AP, GP, HP

1 Arithmetic Progression

1.1 Introduction

For starting term a, common difference d, an A.P. of n terms could be formed as:

$$a,\ a+d,\ a+2d,\ \cdots,\ a+(n-1)d$$

Properties:

- General term: $a_n = a + (n-1)d$
  (Also called last term l)
- Sum of an A.P.:
  $S_n = \frac{n}{2}\{2a + (n-1)d\}$
  $S_n = \frac{n}{2}\{a + l\}$
- In A.P. of m terms:
  n^{th} term from end $= (m-n+1)^{th}$ term from start
  n^{th} term from end: $a + (m-n)d$
  n^{th} term from end: $l - (n-1)d$
- If a, b, c are in A.P.:
  $2b = a + c$
  $a_1 + a_n = a_2 + a_{n-1} = \cdots$
  $2a_n = a_{n+k} + a_{n-k}$
- Middle term of A.P.:
  ODD $\Rightarrow \frac{n+1}{2}$, CD $\rightarrow d$
  EVEN $\Rightarrow \frac{n}{2}$ or $\frac{n}{2}+1$, CD $\rightarrow 2d$

Results of sum of AP :

- Seq is AP $\rightarrow$ if sum of n terms form $An^2 + Bn$
- if ratio of sum is given, then ratio of n^{th} term is
  $\rightarrow$ replace n by $2n-1$
- if ratio of n^{th} term is given, then ratio of sum is
  $\rightarrow$ replace n by $\frac{n+1}{2}$

1.2 Arithmetic Mean (AM)

Arithmetic mean between a and b be:

$$A = \frac{a+b}{2}$$

AM of n terms: $\frac{1}{n}[a_1 + a_2 + a_3 + \cdots + a_n]$

n AM between two numbers:
For a and b, n AMs are:

$$a,\ A_1,\ A_2,\ A_3,\ \cdots,\ A_n,\ b$$

- number of terms: $n+2$
- common difference: $\frac{b-a}{n+1}$
- n^{th} term of n AM: $a + nd$

1.3 Sum of some Sequence

First n natural number : $\frac{n(n+1)}{2}$

First n odd number : n^2

First n even number : $n(n+1)$

Square of first n number : $\frac{n(n+1)(2n+1)}{6}$

Cube of first n number : $\left\{\frac{n(n+1)}{2}\right\}^2$

4^{th} power of n no : $\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

2 Geometric Progression

2.1 Introduction

For starting term a, common difference r, an G.P. could be formed as:

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots,\ ar^{n-1},\ ar^n,\ \cdots$$

Properties:

- General term: $a_n = ar^{n-1}$
  (Also called last term l)
- Sum of n terms:
  $S_n = \begin{cases} a\left(\frac{1-r^n}{1-r}\right), & r \neq 1 \\ na, & r = 1 \end{cases}$
  $S_n = \frac{a - lr}{1-r} = \frac{lr - a}{r-1}$
- Sum of ∞ terms:
  $S_\infty = \begin{cases} \frac{a}{1-r} & |r| < 1 \\ \infty & |r| > 1 \end{cases}$
- In G.P. of m terms:
  n^{th} term from end: $a_n = ar^{m-n}$
  n^{th} term from end: $a_n = l\left(\frac{1}{r}\right)^{n-1}$

- If a, b, c are in G.P.:
  $b^2 = ac$
  $a_1 a_n = a_2 a_{n-1} = \cdots = a_k a_{n-k+1}$
- GP divided/multiplied by constant, stays GP
- reciprocal of GP, is GP
- if $a_1, a_2 \ldots a_n \Rightarrow$ GP
  $\log a_1, \log a_2 \ldots \log a_n \Rightarrow$ AP and vice-versa

2.2 Geometric Mean (GM)

Geometric mean between a and b be:

$$G = \sqrt{ab}$$

GM of n terms: $(a_1 \cdot a_2 \cdot a_3 \cdots a_n)^{\frac{1}{n}}$
n GM between two numbers:
For a and b, n GMs are:

$$a,\, G_1,\, G_2,\, G_3,\, \cdots,\, G_n,\, b$$

- number of terms: $n+2$
- common difference: $r = \left(\dfrac{b}{a}\right)^{\frac{1}{n+1}}$
- n^{th} term of n GM: ar^n

3 Harmonic Progression

3.1 Introduction

A series $H = a_1, a_2, a_3, \cdots, a_n$ is said to be in H.P., iff $\dfrac{1}{a_1}, \dfrac{1}{a_2}, \dfrac{1}{a_3}, \cdots, \dfrac{1}{a_n}$ is in an arithmetic progression.

Example: $2,\, 3,\, 6 \Leftarrow$ H.P.
$\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{6} \Leftarrow$ A.P.

Properties:

- Common difference: $d = \dfrac{1}{a_2} - \dfrac{1}{a_1}$
- General term:
  If a and d are two terms of A.P.: $h_n = \dfrac{1}{a+(n-1)d}$
  If a and d are two terms of H.P.: $h_n = \dfrac{1}{\frac{1}{a} + (n-1)\left(\frac{1}{b} - \frac{1}{a}\right)}$

3.2 Harmonic Mean

Harmonic mean between a and b be:

$$H = \frac{2ab}{a+b}$$

HM of n terms: $\dfrac{n}{\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \cdots, \frac{1}{a_n}}$
n HM between two numbers:
For a and b, n HMs are:

$$a,\, H_1,\, H_2,\, H_3,\, \cdots,\, H_n,\, b$$

- number of terms: $n+2$
- common difference: $\dfrac{a-b}{(n+1)ab}$
- n^{th} term of n HM: $\dfrac{1}{a} + (n+1)D$

4 Relation between AP, GP and HP

- **A,G and H between 2 numbers(a and b):**

$$A = \frac{a+b}{2} \qquad G = \sqrt{AB} \qquad H = \frac{2ab}{a+b}$$

- $A \geq G \geq H$
- quadratic equation having a and b as its roots
  $$x^2 - 2Ax + G^2 = 0$$
- the two numbers (a,b) are $A \pm \sqrt{A^2 - G^2}$
- if A and G are in the ratio $m:n$, then the number(a,b) are in ratio
  $$m + \sqrt{m^2 - n^2} : m - \sqrt{m^2 - n^2}$$
- **A,G and H between 3 numbers $(\mathbf{a}, \mathbf{b}, \mathbf{c})$:**

$$A = \frac{a+b+c}{3} \quad G = \sqrt[3]{abc} \quad \frac{1}{H} = \frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{3}$$

- cubic equation where a,b,c are the roots
  $$x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$$

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- **Example:** for 1 and 9

$$A = 5 \quad G = 3 \quad H = \frac{9}{5}$$

$$G^2 = AH \Rightarrow 9 = 5 \cdot \frac{9}{5}$$

5 Arithmetico-geometric Progression (AGP)

$a, (a+d)r, (a+2d)r^2, \cdots, (a+nd)r^n$ is a AGP sequence

n^{th} **term:** $a_n = \{a + (n-1)d\}.r^{n-1}$

Sum of ∞ term:

$$S_\infty = \begin{cases} \dfrac{a}{1-r} + \dfrac{d.r}{(1-r)^2}, & |r| < 1 \\ \infty, & |r| > 1 \end{cases}$$

Sum of n terms:

$$S_n = \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{a - [a + (n-1)d]r^n}{1-r}$$

Tips and Tricks

(black space for tips,tricks and important question)

Finding sum of AGP through diference method

1. multiply r in the series
2. subtract new series from old series $S_n - rS_n$
3. Now you are left with an AP, find sum as of an AP