GATE EE 2023

EE:1205 Signals and System
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Question: The Z-transform of a discrete signal $x(n)$ is

$$X(z) = \frac{4z}{\left(z-\frac{1}{5}\right)\left(z-\frac{2}{3}\right)(z-3)} \text{ with ROC= } R \quad (1)$$

Which one of the following statements is TRUE?

(a) Discrete time Fourier transform of $x[n]$ converges if R is $|z| > 3$

(b) Discrete time Fourier transform of $x[n]$ converges if R is $\frac{2}{3} < |z| < 3$

(c) Discrete time Fourier transform of $x[n]$ converges if R is such that $x[n]$ is a left-sided sequence.

(d) Discrete time Fourier transform of $x[n]$ converges if R is such that $x[n]$is a right-sided sequence.

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Solution:
The Z-transform of a sequence is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad (2)$$

The ROC for a given $x[n]$, is defined as the range of z for which the Z-transform converges.

$$\sum_{n=-\infty}^{\infty} |x[n]\, z^{-n}| < \infty \quad (3)$$

The ROC cannot contain any poles.By definition a pole is a where $X(z)$ is infinite. Since $X(z)$ must be finite for all z for convergence, there cannot be a pole in the ROC
If $x[n]$ is a finite-duration sequence, then the ROC is the entire z-plane, except possibly $z = 0$ or $|z| = \infty$.
If $x[n]$ is a right-sided sequence, then the ROC extends outward from the outermost pole in X(z).
If $x[n]$ is a left-sided sequence, then the ROC extends inward from the innermost pole in X(z).
If $x[n]$ is a two-sided sequence, the ROC will be a ring in the z-plane that is bounded on the interior and exterior by a pole.
Poles of $X(z)$ are located at $z = \frac{1}{5}$, $z = \frac{2}{3}$, and $z = 3$.
For DTFT to converge, the ROC of Z-transform of $x[n]$ should contain unit circle.

(a) If ROC is $|z| > 3$, it does not include unit circle
Option (a) is wrong.

(b) If ROC is $\frac{2}{3} < |z| < 3$, the ROC includes unit circle.
So, option (b) is correct.

(c) If $x(n)$ is a left-sided sequence, then ROC will be $|z| < \frac{1}{5}$, which does not include the unit circle.
Option (c) is wrong.

(d) If $x(n)$ is a right-sided sequence, then the ROC is $|z| > 3$, which does not include the unit circle.
Option (d) is wrong.

Hence, the correct option is (b).