# Unit 5

# Counting and Discrete Probability

## 5.1 Counting

### 5.1.1 Basics of Counting

We study two of the most elementary methods of counting: the product rule and the sum rule. These rule are applied when we can divide the sets or tasks under study into a finite number of subsets or subtasks.

**I. The Product Rule:** Suppose that a task $T$ can be broken down into $n$ subtasks $T_1, T_2, \cdots, T_n$ and suppose that the task $T$ is completed when all the subtasks $T_1, \cdots, T_n$ are completed. If the task $T_i$ can be completed in $k_i$ ways for each $i = 1, \cdots, n$, then the product rule says that the task $T$ can be completed in $k_1 k_2 \cdots k_n$ ways.

**Examples:**

1. If two cards are drawn, each from a separate deck of $52$ cards, then in how many ways can they be selected?
**Solution:** Here,

$$\text{number of ways to draw the first card} = k_1 = 52$$
$$\text{number of ways to draw the second card} = k_2 = 52.$$

Since both the first and the second cards have to be drawn, so by product rule, two cards can be drawn in $k_1 \times k_2 = 52 \times 52 = 2704$ ways.

2. How many $3$-digit numbers can be formed using the digits $1, 3, 4, 5, 6, 8$ and $9$ with repetition allowed and not allowed?
**Solution:** When repetition is allowed,

$$\text{number of ways to select the first digit} = k_1 = 7$$
$$\text{number of ways to select the second digit} = k_2 = 7$$
$$\text{number of ways to select the third digit} = k_3 = 7.$$

Since all three digits have to be selected to form 3-digit number, so by product rule, a 3-digit number with repetition allowed can be formed in $k_1k_2k_3 = 7 \times 7 \times 7 = 343$ ways. When repetition is not allowed,

$$\text{number of ways to select the first digit} = k_1 = 7$$
$$\text{number of ways to select the second digit} = k_2 = 6$$
$$\text{number of ways to select the third digit} = k_3 = 5.$$

So again by product rule, a 3-digit number without repetition can be formed in $k_1k_2k_3 = 7 \times 6 \times 5 = 210$ ways.

3. How many different bit strings are there of length six?
**Solution:** Here,

$$\text{number of ways to select the } 1^{st} \text{ digit} = k_1 = 2$$
$$\text{number of ways to select the } 2^{nd} \text{ digit} = k_2 = 2.$$
$$\text{number of ways to select the } 3^{rd} \text{ digit} = k_3 = 2.$$
$$\text{number of ways to select the } 4^{th} \text{ digit} = k_4 = 2.$$
$$\text{number of ways to select the } 5^{th} \text{ digit} = k_5 = 2.$$
$$\text{number of ways to select the } 6^{th} \text{ digit} = k_6 = 2.$$

So by the product rule, there are

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$$

bit strings of length six.

4. How many bit strings of length ten begin and end with a 1?
**Solution:** To form a bit string of length ten that ends with a 1, there are 2 ways to select the second to the ninth digit and 1 way to select the first and tenth digit. So using the product rule, there are

$$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 1 = 256$$

bit strings of length ten that begin and end with a 1.

**II. The Sum Rule:** Suppose that a task $T$ can be broken into $n$ disjoint subtasks $T_1, T_2, \cdots, T_n$ i.e., none of the subtasks in $T_1, \cdots, T_n$ are same. Also suppose that the task $T$ is completed when exactly one among the subtasks $T_1, \cdots, T_n$ is completed. If the task $T_i$ can be completed in $k_i$ ways for each $i = 1, \cdots, n$, then the sum rule says that the task $T$ can be completed in $k_1 + k_2 + \cdots + k_n$ ways.

**Examples:**

1. In how many ways can one representative be chosen from classes 11 and 12 consisting of 49 and 34 students respectively?
**Solution:** Here

$$\text{number of ways to select one representative from class } 11 = k_1 = 49$$
$$\text{number of ways to select one representative from class } 12 = k_2 = 34.$$

So by sum rule, number of ways to select one representative either from class 11 or class 12 is $k_1 + k_2 = 49 + 34 = 83$ ways.

2. (Example using the combined rules of sum and product) How many license plates can be made using either two letters followed by four digits or two digits followed by four letters?
   **Solution:** By the product rule, the number of license plates with two letters followed by four digits is
   $$26 \times 26 \times 10 \times 10 \times 10 \times 10 = 6760000$$
   and the number of license plates with two digits followed by four letters is
   $$10 \times 10 \times 26 \times 26 \times 26 \times 26 = 45697600.$$
   Since license plates can be one of these two types, so by the sum rule, the total number of such license plates is
   $$6760000 + 45697600 = 52457600.$$

### 5.1.2 Pigeonhole Principle

**Theorem (The Pigeonhole Principle):** If $k+1$ or more objects are placed into $k$ boxes, then there is at least one box containing two or more of the objects.

**Proof:** Suppose that $k+1$ or more objects are placed into $k$ boxes but that none of the boxes contain two or more of the objects i.e., all of the $k$ boxes contain at most one object. This implies that there are at most $k$ objects which is a contradiction because there were at least $k+1$ objects. Therefore, at least one box must contain two or more objects. □

**Examples:**

1. If a classroom has 8 students, then at least two of them must have birthdays on the same weekday. To see why, consider the students as the objects and the 7 weekdays as the boxes. Since the birthday of each of the 8 students must be on one of these 7 weekdays and there are more students than number of weekdays, so by the pigeonhole principle, there must be at least one weekday in which two of more students have their birthdays.

2. If any five distinct numbers are chosen from 1 to 8, then two of them must add to 9.

   Explanation: There are only 4 ways to choose a pair of numbers from 1 to 8 so that they add up to 9: $\{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}$. Consider these as boxes. Now choose any five numbers from 1 to 8. Consider these as objects. So there are more number of objects than boxes. Since each of the five numbers chosen must belong to one of the above four pairs, the pigeonhole principle tells us that two of the chosen numbers must belong to the same pair i.e., the sum of those two numbers must be 9.

3. If $f : S \to T$ is a function where $|S| > |T|$, then $f$ cannot be one-to-one.

   Explanation: Consider the elements of $S$ as the objects and the elements of $T$ as the boxes. Since $|S| > |T|$, so the number of objects is greater than the number of boxes. Since the function $f$ assigns the elements of $S$ to the elements of $T$ and there are more elements in $S$ than in $T$, so by the pigeonhole principle, $f$ must assign at least two elements of $S$ to the same element of $T$. Thus $f$ cannot be one-to-one.

**Theorem (The Generalized Pigeonhole Principle):** If $N$ objects are placed in $k$ boxes, then there is at least one box containing $\left\lceil \frac{N}{k} \right\rceil$ or more objects.

**Proof:** The proof is by contradiction. Suppose $N$ objects are placed in $k$ boxes but that none of the boxes contain $\left\lceil \frac{N}{k} \right\rceil$ or more objects i.e., all the boxes contains at most $\left\lceil \frac{N}{k} \right\rceil - 1$ objects. So the total number of objects is at most $k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right)$. But since $\lceil x \rceil < x + 1$ so

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N.$$

That is, the total number of objects is less than $N$ which is false because there were $N$ number of objects given. Therefore, at least one box must contain $\left\lceil \frac{N}{k} \right\rceil$ or more objects. $\square$

**Note:** If a number of objects are to be distributed in $k$ boxes such that there is guarantee that at least one box has $m$ or more objects, then the minimum number of objects required is

$$k(m-1) + 1.$$

**Examples:**

1. If 30 dictionaries in a library contains a total of 61327 pages, then how many minimum pages at least one dictionary must have?
   **Solution:** Here

   $$\text{total number of pages} = N = 61327$$
   $$\text{total number of dictionaries} = k = 30$$

   So by generalized pigeonhole principle, at least one dictionary must have a minimum of

   $$\left\lceil \frac{61327}{30} \right\rceil = \lceil 2044.23 \rceil = 2045$$

   pages.

2. In a group of 61 people, at least how many of them must have birthdays on the same month?
   **Solution:** Here,

$$\text{total number of people} = N = 61$$
$$\text{total number of months} = k = 12.$$

So by generalized pigeonhole principle, at least

$$\left\lceil \frac{61}{12} \right\rceil = \lceil 5.08 \rceil = 6$$

people must have their birthdays on the same month.

3. How many students must be in a class to guarantee that at least two students receive the same score on the final exam if the exam is graded on a scale from $0$ to $100$ points?
**Solution:** Here,

$$\text{total number of possible scores} = k = 101$$
$$\text{minimum number of students to receive the same score} = m = 2.$$

So to guarantee that at least two students receive the same score, the class must have minimum of $k(m-1)+1 = 101(2-1)+1 = 102$ students.

4. How many numbers must be selected from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ to guarantee that at least one pair of these numbers add up to $11$?
**Solution:** The pair of numbers that add up to $11$ are $\{1, 10\}, \{2, 9\}, \{3, 8\}, \{4, 7\}, \{5, 6\}$. So

$$\text{total number of possible pairs of numbers} = k = 5$$
$$\text{number of numbers to be in same pair} = m = 2.$$

So to guarantee that at least one pair of these numbers add up to $11$, we must select minimum of $k(m-1)+1 = 5(2-1)+1 = 6$ numbers.

5. Find the minimum number of people in a group to be sure that three of them are born on the same month.
**Solution:** Here,

$$\text{total number of months} = k = 12$$
$$\text{number of people to be born on same month} = m = 3$$

So minimum number of people required in the group is

$$k(m-1)+1 = 12(3-1)+1 = 25.$$

6. Find the minimum number of students to guarantee that $50$ of them obtain a same grade where there are altogether $20$ grades.
**Solution:** Here,

$$\text{total number of grades available} = k = 20$$
$$\text{minimum number of students to get same grade} = m = 50.$$

So the required minimum number of students in the class is $k(m-1)+1 = 20(50-1)+1 = 981$.

### 5.1.3 Permutations and Combinations

**Permutation:** The permutation of $n$ distinct objects taken $r$ at a time (also called an $r$-permutation of $n$ objects) is an ordered arrangement of $r$ of the objects. The number of $r$-permutations of $n$-objects is denoted by $P(n, r)$.

For example, in the set $\{a, b, c\}$ consisting of three distinct objects, the 2-permutations are $ab, ba, bc, cb, ac, ca$. So $P(3, 2) = 6$.

**Theorem:** The number of $r$-permutations of a set with $n$ distinct elements is

$$P(n, r) = \frac{n!}{(n-r)!}.$$

**Proof:** Since there are $n$ distinct elements, the first element can be chosen in $n$ ways. Then there are $n-1$ elements left from which the second element can be chosen i.e., there are $n-1$ ways to choose the second element. Similarly, there are $n-2$ ways to choose the third element and proceeding similarly, there are $n-r+1$ ways to choose the $r^{th}$ element. Therefore by product rule, there are $n(n-1)(n-2)\cdots(n-r+1)$ ways to choose $r$ elements from the set of $n$ elements i.e.,

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$
$$= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

□

**Examples:**

1. How many license plates of 4 different digits can be formed?

   **Solution:** We have

   number of digits $= n = 10$,
   number of digits to be selected $= r = 4$.

   So the number of license plates of 4 different digits is

   $$P(10, 4) = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 10 \times 9 \times 8 \times 7 = 5040.$$

2. How many permutation of the letters A, B, C, D, E, F, G, H contain

   a. the string CDE?

   **Solution:** Since CDE must appear as a block, it can be treated as a single letter. So the total number of letters $n = 6$ (which are A, B, CDE, F, G, H) and the letters to be selected $r = 6$. This can be done in

   $$P(6, 6) = \frac{6!}{(6-6)!} = \frac{720}{1} = 720$$

   ways.

b. the strings BA and FGH?

**Solution:** Since BA must appear as a block, it can be treated as a single letter and for the same reason, FGH can be treated as a single letter as well. So the problem reduces to finding the number of permutations of $n = 5$ letters, namely BA, C, D, E, FGH taken $r = 5$ at a time. This can be done in

$$P(5, 5) = 5! = 120$$

ways.

3. How many permutations of the letters A, B, C, D, E, F, G, H are possible so that the letters D, E and F are adjacent?

**Solution:** There are $P(3, 3) = 3!$ ways of arranging the letters D, E and F. For each such arrangement, there are $P(6, 6) = 6!$ ways of arranging the letters A, B, C, G, H, X where X denotes the arrangement of D, E, F. So there are a total of

$$3! \times 6! = 6 \times 720 = 4320$$

permutations of the letters A, B, C, D, E, F, G, H so that the letters D, E and F are adjacent.

**Combination:** The combination of $n$ distinct objects taken $r$ at a time (also called an $r$-combination of $n$ objects) is an unordered selection of $r$ of the objects. The number of $r$-combinations of $n$ objects is denoted by $C(n, r)$ or $\begin{pmatrix} n \\ r \end{pmatrix}$.

Since $r$-combinations differ from one another only when the objects in them are different, so it can be interpreted as an $r$ element subset of a set consisting of $n$ elements and $C(n, r)$ can be interpreted as the number of $r$ element subsets of a set containing $n$ elements.

For example, in the set $\{a, b, c\}$ consisting of three distinct objects, the 2-combinations are $ab, bc, ca$. So $C(3, 2) = 3$.

**Theorem:** The number of $r$-combinations of a set with $n$ distinct elements is

$$C(n, r) = \frac{n!}{r!(n-r)!}.$$

**Proof:** The $r$-permutations of the set can be obtained by first forming the $C(n, r)$ number of $r$-combinations of the set and then ordering the elements in each $r$-combination which can be done in $P(r, r)$ ways. Therefore

$$P(n, r) = C(n, r)P(n, r)$$

$$\text{or, } C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{\dfrac{n!}{(n-r)!}}{\dfrac{r!}{(r-r)!}} = \frac{n!}{r!(n-r)!}$$

Therefore, $C(n, r) = \dfrac{n!}{r!(n-r)!}$. □

**Corollary:** $C(n, 0) = C(n, n) = 1$ for any integer $n \geq 1$.

**Proof:**

$$C(n, 0) = \frac{n!}{0!(n-0)!} = \frac{n!}{0!n!} = \frac{n!}{n!} = 1$$

$$C(n, n) = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = \frac{n!}{n!} = 1$$

□

**Corollary:** $C(n, r) = C(n, n-r)$.

**Proof:** From the theorem, we have

$$C(n, r) = \frac{n!}{r!(n-r)!} = \frac{n!}{[n-(n-r)]!(n-r)!} = \frac{n!}{(n-r)![n-(n-r)]!} = C(n, n-r).$$

□

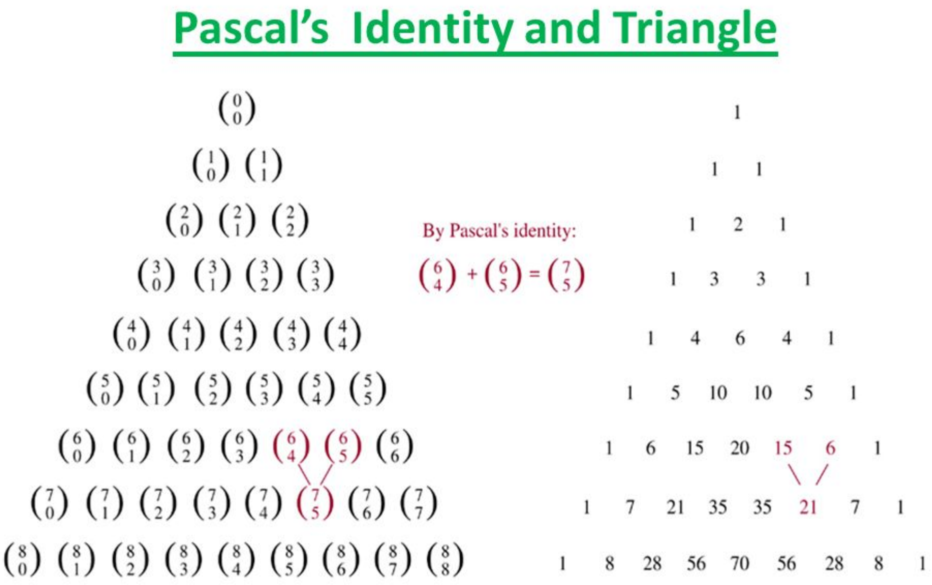
**Pascal's Identity:** $C(n, r) + C(n, r-1) = C(n+1, r)$.

**Proof:** We have

$$\begin{aligned}
C(n, r) + C(n, r-1) &= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!} \\
&= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{(n-r+1)}\right] \\
&= \frac{n!}{(r-1)!(n-r)!}\left[\frac{n-r+1+r}{r(n-r+1)}\right] \\
&= \frac{n!}{(r-1)!(n-r)!} \times \frac{n+1}{r(n-r+1)} \\
&= \frac{(n+1)!}{r!(n-r+1)!} \\
&= \frac{(n+1)!}{r!(n+1-r)!} \\
&= C(n+1, r)
\end{aligned}$$

□

**Pascal's Triangle:** The arrangement of the binomial coefficients in triangular form as below is called Pascal's triangle.




(a) **FIGURE 1** Pascal's Triangle. (b)

As is clearly seen in the figure above, the inner elements in each row of Pascal's triangle is obtained by applying the Pascal's identity on the two elements above it.

**Example:**

1. How many ways are there to form a 6-member committee from 20 people?

   **Solution:** We have

   $$\text{total number of people} = n = 20,$$
   $$\text{number of members to be selected} = r = 6.$$

   So the number of 6-member committee that can be formed is

   $$C(20, 6) = \frac{20!}{6!(20-6)!} = \frac{20!}{6!14!} = 38760.$$

2. How many ways are there to form a 7-member committee containing 3 women and 4 men from an available set of 20 women and 30 men?

   **Solution:** We have,

   $$\text{number of ways to select 3 women from 20 women} = C(20, 3) = 1140$$
   $$\text{number of ways to select 4 men from 30 men} = C(30, 4) = 27405.$$

   So by the product rule, there are

   $$1140 \times 27405 = 31241700$$

   ways to form the required 7-member committee.

3. How many bit strings of length 10 contain

   a. exactly four 1's?

   **Solution:** We number the bit positions of a 10-bit string by numbers 1 to 10 and select any 4 of those positions. In these selected positions, we fill in 1 and in others we fill 0. So we see that the number of bit strings containing exactly four 1's is equal to $C(10, 4) = 210$.

   b. at most four 1's?

   **Solution:** A length 10 bit string that has

$$\begin{gathered}0 \text{ number of 1's is } C(10,0) = 1,\\ 1 \text{ number of 1's is } C(10,1) = 10,\\ 2 \text{ number of 1's is } C(10,2) = 45,\\ 3 \text{ number of 1's is } C(10,3) = 120,\\ 4 \text{ number of 1's is } C(10,4) = 210.\end{gathered}$$

   Therefore the total number of bit strings of length 10 having at most four 1's is

$$1 + 10 + 45 + 120 + 210 = 386.$$

   c. at least four 1's?

   **Solution:** A length 10 bit string that has

$$\begin{gathered}4 \text{ number of 1's is } C(10,4) = 210,\\ 5 \text{ number of 1's is } C(10,5) = 252,\\ 6 \text{ number of 1's is } C(10,6) = 210,\\ 7 \text{ number of 1's is } C(10,7) = 120,\\ 8 \text{ number of 1's is } C(10,8) = 45,\\ 9 \text{ number of 1's is } C(10,9) = 10,\\ 10 \text{ number of 1's is } C(10,10) = 1.\end{gathered}$$

   Therefore the total number of bit strings of length 10 having at least four 1's is

$$210 + 252 + 210 + 120 + 45 + 10 + 1 = 848.$$

   d. an equal number of 0's and 1's?

   **Solution:** A length 10 bit string that has equal number of 0's and 1's has exactly five 0's. The number of such bit strings is

$$C(10, 5) = 252.$$

### 5.1.4 Binomial Coefficients

**Binomial Theorem:** Let $x$ and $y$ be variables and let $n \geq 1$ be an integer. Then

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r.$$

**$k$-th term in a binomial expansion of $(x+y)^n$:** There are altogether $n+1$ terms in the binomial expansion of $(x+y)^n$. The general term of this expansion is of the form

$$\binom{n}{r} x^{n-r} y^r$$

where $r$ can change from 0 to $n$. When $r = 0$, one gets the first term; when $r = 1$, one gets the second term and so on. So the $k$ term is obtained when $r = k - 1$ which is

$$\binom{n}{k-1} x^{n-(k-1)} y^{k-1}.$$

**Middle terms in the binomial expansion of $(x+y)^n$:** If $n$ is even, then there is exactly one middle term in the expansion of $(x+y)^n$, namely the $\frac{n}{2}+1$-th term: $\binom{n}{\frac{n}{2}} x^{\frac{n}{2}} y^{\frac{n}{2}}$.

If $n$ is odd, then there are two middle terms in the expansion of $(x+y)^n$, namely the $\frac{n-1}{2}+1$-th term and $\frac{n+1}{2}+1$-th terms:

$$\binom{n}{\frac{n-1}{2}} x^{n-\frac{n-1}{2}} y^{\frac{n-1}{2}} = \binom{n}{\frac{n-1}{2}} x^{\frac{n+1}{2}} y^{\frac{n-1}{2}}$$

and

$$\binom{n}{\frac{n+1}{2}} x^{n-\frac{n+1}{2}} y^{\frac{n+1}{2}} = \binom{n}{\frac{n+1}{2}} x^{\frac{n-1}{2}} y^{\frac{n+1}{2}}.$$

**Binomial Coefficients:** The number of $r$-combinations of $n$ elements i.e., $\binom{n}{r}$ occur as coefficients in the expansion of the binomial expressions such as $(a+b)^n$. Therefore such numbers are called binomial coefficients.

**Corollary 1:** Let $n$ be a nonnegative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

**Proof:** From the binomial theorem, we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Let $x = 1$ and $y = 1$. Then

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k.$$

$$\text{or, } 2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

□

**Corollary 2:** Let $n$ be a positive integer. Then $\sum_{k=0}^{n}(-1)^k \binom{n}{k} = 0$.

**Proof:** From the binomial theorem, we know

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k.$$

Let $x = 1$ and $y = -1$. Then

$$(1+(-1))^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k}(-1)^k$$

$$\text{or, } 0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

Therefore, $\sum_{k=0}^{n}(-1)^k \binom{n}{k} = 0$. □

**Corollary 3:** Let $n$ be a nonnegative integer. Then $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$.

**Proof:** From the binomial theorem, we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k.$$

Let $x = 1$ and $y = 2$. Then

$$(1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k}2^k$$

$$\text{or, } 3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$$

Therefore, $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$. □

### 5.1.5 Generalized Permutations and Combinations

**Permutation with repetition:** Given $n$ distinct objects, one can take ordered arrangement of $r$ of those objects where those $r$ objects are allowed to repeat. Such kind of permutation is known as $r$-**permutation with repetition**.

For example, if $a, b, c$ are three objects, then the 2-permutations of these objects with repetition allowed are $aa, ab, ba, bb, ca, cc, ac, bc, cb$. These are $3^2 = 9$ in number.

**Theorem 1:** The number of $r$-permutations of a set of $n$ objects with repetition allowed is $n^r$.

**Example:**

1. How many license plates of 4 digits can be formed?

   **Solution:** We have

$$\text{number of digits} = n = 10,$$
$$\text{number of digits to be selected} = r = 4.$$

   So the number of license plates of 4 digits is

$$n^r = 10^4 = 10000.$$

**Combination with repetition:** Given $n$ distinct objects, one can take unordered arrangement of $r$ of those objects where those $r$ objects are allowed to repeat. Such kind of combination is known as $r$-combination with repetition.

For example, given three distinct objects $a, b, c$, the 2-combinations of these objects with repetition allowed are $aa, ab, bb, ba, cc, ca$. These are $C(3+2-1, 2) = C(4, 2) = 6$ in number.

**Theorem 2:** The number of $r$-combinations of a set of $n$ objects with repetition allowed is $C(n+r-1, r)$.

**Example:**

1. Suppose that a library contains Maths, Physics and Chemistry books. If a student has to select 6 books from the library, in how many ways can he/she do this without concerning about the order of chosen books assuming that there are at least 6 copies of each book?

   **Solution:** Here,

$$\text{number of distinct objects, i.e., books available is } n = 3$$
$$\text{number of book to select } r = 6$$

   This can be done in

$$C(n+r-1, r) = C(3+6-1, 6) = C(8, 6) = 28$$

   ways.

**Permutation with indistinguishable objects:** When there are $n$ given objects which are not necessarily distinct, then the permutation of those objects is called permutation with indistinguishable objects. The number of possible permutations of those objects is given by the following theorem:

**Theorem 3:** The number of different permutations of $n$ objects where there are $n_1$ indistinguishable objects of type 1, $n_2$ indistinguishable objects of type 2 and so on $n_k$ indistinguishable objects of type $k$, is given by

$$\frac{n!}{n_1! n_2! \cdots n_k!}.$$

**Example:**

1. How many distinct permutations can be formed from all the letters of the word $EVERGREEN$?

   **Solution:** The total number of letters in the word EVERGREEN is $n = 9$ where

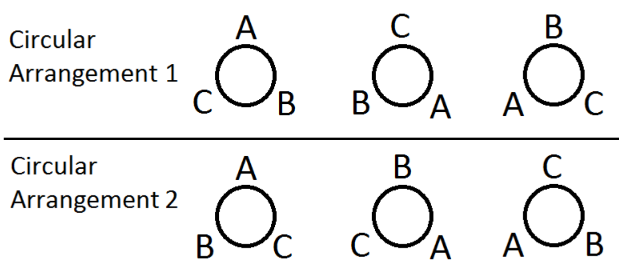
   the number of letters $E$ = $n_1 = 4$
   the number of letters $V$ = $n_2 = 1$
   the number of letters $R$ = $n_3 = 2$
   the number of letters $G$ = $n_4 = 1$
   the number of letters $N$ = $n_5 = 1$

   Therefore the number of distinct permutations is

   $$\frac{9!}{4!\ 1!\ 2!\ 1!\ 1!} = 7560.$$

**Circular Permutation:** The circular permutation of $n$ distinct objects is an ordered arrangement (without repetition) of those objects in a circle.

In an ordinary (or linear) permutation, we get different permutations when the objects are arranged in different order. But in circular permutation, different permutations are obtained only when relative order of the objects are changed in the arrangement. For example, if three objects $A, B, C$ are taken, then their circular permutations are $ABC$ and $ACB$ only. The arrangements $BCA$ and $CAB$ are the same as the arrangement $ABC$ when viewed as circular permutations. Similarly, $BAC$ and $CBA$ are same as $ACB$ when viewed as circular permutations.




**Theorem 4:** The number of circular permutations of $n$ distinct objects in $(n-1)!$.

**Example:**

1. In how many ways can 6 people be seated in a round table?

   **Solution:** The number of people to be seated in a round table is $n = 6$. This can be done in

   $$(n-1)! = (6-1)! = 5! = 120$$

   ways.

## 5.2 Advanced Counting

### 5.2.1 Recurrence Relations

**Recurrence relation:** A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses $a_n$, where $n \geq n_0$ for some nonnegative integer $n_0$, in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \cdots, a_{n-1}$.

For example, the sequence $1, 4, 7, 10, 13, \cdots$ can be expressed using a recursive relation as $a_n = a_{n-1} + 3$, $n \geq 1$, $a_0 = 1$.

**Solution of a recurrence relation:** A sequence whose terms satisfies the recurrence relation is called a solution of that recurrence relation.

For example, the sequence $\{a_n\}$ where $a_n = 3^n$, satisfies the recurrence relation $a_n = 3a_{n-1}$ because $a_n = 3^n = 3.3^{n-1} = 3a_{n-1}$ i.e., the given sequence satisfies the given recurrence relation.

**Problems:**

1. Find the first five terms of the sequence defined by the recurrence relation $a_n = na_{n-1} + a_{n-2}^2$, $a_0 = -1$, $a_1 = 0$.

   **Solution:** Here, the given recurrence relation is

   $$a_n = na_{n-1} + a_{n-2}^2,\ a_0 = -1,\ a_1 = 0$$

   So

   $$\begin{aligned} a_2 &= 2a_{2-1} + a_{2-2}^2 = 2a_1 + a_0^2 = 1 \\ a_3 &= 3a_{3-1} + a_{3-2}^2 = 3a_2 + a_1^2 = 3 \\ a_4 &= 4a_{4-1} + a_{4-2}^2 = 4a_3 + a_2^2 = 13 \\ a_5 &= 5a_{5-1} + a_{5-2}^2 = 5a_4 + a_3^2 = 74 \end{aligned}$$

2. Determine whether the sequences $\{a_n\}$ given below are the solutions of the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$.

   a. $a_n = 4^n$ b. $a_n = (-4)^n$ c. $a_n = 2.4^n + 3n4^n$

   **Solution:**

   a. We have $a_n = 4^n$. So $a_{n-1} = 4^{n-1}$ and $a_{n-2} = 4^{n-2}$. Therefore

   $$8a_{n-1} - 16a_{n-2} = 8.4^{n-1} - 16.4^{n-2} = 4^{n-2}(8.4 - 16) = 4^{n-2}.16 = 4^n = a_n.$$

   So $a_n = 4^n$ is a solution of the given recurrence relation.

   b. When $a_n = (-4)^n$, then $a_{n-1} = (-4)^{n-1}$ and $a_{n-2} = (-4)^{n-2}$. Therefore

   $$8a_{n-1} - 16a_{n-2} = 8.(-4)^{n-1} - 16.(-4)^{n-2} = (-4)^{n-2}(8. - 4 - 16)$$

   $$= (-4)^{n-2}(-32 - 16) = (-4)^{n-2}. - 48 = -3(-4)^{n-2}.16 = -3(-4)^n \neq a_n.$$

So $a_n = (-4)^n$ is not a solution of the given recurrence relation.

c. When $a_n = 4^n(2+3n)$, we have $a_{n-1} = 4^{n-1}(2+3(n-1)) = (3n-1)4^{n-1}$ and $a_{n-2} = 4^{n-2}(2+3(n-2)) = (3n-4)4^{n-2}$. So
$RHS = 8a_{n-1} - 16a_{n-2} = 8(3n-1)4^{n-1} - 16(3n-4)4^{n-2}$
$= 2(3n-1)4^n - (3n-4)4^n = 4^n(6n-2-3n+4) = 4^n(2+3n) = a_n = LHS.$
Therefore $a_n = 4^n(2+3n)$ is a solution of the given recurrence relation.

3. Find a recurrence relation satisfied by the following sequence:
(A) $3, 8, 13, 18, 23, \cdots$
**Solution:** We have

$$\begin{aligned}
a_1 &= 3 \\
a_2 &= 8 = 3+5 = a_1+5 \\
a_3 &= 13 = 8+5 = a_2+5 \\
a_4 &= 18 = 13+5 = a_3+5 \\
a_5 &= 23 = 18+5 = a_4+5
\end{aligned}$$

Therefore the required recurrence relation is $a_n = a_{n-1} + 5$, with $a_1 = 3$.

(B) $a_n = n^2$
**Solution:** We have $a_n = n^2$. So

$$a_{n-1} = (n-1)^2 = n^2 - 2n + 1 = a_n - 2n + 1$$

and therefore

$$a_n = a_{n-1} + 2n - 1$$

is the required recurrence relation.

4. Solve the following recurrence relations using iterative methods:

a. $a_n = a_{n-1} + 3$, $a_0 = 1$
**Solution:** We have,

$$\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2}+3) + 3 = a_{n-2} + 2 \times 3 \\
&= (a_{n-3}+3) + 2 \times 3 = a_{n-3} + 3 \times 3 \\
&\vdots \\
&= a_{n-n} + n \times 3 \\
&= a_0 + 3n = 1 + 3n
\end{aligned}$$

Hence $a_n = 1 + 3n$, $n \geq 0$ is the solution of the given recurrence relation.

b. $a_n = 2na_{n-1}$, $a_0 = 1$

**Solution:** We have,

$$\begin{aligned} a_n &= 2na_{n-1} \\ &= 2n(2(n-1)a_{n-2}) = 2^2n(n-1)a_{n-2} \\ &= 2^2n(n-1)(2(n-2)a_{n-3}) = 2^3n(n-1)(n-2)a_{n-3} \\ &\vdots \\ &= 2^n n(n-1)(n-2)\cdots[n-(n-1)]a_{n-n} \\ &= 2^n n(n-1)(n-2)\cdots 3.2.1.a_0 \\ &= 2^n n!.1 = 2^n n! \end{aligned}$$

Therefore, $a_n = 2^n n!$, $n \geq 0$ is the solution of the given recurrence relation.

c. $a_n = na_{n-1}$, $a_0 = 5$

d. $a_n = 2a_{n-1} - 3$, $a_0 = -1$

e. $a_n = a_{n-1} + n^2$, $a_0 = 7$

f. $a_n = a_{n-1} + 3^n$, $a_0 = 1$

g. $a_n = \dfrac{a_{n-1}}{n}$, $a_1 = 1$

**Modeling with recurrence relations:**

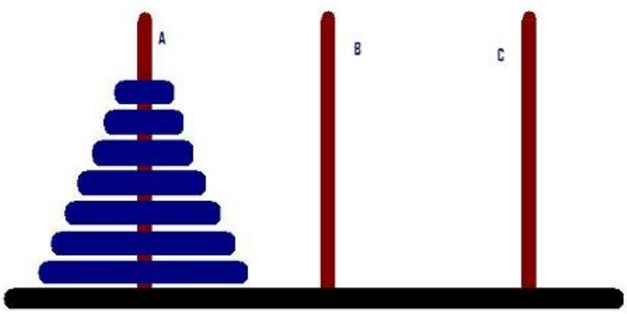
Recurrence relations can be used to solve seemingly difficult counting problems which occur frequently in many situations. To solve such problems, the problem is first of all modeled using a recurrence relation and that recurrence relation is then solved to obtain the required answer.

1. **The tower of Hanoi problem:** There are three poles and $n$ disks of different diameters placed on the pole $A$. These $n$ disks are to be moved to the pole $C$ with same order of arrangement of disks and obeying the following rules:

   i. only one disk may be moved at a time

   ii. pole $B$ may be used for temporary storage of disks

   iii. no disk should ever be placed on top of a disk of smaller diameter



Find the number of moves required to move all $n$ disks from pole $A$ to pole $C$.

**Solution:** Let $H_n$ denote the number of moves needed to solve the tower of Hanoi problem with $n$ disks. Then to transfer $n$ disks to pole $C$, we must first transfer the top $n-1$ disks to pole $B$ (this requires $H_{n-1}$ moves), transfer the remaining largest disk from pole $A$ to pole $C$ (this requires 1 move) and then transfer $n-1$ disks from pole $B$ to pole $C$ (this again requires $H_{n-1}$ moves). Therefore,

$$H_n = H_{n-1} + 1 + H_{n-1} = 2h_{n-1} + 1.$$

Also, the number of moves required to transfer 1 disk from pole $A$ to pole $B$ is 1 i.e., $H_1 = 1$. So we obtain the recurrence relation

$$H_n = 2H_{n-1} + 1, \qquad H_1 = 1.$$

We solve this recurrence relation iteratively as follows:

$$\begin{aligned}
H_n &= 2H_{n-1} + 1 \\
&= 2[2H_{n-2} + 1] + 1 = 2^2 H_{n-2} + 2 + 1 \\
&= 2^2[2H_{n-3} + 1] + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\
&\vdots \\
&= 2^{n-1} H_{n-(n-1)} + 2^{n-2} + \cdots + 2 + 1 \\
&= 2^{n-1} H_1 + 2^{n-2} + \cdots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 \\
&= \frac{2^n - 1}{2 - 1} = 2^n - 1
\end{aligned}$$

Therefore $H_n = 2^n - 1$.

2. Find a recurrence relation and give initial conditions for the number of bit strings of length $n$ that do not have two consecutive zeros.

**Solution:** Let $a_n$ denote the number of bit strings of length $n$ that do not have two consecutive zeros. Then $a_n = b_n + c_n$ where $b_n$ is the number of bit strings of length $n$ that do not have two consecutive zeros and ends with a 1 and $c_n$ is the number of bit strings of length $n$ that do not have two consecutive zeros and ends with a $0$.

Now $x = x_1 x_2 \cdots x_{n-1} x_n$ is a bit string of length $n$ ending with a $1$ that does not contain two consecutive zeros if and only if $x_1 x_2 \cdots x_{n-1}$ does not contain two consecutive zeros. Therefore $b_n = a_{n-1}$. Also $x = x_1 x_2 \cdots x_{n-1} x_n$ is a bit string of length $n$ ending with a $0$ that do not have two consecutive zeros if and only if $x_{n-1} = 1$ and $x_1 x_2 \cdots x_{n-2}$ does not have two consecutive zeros. Therefore $c_n = b_{n-1} = a_{n-2}$. Hence $a_n = b_n + c_n = a_{n-1} + a_{n-2}$. Clearly, $a_1 = 2$ and $a_2 = 3$ which are the required initial conditions.

3. **Compound Interest Problem:** If a principal of amount $P_0 = 10000$ is deposited in an account which gives an annual compound interest rate of $11\%$, then what would be the amount after $N$ years?

   **Solution:** If $P_N$ denotes the amount after $N$ number of years, then this amount is what we get after adding $11\%$ interest to the amount that was at the end of $N-1$ number of years i.e.,

$$P_N = P_{N-1} + \frac{11}{100}P_{N-1} = \frac{111}{100}P_{N-1}$$

   which is a recurrence relation with initial condition $P_0 = 10000$. We can solve this recurrence relation using the back substitution method as follows:

$$P_N = \frac{111}{100}P_{N-1} = \left(\frac{111}{100}\right)^2 P_{N-2} = \cdots = \left(\frac{111}{100}\right)^N P_{N-N} = \left(\frac{111}{100}\right)^N P_0 = \left(\frac{111}{100}\right)^N 10000.$$

### 5.2.2 Solving Homogeneous Recurrence Relations

**Linear homogeneous recurrence relation:** A linear homogeneous recurrence relation of degree $k$ with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

where $c_1, \cdots, c_k$ are constants and $c_k \neq 0$.

For example, $a_n = a_{n-1} + 2a_{n-2}$ is a linear homogeneous recurrence relation of degree 2, $a_n = 4a_{n-2}$ is also a linear homogeneous recurrence relation of degree 2. But $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ is a linear homogeneous recurrence relation of degree 3.

**Characteristic equations and characteristic roots:** If

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}, c_k \neq 0$$

is a linear homogeneous recurrence relation of degree $k$, then the polynomial

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

is called the characteristic equation of this recurrence relation.

The solutions of the characteristic equation are called the characteristic roots of this recurrence relation.

For example, $a_n = a_{n-1} + 2a_{n-2}$ has characteristic equation $r^2 - r - 2 = 0$ with characteristic roots $r_1 = 2$ and $r_2 = -1$, $a_n = 4a_{n-2}$ has characteristic equation $r^2 - 4 = 0$ with characteristic roots $r_1 = 2$ and $r_2 = -2$ and $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ has characteristic equation $r^3 - 3r^2 + 3r - 1 = 0$ with characteristic roots $r_1 = 1$, $r_2 = 1$ and $r_3 = 1$.

**Theorem 1:**(Statement only) Let $c_1$ and $c_2$ be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots $r_1$ and $r_2$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

for $n = 0, 1, 2, \cdots$ where $\alpha_1$ and $\alpha_2$ are constants.

**Examples illustrating the use of Theorem 1:**

1. Find the solution of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$, with $a_0 = 1$ and $a_1 = 0$.

   **Solution:** Here the given recurrence relation is

   $$a_n = 5a_{n-1} - 6a_{n-2}, a_0 = 1 \text{ and } a_1 = 0 \cdots\cdots (1)$$

   The characteristic equation of this recurrence relation is $r^2 - 5r + 6 = 0$. Solutions of this characteristic equation are $r_1 = 2$ and $r_2 = 3$. Therefore the solution $\{a_n\}$ of the recurrence relation (1) must be of the form

   $$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

   $$\text{or, } a_n = \alpha_1 2^n + \alpha_2 3^n \cdots\cdots (2)$$

   where $\alpha_1$ and $\alpha_2$ are some constants.

   When $n = 0$, equation (2) give

   $$a_0 = \alpha_1 + \alpha_2$$

   $$\text{or, } 1 = \alpha_1 + \alpha_2 \cdots\cdots (3)$$

   When $n = 1$, equation (2) gives

   $$a_1 = 2\alpha_1 + 3\alpha_2$$

   $$\text{or, } 0 = 2\alpha_1 + 3\alpha_2 \cdots\cdots (4)$$

   Solving equations (3) and (4), we get, $\alpha_1 = 3$ and $\alpha_2 = -2$. (DO THE DETAILS YOURSELF) Therefore the solution $\{a_n\}$ of the given recurrence relation is of the form $a_n = 3.2^n - 2.3^n$.

2. Find an explicit formula for Fibonacci numbers. OR Solve the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with $f_0 = 0$ and $f_1 = 1$.

   **Solution:** The recurrence relation representing the Fibonacci numbers is

   $$f_n = f_{n-1} + f_{n-2}, \quad f_0 = 0, f_1 = 1 \cdots\cdots (1)$$

   The characteristic equation associated with this recurrence relation is $r^2 - r - 1 = 0$ whose solutions are $r_1 = \dfrac{1+\sqrt{5}}{2}$ and $r_2 = \dfrac{1-\sqrt{5}}{2}$. Therefore the solution $\{a_n\}$ of this recurrence relation is of the form

   $$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \cdots\cdots (2)$$

where $\alpha_1$ and $\alpha_2$ are constants.

When $n = 0$, then equation (2) gives

$$f_0 = \alpha_1 + \alpha_2$$

$$\text{or, } 0 = \alpha_1 + \alpha_2 \cdots\cdots (3)$$

and when $n = 1$, equation (2) gives

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)$$

$$\text{or, } 1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) \cdots\cdots (4)$$

Solving (3) and (4), we get $\alpha_1 = \dfrac{1}{\sqrt{5}}$ and $\alpha_2 = \dfrac{-1}{\sqrt{5}}$. (DO THE DETAILS YOURSELF)

Hence the explicit formula for the Fibonacci numbers is

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

3. Find the solution of following recurrence relations:

   a. $a_n = a_{n-1} + 6a_{n-2}, a_0 = 3, a_1 = 6$

   b. $a_n = 7a_{n-1} - 10a_{n-2}, a_0 = 2, a_1 = 1$

   c. $a_n = 4a_{n-2}, a_0 = 0, a_1 = 4$

**Theorem 2 (Generalization of Theorem 1):** (Statement only) Let $c_1, c_2, \cdots, c_k$ be real numbers. Suppose that the characteristic equation $r^k - c_1 r^{k-1} - \cdots - c_k = 0$ has $k$ distinct roots $r_1, r_2, \cdots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-l}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \cdots$ where $\alpha_1, \alpha_2, \cdots, \alpha_k$ are constants.

**Examples illustrating the use of Theorem 2:**

1. Solve the recurrence relation $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ with $a_0 = 3, a_1 = 6$ and $a_2 = 0$.

   **Solution:** The given recurrence relation is

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}, a_0 = 3, a_1 = 6, a_2 = 0 \cdots\cdots (1)$$

The characteristic equation of this recurrence relation is

$$r^3 - 2r^2 - r + 2 = 0$$

$$\text{or, } r^2(r-2) - 1(r-2) = 0$$

$$\text{or, } (r^2-1)(r-2) = 0$$

$$\text{or, } (r-1)(r+1)(r-2) = 0$$

So the characteristic roots are $r_1 = -1, r_2 = 1$ and $r_3 = 2$. Therefore the solution $\{a_n\}$ of the recurrence relation must be of the form

$$a_n = \alpha_1(-1)^n + \alpha_2 1^n + \alpha_3 2^n$$

$$\text{or, } a_n = \alpha_1(-1)^n + \alpha_2 + \alpha_3 2^n \cdots\cdots (2)$$

where $\alpha_1, \alpha_2$ and $\alpha_3$ are constants.

When $n = 0$, equation (2) becomes

$$3 = \alpha_1 + \alpha_2 + \alpha_3 \cdots\cdots (3)$$

When $n = 1$, equation (2) becomes

$$6 = -\alpha_1 + \alpha_2 + 2\alpha_3 \cdots\cdots (4)$$

When $n = 2$, equation (2) becomes

$$0 = \alpha_1 + \alpha_2 + 4\alpha_3 \cdots\cdots (5)$$

Solving (3), (4) and (5), we get $\alpha_1 = -2$, $\alpha_2 = 6$ and $\alpha_3 = -1$. (DO THE DETAILS YOURSELF) Hence the solution $\{a_n\}$ of the recurrence relation (1) is of the form

$$a_n = -2(-1)^n + 6 - 2^n = 2(-1)^{n+1} - 2^n + 6.$$

2. Solve the following recurrence relations:

   a. $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with $a_0 = 2, a_1 = 5, a_2 = 15$.
   Hint: $r^3 - 6r^2 + 11r - 6 = r^3 - 5r^2 + 6r - r^2 + 5r - 6 = (r-1)(r^2 - 5r + 6)$

   b. $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9, a_1 = 10, a_2 = 32$.
   Hint: $r^3 - 7r - 6 = r^3 - r^2 - 6r + r^2 - r - 6 = (r+1)(r^2 - r - 6)$

**Theorem 3:**(Statement only) Let $c_1$ and $c_2$ be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root $r_0$. A sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if

$$a_n = r_0^n(\alpha_1 + \alpha_2 n)$$

for $n = 0, 1, 2, \cdots$ where $\alpha_1$ and $\alpha_2$ are constants.

**Examples illustrating the use of Theorem 3:**

1. Find the solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ with $a_0 = 4, a_1 = 1$.

   **Solution:** The given recurrence relation is

   $$a_n = 2a_{n-1} - a_{n-2}, a_0 = 4, a_1 = 1 \cdots\cdots (1)$$

   The characteristic equation of this recurrence relation is $r^2 - 2r + 1 = 0$ whose only solution is $r_0 = 1$. Therefore the solution of this recurrence relation is of the form

   $$a_n = r_0^n(\alpha_1 + \alpha_2 n)$$
   $$\text{or, } a_n = 1^n(\alpha_1 + \alpha_2 n)$$
   $$\text{or, } a_n = \alpha_1 + \alpha_2 n \cdots\cdots (2)$$

   where $\alpha_1$ and $\alpha_2$ are some constants.

   When $n = 0$, equation (2) becomes

   $$a_0 = \alpha_1 + \alpha_2.0$$
   $$\text{or, } 4 = \alpha_1 \cdots\cdots (3)$$

   and when $n = 1$, equation (2) becomes

   $$a_1 = \alpha_1 + \alpha_2.1$$
   $$\text{or, } 1 = \alpha_1 + \alpha_2 \cdots\cdots (4)$$

   Solving (3) and (4), we get $\alpha_1 = 4$ and $\alpha_2 = -3$. (DO THE DETAILS YOURSELF) Hence the solution $\{a_n\}$ of this recurrence relation is of the form $a_n = 4 - 3n$.

2. Find the solution of the following recurrence relations:

   a. $a_n = -6a_{n-1} - 9a_{n-2}, a_0 = 3, a_1 = -3$

   b. $a_n = 4a_{n-1} - 4a_{n-2}, a_0 = \dfrac{5}{2}, a_1 = 8$

**Theorem 4 (Generalization of Theorem 3):** (Statement only) Let $c_1, c_2, \cdots, c_k$ be real numbers. Suppose that the characteristic equation $r^k - c_1 r^{k-1} - \cdots - c_k = 0$ has $t$ distinct roots $r_1, r_2, \cdots, r_t$ with multiplicities $m_1, m_2, \cdots, m_t$ respectively, so that $m_i \geq 1$ for $i = 1, 2, \cdots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = &(\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ &+(\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ &+ \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \cdots$ where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

**Examples illustrating the use of Theorem 4:**

1. Solve the recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ with $a_0 = -5, a_1 = 4$ and $a_2 = 88$.

   **Solution:** The given recurrence relation is

$$a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}, a_0 = -5, a_1 = 4, a_2 = 88 \cdots\cdots (1)$$

   The characteristic equation of this recurrence relation is

$$r^3 - 6r^2 + 12r - 8 = 0$$

$$\text{or, } r^3 - 3.2r^2 + 3.2^2r - 2^3 = 0$$

$$\text{or, } (r-2)^3 = 0$$

   Therefore, the solution of the characteristic equation is $r_1 = 2$ with multiplicity $m_1 = 3$. Hence the solution $\{a_n\}$ of the recurrence relation must be of the form

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2)2^n \cdots\cdots (2)$$

   where $\alpha_{1,0}, \alpha_{1,1}$ and $\alpha_{1,2}$ are constants.

   When $n = 0$, then from equation (2), we get

$$a_0 = (\alpha_{1,0} + \alpha_{1,1}.0 + \alpha_{1,2}.0^2)2^0$$

$$\text{or, } -5 = \alpha_{1,0} \cdots\cdots (3)$$

   When $n = 1$, then from equation (2), we get

$$a_1 = (\alpha_{1,0} + \alpha_{1,1}.1 + \alpha_{1,2}.1^2)2^1$$

$$\text{or, } 4 = 2\alpha_{1,0} + 2\alpha_{1,1} + 2\alpha_{1,2}$$

$$\text{or, } 2 = \alpha_{1,0} + \alpha_{1,1} + \alpha_{1,2} \cdots\cdots (4)$$

   When $n = 2$, then from equation (2), we get

$$a_2 = (\alpha_{1,0} + \alpha_{1,1}.2 + \alpha_{1,2}.2^2)2^2$$

$$\text{or, } 88 = (\alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2})4$$

$$\text{or, } 22 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2} \cdots\cdots (5)$$

   Solving equations (3), (4) and (5), we get $\alpha_{1,0} = -5$, $\alpha_{1,1} = \frac{1}{2}$, $\alpha_{1,2} = \frac{13}{2}$. (DO THE DETAILS YOURSELF) Therefore, the solution $\{a_n\}$ of the recurrence relation (1) is of the form

$$a_n = \left(-5 + \frac{1}{2}n + \frac{13}{2}n^2\right)2^n.$$

2. Find the solution of the following recurrence relations:

   a. $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}, a_0 = 5, a_1 = -9, a_2 = 15$

b. $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}, a_0 = 1, a_1 = 4, a_2 = 8$
Hint: $r^3 - 7r^2 + 16r - 12 = r^3 - 7r^2 + 12r + 4r - 12 = r(r^2 - 7r + 12) + 4(r - 3) = r(r-3)(r-4) + 4(r-3) = (r-3)(r-2)^2 = 0$. Therefore $r_1 = 3, m_1 = 1$, $r_2 = 2, m_2 = 2$. So $a_n = \alpha_{1,0}3^n + (\alpha_{2,0} + \alpha_{2,1}n)2^n$.

**Summary on solution of linear homogeneous recurrence relations:**

1. **Solving linear homogeneous recurrence relations of degree $2$:**

   To solve a linear homogeneous recurrence relations of degree $2$ of the form

$$a_n = c_1a_{n-1} + c_2a_{n-2}, a_0 = d_1, a_1 = d_2$$

   we follow the following four steps:

STEP I: Form the characteristic equation

$$r^2 - c_1r - c_2 = 0.$$

STEP II: Solve the characteristic equation to find the characteristic roots $r_1$ and $r_2$.

STEP III: There are two cases that arises depending upon the nature of the roots $r_1$ and $r_2$.

Case A: $r_1 \neq r_2$
In this case, the solution of the recurrence relation is of the form

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

where $\alpha_1$ and $\alpha_2$ are constants.

Case B: $r_1 = r_2$
In this case, the solution of the recurrence relation is of the form

$$a_n = (\alpha_1 + \alpha_2 n)r_1^n$$

where $\alpha_1$ and $\alpha_2$ are constants.

STEP IV: Use the initial conditions, $a_0 = d_1$ and $a_1 = d_2$ to obtain the values of the constants $\alpha_1$ and $\alpha_2$.

2. **Solving linear homogeneous recurrence relations of degree $3$:**

   To solve a linear homogeneous recurrence relations of degree $3$ of the form

$$a_n = c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3}, a_0 = d_1, a_1 = d_2, a_2 = d_3$$

   we follow the following four steps:

STEP I: Form the characteristic equation

$$r^3 - c_1r^2 - c_2r - c_3 = 0.$$

STEP II: Solve the characteristic equation to find the characteristic roots $r_1$, $r_2$ and $r_3$.

STEP III: There are three cases that arises depending upon the nature of the roots $r_1$, $r_2$ and $r_3$.

Case A: $r_1, r_2$ and $r_3$ are all distinct:
In this case, the solution of the recurrence relation is of the form

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n$$

where $\alpha_1$, $\alpha_2$ and $\alpha_3$ are constants.

Case B: $r_1 = r_2 = r_3$
In this case, the solution of the recurrence relation is of the form

$$a_n = (\alpha_1 + \alpha_2 n + \alpha_3 n^2) r_1^n$$

where $\alpha_1$, $\alpha_2$ and $\alpha_3$ are constants.

Case C: $r_1 = r_2$ and $r_3$ is distinct:
In this case, the solution of the recurrence relation is of the form

$$a_n = (\alpha_1 + \alpha_2 n) r_1^n + \alpha_3 r_3^n$$

where $\alpha_1$, $\alpha_2$ and $\alpha_3$ are constants.

STEP IV: Use the initial conditions, $a_0 = d_1$, $a_1 = d_2$ and $a_2 = d_3$ to obtain the values of the constants $\alpha_1$, $\alpha_2$ and $\alpha_3$.

### 5.2.3 Solving Nonhomogeneous Recurrence Relation

**Linear nonhomogeneous recurrence relation:** A linear nonhomogeneous recurrence relation with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

where $c_1, c_2, \cdots, c_k$ are real numbers and $F(n)$ is a function not identically zero depending only on $n$.

For example, $a_n = a_{n-1} + 2a_{n-2} + n^2 + n$ is a linear nonhomogeneous recurrence relation.

**Associated homogeneous recurrence relation:** Given a linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is called the associated homogeneous recurrence relation.

For example, $a_n = a_{n-1} + 2a_{n-2}$ is the associated homogeneous recurrence relation of the linear nonhomogeneous recurrence relation $a_n = a_{n-1} + 2a_{n-2} + n^2 + n$.

**Theorem 5:** (Statement only) If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$ where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$.

The particular solution $a_n^{(p)}$ of a nonhomogeneous recurrence relation depends upon the function $F(n)$. We consider the cases when $F(n)$ is either a polynomial function or an exponential function or their product. The particular solutions in each of these cases is given below:

**Finding particular solution $a_n^{(p)}$ of a linear nonhomogeneous recurrence relation:**

I: If $F(n) = p(n)$ where $p(n)$ is a polynomial of degree $m$, then the particular solution $a_n^{(p)}$ is a complete polynomial (i.e., containing all the terms) of degree $m$ as well. For example, if $F(n) = n^3 + 2$, then $F(n)$ is a polynomial of degree 3 and therefore $a_n^{(p)} = c_1 n^3 + c_2 n^2 + c_3 n + c_4$ is the particular solution where $c_1, c_2, c_3, c_4$ are some constants.

II: If $F(n) = a^n$ is an exponential function of $n$, then the form of particular solution $a_n^{(p)}$ depends upon two cases as below:

Case A: If $a$ is not a characteristic root of the associated homogeneous recurrence relation then $a_n^{(p)} = ca^n$ where $c$ is some constant.

Case B: If $a$ is a characteristic root of the associated homogeneous recurrence relation which is repeated $s$ times, then $a_n^{(p)} = cn^s a^n$ where $c$ is some constant.

III: If $F(n) = p(n)a^n$ where $p(n)$ is a polynomial of degree $m$, then the form of particular solution $a_n^{(p)}$ again depends upon two cases as below:

Case A: If $a$ is not a characteristic root of the associated homogeneous recurrence relation then $a_n^{(p)} = q(n)a^n$ where $q(n)$ is a complete polynomial of degree $m$.

Case B: If $a$ is a characteristic root of the associated homogeneous recurrence relation which is repeated $s$ times then $a_n^{(p)} = q(n)n^s a^n$ where $q(n)$ is a complete polynomial of degree $m$.

**Examples:**

1. Find all solutions of the recurrence relation

$$a_n = -5a_{n-1} - 6a_{n-2} + 3n^2 \cdots\cdots (1)$$

**Solution:** The associated homogeneous recurrence relation of the given recurrence relation is

$$a_n = -5a_{n-1} - 6a_{n-2} \cdots\cdots (2)$$

The characteristic equation of this recurrence relation is $r^2 + 5r + 6 = 0$ whose solutions are $r_1 = -2$ and $r_2 = -3$. So the solution $\{a_n^{(h)}\}$ of the recurrence relation (2) is of the form

$$a_n^{(h)} = \alpha_1(-2)^n + \alpha_2(-3)^n$$
$$\text{or, } a_n^{(h)} = (-1)^n(\alpha_1 2^n + \alpha_2 3^n)$$

Since $F(n) = 3n^2$ is a polynomial of degree 2, assume that $a_n^{(p)} = c_1n^2 + c_2n + c_3$ is one particular solution of (1). Then from (1)

$$c_1n^2 + c_2n + c_3 = -5[c_1(n-1)^2 + c_2(n-1) + c_3] - 6[c_1(n-2)^2 + c_2(n-2) + c_3] + 3n^2$$
$$\text{or, } 12c_1n^2 - (34c_1 - 12c_2)n + (29c_1 - 17c_2 + 12c_3) = 3n^2$$

Comparing the LHS and RHS, we get

$$12c_1 = 3, -(34c_1 - 12c_2) = 0, 29c_1 - 17c_2 + 12c_3 = 0.$$

Solving the above equations, we get $c_1 = \frac{1}{4}$, $c_2 = \frac{17}{24}$, $c_3 = \frac{115}{288}$. (DO THE DETAILS YOURSELF) So the particular solution $\{a_n^{(p)}\}$ of (1) is of the form

$$a_n^{(p)} = \frac{1}{4}n^2 + \frac{17}{24}n + \frac{115}{288}.$$

Hence all the solutions of (1) are of the form

$$a_n = a_n^{(h)} + a_n^{(p)} = (-1)^n(\alpha_1 2^n + \alpha_2 3^n) + \frac{1}{4}n^2 + \frac{17}{24}n + \frac{115}{288}.$$

2. Find all solutions of the recurrence relation

$$a_n = -5a_{n-1} - 6a_{n-2} + 7^n \cdots\cdots (1)$$

**Solution:** The associated homogeneous recurrence relation of the given recurrence relation is $a_n = -5a_{n-1} - 6a_{n-2}$ whose solution $\{a_n^{(h)}\}$ is of the form $a_n^{(h)} = (-1)^n(\alpha_1 2^n + \alpha_2 3^n)$.

Since $F(n) = 7^n$ and 7 is not a characteristic root of the associated homogeneous recurrence relation, we let $a_n^{(p)} = c7^n$ be the particular solution of (1) where $c$ is some constant. Then from (1), we get

$$c7^n = -5c.7^{n-1} - 6c.7^{n-2} + 7^n$$
$$\text{or, } c7^2 = -5c.7 - 6c + 7^2$$
$$\text{or, } 49c = -35c - 6c + 49$$
$$\text{or, } 49c + 41c = 49$$
$$\text{or, } c = \frac{49}{90}$$

So $a_n^{(p)} = \frac{49}{90}7^n = \frac{7^{n+2}}{90}$ is a particular solution of (1). Hence all solutions of (1) are of the form

$$a_n = (-1)^n(\alpha_1 2^n + \alpha_2 3^n) + \frac{7^{n+2}}{90}.$$

### 5.2.4 Divide-and-Conquer Algorithms and Recurrence Relations

Divide-and-conquer is a strategy of problem solving where one first divides the given problem into one or more instances of similar subproblems but with smaller input size than the original problem. This division is continued until the stage is reached when one can easily solve the individual subproblems. Then the process of "conquering" starts where these solved subproblems are somehow combined to form the solution of the original problem. The algorithms that use the divide-and-conquer strategy to solve problems are called divide-and-conquer algorithms. Binary search algorithm used to locate an element in a given list of $n$ elements is an example of divide-and-conquer algorithm. Similarly the merge sort algorithm used to sort a given list of $n$ elements into ascending or descending order is also an example of divide-and-conquer algorithm.

The computational complexity (i.e. the number of operations required to solve a problem of size $n$) of most of the divide-and-conquer algorithms can be estimated using a recurrence relation. If an algorithm divides a problem of size $n$ into $a$ similar subproblems each of size $n/b$ and $g(n)$ denotes the number of extra operations required to combine the solution of subproblems in the "conquer" phase of the algorithm then the number of operations required to solve this problem, denoted by $f(n)$, can be represented by a recurrence relation as

$$f(n) = af\left(\frac{n}{b}\right) + g(n).$$

This recurrence relation is called a divide-and-conquer recurrence relation. Solving this recurrence relation will help us to estimate the number of operations to solve the given problem.

## 5.3 Exercise

### 5.3.1 Basics of Counting

1. How many different bit strings of length $9$ are there that begins with a $1$ and ends in $00$?
2. How many different bit strings are there of length $n$ that starts and ends in $1$?
3. How many bit strings of length $11$ are there that starts and ends in $01$?
4. How many ways are there to select first, second and third prize winners from $10$ different students?
5. Determine the number of ways that a football team can be formed from a pool of $40$ players.
6. Find the number of different license plates if it contains a sequence of three letters followed by three digits.

### 5.3.2 Pigeonhole Principle

1. How many numbers must be selected from the numbers $1$ to $10$ to guarantee that the sum of at least one pair of selected numbers is $11$?
2. In a group of $100$ people, find the minimum number of people who were born in the same month.
3. Find the minimum number of students in a class to be sure that five of them are born in the same month.
4. Find the minimum number of people in a room for $6$ of them to be born in the same weekday.
5. Use pigeonhole principle to find the minimum number of students in a class to guarantee that at least two students receive the same score on the exams if the exam is graded from $0$ to $60$.
6. Find the minimum number of students to guarantee that $50$ of them obtain same grade where there are altogether $20$ different grades.
7. How many cards must be selected from a deck of $52$ cards to ensure that at least $3$ cards of the same suite are chosen?

### 5.3.3 Permutation and Combination

1. Find the values of $P(10, 4)$ and $P(10, 6)$.
2. How many permutations of the letters B, C, D, E, F, G, H, I contain the string CGI?
3. How many $4$-letter words are possible from the vowel letters?
4. In how many ways can the letters of the word MISSISSIPPI be arranged?

5. In how many ways can the letters of the word COCACOLA be arranged?
6. In how many ways can numbers on the clock face be arranged?
7. Find the number of ways in which a group of eight people can arrange themselves around a circular table if two of them must sit together.
8. A competition distributes prizes to the top $5$ players among $15$ players. If two of the players are guaranteed to be among the top five, then find the number of ways in which the prize can be distributed among $15$ players. What if two players are definitely not going to be among the top five?
9. Compute the values of $C(10, 4)$ and $C(10, 6)$ using the formula.
10. Calculate $C(15, 4)$. Hence find the value of $C(15, 11)$ without using any calculation.
11. How many subsets of $5$ elements can be made from a set with $11$ distinct elements?
12. Determine the number of ways that a football team can be formed from a pool of $40$ players.
13. If $n$ is an even integer, how many bit strings are there of length $n$ with equal number of 0's and 1's?
14. In how many ways can a committee consisting of three men and three women be formed from seven men and five women?
15. In how many ways can $5$ balls be chosen from an urn containing $8$ red and $7$ black balls so that (i) they must all be of same color? (ii) three are red and two are black?
16. Prove that $kC(n, k) = nC(n - 1, k - 1)$.
17. In how many ways can $5$ balls be selected from a box containing red, green and blue balls if the order in which the balls are selected does not matter and there are at least $5$ balls of each color?

### 5.3.4 Binomial Coefficients

1. Find the general term in the expansion of $\left(x^2 + \dfrac{a^2}{x}\right)^5$.
2. Find the $7^{th}$ term of $\left(x + \dfrac{1}{x}\right)^{10}$.
3. Find the $5^{th}$ term of $\left(x - \dfrac{2}{x}\right)^7$.
4. Find the coefficient of $x^5$ in the expansion of $\left(x + \dfrac{1}{2x}\right)^7$.
5. Find the coefficient of $x^2y^4$ in the expansion of $(x + y)^6$.
6. What is the coefficient of $x^3y^4$ in the expansion of $(x + 3y)^7$?
7. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

8. Find the general term and the middle terms in the expansion of $(2a+3b)^{12}$.
9. Find the middle term(s) in the expansion of $\left(2x+\dfrac{1}{3x^2}\right)^9$.
10. Find the middle terms of $\left(2x^2-\dfrac{a^2}{x}\right)^{15}$.
11. Find the term independent of $x$ in the expansion of $\left(x^2+\dfrac{1}{x}\right)^{12}$.
12. Find the term independent of $x$ in the expansion of $\left(2x+\dfrac{1}{3x^2}\right)^9$.
13. Find the term independent of $x$ in the expansion of $\left(\dfrac{3x^2}{2}-\dfrac{1}{3x}\right)^9$.
14. Use the binomial theorem to prove $\sum_{k=0}^{n}\binom{n}{k}2^k=3^n$.
15. Find the expansion of $(1+2x)^6$ using Pascal's triangle.
16. Find the expansion of $(2x-3)^5$ using Pascal's triangle.

### 5.3.5 Discrete Probability

### 5.3.6 Advanced Counting

1. What is the form of the solution of a linear homogeneous recurrence relation if its characteristic roots are $2, 2, 3, 3, 3, -1$?
2. What is the form of the solution of a linear homogeneous recurrence relation if its characteristic roots are $5, 5, 5, -2, -2, 1, 1, 1, 4$?
3. Determine whether the sequence $a_n = 2n$ is a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ or not.
4. Determine whether $a_n = -n + 2$ is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$.
5. Determine whether $a_n = -6n + 2$ is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$.
6. Determine whether $a_n = 2^n$ is a solution of the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$.
7. Let $a_n = a_{n-1} - a_{n-2}$ for $n \geq 2$ with $a_0 = 3$ and $a_1 = 5$. Find the values of $a_2$ and $a_3$.
8. Solve the recurrence relations using the iterative approach:

   (a) $a_n = \dfrac{a_{n-1}}{n}$ with $a_1 = 1$.

   (b) $a_n = 2a_{n-1}$, $a_0 = 1$.

   (c) $a_n = 2a_{n-1} - 3$, $a_0 = -1$.

(d) $a_n = na_{n-1}, a_0 = 5$.

(e) $a_n = a_{n-1} + 2, a_0 = 3$

(f) $a_n = a_{n-1} + 3^n$, $a_0 = 1$.

(g) $a_n = a_{n-1} + n^2$, $a_0 = 7$

(h) $a_n = ra_{n-1} + s$ where $r$ and $s$ are constants.

(i) $P_n = \dfrac{111}{100}P_{n-1}$, $P_0 = 10,000$

(j) $H_n = 2H_{n-1} + 1$, $H_1 = 1$

9. Solve the following recurrence relations:

(a) $a_n = a_{n-1} + 6a_{n-2}$, with the initial conditions $a_0 = 3, a_1 = 6$.

(b) $a_n = -6a_{n-1} - 9a_{n-2}$ with the initial conditions $a_0 = 3$ and $a_1 = -3$.

(c) $a_n = 6a_{n-1} - 9a_{n-2}$ with the initial conditions $a_0 = 3$ and $a_1 = -3$.

(d) $a_n = 7a_{n-1} - 10a_{n-2}$ with initial conditions $a_0 = 2$ and $a_1 = 1$.

(e) $a_n = a_{n-1} + 2a_{n-2}$ with initial conditions $a_0 = 2$ and $a_1 = 7$.

(f) $a_n = 8a_{n-1} - 16a_{n-2}$ with initial conditions, $a_0 = 5$ and $a_1 = 8$.

(g) $a_n = a_{n-1} + a_{n-2}$ with initial conditions $a_1 = 2$ and $a_2 = 3$.

(h) $a_n = 3a_{n-1} + 4a_{n-2}$ with initial conditions $a_0 = a_1 = 1$.

(i) $a_n = 6a_{n-1} - 8a_{n-2}$ with initial conditions $a_0 = 4, a_1 = 10$.

(j) $a_n = 4a_{n-2}$, with initial conditions $a_0 = 0, a_1 = 4$.

(k) $a_n = 4a_{n-1} - 4a_{n-2}$, with the initial conditions $a_0 = 6, a_1 = 8$.

(l) $a_n = -4a_{n-1} - 4a_{n-2}$, with the initial conditions $a_0 = 0, a_1 = 4$.

(m) $a_n = 6a_{n-1} - 9a_{n-2}$, with initial conditions $a_0 = 1$, $a_1 = 6$.

(n) $a_n = -4a_{n-1} - 4a_{n-2}$, with the initial conditions $a_0 = \dfrac{5}{2}, a_1 = 8$.

(o) $a_n = \dfrac{a_{n-2}}{4}$ with initial conditions $a_0 = 2, a_1 = 8$.

(p) $a_n = a_{n-2}$, with initial conditions $a_0 = 5, a_1 = -1$.

(q) $a_n = -4a_{n-1} + 5a_{n-2}$, with initial conditions $a_0 = 2, a_1 = 8$.

10. For the Fibonacci sequence $\{f_n\}$, prove that $f_{n+1}^2 - f_n^2 = f_{n-1}f_{n+2}$ for $n \geq 2$.

11. Find an explicit formula for the sequence $2, 5, 7, 12, 19, 31, \cdots$.

12. Solve the following recurrence relations:

(a) $a_n = 6a_{n-1} - 11a_{n-1} + 6a_{n-3}$ with the initial conditions $a_0 = 2, a_1 = 5, a_2 = 15$.

(b) $a_n = 7a_{n-2} + 6a_{n-3}$ with the initial conditions $a_0 = 9, a_1 = 10, a_2 = 32$.

(c) $a_n = 5a_{n-2} - 4a_{n-4}$ with initial conditions $a_0 = 3, a_1 = 2, a_2 = 6, a_3 = 8$.

(d) $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}$ with the initial conditions $a_0 = 1, a_1 = 4, a_2 = 8$.

(e) $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with initial conditions $a_0 = 5, a_1 = -9$ and $a_2 = 15$.

(f) $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ with initial conditions $a_0 = 7, a_1 = -4, a_2 = 8$.

(g) $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ with initial conditions $a_0 = 3$, $a_1 = 6$, $a_2 = 0$.

(h) $a_n = 8a_{n-1} - 21a_{n-2} + 18a_{n-3}$.

(i) $a_n = a_{n-2} - 16a_{n-4}$.

(j) $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + 3^n$.

(k) $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + 2^n$.

(l) $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + n^2$.

(m) $a_n = 2a_{n-1} + 4^n$.

(n) $a_n = 2a_{n-1} + n^2 + n$.

(o) $a_n = 8a_{n-1} - 16a_{n-2} + n^2$.

(p) $a_n = 7a_{n-1} - 10a_{n-2} + 4^n$.

(q) $a_n = -5a_{n-1} - 6a_{n-2} + 5^n$.

(r) $a_n = 5a_{n-1} - 6a_{n-2} + 7n + 2$.

(s) $a_n = 5a_{n-1} - 6a_{n-2} + n^2 - n$.

(t) $a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n$.