Algorithm 23.4.

```
1 procedure sifter(p, r)
2     with probability p do
3         r ← id
4         return ⊥
5     else
6         return r
```

Algorithm 23.4: A sifter

To use a sifter effectively, p should be tuned to match the number of processes that are likely to use it. This is because of the following lemma:

Lemma 23.6.1. *Fix p, and let X processes executed a sifter with parameter p. Let Y be the number of processes for which the sifter returns $\perp$. Then*

$$\mathrm{E}[X \mid Y] \leq pX + \frac{1}{p}. \tag{23.6.1}$$

Proof. In order to return $\perp$, a process must either (a) write to r, which occurs with probability p, or (b) read r before any other process writes to it. The expected number of writers, conditioned on X, is exactly pX. The expected number of readers before the first write has a geometric distribution truncated by X. Removing the truncation gives exactly $\frac{1}{p}$ expected readers, which is an upper bound on the correct value. □

For n initial processes, the choice of p that minimizes the bound in (23.6.1) is $\frac{1}{\sqrt{n}}$, giving at most $2\sqrt{n}$ expected survivors. Iterating this process with optimal p at each step gives a sequence of at most n, $2\sqrt{n}$, $2\sqrt{2\sqrt{n}}$, etc., expected survivors after each sifter. The twos are a little annoying, but a straightforward induction bounds the expected survivors after i rounds by $4 \cdot n^{2^{-i}}$. In particular, we get at most 8 expected survivors after $\lceil \lg \lg n \rceil$ rounds.

At this point it makes sense to switch to a fixed p and a different analysis. For $p = 1/2$, the first process to access r always survives, and each subsequent process survives with probability at most $3/4$ (because it leaves if the first process writes and it reads). So the number of "excess" processes drops as $(3/4)^i$, and an additional $\left\lceil \log_{4/3}(7/\epsilon) \right\rceil$ rounds are enough to reduce the expected number of survivors from $1 + 7$ to $1 + \epsilon$ for any fixed ϵ.[3]

It follows that

Theorem 23.6.2. *An initial set of n processes can be reduced to 1 with probability at least $1 - \epsilon$ using $O(\log \log n + \log(1/\epsilon))$ rounds of sifters.*

Proof. Let X be the number of survivors after $\lceil \lg \lg n \rceil + \left\lceil \log_{4/3}(7/\epsilon) \right\rceil$ rounds of sifters, with probabilities tuned as described above. We've shown that $\mathrm{E}[X] \leq 1 + \epsilon$, so $\mathrm{E}[X - 1] \leq \epsilon$. Since $X - 1 \geq 0$, from Markov's inequality we have $\Pr[X \geq 2] = \Pr[X - 1 \geq 1] \leq \mathrm{E}[X - 1]/1 \leq \epsilon$. □

[3] This argument essentially follows the proof of [Asp12a, Theorem 2], which, because of neglecting to subtract off a 1 at one point, ends up with $8/\epsilon$ instead of $7/\epsilon$.

23.6.1 Test-and-set using sifters

Sifters were initially designed to be used for test-and-set. For this purpose, we treat a return value of $\perp$ as "keep going" and anything else as "leave with value 1." Using $O(\log \log n)$ rounds of sifters, we can get down to one process that hasn't left with probability at least $1 - \log^{-c} n$ for any fixed constant c. We then need a fall-back TAS to handle the $\log^{-c} n$ chance that we get more than one such survivor.

Alistarh and Aspnes [AA11] used the `RatRace` algorithm of Alistarh *et al.* [AAG+10] for this purpose. This is an adaptive randomized test-and-set built from splitters and two-process consensus objects that runs in $O(\log k)$ expected time, where k is the number of processes that access the test-and-set; a sketch of this algorithm is given in §24.5.2. If we want to avoid appealing to this algorithm, a somewhat simpler approach is to use an approach similar to the Lamport's fast-path mutual exclusion algorithm (described in §17.4.2): any process that survives the splitters tries to rush to a two-process TAS at the top of a tree of two-processes TASes by winning a splitter, and if it doesn't win the splitter, it enters at a leaf and pays $O(\log n)$ expected steps. By setting $\epsilon = 1/\log n$, the overall expected cost of this final stage is $O(1)$.

This algorithm does not guarantee linearizability. I might lose a sifter early on only to have a later process win all the sifters (say, by writing to each one) and return 0. A `gate` bit as in Algorithm 22.3 solves this problem. The full code is given in Algorithm 23.5.

23.6.2 Consensus using sifters

With some trickery, the sifter mechanism can be adapted to solve consensus, still in $O(\log \log n)$ expected individual work [Asp12a]. The main difficulty is that a process can no longer drop out as soon as it knows that it lost: it still needs to figure out who won, and possible help that winner over the finish line.

The basic idea is that when a process p loses a sifter to some other process q, p will act like a clone of q from that point on. In order to make this work, each process writes down at the start of the protocol all of the coin-flips it intends to use to decide whether to read or write at each round of sifting. Together with its input, these coin-flips make up the process's **persona**. In analyzing the progress of the sifter, we count surviving personae (with multiple copies of the same persona counting as one) instead of surviving processes.

```
1  if gate ≠ ⊥ then
2  |  return 1
3  else
4  |  gate ← myId
5  |  for i ← 1 . . . ⌈log log n⌉ + ⌈log_{4/3}(7 log n)⌉ do
6  |  |  with probability min(1/2, 2^{1−2^{−i+1}}) do
7  |  |  |  r_i ← myId
8  |  |  else
9  |  |  |  w ← r_i
10 |  |  |  if w ≠ ⊥ then
11 |  |  |  |  return 1
12 if splitter() = stop then
13 |  return 0
14 else
15 |  return AWWTAS()
```

Algorithm 23.5: Test-and-set in $O(\log \log n)$ expected time

Pseudocode for this algorithm is given in Algorithm 23.6. Note that the loop body is essentially the same as the code in Algorithm 23.4, except that the random choice is replaced by a lookup in persona.chooseWrite.

To show that this works, we need to argue that having multiple copies of a persona around doesn't change the behavior of the sifter. In each round, we will call the first process with a given persona p to access r_i the **representative** of p, and argue that a persona survives round i in this algorithm precisely when its representative would survive round i in a corresponding test-and-set sifter with the schedule restricted only to the representatives.

There are three cases:

1. The representative of p writes. Then at least one copy of p survives.
2. The representative of p reads a null value. Again at least one copy of p survives.
3. The representative of p reads a non-null value. Then no copy of p survives: all subsequent reads by processes carrying p also read a non-null value and discard p, and since no process with p writes, no other process adopts p.

```
1  procedure conciliator(input)
2      Let R = ⌈log log n⌉ + ⌈log_{4/3}(7/ε)⌉
3      Let chooseWrite be a vector of R independent random Boolean
       variables with Pr[chooseWrite[i] = 1] = p_i, where
       p_i = 2^{1−2^{−i+1}} (n)^{−2^{−i}} for i ≤ ⌈log log n⌉ and p_i = 1/2 for larger i.
4      persona ← ⟨input, chooseWrite, myId⟩
5      for i ← 1 . . . R do
6          if persona.chooseWrite[i] = 1 then
7              r_i ← persona
8          else
9              v ← r_i
10             if v ≠ ⊥ then
11                 persona ← v
12     return persona.input
```

Algorithm 23.6: Sifting conciliator (from [Asp12a])

From the preceding analysis for test-and-set, we have that after $O(\log\log n + \log 1/\epsilon)$ rounds with appropriate probabilities of writing, at most $1+\epsilon$ values survive on average. This gives a probability of at most ϵ of disagreement. By alternating these conciliators with adopt-commit objects, we get agreement in $O(\log\log n + \log m/\log\log m)$ expected time, where m is the number of possible input values.

I don't think the $O(\log\log n)$ part of this expression is optimal, but I don't know how to do better.

23.7 $O(\log^* n)$ Randomized test-and-set

A more sophisticated sifter due to Giakkoupis and Woelfel [GW12a] removes all but $O(\log n)$ processes, on average, using two operations for each process. Iterating this sifter reduces the expected survivors to $O(1)$ in $O(\log^* n)$ rounds. A particularly nice feature of the Giakkoupis-Woelfel algorithm is that (if you don't care about space) it doesn't have any parameters that require tuning to n: this means that exactly the same structure can be used in each round. An unfortunate feature is that it's not possible to guarantee that every process that leaves learns the identity of a process that stays: this means that it can't adapted into a consensus protocol using the persona

trick described in §23.6.2.

Pseudocode is given in Algorithm 23.7. In this simplified version, we assume an infinitely long array $A[1\ldots]$, so that we don't need to worry about n. Truncating the array at $\log n$ also works, but the analysis requires handling the last position as a special case, which I am too lazy to do here.

```
1 Choose r ∈ ℤ⁺ such that Pr[r = i] = 2^{-i}
2 A[r] ← 1
3 if A[r + 1] = 0 then
4 |  stay
5 else
6 |  leave
```

Algorithm 23.7: Giakkoupis-Woelfel sifter [GW12a]

Lemma 23.7.1. *In any execution of Algorithm 23.7 with an oblivious adversary and n processes, at least one process stays, and the expected number of processes that stay is $O(\log n)$.*

Proof. For the first part, observe that any process that picks the largest value of r among all processes will survive; since the number of processes is finite, there is at least one such survivor.

For the second part, let X_i be the number of survivors with $r = i$. Then $\mathrm{E}[X_i]$ is bounded by $n \cdot 2^{-i}$, since no process survives with $r = i$ without first choosing $r = i$. But we can also argue that $\mathrm{E}[X_i] \leq 3$ for any value of n, by considering the sequence of write operations in the execution.

Because the adversary is oblivious, the location of these writes is uncorrelated with their ordering. If we assume that the adversary is trying to maximize the number of survivors, its best strategy is to allow each process to read immediately after writing, as delaying this read can only increase the probability that $A[r + 1]$ is nonzero. So in computing X_i, we are counting the number of writes to $A[i]$ before the first write to $A[i + 1]$. Let's ignore all writes to other registers; then the j-th write to either of $A[i]$ or $A[i + 1]$ has a conditional probability of $2/3$ of landing on $A[i]$ and $1/3$ on $A[i + 1]$. We are thus looking at a geometric distribution with parameter $1/3$, which has expectation 3.

Combining these two bounds gives $\mathrm{E}[X_i] \leq \min(3, 2^{-i})$. So then

$$\begin{aligned}\mathrm{E}[\text{survivors}] &\leq \sum_{i=1}^{\infty} \min(3, n \cdot 2^{-i}) \\ &= 3 \lg n + O(1),\end{aligned}$$

because once $n \cdot 2^{-i}$ drops below 3, the remaining terms form a geometric series. □

Like square root, logarithm is concave, so Jensen's inequality applies here as well. So $O(\log^* n)$ rounds of Algorithm 23.7 reduces us to an expected constant number of survivors, which can then be fed to RatRace.

With an adaptive adversary, all of the sifter-based test-and-sets fail badly: in this particular case, an adaptive adversary can sort the processes in order of increasing write location so that every process survives. The best known n-process test-and-set for an adaptive adversary is still a tree of 2-process randomized test-and-sets, as in the Afek *et al.* [AWW93] algorithm described in §22.2. Whether $O(\log n)$ expected steps is in fact necessary is still open (as is the exact complexity of test-and-set with an oblivious adversary).

23.8 Space bounds

A classic result of Fich, Herlihy, and Shavit [FHS98] shows that $\Omega(\sqrt{n})$ registers are needed to solve consensus even under the very weak requirement of **nondeterministic solo termination**, which says that for every reachable configuration and every process p, there exists some continuation of the execution in which the protocol terminates with only p running. The best known upper bound is the trivial n—one single-writer register per process—since any multi-writer register algorithm can be translated into a single-writer algorithm and (assuming wide enough registers) multiple registers of a single process can be combined into one.

There has been very little progress in closing the gap between these two bounds since the original conference version of the FHS paper from 1993, although very recently, Giakkoupis *et al.* [GHHW13] have shown a surprising $O(\sqrt{n})$-space algorithm for the closely related problem of leader election, which is basically test-and-set without guaranteeing linearizability. The main difference between leader election and consensus is that in consensus every process learns the identity of the winner, instead of just whether

it personally won or lost. It is not clear whether the techniques used for this problem could carry across to consensus.

Chapter 24

Renaming

We will start by following the presentation in [AW04, §16.3]. This mostly describes results of the original paper of Attiya *et al.* [ABND+90] that defined the renaming problem and gave a solution for message-passing; however, it's now more common to treat renaming in the context of shared-memory, so we will follow Attiya and Welch's translation of these results to a shared-memory setting.

24.1 Renaming

In the **renaming** problem, we have n processes, each starts with a name from some huge namespace, and we'd like to assign them each unique names from a much smaller namespace. The main application is allowing us to run algorithms that assume that the processes are given contiguous numbers, e.g. the various collect or atomic snapshot algorithms in which each process is assigned a unique register and we have to read all of the registers. With renaming, instead of reading a huge pile of registers in order to find the few that are actually used, we can map the processes down to a much smaller set.

Formally, we have a decision problem where each process has input x_i (its original name) and output y_i, with the requirements:

Termination Every nonfaulty process eventually decides.

Uniqueness If $p_i \neq p_j$, then $y_i \neq y_j$.

Anonymity The code executed by any process depends only on its input x_i: for any execution of processes $p_1 \dots p_n$ with inputs $x_1 \dots x_n$, and

any permutation π of $[1 \dots n]$, there is a corresponding execution of processes $p_{\pi(1)} \dots p_{\pi(n)}$ with inputs $x_1 \dots x_n$ in which $p_{\pi(i)}$ performs exactly the same operations as p_i and obtains the same output y_i.

The last condition is like non-triviality for consensus: it excludes algorithms where p_i just returns i in all executions. Typically we do not have to do much to prove anonymity other than observing that all processes are running the same code.

We will be considering renaming in a shared-memory system, where we only have atomic registers to work with.

24.2 Performance

Conventions on counting processes:

- N = number of possible original names.
- n = maximum number of processes.
- k = number of processes that actually execute the algorithm.

Ideally, we'd like any performance measures we get to depend on k alone if possible (giving an **adaptive** algorithm). Next best would be something polynomial in n and k. Anything involving N is bad.

We'd also like to minimize the size of the output namespace. How well we can do this depends on what assumptions we make. For deterministic algorithms using only read-write registers, a lower bound due to Herlihy and Shavit [HS99] shows that we can't get fewer than $2n - 1$ names for general n.[1] Our target thus will be exactly $2n-1$ output names if possible, or $2k-1$ if we are trying to be adaptive. For randomized algorithm, it is possible to solve **strong** or **tight** renaming, where the size of the namespace is exactly k; we'll see how to do this in §24.5.

A small note on bounds: There is a lot of variation in the literature on how bounds on the size of the output namespace are stated. The original Herlihy-Shavit lower bound [HS99] says that there is no general renaming algorithm that uses $2n$ names for $n + 1$ processes; in other words, any n-process algorithm uses at least $2n - 1$ names. Many subsequent papers

[1] This lower bound was further refined by Castañeda and Rajsbaum [CR08], who show that $2n - 2$ (but no less!) is possible for certain special values of n; all of these lower bounds make extensive use of combinatorial topology, so we won't try to present them here.

discussing lower bounds on the namespace follow the approach of Herlihy and Shavit and quote lower bounds that are generally 2 higher than the minimum number of names needed for n processes. This requires a certain amount of translation when comparing these lower bounds with upper bounds, which use the more natural convention.

24.3 Order-preserving renaming

Before we jump into upper bounds, let's do an easy lower bound from the Attiya *et al.* paper [ABND$^+$90]. This bound works on a variant of renaming called **order-preserving renaming**, where we require that $y_i < y_j$ whenever $x_i < x_j$. Unfortunately, this requires a very large output namespace: with t failures, any asynchronous algorithm for order-preserving renaming requires $2^t(n-t+1)-1$ possible output names. This lower bound applies regardless of the model, as long as some processes may start after other processes have already been assigned names.

For the wait-free case, we have $t = n-1$, and the bound becomes just 2^n-1. This is a simpler case than the general t-failure case, but the essential idea is the same: if I've only seen a few of the processes, I need to leave room for the others.

Theorem 24.3.1. *There is no order-preserving renaming algorithm for n processes using fewer than 2^n-1 names.*

Proof. By induction on n. For $n = 1$, we use $2^1 - 1 = 1$ names; this is the base case. For larger n, suppose we use m names, and consider an execution in which one process p_n runs to completion first. This consumes one name y_n and leaves k names less than y_n and $m-k-1$ names greater than y_n. By setting all the inputs x_i for $i < n$ either less than x_n or greater than x_n, we can force the remaining processes to choose from the remaining k or $m-k-1$ names. Applying the induction hypothesis, this gives $k \geq 2^{n-1}-1$ and $m-k-1 \geq 2^{n-1}-1$, so $m = k+(m-k-1)+1 \geq 2(2^{n-1}-1)+1 = 2^n-1$. □

24.4 Deterministic renaming

In **deterministic renaming**, we can't use randomization, and may or may not have any primitives stronger than atomic registers. With just atomic registers, we can only solve loose renaming; with test-and-set, we can solve tight renaming. In this section, we describe some basic algorithms for deterministic renaming.

24.4.1 Wait-free renaming with $2n-1$ names

Here we use Algorithm 55 from [AW04], which is an adaptation to shared memory of the message-passing renaming algorithm of [ABND+90]. One odd feature of the algorithm is that, as written, it is not anonymous: processes communicate using an atomic snapshot object and use their process ids to select which component of the snapshot array to write to. But if we think of the process ids used in the algorithm as the inputs x_i rather than the actual process ids i, then everything works. The version given in Algorithm 24.1 makes this substitution explicit, by treating the original name i as the input.

```
1  procedure getName()
2      s ← 1
3      while true do
4          a[i] ← s
5          view ← snapshot(a)
6          if view[j] = s for some j then
7              r ← |{j : view[j] ≠ ⊥ ∧ j ≤ i}|
8              s ← r-th positive integer not in
                   {view[j] : j ≠ i ∧ view[j] = ⊥}
9          else
10             return s
```

Algorithm 24.1: Wait-free deterministic renaming

The array a holds proposed names for each process (indexed by the original names), or $\perp$ for processes that have not proposed a name yet. If a process proposes a name and finds that no other process has proposed the same name, it takes it; otherwise it chooses a new name by first computing its rank r among the active processes and then choosing the r-th smallest name that hasn't been proposed by another process. Because the rank is at most n and there are at most $n-1$ names proposed by the other processes, this always gives proposed names in the range $[1 \ldots 2n-1]$. But it remains to show that the algorithm satisfies uniqueness and termination.

For uniqueness, consider two process with original names i and j. Suppose that i and j both decide on s. Then i sees a view in which $a[i] = s$ and $a[j] \neq s$, after which it no longer updates $a[i]$. Similarly, j sees a view in which $a[j] = s$ and $a[i] \neq s$, after which it no longer updates $a[j]$. If i's view is obtained first, then j can't see $a[i] \neq s$, but the same holds if j's view is

obtained first. So in either case we get a contradiction, proving uniqueness.

Termination is a bit trickier. Here we argue that no process can run forever without picking a name, by showing that if we have a set of processes that are doing this, the one with smallest original name eventually picks a name. More formally, call a process *trying* if it runs for infinitely many steps without choosing a name. Then in any execution with at least one trying process, eventually we reach a configuration where all processes have either finished or are trying. In some subsequent configuration, all the processes have written to the a array at least once; from this point on, the set of non-null positions in a—and thus the rank each process computes for itself—is stable.

Starting from some such stable configuration, look at the trying process i with the smallest original name, and suppose it has rank r. Let $F = \{z_1 < z_2 \dots\}$ be the set of "free names" that are not proposed in a by any of the finished processes. Observe that no trying process $j \neq i$ ever proposes a name in $\{z_1 \dots z_r\}$, because any such process has rank greater than r. This leaves z_r open for i to claim, provided the other names in $\{z_1 \dots z_r\}$ eventually become free. But this will happen, because only trying processes may have proposed these names (early on in the execution, when the finished processes hadn't finished yet), and the trying processes eventually propose new names that are not in this range. So eventually process i proposes z_r, sees no conflict, and finishes, contradicting the assumption that it is trying.

Note that we haven't proved any complexity bounds on this algorithm at all, but we know that the snapshot alone takes at least $\Omega(N)$ time and space. Brodksy *et al.* [BEW11] cite a paper of Bar-Noy and Dolev [BND89] as giving a shared-memory version of [ABND$^+$90] with complexity $O(n \cdot 4^n)$; they also give algorithms and pointers to algorithms with much better complexity.

24.4.2 Long-lived renaming

In **long-lived renaming** a process can release a name for later use by other processes (or the same process, if it happens to run choose-name again). Now the bound on the number of names needed is $2k-1$, where k is the maximum number of concurrently active processes. Algorithm 24.1 can be converted to a long-lived renaming algorithm by adding the `releaseName` procedure given in Algorithm 24.2. This just erases the process's proposed name, so that some other process can claim it.

Here the termination requirement is weakened slightly, to say that some process always makes progress in `getName`. It may be, however, that there is some process that never successfully obtains a name, because it keeps

```
1 procedure releaseName()
2   a[i] ← ⊥
```

Algorithm 24.2: Releasing a name

getting stepped on by other processes zipping in and out of `getName` and `releaseName`.

24.4.3 Renaming without snapshots

Moir and Anderson [MA95] give a renaming protocol that is somewhat easier to understand and doesn't require taking snapshots over huge arrays. A downside is that the basic version requires $k(k+1)/2$ names to handle k active processes.

24.4.3.1 Splitters

The Moir-Anderson renaming protocol uses a network of **splitters**, which we last saw providing a fast path for mutual exclusion in §17.4.2. Each splitter is a widget, built from a pair of atomic registers, that assigns to each processes that arrives at it the value right, down, or stop. As discussed previously, the useful properties of splitters are that if at least one process arrives at a splitter, then (a) at least one process returns right or stop; and (b) at least one process returns down or stop; (c) at most one process returns stop; and (d) any process that runs by itself returns stop.

We proved the last two properties in §17.4.2; we'll prove the first two here. Another way of describing these properties is that of all the processes that arrive at a splitter, some process doesn't go down and some process doesn't go right. By arranging splitters in a grid, this property guarantees that every row or column that gets at least one process gets to keep it—which means that with k processes, no process reaches row $k+1$ or column $k+1$.

Algorithm 24.3 gives the implementation of a splitter (it's identical to Algorithm 17.5, but it will be convenient to have another copy here).

Lemma 24.4.1. *If at least one process completes the splitter, at least one process returns* stop *or* right.

Proof. Suppose no process returns right; then every process sees open in door, which means that every process writes its id to race before any process

```
   shared data:
 1 atomic register race, big enough to hold an id, initially ⊥
 2 atomic register door, big enough to hold a bit, initially open
 3 procedure splitter(id)
 4     race ← id
 5     if door = closed then
 6         return right
 7     door ← closed
 8     if race = id then
 9         return stop
10     else
11         return down
```

Algorithm 24.3: Implementation of a splitter

closes the door. Some process writes its id last: this process will see its own id in race and return stop. □

Lemma 24.4.2. *If at least one process completes the splitter, at least one process returns* stop *or* down*.*

Proof. First observe that if no process ever writes to door, then no process completes the splitter, because the only way a process can finish the splitter without writing to door is if it sees closed when it reads door (which must have been written by some other process). So if at least one process finishes, at least one process writes to door. Let p be any such process. From the code, having written door, it has already passed up the chance to return right; thus it either returns stop or down. □

24.4.3.2 Splitters in a grid

Now build an m-by-m triangular grid of splitters, arranged as rows $0 \ldots m-1$ and columns $0 \ldots m-1$, where a splitter appears in each position (r, c) with $r + c \leq m - 1$ (see Figure 24.1 for an example; this figure is taken from [Asp10]). Assign a distinct name to each of the $\binom{m}{2}$ splitters in this grid. To obtain a name, a process starts at $(r, c) = (0, 0)$, and repeatedly executes the splitter at its current position (r, c). If the splitter returns right, it moves to $(r, c + 1)$; if down, it moves to $(r + 1, c)$; if stop, it stops, and returns the name of its current splitter. This gives each name to at most

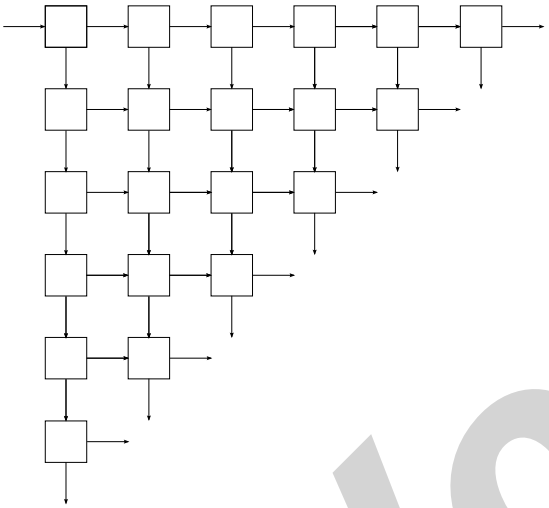


Figure 24.1: A 6×6 Moir-Anderson grid

one process (by Lemma 17.4.3); we also have to show that if at most m processes enter the grid, every process stops at some splitter.

The argument for this is simple. Suppose some process p leaves the grid on one of the $2m$ output wires. Look at the path it takes to get there (see Figure 24.2, also taken from [Asp10]). Each splitter on this path must handle at least two processes (or p would have stopped at that splitter, by Lemma 17.4.4). So some other process leaves on the other output wire, either right or down. If we draw a path from each of these wires that continues right or down to the end of the grid, then along each of these m disjoint paths either some splitter stops a process, or some process reaches a final output wire, each of which is at a distinct splitter. But this gives m processes in addition to p, for a total of $m + 1$ processes. It follows that:

Theorem 24.4.3. *An $m \times m$ Moir-Anderson grid solves renaming for up to m processes.*

The time complexity of the algorithm is $O(m)$: Each process spends at most 4 operations on each splitter, and no process goes through more than $2m$ splitters. In general, any splitter network will take at least n steps to stop n processes, because the adversary can run them all together in a horde that drops only one process at each splitter.

If we don't know k in advance, we can still guarantee names of size $O(k^2)$ by carefully arranging them so that each k-by-k subgrid contains the first $\binom{k}{2}$ names. This gives an adaptive renaming algorithm (although the namespace

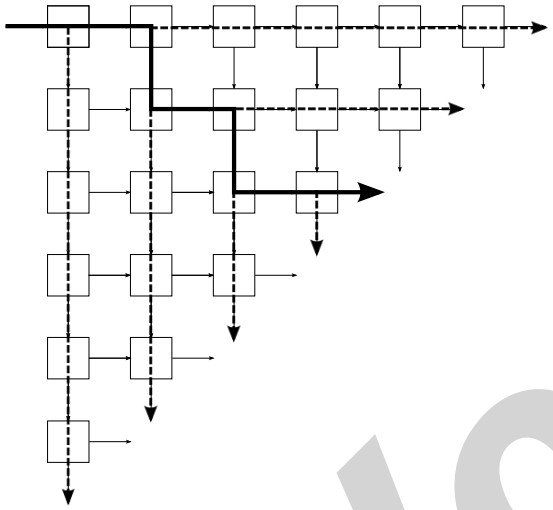


Figure 24.2: Path taken by a single process through a 6×6 Moir-Anderson grid (heavy path), and the 6 disjoint paths it spawns (dashed paths).

size is pretty high). We still have to choose our grid to be large enough for the largest k we might actually encounter; the resulting space complexity is $O(n^2)$.

With a slightly more clever arrangement of the splitters, it is possible to reduce the space complexity to $O(n^{3/2})$ [Asp10]. Whether further reductions are possible is an open problem. Note however that linear time complexity makes splitter networks uncompetitive with much faster randomized algorithms (as we'll see in §24.5), so this may not be a very important open problem.

24.4.4 Getting to $2n - 1$ names in polynomial space

From before, we have an algorithm that will get $2n - 1$ names for n processes out of N possible processes when run using $O(N)$ space (for the enormous snapshots). To turn this into a bounded-space algorithm, run Moir-Anderson first to get down to $\Theta(k^2)$ names, then run the previous algorithm (in $\Theta(n^2)$ space) using these new names as the original names.

Since we didn't prove anything about time complexity of the humongous-snapshot algorithm, we can't say much about the time complexity of this combined one. Moir and Anderson suggest instead using an $O(Nk^2)$ algorithm of Borowsky and Gafni to get $O(k^4)$ time for the combined algorithm.

This is close to the best known: a later paper by Afek and Merritt [AM99]

holds the current record for deterministic adaptive renaming into $2k - 1$ names at $O(k^2)$ individual steps. On the lower bound side, it is known that $\Omega(k)$ is a lower bound on the individual steps of any renaming protocol with a polynomial output namespace [AAGG11].

24.4.5 Renaming with test-and-set

Moir and Anderson give a simple renaming algorithm based on test-and-set that is **strong** (k processes are assigned exactly the names $1 \dots k$), **adaptive** (the time complexity to acquire a name is $O(k)$), and **long-lived**, which means that a process can release its name and the name will be available to processes that arrive later. In fact, the resulting algorithm gives **long-lived strong renaming**, meaning that the set of names in use will always be no larger than the set of processes that have started to acquire a name and not yet finished releasing one; this is a little stronger than just saying that the algorithm is strong and that it is long-lived separately.

The algorithm is simple: we have a line of test-and-set bits $T[1] \dots T[n]$. To acquire a name, a process starts at $T[1]$ and attempts to win each test-and-set until it succeeds; whichever $T[i]$ it wins gives it name i. To release a name, a process releases the test-and-set.

Without the releases, the same mechanism gives fetch-and-increment [AWW93]. Fetch-and-increment by itself solves tight renaming (although not long-lived renaming, since there is no way to release a name).

24.5 Randomized renaming

With randomization, we can beat both the $2k-1$ lower bound on the size of the output namespace from [HS99] and the $\Omega(k)$ lower bound on individual work from [AAGG11], achieving strong renaming with $O(\log k)$ expected individual work [AACH+11].

The basic idea is that we can use randomization for **load balancing**, where we avoid the problem of having an army of processes marching together with only a few peeling off at a time (as in splitter networks) by having the processes split up based on random choices. For example, if each process generates a random name consisting of $2 \lceil \lg n \rceil$ bits, then it is reasonably likely that every process gets a unique name in a namespace of size $O(n^2)$ (we can't hope for less than $O(n^2)$ because of the **birthday paradox**). But we want all processes to be guaranteed to have unique names, so we need some more machinery.

We also need the processes to have initial names; if they don't, there is always some nonzero probability that two identical processes will flip their coins in exactly the same way and end up with the same name. This observation was formalized by Buhrman, Panconesi, Silvestri, and Vitányi [BPSV06].

24.5.1 Randomized splitters

Attiya, Kuhn, Plaxton, Wattenhofer, and Wattenhofer [AKP+06] suggested the use of **randomized splitters** in the context of another problem (**adaptive collect**) that is closely related to renaming.

A randomized splitter is just like a regular splitter, except that if a process doesn't stop it flips a coin to decide whether to go right or down. Randomized splitters are nice because they usually split better than deterministic splitters: if k processes reach a randomized splitter, with high probability no more than $k/2 + O(\sqrt{k \log k})$ will leave on either output wire.

It's not hard to show that a binary tree of these things of depth $2\lceil \lg n \rceil$ stops all but a constant expected number of processes on average;[2] processes that don't stop can be dropped into a backup renaming algorithm (Moir-Anderson, for example) only a constant increase in expected individual work.

Furthermore, the binary tree of randomized splitters is adaptive; if only k processes show up, we only need $O(\log k)$ levels levels on average to split them up. This gives renaming into a namespace with expected size $O(k^2)$ in $O(\log k)$ expected individual steps.

24.5.2 Randomized test-and-set plus sampling

Subsequent work by Alistarh *et al.* [AAG+10] showed how some of the same ideas could be used to get strong renaming, where the output namespace has size exactly n (note this is not adaptive; another result in the same paper gives adaptive renaming, but it's not strong). There are two pieces to this result: an implementation of randomized test-and-set called `RatRace`, and a sampling procedure for getting names called `ReShuffle`.

The `RatRace` protocol implements a randomized test-and-set with $O(\log k)$ expected individual work. The essential idea is to use a tree of randomized splitters to assign names, then have processes walk back up the same tree

[2] The proof is to consider the expected number of pairs of processes that flip their coins the same way for all $2\lceil \lg n \rceil$ steps. This is at most $\binom{n}{2} n^{-2} < 1/2$, so on average at most 1 process escapes the tree, giving (by symmetry) at most a $1/n$ chance that any particular process escapes. Making the tree deeper can give any polynomial fraction of escapees while still keeping $O(\log n)$ layers.

attempting to win a 3-process randomized test-and-set at each node (there are 3 processes, because in addition to the winners of each subtree, we may also have a process that stopped on that node in the renaming step); this test-and-set is just a very small binary tree of 2-process test-and-sets implemented using the algorithm of Tromp and Vitányi [TV02]. A `gate` bit is added at the top as in the test-and-set protocol of Afek *et al.* [AGTV92] to get linearizability.

Once we have test-and-set, we could get strong renaming using a linear array of test-and-sets as suggested by Moir and Anderson [MA95], but it's more efficient to use the randomization to spread the processes out. In the `ReShuffle` protocol, each process chooses a name in the range $[1 \dots n]$ uniformly at random, and attempts to win a test-and-set guarding that name. If it doesn't work, it tries again. Alistarh *et al.* show that this method produces unique names for everybody in $O(n \log^4 n)$ total steps with high probability. The individual step complexity of this algorithm, however, is not very good: there is likely to be some unlucky process that needs $\Omega(n)$ probes (at an expected cost of $\Theta(\log n)$ steps each) to find an empty slot.

24.5.3 Renaming with sorting networks

A later paper by Alistarh *et al.* [AACH+11] reduces the cost of renaming still further, getting $O(\log k)$ expected individual step complexity for acquiring a name. The resulting algorithm is both adaptive and strong: with k processes, only names 1 through k are used. We'll describe the non-adaptive version here.

The basic idea is to build a **sorting network** out of test-and-sets; the resulting structure, called a **renaming network**, routes each process through a sequence of test-and-sets to a unique output wire. Unlike a splitter network, a renaming network uses the stronger properties of test-and-set to guarantee that (once the dust settles) only the lowest-numbered output wires are chosen; this gives strong renaming.

24.5.3.1 Sorting networks

A sorting network is a kind of parallel sorting algorithm that proceeds in synchronous rounds, where in each round the elements of an array at certain fixed positions are paired off and swapped if they are out of order. The difference between a sorting network and a standard comparison-based sort is that the choice of which positions to compare at each step is static, and doesn't depend on the outcome of previous comparisons; also, the only effect

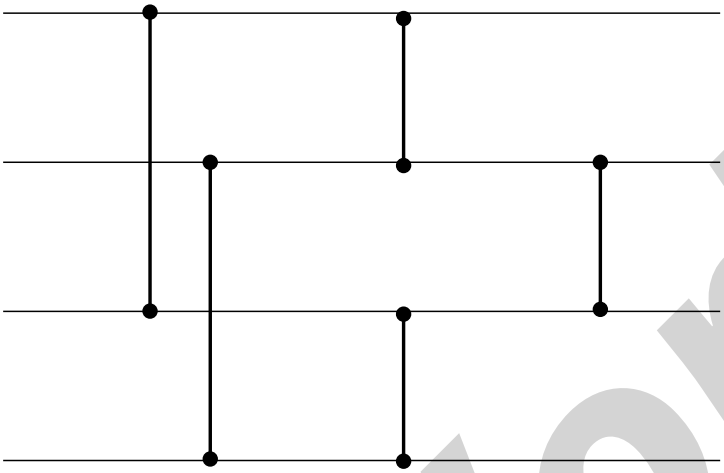


Figure 24.3: A sorting network

of a comparison is possibly swapping the two values that were compared.

Sorting networks are drawn as in Figure 24.3. Each horizontal line or **wire** corresponds to a position in the array. The vertical lines are **comparators** that compare two values coming in from the left and swap the larger value to the bottom. A network of comparators is a sorting network if the sequences of output values is always sorted no matter what the order of values on the inputs is.

The **depth** of a sorting network is the maximum number of comparators on any path from an input to an output. The **width** is the number of wires; equivalently, the number of values the network can sort. The sorting network in Figure 24.3 has depth 3 and width 4.

Explicit constructions of sorting networks with width n and depth $O(\log^2 n)$ are known [Bat68]. It is also known that sorting networks with depth $O(\log n)$ exist [AKS83], but no explicit construction of such a network is known.

24.5.3.2 Renaming networks

To turn a sorting network into a renaming network, we replace the comparators with test-and-set bits, and allow processes to walk through the network asynchronously. This is similar to an earlier mechanism called a **counting network** [AHS94], which used certain special classes of sorting networks as counters, but here any sorting network works.

Each process starts on a separate input wire, and we maintain the in-

variant that at most one process ever traverses a wire. It follows that each test-and-set bit is only used by two processes. The first process to reach the test-and-set bit is sent out the lower output, while the second is sent out the upper output. If we imagine each process that participates in the protocol as a one and each process that doesn't as a zero, the test-and-set bit acts as a comparator: if no processes show up on either input (two zeros), no processes leave (two zeros again); if processes show up on both inputs (two ones), processes leave on both (two ones again); and if only one process ever shows up (a zero and a one), it leaves on the bottom output (zero and one, sorted). Because the original sorting network sorts all the ones to the bottom output wires, the corresponding renaming network sorts all the processes that arrive to the bottom outputs. Label these outputs starting at 1 at the bottom to get renaming.

Since each test-and-set involves at most two processes, we can carry them out in $O(1)$ expected register operations using, for example, the protocol of Tromp and Vitányi [TV02]. The expected cost for a process to acquire a name is then $O(\log n)$ (using an AKS sorting network). A more complicated construction in the Alistarh *et al.* paper shows how to make this adaptive, giving an expected cost of $O(\log k)$ instead.

The use of test-and-sets to route processes to particular names is similar to the line of test-and-sets proposed by Moir and Anderson [MA95] as described in §24.4.5. Some differences between that protocol and renaming networks is that renaming networks do not by themselves give fetch-and-increment (although Alistarh *et al.* show how to build fetch-and-increment on top of renaming networks at a small additional cost), and renaming networks do not provide any mechanism for releasing names. The question of whether it is possible to get cheap long-lived strong renaming is still open.

24.5.4 Randomized loose renaming

Loose renaming should be easier than strong renaming, and using a randomized algorithm it essentially reduces to randomized load balancing. A basic approach is to use $2n$ names, and guard each with a test-and-set; because less than half of the names are taken at any given time, each process gets a name after $O(1)$ tries and the most expensive renaming operation over all n processes takes $O(\log n)$ expected steps.

A more sophisticated version of this strategy, which appears in [AAGW13], uses $n(1+\epsilon)$ output names to get $O(\log \log n)$ maximum steps. The intuition for why this works is if n processes independently choose one of cn names uniformly at random, then the expected number of collisions—pairs

of processes that choose the same name—is $\binom{n}{2}/cn$, or about $n/2c$. This may seem like only a constant-factor improvement, but if we instead look at the ratio between the survivors $n/2c$ and the number of allocated names cn, we have now moved from $1/c$ to $1/2c^2$. The 2 gives us some room to reduce the number of names in the next round, to $cn/2$, say, while still keeping a $1/c^2$ ratio of survivors to names.

So the actual renaming algorithm consists of allocating $cn/2^i$ names to round i, and squaring the ratio of survivors to names in each rounds. It only takes $O(\log \log n)$ rounds to knock the ratio of survivors to names below $1/n$, so at this point it is likely that all processes will have finished. At the same time, the sum over all rounds of the allocated names forms a geometric series, so only $O(n)$ names are needed altogether.

Swept under the carpet here is a lot of careful analysis of the probabilities. Unlike what happens with sifters (see §23.6), Jensen's inequality goes the wrong way here, so some additional technical tricks are needed (see the paper for details). But the result is that only $O(\log \log n)$ rounds are to assign every process a name with high probability, which is the best value currently known.

There is a rather weak lower bound in the Alistarh *et al.* paper that shows that $\Omega(\log \log n)$ steps are needed for some process in the worst case, under the assumption that the renaming algorithm uses only test-and-set objects and that a process acquires a name as soon as it wins some test-and-set object. This does not give a lower bound on the problem in general, and indeed the renaming-network based algorithms discussed previously do not have this property. So the question of the exact complexity of randomized loose renaming is still open.

Chapter 25

Software transactional memory

1

Software transactional memory, or **STM** for short, goes back to Shavit and Touitou [ST97] based on earlier proposals for hardware support for transactions by Herlihy and Moss [HM93]. Recently very popular in programming language circles. We'll give a high-level description of the Shavit and Touitou results; for full details see the actual paper.

We start with the basic idea of a **transaction**. In a transaction, I read a bunch of registers and update their values, and all of these operations appear to be **atomic**, in the sense that the transaction either happens completely or not at all, and serializes with other transactions as if each occurred instantaneously. Our goal is to implement this with minimal hardware support, and use it for everything.

Generally we only consider **static transactions** where the set of memory locations accessed is known in advance, as opposed to **dynamic transactions** where it may vary depending on what we read (for example, maybe we have to follow pointers through some data structure). Static transactions are easier because we can treat them as multi-word read-modify-write.

Implementations are usually **non-blocking**: some infinite stream of transactions succeed, but not necessarily yours. This excludes the simplest method based on acquiring locks, since we have to keep going even if a lock-holder crashes, but is weaker than wait-freedom since we can have starvation.

25.1 Motivation

Some selling points for software transactional memory:

1. We get atomic operations without having to use our brains much. Unlike hand-coded atomic snapshots, counters, queues, etc., we have a universal construction that converts any sequential data structure built on top of ordinary memory into a concurrent data structure. This is useful since most programmers don't have very big brains. We also avoid burdening the programmer with having to remember to lock things.

2. We can build large shared data structures with the possibility of concurrent access. For example, we can implement atomic snapshots so that concurrent updates don't interfere with each other, or an atomic queue where enqueues and dequeues can happen concurrently so long as the queue always has a few elements in it to separate the enqueuers and dequeuers.

3. We can execute atomic operations that span multiple data structures, even if the data structures weren't originally designed to work together, provided they are all implemented using the STM mechanism. This is handy in classic database-like settings, as when we want to take $5 from my bank account and put it in yours.

On the other hand, we now have to deal with the possibility that operations may fail. There is a price to everything.

25.2 Basic approaches

- Locking (not non-blocking). Acquire either a single lock for all of memory (doesn't allow much concurrency) or a separate lock for each memory location accessed. The second approach can lead to deadlock if we aren't careful, but we can prove that if every transaction acquires locks in the same order (e.g., by increasing memory address), then we never get stuck: we can order the processes by the highest lock acquired, and somebody comes out on top. Note that acquiring locks in increasing order means that I have to know which locks I want before I acquire any of them, which may rule out dynamic transactions.

- Single-pointer compare-and-swap (called "Herlihy's method" in [ST97], because of its earlier use for constructing concurrent data structures

by Herlihy [Her93]). All access to the data structure goes through a pointer in a CAS. To execute a transaction, I make my own copy of the data structure, update it, and then attempt to redirect the pointer. Advantages: trivial to prove that the result is linearizable (the pointer swing is an atomic action) and non-blocking (somebody wins the CAS); also, the method allows dynamic transactions (since you can do anything you want to your copy). Disadvantages: There's a high overhead of the many copies,[1] and the single-pointer bottleneck limits concurrency even when two transactions use disjoint parts of memory.

- Multiword RMW: This is the approach suggested by Shavit and Touitou, which most subsequent work follows. As usually implemented, it only works for static transactions. The idea is that I write down what registers I plan to update and what I plan to do to them. I then attempt to acquire all the registers. If I succeed, I update all the values, store the old values, and go home. If I fail, it's because somebody else already acquired one of the registers. Since I need to make sure that somebody makes progress (I may be the only process left alive), I'll help that other process finish its transaction if possible. Advantages: allows concurrency between disjoint transactions. Disadvantages: requires implementing multi-word RMW—in particular, requires that any process be able to understand and simulate any other process's transactions. Subsequent work often simplifies this to implementing multi-word CAS, which is sufficient to do non-blocking multi-word RMW since I can read all the registers I need (without any locking) and then do a CAS to update them (which fails only if somebody else succeeded).

25.3 Implementing multi-word RMW

We'll give a sketchy description of Shavit and Touitou's method [ST97], which essentially follows the locking approach but allows other processes to help dead ones so that locks are always released.

The synchronization primitive used is **LL/SC**: LL (**load-linked**) reads a register and leaves our id attached to it, SC (**store-conditional**) writes a register only if our id is still attached, and clears any other id's that might

[1] This overhead can be reduced in many cases by sharing components, a subject that has seen much work in the functional programming literature. See for example [Oka99].

also be attached. It's easy to build a 1-register CAS (CAS1) out of this, though Shavit and Touitou exploit some additional power of LL/SC.

25.3.1 Overlapping LL/SC

The particular trick that gets used in the Shavit-Touitou protocol is to use two overlapping LL/SC pairs to do a CAS-like update on one memory location while checking that another memory location hasn't changed. The purpose of this is to allow multiple processes to work on the same transaction (which requires the first CAS to avoid conflicts with other transactions) while making sure that slow processes don't cause trouble by trying to complete transactions that have already finished (the second check).

To see this in action, suppose we have a register r that we want to do a CAS on, while checking that a second register status is $\perp$ (as opposed to success or failure). If we execute the code fragment in Algorithm 25.1, it will succeed only if nobody writes to status between its LL and SC and similarly for r; if this occurs, then at the time of LL(r), we know that status $= \perp$, and we can linearize the write to r at this time if we restrict all access to r to go through LL/SC.

```
1 if LL(status) = ⊥ then
2     if LL(r) = oldValue then
3         if SC(status, ⊥) = true then
4             SC(r, newValue)
```

Algorithm 25.1: Overlapping LL/SC

25.3.2 Representing a transaction

Transactions are represented by records rec. Each such record consists of a status component that describes how far the transaction has gotten (needed to coordinate cooperating processes), a version component that distinguishes between versions that may reuse the same space (and that is used to shut down the transaction when complete), a stable component that indicates when the initialization is complete, an Op component that describes the RMW to be performance, an array addresses[] of pointers to the arguments to the RMW, and an array oldValues[] of old values at these addresses (for the R part of the RWM). These are all initialized by the initiator of the

transaction, who will be the only process working on the transaction until it starts acquiring locks.

25.3.3 Executing a transaction

Here we give an overview of a transaction execution:

1. Initialize the record rec for the transaction. (Only the initiator does this.)

2. Attempt to acquire ownership of registers in addresses[]. See the `AcquireOwnerships` code in the paper for details. The essential idea is that we want to set the field owner[r] for each memory location r that we need to lock; this is done using an overlapping LL/SC as described above so that we only set owner[r] if (a) r is currently unowned, and (b) nothing has happened to rec.status or rec.version. Ownership is acquired in order of increasing memory address; if we fail to acquire ownership for some r, our transaction fails. In case of failure, we set rec.status to failure and release all the locks we've acquired (checking rec.version in the middle of each LL/SC so we don't release locks for a later version using the same record). If we are the initiator of this transaction, we will also go on to attempt to complete the transaction that got in our way.

3. Do a `LL` on rec.status to see if `AcquireOwnerships` succeeded. If so, update the memory, store the old results in oldValues, and release the ownerships. If it failed, release ownership and help the next transaction as described above.

Note that only an initiator helps; this avoids a long chain of helping and limits the cost of each attempted transaction to the cost of doing two full transactions, while (as shown below) still allowing some transaction to finish.

25.3.4 Proof of linearizability

Intuition is:

- Linearizability follows from the linearizability of the locking protocol: acquiring ownership is equivalent to grabbing a lock, and updates occur only when all registers are locked.

- Complications come from (a) two or more processes trying to complete the same transaction and (b) some process trying to complete an old transaction that has already terminated. For the first part we just make sure that the processes don't interfere with each other, e.g. I am happy when trying to acquire a location if somebody else acquires it for the same transaction. For the second part we have to check rec.status and rec.version before doing just about anything. See the pseudocode in the paper for details on how this is done.

25.3.5 Proof of non-blockingness

To show that the protocol is non-blocking we must show that if an unbounded number of transactions are attempted, one eventually succeeds. First observe that in order to fail, a transaction must be blocked by another transaction that acquired ownership of a higher-address location than it did; eventually we run out of higher-address locations, so there is some transaction that doesn't fail. Of course, this transaction may not succeed (e.g., if its initiator dies), but either (a) it blocks some other transaction, and that transaction's initiator will complete it or die trying, or (b) it blocks no future transactions. In the second case we can repeat the argument for the $n-1$ surviving processes to show that some of them complete transactions, ignoring the stalled transaction from case (b).

25.4 Improvements

One downside of the Shavit and Touitou protocol is that it uses LL/SC very aggressively (e.g. with overlapping LL/SC operations) and uses non-trivial (though bounded, if you ignore the ever-increasing version numbers) amounts of extra space. Subsequent work has aimed at knocking these down; for example a paper by Harris, Fraser, and Pratt [HFP02] builds multi-register CAS out of single-register CAS with $O(1)$ extra bits per register. The proof of these later results can be quite involved; Harris et al, for example, base their algorithm on an implementation of 2-register CAS whose correctness has been verified only by machine (which may be a plus in some views).

25.5 Limitations

There has been a lot of practical work on STM designed to reduce overhead on real hardware, but there's still a fair bit of overhead. On the theory side, a lower bound of Attiya, Hillel, and Milani [AHM09] shows that any STM system that guarantees non-interference between non-overlapping RMW transactions has the undesirable property of making read-only transactions as expensive as RMW transactions: this conflicts with the stated goals of many practical STM implementations, where it is assumed that most transactions will be read-only (and hopefully cheap). So there is quite a bit of continuing research on finding the right trade-offs.

Chapter 26

Obstruction-freedom

1

The gold standard for shared-memory objects is **wait-freedom**: I can finish my operation in a bounded number of steps no matter what anybody else does. Like the gold standard in real life, this can be overly constraining. So researchers have developed several weaker progress guarantees that are nonetheless useful. The main ones are:

Lock-freedom An implementation is **lock-free** if infinitely many operations finish in any infinite execution. In simpler terms, somebody always makes progress, but maybe not you. (Also called **non-blocking**.)

Obstruction-freedom An implementation is **obstruction-free** if, starting from any reachable configuration, any process can finish in a bounded number of steps if all of the other processes stop. This definition was proposed in 2003 by Herlihy, Luchangco, and Moir [HLM03]. In lower bounds (e.g., the Jayanti-Tan-Toueg bound described in Chapter 20) essentially the same property is often called **solo-terminating**.

Both of these properties exclude traditional lock-based algorithms, where some process grabs a lock, updates the data structure, and then release the lock; if this process halts, no more operations finish. Both properties are also weaker than wait-freedom. It is not hard to show that lock-freedom is a stronger condition that obstruction-freedom: given a lock-free implementation, if we can keep some single process running forever in isolation, we get an infinite execution with only finitely many completed operations. So we have a hierarchy: wait-free $>$ lock-free $>$ obstruction-free $>$ locking.

26.1 Why build obstruction-free algorithms?

The pitch is similar to the pitch for building locking algorithms: an obstruction-free algorithm might be simpler to design, implement, and reason about than a more sophisticated algorithm with stronger properties. Unlike locking algorithms, an obstruction-free algorithm won't fail because some process dies holding the lock; instead, it fails if more than one process runs the algorithm at the same time. This possibility may be something we can avoid by building a **contention manager**, a high-level protocol that detects contention and delays some processes to avoid it (say, using randomized exponential back-off).

26.2 Examples

26.2.1 Lock-free implementations

Pretty much anything built using compare-and-swap or LL/SC ends up being lock-free. A simple example would be a counter, where an increment operation does

```
1 x ← LL(C)
2 SC(C, x + 1)
```

This is lock-free (the only way to prevent a store-conditional from succeeding is if some other store-conditional succeeds, giving infinitely many successful increments) but not wait-free (I can starve). It's also obstruction-free, but since it's already lock-free we don't care about that.

26.2.2 Double-collect snapshots

Similarly, suppose we are doing atomic snapshots. We know that there exist wait-free implementations of atomic snapshots, but they are subtle and confusing. So we want to do something simpler, and hope that we at least get obstruction-freedom.

If we do double-collects, that is, we have updates just write to a register and have snapshots repeatedly collect until they get two collects in a row with the same values, then any snapshot that finishes is correct (assuming no updaters ever write the same value twice, which we can enforce with nonces). This isn't wait-free, because we can keep a snapshot going forever

by doing a lot of updates. It *is* lock-free, because we have to keep doing updates to make this happen.

We can make this merely obstruction-free if we work hard (there is no reason to do this, but it illustrates the difference between lock-freedom—good—and obstruction-freedom—not so good). Suppose that every process keeps a count of how many collects it has done in a register that is included in other process's collects (but not its own). Then two concurrent scans can stall each other forever (the implementation is not lock-free), but if only one is running it completes two collects in $O(n)$ operations without seeing any changes (it is obstruction-free).

26.2.3 Software transactional memory

Similar things happen with software transactional memory (see Chapter 25). Suppose that I have an implementation of multiword compare-and-swap, and I want to carry out a transaction. I read all the values I need, then execute an MCAS operation that only updates if these values have not changed. The resulting algorithm is lock-free (if my transaction fails, it's because some update succeeded). If however I am not very clever and allow some values to get written outside of transactions, then I might only be obstruction-free.

26.2.4 Obstruction-free test-and-set

Algorithm 26.1 gives an implementation of 2-process test-and-set from atomic registers that is obstruction-free; this demonstrates that obstruction-freedom lets us evade the wait-free impossibility results implied by the consensus hierarchy ([Her91b], discussed in Chapter 18).

The basic idea goes back to the **racing counters** technique used in consensus protocols starting with Chor, Israeli, and Li [CIL94], and there is some similarity to a classic randomized wait-free test-and-set due to Tromp and Vitányi [TV02]. Each process keeps a position x in memory that it also stores from time to time in its register $a[i]$. If a process gets 2 steps ahead of the other process (as observed by comparing x to $a[i-1]$, it wins the test-and-set; if a process falls one or more steps behind, it (eventually) loses. To keep space down and guarantee termination in bounded time, all values are tracked modulo 5.

Why this works: observe that whenever a process computes δ, x is equal to $a[i]$; so δ is always an instantaneous snapshot of $a[i] - a[1-i]$. If I observe $\delta = 2$ and return 0, your next read will either show you $\delta = -2$ or $\delta = -1$ (depending on whether you increment $a[1-i]$ after my read). In the latter

```
1  x ← 0
2  while true do
3  |  δ ← x − a[1 − i]
4  |  if δ = 2 (mod 5) then
5  |  |  return 0
6  |  else if δ = −1 (mod 5) do
7  |  |  return 1
8  |  else
9  |  |  x ← (x + 1) mod 5
10 |  |  a[i] ← x
```

Algorithm 26.1: Obstruction-free 2-process test-and-set

case, you return 1 immediately; in the former, you return after one more increment (and more importantly, you can't return 0). Alternatively, if I ever observe $\delta = -1$, your next read will show you either $\delta = 1$ or $\delta = 2$; in either case, you will eventually return 0. (We chose 5 as a modulus because this is the smallest value that makes the cases $\delta = 2$ and $\delta = -2$ distinguishable.)

We can even show that this is linearizable, by considering a solo execution in which the lone process takes two steps and returns 0 (with two processes, solo executions are the only interesting case for linearizability).

However, Algorithm 26.1 is not wait-free or even lock-free: if both processes run in lockstep, they will see $\delta = 0$ forever. But it is obstruction-free. If I run by myself, then whatever value of δ I start with, I will see -1 or 2 after at most 6 operations.[1]

This gives an **obstruction-free step complexity** of 6, where the obstruction-free step complexity is defined as the maximum number of operations any process can take after all other processes stop. Note that our usual wait-free measures of step complexity don't make a lot of sense for obstruction-free algorithms, as we can expect a sufficiently cruel adversary to be able to run them up to whatever value he likes.

Building a tree of these objects as in §22.2 gives n-process test-and-set with obstruction-free step complexity $O(\log n)$.

[1] The worst case is where an increment by my fellow process leaves $\delta = -1$ just before my increment.

26.2.5 An obstruction-free deque

(We probably aren't going to do this in class.)

So far we don't have any good examples of why we would want to be obstruction-free if our algorithm is based on CAS. So let's describe the case Herlihy et al. suggested.

A **deque** is a generalized queue that supports push and pop at both ends (thus it can be used as either a queue or a stack, or both). A classic problem in shared-memory objects is to build a deque where operations at one end of the deque don't interfere with operations at the other end. While there exist lock-free implementation with this property, there is a particularly simple implementation using CAS that is only obstruction-free.

Here's the idea: we represent the deque as an infinitely-long array of compare-and-swap registers (this is a simplification from the paper, which gives a bounded implementation of a bounded deque). The middle of the deque holds the actual contents. To the right of this region is an infinite sequence of **right null** (RN) values, which are assumed never to appear as a pushed value. To the left is a similar infinite sequence of **left null** (LN) values. Some magical external mechanism (called an **oracle** in the paper) allows processes to quickly find the first null value at either end of the non-null region; the correctness of the protocol does not depend on the properties of the oracle, except that it has to point to the right place at least some of the time in a solo execution. We also assume that each cell holds a version number whose only purpose is to detect when somebody has fiddled with the cell while we aren't looking (if we use LL/SC, we can drop this).

Code for `rightPush` and `rightPop` is given in Algorithm 26.2 (the code for `leftPush` and `leftPop` is symmetric).

It's easy to see that in a solo execution, if the oracle doesn't lie, either operation finishes and returns a plausible value after $O(1)$ operations. So the implementation is obstruction-free. But is it also correct?

To show that it is, we need to show that any execution leaves the deque in a sane state, in particular that it preserves the invariant that the deque consists of left-nulls followed by zero or more values followed by right-nulls, and that the sequence of values in the queue is what it should be.

This requires a detailed case analysis of which operations interfere with each other, which can be found in the original paper. But we can give some intuition here. The two CAS operations in `rightPush` or `rightPop` succeed only if neither register was modified between the preceding read and the CAS. If both registers are unmodified at the time of the second CAS, then the two CAS operations act like a single two-word CAS, which replaces the

```
1  procedure rightPush(v)
2      while true do
3          k ← oracle(right)
4          prev ← a[k − 1]
5          next ← a[k]
6          if prev.value ≠ RN and next.value = RN then
7              if CAS(a[k − 1], prev, [prev.value, prev.version + 1]) then
8                  if CAS(a[k], next, [v, next.version + 1]) then
9                      we win, go home

10 procedure rightPop()
11     while true do
12         k ← oracle(right)
13         cur ← a[k − 1]
14         next ← a[k]
15         if cur.value ≠ RN and next.value = RN then
16             if cur.value = LN and A[k − 1] = cur then
17                 return empty
18             else if CAS(a[k], next, [RN, next.version + 1]) do
19                 if CAS(a[k − 1], cur, [RN, cur.version + 1]) then
20                     return cur.value
```

Algorithm 26.2: Obstruction-free deque

previous values (top, RN) with (top, value) in `rightPush` or (top, value) with (top, RN) in `rightPop`; in either case the operation preserves the invariant. So the only way we get into trouble is if, for example, a `rightPush` does a CAS on $a[k-1]$ (verifying that it is unmodified and incrementing the version number), but then some other operation changes $a[k-1]$ before the CAS on $a[k]$. If this other operation is also a `rightPush`, we are happy, because it must have the same value for k (otherwise it would have failed when it saw a non-null in $a[k-1]$), and only one of the two right-pushes will succeed in applying the CAS to $a[k]$. If the other operation is a `rightPop`, then it can only change $a[k-1]$ after updating $a[k]$; but in this case the update to $a[k]$ prevents the original right-push from changing $a[k]$. With some more tedious effort we can similarly show that any interference from `leftPush` or `leftPop` either causes the interfering operation or the original operation to fail. This covers 4 of the 16 cases we need to consider. The remaining cases will be brushed under the carpet to avoid further suffering.

26.3 Boosting obstruction-freedom to wait-freedom

1

Naturally, having an obstruction-free implementation of some object is not very helpful if we can't guarantee that some process eventually gets its unobstructed solo execution. In general, we can't expect to be able to do this without additional assumptions; for example, if we could, we could solve consensus using a long sequence of adopt-commit objects with no randomization at all.[2] So we need to make some sort of assumption about timing, or find somebody else who has already figured out the right assumption to make.

Those somebodies turn out to be Faith Ellen Fich, Victor Luchangco, Mark Moir, and Nir Shavit, who give an algorithm for boosting obstruction-freedom to wait-freedom [FLMS05]. The timing assumption is **unknown-bound semisynchrony**, which means that in any execution there is some maximum ratio R between the shortest and longest time interval between any two consecutive steps of the same non-faulty process, but the processes don't know what this ratio is.[3] In particular, if I can execute more than R

[2] This fact was observed by Herlihy *et al.* [HLM03] in their original obstruction-free paper; it also implies that there exists a universal obstruction-free implementation of anything based on Herlihy's universal construction.

[3] This is a much older model, which goes back to a famous paper of Dwork, Lynch, and Stockmeyer [DLS88].

steps without you doing anything, I can reasonably conclude that you are dead—the semisynchrony assumption thus acts as a failure detector.

The fact that R is unknown might seem to be an impediment to using this failure detector, but we can get around this. The idea is to start with a small guess for R; if a process is suspected but then wakes up again, we increment the guess. Eventually, the guessed value is larger than the correct value, so no live process will be falsely suspected after this point. Formally, this gives an eventually perfect ($\Diamond P$) failure detector, although the algorithm does not specifically use the failure detector abstraction.

To arrange for a solo execution, when a process detects a conflict (because its operation didn't finish quickly), it enters into a "panic mode" where processes take turns trying to finish unmolested. A fetch-and-increment register is used as a timestamp generator, and only the process with the smallest timestamp gets to proceed. However, if this process is too sluggish, other processes may give up and overwrite its low timestamp with ∞, temporarily ending its turn. If the sluggish process is in fact alive, it can restore its low timestamp and kill everybody else, allowing it to make progress until some other process declares it dead again.

The simulation works because eventually the mechanism for detecting dead processes stops suspecting live ones (using the technique described above), so the live process with the winning timestamp finishes its operation without interference. This allows the next process to proceed, and eventually all live processes complete any operation they start, giving the wait-free property.

The actual code is in Algorithm 26.3. It's a rather long algorithm but most of the details are just bookkeeping.

The preamble before entering PANIC mode is a fast-path computation that allows a process that actually is running in isolation to skip testing any timestamps or doing any extra work (except for the one register read of PANIC). The assumption is that the constant B is set high enough that any process generally will finish its operation in B steps without interference. If there is interference, then the timestamp-based mechanism kicks in: we grab a timestamp out of the convenient fetch-and-add register and start slugging it out with the other processes.

(A side note: while the algorithm as presented in the paper assumes a fetch-and-add register, any timestamp generator that delivers increasing values over time will work. So if we want to limit ourselves to atomic registers, we could generate timestamps by taking snapshots of previous timestamps, adding 1, and appending process ids for tie-breaking.)

Once I have a timestamp, I try to knock all the higher-timestamp pro-

```
1  if ¬PANIC then
2      execute up to B steps of the underlying algorithm
3      if we are done then return
4  PANIC ← true // enter panic mode
5  myTimestamp ← fetchAndIncrement()
6  A[i] ← 1 // reset my activity counter
7  while true do
8      T[i] ← myTimestamp
9      minTimestamp ← myTimestamp; winner ← i
10     for j ← 1...n, j ≠ i do
11         otherTimestamp ← T[j]
12         if otherTimestamp < minTimestamp then
13             T[winner] ← ∞ // not looking so winning any more
14             minTimestamp ← otherTimestamp; winner ← j
15         else if otherTimestamp < ∞ do
16             T[j] ← ∞
17     if i = winner then
18         repeat
19             execute up to B steps of the underlying algorithm
20             if we are done then
21                 T[i] ← ∞
22                 PANIC ← false
23                 return
24             else
25                 A[i] ← A[i] + 1
26                 PANIC ← true
27         until T[i] = ∞
28     repeat
29         a ← A[winner]
30         wait a steps
31         winnerTimestamp ← T[winner]
32     until a = A[winner] or winnerTimestamp ≠ minTimestamp
33     if winnerTimestamp = minTimestamp then
34         T[winner] ← ∞ // kill winner for inactivity
```

Algorithm 26.3: Obstruction-freedom booster from [FLMS05]

cesses out of the way (by writing ∞ to their timestamp registers). If I see a smaller timestamp than my own, I'll drop out myself ($T[i] \leftarrow \infty$), and fight on behalf of its owner instead. At the end of the j loop, either I've decided I am the winner, in which case I try to finish my operation (periodically checking $T[i]$ to see if I've been booted), or I've decided somebody else is the winner, in which case I watch them closely and try to shut them down if they are too slow ($T[\mathsf{winner}] \leftarrow \infty$). I detect slow processes by inactivity in $A[\mathsf{winner}]$; similarly, I signal my own activity by incrementing $A[i]$. The value in $A[i]$ is also used as an increasing guess for the time between increments of $A[i]$; eventually this exceeds the $R(B + O(1))$ operations that I execute between incrementing it.

We still need to prove that this all works. The essential idea is to show that whatever process has the lowest timestamp finishes in a bounded number of steps. To do so, we need to show that other processes won't be fighting it in the underlying algorithm. Call a process *active* if it is in the loop guarded by the "if $i = \mathsf{winner}$" statement. Lemma 1 from the paper states:

Lemma 26.3.1 ([FLMS05, Lemma 1])**.** *If processes i and j are both active, then $T[i] = \infty$ or $T[j] = \infty$.*

Proof. Assume without loss of generality that i last set $T[i]$ to $\mathsf{myTimestamp}$ in the main loop after j last set $T[j]$. In order to reach the active loop, i must read $T[j]$. Either $T[j] = \infty$ at this time (and we are done, since only j can set $T[j] < \infty$), or $T[j]$ is greater than i's timestamp (or else i wouldn't think it's the winner). In the second case, i sets $T[j] = \infty$ before entering the active loop, and again the claim holds. □

The next step is to show that if there is some process i with a minimum timestamp that executes infinitely many operations, it increments $A[i]$ infinitely often (thus eventually making the failure detector stop suspecting it). This gives us Lemma 2 from the paper:

Lemma 26.3.2 ([FLMS05, Lemma 2])**.** *Consider the set of all processes that execute infinitely many operations without completing an operation. Suppose this set is non-empty, and let i hold the minimum timestamp of all these processes. Then i is not active infinitely often.*

Proof. Suppose that from some time on, i is active forever, i.e., it never leaves the active loop. Then $T[i] < \infty$ throughout this interval (or else i leaves the loop), so for any active j, $T[j] = \infty$ by the preceding lemma. It follows that any active $T[j]$ leaves the active loop after $B + O(1)$ steps of j

(and thus at most $R(B+O(1))$ steps of i). Can j re-enter? If j's timestamp is less than i's, then j will set $T[i] = \infty$, contradicting our assumption. But if j's timestamp is greater than i's, j will not decide it's the winner and will not re-enter the active loop. So now we have i alone in the active loop. It may still be fighting with processes in the initial fast path, but since i sets PANIC every time it goes through the loop, and no other process resets PANIC (since no other process is active), no process enters the fast path after some bounded number of i's steps, and every process in the fast path leaves after at most $R(B + O(1))$ of i's steps. So eventually i is in the loop alone forever—and obstruction-freedom means that it finishes its operation and leaves. This contradicts our initial assumption that i is active forever. □

So now we want to argue that our previous assumption that there exists a bad process that runs forever without winning leads to a contradiction, by showing that the particular i from Lemma 26.3.2 actually finishes (note that Lemma 26.3.2 doesn't quite do this—we only show that i finishes if it stays active long enough, but maybe it doesn't stay active).

Suppose i is as in Lemma 26.3.2. Then i leaves the active loop infinitely often. So in particular it increments $A[i]$ infinitely often. After some finite number of steps, $A[i]$ exceeds the limit $R(B+O(1))$ on how many steps some other process can take between increments of $A[i]$. For each other process j, either j has a lower timestamp than i, and thus finishes in a finite number of steps (from the premise of the choice of i), or j has a higher timestamp than i. Once we have cleared out all the lower-timestamp processes, we follow the same logic as in the proof of Lemma 26.3.2 to show that eventually (a) i sets $T[i] < \infty$ and PANIC = true, (b) each remaining j observes $T[i] < \infty$ and PANIC = true and reaches the waiting loop, (c) all such j wait long enough (since $A[i]$ is now very big) that i can finish its operation. This contradicts the assumption that i never finishes the operation and completes the proof.

26.3.1 Cost

If the parameters are badly tuned, the potential cost of this construction is quite bad. For example, the slow increment process for $A[i]$ means that the time a process spends in the active loop even after it has defeated all other processes can be as much as the square of the time it would normally take to complete an operation alone—and every other process may pay R times this cost waiting. This can be mitigated to some extent by setting B high enough that a winning process is likely to finish in its first unmolested pass through the loop (recall that it doesn't detect that the other processes have

reset $T[i]$ until after it makes its attempt to finish). An alternative might be to double $A[i]$ instead of incrementing it at each pass through the loop. However, it is worth noting (as the authors do in the paper) that nothing prevents the underlying algorithm from incorporating its own **contention management** scheme to ensure that most operations complete in B steps and PANIC mode is rarely entered. So we can think of the real function of the construction as serving as a backstop to some more efficient heuristic approach that doesn't necessarily guarantee wait-free behavior in the worst case.

26.4 Lower bounds for lock-free protocols

1

So far we have seen that obstruction-freedom buys us an escape from the impossibility results that plague wait-free constructions, while still allowing practical implementations of useful objects under plausible timing assumptions. Yet all is not perfect: it is still possible to show non-trivial lower bounds on the costs of these implementations in the right model. We will present one of these lower bounds, the linear-contention lower bound of Ellen, Hendler, and Shavit [EHS12].[4] First we have to define what is meant by contention.

26.4.1 Contention

A limitation of real shared-memory systems is that physics generally won't permit more than one process to do something useful to a shared object at a time. This limitation is often ignored in computing the complexity of a shared-memory distributed algorithm (and one can make arguments for ignoring it in systems where communication costs dominate update costs in the shared-memory implementation), but it is useful to recognize it if we can't prove lower bounds otherwise. Complexity measures that take the cost of simultaneous access into account go by the name of **contention**.

The particular notion of contention used in the Ellen *et al.* paper is an adaptation of the contention measure of Dwork, Herlihy, and Waarts [DHW97]. The idea is that if I access some shared object, I pay a price in **memory stalls** for all the other processes that are trying to access it at the same time

[4] The result first appeared in FOCS in 2005 [FHS05], with a small but easily fixed bug in the definition of the class of objects the proof applies to. We'll use the corrected definition from the journal version.

but got in first. In the original definition, given an execution of the form $A\phi_1\phi_2 \ldots \phi_k\phi A'$, where all operations ϕ_i are applied to the same object as ϕ, and the last operation in A is not, then ϕ_k incurs k memory stalls. Ellen *et al.* modify this to only count sequences of *non-trivial* operations, where an operation is non-trivial if it changes the state of the object in some states (e.g., writes, increments, compare-and-swap—but not reads). Note that this change only strengthens the bound they eventually prove, which shows that in the worst case, obstruction-free implementations of operations on objects in a certain class incur a linear number of memory stalls (possibly spread across multiple base objects).

26.4.2 The class G

The Ellen *et al.* bound is designed to be as general as possible, so the authors define a class G of objects to which it applies. As is often the case in mathematics, the underlying meaning of G is "a reasonably large class of objects for which this particular proof works," but the formal definition is given in terms of when certain operations of the implemented object are affected by the presence or absence of other operations—or in other words, when those other operations need to act on some base object in order to let later operations know they occurred.

An object is in **class** G if it has some operation Op and initial state s such that for any two processes p and q and every sequence of operations $A\phi A'$, where

1. ϕ is an instance of Op executed by p,
2. no operation in A or A' is executed by p,
3. no operation in A' is executed by q, and
4. no two operations in A' are executed by the same process;

then there exists a sequence of operations Q by q such that for every sequence $H\phi H'$ where

1. HH' is an interleaving of Q and the sequences $AA'|r$ for each process r,
2. H' contains no operations of q, and
3. no two operations in H' are executed by the same process;

then the return value of ϕ to p changes depending on whether it occurs after $A\phi$ or $H\phi$.

This is where "makes the proof work" starts looking like a much simpler definition. The intuition is that deep in the guts of the proof, we are going to be injecting some operations of q into an existing execution (hence adding Q), and we want to do it in a way that forces q to operate on some object that p is looking at (hence the need for $A\phi$ to return a different value from $H\phi$), without breaking anything else that is going on (all the rest of the conditions). The reason for pulling all of these conditions out of the proof into a separate definition is that we also want to be able to show that particular classes of real objects satisfy the conditions required by the proof, without having to put a lot of special cases into the proof itself.

Lemma 26.4.1. *A mod-m fetch-and-increment object, with $m \geq n$, is in G.*

Proof. This is a classic proof-by-unpacking-the-definition. Pick some execution $A\phi A'$ satisfying all the conditions, and let a be the number of fetch-and-increments in A and a' the number in A'. Note $a' \leq n-2$, since all operations in A' are by different processes.

Now let Q be a sequence of $n-a'-1$ fetch-and-increments by q, and let HH' be an interleaving of Q and the sequences $AA'|r$ for each r, where H' includes no two operation of the same process and no operations at all of q. Let h, h' be the number of fetch-and-increments in H, H', respectively. Then $h + h' = a + a' + (n - a' - 1) = n + a - 1$ and $h' \leq n - 2$ (since H' contains at most one fetch-and-increment for each process other than p and q). This gives $h \geq (n + a + 1) - (n - 2) = a + 1$ and $h \leq n + a - 1$, and the return value of ϕ after $H\phi$ is somewhere in this range mod m. But none of these values is equal to a mod m (that's why we specified $m \geq n$, although as it turns out $m \geq n - 1$ would have been enough), so we get a different return value from $H\phi$ than from $A\phi$. □

As a corollary, we also get stock fetch-and-increment registers, since we can build mod-m registers from them by taking the results mod m.

A second class of class-G objects is obtained from snapshot:

Lemma 26.4.2. *Single-writer snapshot objects are in G.*[5]

[5]For the purposes of this lemma, "single-writer" means that each segment can be written to by only one process, not that there is only one process that can execute update operations.

Proof. Let $A\phi A'$ be as in the definition, where ϕ is a scan operation. Let Q consist of a single update operation by q that changes its segment. Then in the interleaved sequence HH', this update doesn't appear in H' (it's forbidden), so it must be in H. Nobody can overwrite the result of the update (single-writer!), so it follows that $H\phi$ returns a different snapshot from $A\phi$. □

26.4.3 The lower bound proof

Theorem 26.4.3 ([EHS12, Theorem 5.2])**.** *For any obstruction-free implementation of some object in class G from RMW base objects, there is an execution in which some operation incurs $n-1$ stalls.*

We can't do better than $n-1$, because it is easy to come up with implementations of counters (for example) that incur at most $n-1$ stalls. Curiously, we can even spread the stalls out in a fairly arbitrary way over multiple objects, while still incurring at most $n-1$ stalls. For example, a counter implemented using a single counter (which is a RMW object) gets exactly $n-1$ stalls if $n-1$ processes try to increment it at the same time, delaying the remaining process. At the other extreme, a counter implemented by doing a collect over $n-1$ single-writer registers (also RMW objects) gets at least $n-1$ stalls—distributed as one per register—if each register has a write delivered to it while the reader waiting to read it during its collect. So we have to allow for the possibility that stalls are concentrated or scattered or something in between, as long as the total number adds up at least $n-1$.

The proof supposes that the theorem is not true and then shows how to boost an execution with a maximum number $k < n-1$ stalls to an execution with $k+1$ stalls, giving a contradiction. (Alternatively, we can read the proof as giving a mechanism for generating an $(n-1)$-stall execution by repeated boosting, starting from the empty execution.)

This is pretty much the usual trick: we assume that there is a class of bad executions, then look for an extreme member of this class, and show that it isn't as extreme as we thought. In doing so, we can restrict our attention to particularly convenient bad executions, so long as the existence of some bad execution implies the existence of a convenient bad execution.

Formally, the authors define a *k-stall execution* for process p as an execution $E\sigma_1 \dots \sigma_i$ where E and σ_i are sequence of operations such that:

1. p does nothing in E,
2. Sets of processes S_j, $j = 1 \dots i$, whose union $S = \bigcup_{j=1}^{i} S_j$ has size k,

are each covering objects $\mathcal{O}_j$ after E with pending non-trivial operations,

3. Each σ_j consists of p applying events by itself until it is about to apply an event to $\mathcal{O}_j$, after which each process in S_j accesses $\mathcal{O}_j$, after which p accesses $\mathcal{O}_j$.

4. All processes not in S are idle after E,

5. p starts at most one operation of the implemented object in $\sigma_1 \dots \sigma_i$, and

6. In every extension of E in which p and the processes in S don't take steps, no process applies a non-trivial event to any base object accessed in $\sigma_1 \dots \sigma_i$. (We will call this the **weird condition** below.)

So this definition includes both the fact that p incurs k stalls and some other technical details that make the proof go through. The fact that p incurs k stalls follows from observing that it incurs $|S_j|$ stalls in each segment σ_j, since all processes in S_j access $\mathcal{O}_j$ just before p does.

Note that the empty execution is a 0-stall execution (with $i = 0$) by the definition. This shows that a k-stall execution exists for some k.

Note also that the weird condition is pretty strong: it claims not only that there are no non-trivial operation on $\mathcal{O}1 \dots \mathcal{O}i$ in τ, but also that there are no non-trivial operations on *any* objects accessed in $\sigma_1 \dots \sigma_i$, which may include many more objects accessed by p.[6]

We'll now show that if a k-stall execution exists, for $k \leq n - 2$, then a $(k + k')$-stall execution exists for some $k' > 0$. Iterating this process eventually produces an $(n - 1)$-stall execution.

Start with some k-stall execution $E\sigma_1 \dots \sigma_i$. Extend this execution by a sequence of operations σ in which p runs in isolation until it finishes its operation ϕ (which it may start in σ if it hasn't done so already), then each process in S runs in isolation until it completes its operation. Now linearize the high-level operations completed in $E\sigma_1 \dots \sigma_i\sigma$ and factor them as $A\phi A'$ as in the definition of class G.

Let q be some process not equal to p or contained in any S_j (this is where we use the assumption $k \leq n - 2$). Then there is some sequence of high-level operations Q of q such that $H\phi$ does not return the same value as $A\phi$ for any interleaving HH' of Q with the sequences of operations in AA'

[6] And here is where I screwed up in class on 2011-11-14, by writing the condition as the weaker requirement that nobody touches $\mathcal{O}_1 \dots \mathcal{O}_i$.

satisfying the conditions in the definition. We want to use this fact to shove at least one more memory stall into $E\sigma_1 \dots \sigma_i \sigma$, without breaking any of the other conditions that would make the resulting execution a $(k+k')$-stall execution.

Consider the extension τ of E where q runs alone until it finishes every operation in Q. Then τ applies no nontrivial events to any base object accessed in $\sigma_1 \dots \sigma_k$, (from the weird condition on k-stall executions) and the value of each of these base objects is the same after E and $E\tau$, and thus is also the same after $E\sigma_1 \dots \sigma_k$ and $E\tau\sigma_1 \dots \sigma_k$.

Now let σ' be the extension of $E\tau\sigma_1 \dots \sigma_k$ defined analogously to σ: p finishes, then each process in each S_j finishes. Let $H\phi H'$ factor the linearization of $E\tau\sigma_1 \dots \sigma_i\sigma'$. Observe that HH' is an interleaving of Q and the high-level operations in AA', that H' contains no operations by q (they all finished in τ, before ϕ started), and that H' contains no two operations by the same process (no new high-level operations start after ϕ finishes, so there is at most one pending operation per process in S that can be linearized after ϕ).

Now observe that q does some non-trivial operation in τ to some base object accessed by p in σ. If not, then p sees the same responses in σ' and in σ, and returns the same value, contradicting the definition of class G.

So does q's operation in τ cause a stall in σ? Not necessarily: there may be other operations in between. Instead, we'll use the existence of q's operation to demonstrate the existence of at least one operation, possibly by some other process we haven't even encountered yet, that does cause a stall. We do this by considering the set F of all finite extensions of E that are free of p and S operations, and look for an operation that stalls p somewhere in this infinitely large haystack.

Let $\mathcal{O}_{i+1}$ be the first base object accessed by p in σ that is also accessed by some non-trivial event in some sequence in F. We will show two things: first, that $\mathcal{O}_{i+1}$ exists, and second, that $\mathcal{O}_{i+1}$ is distinct from the objects $\mathcal{O}_1 \dots \mathcal{O}_i$. The first part follows from the fact that τ is in F, and we have just shown that τ contains a non-trivial operation (by q) on a base object accessed by p in σ. For the second part, we use the weird condition on k-stall executions again: since every extension of E in F is $(\{p\} \cup S)$-free, no process applies a non-trivial event to any base object accessed in $\sigma_1 \dots \sigma_i$, which includes all the objects $\mathcal{O}_1 \dots \mathcal{O}_i$.

You've probably guessed that we are going to put our stalls in on $\mathcal{O}_{i+1}$. We choose some extension X from F that maximizes the number of processes with simultaneous pending non-trivial operations on $\mathcal{O}_{i+1}$ (we'll call this set of processes S_{i+1} and let $|S_{i+1}|$ be the number $k' > 0$ we've been waiting for),

and let E' be the minimum prefix of X such that these pending operations are still pending after EE'.

We now look at the properties of EE'. We have:

- EE' is p-free (follows from E being p-free and $E' \in F$, since everything in F is p-free).
- Each process in S_j has a pending operation on $\mathcal{O}_j$ after EE' (it did after E, and didn't do anything in E').

This means that we can construct an execution $EE'\sigma_1 \dots \sigma_i\sigma_{i+1}$ that includes $k + k'$ memory stalls, by sending in the same sequences $\sigma_1 \dots \sigma_i$ as before, then appending a new sequence of events where (a) p does all of its operations in σ up to its first operation on $\mathcal{O}_{i+1}$; then (b) all the processes in the set S_{i+1} of processes with pending events on $\mathcal{O}_{i+1}$ execute their pending events on $\mathcal{O}_{i+1}$; then (c) p does its first access to $\mathcal{O}_{i+1}$ from σ. Note that in addition to giving us $k + k'$ memory stalls, σ_{i+1} also has the right structure for a $(k + k')$-stall execution. But there is one thing missing: we have to show that the weird condition on further extensions still holds.

Specifically, letting $S' = S \cup S_{i+1}$, we need to show that any $(\{p\} \cup S')$-free extension α of EE' includes a non-trivial access to a base object accessed in $\sigma_1 \dots \sigma_{i+1}$. Observe first that since α is $(\{p\} \cup S')$-free, then $E'\alpha$ is $(\{p\} \cup S)$-free, and so it's in F: so by the weird condition on $E\sigma_1 \dots \sigma_i$, $E'\alpha$ doesn't have any non-trivial accesses to any object with a non-trivial access in $\sigma_1 \dots \sigma_i$. So we only need to squint very closely at σ_{i+1} to make sure it doesn't get any objects in there either.

Recall that σ_{i+1} consists of (a) a sequence of accesses by p to objects already accessed in $\sigma_1 \dots \sigma_i$ (already excluded); (b) an access of p to $\mathcal{O}_{i+1}$; and (c) a bunch of accesses by processes in S_{i+1} to $\mathcal{O}_{i+1}$. So we only need to show that α includes no non-trivial accesses to $\mathcal{O}_{i+1}$. Suppose that it does: then there is some process that eventually has a pending non-trivial operation on $\mathcal{O}_{i+1}$ somewhere in α. If we stop after this initial prefix α' of α, we get $k' + 1$ processes with pending operations on $\mathcal{O}_{i+1}$ in $EE'\alpha'$. But then $E'\alpha'$ is an extension of E with $k' + 1$ processes with a simultaneous pending operation on $\mathcal{O}_{i+1}$. This contradicts the choice of X to maximize k'. So if our previous choice was in fact maximal, the weird condition still holds, and we have just constructed a $(k + k')$-stall execution. This concludes the proof.

26.4.4 Consequences

We've just shown that counters and snapshots have $(n-1)$-stall executions, because they are in the class G. A further, rather messy argument (given in the Ellen *et al.* paper) extends the result to stacks and queues, obtaining a slightly weaker bound of n total stalls and operations for some process in the worst case.[7] In both cases, we can't expect to get a sublinear worst-case bound on time under the reasonable assumption that both a memory stall and an actual operation takes at least one time unit. This puts an inherent bound on how well we can handle hot spots for many practical objects, and means that in an asynchronous system, we can't solve contention at the object level in the worst case (though we may be able to avoid it in our applications).

But there might be a way out for some restricted classes of objects. We saw in Chapter 21 that we could escape from the Jayanti-Tan-Toueg [JTT00] lower bound by considering bounded objects. Something similar may happen here: the Fich-Herlihy-Shavit bound on fetch-and-increments requires executions with $n(n-1)^d + n$ increments to show $n-1$ stalls for some fetch-and-increment if each fetch-and-increment only touches d objects, and even for $d = \log n$ this is already superpolynomial. The max-register construction of a counter [AAC09] doesn't help here, since everybody hits the switch bit at the top of the max register, giving $n-1$ stalls if they all hit it at the same time. But there might be some better construction that avoids this.

26.4.5 More lower bounds

There are many more lower bounds one can prove on lock-free implementations, many of which are based on previous lower bounds for stronger models. We won't present these in class, but if you are interested, a good place to start is [AGHK06].

26.5 Practical considerations

Also beyond the scope of what we can do, there is a paper by Fraser and Harris [FH07] that gives some nice examples of the practical trade-offs in choosing between multi-register CAS and various forms of software transactional memory in implementing lock-free data structures.

[7] This is out of date: Theorem 6.2 of [EHS12] gives a stronger result than what's in [FHS05].

Chapter 27

BG simulation

The **Borowsky-Gafni simulation** [BG93], or **BG simulation** for short, is a deterministic, wait-free algorithm that allows $t+1$ processes to collectively construct a simulated execution of a system of $n > t$ processes of which t may crash. For both the simulating and simulated system, the underlying shared-memory primitives are atomic snapshots; these can be replaced by atomic registers using any standard snapshot algorithm. The main consequence of the BG simulation is that the question of what decision tasks can be computed deterministically by an asynchronous shared-memory system that tolerates t crash failures reduces to the question of what can be computed by a wait-free system with exactly $t+1$ processes. This is an easier problem, and in principle can be determined exactly using the topological approach described in Chapter 28.

The intuition for how this works is that the $t+1$ simulating processes solve a sequence of agreement problems to decide what the n simulated processes are doing; these agreement problems are structured so that the failure of a simulator stops at most one agreement. So if at most t of the simulating processes can fail, only t simulated processes get stuck as well.

We'll describe here a version of the BG simulation that appears in a follow-up paper by Borowsky, Gafni, Lynch, and Rajsbaum [BGLR01]. This gives a more rigorous presentation of the mechanisms of the original Borowsky-Gafni paper, and includes a few simplifications.

27.1 Safe agreement

The **safe agreement** mechanism performs agreement without running into the FLP bound, by using termination condition: it is guaranteed to termi-

nate only if there are no failures by any process during an initial **unsafe** section of its execution. Each process i starts the agreement protocol with a $\texttt{propose}_i(v)$ event for its input value v. At some point during the execution of the protocol, the process receives a notification $\texttt{safe}_i$, followed later (if the protocol finishes) by a second notification $\texttt{agree}_i(v')$ for some output value v'. It is guaranteed that the protocol terminates as long as all processes continue to take steps until they receive the `safe` notification, and that the usual validity (all outputs equal some input) and agreement (all outputs equal each other) conditions hold. There is also a wait-free progress condition that the $\texttt{safe}_i$ notices do eventually arrive for any process that doesn't fail, no matter what the other processes do (so nobody gets stuck in their unsafe section).

Pseudocode for a safe agreement object is given in Algorithm 27.1. This is a translation of the description of the algorithim in [BGLR01], which is specified at a lower level using I/O automata.

```
   // propose_i(v)
1  A[i] ← ⟨v, i⟩
2  if snapshot(A) contains ⟨j, 2⟩ for some j ≠ i then
       // Back off
3      A[i] ← ⟨v, 0⟩
4  else
       // Advance
5      A[i] ← ⟨v, 2⟩
   // safe_i
6  repeat
7      s ← snapshot(A)
8  until s does not contain ⟨j, 1⟩ for any j
   // agree_i
9  return s[j].value where j is smallest index with s[j].level = 2
```

Algorithm 27.1: Safe agreement (adapted from [BGLR01])

The communication mechanism is a snapshot object containing a pair $A[i] = \langle \mathsf{value}_i, \mathsf{level}_i \rangle$ for each process i, initially $\langle \perp, 0 \rangle$. When a process carries out $\texttt{propose}_i(v)$, it sets $A[i]$ to $\langle v, 1 \rangle$, advancing to level 1. It then looks around to see if anybody else is at level 2; if so, it backs off to 0, and if not, it advances to 2. In either case it then spins until it sees a snapshot with nobody at level 1, and agrees on the level-2 value with the smallest index i.

The $\texttt{safe}_i$ transition occurs when the process leaves level 1 (no matter which way it goes). This satisfies the progress condition, since there is no loop before this, and guarantees termination if all processes leave their unsafe interval, because no process can then wait forever for the last 1 to disappear.

To show agreement, observe that at least one process advances to level 2 (because the only way a process doesn't is if some other process has already advanced to level 2), so any process i that terminates observes a snapshot s that contains at least one level-2 tuple and no level-1 tuples. This means that any process j whose value is not already at level 2 in s can at worst reach level 1 after s is taken. But then j sees a level-2 tuples and backs off. It follows that any other process i' that takes a later snapshot s' that includes no level-1 tuples sees the same level-2 tuples as i, and computes the same return value. (Validity also holds, for the usual trivial reasons.)

27.2 The basic simulation algorithm

The basic BG simulation uses a single snapshot object A with $t+1$ components (one for each simulating process) and an infinite array of safe agreement objects S_{jr}, where the output of S_{jr} represents the value s_{jr} of the r-th snapshot performed by simulated process j. Each component $A[i]$ of A is itself a vector of n components $A[i][j]$, each of which is a tuple $\langle v, r\rangle$ representing the value v that process i determines process j would have written after taking its r-th snapshot.[1]

Each simulating process i cycles through all simulated processes j. Simulating one round of a particular process j involves four phases:

1. The process makes an initial guess for s_{jr} by taking a snapshot of A and taking the value with the largest round number for each component $A[-][k]$.

2. The process initiates the safe agreement protocol S_{jr} using this guess. It continues to run S_{jr} until it leaves the unsafe interval.

[1]The underlying assumption is that all simulated processes alternate between taking snapshots and doing updates. This assumption is not very restrictive, because two snapshots with no intervening update are equivalent to two snapshots separated by an update that doesn't change anything, and two updates with no intervening snapshot can be replaced by just the second update, since the adversary could choose to schedule them back-to-back anyway.

3. The process attempts to finish S_{jr}, by performing one iteration of the loop from Algorithm 27.1. If this iteration doesn't succeed, it moves on to simulating $j+1$ (but will come back to this phase for j eventually).

4. If S_{jr} terminates, the process computes a new value v_{jr} for j to write based on the simulated snapshot returned by S_{jr}, and updates $A[i][j]$ with $\langle v_{jr}, r\rangle$.

Actually implementing this while maintaining an abstraction barrier around safe agreement is tricky. One approach might be to have each process i manage a separate thread for each simulated process j, and wrap the unsafe part of the safe agreement protocol inside a mutex just for threads of i. This guarantees that i enters the unsafe part of any safe agreement object on behalf of only one simulated j at a time, while preventing delays in the safe part of S_{jr} from blocking it from finishing some other $S_{j'r'}$.

27.3 Effect of failures

So now what happens if a simulating process i fails? This won't stop any other process i' from taking snapshots on behalf of j, or from generating its own values to put in $A[i'][j]$. What it may do is prevent some safe agreement object S_{jr} from terminating. The termination property of S_{jr} means that this can only occur if the failure occurs while i is in the unsafe interval for S_{jr}—but since i is only in the unsafe interval for at most one S_{jr} at a time, this stalls only one simulated process j. It doesn't block any i', because any other i' is guaranteed to leave its own unsafe interval for S_{jr} after finitely many steps, and though it may waste some effort waiting for S_{jr} to finish, once it is in the safe interval it doesn't actually wait for it before moving on to other simulated j'.

It follows that each failure of a simulating process knocks out at most one simulated process. So a wait-free system with $t+1$ processes—and thus at most t failures in the executions we care about—will produces at most t failures inside the simulation.

27.4 Inputs and outputs

Two details not specified in the description above are how i determines j's initial input and how i determines its own outputs from the outputs of the simulated processes. For the basic BG simulation, this is pretty straightforwards: we use the safe agreement objects S_{j0} to agree on j's

input, after each i proposes its own input vector for all j based on its own input to the simulator protocol. For outputs, i waits for at least $n-t$ of the simulated processes to finish, and computes its own output based on what it sees.

One issue that arises here is that we can only use the simulation to solve **colorless tasks**, which are decision problems where any process can take the output of any other process without causing trouble.[2] This works for consensus or k-set agreement, but fails pretty badly for renaming. The **extended BG simulation**, due to Gafni [Gaf09], solves this problem by mapping each simulating process p to a specific simulated process q_p, and using a more sophisticated simulation algorithm to guarantee that q_p doesn't crash unless p does. Details can be found in Gafni's paper; there is also a later paper by Imbs and Raynal [IR09] that simplifies some details of the construction. Here, we will limit ourselves to the basic BG simulation.

27.5 Correctness of the simulation

To show that the simulation works, observe that we can extract a simulated execution by applying the following rules:

1. The round-r write operation of j is represented by the first write tagged with round r performed for j.

2. The round-r snapshot operation of j is represented by whichever snapshot operation wins S_{jr}.

The simulated execution then consists of a sequence of write and snapshot operations, with order of the operations determined by the order of their representatives in the simulating execution, and the return values of the snapshots determined by the return values of their representatives.

Because all processes that simulate a write for j in round r use the same snapshots to compute the state of j, they all write the same value. So the only way we get into trouble is if the writes included in our simulated snapshots are inconsistent with the ordering of the simulated operations defined above. Here the fact that each simulated snapshot corresponds to a real snapshot makes everything work: when a process performs a snapshot for S_{jr}, then it includes all the simulated write operations that happen

[2] The term "colorless" here comes from use of colors to represent process ids in the topological approach described in Chapter 28. These colors aren't really colors, but topologists like coloring nodes better than assigning them ids.

before this snapshot, since the s-th write operation by k will be represented in the snapshot if and only if the first instance of the s-th write operation by k occurs before it. The only tricky bit is that process i's snapshot for S_{jr} might include some operations that can't possibly be included in S_{jr}, like j's round-r write or some other operation that depends on it. But this can only occur if some other process finished S_{jr} before process i takes its snapshot, in which case i's snapshot will not win S_{jr} and will be discarded.

27.6 BG simulation and consensus

BG simulation was originally developed to attack k-set agreement, but (as pointed out by Gafni [Gaf09]) it gives a particularly simple proof of the impossibility of consensus with one faulty process. Suppose that we had a consensus protocol that solved consensus for $n > 1$ processes with one crash failure, using only atomic registers. Then we could use BG simulation to get a wait-free consensus protocol for two processes. But it's easy to show that atomic registers can't solve wait-free consensus, because (following [LAA87]), we only need to do the last step of FLP that gets a contradiction when moving from a bivalent C to 0-valent Cx or 1-valent Cy. We thus avoid the complications that arise in the original FLP proof from having to deal with fairness.

More generally, BG simulation means that increasing the number of processes while keeping the same number of crash failures doesn't let us compute anything we couldn't before. This gives a formal justification for the slogan that the difference between distributed computing and parallel computing is that in a distributed system, more processes can only make things worse.

Chapter 28

Topological methods

Here we'll describe some results applying topology to distributed computing, mostly following a classic paper of Herlihy and Shavit [HS99]. This was one of several papers [BG93, SZ00] that independently proved lower bounds on **k-set agreement** [Cha93], which is a relaxation of consensus where we require only that there are at most k distinct output values (consensus is 1-set agreement). These lower bounds had failed to succumb to simpler techniques.

28.1 Basic idea

- Represent indistinguishability proofs using tools from topology.
- Typical indistinguishability proof:
  - Show certain executions are indistinguishable to some process (and thus that process produces same output in both executions).
  - In general case, have a chain of schedules $S_1, S_2, \ldots, S_k$ such that for each i there is some p with $S_i|p = S_{i+1}|p$. The restriction to p acts as an edge between points representing executions, and we use the existence of a path of such edges as a proof that the decision value in S_1 is the same as in S_k, assuming all processes must agree on the decision value.
- Topological version:
  - Essentially the dual of the above: points are now individual process states (or histories), and edges (and higher-dimensional

structures) represent consistent states of different processes (i.e., executions in which both states occur).

- Considering many possible states produces a **simplicial complex**, a finite combinatorial structure used in topology to model continuous surfaces.
- Properties of the simplicial complex resulting from some protocol or problem specification can then be used to determine properties of the underlying protocol or problem.
- Topologists know a lot of properties to look at.

28.2 k-set agreement

The motivating problem for much of this work was getting impossibility results for **k-set agreement**, proposed by Chaudhuri [Cha93]. The k-set agreement problem is similar to consensus, where each process starts with an input and eventually returns a decision value that must be equal to some process's input, but the agreement condition is relaxed to require only that the set of decision values include at most k values.

With $k-1$ crash failures, it's easy to build a k-set agreement algorithm: wait until you seen $n-k+1$ input values, then choose the smallest one you see. This works because any value a process returns is necessarily among the k smallest input values (including the $k-1$ it didn't see). Chaudhuri conjectured that k-set agreement was not solvable with k failures, and gave a proof of a partial result (analogous to the existence of an initial bivalent configuration for consensus) based on Sperner's Lemma [Spe28]. This is a classic result in topology that says that certain colorings of the vertices of a graph in the form of a triangle that has been divided into smaller triangles necessarily contain a small triangle with three different colors on its corners. This connection between k-set renaming and Sperner's Lemma became the basic idea behind each the three independent proofs of the conjecture that appeared shortly thereafter [HS99, BG93, SZ00].

Our plan is to give a sufficient high-level description of the topological approach that the connection between k-set agreement and Sperner's Lemma becomes obvious. It is possible to avoid this by approaching the problem purely combinatorially, as is done in Section 16.3 of [AW04]. The presentation there is obtained by starting with a topological argument and getting rid of the topology (in fact, the proof in [AW04] contains a proof of Sperner's Lemma with the serial numbers filed off). The disadvantage of this approach is that it obscures what is really going in and makes it harder

to obtain insight into how topological techniques might help for other problems. The advantage is that (unlike these notes) the resulting text includes actual proofs instead of handwaving.

28.3 Representing distributed computations using topology

Topology is the study of properties of shapes that are preserved by continuous functions between their points that have continuous inverses, which get the rather fancy name of **homeomorphisms**. A continuous function[1] is one that maps nearby points to nearby points. A homeomorphism is continuous in both directions: this basically means that you can stretch and twist and otherwise deform your object however you like, as long as you don't tear it (which would map nearby points on opposite sides of the tear to distant points) or glue bits of it together (which turns into tearing when we look at the inverse function). Topologists are particularly interested in showing when there is no homeomorphism between two objects; the classic example is that you can't turn a sphere into a donut without damaging it, but you can turn a donut into a coffee mug (with a handle).

Working with arbitrary objects embedded in umpteen-dimensional spaces is messy, so topologists invented a finite way of describing certain well-behaved objects combinatorially, by replacing ugly continuous objects like spheres and coffee mugs with simpler objects pasted together in complex ways. The simpler objects are **simplexes**, and the more complicated pasted-together objects are called **simplicial complexes**. The nifty thing about simplicial complexes is that they give a convenient tool for describing what states or outputs of processes in a distributed algorithm are "compatible" in some sense, and because topologists know a lot about simplicial complexes, we can steal their tools to describe distributed algorithms.

28.3.1 Simplicial complexes and process states

The formal definition of a k-dimensional **simplex** is the convex closure of $(k+1)$ points $\{x_1 \dots x_{k+1}\}$ in general position; the convex closure part means the set of all points $\sum a_i x_i$ where $\sum a_i = 1$ and each $a_i \geq 0$, and the general position part means that the x_i are not all contained in some

[1] Strictly speaking, a continuous function between metric spaces; there is an even more general definition of continuity that holds for spaces that are too strange to have a consistent notion of distance.

subspace of dimension $(k-1)$ or smaller (so that the simplex isn't squashed flat somehow). What this gives us is a body with $(k+1)$ corners and $(k+1)$ faces, each of which is a $(k-1)$-dimensional simplex (the base case is that a 0-dimensional simplex is a point). Each face includes all but one of the corners, and each corner is on all but one of the faces. So we have:

- 0-dimensional simplex: point.[2]
- 1-dimensional simplex: line segment with 2 endpoints (which are both corners and faces).
- 2-dimensional simplex: triangle (3 corners with 3 1-dimensional simplexes for sides).
- 3-dimensional simplex: tetrahedron (4 corners, 4 triangular faces).
- 4-dimensional simplex: 5 corners, 5 tetrahedral faces. It's probably best not to try to visualize this.

A simplicial complex is a bunch of simplexes stuck together; formally, this means that we pretend that some of the corners (and any faces that include them) of different simplexes are identical points. There are ways to do this right using equivalence relations. But it's easier to abstract out the actual geometry and go straight to a combinatorial structure.

An (abstract) simplicial complex is just a collection of sets with the property that if A is a subset of B, and B is in the complex, then A is also in the complex (this means that if some simplex is included, so are all of its faces, their faces, etc.). This combinatorial version is nice for reasoning about simplicial complexes, but is not so good for drawing pictures.

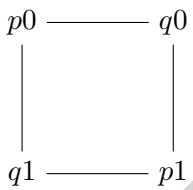
The trick to using this for distributed computing problems is that we are going to build simplicial complexes by letting points be process states (or sometimes process inputs or outputs), each labeled with a process id, and letting the sets that appear in the complex be those collections of states/inputs/outputs that are compatible with each other in some sense. For states, this means that they all appear in some global configuration in some admissible execution of some system; for inputs and outputs, this means that they are permitted combinations of inputs or outputs in the specification of some problem.

[2]For consistency, it's sometimes convenient to define a point as having a single (-1)-dimensional face defined to be the empty set. We won't need to bother with this, since 0-dimensional simplicial complexes correspond to 1-process distributed systems, which are amply covered in almost every other Computer Science class you have ever taken.

Example: For 2-process binary consensus with processes 0 and 1, the **input complex**, which describes all possible combinations of inputs, consists of the sets

$$\{\{\}, \{p0\}, \{q0\}, \{p1\}, \{q1\}, \{p0, q0\}, \{p0, q1\}, \{p1, q0\}, \{p1, q1\}\},$$

which we might draw like this:

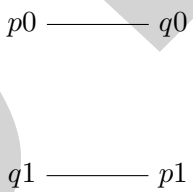


Note that there are no edges from $p0$ to $p1$ or $q0$ to $q1$: we can't have two different states of the same process in the same global configuration.

The **output complex**, which describes the permitted outputs, is

$$\{\{\}, \{p0\}, \{q0\}, \{p1\}, \{q1\}, \{p0, q0\}, \{p1, q1\}\}.$$

As a picture, this omits two of the edges (1-dimensional simplexes) from the input complex:

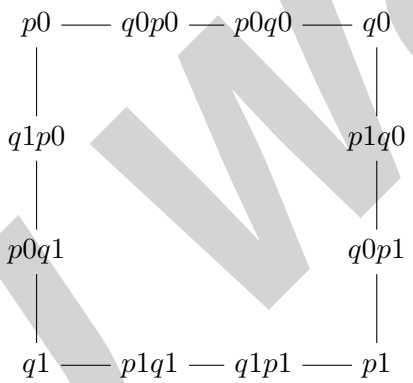


One thing to notice about this output complex is that it is not **connected**: there is no path from the $p0$–$q0$ component to the $q1$–$p1$ component.

Here is a simplicial complex describing the possible states of two processes p and q, after each writes 1 to its own bit then reads the other process's bit. Each node in the picture is labeled by a sequence of process ids. The first id in the sequence is the process whose view this node represents; any other process ids are processes this first process sees (by seeing a 1 in the other process's register). So p is the view of process p running by itself, while pq is the view of process p running in an execution where it reads q's register after q writes it.

The edges express the constraint that if we both write before we read, then if I don't see your value you must see mine (which is why there is no p–q edge), but all other combinations are possible. Note that this complex *is* connected: there is a path between any two points.

Here's a fancier version in which each process writes its input (and remembers it), then reads the other process's register (i.e., a one-round full-information protocol). We now have final states that include the process's own id and input first, then the other process's id and input if it is visible. For example, $p1$ means p starts with 1 but sees a null and $q0p1$ means q starts with 0 but sees p's 1. The general rule is that two states are compatible if p either sees nothing or q's actual input and similarly for q, and that at least one of p or q must see the other's input. This gives the following simplicial complex:



Again, the complex is connected.

The fact that this looks like four copies of the p–qp–pq–q complex pasted into each edge of the input complex is not an accident: if we fix a pair of inputs i and j, we get pi–$qjpi$–$piqj$–qj, and the corners are pasted together because if p sees only $p0$ (say), it can't tell if it's in the $p0/q0$ execution or the $p0/q1$ execution.

The same process occurs if we run a two-round protocol of this form, where the input in the second round is the output from the first round. Each round subdivides one edge from the previous round into three edges:

$p - q$

$p - qp - pq - q$

$p - (qp)p - p(qp) - qp - (pq)(qp) - (qp)(pq) - pq - q(pq) - (pq)q - q$

Here $(pq)(qp)$ is the view of p after seeing pq in the first round and seeing that q saw qp in the first round.

28.3.2 Subdivisions

In the simple write-then-read protocol above, we saw a single input edge turn into 3 edges. Topologically, this is an example of a **subdivision**, where we represent a simplex using several new simplexes pasted together that cover exactly the same points.

Certain classes of protocols naturally yield subdivisions of the input complex. The **iterated immediate snapshot** (IIS) model, defined by Borowsky and Gafni [BG97], considers executions made up of a sequence of rounds (the iterated part) where each round is made up of one or more mini-rounds in which some subset of the processes all write out their current views to their own registers and then take snapshots of all the registers (the immediate snapshot part). The two-process protocols of the previous section are special cases of this model.

Within each round, each process p obtains a view v_p that contains the previous-round views of some subset of the processes. We can represent the views as a subset of the processes, which we will abbreviate in pictures by putting the view owner first: pqr will be the view $\{p, q, r\}$ as seen by p, while qpr will be the same view as seen by q. The requirements on these views are that (a) every process sees its own previous view: $p \in v_p$ for all p; (b) all views are comparable: $v_p \subseteq v_q$ or $v_q \subseteq v_p$; and (c) if I see you, then I see everything you see: $q \in v_p$ implies $v_q \subseteq v_p$. This last requirement is called **immediacy** and follows from the assumption that writes and snapshots are done in the same mini-round: if I see your write, then your snapshot takes place no later than mine does.

The IIS model does not correspond exactly to a standard shared-memory model (or even a standard shared-memory model augmented with cheap snapshots). There are two reasons for this: standard snapshots don't provide

immediacy, and standard snapshots allow processes to go back an perform more than one snapshot on the same object. The first issue goes away if we are looking at impossibility proofs, because the adversary can restrict itself only to those executions that satisfy immediacy. The second issue is more delicate, but Borowsky and Gafni demonstrate that any decision protocol that runs in the standard model can be simulated in the IIS model, using a variant of BG simulation.

For three processes, one round of immediate snapshots gives rise to the simplicial complex depicted in Figure 28.1. The corners of the big triangle are the solo views of processes that do their snapshots before anybody else shows up. Along the edges of the big triangle are views corresponding to 2-process executions, while in the middle are complete views of processes that run late enough to see everything. Each little triangle corresponds to some execution. For example, the triangle with corners p, qp, rpq corresponds to a sequential execution where p sees nobody, q sees p, and r sees both p and q. The triangle with corners pqr, qpr, and rpq is the maximally-concurrent execution where all three processes write before all doing their snapshots: here everybody sees everybody. It is not terribly hard to enumerate all possible executions and verify that the picture includes all of them. In higher dimension, the picture is more complicated, but we still get a subdivision that preserves the original topological structure [BG97].

Figure 28.2 shows (part of) the next step of this process: here we have done two iterations of immediate snapshot, and filled in the second-round subdivisions for the p–qpr–rpq and pqr–qpr–rpq triangles. (Please imagine similar subdivisions of all the other triangles that I was too lazy to fill in by hand.) The structure is recursive, with each first-level triangle mapping to an image of the entire first-level complex. As in the two-process case, adjacent triangles overlap because the relevant processes don't have enough information; for example, the points on the qpr–rpq edge correspond to views of q or r that don't include p in round 2 and so can't tell whether p saw p or pqr in round 1.

The important feature of the round-2 complex (and the round-k complex in general) is that it's a **triangulation** of the original outer triangle: a partition into little triangles where each corner aligns with corners of other little triangles.

(Better pictures of this process in action can be found in Figures 25 and 26 of [HS99].)

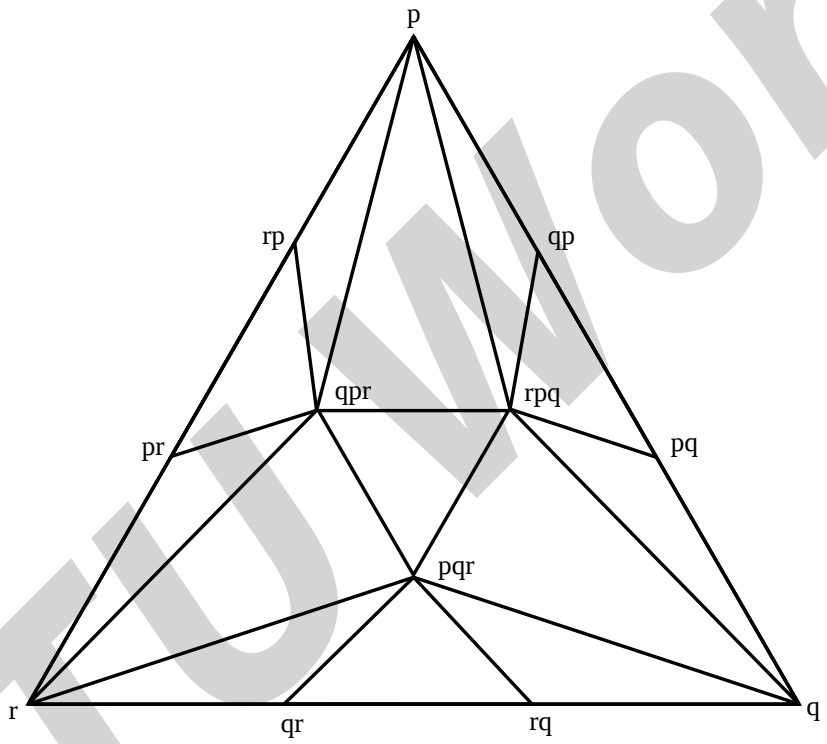


Figure 28.1: Subdivision corresponding to one round of immediate snapshot

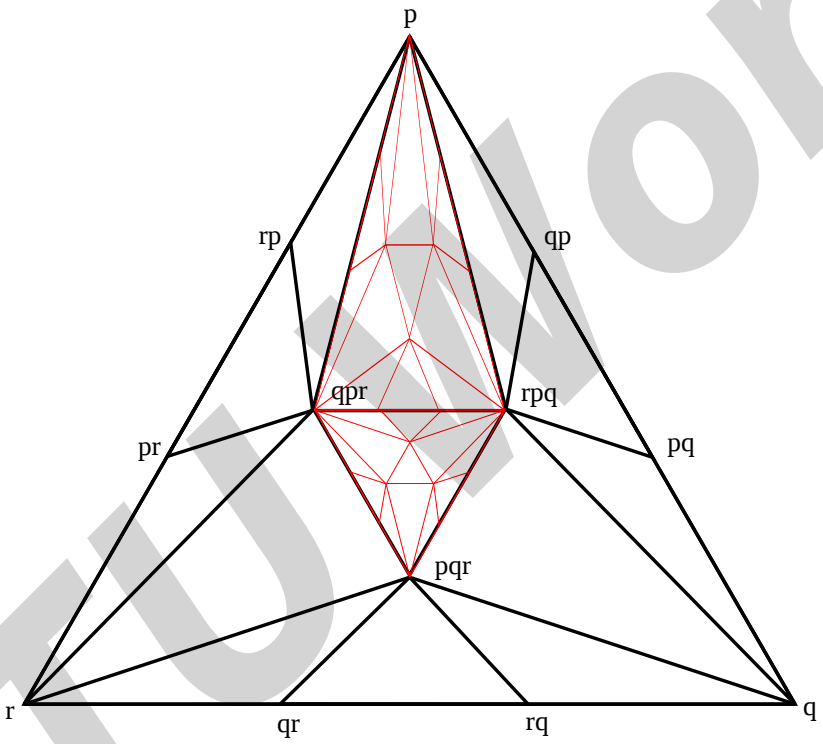


Figure 28.2: Subdivision corresponding to two rounds of immediate snapshot

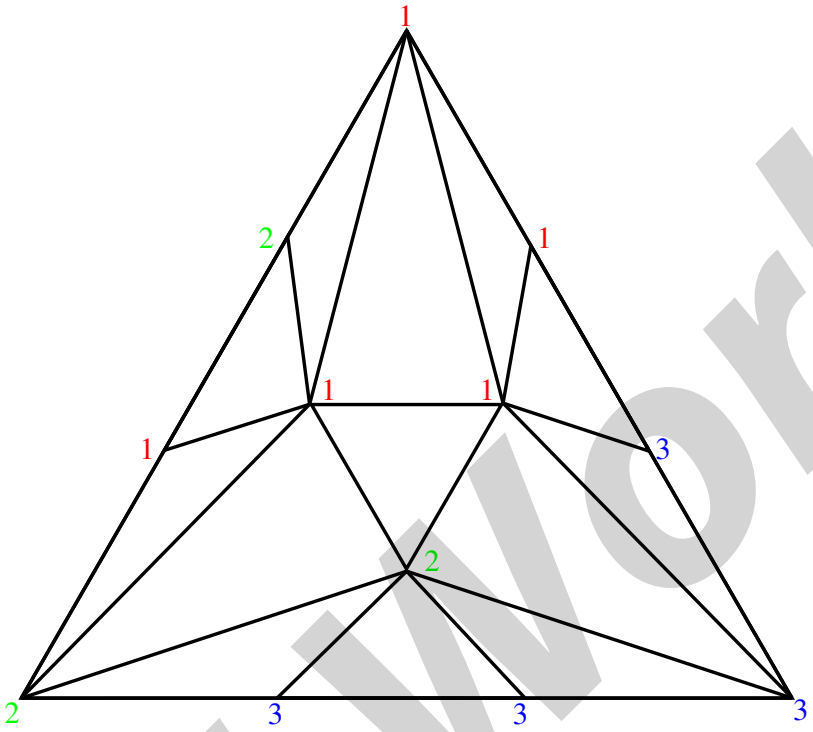


Figure 28.3: An attempt at 2-set agreement

28.4 Impossibility of k-set agreement

Now let's show that there is no way to do k-set agreement with $n = k + 1$ processes in the IIS model.

Suppose that after some fixed number of rounds, each process chooses an output value. This output can only depend on the view of the process, so is fixed for each vertex in the subdivision. Also, the validity condition means that a process can only choose an output that it can see among the inputs in its view. This means that at the corners of the outer triangle (corresponding to views where the process thinks it's alone), a process must return its input, while along the outer edges (corresponding to views where two processes may see each other but not the third), a process must return one of the two inputs that appear in the corners incident to the edge. Internal corners correspond to views that include—directly or indirectly—the inputs of all processes, so these can be labeled arbitrarily. An example is given in Figure 28.3, for a one-round protocol with three processes.

We now run into Sperner's Lemma [Spe28], which says that, for any subdivision of a simplex into smaller simplexes, if each corner of the original simplex has a different color, and each corner that appears on some face of the original simplex has a color equal to the color of one of the corners of that face, then within the subdivision there are an odd number of simplexes whose corners are all colored differently.[3]

How this applies to k-set agreement: Suppose we have $n = k+1$ processes in a wait-free system (corresponding to allowing up to k failures). With the cooperation of the adversary, we can restrict ourselves to executions consisting of ℓ rounds of iterated immediate snapshot for some ℓ (termination comes in here to show that ℓ is finite). This gives a subdivision of a simplex, where each little simplex corresponds to some particular execution and each corner some process's view. Color all the corners of the little simplexes in this subdivision with the output of the process holding the corresponding view. Validity means that these colors satisfy the requirements of Sperner's Lemma. Sperner's Lemma then says that some little simplex has all $k + 1$ colors, giving us a bad execution with more than k distinct output values.

The general result says that we can't do k-set agreement with k failures for any $n > k$. We haven't proved this result, but it can be obtained from the $n = k + 1$ version using a simulation of $k + 1$ processes with k failures by n processes with k failures due to Borowsky and Gafni [BG93].

28.5 Simplicial maps and specifications

Let's step back and look at consensus again.

[3] The proof of Sperner's Lemma is not hard, and is done by induction on the dimension k. For $k = 0$, any subdivision consists of exactly one zero-dimensional simplex whose single corner covers all $k + 1 = 1$ colors. For $k + 1$, suppose that the colors are $\{1, \ldots, k + 1\}$, and construct a graph with a vertex for each little simplex in the subdivision and an extra vertex for the region outside the big simplex. Put an edge in this graph between each pair of regions that share a k-dimensional face with colors $\{1, \ldots, k\}$. The induction hypothesis tells us that there are an odd number of edges between the outer-region vertex and simplexes on the $\{1, \ldots, k\}$-colored face of the big simplex. The Handshaking Lemma from graph theory says that the sum of the degrees of all the nodes in the graph is even. But this can only happen if there are an even number of nodes with odd degree, implying that the are are an odd number of simplexes in the subdivision with an odd number of faces colored $\{1, \ldots, k\}$.

Now suppose we have a simplex with at least one face f colored $\{1, \ldots, k\}$. If the opposite corner is colored $c \neq k + 1$, then it has exactly two faces colored $\{1, \ldots, k\}$: f, and the face that replaces f's c-colored corner with the opposite corner. So the only way to get an odd number of $\{1, \ldots, k\}$-colored faces is to have all $k+1$ colors. It follows that there are an odd number of $(k + 1)$-colored simplexes.

One thing we could conclude from the fact that the output complex for consensus was not connected but the ones describing our simple protocols were was that we can't solve consensus (non-trivially) using these protocols. The reason is that to solve consensus using such a protocol, we would need to have a mapping from states to outputs (this is just whatever rule tells each process what to decide in each state) with the property that if some collection of states are consistent, then the outputs they are mapped to are consistent.

In simplical complex terms, this means that the mapping from states to outputs is a **simplicial map**, a function f from points in one simplicial complex C to points in another simplicial complex D such that for any simplex $A \in C$, $f(A) = \{f(x)|x \in A\}$ gives a simplex in D. (Recall that consistency is represented by including a simplex, in both the state complex and the output complex.) A mapping from states to outputs that satisfies the consistency requirements encoded in the output complex s always a simplicial map, with the additional requirement that it preserves process ids (we don't want process p to decide the output for process q). Conversely, any id-preserving simplicial map gives an output function that satisfies the consistency requirements.

Simplicial maps are examples of **continuous functions**, which have all sorts of nice topological properties. One nice property is that a continuous function can't separate a connected space into disconnected components. We can prove this directly for simplical maps: if there is a path of 1-simplexes $\{x_1, x_2\}, \{x_2, x_3\}, \ldots \{x_{k-1}, x_k\}$ from x_1 to x_k in C, and $f : C \to D$ is a simplicial map, then there is a path of 1-simplexes $\{f(x_1), f(x_2)\}, \ldots$ from $f(x_1)$ to $f(x_k)$. Since being connected just means that there is a path between any two points,[4] if C is connected we've just shown that $f(C)$ is as well.

Getting back to our consensus example, it doesn't matter what simplicial map f you pick to map process states to outputs; since the state complex C is connected, so is $f(C)$, so it lies entirely within one of the two connected components of the output complex. This means in particular that everybody always outputs 0 or 1: the protocol is trivial.

28.5.1 Mapping inputs to outputs

For general decision tasks, it's not enough for the outputs to be consistent with each other. They also have to be consistent with the inputs. This can

[4] Technically, this is the definition of **path-connected**, which is the same as connected for well-behaved topological spaces.

be expressed by a relation Δ between input simplexes and output simplexes.

Formally, a decision task is modeled by a triple (I, O, Δ), where I is the input complex, O is the output complex, and $(A, B) \in \Delta$ if and only if B is a permissible output given input I. Here there are no particular restrictions on Δ (for example, it doesn't have to be a simplicial map or even a function), but it probably doesn't make sense to look at decision tasks unless there is at least one permitted output simplex for each input simplex.

28.6 The asynchronous computability theorem

Given a decision task specified in this way, there is a topological characterization of when it has a wait-free solution. This is given by the **Asynchronous Computability Theorem** (Theorem 3.1 in [HS99]), which says:

Theorem 28.6.1. *A decision task (I, O, Δ) has a wait-free protocol using shared memory if and only if there exists a chromatic subdivision σ of I and a color-preserving simplicial map $\mu : \sigma(I) \to O$ such that for each simplex s in $\sigma(I)$, $\mu(S) \in \Delta(\text{carrier}(S, I))$.*

To unpack this slightly, a **chromatic subdivision** is a subdivision where each vertex is labeled by a process id (a color), and no simplex has two vertices with the same color. A color-preserving simplicial map is a simplicial map that preserves ids. The carrier of a simplex in a subdivision is whatever original simplex it is part of. So the theorem says that I can only solve a task if I can find a simplicial map from a subdivision of the input complex to the output complex that doesn't do anything strange to process ids and that is consistent with Δ.

Looking just at the theorem, one might imagine that the proof consists of showing that the **protocol complex** defined by the state complex after running the protocol to completion is a subdivision of the input complex, followed by the same argument we've seen already about mapping the state complex to the output complex. This is almost right, but it's complicated by two inconvenient facts: (a) the state complex generally isn't a subdivision of the input complex, and (b) if we have a map from an arbitrary subdivision of the input complex, it is not clear that there is a corresponding protocol that produces this particular subdivision.

So instead the proof works like this:

Protocol implies map Even though we don't get a subdivision with the full protocol, there is a restricted set of executions that does give a

subdivision. So if the protocol works on this restricted set of executions, an appropriate map exists. There are two ways to prove this: Herlihy and Shavit do so directly, by showing that this restricted set of executions exists, and Borowksy and Gafni [BG97] do so indirectly, by showing that the IIS model (which produces exactly the standard chromatic subdivision used in the ACT proof) can simulate an ordinary snapshot model. Both methods are a bit involved, so we will skip over this part.

Map implies protocol This requires an algorithm. The idea here is that that **participating set** algorithm, originally developed to solve k-set agreement [BG93], produces precisely the standard chromatic subdivision used in the ACT proof. In particular, it can be used to solve the problem of **simplex agreement**, the problem of getting the processes to agree on a particular simplex contained within the subdivision of their original common input simplex. This is a little easier to explain, so we'll do it.

28.6.1 The participating set protocol

Algorithm 28.1 depicts the participating set protocol; this first appeared in [BG93], although the presentation here is heavily influenced by the version in Elizabeth Borowsky's dissertation [Bor95]. The shared data consists of a snapshot object level, and processes start at a high level and float down until they reach a level i such that there are already i processes at this level or below. The set returned by a process consists of all processes it sees at its own level or below, and it can be shown that this in fact implements a one-shot immediate snapshot. Since immediate snapshots yield a standard subdivision, this gives us what we want for converting a color-preserving simplicial map to an actual protocol.

```
1 Initially, level[i] = n + 2 for all i.
2 repeat
3 |   level[i] ← level[i] − 1
4 |   v ← snapshot(level)
5 |   S ← {j | v[j] ≤ level[i]}
6 until |S| ≥ level[i]
7 |S| ≥ level[i] return S
```

Algorithm 28.1: Participating set

The following theorem shows that the return values from participating set have all the properties we want for iterated immediate snapshot:

Theorem 28.6.2. *Let S_i be the output of the participating set algorithm for process i. Then all of the following conditions hold:*

1. *For all i, $i \in S_i$. (Self-containment.)*
2. *For all i, j, $S_i \subseteq S_j$ or $S_j \subseteq S_i$. (Atomic snapshot.)*
3. *For all i, j, if $i \in S_j$, then $S_i \subseteq S_j$. (Immediacy.)*

Proof. Self-inclusion is trivial, but we will have to do some work for the other two properties.

The first step is to show that Algorithm 28.1 neatly sorts the processes out into levels, where each process that returns at level k returns precisely the set of processes at level k and below.

For each process i, let S_i be defined as above, let ℓ_i be the final value of $\mathsf{level}[i]$ when i returns, and let $S'_i = \{j \mid \ell_j \leq S_i\}$. Our goal is to show that $S'_i = S_i$, justifying the above claim.

Because no process ever increases its level, if process i observes $\mathsf{level}[j] \leq \ell_i$ in its last snapshot, then $\ell_j \leq \mathsf{level}[j] \leq \ell_i$. So S'_i is a superset of S_i. We thus need to show only that no extra processes sneak in; in particular, we will to show that $S_i = S'_i$, by showing that both equal ℓ_i.

The first step is to show that $|S'_i| \geq |S_i| \geq \ell_i$. The first inequality follows from the fact that $S'_i \supseteq S_i$; the second follows from the code (if not, i would have stayed in the loop).

The second step is to show that $|S'_i| \leq \ell_i$. Suppose not; that is, suppose that $|S'_i| > \ell_i$. Then there are at least $\ell_i + 1$ processes with level ℓ_i or less, all of which take a snapshot on level $\ell_i + 1$. Let i' be the last of these processes to take a snapshot while on level $\ell_i + 1$. Then i' sees at least $\ell_i + 1$ processes at level $\ell_i + 1$ or less and exits, contradicting the assumption that it reaches level ℓ_i. So $|S'_i| \leq \ell_i$.

The atomic snapshot property follows immediately from the fact that if $\ell_i \leq \ell_j$, then $\ell_k \leq \ell_i$ implies $\ell_k \leq \ell_j$, giving $S_i = S'_i \subseteq S'_j = S_j$. Similarly, for immediacy we have that if $i \in S_j$, then $\ell_i \leq \ell_j$, giving $S_i \leq S_j$ by the same argument. □

The missing piece for turning this into IIS is that in Algorithm 28.1, I only learn the identities of the processes I am supposed to include but not their input values. This is easily dealt with by adding an extra register for each process, to which it writes its input before executing participating set.

28.7 Proving impossibility results

To show something is impossible using the ACT, we need to show that there is no color-preserving simplicial map from a subdivision of I to O satisfying the conditions in Δ. This turns out to be equivalent to showing that there is no continuous function from I to O with the same properties, because any such simplicial map can be turned into a continuous function (on the geometric version of I, which includes the intermediate points in addition to the corners). Fortunately, topologists have many tools for proving non-existence of continuous functions.

28.7.1 k-connectivity

Define the m-dimensional **disk** to be the set of all points at most 1 unit away from the origin in $\mathbb{R}^m$, and the m-dimensional **sphere** to be the surface of the $(m+1)$-dimensional disk (i.e., all points exactly 1 unit away from the origin in $\mathbb{R}^{m+1}$). Note that what we usually think of as a sphere (a solid body), topologists call a disk, leaving the term sphere for just the outside part.

An object is **k-connected** if any continuous image of an m-dimensional sphere can be extended to a continuous image of an $(m+1)$-dimensional disk, for all $m \leq k$.[5] This is a roundabout way of saying that if we can draw something that looks like a deformed sphere inside our object, we can always include the inside as well: there are no holes that get in the way. The punch line is that continuous functions preserve k-connectivity: if we map an object with no holes into some other object, the image had better not have any holes either.

Ordinary path-connectivity is the special case when $k = 0$; here, the 0-sphere consists of two points and the 1-disk is the path between them. So 0-connectivity says that for any two points, there is a path between them.

For 1-connectivity, if we draw a loop (a path that returns to its origin), we can include the interior of the loop somewhere. One way to thinking about this is to say that we can shrink the loop to a point without leaving the object (the technical term for this is that the path is **null-homotopic**, where a **homotopy** is a way to transform one thing continuously into another thing over time and the **null path** sits on a single point). An object that is 1-connected is also called **simply connected**.

[5] This definition is for the topological version of k-connectivity. It is not related in any way to the definition of k-connectivity in graph theory, where a graph is k-connected if there are k disjoint paths between any two points.

For 2-connectivity, we can't contract a sphere (or box, or the surface of a 2-simplex, or anything else that looks like a sphere) to a point.

The important thing about k-connectivity is that it is possible to prove that any subdivision of a k-connected simplicial complex is also k-connected (sort of obvious if you think about the pictures, but it can also be proved formally), and that k-connectivity is preserved by simplicial maps (if not, somewhere in the middle of all the k-simplexes representing our surface is a $(k+1)$-simplex in the domain that maps to a hole in the range, violating the rule that simplicial maps map simplexes to simplexes). So a quick way to show that the Asynchronous Computability Theorem implies that something is not asynchronously computable is to show that the input complex is k-connected and the output complex isn't.

28.7.2 Impossibility proofs for specific problems

Here are some applications of the Asynchronous Computability Theorem and k-connectivity:

Consensus There is no nontrivial wait-free consensus protocol for $n \geq 2$ processes. Proof: The input complex is 1-connected, but the output complex is not, and we need a map that covers the entire output complex (by nontriviality).

k-set agreement There is no wait-free k-set agreement for $n \geq k+1$ processes. Proof: The output complex for k-set agreement is not k-connected, because buried inside it are lots of $(k+1)$-dimensional holes corresponding to missing simplexes where all $k+1$ processes choose different values. But these holes aren't present in the input complex—it's OK if everybody starts with different inputs—and the validity requirements for k-set agreement force us to map the surfaces of these non-holes around holes in the output complex. (This proof actually turns into the Sperner's Lemma proof if we fully expand the claim about having to map the input complex around the hole.)

Renaming There is no wait-free renaming protocol with less than $2n-1$ output names for all n. The general proof of this requires showing that with fewer names we get holes that are too big (and ultimately reduces to Sperner's Lemma); for the special case of $n = 3$ and $m = 4$, see Figure 28.4, which shows how the output complex of renaming folds up into the surface of a torus. This means that renaming for $n = 3$

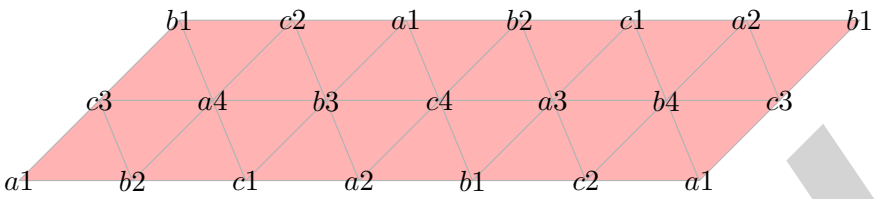


Figure 28.4: Output complex for renaming with $n = 3$, $m = 4$. Each vertex is labeled by a process id (a, b, c) and a name $(1, 2, 3, 4)$. Observe that the left and right edges of the complex have the same sequence of labels, as do the top and bottom edges; the complex thus folds up into a (twisted) torus. (This is a poor imitation of part of [HS99, Figure 9].)

and $m = 4$ is *exactly the same* as trying to stretch a basketball into an inner tube.

Chapter 29

Approximate agreement

1

The **approximate agreement** [DLP+86] or **ϵ-agreement** problem is another relaxation of consensus where input and output values are real numbers, and a protocol is required to satisfy modified validity and agreement conditions.

Let x_i be the input of process i and y_i its output. Then a protocol satisfies approximate agreement if it satisfies:

Termination Every nonfaulty process eventually decides.

Validity Every process returns an output within the range of inputs. Formally, for all i, it holds that $(\min_j x_j) \leq y_i \leq (\max_j x_j)$.

ϵ-agreement For all i and j, $|i - j| \leq \epsilon$.

Unlike consensus, approximate agreement has wait-free algorithms for asynchronous shared memory, which we'll see in §29.1). But a curious property of approximate agreement is that it has no **bounded wait-free** algorithms, even for two processes (see §29.2)

29.1 Algorithms for approximate agreement

Not only is approximate agreement solvable, it's actually easily solvable, to the point that there are many known algorithms for solving it.

We'll use the algorithm of Moran [Mor95], mostly as presented in [AW04,

Algorithm 54] but with a slight bug fix;[1] pseudocode appears in Algorithm 29.1.[2]

The algorithm carries out a sequence of asynchronous rounds in which processes adopt new values, such that the **spread** of the vector of all values V_r in round r, defined as spread $V_r = \max V_r - \min V_r$, drops by a factor of 2 per round. This is done by having each process choose a new value in each round by taking the midpoint (average of min and max) of all the values it sees in the previous round. Slow processes will jump to the maximum round they see rather than propagating old values up from ancient rounds; this is enough to guarantee that latecomer values that arrive after some process writes in round 2 are ignored.

The algorithm uses a single snapshot object A to communicate, and each process stores its initial input and a round number along with its current preference. We assume that the initial values in this object all have round number 0, and that $\log_2 0 = -\infty$ (which avoids a special case in the termination test).

```
1 A[i] ← ⟨x_i, 1, x_i⟩
2 repeat
3 |   ⟨x'_1, r_1, v_1⟩ ... ⟨x'_n, r_n, v_n⟩ ← snapshot(A)
4 |   r_max ← max_j r_j
5 |   v ← midpoint{v_j | r_j = r_max}
6 |   A[i] ← ⟨x_i, r_max + 1, v⟩
7 until r_max ≥ 2 and r_max ≥ log_2(spread({x'_j})/ε)
8 return v
```

Algorithm 29.1: Approximate agreement

To show this works, we want to show that the midpoint operation guarantees that the spread shrinks by a factor of 2 in each round. Let V_r be the set of all values v that are ever written to the snapshot object with round

[1] The original algorithm from [AW04] does not include the test $r_{\max} \geq 2$. This allows for bad executions in which process 1 writes its input of 0 in round 1 and takes a snapshot that includes only its own input, after which process 2 runs the algorithm to completion with input 1. Here process 2 will see 0 and 1 in round 1, and will write $(1/2, 2, 1)$ to $A[2]$; on subsequent iterations, it will see only the value 1/2 in the maximum round, and after $\lceil \log_2(1/\epsilon) \rceil$ rounds it will decide on 1/2. But if we now wake process 1 up, it will decided 0 immediately based on its snapshot, which includes only its own input and gives spread$(x) = 0$. Adding the extra test prevents this from happening, as new values that arrive after somebody writes round 2 will be ignored.

[2] Showing that this particular algorithm works takes a lot of effort. If I were to do this over, I'd probably go with a different algorithm due to Schenk [Sch95].

number r. Let $U_r \subseteq V_r$ be the set of values that are ever written to the snapshot object with round number r before some process writes a value with round number $r+1$ or greater; the intuition here is that U_r includes only those values that might contribute to the computation of some round-$(r+1)$ value.

Lemma 29.1.1. *For all r for which V_{r+1} is nonempty,*

$$\text{spread}(V_{r+1}) \leq \text{spread}(U_r)/2.$$

Proof. Let U_r^i be the set of round-r values observed by a process i in the iteration in which it sees $r_{\max} = r$ in some iteration, if such an iteration exists. Note that $U_r^i \subseteq U_r$, because if some value with round $r+1$ or greater is written before i's snapshot, then i will compute a larger value for $r_{\max}$.

Given two processes i and j, we can argue from the properties of snapshot that either $U_r^i \subseteq U_r^j$ or $U_r^j \subseteq U_r^i$. The reason is that if i's snapshot comes first, then j sees at least as many round-r values as i does, because the only way for a round-r value to disappear is if it is replaced by a value in a later round. But in this case, process j will compute a larger value for $r_{\max}$ and will not get a view for round r. The same holds in reverse if j's snapshot comes first.

Observe that if $U_r^i \subseteq U_r^j$, then

$$\left|\text{midpoint}(U_r^i) - \text{midpoint}(U_r^j)\right| \leq \text{spread}(U_r^j)/2.$$

This holds because $\text{midpoint}(U_r^i)$ lies within the interval $\left[\min U_r^j, \max U_r^j\right]$, and every point in this interval is within $\text{spread}(U_r^j)/2$ of $\text{midpoint}(U_r^j)$. The same holds if $U_r^j \subseteq U_r^i$. So any two values written in round $r+1$ are within $\text{spread}(U_r)/2$ of each other.

In particular, the minimum and maximum values in V_{r+1} are within $\text{spread}(U_r)/2$ of each other, so $\text{spread}(V_{r+1}) \leq \text{spread}(U_r)/2$. □

Corollary 29.1.2. *For all $r \geq 2$ for which V_r is nonempty,*

$$\text{spread}(V_r) \leq \text{spread}(U_1)/2^{r-1}.$$

Proof. By induction on r. For $r = 2$, this is just Lemma 29.1.1. For larger r, use the fact that $U_{r-1} \subseteq V_{r-1}$ and thus $\text{spread}(U_{r-1}) \leq \text{spread}(V_{r-1})$ to compute

$$\begin{aligned}
\text{spread}(V_r) &\leq \text{spread}(U_{r-1})/2 \\
&\leq \text{spread}(V_{r-1})/2 \\
&\leq (\text{spread}(U_1)/2^{r-2})/2 \\
&= \text{spread}(U_1)/2^{r-1}.
\end{aligned}$$

□

Let i be some process that finishes in the fewest number of rounds. Process i can't finish until it reaches round $r_{\max}+1$, where $r_{\max} \geq \log_2(\text{spread}(\{x'_j\})/\epsilon)$ for a vector of input values x' that it reads after some process writes round 2 or greater. We have $\text{spread}(\{x'_j\}) \geq \text{spread}(U_1)$, because every value in U_1 is included in x'. So $r_{\max} \geq \log_2(\text{spread}(U_1)/\epsilon)$ and $\text{spread}(V_{r_{\max}+1}) \leq \text{spread}(U_1)/2^{r_{\max}} \leq \text{spread}(U_1)/(\text{spread}(U_1)/\epsilon) = \epsilon$. Since any value returned is either included in $V_{r_{\max}+1}$ or some later $V_{r'} \subseteq V_{r_{\max}+1}$, this gives us that the spread of all the outputs is less than ϵ: Algorithm 29.1 solves approximate agreement.

The cost of Algorithm 29.1 depends on the cost of the snapshot operations, on ϵ, and on the initial input spread D. For linear-cost snapshots, this works out to $O(n \log(D/\epsilon))$.

29.2 Lower bound on step complexity

The dependence on D/ϵ is necessary, at least for deterministic algorithms. Here we give a lower bound due to Herlihy [Her91a], which shows that any deterministic approximate agreement algorithm takes at least $\log_3(D/\epsilon)$ total steps even with just two processes.

Define the **preference** of a process in some configuration as the value it will choose if it runs alone starting from this configuration. The preference of a process p is well-defined because the process is deterministic; it also can only change as a result of a write operation by another process q (because no other operations are visible to p, and p's own operations can't change its preference). The validity condition means that in an initial state, each process's preference is equal to its input.

Consider an execution with two processes p and q, where p starts with preference p_0 and q starts with preference q_0. Run p until it is about to perform a write that would change q's preference. Now run q until it is about to change p's preference. If p's write no longer changes q's preference, start p again and repeat until both p and q have pending writes that will change the other process's preference. Let p_1 and q_1 be the new preferences that result from these operations. The adversary can now choose between running P only and getting to a configuration with preferences p_0 and q_1, Q only and getting p_1 and q_0, or both and getting p_1 and q_1; each of these choices incurs at least one step. By the triangle inequality, $|p_0 - q_0| \leq |p_0 - q_1| + |q_1 - p_1| + |p_1 - q_0|$, so at least on of these configurations has a spread between preferences that is at least 1/3 of the initial spread. It

follows that after k steps the best spread we can get is $D/3^k$, requiring $k \geq \log_3(D/\epsilon)$ steps to get ϵ-agreement.

Herlihy uses this result to show that there are decisions problems that have wait-free but not bounded wait-free deterministic solutions using registers. Curiously, the lower bound says nothing about the dependence on the number of processes; it is conceivable that there is an approximate agreement protocol with running time that depends only on D/ϵ and not n.

Appendix

Appendix A

Assignments

Assignments are typically due Wednesdays at 5:00pm. Assignments can be turned in to Ennan Zhai's mailbox on the first floor of AKW.

A.1 Assignment 1: due Wednesday, 2014-01-29, at 5:00pm

Bureaucratic part

Send me email! My address is `james.aspnes@gmail.com`.

In your message, include:

1. Your name.
2. Your status: whether you are an undergraduate, grad student, auditor, etc.
3. Anything else you'd like to say.

(You will not be graded on the bureaucratic part, but you should do it anyway.)

A.1.1 Counting evil processes

A connected bidirectional asynchronous network of n processes with identities has diameter D and may contain zero or more evil processes. Fortunately, the evil processes, if they exist, are not Byzantine, fully conform to RFC 3514 [Bel03], and will correctly execute any code we provide for them.

Suppose that all processes wake up at time 0 and start whatever protocol we have given them. Suppose that each process initially knows whether it is evil, and knows the identities of all of its neighbors. However, the processes do not know the number of processes n or the diameter of the network D.

Give a protocol that allows every process to correctly return the number of evil processes no later than time D. Your protocol should only return a value once for each process (no converging to the correct answer after an initial wrong guess).

Solution

There are a lot of ways to do this. Since the problem doesn't ask about message complexity, we'll do it in a way that optimizes for algorithmic simplicity.

At time 0, each process initiates a separate copy of the flooding algorithm (Algorithm 4.1). The message $\langle p, N(p), e\rangle$ it distributes consists of its own identity, the identities of all of its neighbors, and whether or not it is evil.

In addition to the data for the flooding protocol, each process tracks a set I of all processes it has seen that initiated a protocol and a set N of all processes that have been mentioned as neighbors. The initial values of these sets for process p are $\{p\}$ and $N(p)$, the neighbors of p.

Upon receiving a message $\langle q, N(q), e\rangle$, a process adds q to I and $N(q)$ to N. As soon as $I = N$, the process returns a count of all processes for which $e =$ **true**.

Termination by D: Follows from the same analysis as flooding. Any process at distance d from p has $p \in I$ by time d, so I is complete by time D.

Correct answer: Observe that $N = \bigcup_{i \in I} N(i)$ always. Suppose that there is some process q that is not in I. Since the graph is connected, there is a path from p to q. Let r be the last node in this path in I, and let s be the following node. Then $s \in N \setminus I$ and $N \neq I$. By contraposition, if $I = N$ then I contains all nodes in the network, and so the count returned at this time is correct.

A.1.2 Avoiding expensive processes

Suppose that you have a bidirectional but not necessarily complete asynchronous message-passing network represented by a graph $G = (V, E)$ where each node in V represents a process and each edge in E connects two processes that can send messages to each other. Suppose further that each

process is assigned a weight 1 or 2. Starting at some initiator process, we'd like to construct a shortest-path tree, where each process points to one of its neighbors as its parent, and following the parent pointers always gives a path of minimum total weight to the initiator.[1]

Give a protocol that solves this problem with reasonable time, message, and bit complexity, and show that it works.

Solution

There's an ambiguity in the definition of total weight: does it include the weight of the initiator and/or the initial node in the path? But since these values are the same for all paths to the initiator from a given process, they don't affect which is lightest.

If we don't care about bit complexity, there is a trivial solution: Use an existing BFS algorithm followed by convergecast to gather the entire structure of the network at the initiator, run your favorite single-source shortest-path algorithm there, then broadcast the results. This has time complexity $O(D)$ and message complexity $O(DE)$ if we use the BFS algorithm from §5.3. But the last couple of messages in the convergecast are going to be pretty big.

A solution by reduction: Suppose that we construct a new graph G' where each weight-2 node u in G is replaced by a clique of nodes $u_1, u_2, \ldots u_k$, with each node in the clique attached to a different neighbor of u. We then run any breadth-first search protocol of our choosing on G', where each weight-2 node simulates all members of the corresponding clique. Because any path that passes through a clique picks up an extra edge, each path in the breadth-first search tree has a length exactly equal to the sum of the weights of the nodes other than its endpoints.

A complication is that if I am simulating k nodes, between them they may have more than one parent pointer. So we define $u.\mathsf{parent}$ to be $u_i.\mathsf{parent}$ where u_i is a node at minimum distance from the initiator in G'. We also re-route any incoming pointers to $u_j \neq u_i$ to point to u_i instead. Because u_i was chosen to have minimum distance, this never increases the length of any path, and the resulting modified tree is a still a shortest-path tree.

Adding nodes blows up $|E'|$, but we don't need to actually send messages between different nodes u_i represented by the same process. So if we use the

[1] Clarification added 2014-01-26: The actual number of hops is not relevant for the construction of the shortest-path tree. By shortest path, we mean path of minimum total weight.

§5.3 algorithm again, we only send up to D messages per real edge, giving $O(D)$ time and $O(DE)$ messages.

If we don't like reductions, we could also tweak one of our existing algorithms. Gallager's layered BFS (§5.2) is easily modified by changing the depth bound for each round to a total-weight bound. The synchronizer-based BFS can also be modified to work, but the details are messy.

A.2 Assignment 2: due Wednesday, 2014-02-12, at 5:00pm

A.2.1 Synchronous agreement with weak failures

Suppose that we modify the problem of synchronous agreement with crash failures from Chapter 7 so that instead of crashing a process forever, the adversary may jam some or all of its outgoing messages for a single round. The adversary has limited batteries on its jamming equipment, and can only cause f such one-round faults. There is no restriction on when these one-round jamming faults occur: the adversary might jam f processes for one round, one process for f rounds, or anything in between, so long as the sum over all rounds of the number of processes jammed in each round is at most f. For the purposes of agreement and validity, assume that a process is non-faulty if it is never jammed.[2]

As a function of f and n, how many rounds does it take to reach agreement in the worst case in this model, under the usual assumptions that processes are deterministic and the algorithm must satisfy agreement, termination, and validity? Give the best upper and lower bounds that you can.

Solution

The par solution for this is an $\Omega(\sqrt{f})$ lower bound and $O(f)$ upper bound. I don't know if it is easy to do better than this.

For the lower bound, observe that the adversary can simulate an ordinary crash failure by jamming a process in every round starting in the round it crashes in. This means that in an r-round protocol, we can simulate k crash failures with kr jamming faults. From the Dolev-Strong lower bound [DS83]

[2] Clarifications added 2014-02-10: We assume that processes don't know that they are being jammed or which messages are lost (unless the recipient manages to tell them that a message was not delivered). As in the original model, we assume a complete network and that all processes have known identities.

(see also Chapter 7), we know that there is no r-round protocol with $k = r$ crash failures faults, so there is no r-round protocol with r^2 jamming faults. This gives a lower bound of $\lfloor \sqrt{f} \rfloor + 1$ on the number of rounds needed to solve synchronous agreement with f jamming faults.[3]

For the upper bound, have every process broadcast its input every round. After $f+1$ rounds, there is at least one round in which no process is jammed, so every process learns all the inputs and can take, say, the majority value.

A.2.2 Byzantine agreement with contiguous faults

Suppose that we restrict the adversary in Byzantine agreement to corrupting a connected subgraph of the network; the idea is that the faulty nodes need to coordinate, but can't relay messages through the non-faulty nodes to do so.

Assume the usual model for Byzantine agreement with a network in the form of an $m \times m$ torus. This means that each node has a position (x, y) in $\{0, \ldots, m-1\} \times \{0, \ldots, m-1\}$, and its neighbors are the four nodes $(x+1 \bmod m, y)$, $(x-1 \bmod m, y)$, $(x, y+1 \bmod m)$, and $(x, y-1 \bmod m)$.

For sufficiently large m,[4] what is the largest number of faults f; that this system can tolerate and still solve Byzantine agreement?

Solution

The relevant bound here is the requirement that the network have enough connectivity that the adversary can't take over half of a vertex cut (see §8.1.3). This is complicated slightly by the requirement that the faulty nodes be contiguous.

The smallest vertex cut in a sufficiently large torus consists of the four neighbors of a single node; however, these nodes are not connected. But we can add a third node to connect two of them (see Figure A.1).

By adapting the usual lower bound we can use this construction to show that $f = 3$ faults are enough to prevent agreement when $m \geq 3$. The question then is whether $f = 2$ faults is enough.

By a case analysis, we can show that any two nodes in a sufficiently large torus are either adjacent themselves or can be connected by three paths, where no two paths have adjacent vertices. Assume without loss of

[3] Since Dolev-Strong only needs to crash one process per round, we don't really need the full r jamming faults for processes that crash late. This could be used to improve the constant for this argument.

[4] Problem modified 2014-02-03. In the original version, it asked to compute f for all m, but there are some nasty special cases when m is small.

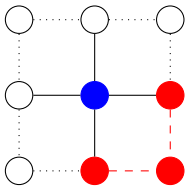


Figure A.1: Connected Byzantine nodes take over half a cut

generality that one of the nodes is at position $(0, 0)$. Then any other node is covered by one of the following cases:

1. Nodes adjacent to $(0, 0)$. These can communicate directly.

2. Nodes at $(0, i)$ or $(i, 0)$. These cases are symmetric, so we'll describe the solution for $(0, i)$. Run one path directly north: $(0, 1), (0, 2), \ldots, (0, i-1)$. Similarly, run a second path south: $(0, -1), (0, -2), \ldots (0, i + 1)$. For the third path, take two steps east and then run north and back west: $(1, 0), (2, 0), (2, 1), (2, 2), \ldots, (2, i), (1, i)$. These paths are all non-adjacent as long as $m \geq 4$.

3. Nodes at $(\pm 1, i)$ or $(i, \pm 1)$, where i is not -1, 0, or 1. Suppose the node is at $(1, i)$. Run one path east then north through $(1, 0), (1, 1), \ldots, (1, i-1)$. The other two paths run south and west, with a sideways jog in the middle as needed. This works for m sufficiently large to make room for the sideways jogs.

4. Nodes at $(\pm 1, \pm 1)$ or (i, j) where neither of i or j is -1, 0, or 1. Now we can run one path north then east, one east then north, one south then west, and one west then south, creating four paths in a figure-eight pattern centered on $(0, 0)$.

A.3 Assignment 3: due Wednesday, 2014-02-26, at 5:00pm

A.3.1 Among the elect

The adversary has decided to be polite and notify each non-faulty processes when he gives up crashing it. Specifically, we have the usual asynchronous message-passing system with up to f faulty processes, but every non-faulty process is eventually told that it is non-faulty. (Faulty processes are told nothing.)

For what values of f can you solve consensus in this model?

Solution

We can tolerate $f < n/2$, but no more.

If $f < n/2$, the following algorithm works: Run Paxos, where each process i waits to learn that it is non-faulty, then acts as a proposer for proposal number i. The highest-numbered non-faulty process then carries out a proposal round that succeeds because no higher proposal is ever issued, and both the proposer (which is non-faulty) and a majority of accepters participate.

If $f \geq n/2$, partition the processes into two groups of size $\lfloor n/2 \rfloor$, with any leftover process crashing immediately. Make all of the processes in both groups non-faulty, and tell each of them this at the start of the protocol. Now do the usual partitioning argument: Run group 0 with inputs 0 with no messages delivered from group 1 until all processes decide 0 (we can do this because the processes can't distinguish this execution from one in which the group 1 processes are in fact faulty). Run group 1 similarly until all processes decide 1. We have then violated agreement, assuming we didn't previously violate termination of validity.

A.3.2 Failure detectors on the cheap

Suppose we do not have the budget to equip all of our machines with failure detectors. Instead, we order an eventually strong failure detector for k machines, and the remaining $n - k$ machines get fake failure detectors that never suspect anybody. The choice of which machines get the real failure detectors and which get the fake ones is under the control of the adversary.

This means that every faulty process is eventually permanently suspected by every non-faulty process with a real failure detector, and there is at least one non-faulty process that is eventually permanently not suspected by anybody. Let's call the resulting failure detector $\Diamond S_k$.

Let f be the number of actual failures. Under what conditions on f, k, and n can you still solve consensus in the usual deterministic asynchronous message-passing model using $\Diamond S_k$?

Solution

First observe that $\Diamond S$ can simulate $\Diamond S_k$ for any k by having $n - k$ processes ignore the output of their failure detectors. So we need $f < n/2$ by the usual lower bound on $\Diamond S$.

If $f \geq k$, we are also in trouble. The $f > k$ case is easy: If there exists a consensus protocol for $f > k$, then we can transform it into a consensus protocol for $n - k$ processes and $f - k$ failures, with no failure detectors at all, by pretending that there are an extra k processes with real failure detectors that crash immediately. The FLP impossibility result rules this out.

If $f = k$, we have to be a little more careful. By immediately crashing $f - 1$ processes with real failure detectors, we can reduce to the $f = k = 1$ case. Now the adversary runs the FLP strategy. If no processes crash, then all $n - k + 1$ surviving process report no failures; if it becomes necessary to crash a process, this becomes the one remaining process with the real failure detector. In either case the adversary successfully prevents consensus.

So let $f < k$. Then we have weak completeness, because every faulty process is eventually permanently suspected by at least $k - f > 0$ processes. We also have weak accuracy, because it is still the case that some process is eventually permanently never suspected by anybody. By boosting weak completeness to strong completeness as described in §11.2.3, we can turn out failure detector into $\Diamond S$, meaning we can solve consensus precisely when $f < \min(k, n/2)$.

A.4 Assignment 4: due Wednesday, 2014-03-26, at 5:00pm

A.4.1 A global synchronizer with a global clock

Consider an asynchronous message-passing system with n processes in a bidirectional ring with no failures. Suppose that the processes are equipped with a global clock, which causes a local event to occur simultaneously at each process every c time units, where as usual 1 is the maximum message delay. We would like to use this global clock to build a global synchronizer. Provided c is at least 1, a trivial approach is to have every process advance to the next round whenever the clock pulse hits. This gives one synchronous round every c time units.

Suppose that c is greater than 1 but still $o(n)$. Is it possible to build a global synchronizer in this model that runs more than a constant ratio faster than this trivial global synchronizer in the worst case?

A.4.2 A message-passing counter

A **counter** is a shared object that support operations `inc` and `read`, where `read` returns the number of previous `inc` operations.

Algorithm A.1 purports to implement a counter in an asynchronous message-passing system subject to $f < n/2$ crash failures. In the algorithm, each process i maintains a vector c_i of contributions to the counter from all the processes, as well as a nonce r_i used to distinguish responses to different read operations from each other. All of these values are initially zero.

Show that the implemented counter is linearizable, or give an example of an execution where it isn't.

```
1  procedure inc
2      c_i[i] ← c_i[i] + 1
3      Send c_i[i] to all processes.
4      Wait to receive ack(c_i[i]) from a majority of processes.
5  upon receiving c from j do
6      c_i[j] ← max(c_i[j], c)
7      Send ack(c) to j.
8  procedure read
9      r_i ← r_i + 1
10     Send read(r_i) to all processes.
11     Wait to receive respond(r_i, c_j) from a majority of processes j.
12     return Σ_k max_j c_j[k]
13 upon receiving read(r) from j do
14     Send respond(r, c_i) to j
```

Algorithm A.1: Counter algorithm for Problem A.4.2.

A.5 Assignment 5: due Wednesday, 2014-04-09, at 5:00pm

A.5.1 A concurrency detector

Consider the following optimistic mutex-like object, which we will call a **concurrency detector**. A concurrency detector supports two operations for each process i, `enter`$_i$ and `exit`$_i$. These operations come in pairs: a

process enters a critical section by executing $\texttt{enter}_i$, and leaves by executing $\texttt{exit}_i$. The behavior of the object is undefined if a process calls $\texttt{enter}_i$ twice without an intervening $\texttt{exit}_i$, or calls $\texttt{exit}_i$ without first calling $\texttt{enter}_i$.

Unlike mutex, a concurrency detector does not enforce that only one process is in the critical section at a time; instead, $\texttt{exit}_i$ returns 1 if the interval between it and the previous $\texttt{enter}_i$ overlaps with some interval between a $\texttt{enter}_j$ and corresponding $\texttt{exit}_j$ for some $j \neq i$, and returns 0 if there is no overlap.

Is there a deterministic linearizable wait-free implementation of a concurrency detector from atomic registers? If there is, give an implementation. If there is not, give an impossibility proof.

Solution

It is not possible to implement this object using atomic registers.

Suppose that there were such an implementation. Algorithm A.2 implements two-process consensus using a two atomic registers and a single concurrency detector, initialized to the state following $\texttt{enter}_1$.

```
1  procedure consensus1(v)
2      r1 ← v
3      if exit1() = 1 then
4          return r2
5      else
6          return v
7  procedure consensus2(v)
8      r2 ← v
9      enter2()
10     if exit2() = 1 then
11         return v
12     else
13         return r1
```

Algorithm A.2: Two-process consensus using the object from Problem A.5.1

Termination is immediate from the absence of loops in the code.

To show validity and termination, observe that one of two cases holds:

1. Process 1 executes $\texttt{exit}_1$ before process 2 executes $\texttt{enter}_2$. In this

case there is no overlap between the interval formed by the implicit $\texttt{enter}_1$ and $\texttt{exit}_1$ and the interval formed by $\texttt{enter}_2$ and $\texttt{exit}_2$. So the $\texttt{exit}_1$ and $\texttt{exit}_2$ operations both return 0, causing process 1 to return its own value and process 2 to return the contents of r_1. These will equal process 1's value, because process 2's read follows its call to $\texttt{enter}_2$, which follows $\texttt{exit}_1$ and thus process 1's write to r_1.

2. Process 1 executes $\texttt{exit}_1$ after process 2 executes $\texttt{enter}_2$. Now both `exit` operations return 1, and so process 2 returns its own value while process 1 returns the contents of r_2, which it reads after process 2 writes its value there.

In either case, both processes return the value of the first process to access the concurrency detector, satisfying both agreement and validity. This would give a consensus protocol for two processes implemented from atomic registers, contradicting the impossibility result of Loui and Abu-Amara [LAA87].

A.5.2 Two-writer sticky bits

A **two-writer sticky bit** is a sticky bit that can be read by any process, but that can only be written to by two specific processes.

Suppose that you have an unlimited collection of two-writer sticky bits for each pair of processes, plus as many ordinary atomic registers as you need. What is the maximum number of processes for which you can solve wait-free binary consensus?

Solution

If $n = 2$, then a two-writer sticky bit is equivalent to a sticky bit, so we can solve consensus.

If $n \geq 3$, suppose that we maneuver our processes as usual to a bivalent configuration C with no bivalent successors. Then there are three pending operations x, y, and z, that among them produce both 0-valent and 1-valent configurations. Without loss of generality, suppose that Cx and Cy are both 0-valent and Cz is 1-valent. We now consider what operations these might be.

1. If x and z apply to different objects, then $Cxz = Czx$ must be both 0-valent and 1-valent, a contradiction. Similarly if y and z apply to different objects. This shows that all three operations apply to the same object O.

2. If O is a register, then the usual case analysis of Loui and Abu-Amara [LAA87] gives us a contradiction.

3. If O is a two-writer sticky bit, then we can split cases further based on z:

   (a) If z is a read, then either:

      i. At least one of x and y is a read. But then $Cxz = Czx$ or $Cyz = Czy$, and we are in trouble.

      ii. Both x and y are writes. But then Czx (1-valent) is indistinguishable from Cx (0-valent) by the two processes that didn't perform z: more trouble.

   (b) If z is a write, then at least one of x or y is a read; suppose it's x. Then Cxz is indistinguishable from Cz by the two processes that didn't perform x.

Since we reach a contradiction in all cases, it must be that when $n \geq 3$, every bivalent configuration has a bivalent successor, which shows that we can't solve consensus in this case. The maximum value of n for which we can solve consensus is 2.

A.6 Assignment 6: due Wednesday, 2014-04-23, at 5:00pm

A.6.1 A rotate register

Suppose that you are asked to implement a concurrent m-bit register that supports in addition to the usual `read` and `write` operations a `RotateLeft` operation that rotates all the bits to the left; this is equivalent to doing a left shift (multiplying the value in the register by two) followed by replacing the lowest-order bit with the previous highest-order bit.

For example, if the register contains 1101, and we do `RotateLeft`, it now contains 1011.

Show that if m is sufficiently large as a function of the number of processes n, $\Theta(n)$ steps per operation in the worst case are necessary and sufficient to implement a linearizable wait-free m-bit shift register from atomic registers.

Solution

The necessary part is easier, although we can't use JTT (Chapter 20) directly because having write operations means that our rotate register is not perturbable. Instead, we argue that if we initialize the register to 1, we get a mod-m counter, where increment is implemented by `RotateLeft` and read is implemented by taking the log of the actual value of the counter. Letting $m \geq 2n$ gives the desired $\Omega(n)$ lower bound, since a mod-$2n$ counter *is* perturbable.

For sufficiency, we'll show how to implement the rotate register using snapshots. This is pretty much a standard application of known techniques [AH90b, AM93], but it's not a bad exercise to write it out.

Pseudocode for one possible solution is given in Algorithm A.3.

The register is implemented using a single snapshot array A. Each entry in the snapshot array holds four values: a timestamp and process id indicating which write the process's most recent operations apply to, the initial write value corresponding to this timestamp, and the number of rotate operations this process has applied to this value. A write operation generates a new timestamp, sets the written value to its input, and resets the rotate count to 0. A rotate operation updates the timestamp and associated write value to the most recent that the process sees, and adjusts the rotate count as appropriate. A read operation combines all the rotate counts associated with the most recent write to obtain the value of the simulated register.

Since each operation requires one snapshot and at most one update, the cost is $O(n)$ using the linear-time snapshot algorithm of Inoue *et al.* [IMCT94]. Linearizability is easily verified by observing that ordering all operations by the maximum timestamp/process tuple that they compute and then by the total number of rotations that they observe produces an ordering consistent with the concurrent execution for which all return values of reads are correct.

A.6.2 A randomized two-process test-and-set

Algorithm A.4 gives pseudocode for a protocol for two processes p_0 and p_1. It uses two shared unbounded single-writer atomic registers r_0 and r_1, both initially 0. Each process also has a local variable s.

1. Show that any return values of the protocol are consistent with a linearizable, single-use test-and-set.

```
1  procedure write(A, v)
2      s ← snapshot(A)
3      A[id] ← ⟨max_i s[i].timestamp + 1, id, v, 0⟩
4  procedure RotateLeft(A)
5      s ← snapshot(A)
6      Let i maximize ⟨s[i].timestamp, s[i].process⟩
7      if s[i].timestamp = A[id].timestamp and
       s[i].process = A[id].process then
           // Increment my rotation count
8          A[id].rotations ← A[id].rotations + 1
9      else
           // Reset and increment my rotation count
10         A[id] ← ⟨s[i].timestamp, s[i].process, s[i].value, 1⟩
11 procedure read(A)
12     s ← snapshot(A)
13     Let i maximize ⟨s[i].timestamp, s[i].process⟩
14     Let
       r = ∑_{j,s[j].timestamp=s[i].timestamp∧s[j].process=s[i].process} s[j].rotations
15     return s[i].value rotated r times.
```

Algorithm A.3: Implementation of a rotate register

```
1  procedure TAS_i()
2      while true do
3          with probability 1/2 do
4              r_i ← r_i + 1
5          else
6              r_i ← r_i
7          s ← r_{¬i}
8          if s > r_i then
9              return 1
10         else if s < r_i − 1 do
11             return 0
```

Algorithm A.4: Randomized two-process test-and-set for A.6.2

2. Will this protocol always terminate with probability 1 assuming an oblivious adversary?

3. Will this protocol always terminate with probability 1 assuming an adaptive adversary?

Solution

1. To show that this implements a linearizable test-and-set, we need to show that exactly one process returns 0 and the other 1, and that if one process finishes before the other starts, the first process to go returns 1.

   Suppose that p_i finishes before $p_{\neg i}$ starts. Then p_i reads only 0 from $r_{\neg i}$, and cannot observe $r_i < r_{\neg i}$: p_i returns 0 in this case.

   We now show that the two processes cannot return the same value. Suppose that both processes terminate. Let i be such that p_i reads $r_{\neg i}$ for the last time before $p_{\neg i}$ reads r_i for the last time. If p_i returns 0, then it observes $r_i \geq r_{\neg i} + 2$ at the time of its read; $p_{\neg i}$ can increment $r_{\neg i}$ at most once before reading r_i again, and so observed $r_{\neg i} < r_i$ and returns 1.

   Alternatively, if p_i returns 1, it observed $r_i < r_{\neg i}$. Since it performs no more increments on r_i, p_i also observes $r_i < r_{\neg i}$ in all subsequent reads, and so cannot also return 1.

2. Let's run the protocol with an oblivious adversary, and track the value of $r_0^t - r_1^t$ over time, where r_i^t is the value of r_i after t writes (to either register). Each write to r_0 increases this value by $1/2$ on average, with a change of 0 or 1 equally likely, and each write to r_1 decreases it by $1/2$ on average.

   To make things look symmetric, let Δ^t be the change caused by the t-th write and write Δ^t as $c^t + X^t$ where $c^t = \pm 1/2$ is a constant determined by whether p_0 or p_1 does the t-th write and $X^t = \pm 1/2$ is a random variable with expectation 0. Observe that the X^t variables are independent of each other and the constants c^t (which depend only on the schedule).

   For the protocol to run forever, at every time t it must hold that $|r_0^t - r_1^t| \leq 3$; otherwise, even after one or both processes does its next write, we will have $\left|r_0^{t'} - r_1^{t'}\right|$ and the next process to read will

terminate. But

$$\begin{aligned}\left|r_0^t - r_1^t\right| &= \left|\sum_{s=1}^{t} \Delta^s\right| \\ &= \left|\sum_{s=1}^{t} (c_s + X_s)\right| \\ &= \left|\sum_{s=1}^{t} c_s + \sum_{s=1}^{t} X_s\right|.\end{aligned}$$

The left-hand sum is a constant, while the right-hand sum has a binomial distribution. For any fixed constant, the probability that a binomial distribution lands within ± 2 of the constant goes to zero in the limit as $t \to \infty$, so with probability 1 there is some t for which this event does not occur.

3. For an adaptive adversary, the following strategy prevents agreement:

   (a) Run p_0 until it is about to increment r_0.

   (b) Run p_1 until it is about to increment r_1.

   (c) Allow both increments to proceed and repeat.

   The effect is that both processes always observe $r_0 = r_1$ whenever they do a read, and so never finish. This works because the adaptive adversary can see the coin-flips done by the processes before they act on them; it would not work with an oblivious adversary or in a model that supported probabilistic writes.

A.7 CS465/CS565 Final Exam, May 2nd, 2014

Write your answers in the blue book(s). Justify your answers. Work alone. Do not use any notes or books.

There are four problems on this exam, each worth 20 points, for a total of 80 points. You have approximately three hours to complete this exam.

A.7.1 Maxima (20 points)

Some deterministic processes organized in an anonymous, synchronous ring are each given an integer input (which may or may not be distinct from other processes' inputs), but otherwise run the same code and do not know the

size of the ring. We would like the processes to each compute the maximum input. As usual, each process may only return an output once, and must do so after a finite number of rounds, although it may continue to participate in the protocol (say, by relaying messages) even after it returns an output.

Prove or disprove: It is possible to solve this problem in this model.

Solution

It's not possible.

Consider an execution with $n = 3$ processes, each with input 0. If the protocol is correct, then after some finite number of rounds t, each process returns 0. By symmetry, the processes all have the same states and send the same messages throughout this execution.

Now consider a ring of size $2(t + 1)$ where every process has input 0, except for one process p that has input 1. Let q be the process at maximum distance from p. By induction on r, we can show that after r rounds of communication, every process that is more than $r+1$ hops away from p has the same state as all of the processes in the 3-process execution above. So in particular, after t rounds, process q (at distance $t+1$) is in the same state as it would be in the 3-process execution, and thus it returns 0. But—as it learns to its horror, one round too late—the correct maximum is 1.

A.7.2 Historyless objects (20 points)

Recall that a shared-memory object is **historyless** if any operation on the object either (a) always leaves the object in the same state as before the operation, or (b) always leaves the object in a new state that doesn't depend on the state before the operation.

What is the maximum possible consensus number for a historyless object? That is, for what value n is it possible to solve wait-free consensus for n processes using some particular historyless object but not possible to solve wait-free consensus for $n + 1$ processes using any historyless object?

Solution

Test-and-sets are (a) historyless, and (b) have consensus number 2, so n is at least 2.

To show that no historyless object can solve wait-free 3-process consensus, consider an execution that starts in a bivalent configuration and runs to a configuration C with two pending operations x and y such that Cx is 0-valent and Cy is 1-valent. By the usual arguments x and y must both be

operations on the same object. If either of x and y is a read operation, then (0-valent) Cxy and (1-valent) Cyx are indistinguishable to a third process p_z if run alone, because the object is left in the same state in both configurations; whichever way p_z decides, it will give a contradiction in an execution starting with one of these configurations. If neither of x and y is a read, then x overwrites y, and Cx is indistinguishable from $Cyx$to p_z if p_z runs alone; again we get a contradiction.

A.7.3 Hams (20 points)

Hamazon, LLC, claims to be the world's biggest delivery service for canned hams, with guaranteed delivery of a canned ham to your home anywhere on Earth via suborbital trajectory from secret launch facilities at the North and South Poles. Unfortunately, these launch facilities may be subject to crash failures due to inclement weather, trademark infringement actions, or military retaliation for misdirected hams.

For this problem, you are to evaluate Hamazon's business model from the perspective of distributed algorithms. Consider a system consisting of a client process and two server processes (corresponding to the North and South Pole facilities) that communicate by means of asynchronous message passing. In addition to the usual message-passing actions, each server also has an irrevocable `launch` action that launches a ham at the client. As with messages, hams are delivered asynchronously: it is impossible for the client to tell if a ham has been launched until it arrives.

A ham protocol is correct provided (a) a client that orders no ham receives no ham; and (b) a client that orders a ham receives exactly one ham. Show that there can be no correct deterministic protocol for this problem if one of the servers can crash.

Solution

Consider an execution in which the client orders ham. Run the northern server together with the client until the server is about to issue a `launch` action (if it never does so, the client receives no ham when the southern server is faulty).

Now run the client together with the southern server. There are two cases:

1. If the southern server ever issues `launch`, execute both this and the northern server's `launch` actions: the client gets two hams.

2. If the southern server never issues `launch`, never run the northern server again: the client gets no hams.

In either case, the one-ham rule is violated, and the protocol is not correct.[5]

A.7.4 Mutexes (20 points)

A swap register s has an operation `swap`(s, v) that returns the argument to the previous call to `swap`, or $\perp$ if it is the first such operation applied to the register. It's easy to build a mutex from a swap register by treating it as a test-and-set: to grab the mutex, I swap in 1, and if I get back $\perp$ I win (and otherwise try again); and to release the mutex, I put back $\perp$.

Unfortunately, this implementation is not starvation-free: some other process acquiring the mutex repeatedly might always snatch the $\perp$ away just before I try to swap it out. Algorithm A.5 uses a swap object s along with an atomic register r to try to fix this.

```
1 procedure mutex()
2     predecessor ← swap(s, myId)
3     while r ≠ predecessor do
4         try again
      // Start of critical section
5     ...
      // End of critical section
6     r ← myId
```

Algorithm A.5: Mutex using a swap object and register

Prove that Algorithm A.5 gives a starvation-free mutex, or give an example of an execution where it fails. You should assume that s and r are both initialized to $\perp$.

[5] It's tempting to try to solve this problem by reduction from a known impossibility result, like Two Generals or FLP. For these specific problems, direct reductions don't appear to work. Two Generals assumes message loss, but in this model, messages are not lost. FLP needs any process to be able to fail, but in this model, the client never fails. Indeed, we can solve consensus in the Hamazon model by just having the client transmit its input to both servers.

Solution

Because processes use the same id if they try to access the mutex twice, the algorithm doesn't work.

Here's an example of a bad execution:

1. Process 1 swaps 1 into s and gets $\perp$, reads $\perp$ from r, performs its critical section, and writes 1 to r.
2. Process 2 swaps 2 into s and gets 1, reads 1 from r, and enters the critical section.
3. Process 1 swaps 1 into s and gets 2, and spins waiting to see 2 in r.
4. Process 3 swaps 3 into s and gets 1. Because r is still 1, process 3 reads this 1 and enters the critical section. We now have two processes in the critical section, violating mutual exclusion.

I believe this works if each process adopts a new id every time it calls `mutex`, but the proof is a little tricky.[6]

[6]The simplest proof I can come up with is to apply an invariant that says that (a) the processes that have executed `swap`$(s,$ myId$)$ but have not yet left the while loop have predecessor values that form a linked list, with the last pointer either equal to $\perp$ (if no process has yet entered the critical section) or the last process to enter the critical section; (b) r is $\perp$ if no process has yet left the critical section, or the last process to leave the critical section otherwise; and (c) if there is a process that is in the critical section, its predecessor field points to the last process to leave the critical section. Checking the effects of each operation shows that this invariant is preserved through the execution, and (a) combined with (c) show that we can't have two processes in the critical section at the same time. Additional work is still needed to show starvation-freedom. It's a good thing this algorithm doesn't work as written.

Appendix B

Sample assignments from Fall 2011

B.1 Assignment 1: due Wednesday, 2011-09-28, at 17:00

Bureaucratic part

Send me email! My address is `aspnes@cs.yale.edu`.

In your message, include:

1. Your name.
2. Your status: whether you are an undergraduate, grad student, auditor, etc.
3. Anything else you'd like to say.

(You will not be graded on the bureaucratic part, but you should do it anyway.)

B.1.1 Anonymous algorithms on a torus

An $n \times m$ **torus** is a two-dimensional version of a ring, where a node at position (i, j) has a neighbor to the north at $(i, j-1)$, the east at $(i+1, j)$, the south at $(i, j+1)$, and the west at $(i-1, j)$. These values wrap around modulo n for the first coordinate and modulo m for the second; so $(0, 0)$ has neighbors $(0, m-1)$, $(1, 0)$, $(0, 1)$, and $(n-1, 0)$.

Suppose that we have a synchronous message-passing network in the form of an $n \times m$ torus, consisting of anonymous, identical processes that do not know n, m, or their own coordinates, but do have a sense of direction (meaning they can tell which of their neighbors is north, east, etc.).

Prove or disprove: Under these conditions, there is a deterministic[1] algorithm that computes whether $n > m$.

Solution

Disproof: Consider two executions, one in an $n \times m$ torus and one in an $m \times n$ torus where $n > m$ and both n and m are at least 2.[2] Using the same argument as in Lemma 6.1.1, show by induction on the round number that, for each round r, all processes in both executions have the same state. It follows that if the processes correctly detect $n > m$ in the $n \times m$ execution, then they incorrectly report $m > n$ in the $m \times n$ execution.

B.1.2 Clustering

Suppose that k of the nodes in an asynchronous message-passing network are designated as cluster heads, and we want to have each node learn the identity of the nearest head. Given the most efficient algorithm you can for this problem, and compute its worst-case time and message complexities.

You may assume that processes have unique identifiers and that all processes know how many neighbors they have.[3]

Solution

The simplest approach would be to run either of the efficient distributed breadth-first search algorithms from Chapter 5 simultaneously starting at all cluster heads, and have each process learn the distance to all cluster heads at once and pick the nearest one. This gives $O(D^2)$ time and $O(k(E+VD))$ messages if we use layering and $O(D)$ time and $O(kDE)$ messages using local synchronization.

We can get rid of the dependence on k in the local-synchronization algorithm by running it almost unmodified, with the only difference being the attachment of a cluster head id to the exactly messages. The simplest way to show that the resulting algorithm works is to imagine coalescing all cluster

[1] Clarification added 2011-09-28.

[2] This last assumption is not strictly necessary, but it avoids having to worry about what it means when a process sends a message to itself.

[3] Clarification added 2011-09-26.

heads into a single initiator; the clustering algorithm effectively simulates the original algorithm running in this modified graph, and the same proof goes through. The running time is still $O(D)$ and the message complexity $O(DE)$.

B.1.3 Negotiation

Two merchants A and B are colluding to fix the price of some valuable commodity, by sending messages to each other for r rounds in a synchronous message-passing system. To avoid the attention of antitrust regulators, the merchants are transmitting their messages via carrier pigeons, which are unreliable and may become lost. Each merchant has an initial price p_A or p_B, which are integer values satisfying $0 \leq p \leq m$ for some known value m, and their goal is to choose new prices p'_A and p'_B, where $|p'_A - p'_B| \leq 1$. If $p_A = p_B$ and no messages are lost, they want the stronger goal that $p'_A = p'_B = p_A = p_B$.

Prove the best lower bound you can on r, as a function of m, for all protocols that achieve these goals.

Solution

This is a thinly-disguised version of the Two Generals Problem from Chapter 3, with the agreement condition $p'_A = p'_B$ replaced by an **approximate agreement** condition $|p'_A - p'_B| \leq 1$. We can use a proof based on the indistinguishability argument in §3.2 to show that $r \geq m/2$.

Fix r, and suppose that in a failure-free execution both processes send messages in all rounds (we can easily modify an algorithm that does not have this property to have it, without increasing r). We will start with a sequence of executions with $p_A = p_B = 0$. Let X_0 be the execution in which no messages are lost, X_1 the execution in which A's last message is lost, X_2 the execution in which both A and B's last messages are lost, and so on, with X_k for $0 \leq k \leq 2r$ losing k messages split evenly between the two processes, breaking ties in favor of losing messages from A.

When i is even, X_i is indistinguishable from X_{i+1} by A; it follows that p'_A is the same in both executions. Because we no longer have agreement, it may be that $p'_B(X_i)$ and $p'_B(X_{i+1})$ are not the same as p'_A in either execution; but since both are within 1 of p'_A, the difference between them is at most 2. Next, because X_{i+1} to X_{i+2} are indistinguishable to B, we have $p'_B(X_{i+1}) = p'_B(X_{i+2})$, which we can combine with the previous claim to get $|p'_B(X_i) - p'_B(X_{i+2})|$. A simple induction then gives $p'_B(X_{2r}) \leq 2r$, where

X_{2r} is an execution in which all messages are lost.

Now construct executions X_{2r+1} and X_{2r+2} by changing p_A and p_B to m one at a time. Using essentially the same argument as before, we get $|p'_B(X_{2r}) - p'_B(X_{2r+2})| \leq 2$ and thus $p'_B(X_{2r+2}) \leq 2r + 2$.

Repeat the initial $2r$ steps backward to get to an execution X_{4r+2} with $p_A = p_B = m$ and no messages lost. Applying the same reasoning as above shows $m = p'_B(X_{4r+2}) \leq 4r + 2$ or $r \geq \frac{m-2}{4} = \Omega(m)$.

Though it is not needed for the solution, it is not too hard to unwind the lower bound argument to extract an algorithm that matches the lower bound up to a small constant factor. For simplicity, let's assume m is even.

The protocol is to send my input in the first message and then use $m/2-1$ subsequent acknowledgments, stopping immediately if I ever fail to receive a message in some round; the total number of rounds r is exactly $m/2$. If I receive s messages in the first s rounds, I decide on $\min(p_A, p_B)$ if that value lies in $[m/2 - s, m/2 + s]$ and the nearest endpoint otherwise. (Note that if $s = 0$, I don't need to compute $\min(p_A, p_B)$, and if $s > 0$, I can do so because I know both inputs.)

This satisfies the approximate agreement condition because if I see only s messages, you see at most $s + 1$, because I stop sending once I miss a message. So either we both decide $\min(p_A, p_B)$ or we choose endpoints $m/2 \pm s_A$ and $m/2 \pm s_B$ that are within 1 of each other. It also satisfies the validity condition $p'_A = p'_B = p_A = p_B$ when both inputs are equal and no messages are lost (and even the stronger requirement that $p'_A = p'_B$ when no messages are lost), because in this case $[m/2 - s, m/2 + s]$ is exactly $[0, m]$ and both processes decide $\min(p_A, p_B)$.

There is still a factor-of-2 gap between the upper and lower bounds. My guess would be that the correct bound is very close to $m/2$ on both sides, and that my lower bound proof is not quite clever enough.

B.2 Assignment 2: due Wednesday, 2011-11-02, at 17:00

B.2.1 Consensus with delivery notifications

The FLP bound (Chapter 9) shows that we can't solve consensus in an asynchronous system with one crash failure. Part of the reason for this is that only the recipient can detect when a message is delivered, so the other processes can't distinguish between a configuration in which a message has or has not been delivered to a faulty process.

Suppose that we augment the system so that senders are notified immediately when their messages are delivered. We can model this by making the delivery of a single message an event that updates the state of both sender and recipient, both of which may send additional messages in response. Let us suppose that this includes attempted deliveries to faulty processes, so that any non-faulty process that sends a message m is eventually notified that m has been delivered (although it might not have any effect on the recipient if the recipient has already crashed).

1. Show that this system can solve consensus with one faulty process when $n = 2$.
2. Show that this system cannot solve consensus with two faulty processes when $n = 3$.

Solution

1. To solve consensus, each process sends its input to the other. Whichever input is delivered first becomes the output value for both processes.
2. To show impossibility with $n = 3$ and two faults, run the usual FLP proof until we get to a configuration C with events e' and e such that Ce is 0-valent and $Ce'e$ is 1-valent (or vice versa). Observe that e and e' may involve two processes each (sender and receiver), for up to four processes total, but only a process that is involved in both e and e' can tell which happened first. There can be at most two such processes. Kill both, and get that $Ce'e$ is indistinguishable from Cee' for the remaining process, giving the usual contradiction.

B.2.2 A circular failure detector

Suppose we equip processes $0 \dots n-1$ in an asynchronous message-passing system with n processes subject to crash failures with a failure detector that is strongly accurate (no non-faulty process is ever suspected) and causes process $i+1 \pmod n$ to eventually permanently suspect process i if process i crashes. Note that this failure detector is not even weakly complete (if both i and $i+1$ crash, no non-faulty process suspects i). Note also that the ring structure of the failure detector doesn't affect the actual network: even though only process $i+1 \pmod n$ may suspect process i, any process can send messages to any other process.

Prove the best upper and lower bounds you can on the largest number of failures f that allows solving consensus in this system.

Solution

There is an easy reduction to FLP that shows $f \leq n/2$ is necessary (when n is even), and a harder reduction that shows $f < 2\sqrt{n} - 1$ is necessary. The easy reduction is based on crashing every other process; now no surviving process can suspect any other survivor, and we are back in an asynchronous message-passing system with no failure detector and 1 remaining failure (if f is at least $n/2 + 1$).

The harder reduction is to crash every $(\sqrt{n})$-th process. This partitions the ring into $\sqrt{n}$ segments of length $\sqrt{n} - 1$ each, where there is no failure detector in any segment that suspects any process in another segment. If an algorithm exists that solves consensus in this situation, then it does so even if (a) all processes in each segment have the same input, (b) if any process in one segment crashes, all $\sqrt{n} - 1$ process in the segment crash, and (c) if any process in a segment takes a step, all take a step, in some fixed order. Under this additional conditions, each segment can be simulated by a single process in an asynchronous system with no failure detectors, and the extra $\sqrt{n} - 1$ failures in $2\sqrt{n} - 1$ correspond to one failure in the simulation. But we can't solve consensus in the simulating system (by FLP), so we can't solve it in the original system either.

On the other side, let's first boost completeness of the failure detector, by having any process that suspects another transmit this submission by reliable broadcast. So now if any non-faulty process i suspects $i + 1$, all the non-faulty processes will suspect $i + 1$. Now with up to t failures, whenever I learn that process i is faulty (through a broadcast message passing on the suspicion of the underlying failure detector, I will suspect processes $i + 1$ through $i + t - f$ as well, where f is the number of failures I have heard about directly. I don't need to suspect process $i + t - f + 1$ (unless there is some intermediate process that has also failed), because the only way that this process will not be suspected eventually is if every process in the range i to $i + t - f$ is faulty, which can't happen given the bound t.

Now if t is small enough that I can't cover the entire ring with these segments, then there is some non-faulty processes that is far enough away from the nearest preceding faulty process that it is never suspected: this gives us an eventually strong failure detector, and we can solve consensus using the standard Chandra-Toueg $\Diamond S$ algorithm from §11.4 or [CT96]. The inequality I am looking for is $f(t - f) < n$, where the left-hand side is maximized by setting $f = t/2$, which gives $t^2/4 < n$ or $t < \sqrt{2n}$. This leaves a gap of about $\sqrt{2}$ between the upper and lower bounds; I don't know which one can be improved.

I am indebted to Hao Pan for suggesting the $\Theta(\sqrt{n})$ upper and lower bounds, which corrected an error in my original draft solution to this problem.

B.2.3 An odd problem

Suppose that each of n processes in a message-passing system with a complete network is attached to a sensor. Each sensor has two states, *active* and *inactive*; initially, all sensors are off. When the sensor changes state, the corresponding process is notified immediately, and can update its state and send messages to other processes in response to this event. It is also guaranteed that if a sensor changes state, it does not change state again for at least two time units. We would like to detect when an odd number of sensors are active, by having at least one process update its state to set off an alarm at a time when this condition holds.

A correct protocol for this problem should satisfy two conditions:

No false positives If a process sets of an alarm, then an odd number of sensors are active.

Termination If at some time an odd number of sensors are active, and from that point on no sensor changes its state, then some process eventually sets off an alarm.

For what values of n is it possible to construct such a protocol?

Solution

It is feasible to solve the problem for $n < 3$.

For $n = 1$, the unique process sets off its alarm as soon as its sensor becomes active.

For $n = 2$, have each process send a message to the other containing its sensor state whenever the sensor state changes. Let s_1 and s_2 be the state of the two process's sensors, with 0 representing inactive and 1 active, and let p_i set off its alarm if it receives a message s such that $s \oplus s_i = 1$. This satisfies termination, because if we reach a configuration with an odd number of active sensors, the last sensor to change causes a message to be sent to the other process that will cause it to set off its alarm. It satisfies no-false-positives, because if p_i sets off its alarm, then $s_{\neg i} = s$ because at most one time unit has elapsed since $p_{\neg i}$ sent s; it follows that $s_{\neg i} \oplus s_i = 1$ and an odd number of sensors are active.

No such protocol is possible for $n \geq 3$. Make p_1's sensor active. Run the protocol until some process p_i is about to enter an alarm state (this occurs eventually because otherwise we violate termination). Let p_j be one of p_2 or p_3 with $j \neq i$, activate p_j's sensor (we can do this without violating the once-per-time-unit restriction because it has never previously been activated) and then let p_i set off its alarm. We have now violated no-false-positives.

B.3 Assignment 3: due Friday, 2011-12-02, at 17:00

B.3.1 A restricted queue

Suppose you have an atomic queue Q that supports operations `enq` and `deq`, restricted so that:

- `enq`(Q) always pushes the identity of the current process onto the tail of the queue.
- `deq`(Q) tests if the queue is nonempty and its head is equal to the identity of the current process. If so, it pops the head and returns **true**. If not, it does nothing and returns **false**.

The rationale for these restrictions is that this is the minimal version of a queue needed to implement a starvation-free mutex using Algorithm 17.2.

What is the consensus number of this object?

Solution

The restricted queue has consensus number 1.

Suppose we have 2 processes, and consider all pairs of operations on Q that might get us out of a bivalent configuration C. Let x be an operation carried out by p that leads to a b-valent state, and y an operation by q that leads to a $(\neg b)$-valent state. There are three cases:

- Two `deq` operations. If Q is empty, the operations commute. If the head of the Q is p, then y is a no-op and p can't distinguish between Cx and Cyx. Similarly for q if the head is q.
- One `enq` and one `deq` operation. Suppose x is an `enq` and y a `deq`. If Q is empty or the head is not q, then y is a no-op: p can't distinguish Cx from Cyx. If the head is q, then x and y commute. The same holds in reverse if x is a `deq` and y an `enq`.

- Two `enq` operations. This is a little tricky, because Cxy and Cyx are different states. However, if Q is nonempty in C, whichever process isn't at the head of Q can't distinguish them, because any `deq` operation returns false and never reaches the newly-enqueued values. This leaves the case where Q is empty in C. Run p until it is poised to do $x' =$ `deq`(Q) (if this never happens, p can't distinguish Cxy from Cyx); then run q until it is poised to do $y' =$ `deq`(Q) as well (same argument as for p). Now allow both `deq` operations to proceed in whichever order causes them both to succeed. Since the processes can't tell which `deq` happened first, they can't tell which `enq` happened first either. Slightly more formally, if we let α be the sequence of operations leading up to the two `deq` operations, we've just shown $Cxy\alpha x'y'$ is indistinguishable from $Cyx\alpha y'x'$ to both processes.

In all cases, we find that we can't escape bivalence. It follows that Q can't solve 2-process consensus.

B.3.2 Writable fetch-and-increment

Suppose you are given an unlimited supply of atomic registers and fetch-and-increment objects, where the fetch-and-increment objects are all initialized to 0 and supply *only* a fetch-and-increment operation that increments the object and returns the old value. Show how to use these objects to construct a wait-free, linearizable implementation of an augmented fetch-and-increment that also supports a `write` operation that sets the value of the fetch-and-increment and returns nothing.

Solution

We'll use a snapshot object a to control access to an infinite array f of fetch-and-increments, where each time somebody writes to the implemented object, we switch to a new fetch-and-increment. Each cell in a holds (timestamp, base), where base is the starting value of the simulated fetch-and-increment. We'll also use an extra fetch-and-increment T to hand out timestamps.

Code is in Algorithm B.1.

Since this is all straight-line code, it's trivially wait-free.

Proof of linearizability is by grouping all operations by timestamp, using $s[i]$.timestamp for `FetchAndIncrement` operations and t for `write` operations, then putting `write` before `FetchAndIncrement`, then ordering `FetchAndIncrement` by return value. Each group will consist of a `write`(v) for some v followed by zero or more `FetchAndIncrement` operations, which will return increasing

```
1 procedure FetchAndIncrement()
2     s ← snapshot(a)
3     i ← argmax_i(s[i].timestamp)
4     return f[s[i].timestamp] + s[i].base

5 procedure write(v)
6     t ← FetchAndIncrement(T)
7     a[myId] ← (t, v)
```

Algorithm B.1: Resettable fetch-and-increment

values starting at v since they are just returning values from the underlying `FetchAndIncrement` object; the implementation thus meets the specification.

To show consistency with the actual execution order, observe that timestamps only increase over time and that the use of snapshot means that any process that observes or writes a timestamp t does so at a time later than any process that observes or writes any $t' < t$; this shows the group order is consistent. Within each group, the `write` writes $a[\mathsf{myId}]$ before any `FetchAndIncrement` reads it, so again we have consistency between the `write` and any `FetchAndIncrement` operations. The `FetchAndIncrement` operations are linearized in the order in which they access the underlying $f[\dots]$ object, so we win here too.

B.3.3 A box object

Suppose you want to implement an object representing a $w \times h$ box whose width (w) and height (h) can be increased if needed. Initially, the box is 1×1, and the coordinates can be increased by 1 each using `IncWidth` and `IncHeight` operations. There is also a `GetArea` operation that returns the area $w \cdot h$ of the box.

Give an obstruction-free deterministic implementation of this object from atomic registers that optimizes the worst-case individual step complexity of `GetArea`, and show that your implementation is optimal by this measure up to constant factors.

Solution

Let b be the box object. Represent b by a snapshot object a, where $a[i]$ holds a pair $(\Delta w_i, \Delta h_i)$ representing the number of times process i has

executed `IncWidth` and `IncHeight`; these operations simply increment the appropriate value and update the snapshot object. Let `GetArea` take a snapshot and return $(\sum_i \Delta w_i)(\sum_i \Delta h_i)$; the cost of the snapshot is $O(n)$.

To see that this is optimal, observe that we can use `IncWidth` and `GetArea` to represent `inc` and `read` for a standard counter. The Jayanti-Tan-Toueg bound applies to counters, giving a worst-case cost of $\Omega(n)$ for `GetArea`.

B.4 CS465/CS565 Final Exam, December 12th, 2011

Write your answers in the blue book(s). Justify your answers. Work alone. Do not use any notes or books.

There are four problems on this exam, each worth 20 points, for a total of 80 points. You have approximately three hours to complete this exam.

General clarifications added during exam Assume all processes have unique ids and know n. Assume that the network is complete in the message-passing model.

B.4.1 Lockable registers (20 points)

Most memory-management units provide the ability to control access to specific memory pages, allowing a page to be marked (for example) read-only. Suppose that we model this by a **lockable register** that has the usual register operations `read`(r) and `write`(r, v) plus an additional operation `lock`(r). The behavior of the register is just like a normal atomic register until somebody calls `lock`(r); after this, any call to `write`(r) has no effect.

What is the consensus number of this object?

Solution

The consensus number is ∞; a single lockable register solves consensus for any number of processes. Code is in Algorithm B.2.

```
1 write(r, input)
2 lock(r)
3 return read(r)
```

Algorithm B.2: Consensus using a lockable register

Termination and validity are trivial. Agreement follows from the fact that whatever value is in r when `lock`(r) is first called will never change, and thus will be read and returned by all processes.

B.4.2 Byzantine timestamps (20 points)

Suppose you have an asynchronous message passing system with exactly one Byzantine process.

You would like the non-faulty processes to be able to acquire an increasing sequence of timestamps. A process should be able to execute the timestamp protocol as often as it likes, and it should be guaranteed that when a process is non-faulty, it eventually obtains a timestamp that is larger than any timestamp returned in any execution of the protocol by a non-faulty process that finishes before the current process's execution started.

Note that there is no bound on the size of a timestamp, so having the Byzantine process run up the timestamp values is not a problem, as long as it can't cause the timestamps to go down.

For what values of n is it possible to solve this problem?

Solution

It is possible to solve the problem for all n except $n = 3$. For $n = 1$, there are no non-faulty processes, so the specification is satisfied trivially. For $n = 2$, there is only one non-faulty process: it can just keep its own counter and return an increasing sequence of timestamps without talking to the other process at all.

For $n = 3$, it is not possible. Consider an execution in which messages between non-faulty processes p and q are delayed indefinitely. If the Byzantine process r acts to each of p and q as it would if the other had crashed, this execution is indistinguishable to p and q from an execution in which r is correct and the other is faulty. Since there is no communication between p and q, it is easy to construct and execution in which the specification is violated.

For $n \geq 4$, the protocol given in Algorithm B.3 works.

The idea is similar to the Attiya, Bar-Noy, Dolev distributed shared memory algorithm [ABND95]. A process that needs a timestamp polls $n-1$ other processes for the maximum values they've seen and adds 1 to it; before returning, it sends the new timestamp to all other processes and waits to receive $n-1$ acknowledgments. The Byzantine process may choose not to answer, but this is not enough to block completion of the protocol.

```
1  procedure getTimestamp()
2      c_i ← c_i + 1
3      send probe(c_i) to all processes
4      wait to receive response(c_i, v_j) from n − 1 processes
5      v_i ← (max_j v_j) + 1
6      send newTimestamp(c_i, v_i) to all processes
7      wait to receive ack(c_i) from n − 1 processes
8      return v_i

9  upon receiving probe(c_j) from j do
10     send response(c_j, v_i) to j

11 upon receiving newTimestamp(c_j, v_j) from j do
12     v_i ← max(v_i, v_j)
13     send ack(c_j) to j
```

Algorithm B.3: Timestamps with $n \geq 3$ and one Byzantine process

To show the timestamps are increasing, observe that after the completion of any call by i to `getTimestamp`, at least $n-2$ non-faulty processes j have a value $v_j \geq v_i$. Any call to `getTimestamp` that starts later sees at least $n-3 > 0$ of these values, and so computes a max that is at least as big as v_i and then adds 1 to it, giving a larger value.

B.4.3 Failure detectors and k-set agreement (20 points)

Recall that in the k-set agreement problem we want each of n processes to choose a decision value, with the property that the set of decision values has at most k distinct elements. It is known that k-set agreement cannot be solved deterministically in an asynchronous message-passing or shared-memory system with k or more crash failures.

Suppose that you are working in an asynchronous message-passing system with an eventually strong ($\Diamond S$) failure detector. Is it possible to solve k-set agreement deterministically with f crash failures, when $k \leq f < n/2$?

Solution

Yes. With $f < n/2$ and $\Diamond S$, we can solve consensus using Chandra-Toueg [CT96]. Since this gives a unique decision value, it solves k-set agree-

ment for any $k \geq 1$.

B.4.4 A set data structure (20 points)

Consider a data structure that represents a set S, with an operation $\texttt{add}(S, x)$ that adds x to S by setting $S \leftarrow S \cup \{x\}$), and an operation $\texttt{size}(S)$ that returns the number of distinct[4] elements $|S|$ of S. There are no restrictions on the types or sizes of elements that can be added to the set.

Show that any deterministic wait-free implementation of this object from atomic registers has individual step complexity $\Omega(n)$ for some operation in the worst case.

Solution

Algorithm B.4 implements a counter from a set object, where the counter read consists of a single call to $\texttt{size}(S)$. The idea is that each increment is implemented by inserting a new element into S, so $|S|$ is always equal to the number of increments.

```
1 procedure inc(S)
2     nonce ← nonce + 1
3     add(S, ⟨myId, nonce⟩).

4 procedure read(S)
5     return size(S)
```

Algorithm B.4: Counter from set object

Since the Jayanti-Tan-Toueg lower bound [JTT00] gives a lower bound of $\Omega(n)$ on the worst-case cost of a counter read, there exists an execution in which $\texttt{size}(S)$ takes $\Omega(n)$ steps.

(We could also apply JTT directly by showing that the set object is perturbable; this follows because adding an element not added by anybody else is always visible to the reader.)

[4] Clarification added during exam.

Appendix C

Additional sample final exams

This appendix contains final exams from previous times the course was offered, and is intended to give a rough guide to the typical format and content of a final exam. Note that the topics covered in past years were not necessarily the same as those covered this year.

C.1 CS425/CS525 Final Exam, December 15th, 2005

Write your answers in the blue book(s). Justify your answers. Work alone. Do not use any notes or books.

There are three problems on this exam, each worth 20 points, for a total of 60 points. You have approximately three hours to complete this exam.

C.1.1 Consensus by attrition (20 points)

Suppose you are given a **bounded fetch-and-subtract** register that holds a non-negative integer value and supports an operation fetch-and-subtract(k) for each $k > 0$ that (a) sets the value of the register to the previous value minus k, or zero if this result would be negative, and (b) returns the previous value of the register.

Determine the consensus number of bounded fetch-and-subtract under the assumptions that you can use arbitrarily many such objects, that you can supplement them with arbitrarily many multiwriter/multireader read/write registers, that you can initialize all registers of both types to initial values

of your choosing, and that the design of the consensus protocol can depend on the number of processes N.

Solution

The consensus number is 2.

To implement 2-process wait-free consensus, use a single fetch-and-subtract register initialized to 1 plus two auxiliary read/write registers to hold the input values of the processes. Each process writes its input to its own register, then performs a fetch-and-subtract(1) on the fetch-and-subtract register. Whichever process gets 1 from the fetch-and-subtract returns its own input; the other process (which gets 0) returns the winning process's input (which it can read from the winning process's read/write register.)

To show that the consensus number is at most 2, observe that any two fetch-and-subtract operations commute: starting from state x, after fetch-and-subtract(k_1) and fetch-and-subtract(k_2) the value in the fetch-and-subtract register is $\max(0, x - k_1 - k_2)$ regardless of the order of the operations.

C.1.2 Long-distance agreement (20 points)

Consider an asynchronous message-passing model consisting of N processes $p_1 \dots p_N$ arranged in a line, so that each process i can send messages only to processes $i-1$ and $i+1$ (if they exist). Assume that there are no failures, that local computation takes zero time, and that every message is delivered at most 1 time unit after it is sent no matter how many messages are sent on the same edge.

Now suppose that we wish to solve agreement in this model, where the agreement protocol is triggered by a local *input* event at one or more processes and it terminates when every process executes a local *decide* event. As with all agreement problems, we want Agreement (all processes decide the same value), Termination (all processes eventually decide), and Validity (the common decision value previously appeared in some input). We also want no false starts: the first action of any process should either be an *input* action or the receipt of a message.

Define the time cost of a protocol for this problem as the worst-case time between the first *input* event and the last *decide* event. Give the best upper and lower bounds you can on this time as function of N. Your upper and lower bounds should be *exact*: using no asymptotic notation or hidden constant factors. Ideally, they should also be equal.

Solution

Upper bound

Because there are no failures, we can appoint a leader and have it decide. The natural choice is some process near the middle, say $p_{\lfloor (N+1)/2 \rfloor}$. Upon receiving an input, either directly through an *input* event or indirectly from another process, the process sends the input value along the line toward the leader. The leader takes the first input it receives and broadcasts it back out in both directions as the decision value. The worst case is when the protocol is initiated at p_N; then we pay $2(N - \lfloor (N+1)/2 \rfloor)$ time to send all messages out and back, which is N time units when N is even and $N-1$ time units when N is odd.

Lower bound

Proving an almost-matching lower bound of $N-1$ time units is trivial: if p_1 is the only initiator and it starts at time t_0, then by an easy induction argument,in the worst case p_i doesn't learn of any input until time $t_0+(i-1)$, and in particular p_N doesn't find out until after $N-1$ time units. If p_N nonetheless decides early, its decision value will violate validity in some executions.

But we can actually prove something stronger than this: that N time units are indeed required when N is odd. Consider two slow executions Ξ_0 and Ξ_1, where (a) all messages are delivered after exactly one time unit in each execution; (b) in Ξ_0 only p_1 receives an input and the input is 0; and (c) in Ξ_1 only p_N receives an input and the input is 1. For each of the executions, construct a causal ordering on events in the usual fashion: a send is ordered before a receive, two events of the same process are ordered by time, and other events are partially ordered by the transitive closure of this relation.

Now consider for Ξ_0 the set of all events that precede the *decide(0)* event of p_1 and for Ξ_1 the set of all events that precede the *decide(1)* event of p_N. Consider further the sets of processes S_0 and S_1 at which these events occur; if these two sets of processes do not overlap, then we can construct an execution in which both sets of events occur, violating Agreement.

Because S_0 and S_1 overlap, we must have $|S_0| + |S_1| \geq N+1$, and so at least one of the two sets has size at least $\lceil (N+1)/2 \rceil$, which is $N/2+1$ when N is even. Suppose that it is S_0. Then in order for any event to occur at $p_{N/2+1}$ at all some sequence of messages must travel from the initial input to p_1 to process $p_{N/2+1}$ (taking $N/2$ time units), and the causal ordering

implies that an additional sequence of messages travels back from $p_{N/2+1}$ to p_1 before p_1 decides (taking and additional $N/2$ time units). The total time is thus N.

C.1.3 Mutex appendages (20 points)

An **append** register supports standard read operations plus an append operation that appends its argument to the list of values already in the register. An **append-and-fetch** register is similar to an append register, except that it returns the value in the register after performing the append operation. Suppose that you have an failure-free asynchronous system with anonymous deterministic processes (i.e., deterministic processes that all run exactly the same code). Prove or disprove each of the following statements:

1. It is possible to solve mutual exclusion using only append registers.
2. It is possible to solve mutual exclusion using only append-and-fetch registers.

In either case, the solution should work for arbitrarily many processes—solving mutual exclusion when $N = 1$ is not interesting. You are also not required in either case to guarantee lockout-freedom.

Clarification given during exam

1. If it helps, you may assume that the processes know N. (It probably doesn't help.)

Solution

1. Disproof: With append registers only, it is not possible to solve mutual exclusion. To prove this, construct a failure-free execution in which the processes never break symmetry. In the initial configuration, all processes have the same state and thus execute either the same read operation or the same append operation; in either case we let all N operations occur in some arbitrary order. If the operations are all reads, all processes read the same value and move to the same new state. If the operations are all appends, then no values are returned and again all processes enter the same new state. (It's also the case that the processes can't tell from the register's state which of the identical append operations went first, but we don't actually need to use this fact.)

Since we get a fair failure-free execution where all processes move through the same sequence of states, if any process decides it's in its critical section, all do. We thus can't solve mutual exclusion in this model.

2. Since the processes are anonymous, any solution that depends on them having identifiers isn't going to work. But there is a simple solution that requires only appending single bits to the register.

   Each process trying to enter a critical section repeatedly executes an append-and-fetch operation with argument 0; if the append-and-fetch operation returns either a list consisting only of a single 0 or a list whose second-to-last element is 1, the process enters its critical section. To leave the critical section, the process does append-and-fetch(1).

C.2 CS425/CS525 Final Exam, May 8th, 2008

Write your answers in the blue book(s). Justify your answers. Work alone. Do not use any notes or books.

There are four problems on this exam, each worth 20 points, for a total of 80 points. You have approximately three hours to complete this exam.

C.2.1 Message passing without failures (20 points)

Suppose you have an asynchronous message-passing system with a complete communication graph, unique node identities, and no failures. Show that any deterministic atomic shared-memory object can be simulated in this model, or give an example of a shared-memory object that can't be simulated.

Solution

Pick some leader node to implement the object. To execute an operation, send the operation to the leader node, then have the leader carry out the operation (sequentially) on its copy of the object and send the results back.

C.2.2 A ring buffer (20 points)

Suppose you are given a **ring buffer object** that consists of $k \geq 1$ memory locations $a[0] \ldots a[k-1]$ with an atomic *shift-and-fetch* operation that takes an argument v and (a) shifts v into the buffer, so that $a[i] \leftarrow a[i+1]$ for

each i less than $k-1$ and $a[k-1] \leftarrow v$; and (b) returns a snapshot of the new contents of the array (after the shift).

What is the consensus number of this object as a function of k?

Solution

We can clearly solve consensus for at least k processes: each process calls shift-and-fetch on its input, and returns the first non-null value in the buffer.

So now we want to show that we can't solve consensus for $k+1$ processes. Apply the usual FLP-style argument to get to a bivalent configuration C where each of the $k+1$ processes has a pending operation that leads to a univalent configuration. Let e_0 and e_1 be particular operations leading to 0-valent and 1-valent configurations, respectively, and let $e_2 \dots e_k$ be the remaining $k-1$ pending operations.

We need to argue first that no two distinct operations e_i and e_j are operations of different objects. Suppose that Ce_i is 0-valent and Ce_j is 1-valent; then if e_i and e_j are on different objects, Ce_ie_j (still 0-valent) is indistinguishable by all processes from Ce_je_i (still 1-valent), a contradiction. Alternatively, if e_i and e_j are both b-valent, there exists some $(1-b)$-valent e_k such that e_i and e_j both operate on the same object as e_k, by the preceding argument. So all of $e_0 \dots e_k$ are operations on the same object.

By the usual argument we know that this object can't be a register. Let's show it can't be a ring buffer either. Consider the configurations $Ce_0e_1 \dots e_k$ and $Ce_1 \dots e_k$. These are indistinguishable to the process carrying out e_k (because its sees only the inputs to e_1 through e_k in its snapshot). So they must have the same valence, a contradiction.

It follows that the consensus number of a k-element ring buffer is exactly k.

C.2.3 Leader election on a torus (20 points)

An $n \times n$ torus is a graph consisting of n^2 nodes, where each node (i, j), $0 \leq i, j \leq n-1$, is connected to nodes $(i-1, j), (i+1, j), (i, j-1)$, and $(i, j+1)$, where all computation is done mod n.

Suppose you have an asynchronous message-passing system with a communication graph in the form of an $n \times n$ torus. Suppose further that each node has a unique identifier (some large natural number) but doesn't know the value of n. Give an algorithm for leader election in this model with the best message complexity you can come up with.

Solution

First observe that each row and column of the torus is a bidirectional ring, so we can run e.g. Hirschbirg and Sinclair's $O(n \log n)$-message protocol within each of these rings to find the smallest identifier in the ring. We'll use this to construct the following algorithm:

1. Run Hirschbirg-Sinclair in each row to get a local leader for each row; this takes $n \times O(n \log n) = O(n^2 \log n)$ messages. Use an additional n messages per row to distribute the identifier for the row leader to all nodes and initiate the next stage of the protocol.

2. Run Hirschbirg-Sinclair in each column with each node adopting the row leader identifier as its own. This costs another $O(n^2 \log n)$ messages; at the end, every node knows the minimum identifier of all nodes in the torus.

The total message complexity is $O(n^2 \log n)$. (I suspect this is optimal, but I don't have a proof.)

C.2.4 An overlay network (20 points)

A collection of n nodes—in an asynchronous message-passing system with a connected, bidirectional communications graph with $O(1)$ links per node—wish to engage in some strictly legitimate file-sharing. Each node starts with some input pair (k, v), where k is a key and v is a value, and the search problem is to find the value v corresponding to a particular key k.

1. Suppose that we can't do any preparation ahead of time. Give an algorithm for searching with the smallest asymptotic worst-case message complexity you can find as a function of n. You may assume that there are no limits on time complexity, message size, or storage space at each node.

2. Suppose now that some designated leader node can initiate a protocol ahead of time to pre-process the data in the nodes before any query is initiated. Give a pre-processing algorithm (that does not depend on which key is eventually searched for) and associated search algorithm such that the search algorithm minimizes the asymptotic worst-case message complexity. Here you may assume that there are no limits on time complexity, message size, or storage space for either algorithm, and that you don't care about the message complexity of the pre-processing algorithm.

3. Give the best lower bound you can on the total message complexity of the pre-processing and search algorithms in the case above.

Solution

1. Run depth-first search to find the matching key and return the corresponding value back up the tree. Message complexity is $O(|E|) = O(n)$ (since each node has only $O(1)$ links).

2. Basic idea: give each node a copy of all key-value pairs, then searches take zero messages. To give each node a copy of all key-value pairs we could do convergecast followed by broadcast ($O(n)$ message complexity) or just flood each pair $O(n^2)$. Either is fine since we don't care about the message complexity of the pre-processing stage.

3. Suppose the total message complexity of both the pre-processing stage and the search protocol is less than $n - 1$. Then there is some node other than the initiator of the search that sends no messages at any time during the protocol. If this is the node with the matching key-value pair, we don't find it. It follows that any solution to the search problem. requires a total of $\Omega(n)$ messages in the pre-processing and search protocols.

C.3 CS425/CS525 Final Exam, May 10th, 2010

Write your answers in the blue book(s). Justify your answers. Work alone. Do not use any notes or books.

There are four problems on this exam, each worth 20 points, for a total of 80 points. You have approximately three hours to complete this exam.

C.3.1 Anti-consensus (20 points)

A wait-free **anti-consensus** protocol satisfies the conditions:

Wait-free termination Every process decides in a bounded number of its own steps.

Non-triviality There is at least one process that decides different values in different executions.

Disagreement If at least two processes decide, then some processes decide on different values.

Show that there is no deterministic wait-free anti-consensus protocol using only atomic registers for two processes and two possible output values, but there is one for three processes and three possible output values.

Clarification: You should assume processes have distinct identities.

Solution

No protocol for two: turn an anti-consensus protocol with outputs in $\{0, 1\}$ into a consensus protocol by having one of the processes always negate its output.

A protocol for three: Use a splitter.

C.3.2 Odd or even (20 points)

Suppose you have a protocol for a synchronous message-passing ring that is anonymous (all processes run the same code) and uniform (this code is the same for rings of different sizes). Suppose also that the processes are given inputs marking some, but not all, of them as leaders. Give an algorithm for determining if the size of the ring is odd or even, or show that no such algorithm is possible.

Clarification: Assume a bidirectional, oriented ring and a deterministic algorithm.

Solution

Here is an impossibility proof. Suppose there is such an algorithm, and let it correctly decide "odd" on a ring of size $2k + 1$ for some k and some set of leader inputs. Now construct a ring of size $4k + 2$ by pasting two such rings together (assigning the same values to the leader bits in each copy) and run the algorithm on this ring. By the usual symmetry argument, every corresponding process sends the same messages and makes the same decisions in both rings, implying that the processes incorrectly decide the ring of size $4k + 2$ is odd.

C.3.3 Atomic snapshot arrays using message-passing (20 points)

Consider the following variant of Attiya-Bar-Noy-Dolev for obtaining snapshots of an array instead of individual register values, in an asynchronous message-passing system with $t < n/4$ crash failures. The data structure we

are simulating is an array a consisting of an atomic register $a[i]$ for each process i, with the ability to perform atomic snapshots.

Values are written by sending a set of $\langle i, v, t_i \rangle$ values to all processes, where i specifies the segment $a[i]$ of the array to write, v gives a value for this segment, and t_i is an increasing timestamp used to indicate more recent values. We use a set of values because (as in ABD) some values may be obtained indirectly.

To update segment $a[i]$ with value v, process i generates a new timestamp t_i, sends $\{\langle i, v, t_i \rangle\}$ to all processes, and waits for acknowledgments from at least $3n/4$ processes.

Upon receiving a message containing one or more $\langle i, v, t_i \rangle$ triples, a process updates its copy of $a[i]$ for any i with a higher timestamp than previously seen, and responds with an acknowledgment (we'll assume use of nonces so that it's unambiguous which message is being acknowledged).

To perform a snapshot, a process sends SNAPSHOT to all processes, and waits to receive responses from at least $3n/4$ processes, which will consist of the most recent values of each $a[i]$ known by each of these processes together with their timestamps (it's a set of triples as above). The snapshot process then takes the most recent versions of $a[i]$ for each of these responses and updates its own copy, then sends its entire snapshot vector to all processes and waits to receive at least $3n/4$ acknowledgments. When it has received these acknowledgments, it returns its own copy of $a[i]$ for all i.

Prove or disprove: The above procedure implements an atomic snapshot array in an asynchronous message-passing system with $t < n/4$ crash failures.

Solution

Disproof: Let s_1 and s_2 be processes carrying out snapshots and let w_1 and w_2 be processes carrying out writes. Suppose that each w_i initiates a write of 1 to $a[w_i]$, but all of its messages to other processes are delayed after it updates its own copy $a_{w_i}[w_i]$. Now let each s_i receive responses from $3n/4 - 1$ processes not otherwise mentioned plus w_i. Then s_1 will return a vector with $a[w_1] = 1$ and $a[w_2] = 0$ while s_2 will return a vector with $a[w_1] = 0$ and $a[w_2] = 1$, which is inconsistent. The fact that these vectors are also disseminated throughout at least $3n/4$ other processes is a red herring.

C.3.4 Priority queues (20 points)

Let Q be a priority queue whose states are multisets of natural numbers and that has operations `enq`(v) and `deq`(), where `enq`(p) adds a new value v to the queue, and `deq`() removes and returns the smallest value in the queue, or returns null if the queue is empty. (If there is more than one copy of the smallest value, only one copy is removed.)

What is the consensus number of this object?

Solution

The consensus number is 2. The proof is similar to that for a queue.

To show we can do consensus for $n = 2$, start with a priority queue with a single value in it, and have each process attempt to dequeue this value. If a process gets the value, it decides on its own input; if it gets null, it decides on the other process's input.

To show we can't do consensus for $n = 3$, observe first that starting from any states C of the queue, given any two operations x and y that are both enqueues or both dequeues, the states Cxy and Cyx are identical. This means that a third process can't tell which operation went first, meaning that a pair of enqueues or a pair of dequeues can't get us out of a bivalent configuration in the FLP argument. We can also exclude any split involving two operations on different queues (or other objects) But we still need to consider the case of a dequeue operation d and an enqueue operation e on the same queue Q. This splits into several subcases, depending on the state C of the queue in some bivalent configuration:

1. $C = \{\}$. Then $Ced = Cd = \{\}$, and a third process can't tell which of d or e went first.

2. C is nonempty and $e =$ `enq`(v), where v is greater than or equal to the smallest value in C. Then Cde and Ced are identical, and no third process can tell which of d or e went first.

3. C is nonempty and $e =$ `enq`(v), where v is less than any value in C. Consider the configurations Ced and Cde. Here the process p_d that performs d can tell which operation went first, because it either obtains v or some other value $v' \neq v$. Kill this process. No other process in Ced or Cde can distinguish the two states without dequeuing whichever of v or v' was not dequeued by p_d. So consider two parallel executions $Ced\sigma$ and $Cde\sigma$ where σ consists of an arbitrary sequence of operations ending with a `deq` on Q by some process p (if no process ever attempts

to dequeue from Q, then we have already won, since the survivors can't distinguish Ced from Cde). Now the state of all objects is the same after $Ced\sigma$ and $Cde\sigma$, and only p_d and p have different states in these two configurations. So any third process is out of luck.

Appendix D

I/O automata

D.1 Low-level view: I/O automata

An **I/O automaton** A is an automaton where transitions are labeled by **actions**, which come in three classes: **input actions**, triggered by the outside world; **output actions** triggered by the automaton and visible to the outside world; and **internal actions**, triggered by the automaton but not visible to the outside world. These classes correspond to inputs, outputs, and internal computation steps of the automaton; the latter are provided mostly to give merged input/output actions a place to go when automata are composed together. A **transition relation** $\texttt{trans}(A)$ relates $\texttt{states}(A) \times \texttt{acts}(A) \times \texttt{states}(A)$; if (s, a, s') is in $\texttt{trans}(A)$, it means that A can move from state s to state s' by executing action a.

There is also an equivalence relation $\texttt{task}(A)$ on the output and internal actions, which is used for enforcing fairness conditions—the basic idea is that in a fair execution some action in each equivalence class must be executed eventually (a more accurate definition will be given below).

The I/O automaton model carries with it a lot of specialized jargon. We'll try to avoid it as much as possible. One thing that will be difficult to avoid in reading [Lyn96] is the notion of a **signature**, which is just the tuple $\texttt{sig}(A) = (\texttt{in}(A), \texttt{out}(A), \texttt{int}(A))$ describing the actions of an automaton A.

D.1.1 Enabled actions

An action a is **enabled** in some state s if $\texttt{trans}(A)$ contains at least one transition (s, a, s'). Input actions are *always* enabled—this is a requirement of the model. Output and internal actions—the "locally controlled"

actions—are not subject to this restriction. A state s is **quiescent** if only input actions are enabled in s.

D.1.2 Executions, fairness, and traces

An **execution** of A is a sequence $s_0 a_0 s_1 a_1 \ldots$ where each triple $(s_i, a_i s_{i+1})$ is in $\texttt{trans}(A)$. Executions may be finite or infinite; if finite, they must end in a state.

A **trace** of A is a subsequence of some execution consisting precisely of the external (i.e., input and output) actions, with states and internal actions omitted. If we don't want to get into the guts of a particular I/O automaton—and we usually don't, unless we can't help it because we have to think explicitly about states for some reason—we can describe its externally-visible behavior by just giving its set of traces.

D.1.3 Composition of automata

Composing a set of I/O automata yields a new super-automaton whose state set is the Cartesian product of the state sets of its components and whose action set is the union of the action sets of its components. A transition with a given action a updates the states of all components that have a as an action and has no effect on the states of other components. The classification of actions into the three classes is used to enforce some simple compatibility rules on the component automata; in particular:

1. An internal action of a component is never an action of another component—internal actions are completely invisible.

2. No output action of a component can be an output action of another component.

3. No action is shared by infinitely many components.[1] In practice this means that no action can be an input action of infinitely many components, since the preceding rules mean that any action is an output or internal action of at most one component.

All output actions of the components are also output actions of the composition. An input action of a component is an input of the composition only if some other component doesn't supply it as an output; in this case

[1] Note that infinite (but countable) compositions *are* permitted.

it becomes an output action of the composition. Internal actions remain internal (and largely useless, except for bookkeeping purposes).

The `task` equivalence relation is the union of the `task` relations for the components: this turns out to give a genuine equivalence relation on output and internal actions precisely because the first two compatibility rules hold.

Given an execution or trace X of a composite automaton that includes A, we can construct the corresponding execution or trace $X|A$ of A which just includes the states of A and the actions visible to A (events that don't change the state of A drop out). The definition of composition is chosen so that $X|A$ is in fact an execution/trace of A whenever X is.

D.1.4 Hiding actions

Composing A and B continues to expose the outputs of A even if they line up with inputs of B. While this may sometimes be desirable, often we want to shove such internal communication under the rug. The model lets us do this by redefining the signature of an automaton to make some or all of the output actions into internal actions.

D.1.5 Fairness

I/O automata come with a built-in definition of **fair executions**, where an execution of A is **fair** if, for each equivalence class C of actions in `task`(A),

1. the execution is finite and no action in C is enabled in the final state, or
2. the execution is infinite and there are infinitely many occurrences of actions in C, or
3. the execution is infinite and there are infinitely many states in which no action in C is enabled.

If we think of C as corresponding to some thread or process, this says that C gets infinitely many chances to do something in an infinite execution, but may not actually do them if it gives ups and stops waiting (the third case). The finite case essentially says that a finite execution isn't fair unless nobody is waiting at the end. The motivation for this particular definition is that it guarantees (a) that any finite execution can be extended to a fair execution and (b) that the restriction $X|A$ of a fair execution or trace X is also fair.

Fairness is useful e.g. for guaranteeing message delivery in a message-passing system: make each message-delivery action its own task class and each message will eventually be delivered; similarly make each message-sending action its own task class and a process will eventually send every message it intends to send. Tweaking the task classes can allow for possibilities of starvation, e.g. if all message-delivery actions are equivalent then a spammer can shut down the system in a "fair" execution where only his (infinitely many) messages are delivered.

D.1.6 Specifying an automaton

The typical approach is to write down preconditions and effects for each action (for input actions, the preconditions are empty). An example would be the spambot in Algorithm D.1.

```
1 input action setMessage(m)
2     effects
3         state ← m
4 output action spam(m)
5     precondition
6         spam = m
7     effects
8         none (keep spamming)
```

Algorithm D.1: Spambot as an I/O automaton

(Plus an initial state, e.g. state $= \perp$, where $\perp$ is not a possible message, and a task partition, of which we will speak more below when we talk about liveness properties.)

D.2 High-level view: traces

When studying the behavior of a system, traces are what we really care about, and we want to avoid talking about states as much as possible. So what we'll aim to do is to get rid of the states early by computing the set of traces (or fair traces) of each automaton in our system, then compose traces to get traces for the system as a whole. Our typical goal will be to show that the resulting set of traces has some desirable properties, usually of the form (1) nothing bad happens (a **safety property**); (2) something good

eventually happens (a **liveness property**); or (3) the horribly complex composite automaton representing this concrete system acts just like that nice clean automaton representing a specification (a **simulation**).

Very formally, a **trace property** specifies both the signature of the automaton and a set of traces, such that all traces (or perhaps fair traces) of the automata appear in the set. We'll usually forget about the first part.

Tricky detail: It's OK if not all traces in P are generated by A (we want $\texttt{trace}(A) \subseteq P$, but not necessarily $\texttt{trace}(A) = P$). But $\texttt{trace}(A)$ will be pretty big (it includes, for example, all finite sequences of input actions) so hopefully the fact that A has to do something with inputs will tell us something useful.

D.2.1 Example

A property we might demand of the spambot above (or some other abstraction of a message channel) is that it only delivers messages that have previously been given to it. As a trace property this says that in any trace t, if $t_k = \mathsf{spam}(m)$, then $t_j = \mathsf{setMessage}(m)$ for some $j < k$. (As a set, this is just the set of all sequences of external spambot-actions that have this property.) Call this property P.

To prove that the spambot automaton given above satisfies P, we might argue that for any execution $s_0 a_0 s_1 a_1 \ldots$, that $s_i = m$ in the last $\mathsf{setMessage}$ action preceding s_i, or $\perp$ if there is no such action. This is easily proved by induction on i. It then follows that since $\mathsf{spam}(m)$ can only transmit the current state, that if $\mathsf{spam}(m)$ follows $s_i = m$ that it follows some earlier $\mathsf{setMessage}(m)$ as claimed.

However, there are traces that satisfy P that don't correspond to executions of the spambot; for example, consider the trace $\mathsf{setMessage}(0)\mathsf{setMessage}(1)\mathsf{spam}(0)$. This satisfies P (0 was previously given to the automaton $\mathsf{spam}(0)$), but the automaton won't generate it because the 0 was overwritten by the later $\mathsf{setMessage}(1)$ action. Whether this is indicates a problem with our automaton not being nondeterministic enough or our trace property being too weak is a question about what we really want the automaton to do.

D.2.2 Types of trace properties

D.2.2.1 Safety properties

P is a **safety property** if

1. P is nonempty.

2. P is **prefix-closed**, i.e. if xy is in P then x is in P.

3. P is **limit-closed**, i.e. if $x_1, x_1x_2, x_1x_2x_3, \ldots$ are all in P, then so is the infinite sequence obtained by taking their limit.

Because of the last restrictions, it's enough to prove that P holds for all finite traces of A to show that it holds for all traces (and thus for all fair traces), since any trace is a limit of finite traces. Conversely, if there is some trace or fair trace for which P fails, the second restriction says that P fails on any finite prefix of P, so again looking at only finite prefixes is enough. The spambot property mentioned above is a safety property.

Safety properties are typically proved using **invariants**, properties that are shown by induction to hold in all reachable states.

D.2.2.2 Liveness properties

P is a **liveness property** of A if any finite sequence of actions in $\texttt{acts}(A)$ has an extension in P. Note that liveness properties will in general include many sequences of actions that aren't traces of A, since they are extensions of finite sequences that A can't do (e.g. starting the execution with an action not enabled in the initial state). If you want to restrict yourself only to proper executions of A, use a safety property. (It's worth noting that the same property P can't do both: any P that is both a liveness and a safety property includes all sequences of actions because of the closure rules.)

Liveness properties are those that are always eventually satisfiable; asserting one says that the property is eventually satisfied. The typical way to prove a liveness property is with a **progress function**, a function f on states that (a) drops by at least 1 every time something that happens infinitely often happens (like an action from an always-enabled task class) and (b) guarantees P once it reaches 0.

An example would be the following property we might demand of our spambot: any trace with at least one $\mathsf{setMessage}(\ldots)$ action contains infinitely many $\mathsf{spam}(\ldots)$ actions. Whether the spambot automaton will satisfy this property (in fair traces) depends on its task partition. If all $\mathsf{spam}(\ldots)$ actions are in the same equivalence class, then any execution with at least one $\mathsf{setMessage}$ will have some spam $(\ldots)$ action enabled at all times thereafter, so a fair trace containing a $\mathsf{setMessage}$ can't be finite (since spam is enabled in the last state) and if infinite contains infinitely many spam messages (since spam messages of some sort are enabled in all but an initial finite prefix). On the other hand, if $\mathsf{spam}(m_1)$ and $\mathsf{spam}(m_2)$ are not equivalent in

$\texttt{task}(A)$, then the spambot doesn't satisfy the liveness property: in an execution that alternates $\mathsf{setMessage}(m_1)\mathsf{setMessage}(m_2)\mathsf{setMessage}(m_1)\mathsf{setMessage}(m_2)\ldots$ there are infinitely many states in which $\mathsf{spam}(m_1)$ is not enabled, so fairness doesn't require doing it even once, and similarly for $\mathsf{spam}(m_2)$.

D.2.2.3 Other properties

Any other property P can be expressed as the intersection of a safety property (the closure of P) and a liveness property (the union of P and the set of all finite sequences that aren't prefixes of traces in P). The intuition is that the safety property prunes out the excess junk we threw into the liveness property to make it a liveness property, since any sequence that isn't a prefix of a trace in P won't go into the safety property. This leaves only the traces in P.

Example: Let $P = \{0^n 1^\infty\}$ be the set of traces where we eventually give up on our pointless 0-action and start doing only 1-actions forever. Then P is the intersection of the safety property $S = \{0^n 1^m\} \cup P$ (the extra junk is from prefix-closure) and the liveness property $L = \{0^n 11^m 0x | xin\{0,1\}^*\} \cup P$. Property S says that once we do a 1 we never do a 0, but allows finite executions of the form 0^n where we never do a 1. Property L says that we eventually do a 1-action, but that we can't stop unless we later do at least one 0-action.

D.2.3 Compositional arguments

The **product** of trace properties $P_1, P_2 \ldots$ is the trace property P where T is in P if and only if $T|\texttt{sig}(P_i)$ is in P_i for each i. If the $\{A_i\}$ satisfy corresponding propertties $\{P_i\}$ individually, then their composition satisfies the product property. (For safety properties, often we prove something weaker about the A_i, which is that each A_i individually is not the first to violate P—i.e., it can't leave P by executing an internal or output action. In an execution where inputs by themselves can't violate P, P then holds.)

Product properties let us prove trace properties by smashing together properties of the component automata, possibly with some restrictions on the signatures to get rid of unwanted actions. The product operation itself is in a sense a combination of a Cartesian product (pick traces t_i and smash them together) filtered by a consistency rule (the smashed trace must be consistent); it acts much like intersection (and indeed can be made identical to intersection if we treat a trace property with a given signature as a way of describing the set of all T such that $T|\texttt{sig}(P_i)$ is in P_i).

D.2.3.1 Example

Consider two spambots A_1 and A_2 where we identify the $\mathsf{spam}(m)$ operation of A_1 with the $\mathsf{setMessage}(m)$ operation of A_2; we'll call this combined action $\mathsf{spam}_1(m)$ to distinguish it from the output actions of A_2. We'd like to argue that the composite automaton $A_1 + A_2$ satisfies the safety property (call it P_m) that any occurrence of $\mathsf{spam}(m)$ is preceded by an occurrence of $\mathsf{setMessage}(m)$, where the signature of P_m includes $\mathsf{setMessage}(m)$ and $\mathsf{spam}(m)$ for some specific m but no other operations. (This is an example of where trace property signatures can be useful without being limited to actions of any specific component automaton.)

To do so, we'll prove a stronger property P'_m, which is P_m modified to include the $\mathsf{spam}_1(m)$ action in its signature. Observe that P'_m is the product of the safety properties for A_1 and A_2 restricted to $\mathtt{sig}(P'_m)$, since the later says that any trace that includes $\mathsf{spam}(m)$ has a previous $\mathsf{spam}_1(m)$ and the former says that any trace that includes $\mathsf{spam}_1(m)$ has a previous $\mathsf{setMessage}(m)$. Since these properties hold for the individual A_1 and A_2, their product, and thus the restriction P'_m, holds for $A_1 + A_2$, and so P_m (as a further restriction) holds for $A_1 + A_2$ as well.

Now let's prove the liveness property for $A_1 + A_2$, that at least one occurrence of $\mathsf{setMessage}$ yields infinitely many spam actions. Here we let $L_1 = \{$at least one $\mathsf{setMessage}$ action $\Rightarrow$ infinitely many spam_1 actions$\}$ and $L_2 = \{$at least one spam_1 action $\Rightarrow$ infinitely many spam actions$\}$. The product of these properties is all sequences with (a) no $\mathsf{setMessage}$ actions or (b) infinitely many spam actions, which is what we want. This product holds if the individual properties L_1 and L_2 hold for $A_1 + A_2$, which will be the case if we set $\mathtt{task}(A_1)$ and $\mathtt{task}(A_2)$ correctly.

D.2.4 Simulation arguments

Show that $\mathtt{traces}(A)$ is a subset of $\mathtt{traces}(B)$ (possibly after hiding some actions of A) by showing a **simulation relation** $f : \mathtt{states}(A) \to \mathtt{states}(B)$ between states of A and states of B. Requirements on f are

1. If s is in $\mathtt{start}(A)$, then $f(s)$ includes some element of $\mathtt{start}(B)$.
2. If (s, a, s') is in $\mathtt{trans}(A)$ and s is reachable, then for any reachable u in $f(s)$, there is a sequence of actions x that takes u to some v in $f(s')$ with $\mathtt{trace}(x) = \mathtt{trace}(a)$.

Using these we construct an execution of B matching (in trace) an execution of A by starting in $f(s_0)$ and applying the second part of the definition

to each action in the A execution (including the hidden ones!)

D.2.4.1 Example

A single spambot A can simulate the conjoined spambots A_1+A_2. Proof: Let $f(s) = (s, s)$. Then $f(\bot) = (\bot, \bot)$ is a start state of $A_1 + A_2$. Now consider a transition (s, a, s') of A; the action a is either (a) $\mathsf{setMessage}(m)$, giving $s' = m$; here we let $x = \mathsf{setMessage}(m)\mathsf{spam}_1(m)$ with $\mathtt{trace}(x) = \mathtt{trace}(a)$ since $\mathsf{spam}_1(m)$ is internal and $f(s') = (m, m)$ the result of applying x; or (b) $a = \mathsf{spam}(m)$, which does not change s or $f(s)$; the matching x is $\mathsf{spam}(m)$, which also does not change $f(s)$ and has the same trace.

A different proof could take advantage of f being a relation by defining $f(s) = \{(s, s')|s' \in \mathtt{states}(A_2)\}$. Now we don't care about the state of A_2, and treat a $\mathsf{setMessage}(m)$ action of A as the sequence $\mathsf{setMessage}(m)$ in $A_1 + A_2$ (which updates the first component of the state correctly) and treat a $\mathsf{spam}(m)$ action as $\mathsf{spam}_1(m)\mathsf{spam}(m)$ (which updates the second component—which we don't care about—and has the correct trace.) In some cases an approach of this sort is necessary because we don't know which simulated state we are heading for until we get an action from A.

Note that the converse doesn't work: $A_1 + A_2$ don't simulate A, since there are traces of A_1+A_2 (e.g. $\mathsf{setMessage}(0)\mathsf{spam}_1(0)\mathsf{setMessage}(1)\mathsf{spam}(0)$) that don't restrict to traces of A. See [Lyn96, §8.5.5] for a more complicated example of how one FIFO queue can simulate two FIFO queues and vice versa (a situation called **bisimulation**).

Since we are looking at traces rather than fair traces, this kind of simulation doesn't help much with liveness properties, but sometimes the connection between states plus a liveness proof for B can be used to get a liveness proof for A (essentially we have to argue that A can't do infinitely many action without triggering a B-action in an appropriate task class). Again see [Lyn96, §8.5.5].