# Unit 4

# Solution of Linear Algebraic Equations

## 4.1 Review

**System of linear equation:**
A system of $m$ linear equations in $n$ unknowns is defined as the following:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n &= b_m \end{aligned}$$

We can write the above system in matrix notation as $Ax = b$ where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n}, \ x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}, \ b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}_{m\times 1}$$

Here, the matrix $A$ is called the coefficient matrix and the $m \times (n+1)$ matrix $(A, b)$ is called the augmented matrix . By the solution of such system of linear equations, we mean the value of the variables $x_1, x_2, \cdots, x_n$ that satisfies all the equations in the system simultaneously.

**Existence/Uniqueness of solutions:**
Suppose we are given a system of $n$ linear equations in $n$ unknowns which can be written as

$$Ax = b \cdots\cdots (1)$$

where $A$ is an $n \times n$ matrix and $x, b$ are $n \times 1$ matrices. We define the rank of $A$, denoted by rank$(A)$, as the order of the largest submatrix of $A$ which has a nonzero determinant i.e., the order of the largest invertible submatrix of $A$.
We consider the following cases:
Case I: rank$(A) = n$
In this case, the system of linear equations (1) has a unique solution. For example, for the system consisting of linear equations

$$x + 2y = 9$$
$$2x - 3y = 4,$$

the rank of its coefficient matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix}$ is 2 and so the given system has a unique solution $x = 5$, $y = 2$.

Case II: rank$(A) < n$

In this case, the system of linear equations (1) has either no solutions or infinitely many solutions as according to the following subcases:

Case II(a): rank$(A) <$ rank$(A, b)$

In this case the system of linear equations (1) has no solution. Such equations are called inconsistent. For example, for the system consisting of

$$2x - y = 5$$
$$3x - \frac{3}{2} = 4,$$

$1 =$ rank$(A) <$ rank$(A, b) = 2$ and so the system is inconsistent i.e., it does not have a solution.

Case II(b): rank$(A) =$ rank$(A, b)$

In this case, the system of linear equations (1) has infinitely many solutions. Such equations are called dependent. For example, for the system consisting of

$$-2x + 3y = 6$$
$$4x - 6y = -12,$$

rank$(A) =$ rank$(A, b) = 1$ and so the system is dependent i.e., it has infinitely many solutions.

## 4.2 Gaussian Elimination Method

Gaussian elimination method is a method used for solving a system of linear equations

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + a_{nn}x_n = b_n$$

In Gaussian elimination method, the above system is transformed into an upper triangular form using forward-elimination approach which can then be easily solved using the back-substitution method. The detailed method/algorithm is as follows:

1. If $a_{11} \neq 0$, proceed to step 2; otherwise rearrange the equations such that $a_{11} \neq 0$.
2. Eliminate $x_i$ from all but the first equation as follows:

   i. Subtract from the second equation $\dfrac{a_{21}}{a_{11}}$ times the first equation. The second equation then becomes

   $$\left(a_{21} - \frac{a_{21}}{a_{11}}a_{11}\right)x_1 + \left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \cdots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$

   or, $\left(a_{22} - \dfrac{a_{21}}{a_{11}}a_{12}\right)x_2 + \cdots + \left(a_{2n} - \dfrac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \dfrac{a_{21}}{a_{11}}b_1$

   or, $a_{22}^{(1)}x_2 + \cdots\cdots + a_{2n}^{(1)}x_n = b_2^{(1)}$

   where $a_{2i}^{(1)} = a_{2i} - \dfrac{a_{21}}{a_{11}}a_{1i}$ for $i = 2, \cdots, n$ and $b_2^{(1)} = b_2 - \dfrac{a_{21}}{a_{11}}b_1$.

   ii Similarly subtract from the third equation $\dfrac{a_{31}}{a_{11}}$ times the first equation. The resulting equation is written as

   $$a_{32}^{(1)}x_2 + \cdots\cdots + a_{3n}^{(1)}x_n = b_3^{(1)}$$

   We repeat this process till the $n^{th}$ equation is operated on after which we get the following new system of equations:

   $$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n &= b_1 \\ a_{22}^{(1)}x_2 + \cdots\cdots + a_{2n}^{(1)}x_n &= b_2^{(1)} \\ a_{32}^{(1)}x_2 + \cdots\cdots + a_{3n}^{(1)}x_n &= b_3^{(1)} \\ \vdots \qquad\qquad & \quad \vdots \\ a_{n2}^{(1)}x_2 + \cdots\cdots + a_{nn}^{(1)}x_n &= b_n^{(1)} \end{aligned}$$

3. Now eliminate $x_2$ from the third to the last equation in the new set as follows:

   i. Assume $a_{22}^{(1)} \neq 0$, otherwise rearrange the equations so that $a_{22}^{(1)} \neq 0$.

   ii. Subtract from the third equation $\dfrac{a_{32}}{a_{22}}$ times the second equation.

   iii. Subtract from the fourth equation $\dfrac{a_{42}}{a_{22}}$ times the second equation and so on.

4. This process is continued till the last equation contains only one variable $x_n$. The final set of equations will be as follows:

   $$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n &= b_1 \\ a_{22}^{(1)}x_2 + \cdots\cdots + a_{2n}^{(1)}x_n &= b_2^{(1)} \\ \ddots \qquad \vdots \qquad & \quad \vdots \\ a_{nn}^{(n-1)}x_n &= b_n^{(n-1)} \end{aligned}$$

5. Lastly, obtain the solution by back-substitution as follows: we get from the $n^{th}$ equation $x_n = \dfrac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$. This is substituted back in $(n-1)^{th}$ equation to obtain the value of $x_{n-1}$. This is continued till we get the values for all $x_1, x_2, \cdots, x_n$.

The $k^{th}$ equation which is multiplied by the factor $\dfrac{a_{ik}}{a_{kk}}$ is called the **pivot equation** and $a_{kk}$ is called the **pivot element**. Obviously, pivot elements have to be chosen so that they are nonzero for the Gaussian elimination to work correctly.

**Note:** Interpreting the above process in terms of matrices, the Gaussian elimination method transforms the augmented matrix $(A, b)$ into $(A', b')$ where $A'$ is an upper triangular matrix and then uses the back-substitution method to find the values of the variables.

### 4.2.1 Exercises

Solve the following system of linear equations by Gaussian elimination method.

1. $2x_1 + 3x_2 = 1$
   $10x_1 + 9x_2 = 11$

   **Solution:** Subtracting from the second equation $\dfrac{10}{2} = 5$ times the first equation, we get the system

   $$\begin{aligned} 2x_1 + 3x_2 &= 1 \\ -6x_2 &= 6 \end{aligned}$$

   Now solving the above resulting system by back-substitution, we have

   $$x_2 = -1$$

   and

   $$x_1 = \frac{1}{2}(1 - 3x_2) = \frac{1}{2}(1 - 3(-1)) = 2.$$

2. $2x_1 + x_2 + x_3 = 5$
   $4x_1 - 6x_2 = -2$
   $-2x_1 + 7x_2 + 2x_3 = 9$

   **Solution:** Subtracting from the second equation $\dfrac{4}{2} = 2$ times the first equation and subtracting from the third equation $\dfrac{-2}{2} = -1$ times the first equation, we have the system

   $$\begin{aligned} 2x_1 + x_2 + x_3 &= 5 \\ 4x_2 + x_3 &= 6 \\ 8x_2 + 3x_3 &= 14 \end{aligned}$$

   Subtracting from the third equation $\dfrac{8}{4} = 2$ times the second equation, we get the system

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 5 \\ 4x_2 + x_3 &= 6 \\ x_3 &= 2 \end{aligned}$$

Now solving the above resulting system by back-substitution, we have

$$x_3 = 2,$$

$$x_2 = \frac{1}{4}(6 - x_3) = \frac{1}{4}(6 - 2) = 1$$

and

$$x_1 = \frac{1}{2}(5 - x_2 - x_3) = \frac{1}{2}(5 - 1 - 2) = 1.$$

3. $4x_1 - 2x_2 + x_3 = 15$
   $-3x_1 - x_2 + 4x_3 = 8$
   $x_1 - x_2 + 3x_3 = 13$

   **Solution:** Subtracting from the second equation $\frac{-3}{4}$ times the first equation and subtracting from the third equation $\frac{1}{4}$ times the first equation, we get the system

$$\begin{aligned} 4x_1 - 2x_2 + x_3 &= 15 \\ -10x_2 + 19x_3 &= 77 \\ -2x_2 + 11x_3 &= 37 \end{aligned}$$

   Subtracting from the third equation $\frac{-2}{-10} = \frac{1}{5}$ times the second equation, we get the system

$$\begin{aligned} 4x_1 - 2x_2 + x_3 &= 15 \\ -10x_2 + 19x_3 &= 77 \\ -72x_3 &= -216 \end{aligned}$$

   Now solving the above resulting system by back-substitution, we get

$$x_3 = \frac{-216}{-72} = 3,$$

$$x_2 = \frac{1}{-10}(77 - 19x_3) = \frac{1}{-10}(77 - 57) = -2$$

   and

$$x_1 = \frac{1}{4}(15 + 2x_2 - x_3) = \frac{1}{4}(15 - 4 - 3) = 2.$$

4. $3x_1 + 6x_2 + x_3 = 16$
   $2x_1 + 4x_2 + 3x_3 = 13$
   $x_1 + 3x_2 + 2x_3 = 9$

   **Solution:** Subtracting from the second equation $\frac{2}{3}$ times the first equation and subtracting from the third equation $\frac{1}{3}$ times the first equation, we get

$$3x_1 + 6x_2 + x_3 = 16$$
$$7x_3 = 7$$
$$3x_2 + 5x_3 = 11$$

Interchanging the second and third equations, we get,

$$3x_1 + 6x_2 + x_3 = 16$$
$$3x_2 + 5x_3 = 11$$
$$7x_3 = 7$$

Solving this system by back-substitution, we get

$$x_3 = 1,$$

$$x_2 = \frac{1}{3}(11 - 5x_3) = 2$$

and

$$x_1 = \frac{1}{3}(16 - 6x_2 - x_3) = 1.$$

5. $2x_1 + 3x_2 = 0$
   $4x_1 + 5x_2 + x_3 = 3$
   $2x_1 - x_2 - 3x_3 = 5$

6. $x_1 + x_2 + x_3 = 6$
   $x_1 + 2x_2 + 2x_3 = 11$
   $2x_1 + 3x_2 - 4x_3 = 3$

7. $x_1 + 4x_2 + 2x_3 = -2$
   $-2x_1 - 8x_2 + 3x_3 = 32$
   $x_2 + x_3 = 1$

8. $6x_1 - 2x_2 + 2x_3 + 4x_4 = 16$
   $12x_1 - 8x_2 + 6x_3 + 10x_4 = 26$
   $3x_1 - 13x_2 + 9x_3 + 3x_4 = -19$
   $-6x_1 + 4x_2 + x_3 - 18x_4 = -34$

9. $\begin{pmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$

## 4.3 Gauss-Jordan Method

Gauss-Jordan method is also a method for solving a system of linear equations which, like Gaussian elimination method, proceeds by elimination of variables but, unlike Gaussian elimination method, a variable is eliminated not only from the rows below the pivot equation but also from the rows above the pivot equation. Furthermore, all rows are normalized by dividing them by their pivot elements.

The Gauss-Jordon algorithm/method is as follows:

1. Normalize the first equation by dividing by its pivot element.
2. Eliminate variable $x_1$ from all other equations.
3. Normalize the second equation by dividing by its pivot element.
4. Eliminate variable $x_2$ from all other equations above and below the normalized pivot equation.
5. Repeat this process until $x_n$ is eliminated from all but the last equation.
6. Obtain as solution the resulting vector $b$.

**Note:** Interpreting the above process in terms of matrices, the Gauss-Jordan method transforms the augmented matrix $(A, b)$ into $(I, b')$ where $I$ is an identity matrix and the solution vector $b'$.

**Note:** Although Gauss-Jordan method does not require back-substitution, it requires almost $50\%$ more arithmetic operations than Gaussian elimination method. Hence this method is rarely used. However, if we have to solve a system of linear equations $Ax = b$ for different values of right hand side vector $b$ but the same coefficient matrix $A$, then we can use Gauss-Jordan method to obtain the solutions simultaneously by reducing the matrix $A$ to the identity matrix $I$.

### 4.3.1 Exercise

1. Solve using Gauss-Jordan method:

$$
\begin{aligned}
4x_1 - 2x_2 + x_3 &= 15 \\
-3x_1 - x_2 + 4x_3 &= 8 \\
x_1 - x_2 + 3x_3 &= 13
\end{aligned}
$$

**Solution:** Normalizing the first equation by dividing it by the pivot element $4$, we get,

$$
\begin{aligned}
x_1 - \frac{1}{2}x_2 + \frac{1}{4}x_3 &= \frac{15}{4} \\
-3x_1 - x_2 + 4x_3 &= 8 \\
x_1 - x_2 + 3x_3 &= 13
\end{aligned}
$$

Eliminating $x_1$ from the second and third equations, we get

$$x_1 - \frac{1}{2}x_2 + \frac{1}{4}x_3 = \frac{15}{4}$$
$$-5x_2 + \frac{19}{2}x_3 = \frac{77}{2}$$
$$-x_2 + \frac{11}{2}x_3 = \frac{37}{2}$$

Normalizing the second equation by dividing it by the pivot element $-5$, we get,

$$x_1 - \frac{1}{2}x_2 + \frac{1}{4}x_3 = \frac{15}{4}$$
$$x_2 - \frac{19}{10}x_3 = \frac{-77}{10}$$
$$-x_2 + \frac{11}{2}x_3 = \frac{37}{2}$$

Eliminating $x_2$ from the first and third equations, we get

$$x_1 - \frac{14}{20}x_3 = \frac{-1}{10}$$
$$x_2 - \frac{19}{10}x_3 = \frac{-77}{10}$$
$$18x_3 = 54$$

Normalizing the third equation by dividing it by the pivot element $18$, we get,

$$x_1 - \frac{14}{20}x_3 = \frac{-1}{10}$$
$$x_2 - \frac{19}{10}x_3 = \frac{-77}{10}$$
$$x_3 = 3$$

Eliminating $x_3$ from first and third equations, we get,

$$x_1 = 2$$
$$x_2 = -2$$
$$x_3 = 3$$

Therefore, $x_1 = 2$, $x_2 = -2$ and $x_3 = 3$ is the required solution.

2. Solve the following using Gauss-Jordan method:

$$2x_1 + x_2 + x_3 = 5$$
$$4x_1 - 6x_2 = -2$$
$$-2x_1 + 7x_2 + 2x_3 = 9$$

**Solution:** The augmented matrix of the given linear system is

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right)$$

Normalizing the first row by performing $R_1 \longrightarrow \dfrac{R_1}{2}$, we get

$$\left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right)$$

Performing $R_2 \longrightarrow R_2 - 4R_1$ and $R_3 \longrightarrow R_3 - (-2)R_1 = R_3 + 2R_1$, we get

$$\left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{array}\right)$$

Normalizing the second row by performing $R_2 \longrightarrow \dfrac{R_2}{-8}$, we get

$$\left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ 0 & 1 & \frac{1}{4} & \frac{3}{2} \\ 0 & 8 & 3 & 14 \end{array}\right)$$

Performing $R_1 \longrightarrow R_1 - \dfrac{1}{2}R_2$ and $R_3 \longrightarrow R_3 - 8R_2$, we get

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{3}{8} & \frac{7}{4} \\ 0 & 1 & \frac{1}{4} & \frac{3}{2} \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Since the third row is already normalized, we now perform $R_1 \longrightarrow R_1 - \dfrac{3}{8}R_3$ and $R_2 \longrightarrow R_2 - \dfrac{1}{4}R_3$ to get

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Thus the solution of the given linear system is $x_1 = 1$, $x_2 = 1$ and $x_3 = 2$.

3. Solve all the previous problems using the Gauss-Jordan method.

## 4.4 Pivoting

In the Gaussian elimination method or the Gauss-Jordan method of solving the system of linear equations, the process of choosing a pivot element is called pivoting . Since we need to perform divisions by pivot elements, so pivot elements always have to be nonzero. So if any of the pivot elements are zero, we rearrange the equations so that a nonzero element becomes a pivot element.
However, even if the pivot element is not zero, it is sometimes preferable to perform some kind of pivoting. This way, one can reduce the roundoff errors that would otherwise occur.

There are two commonly used methods of pivoting:

**1. Partial pivoting:** In partial pivoting , at each step, from the remaining equations, we choose as pivot element the one with the largest absolute value among the coefficient in the first column.

$$
\begin{array}{ccccccc}
a_{11} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1n} \\
 & a_{22} & \cdots & \cdots & \cdots & \cdots & a_{2n} \\
 & & \ddots & & & & \vdots \\
\cline{4-4}
 & & & a_{kk} & \cdots & \cdots & a_{kn} \\
\text{search} & \text{area} & \rightarrow & \vdots & \cdots & \cdots & \vdots \\
 & & & a_{nk} & \cdots & \cdots & a_{nn} \\
\cline{4-4}
\end{array}
$$

In partial pivoting, it may be necessary to interchange the rows to place the desired element in pivot position.

**2. Complete pivoting:** In complete pivoting , at each step, from the remaining equations, we choose as pivot element the one with the largest absolute value among all the coefficients.

$$
\begin{array}{ccccccc}
a_{11} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1n} \\
 & a_{22} & \cdots & \cdots & \cdots & \cdots & a_{2n} \\
 & & \ddots & & & & \vdots \\
\cline{4-7}
 & & & a_{kk} & \cdots & \cdots & a_{kn} \\
\text{search} & \text{area} & \rightarrow & \vdots & \cdots & \cdots & \vdots \\
 & & & a_{nk} & \cdots & \cdots & a_{nn} \\
\cline{4-7}
\end{array}
$$

In complete pivoting, it may be necessary to interchange the rows as well as the columns to place the desired element in pivot position.

**Algorithm (Gaussian Elimination with Partial Pivoting):**

INPUT: A system of $n$ linear equations $Ax = b$.

PROCESS:

FOR $j = 1$ TO $n - 1$ {

pvt $= |a_{jj}|$
pivot[j] $= j$

ipvt-temp $= j$
FOR $i = j + 1$ TO n {
  IF $|a_{jj}|$ $>$pvt THEN {
    pvt $= |a_{jj}|$
    ipvt-temp $= i$
  }
}
IF pivot[j] $\neq$ ipvt-temp THEN switchrows($j$, ipvt-temp)
FOR $i = j + 1$ TO $n$, $a_{ij} = \dfrac{a_{ij}}{a_{jj}}$
FOR $i = j + 1$ TO $n$ {
  FOR $k = j + 1$ TO $n$, $a_{ik} = a_{ik} - a_{ij} * a_{jk}$
  $b_i = b_i - a_{ij} * b_j$
}

}

$x_n = \dfrac{b_n}{a_{nn}}$

FOR $j = n - 1$ TO $1$ {

  $x_j = b_j$
  FOR $k = n$ TO $j + 1$, $x_j = x_j - x_k * a_{jk}$
  $x_j = \dfrac{x_j}{a_{jj}}$

}

OUTPUT: Solution $x_1, \cdots, x_n$ of the linear system $Ax = b$.

**Ill-Conditioned system:**

Some system of linear equations have coefficients such that the solutions are particularly sensitive to roundoff errors. Such systems are called ill-conditioned systems . For the numerical stability of such ill-conditioned systems, we perform partial or complete pivoting even if the pivot element is not zero.

### 4.4.1 Exercise

Solve the following by Gaussian elimination method with partial pivoting.

1. $2x_1 + x_2 + x_3 = 5$
   $4x_1 - 6x_2 = -2$
   $-2x_1 + 7x_2 + 2x_3 = 9$

   **Solution:** Interchanging the first and second equations, we get

$$\begin{aligned} 4x_1 - 6x_2 &= -2 \\ 2x_1 + x_2 + x_3 &= 5 \\ -2x_1 + 7x_2 + 2x_3 &= 9 \end{aligned}$$

Subtracting from the second equation $\frac{2}{4} = \frac{1}{2}$ times the first equation and subtracting from the third equation $-\frac{2}{4} = -\frac{1}{2}$ times the first equation, we get,

$$\begin{aligned} 4x_1 - 6x_2 &= -2 \\ 4x_2 + x_3 &= 6 \\ 4x_2 + 2x_3 &= 8 \end{aligned}$$

Subtracting from the third equation $\frac{4}{4} = 1$ times the second equation we get,

$$\begin{aligned} 4x_1 - 6x_2 &= -2 \\ 4x_2 + x_3 &= 6 \\ x_3 &= 2 \end{aligned}$$

Now solving by back-substitution method, we get

$$x_3 = 2,$$

$$x_2 = \frac{1}{4}(6 - x_3) = 1$$

and

$$x_1 = \frac{1}{4}(-2 + 6x_2) = 1.$$

2. $6x_1 - 2x_2 + 2x_3 + 4x_4 = 16$
$12x_1 - 8x_2 + 6x_3 + 10x_4 = 26$
$3x_1 - 13x_2 + 9x_3 + 3x_4 = -19$
$-6x_1 + 4x_2 + x_3 - 18x_4 = -34$

**Solution:** Interchanging the first and second equations, we get

$$\begin{aligned} 12x_1 - 8x_2 + 6x_3 + 10x_4 &= 26 \\ 6x_1 - 2x_2 + 2x_3 + 4x_4 &= 16 \\ 3x_1 - 13x_2 + 9x_3 + 3x_4 &= -19 \\ -6x_1 + 4x_2 + x_3 - 18x_4 &= -34 \end{aligned}$$

Subtracting from the second equation $\frac{6}{12} = \frac{1}{2}$ times the first equation, subtracting from the third equation $\frac{3}{12} = \frac{1}{4}$ times the first equation and subtracting from the fourth equation $-\frac{6}{12} = -\frac{1}{2}$ times the first equation, we get

$$\begin{gathered}12x_1 - 8x_2 + 6x_3 + 10x_4 = 26\\ 2x_2 - x_3 - x_4 = 3\\ -11x_2 + \tfrac{15}{3}x_3 + \tfrac{1}{2}x_4 = -\tfrac{51}{2}\\ 4x_3 - 13x_4 = -21\end{gathered}$$

Interchanging the second and third equations, we get

$$\begin{gathered}12x_1 - 8x_2 + 6x_3 + 10x_4 = 26\\ -11x_2 + \tfrac{15}{3}x_3 + \tfrac{1}{2}x_4 = -\tfrac{51}{2}\\ 2x_2 - x_3 - x_4 = 3\\ 4x_3 - 13x_4 = -21\end{gathered}$$

Subtracting from the third equation $\dfrac{-2}{11}$ times the second equation, we get

$$\begin{gathered}12x_1 - 8x_2 + 6x_3 + 10x_4 = 26\\ -11x_2 + \tfrac{15}{3}x_3 + \tfrac{1}{2}x_4 = -\tfrac{51}{2}\\ 4x_3 - 10x_4 = -18\\ 4x_3 - 13x_4 = -21\end{gathered}$$

Subtracting from the fourth equation $\dfrac{4}{4} = 1$ times the third equation, we get

$$\begin{gathered}12x_1 - 8x_2 + 6x_3 + 10x_4 = 26\\ -11x_2 + \tfrac{15}{3}x_3 + \tfrac{1}{2}x_4 = -\tfrac{51}{2}\\ 4x_3 - 10x_4 = -18\\ -3x_4 = -3\end{gathered}$$

Now solving the above equation by back substitution, we get

$$x_4 = 1,$$

$$x_3 = \frac{1}{4}(-18 + 10x_4) = -2,$$

$$x_2 = \frac{-1}{11}(-\frac{51}{2} - \frac{15}{2}x_3 - \frac{1}{2}x_4) = 1$$

and

$$x_1 = \frac{1}{12}(26 + 8x_2 - 6x_3 - 10x_4) = 3.$$

3. Solve all the previous problems using partial pivoting with Gaussian elimination.

## 4.5 Matrix Inversion

Let $A = (a_{ij})$ be an invertible order $n$ matrix and let $X = (x_{ij})$ be its inverse. Then

$$AX = I$$

where $I$ is an order $n$ identity matrix. This is equivalent to $n$ equations $Ax_k = e_k$, $k = 1, 2, \cdots, n$ where $x_k$ and $e_k$ are the $k^{th}$ columns of $X$ and $I$ respectively. Solving the above $n$ linear systems by Gaussian elimination method or Gauss-Jordan method, we get the columns $x_1, x_2, \cdots, x_n$ of $X$ and hence the inverse $X$ of $A$.
For example, let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

be an invertible $3 \times 3$ matrix and let

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

be its inverse. Then,

$$AX = I$$

$$\text{or,} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is equivalent to the following three systems of linear equations:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Solving these three systems, we can find $X$. Since the coefficient matrix of all three systems is the same matrix $A$, we can solve all three systems simultaneously using Gauss-Jordan method as below:

1. Augment the coefficient matrix $A$ with an identity matrix as below:

$$\left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right)$$

2. Apply the Gauss-Jordan method to the augmented matrix to reduce $A$ to an identity matrix as below:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & a'_{11} & a'_{12} & a'_{13} \\ 0 & 1 & 0 & a'_{21} & a'_{22} & a'_{23} \\ 0 & 0 & 1 & a'_{31} & a'_{32} & a'_{33} \end{array}\right)$$

Then the matrix on the right is the inverse of $A$ i.e.,

$$X = \begin{pmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{pmatrix}$$

**Note:** Given a system of linear equations $Ax = b$, the solution $x$ can be computed as $x = A^{-1}b$ after computing the inverse matrix $A^{-1}$ of $A$.

### 4.5.1 Exercise

Find the inverse of the following matrices using Gauss-Jordan elimination technique:

1. $A = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 2 & 3 \\ 3 & 4 & 2 \end{pmatrix}$

   **Solution:** Augmenting $A$ with the identity matrix, we get,

$$\left(\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 4 & 2 & 3 & 0 & 1 & 0 \\ 3 & 4 & 2 & 0 & 0 & 1 \end{array}\right)$$

   Normalize first row:

$$\left(\begin{array}{ccc|ccc} 1 & \frac{3}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 4 & 2 & 3 & 0 & 1 & 0 \\ 3 & 4 & 2 & 0 & 0 & 1 \end{array}\right) \{R_1 \longrightarrow \frac{R_1}{2}\}$$

   Reduce first column:

$$\left(\begin{array}{ccc|ccc} 1 & \frac{3}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & -4 & -5 & -2 & 1 & 0 \\ 0 & \frac{-1}{2} & -4 & \frac{-3}{2} & 0 & 1 \end{array}\right) \{R_2 \longrightarrow R_2 - 4R_1, R_3 \longrightarrow R_3 - 3R_1\}$$

Normalize second row:

$$\left(\begin{array}{ccc|ccc} 1 & \frac{3}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{4} & \frac{1}{2} & \frac{-1}{4} & 0 \\ 0 & \frac{-1}{2} & -4 & \frac{-3}{2} & 0 & 1 \end{array}\right) \{R_2 \longrightarrow \frac{R_2}{-4}\}$$

Reduce second column:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{8} & \frac{-1}{4} & \frac{3}{8} & 0 \\ 0 & 1 & \frac{5}{4} & \frac{1}{2} & \frac{-1}{4} & 0 \\ 0 & 0 & \frac{-27}{8} & \frac{-5}{4} & \frac{-1}{8} & 1 \end{array}\right) \{R_1 \longrightarrow R_1 - \frac{3}{2}R_2, R_3 \longrightarrow R_3 + \frac{1}{2}R_2\}$$

Normalize third row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{8} & \frac{-1}{4} & \frac{3}{8} & 0 \\ 0 & 1 & \frac{5}{4} & \frac{1}{2} & \frac{-1}{4} & 0 \\ 0 & 0 & 1 & \frac{10}{27} & \frac{1}{27} & \frac{-8}{27} \end{array}\right) \{R_3 \longrightarrow \frac{-8}{27}R_3\}$$

Reduce third column:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-8}{27} & \frac{10}{27} & \frac{1}{27} \\ 0 & 1 & 0 & \frac{1}{27} & \frac{-8}{27} & \frac{10}{27} \\ 0 & 0 & 1 & \frac{10}{27} & \frac{1}{27} & \frac{-8}{27} \end{array}\right) \{R_1 \longrightarrow R_1 - \frac{1}{8}R_3, R_2 \longrightarrow R_2 - \frac{5}{4}R_3\}$$

Hence $A^{-1} = \begin{pmatrix} \frac{-8}{27} & \frac{10}{27} & \frac{1}{27} \\ \frac{1}{27} & \frac{-8}{27} & \frac{10}{27} \\ \frac{10}{27} & \frac{1}{27} & \frac{-8}{27} \end{pmatrix}$.

2. $A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix}$
(Det$A = 0$ so inverse does not exist.)

4. $A = \begin{pmatrix} -2 & 4 & -1 \\ -2 & 3 & 0 \\ 7 & -12 & 2 \end{pmatrix}$

5. $A = \begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{pmatrix}$

## 4.6 Matrix Factorization Methods

Suppose that we are given a system of linear equations as

$$Ax = b \cdots\cdots (1)$$

To solve (1), we first factorize the matrix $A$ as

$$A = LU$$

where $L$ is a lower triangular matrix and $U$ is an upper triangular matrix. Then (1) becomes

$$(LU)x = b$$
$$\text{or, } L(Ux) = b \cdots\cdots (2)$$

Let $Ux = y$. Then (2) becomes

$$Ly = b \cdots\cdots (3)$$

Since $L$ is a lower triangular matrix, the system (3) can be easily solved for $y$ by the method of forward-substitution. Once $y$ is known, then from the system $Ux = y$, we can solve it easily for $x$ by the method of back-substitution because $U$ is an upper triangular matrix. Therefore we can solve the system (1) in two stages:

(I) Solve the system $Ly = b$ for $y$ by forward-substitution.

(II) Solve the system $Ux = y$ for $x$ by back-substitution.

The problem that remains now is to find a $LU$ factorization of $A$. This can be accomplished using various methods. We study two such methods:

1. Dolittle factorization
2. Cholesky factorization

### 4.6.1 Dolittle Factorization

Suppose we are given a system of linear equations as

$$Ax = y \cdots\cdots (1)$$

where $A = (a_{ij})$ is an $n \times n$ matrix. In Dolittle factorization method, we factorize the matrix $A$ as $A = LU$ where $L$ is a unit lower triangular matrix (i.e. a lower triangular matrix with all diagonal entries 1) and $U$ is an upper triangular matrix. So we have

$$LU = A$$

$$\text{or, } \begin{pmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$\text{or, } \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & \cdots & l_{21}u_{1n} + u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1}u_{11} & l_{n1}u_{12} + l_{n2}u_{22} & \cdots & l_{n1}u_{1n} + \cdots + u_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Comparing the left and right matrices above element-wise, we can compute the elements of $L$ and $U$ as below:

$1^{st}$ **row of** $U$: Comparing the first row of left and right side matrix, we have

$$u_{11} = a_{11}, u_{12} = a_{12}, \cdots, u_{1n} = a_{1n}.$$

$1^{st}$ **column of** $L$: Comparing the first column of left and right side matrix, we have

$$l_{21}u_{11} = a_{21}, l_{31}u_{11} = a_{31}, \cdots, l_{n1}u_{11} = a_{n1}.$$

So

$$l_{21} = \frac{a_{21}}{a_{11}}, l_{31} = \frac{a_{31}}{a_{11}}, \cdots, l_{n1} = \frac{a_{n1}}{a_{11}}.$$

$2^{nd}$ **row of** $U$**:** Comparing the second row of left and right side matrix, we have

$$l_{21}u_{12} + u_{22} = a_{22}, l_{21}u_{13} + u_{23} = a_{23}, \cdots, l_{21}u_{1n} + u_{2n} = a_{2n}.$$

So

$$u_{22} = a_{22} - l_{21}u_{12}, u_{23} = a_{23} - l_{21}u_{13}, \cdots, u_{2n} = a_{2n} - l_{21}u_{1n}.$$

Thus continuing similarly, we can calculate alternately a row of $U$ and a column of $L$ as above.

### 4.6.2 Exercise

Solve the following using Dolittle decomposition method:

1. $2x_1 + x_2 + x_3 = 5$
   $4x_1 - 6x_2 = -2$
   $-2x_1 + 7x_2 + 2x_3 = 9$

   **Solution:** We have $A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}$.

   Let $L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}$ and $U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$ so that $LU = A$. That is

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}.$$

Then

<u>$1^{st}$ row of $U$:</u> Multiplying the first row of $L$ with the columns of $U$, we have $u_{11} = 2, u_{12} = 1, u_{13} = 1$.

<u>$1^{st}$ column of $L$:</u> Multiplying the rows of $L$ with the first column of $U$, we have $l_{11} = 1$, $l_{21}u_{11} = 4 \Rightarrow l_{21} = \frac{4}{2} = 2$, $l_{31}u_{11} = -2 \Rightarrow l_{31} = \frac{-2}{2} = -1$.

<u>$2^{nd}$ row of $U$:</u> Multiplying the second row of $L$ with the columns of $U$, we have $u_{21} = 0$, $l_{21}u_{12} + u_{22} = -6 \Rightarrow u_{22} = -6 - 2 = -8$, $l_{21}u_{13} + u_{23} = 0 \Rightarrow u_{23} = -2$.

<u>$2^{nd}$ column of $L$:</u> Multiplying the rows of $L$ with the second column of $U$, we have $l_{12} = 0$, $l_{22} = 1$, $l_{31}u_{12} + l_{32}u_{22} = 7 \Rightarrow l_{32} = \frac{7+1}{-8} = -1$.

<u>$3^{rd}$ row of $U$:</u> Multiplying the third row of $L$ with the columns of $U$, we have $u_{31} = 0$, $u_{32} = 0$, $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2 \Rightarrow u_{33} = 2 + 1 - 2 = 1$.

<u>$3^{rd}$ column of $L$:</u> Multiplying the rows of $L$ with the third column of $U$, we have $l_{13} = 0$, $l_{23} = 0$, $l_{33} = 1$.

Hence $A = LU$ where $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{pmatrix}$. So the given system of equations can be written as

$$Ax = b$$

$$\text{or, } LUx = b$$

$$\text{or, } \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix} \cdots\cdots (1)$$

Let

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \cdots\cdots (2)$$

Then from (1) we get,

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}$$

Solving by forward-substitution, we get, $y_1 = 5$, $y_2 = -12$, $y_3 = 2$. So from (2),

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -12 \\ 2 \end{pmatrix}$$

Solving by back-substitution, we get $x_3 = 2$, $x_2 = 1$ and $x_1 = 1$.

2. $2x_1 + 3x_2 + x_3 = 9$
   $x_1 + 2x_2 + 3x_3 = 6$
   $3x_1 + x_2 + 2x_3 = 8$

   **Solution:** We have $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$.

   Let $L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}$ and $U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$ so that $LU = A$. That is

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

   Then

   $1^{st}$ row of $U$: Multiplying the first row of $L$ with the columns of $U$, we have $u_{11} = 2, u_{12} = 3, u_{13} = 1$.

   $1^{st}$ column of $L$: Multiplying the rows of $L$ with the first column of $U$, we have $l_{11} = 1$, $l_{21}u_{11} = 1 \Rightarrow l_{21} = \frac{1}{2}$, $l_{31}u_{11} = 3 \Rightarrow l_{31} = \frac{3}{2}$.

   $2^{nd}$ row of $U$: Multiplying the second row of $L$ with the columns of $U$, we have $u_{21} = 0$, $l_{21}u_{12} + u_{22} = 2 \Rightarrow u_{22} = \frac{1}{2}$, $l_{21}u_{13} + u_{23} = 3 \Rightarrow u_{23} = \frac{5}{2}$.

   $2^{nd}$ column of $L$: Multiplying the rows of $L$ with the second column of $U$, we have $l_{12} = 0, l_{22} = 1, l_{31}u_{12} + l_{32}u_{22} = 1 \Rightarrow l_{32} = -7$.

   $3^{rd}$ row of $U$: Multiplying the third row of $L$ with the columns of $U$, we have $u_{31} = 0$, $u_{32} = 0, l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2 \Rightarrow u_{33} = 18$.

   $3^{rd}$ column of $L$: Multiplying the rows of $L$ with the third column of $U$, we have $l_{13} = 0$, $l_{23} = 0, l_{33} = 1$.

   Hence $A = LU$ where $L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{pmatrix}$. So the given system of equations can be written as

$$Ax = b$$

$$\text{or, } LUx = b$$

$$\text{or, } \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 8 \end{pmatrix} \cdots\cdots (1)$$

Let

$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \cdots\cdots (2)$$

Then from (1) we get,

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 8 \end{pmatrix}$$

Solving by forward-substitution, we get, $y_1 = 9$, $\frac{1}{2}y_1 + y_2 = 6 \Rightarrow y_2 = \frac{3}{2}$, $\frac{3}{2}y_1 - 7y_2 + y_3 = 8 \Rightarrow y_3 = 5$. So from (2),

$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ \frac{3}{2} \\ 5 \end{pmatrix}$$

Solving by back-substitution, we get $x_3 = \dfrac{5}{18}$, $x_2 = \dfrac{29}{18}$ and $x_1 = \dfrac{35}{18}$.

3. $5x_1 - 2x_2 + x_3 = 4$
   $7x_1 + x_2 - 5x_3 = 8$
   $3x_1 + 7x_2 + 4x_3 = 10$

4. $3x_1 + 2x_2 + x_3 = 10$
   $2x_1 + 3x_2 + 2x_3 = 14$
   $x_1 + 2x_2 + 3x_3 = 14$
   $L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{4}{5} & 1 \end{pmatrix}$ $U = \begin{pmatrix} 3 & 2 & 1 \\ 0 & \frac{5}{3} & \frac{4}{3} \\ 0 & 0 & \frac{24}{15} \end{pmatrix}$

5. $x_1 + 2x_2 - 3x_3 = 4$
   $2x_1 + 4x_2 - 6x_3 = 8$
   $x_1 - 2x_2 + 5x_3 = 4$

6. $2x_1 + x_2 + x_3 = 7$
   $4x_1 + 2x_2 + 3x_3 = 4$
   $x_1 - x_2 + x_3 = 0$

7. $2x_1 + x_2 + x_3 - 2x_4 = 0$
   $4x_1 \quad\quad + 2x_3 + x_4 = 8$
   $3x_1 + 2x_2 + 2x_3 \quad\quad = 7$
   $x_1 + 3x_2 + 2x_3 \quad\quad = 3$

8. $x_1 + x_2 - 2x_3 = 3$
   $4x_1 - 2x_2 + x_3 = 5$
   $3x_1 - x_2 + 3x_3 = 8$

### 4.6.3 Cholesky Factorization

**Positive definite matrix:** An $n \times n$ matrix $A$ is said to be positive definite if it is symmetric and $x^T Ax > 0$ for all nonzero column vectors $x \in \mathbb{R}^n$.

**Note:**

(1) If a matrix $A$ is such that all its upper left square submatrices have positive determinants, then $A$ is positive definite.

(2) A positive definite matrix $A$ has all its diagonal elements positive.

**Cholesky Factorization Method:**

Suppose we are given a system of linear equations as

$$Ax = b \cdots\cdots (1)$$

where $A = (a_{ij})$ is an $n \times n$ positive definite matrix. In Cholesky factorization method, we factorize the matrix $A$ as $A = LL^T$ where $L$ is a lower triangular matrix. Then its transpose $L^T$ is an upper triangular matrix. So we have,

$$LL^T = A$$

$$\text{or,} \begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & \cdots & l_{n1} \\ 0 & l_{22} & \cdots & l_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & l_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

$$\text{or,} \begin{pmatrix} l_{11}^2 & l_{11}l_{21} & \cdots & l_{11}l_{n1} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & \cdots & l_{21}l_{n1} + l_{22}l_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1}l_{11} & l_{n1}l_{21} + l_{n2}l_{22} & \cdots & l_{n1}^2 + \cdots + l_{nn}^2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

Comparing the left and right matrices above element-wise after multiplication, we can compute the elements of $L$ as follows:

$1^{st}$ **column of** $L$**:** Comparing the first column of left and right side matrix, we have

$$l_{11}^2 = a_{11}, l_{21}l_{11} = a_{12}, \cdots, l_{n1}l_{11} = a_{1n}.$$

So,

$$l_{11} = \sqrt{a_{11}}, l_{21} = \frac{a_{12}}{l_{11}}, \cdots, l_{n1} = \frac{a_{1n}}{l_{11}}.$$

$2^{nd}$ **column of** $L$**:** Comparing the second column of left and right side matrix, we have

$$l_{21}^2 + l_{22}^2 = a_{22}, l_{31}l_{21} + l_{32}l_{22} = a_{23}, \cdots, l_{n1}l_{21} + l_{n2}l_{22} = a_{2n}.$$

So

$$l_{22} = \sqrt{a_{22} - l_{21}^2}, l_{32} = \frac{1}{l_{22}}(a_{23} - l_{31}l_{21}), \cdots, l_{n2} = \frac{1}{l_{22}}(a_{2n} - l_{n1}l_{21}).$$

Similarly, we calculate the elements of $L$ column-wise.

### 4.6.4 Exercise

Solve the following system using Cholesky factorization:

1. $4x_1 + 2x_2 + 14x_3 = 14$
   $2x_1 + 17x_2 - 5x_3 = -101$
   $14x_1 - 5x_2 + 83x_3 = 155$

   **Solution:** Let

$$A = LL^T$$

$$\text{or, } \begin{pmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix}$$

   Then
   <u>$1^{st}$ column of $L$:</u> We have, $l_{11}^2 = 4 \Rightarrow l_{11} = 2$, $l_{11}l_{21} = 2 \Rightarrow l_{21} = 1$ and $l_{11}l_{31} = 14 \Rightarrow l_{31} = 7$.

   <u>$2^{nd}$ column of $L$:</u> We have, $l_{21} = 0$, $l_{21}^2 + l_{22}^2 = 17 \Rightarrow l_{22} = 4$, and $l_{21}l_{31} + l_{22}l_{32} = -5 \Rightarrow l_{32} = -3$.

   <u>$3^{rd}$ column of $L$:</u> We have, $l_{13} = l_{23} = 0$ and $l_{31}^2 + l_{32}^2 + l_{33}^2 = 83 \Rightarrow l_{33} = 5$.

   Hence

$$\begin{pmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{pmatrix}$$

   So the given system of equations can be written as

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 14 \\ -101 \\ 155 \end{pmatrix} \cdots\cdots (1)$$

   Let

$$\begin{pmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \cdots\cdots (2)$$

   Then from (1)

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 14 \\ -101 \\ 155 \end{pmatrix}$$

Solving this system by forward substitution, we get $y_1 = 7$, $y_2 = -27$ and $y_3 = 5$.
Then from (2)

$$\begin{pmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -27 \\ 5 \end{pmatrix}$$

Solving above by back substitution, we get $x_1 = 3$, $x_2 = -6$ and $x_3 = 1$.

2. $9x_1 + 6x_2 + 12x_3 = 17.4$
   $6x_1 + 13x_2 + 11x_3 = 23.6$
   $12x_1 + 11x_2 + 26x_3 = 30.8$
   (Ans: $x_1 = 0.6, x_2 = 1.2, x_3 = 0.4$)

3. $4x_1 + 6x_2 + 8x_3 = 0$
   $6x_1 + 34x_2 + 52x_3 = -160$
   $8x_1 + 52x_2 + 129x_3 = -452$
   (Ans: $x_1 = 8, x_2 = 0, x_3 = -4$)

4. $x_1 - x_2 + 3x_3 + 2x_4 = 15$
   $-x_1 + 5x_2 - 5x_3 - 2x_4 = -35$
   $3x_1 - 5x_2 + 19x_3 + 3x_4 = 94$
   $2x_1 - 2x_2 + 3x_3 + 21x_4 = 1$

5. $4x_1 + 2x_2 + 4x_3 = 10$
   $2x_1 + 2x_2 - 3x_3 + 2x_4 = 18$
   $4x_1 3x_2 + 6x_3 + 3x_4 = 30$
   $2x_2 + 3x_3 + 9x_4 = 61$
   (Ans: $x_1 = 3, x_2 = -1, x_3 = 0, x_4 = 7$)

## 4.7 Iterative Methods

The methods studied so far for the solution of a system of linear equations $Ax = b$ (eg. Gaussian elimination method, Gauss-Jordan method, matrix factorization method) are known as direct methods. In these methods, one proceeds directly to determine the exact solution of the system. There is no provision for stopping midway and getting an approximate solution.

There is another class of methods known as iterative methods for the solution of the system of linear equations $Ax = b$. In this method, beginning with an initial approximate solution vector $x^{(0)}$ one generates a sequence of vectors $x^{(0)}, x^{(1)}, x^{(2)}, \cdots$ which converges towards the actual solution $x$ under certain conditions. The process can be stopped when the desired level of accuracy is obtained.

We study the following iterative methods:

1. Jacobi or Total-step method
2. Gauss-Seidel or Single-step method

### 4.7.1 Jacobi Iteration Method

Suppose that the following is a system of $n$ linear equations:

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + a_{nn}x_n = b_n$$

Then we can rewrite the above system as

$$\begin{aligned}
x_1 &= \frac{1}{a_{11}}[b_1 - (a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n)] \\
x_2 &= \frac{1}{a_{22}}[b_2 - (a_{21}x_1 + a_{23}x_3 + \cdots + a_{2n}x_n)] \\
\vdots & \qquad\quad \vdots \\
x_n &= \frac{1}{a_{nn}}[b_n - (a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{n,n-1}x_{n-1})]
\end{aligned}$$

If $x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)}$ are the initial approximation to $x_1, x_2, \cdots, x_n$ respectively, then we calculate the next approximation $x_1^{(1)}, x_2^{(1)}, \cdots, x_n^{(1)}$ using the above equations as follows:

$$\begin{aligned}
x_1^{(1)} &= \frac{1}{a_{11}}[b_1 - (a_{12}x_2^{(0)} + a_{13}x_3^{(0)} + \cdots + a_{1n}x_n^{(0)})] \\
x_2^{(1)} &= \frac{1}{a_{22}}[b_2 - (a_{21}x_1^{(0)} + a_{23}x_3^{(0)} + \cdots + a_{2n}x_n^{(0)})] \\
\vdots & \qquad\quad \vdots \\
x_n^{(1)} &= \frac{1}{a_{nn}}[b_n - (a_{n1}x_1^{(0)} + a_{n2}x_2^{(0)} + \cdots + a_{nn-1}x_{n-1}^{(0)})]
\end{aligned}$$

In general, if $x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)}$ are the approximations of $x_1, x_2, \cdots, x_n$ at any stage, then we calculate the next approximation $x_1^{(k+1)}, x_2^{(k+1)}, \cdots, x_n^{(k+1)}$ using $x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)}$ by following iteration:

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}x_j^{(k)}\right]$$

for $i = 1, 2, \cdots, n$.
We continue to calculate new approximations till we obtain a desired level of accuracy. This iterative method is known as Jacobi iterative method . In this method, the initial approximations $x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)}$ are normally taken as zero.

**Algorithm (Jacobi Iterative Method):**

INPUT: A diagonally dominant system of linear equations $Ax = b$ and a termination parameter $\varepsilon$.

PROCESS:

FOR $i = 1$ TO $n$ SET $xo_i = 0$

DO {

SET $key = 0$

FOR $i = 1$ TO $n$ {

SET $sum = b_i$

FOR $j = 1$ TO $n$ AND $j \neq i$ SET $sum = sum - (a_{ij} * xo_j)$

SET $x_i = \dfrac{sum}{a_{ii}}$

IF $key = 0$ AND $\left|\dfrac{x_i - xo_i}{x_i}\right| > \varepsilon$ THEN SET $key = 1$

}

FOR $i = 1$ TO $n$, SET $xo_i = x_i$

} WHILE $(key == 1)$

OUTPUT: Approximate solution $x_i, i = 1, 2, \cdots, n$ of $Ax = b$.

### 4.7.2 Exercise

Solve by Jacobi method the following system:

1. $5x_1 + 2x_2 + x_3 = 12 \cdots\cdots (1)$
   $x_1 + 4x_2 + 2x_3 = 15 \cdots\cdots (2)$
   $x_1 + 2x_2 + 5x_3 = 20 \cdots\cdots (3)$
   (Exact Solution: $x_1 = 1, x_2 = 2, x_3 = 3$)

   **Solution:** From (1), (2) and (3)

$$x_1 = \frac{1}{5}(12 - 2x_2 - x_3) \cdots\cdots (4)$$
$$x_2 = \frac{1}{4}(15 - x_1 - 2x_3) \cdots\cdots (5)$$
$$x_3 = \frac{1}{5}(20 - x_1 - 2x_2) \cdots\cdots (6)$$

   If the initial approximate solution is $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$, then from (4), (5) and (6) we get the first approximation as $x_1^{(1)} = \frac{12}{5} = 2.4$, $x_2^{(1)} = \frac{15}{4} = 3.75$ and $x_3^{(1)} = \frac{20}{5} = 4$. For the second approximation we have

$$x_1^{(2)} = \frac{1}{5}(12 - 2 \times 3.75 - 4) = 0.1$$
$$x_2^{(2)} = \frac{1}{4}(15 - 2.4 - 2 \times 4) = 1.15$$
$$x_3^{(2)} = \frac{1}{5}(20 - 2.4 - 2 \times 3.75) = 2.02$$

Continuing this process, we get

Recheck these values!

$$\begin{array}{ccc} x_1^{(3)} = 1.54 & x_2^{(3)} = 1.72 & x_3^{(3)} = 3.57 \\ x_1^{(4)} = 0.61 & x_2^{(4)} = 1.17 & x_3^{(4)} = 2.60 \\ x_1^{(5)} = 1.41 & x_2^{(5)} = 2.29 & x_3^{(5)} = 3.41 \\ x_1^{(6)} = 0.8 & x_2^{(6)} = 1.69 & x_3^{(6)} = 3.20 \\ x_1^{(7)} = 1.08 & x_2^{(7)} = 1.95 & x_3^{(7)} = 3.16 \end{array}$$

We continue this process till we reach the desired level of accuracy.

2. $6x_1 - 2x_2 + x_3 = 11$
   $-2x_1 + 7x_2 + 2x_3 = 5$
   $x_1 + 2x_2 + -5x_3 = -1$
   (Exact Solution: $x_1 = 2, x_2 = 1, x_3 = 1$)

3. $5x_1 + x_2 + 2x_3 = 19$
   $x_1 + 4x_2 - 2x_3 = -2$
   $2x_1 + 3x_2 + 8x_3 = 39$
   (Exact Solution: $x_1 = 2, x_2 = 1, x_3 = 4$)

4. $x_1 + 9x_2 - 2x_3 = 36$
   $2x_1 - x_2 + 8x_3 = 121$
   $6x_1 + x_2 + x_3 = 107$
   (Exact Solution: $x_1 = 15, x_2 = 5, x_3 = 12$; transform into diagonally dominant system first)

5. $3x_1 + 2x_2 + x_3 = 7$
   $x_1 + 3x_2 + 2x_3 = 4$
   $2x_1 + x_2 + 3x_3 = 7$
   (Not diagonally dominant system but but still solvable)

6. $2x_1 + x_2 + x_3 = 5$
   $3x_1 + 5x_2 + 2x_3 = 15$
   $2x_1 + x_2 + 4x_3 = 8$

7. $3x_1 - 6x_2 + 2x_3 = 15$
   $4x_1 - x_2 + x_3 = 2$
   $x_1 - 3x_2 + 7x_3 = 22$

8. $3x_1 - 2x_2 \quad = 5$
   $-x_1 + 2x_2 - x_3 = 0$
   $-2x_2 + x_3 = -1$

9. $2x_1 - 7x_2 - 10x_3 = -17$
   $5x_1 + x_2 + 3x_3 = 14$
   $x_1 + 10x_2 + 9x_3 = 7$

### 4.7.3 Gauss-Seidel Iteration Method

Suppose that the following is a system of $n$ linear equations as follows:

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + a_{nn}x_n = b_n$$

Then we can rewrite the above system as

$$\begin{aligned} x_1 &= \frac{1}{a_{11}}[b_1 - (a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n)] \\ x_2 &= \frac{1}{a_{22}}[b_2 - (a_{21}x_1 + a_{23}x_3 + \cdots + a_{2n}x_n) \\ \vdots & \qquad\qquad \vdots \\ x_n &= \frac{1}{a_{nn}}[b_n - (a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{n,n-1}x_{n-1})] \end{aligned}$$

If $x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)}$ are the initial approximation to $x_1, x_2, \cdots, x_n$ respectively, then we calculate the next approximation $x_1^{(1)}$ of $x_1$ as below:

$$x_1^{(1)} = \frac{1}{a_{11}}[b_1 - (a_{12}x_2^{(0)} + a_{13}x_3^{(0)} + \cdots + a_{1n}x_n^{(0)})]$$

To calculate the next approximation $x_2^{(1)}$ of $x_2$, we use the recently calculated approximation $x_1^{(1)}$ of $x_1$ together with $x_3^{(0)}, \cdots, x_n^{(0)}$ as below:

$$x_2^{(1)} = \frac{1}{a_{22}}[b_2 - (a_{21}x_1^{(1)} + a_{23}x_3^{(0)} + \cdots + a_{2n}x_n^{(0)})$$

Similarly, for calculating $x_3^{(1)}$ we use $x_1^{(1)}, x_2^{(1)}$ together with $x_4^{(0)}, \cdots, x_n^{(0)}$. This idea can be extended to all subsequent computations. In general, the $(k+1)^{th}$ approximation $x_i^{(k+1)}$ of the variable $x_i$ can be calculated as

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right]$$

for $i = 1, 2, \cdots, n$.

We continue to calculate new approximations till we obtain a desired level of accuracy. This iterative method is known as Gauss-Seidel iterative method . In this method, the initial approximations $x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)}$ are normally taken as zero.

**Algorithm (Gauss-Seidel Iterative Method):**

INPUT: A diagonally dominant system of linear equations $Ax = b$ and termination parameter $\varepsilon$.

PROCESS:

FOR $i = 1, 2, \cdots, n$ SET $x_i = 0$
DO {
SET $key = 0$
FOR $i = 1$ TO $n$ {
SET $sum = b_i$
FOR $j = 1$ TO $n$ AND $j \neq i$ SET $sum = sum - (a_{ij} * x_j)$
SET $temp = \dfrac{sum}{a_{ii}}$
IF $key = 0$ AND $\left|\dfrac{temp - x_i}{temp}\right| > \varepsilon$ THEN SET $key = 1$
SET $x_i = temp$
}
} WHILE $(key == 1)$

OUTPUT: Approximate solution $x_i$, $i = 1, 2, \cdots, n$ of $Ax = b$.

### 4.7.4 Exercise

1. Solve by Gauss-Seidel method the following system:

$$\begin{aligned} 10x_1 + x_2 + x_3 &= 12 \cdots\cdots (1) \\ 2x_1 + 10x_2 + x_3 &= 13 \cdots\cdots (2) \\ 2x_1 + 2x_2 + 10x_3 &= 14 \cdots\cdots (3) \end{aligned}$$

**Solution:** From (1), (2) and (3), we get,

$$\begin{aligned} x_1 &= \frac{1}{10}(12 - x_2 - x_3) \cdots\cdots (4) \\ x_2 &= \frac{1}{10}(13 - 2x_1 - x_3) \cdots\cdots (5) \\ x_3 &= \frac{1}{10}(14 - 2x_1 - 2x_2) \cdots\cdots (6) \end{aligned}$$

If the initial approximate solution is $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$, then

$$x_1^{(1)} = \frac{1}{10}(12 - 0 - 0) = \frac{12}{10} = 1.2$$
$$x_2^{(1)} = \frac{1}{10}(13 - 2 \times 1.2 - 0) = 1.06$$
$$x_3^{(1)} = \frac{1}{10}(14 - 2 \times 1.2 - 2 \times 1.06) = 0.948$$

Similarly,

$$x_1^{(2)} = \frac{1}{10}(12 - 1.06 - 0.948) = 0.9992$$
$$x_2^{(2)} = \frac{1}{10}(13 - 2 \times 0.9992 - 0.948) = 1.0054$$
$$x_3^{(2)} = \frac{1}{10}(14 - 2 \times 0.9992 - 2 \times 1.0054) = 0.9991$$

We continue this process till we reach the desired level of accuracy.

2. Solve all Jacobi problems using Gauss-Seidel method.

**Convergence of iterative methods:**

**Diagonally dominant matrix:** An $n \times n$ matrix $A$ is said to be a diagonally dominant matrix if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

for $i = 1, 2, \cdots n$.

For example,

$$\begin{pmatrix} 6 & -2 & 1 \\ 0 & 7 & -3 \\ 1 & 2 & -5 \end{pmatrix}$$

is a diagonally dominant matrix .

**Diagonally dominant system:** A system of linear equations $Ax = b$ is said to be a diagonally dominant system if the coefficient matrix $A$ is a diagonally dominant matrix.

**Theorem:** If $Ax = b$ is a diagonally dominant system, then the Jacobi and Gauss-Seidel methods converge for any initial approximation $x^{(0)}$.

**Note:** From above theorem, we see that when solving a system of linear equations $Ax = b$ using Jacobi or Gauss-Seidel methods, it is preferable to have a diagonally dominant system so that the solutions are guaranteed to converge. However, if the system is not initially in a diagonally dominant form, then it may be possible to rearrange it in such a way that the resulting system is diagonally dominant. before solving it.

For example, the following system is not a diagonally dominant system:

$$\begin{aligned} 6x_1 - 2x_2 + x_3 &= 11 \\ x_1 + 2x_2 - 5x_3 &= -1 \\ -2x_1 + 7x_2 + 2x_3 &= 5 \end{aligned}$$

However this system can be rearranged as below so that it is a diagonally dominant system and then proceed with the solution using either Jacobi or Gauss-Seidel method:

$$\begin{aligned} 6x_1 - 2x_2 + x_3 &= 11 \\ -2x_1 + 7x_2 + 2x_3 &= 5 \\ x_1 + 2x_2 - 5x_3 &= -1 \end{aligned}$$

Notice that the above theorem gives only the sufficient condition for the convergence, not a necessary condition. Therefore, it may be possible for a linear system to have convergent approximations even though it is not in diagonally dominant form.

## 4.8 Eigenvalues and Eigenvectors

Let $A$ be an $n \times n$ matrix. A number $\lambda \in \mathbb{C}$ is called an eigenvalue of the matrix $A$ if there is a vector $x \neq 0$, $x \in \mathbb{C}^n$ such that

$$Ax = \lambda x.$$

Every such nonzero vector $x$ is called an eigenvector of $A$ associated with the eigenvalue $\lambda$.

**Eigenvalue problem:** Given an $n \times n$ matrix $A$, the problem of finding an eigenvalue of $A$ and a corresponding eigenvector is called the eigenvalue problem .

**Characteristic polynomial:** Given a matrix $A$, the polynomial

$$\phi(x) = \det(A - xI)$$

of degree $n$ is called the characteristic polynomial of the matrix $A$.

**Theorem:** A number $\lambda \in \mathbb{C}$ is an eigenvalue of $A$ if and only if it is a zero of the characteristic polynomial of $A$.

**Proof:** We know that $\lambda \in \mathbb{C}$ is an eigenvalue of $A$ if and only if $Ax = \lambda x$ for some nonzero $x \in \mathbb{C}^n$ i.e.,

$$(A - \lambda I)x = 0$$

for some nonzero $x \in \mathbb{C}^n$. In other words, $\lambda$ is an eigenvalue of $A$ if and only if the above homogeneous system of linear equations has a non-trivial solution. This can happen if and only if $A - \lambda I$ is a singular matrix i.e.,

$$\det(A - \lambda I) = 0$$

which is equivalent to saying that $\lambda$ is a solution of the characteristic polynomial

$$\phi(x) = \det(A - \lambda I)$$

of $A$. ■

**Remark:** From the above theorem, it can be seen that one way to find an eigenvalue of a matrix $A$ would be to find the roots of the characteristic polynomial det$(A - xI)$. Such methods of finding eigenvalues are called direct methods. The direct methods of finding eigenvalues are satisfactory when $n$ is small but when $n$ is large, finding eigenvalues by direct method can be very difficult. In these cases, iterative methods such as power methods are generally used.

**Theorem:** If $\lambda$ is an eigenvalue of $A$, then $p(\lambda)$ is an eigenvalue of $p(A)$ for any polynomial $p(x)$.

So by above theorem, if $\lambda$ is an eigenvalue of $A$ then $\lambda^2 - \lambda + 1$ is an eigenvalue of $A^2 - A + I$. In particular, $\lambda^k$ is an eigenvalue of $A^k$ for any $k \geq 1$.

### 4.8.1 Power Method

Suppose $A$ is an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ such that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$$

and corresponding eigenvectors $v_1,\ v_2,\ \cdots,\ v_n$ which are linearly independent. The power method is an iterative method that is used to approximate the eigenvalue having maximum modulus, $\lambda_1$ in this case, and its associated eigenvector $v_1$. The power method proceeds as follows: An initial guess $x_0$ of eigenvector is made which is then normalized i.e. all the components of $x_0$ is divided by the component which has the largest modulus. Denote this normalized vector by the symbol $x_0'$. This is used to obtain the next approximation as $x_1 = Ax_0'$. From $x_1$, we get the normalized vector $x_1'$ and then obtain the next approximation as $x_2 = Ax_1'$. This iterative process can be expressed in general form as

$$x_{n+1} = Ax_n', \quad n = 0, 1, 2, \cdots.$$

Suppose we have computed $x_n$ this way. The approximation of the eigenvalue $\lambda_1$ at this stage is the component of $x_n$ with the largest modulus and the approximation of corresponding eigenvector is the normalized vector $x_n'$. The initial guess $x_0$ is taken as the vector with all components having value 1.

**Convergence of power method:** Let $u^{(i)}$ be a nonzero eigenvector of $A$ associated with the eigenvalue $\lambda_i$ i.e., $Au^{(i)} = \lambda_i u^{(i)}$, $i = 1, 2, \cdots, n$. We further assume that there is a linearly independent set of $n$ eigenvectors $\{u^{(1)}, u^{(2)}, \cdots, u^{(n)}\}$. Since $\mathbb{C}^n$ is an $n$-dimensional vector space, so this set forms a basis of $\mathbb{C}^n$. Let

$$x^{(0)} = u^{(1)} + u^{(2)} + \cdots + u^{(n)} \cdots\cdots (1)$$

Starting with $x^{(0)}$ as the initial vector, we repeatedly carry out the matrix-vector multiplication using matrix $A$ to produce a sequence of vectors as follows:

$$\begin{aligned} x^{(1)} &= Ax^{(0)} \\ x^{(2)} &= Ax^{(1)} = A^2x^{(0)} \\ x^{(3)} &= Ax^{(2)} = A^3x^{(0)} \\ &\vdots \end{aligned}$$

In general we have $x^{(k)} = A^k x^{(0)}$, $k = 1, 2, 3, \cdots$. Now substituting the value of $x^{(0)}$ from (1) in this expression, we get

$$x^{(k)} = A^k(u^{(1)} + u^{(2)} + \cdots + u^{(n)})$$

$$\text{or, } x^{(k)} = A^k u^{(1)} + A^k u^{(2)} + \cdots + A^k u^{(n)}$$

$$\text{or, } x^{(k)} = \lambda_1^k u^{(1)} + \lambda_2^k u^{(2)} + \cdots + \lambda_n^k u^{(n)}$$

$$\text{or, } x^{(k)} = \lambda_1^k \left[ u^{(1)} + \left(\frac{\lambda_2}{\lambda_1}\right)^k u^{(2)} + \cdots + \left(\frac{\lambda_n}{\lambda_1}\right)^k u^{(n)} \right] \cdots\cdots (2)$$

Since $|\lambda_1| > |\lambda_j|$ for all $j = 2, 3, \cdots, n$ so $\left|\frac{\lambda_j}{\lambda_1}\right| < 1$ and hence $\left|\frac{\lambda_j}{\lambda_1}\right|^k \longrightarrow 0$ as $k \longrightarrow \infty$ i.e., $\left(\frac{\lambda_j}{\lambda_1}\right)^k \longrightarrow 0$ as $k \longrightarrow 0$. Hence equation (2) can be written in the form

$$x^{(k)} = \lambda_1^k (u^{(1)} + \varepsilon^{(k)}) \cdots\cdots (3)$$

where

$$\varepsilon^{(k)} = \left(\frac{\lambda_2}{\lambda_1}\right)^k u^{(2)} + \cdots + \left(\frac{\lambda_n}{\lambda_1}\right)^k u^{(n)} \longrightarrow 0$$

as $k \longrightarrow 0$.

Now let $\phi : \mathbb{C}^n \to \mathbb{C}$ be any linear functional such that $\phi(u^{(1)}) \neq 0$. Applying $\phi$ to (3), we get

$$\phi(x^{(k)}) = \phi[\lambda_1^k (u^{(1)} + \varepsilon^{(k)})] = \lambda_1^k [\phi(u^{(1)}) + \phi(\varepsilon^{(k)})].$$

Let

$$r_k = \frac{\phi(x^{(k+1)})}{\phi(x^{(k)})} = \frac{\lambda_1^{k+1}}{\lambda_1^k} \frac{[\phi(u^{(1)}) + \phi(\varepsilon^{(k+1)})]}{[\phi(u^{(1)}) + \phi(\varepsilon^{(k)})]} = \lambda_1 \left[ \frac{\phi(u^{(1)}) + \phi(\varepsilon^{(k+1)})}{\phi(u^{(1)}) + \phi(\varepsilon^{(k)})} \right].$$

Then

$$\lim_{k\to\infty} r_k = \lambda_1 \lim_{k\to\infty} \left[ \frac{\phi(u^{(1)}) + \phi(\varepsilon^{(k+1)})}{\phi(u^{(1)}) + \phi(\varepsilon^{(k)})} \right].$$

Now, as $k \to \infty$, we have $\varepsilon^{(k)}, \varepsilon^{(k+1)} \to 0$ and so $\phi(\varepsilon^{(k)}), \phi(\varepsilon^{(k+1)}) \to 0$ as well.

So

$$\lim_{k\to\infty} r_k = \lambda_1 \left[ \frac{\phi(u^{(1)}) + 0}{\phi(u^{(1)}) + 0} \right] = \lambda_1 \frac{\phi(u^{(1)})}{\phi(u^{(1)})} = \lambda_1$$

since $\phi(u^{(1)}) \neq 0$.

Hence $\lim_{k\to\infty} r_k = \lambda_1$.

**Algorithm (Power Method):**

INPUT: An $n \times n$ matrix $A = [a_{ij}]$ and a termination parameter $\varepsilon$.

PROCESS:

FOR $i = 1$ TO $n$ SET $x_i = 1$

SET $ev = 1$

DO {

FOR $i = 1$ TO $n$ SET $y_i = 0$
SET $ev\_temp = ev$
FOR $i = 1$ TO $n$

FOR $j = 1$ TO $n$

SET $y_i = y_i + a_{ij}x_j$

FOR $i = 1$ TO $n$ SET $x_i = y_i$
SET $temp\_max = |x_1|$, $k = 1$
FOR $j = 2$ TO $n$ {

IF $|x_j| > temp\_max$ THEN {

SET $temp\_max = |x_j|$
SET $k = j$

}

}

SET $ev = x_k$
FOR $i = 1$ TO $n$ SET $x_i = \dfrac{x_i}{ev}$

} WHILE $\left|\dfrac{ev\_temp - ev}{ev}\right| > \varepsilon$

OUTPUT: Approximation $ev$ of eigenvalue with largest modulus and the corresponding eigenvector $[x_1, \cdots, x_n]$.

### 4.8.2 Exercise

Find and eigenvalue and an associated eigenvector of the following matrices using power method.

1. $A = \begin{pmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{pmatrix}$
   (eigenvalues: $5.47735, 2.44807, 0.074577$)

   **Solution:** Let the initial approximation to eigenvector be

$$x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Then

$$x_1 = Ax_0 = \begin{pmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

Normalizing $x_1$, we have

$$x_1 = \begin{pmatrix} 1 \\ -0.5 \\ 0 \end{pmatrix}.$$

Then

$$x_2 = Ax_1 = \begin{pmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 3.5 \\ -4 \\ 0.5 \end{pmatrix}.$$

Normalizing $x_2$, we have

$$x_2 = \begin{pmatrix} -0.875 \\ 1 \\ -0.125 \end{pmatrix}.$$

Then

$$x_3 = Ax_2 = \begin{pmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} -0.875 \\ 1 \\ -0.125 \end{pmatrix} = \begin{pmatrix} -3.625 \\ 6.125 \\ -1.125 \end{pmatrix}.$$

Normalizing $x_3$, we have

$$x_3 = \begin{pmatrix} -0.5918 \\ 1 \\ -0.1837 \end{pmatrix}.$$

Then

$$x_4 = Ax_3 = \begin{pmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} -0.5918 \\ 1 \\ -0.1837 \end{pmatrix} = \begin{pmatrix} -2.7754 \\ 5.7347 \\ -1.1837 \end{pmatrix}.$$

At this point, the approximate eigenvalue is $5.7347$ and the corresponding eigenvector is $\begin{pmatrix} -0.4840 \\ 1 \\ -0.2064 \end{pmatrix}$ which is obtained after normalizing $x_4$.

After $14$ iterations, the approximate eigenvalue is $5.47743$ and the corresponding eigenvector is $\begin{pmatrix} -0.40368 \\ 1 \\ -0.22334 \end{pmatrix}$.

2. $A = \begin{pmatrix} 2 & 3 & 4 \\ 7 & -1 & 3 \\ 1 & -1 & 5 \end{pmatrix}$

**Solution:** Let the initial approximation to eigenvector be

$$x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Then

$$x_1 = Ax_0 = \begin{pmatrix} 2 & 3 & 4 \\ 7 & -1 & 3 \\ 1 & -1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \\ 5 \end{pmatrix}.$$

Normalizing $x_1$, we have

$$x_1 = \begin{pmatrix} 1 \\ 1 \\ 0.5556 \end{pmatrix}.$$

Then

$$x_2 = Ax_1 = \begin{pmatrix} 2 & 3 & 4 \\ 7 & -1 & 3 \\ 1 & -1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0.5556 \end{pmatrix} = \begin{pmatrix} 7.2224 \\ 7.6668 \\ 2,778 \end{pmatrix}.$$

Normalizing $x_2$, we have

$$x_2 = \begin{pmatrix} 0.942 \\ 1 \\ 0.3623 \end{pmatrix}.$$

Then

$$x_3 = Ax_2 = \begin{pmatrix} 2 & 3 & 4 \\ 7 & -1 & 3 \\ 1 & -1 & 5 \end{pmatrix} \begin{pmatrix} 0.942 \\ 1 \\ 0.3623 \end{pmatrix} = \begin{pmatrix} 6.3332 \\ 6.6809 \\ 1.7535 \end{pmatrix}.$$

Normalizing $x_3$, we have

$$x_3 = \begin{pmatrix} 0.9480 \\ 1 \\ 0.2625 \end{pmatrix}.$$

Then

$$x_4 = Ax_3 = \begin{pmatrix} 2 & 3 & 4 \\ 7 & -1 & 3 \\ 1 & -1 & 5 \end{pmatrix} \begin{pmatrix} 0.9480 \\ 1 \\ 0.2625 \end{pmatrix} = \begin{pmatrix} 5.946 \\ 6.4235 \\ 1.2605 \end{pmatrix}.$$

Normalizing $x_4$, we have

$$x_4 = \begin{pmatrix} 0.9257 \\ 1 \\ 0.212 \end{pmatrix}.$$

Then

$$x_5 = Ax_4 = \begin{pmatrix} 2 & 3 & 4 \\ 7 & -1 & 3 \\ 1 & -1 & 5 \end{pmatrix} \begin{pmatrix} 0.9257 \\ 1 \\ 0.212 \end{pmatrix} = \begin{pmatrix} 5.6994 \\ 6.1159 \\ 0.9857 \end{pmatrix}.$$

At this point, the approximate eigenvalue is $6.1159$ and the corresponding eigenvector is $\begin{pmatrix} 0.9319 \\ 1 \\ 0.1612 \end{pmatrix}$ which is obtained after normalizing $x_5$.

3. $A = \begin{pmatrix} 3 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 3 \end{pmatrix}$

4. $A = \begin{pmatrix} 3.6 & -1.8 & 1.8 \\ -1.8 & 2.8 & -2.6 \\ 1.8 & -2.6 & 2.8 \end{pmatrix}$

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