Contents

1

Test Functions and Distributions

1.1 Introduction

This chapter gives an introduction to theory of distributions. Distributions are introduced as functionals on very good function spaces, called test function spaces. Properties of test functions and distributions frequently used in the literature are discussed here without making use of the underlying topological vector space theory; which will be given in Chapter 8.

The n-dimensional notation and terminology used in this book will be that of Schwartz (1978), Zemanian (1965) and Pathak (1997). If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, then $|x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$. If $y \in \mathbb{R}^n$, then the sum $x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$ and the inner product

$$(x, y) = x_1y_1 + x_2y_2 + \ldots + x_ny_n$$

scalar product

$$cx = (cx_1, cx_2, \ldots, cx_n),\ c \in \mathbb{R}$$

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, then $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ and for $x \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}, \ldots x_n^{\alpha_n}$. For $\alpha, \beta \in \mathbb{N}^n$, the notation $\alpha \le \beta$ means $\alpha_j \le \beta_j$, $1 \le j \le n$. We set

$$\alpha! = \alpha_1!\alpha_2! \ldots \alpha_n!$$

and

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\cdots\binom{\alpha_n}{\beta_n} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$$

For $\alpha \in \mathbb{N}_0^n$, we write

$$D^\alpha = (\partial/\partial x_1)^{\alpha_1}(\partial/\partial x_2)^{\alpha_2}\ldots(\partial/\partial x_n)^{\alpha_n}$$

We shall occasionally use the n-dimensional Leibnitz formula:

$$D^{\alpha}(uv) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^{\beta}u\, D^{\alpha-\beta}v \tag{1.1.1}$$

for $u, v \in C^{|\alpha|}(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$.

The proof of (1.1.1) is given by using induction on the order $|\alpha|$ of α. For $|\alpha|=0$ the formula (1.1.1) reduces to the identity $uv=uv$. Let us assume that the formula holds for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| = m$. Let $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{N}^n$ be a multiindex of order $|\tau| = m+1$. Since $|\tau| \ge 1$, one of the components of τ, say $\tau_1 \ge 1$. Let $\alpha \in \mathbb{N}^n$ be such that $\alpha_1 = \tau_1 - 1$, $\alpha_j = \tau_j$ for $2 \le j \le n$. Then $|\alpha| = m$ and $D^{\tau} = (\partial/\partial x_1)\, D^{\alpha}$.

Now, using the induction hypothesis and the formula

$$(\partial/\partial x_1)(uv) = (\partial/\partial x_1)\, u\cdot v + u\cdot(\partial/\partial x_1)v$$

we obtain

$$\begin{aligned}
D^{\tau}(uv) &= (\partial/\partial x_1)\sum_{\beta\le\alpha}\binom{\alpha}{\beta} D^{\beta}u\cdot D^{\alpha-\beta}v\\
&= \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\{(\partial/\partial x_1)D^{\beta}u\cdot D^{\alpha-\beta}v + D^{\beta}u\cdot(\partial/\partial x_1)D^{\alpha-\beta}v\}\\
&= \sum\binom{\alpha_2}{\beta_2}\cdots\binom{\alpha_n}{\beta_n}\Bigg\{\binom{\alpha_1}{\beta_1}(\partial/\partial x_1)^{\beta_1+1}(\partial/\partial x_2)^{\beta_2}\ldots(\partial/\partial x_n)^{\beta_n}u\\
&\qquad \times D^{\alpha-\beta}v + \binom{\alpha_1}{\beta_1} D^{\beta}u\cdot(\partial/\partial x_1)^{\alpha_1-\beta_1+1}\\
&\qquad \times(\partial/\partial x_2)^{\alpha_2-\beta_2}\ldots(\partial/\partial x_n)^{\alpha_n-\beta_n}\Bigg\}\\
&= \sum\binom{\tau_2}{\beta_2}\cdots\binom{\tau_n}{\beta_n}\Bigg\{\binom{\tau_1-1}{\beta_1-1}+\binom{\tau_1-1}{\beta_1}\Bigg\} D^{\beta}uD^{\tau-\beta}v\\
&= \sum_{\beta\le\tau}\binom{\tau}{\beta} D^{\beta}u\cdot D^{\tau-\beta}v
\end{aligned}$$

■

1.2 Test Functions

If ϕ is a function defined on an open subset Ω of $\mathbb{R}^n$, the closure of the set $\{x\in\Omega : \phi(x)\ne 0\}$ is called the *support* of the function ϕ and denoted by supp ϕ.

The set of all complex valued infinitely differentiable functions ϕ defined on Ω and having compact support is denoted by $\mathscr{D}(\Omega)$ (or $C_c^{\infty}(\Omega)$). It is a linear space. Its elements are called *Schwartz test functions*, or simply test functions.

If K is a fixed compact subset of Ω we shall denote by $\mathscr{D}_K(\Omega)$ the space of all $\phi \in \mathscr{D}(\Omega)$ such that supp $\phi \subset K$.

An example of a test function in $\mathscr{D}(\mathbb{R}^n)$ is

$$\rho(x) = \begin{cases} 0 & |x| \geq 1 \\ \exp\left(\dfrac{1}{|x|^2 - 1}\right) & |x| < 1 \end{cases} \tag{1.2.1}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $|x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$.

Using this function we can construct infinitely many test functions. For $\varepsilon > 0$, define

$$\psi_\varepsilon(x) = \frac{\rho(x/\varepsilon)}{\displaystyle\int_{\mathbb{R}^n} \rho(x/\varepsilon)\, dx} \tag{1.2.2}$$

Then $\psi_\varepsilon(x) \in \mathscr{D}(\mathbb{R}^n)$ with supp $\psi_\varepsilon(x) = \{x : |x| \leq \varepsilon\}$. Moreover,

$$\int_{\mathbb{R}^n} \psi_\varepsilon(x)\, dx = 1 \tag{1.2.3}$$

An approximation property of test functions on $\mathbb{R}$ is given by the following.

Theorem 1.2.1 Any complex-valued continuous function on $\mathbb{R}$ having compact support can be approximated by test functions.

Proof Let f be a continuous function on $\mathbb{R}$ which vanishes outside the interval (a, b). For $\varepsilon > 0$ define

$$f_\varepsilon(x) = \int_{-\infty}^{\infty} f(y)\, \psi_\varepsilon(x - y)\, dy \tag{1.2.4}$$

The right-hand side is called *convolution* of f and ψ_ε, denoted by $(f * \psi_\varepsilon)(x)$; see Chapter 4. Then $f_\varepsilon(x)$ is also a test function which can be shown as follows: Since $\psi_\varepsilon(x) = 0$ outside the interval $-\varepsilon < x < \varepsilon$, $\psi_\varepsilon(x - y) = 0$ for x lying outside $y - \varepsilon < x < y + \varepsilon$. But $f(y) = 0$ for $y \leq a$ and also for $y \geq b$. Hence $f_\varepsilon(x) = 0$ for x lying outside $a - \varepsilon < x < b + \varepsilon$.

Furthermore, we can write

$$f_\varepsilon(x) = \int_a^b f(y)\, \psi_\varepsilon(x - y)\, dy \tag{1.2.5}$$

in which the integrand is a continuous function of $(x, y) \in [a - \varepsilon, b + \varepsilon] \times [a, b]$ and hence is uniformly continuous there. Therefore, we can differentiate the integrand partially with respect to x. Since ψ_ε is a test function the partial derivative of the integrand is also continuous on $[a-\varepsilon, b+\varepsilon] \times [a, b]$ and can be differentiated further using the previous argument. The process can be carried out indefinitely. Thus $f_\varepsilon(x) \in \mathscr{D}(\mathbb{R})$.

Now, we show the approximation property. In view of (1.2.3) we have

$$|f(x) - f_\varepsilon(x)| = \left| \int_{-\infty}^{\infty} [f(x) - f(y)]\, \psi_\varepsilon(x-y)\, dy \right|$$

$$\leq \int_{x-\varepsilon}^{x+\varepsilon} |f(x) - f(y)|\, \psi_\varepsilon(x-y)\, dy$$

For x lying outside $(a-\varepsilon, b+\varepsilon)$, $f(x) - f_\varepsilon(x) = 0$. Hence we consider the case $a - \varepsilon < x < b + \varepsilon$. Then

$$|f(x) - f_\varepsilon(x)| \leq \int_{a-2\varepsilon}^{b+2\varepsilon} |f(x) - f(y)|\, \psi_\varepsilon(x-y)\, dy \qquad (1.2.6)$$

Since $f(y)$ is continuous on $[a-2\varepsilon, b+2\varepsilon]$, it is uniformly continuous there. Hence, for given $\eta > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \eta$ for all pairs of x and y in this interval such that $|x - y| < \delta$. So that

$$|f(x) - f_\varepsilon(x)| \leq \eta \int_{a-2\varepsilon}^{b+2\varepsilon} \psi_\varepsilon(x-y)\, dy$$

$$\leq \eta \int_{-\infty}^{\infty} \psi_\varepsilon(x-y)\, dy = \eta$$

for $|x-y| < \delta$. If we choose $\varepsilon < \delta$, then we find that for $|x - y| < \varepsilon$, $|f(x) - f_\varepsilon(x)| < \eta$ and by definition of $\psi_\varepsilon(x)$ the right-hand side of (1.2.6) is zero for $|x-y| \geq \varepsilon$. Thus $\lim_{\varepsilon \to 0} |f(x) - f_\varepsilon(x)| = 0$ uniformly for all x.

■

Remark 1.2.2 Assume that $f(x)$ is a continuous function on $\mathbb{R}$ such that $0 \leq f(x) \leq 1$. Let $f(x)$ equal 1 on $a - \eta \leq x \leq b + \eta$, $\eta > 0$, and $f(x) = 0$ outside some larger finite interval. Then $f_\varepsilon(x)$ defined by (1.2.5) is a test function with the property that $f_\varepsilon(x) = 1$ for $a \leq x \leq b$ and $0 \leq f_\varepsilon(x) \leq 1$.

An L^p-analogue of Theorem 1.2.1 is given by:

Lemma 1.2.3 Let $\psi \in \mathscr{D}(\mathbb{R}^n)$ be such that

$$\psi \geq 0, \int_{\mathbb{R}^n} \psi(x)\, dx = 1$$

and let $\psi_\varepsilon(x) = \varepsilon^{-n} \psi(x/\varepsilon)$, $\varepsilon > 0$. Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and put

$$f_\varepsilon(x) = (f * \psi_\varepsilon)(x) = \int_{\mathbb{R}^n} f(y)\ \psi_\varepsilon(x-y)\ dy \tag{1.2.7}$$

Then $f_\varepsilon \in C^\infty(\mathbb{R}^n)$, and $f_\varepsilon \to f$ in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0$. Also,

$$\| f_\varepsilon \|_p \le \| f \|_p$$

Proof Since $f \in L^p(\mathbb{R}^n)$, $\psi \in \mathscr{D}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ and by Hölder's inequality,

$$\int_{\mathbb{R}^n} |f(y)\psi_\varepsilon(x-y)|\, dy \le \left(\int_{\mathbb{R}^n} |f(y)|^p\, dy\right)^{1/p} \left(\int_{\mathbb{R}^n} |\psi_\varepsilon(x-y)|^q\, dy\right)^{1/q}$$

it follows that the right-hand side of (1.2.7) exists.

Moreover, since $\psi \in \mathscr{D}$, for every multi-index α, $(\partial/\partial x)^\alpha\ \psi_\varepsilon(x-y) \in L^q(\mathbb{R}^n)$, where $1/p + 1/q = 1$. Hence, by the Hölder's inequality, $f * (\partial/\partial x)^\alpha\ \psi_\varepsilon(x)$ exists for every $x \in \mathbb{R}^n$. Therefore, differentiation and integration can be interchanged and we have $f_\varepsilon(x) \in C^\infty(\mathbb{R}^n)$.

Now, again using Hölder's inequality we have

$$|f_\varepsilon(x)| = \left| \int_{\mathbb{R}^n} f(y)\psi_\varepsilon^{1/p}(x-y)\ \psi_\varepsilon^{1/q}(x-y)\, dy \right|$$

$$\le \left(\int_{\mathbb{R}^n} |f(y)|^p\ \psi_\varepsilon(x-y)\, dy\right)^{1/p} \left(\int_{\mathbb{R}^n} \psi_\varepsilon(x-y)\, dy\right)^{1/q}$$

$$\le \left(\int_{\mathbb{R}^n} |f(x-y)|^p\ \psi_\varepsilon(y)\, dy\right)^{1/p}$$

Hence, using Fubini's theorem and interchanging order of integration, we get

$$\int_{\mathbb{R}^n} |f_\varepsilon(x)|^p\, dx \le \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)|^p\ \psi_\varepsilon(y)\, dy\right) dx$$

$$= \int_{\mathbb{R}^n} \psi_\varepsilon(y)\, dy \int_{\mathbb{R}^n} |f(x-y)|^p\, dx$$

Thus $\qquad \| f_\varepsilon \|_p \le \| f \|_p$

A similar calculation yields

$$|f(x) - f_\varepsilon(x)|^p = \left| \int_{\mathbb{R}^n} (f(x) - f(x-y))\, \psi_\varepsilon(y)\, dy \right|^p$$

$$\leq \int_{\mathbb{R}^n} |f(x) - f(x-y)|^p\, \psi_\varepsilon(y)\, dy$$

Again, using Fubini's theorem we have

$$\|f - f_\varepsilon\|_p^p \leq \int_{\mathbb{R}^n} \|\tau_y f - f\|_p^p\, \psi_\varepsilon(y)\, dy$$

$$= \int_{\mathbb{R}^n} \|\tau_{\varepsilon y} f - f\|_p^p\, \psi(y)\, dy \tag{1.2.8}$$

where $\tau_h f = f(x-h)$.

Next we show the continuity of translation in $L^p(\mathbb{R}^n)$. Since the space of continuous functions of compact support $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, for given $\delta > 0$ there exists $f_\delta \in C_c(\mathbb{R}^n)$ such that $\|f - f_\delta\|_p \leq \delta$. For any $h \in \mathbb{R}^n$ we then have

$$\|\tau_h f - f\|_p = \|\tau_h f - \tau_h f_\delta + \tau_h f_\delta + f_\delta - f - f_\delta\|_p$$

$$\leq \|\tau_h f - \tau_h f_\delta\|_p + \|f_\delta - f\|_p + \|\tau_h f_\delta - f_\delta\|_p$$

$$\leq 2\delta + \|\tau_h f_\delta - f_\delta\|_p$$

Now $\|\tau_h f_\delta - f_\delta\|_p \to 0$ as $h \to 0$, by uniform continuity of f_δ. Therefore, $\|\tau_h f - f\|_p \to 0$ as $h \to 0$.

We note that

$$0 \leq \|\tau_{\varepsilon y} f - f\|_p^p\, \psi(y) \leq (\|f\|_p + \|\tau_{\varepsilon y} f\|_p)^p\, \psi(y)$$

$$= 2^p \|f\|_p^p\, \psi(y)$$

Hence we can apply the dominated convergence theorem to (1.2.8) and then by the aforesaid continuity of translation, get

$$\|f - f_\varepsilon\|_p \to 0 \text{ as } \varepsilon \to 0$$ ■

Theorem 1.2.4 $\mathcal{D}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

Proof Let $f \in L^p(\mathbb{R}^n)$ be given, and define f_ε as in (1.2.7). Choose some $\phi \in \mathcal{D}(\mathbb{R}^n)$ such that $0 \leq \phi \leq 1$ and $\phi = 1$ for $|x| \leq 1$. Set $g_\varepsilon(x) = \phi(\varepsilon x) f_\varepsilon(x)$. Then $g_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$, and

$$\|f - g_\varepsilon\|_p = \|f(x) - f(x)\,\phi(\varepsilon x) + f(x)\,\phi(\varepsilon x) - f_\varepsilon(x)\,\phi(\varepsilon x)\|_p$$

$$\leq \| (f - f_\varepsilon)\, \phi(\varepsilon x) \|_p + \| (\phi(\varepsilon x) - 1) f \|_p$$

$$\leq \| f - f_\varepsilon \|_p + \left(\int_{|x| > \varepsilon^{-1}} | f(x) |^p \, dx \right)^{1/p} \to 0 \text{ as } \varepsilon \to 0$$

by Lemma 1.2.3 and the L^p-convergence of f. ■

The following example provides a motivation to the theorem that follows.

Example 1.2.5 Construct a sequence of test functions $\{\phi_n\}_{n \in \mathbb{Z}}$ in $\mathscr{D}(\mathbb{R})$ such that

(i) For each $x \in \mathbb{R}$, the set $\{n \in \mathbb{Z}: \phi_n(x) \neq 0\}$ is finite.

(ii) $\sum_{n=-\infty}^{\infty} \phi_n(x) = 1 \ \forall \ x \in \mathbb{R}$.

Let $\theta \in \mathscr{D}(\mathbb{R})$, supp $\theta \subset [0, 2]$, $\theta \geq 0$, $\theta(x) = 1$ if $\frac{1}{2} \leq x \leq 3/2$. Let us set

$$\psi(x) = \sum_{n=-\infty}^{\infty} \theta(x + n)$$

Then $\psi(x) \geq 1$ and the sum defining $\psi(x)$ is locally finite, so $\psi \in C^\infty(\mathbb{R})$. Therefore $\phi(x) = \dfrac{\theta(x)}{\psi(x)}$ is an element of $\mathscr{D}(\mathbb{R})$. Moreover,

$$\phi_n(x) = \phi(x + n) = \frac{\theta(x + n)}{\psi(x + n)} = \frac{\theta(x + n)}{\psi(x)}$$

since $\quad \psi(x + k) = \sum_{n=-\infty}^{\infty} \theta(x + k + n) = \sum_{m=-\infty}^{\infty} \theta(x + m) = \psi(x)$

Therefore $\quad \sum_{-\infty}^{\infty} \phi_n(x) = \dfrac{\sum \theta(x + n)}{\psi(x)} = \dfrac{\psi(x)}{\psi(x)} = 1$

We give a construction of a *partition of unity* in Ω, an open subset of $\mathbb{R}^n$, which will be used in the sequel.

Theorem 1.2.6 Let K be a compact subset of Ω and let Ω_j, $1 \leq j \leq k$, be a finite open covering of K, i.e. $K \subset \bigcup_{j=1}^{k} \Omega_j$. Then there are functions $\phi_j \in \mathscr{D}(\Omega_j)$ such that $0 \leq \phi_j \leq 1$, $1 \leq j \leq k$, and $\sum_{j=1}^{k} \phi_j = 1$ on neighbourhood of K.

Proof We can find compact sets K_j, $j = 1, \ldots, k$ such that $K_j \subset \Omega_j$ and

$K \subset \bigcup_{j=1}^{k} \dot{K}_j$, where $\dot{K}_j$ denotes the interior of K_j. For every j, let $\psi_j \in \mathscr{D}(\Omega_j)$ be such that $0 \le \psi_j \le 1$ and $\psi_j = 1$ on K_j. Now, set

$$\phi_1 = \psi_1$$

$$\phi_2 = \psi_2(1 - \psi_1), \ldots,$$

$$\phi_k = \psi_k(1 - \psi_1) \ldots (1 - \psi_{k-1})$$

Then $\phi_j \in \mathscr{D}(\Omega_j)$, $0 \le \phi_j \le 1$ for $j = 1, 2, \ldots, k$ and

$$\sum_{j=1}^{k} \phi_j = \psi_1 + \psi_2(1 - \psi_1) + \ldots + \psi_k(1 - \psi_1) \ldots (1 - \psi_{k-1}) = 1$$

on a neighbourhood of K. For instance, if $x \in \dot{K}_3$, then $\psi_3(x) = 1$, and so $\phi_1 + \phi_2 + \phi_3 = \psi_1 + \psi_2(1 - \psi_1) + (1 - \psi_1)(1 - \psi_2) = 1$, but

$$\sum_{j=4}^{k} \phi_j = \sum_{j=4}^{k} \psi_j(1 - \psi_1)(1 - \psi_2)(1 - \psi_3) \ldots (1 - \psi_{j-1}) = 0 \qquad \blacksquare$$

The set of functions $\{\psi_1, \psi_2, \ldots, \psi_k\}$ is called a C^∞ *partition of unity* subordinate to the open cover $\{\Omega_1, \Omega_2, \ldots, \Omega_k\}$ of K.

1.2.1 Convergence in $\mathscr{D}(\Omega)$

Now, we define the convergence in the space $\mathscr{D}(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$. A sequence of test function $\{\phi_n(x)\}_{n\in\mathbb{N}}$ is said to converge in $\mathscr{D}(\Omega)$ to zero if

(i) supports of all $\phi_n(x)$ are contained in a fixed bounded domain in Ω, and

(ii) for each n-tuple $k = (k_1, \ldots, k_n)$ of nonnegative integers the sequence of partial derivatives $D^k\phi_n(x)$ converges uniformly to zero over all of Ω.

For example, the sequence $\phi_n(x) = \frac{1}{n}\rho(x)$, where ρ is given by (1.2.1), belongs to $\mathscr{D}(\mathbb{R}^n)$ and converges in $\mathscr{D}(\mathbb{R}^n)$ to zero as $n \to \infty$. But the sequence $\phi_n(x) = \rho(x/n) \in \mathscr{D}(\mathbb{R}^n)$ does not converge in $\mathscr{D}$ as $n \to \infty$ because there is no fixed bounded domain in $\mathbb{R}^n$ outside which all $\phi_n(x)$ vanish.

1.3 Distributions

A mapping $f : \mathscr{D}(\Omega) \to \mathbb{C}$ is called functional. The number assigned by f to a test function $\phi \in \mathscr{D}(\Omega)$ is denoted by $\langle f, \phi\rangle$. A functional on $\mathscr{D}$ is said to be linear if

$$\langle f, \alpha\phi_1 + \beta\phi_2 \rangle = \alpha \langle f, \phi_1 \rangle + \beta \langle f, \phi_2 \rangle$$

for any two complex numbers α, β and test functions ϕ_1, ϕ_2. The linear functional f is said to be continuous, if for any sequence of test functions $\{\phi_n\}_{n\in \mathbb{N}}$ that converges in $\mathscr{D}$ to zero, the sequence of numbers $\{\langle f, \phi_n \rangle\}_{n\in \mathbb{N}}$ converges to zero.

A continuous linear functional on $\mathscr{D}(\Omega)$ is called a *distribution*. The space of all such distributions (also called Schwartz distributions) is denoted by $\mathscr{D}'(\Omega)$ or simply by $\mathscr{D}'$. The space $\mathscr{D}'$ is called the dual space of $\mathscr{D}$; see Chapter 8.

Now, we give some examples of distributions.

Example 1.3.1 Let f be a locally integrable function on $\mathbb{R}$, i.e., absolutely integrable on every finite interval in $\mathbb{R}$. Then f generates (defines) a distribution as follows:

$$\langle f, \phi \rangle = \int_{-\infty}^{\infty} f(t)\, \phi(t)\, dt,\ \phi \in \mathscr{D}(\mathbb{R}) \tag{1.3.1}$$

The distribution, generated by the function f, is also denoted by f, for the sake of convenience.

Clearly f is linear on $\mathscr{D}$. To show continuity of f, let $\{\phi_n\}_{n\in \mathbb{N}}$ be a sequence in $\mathscr{D}(\mathbb{R})$ that converges to zero in $\mathscr{D}$ such that supp $\phi_n \subset [a, b]$. Then

$$\begin{aligned} |\langle f, \phi_n \rangle| &\le \int_a^b |f(t)\, \phi_n(t)|\, dt \\ &\le \sup_{a\le t\le b} |\phi_n(t)| \int_a^b |f(t)|\, dt \to 0 \text{ as } n \to \infty \end{aligned}$$

since $\{\phi_n\}$ converges to zero in $\mathscr{D}$.

Distributions generated by locally integrable functions are called *regular distributions*.

Example 1.3.2 For complex λ with Re $\lambda > -1$, the functions

$$x_+^\lambda = \begin{cases} x^\lambda & x > 0 \\ 0 & x \le 0 \end{cases}, \quad x_-^\lambda = \begin{cases} |x|^\lambda & x < 0 \\ 0 & x \ge 0 \end{cases}$$

define regular distributions on $\mathbb{R}$ because the functions x_+ and x_- are locally integrable.

Example 1.3.3 The Dirac delta functional is defined by

$$\langle \delta, \phi \rangle = \phi(0)$$

for all $\phi \in \mathscr{D}(\mathbb{R})$. That δ is a distribution can be shown as follows.

Let $\alpha, \beta \in \mathbb{C}$ and $\phi_1, \phi_2 \in \mathscr{D}(\mathbb{R})$. Then $\alpha\phi_1 + \beta\phi_2 \in \mathscr{D}$ and hence

$$\begin{aligned}\langle \delta, \alpha\phi_1 + \beta\phi_2 \rangle &= \alpha\phi_1(0) + \beta\phi_2(0)\\ &= \alpha\langle f, \phi_1\rangle + \beta\langle f, \phi_2\rangle\end{aligned}$$

To show continuity of δ assume that $\{\phi_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathscr{D}(\mathbb{R})$ that converges to zero in the sense of convergence in $\mathscr{D}$. Then

$$\langle \delta, \phi_n\rangle = \phi_n(0) \to 0 \text{ as } n \to \infty$$

Thus δ is a continuous linear functional on $\mathscr{D}$.

It may be remarked that δ is not a regular distribution. For if it were so then we would have

$$\phi(0) = \langle \delta, \phi\rangle = \int_{-\infty}^{\infty} \delta(t)\phi(t)\, dt \ \forall\ \phi \in \mathscr{D}(\mathbb{R}) \tag{1.3.3}$$

Choose $\phi(t) = \rho(t/a)$, $a > 0$, where ρ is given by (1.2.1). Then (1.3.3) gives

$$e^{-1} = \int_{-a}^{a} \delta(t) \exp(-a^2/(a^2 - t^2))\, dt \tag{1.3.4}$$

Since $a^2/(a^2 - t^2) \geq 1$ for $|t| < a$ we have $\exp(-a^2/(a^2 - t^2)) \leq e^{-1}$. Hence if δ were locally integrable then from (1.3.4),

$$e^{-1} \leq e^{-1} \int_{-a}^{a} |\delta(t)|\, dt \to 0 \text{ as } a \to 0+$$

which is impossible. Hence δ is not a regular distribution. Distributions which are not regular called *singular*.

The functional δ defined on $\mathscr{D}(\mathbb{R}^n)$ by

$$\langle \delta, \phi\rangle = \phi(0, 0, \ldots, 0)$$

can similarly be shown to be a distribution in $\mathscr{D}'(\mathbb{R}^n)$

Example 1.3.4 Finite part of divergent integrals

(1) Consider the usually divergent integral

$$\int_0^{\infty} t^{-3/2} \phi(t)\, dt,\ \phi \in \mathscr{D}(0, \infty) \tag{1.3.5}$$

We show that a certain finite part of this divergent integral defines a singular distribution. Let

$$\phi(t) = \phi(0) + t\psi(t)$$

where

$$\psi(t) = \phi'(0) + \frac{t}{2!}\phi''(0) + \frac{t^2}{3!}\phi'''(0) + \ldots + \frac{t^{n-1}}{n!}\phi^{(n)}(\xi), 0 < \xi < 1$$

is a continuous function of $t > 0$. Assume that the supp $\phi(t) \subset [0, b]$, $b > 0$. Then

$$\int_0^\infty t^{-3/2}\,\phi(t)\,dt = \lim_{\varepsilon\to 0^+}\int_\varepsilon^b t^{-3/2}\,\phi(t)\,dt$$

$$= \lim_{\varepsilon\to 0^+}\left[2\phi(0)\,(\varepsilon^{-1/2} - b^{-1/2}) + \int_\varepsilon^b \psi(t)\,t^{-1/2}\,dt\right]$$

The last integral is convergent. If $\phi(0) \neq 0$ we throw away the divergant part $2\phi(0)\ \varepsilon^{-1/2}$. then the *Hadamard's finite part* of the divergent integral (1.3.5) is given by

$$Fp\int_0^\infty t^{-3/2}\,\phi(t)\,dt = \int_0^b \psi(t)\,t^{-1/2}\,dt - 2\phi(0)\,b^{-1/2} \qquad (1.3.6)$$

It can be shown that (1.3.6) defines a distribution. Linearity of the functional is obvious. To show the continuity we note that

$$\left|\int_0^b \psi(t)\,t^{-1/2}\,dt\right| = \left|\int_0^b \frac{\phi(t) - \phi(0)}{t^{3/2}}\,dt\right|$$

$$\leq \left|\int_0^b t^{-3/2}\,dt\int_0^t \phi'(u)\,du\right|$$

$$\leq \sup_{0\leq u\leq b}\,|\,\phi'(u)\,|\,2b^{1/2}$$

Therefore, if $\{\phi_n\}$ converges in $\mathscr{D}$ to zero, then $\phi_n(0) \to 0$ and $\int_0^b \psi_n(t)$ $\times\, t^{-1/2}\,dt \to 0$ as $n \to \infty$. Thus $\left\{Fp\int_0^\infty t^{-1/2}\phi_n(t)\,dt\right\}_{n=1}^\infty$ converges to zero. Therefore (1.3.5) defines a distribution which is denoted by $Pf\,t^{-3/2}\,H(t)$, called a *pseudofunction*. We write

$$\langle Pf\,t^{-3/2}\,H(t),\,\phi(t)\rangle = Fp\int_0^\infty t^{-3/2}\phi(t)\,dt$$

where $H(t)$ is the *Heaviside unit function* defined by

$$H(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

(2) Following the above procedure it can be shown that for $0 < \alpha < 1$,

$$\langle Pf\, t^{-\alpha-1}H(t),\, \phi(t)\rangle = \int_0^\infty t^{-\alpha-1}\,[\phi(t) - \phi(0)]\,dt,\ \phi \in \mathscr{D}(0, \infty) \quad (1.3.7)$$

defines the distribution $Pft^{-\alpha-1}H(t)$.

Example 1.3.5 The *Cauchy principal value* of the (usually) divergent integral

$$\int_{-\infty}^{\infty} \phi(x)/x\,dx,\ \phi \in \mathscr{D}(\mathbb{R})$$

is defined by

$$Pv\int_{-\infty}^{\infty} \phi(x)/x\,dx := \lim_{\varepsilon\to 0+}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right)\phi(x)/x\,dx$$

This defines a distribution in $\mathscr{D}'(\mathbb{R})$ which is denoted by $Pv(1/x)$.

For, let $\phi \in \mathscr{D}(\mathbb{R})$ with supp $\phi \subset [-a, a]$, $a > 0$. Using Taylor expansion we can write

$$\phi(x) = \phi(0) + x\psi(x)$$

where $\psi(x)$ is continuous on $\mathbb{R}$. Then

$$\int_{\varepsilon}^{a} \phi(x)/x\,dx = \phi(0)\int_{\varepsilon}^{a} dx/x + \int_{\varepsilon}^{a} \psi(x)\,dx$$

and

$$\int_{-a}^{-\varepsilon} \phi(x)/x\,dx = \phi(0)\int_{-a}^{-\varepsilon} dx/x + \int_{-a}^{-\varepsilon} \psi(x)\,dx$$

$$= -\phi(0)\int_{\varepsilon}^{a} dx/x + \int_{-a}^{-\varepsilon} \psi(x)\,dx$$

so that

$$\left(\int_{-a}^{-\varepsilon} + \int_{\varepsilon}^{a}\right)\phi(x)/x\,dx = \int_{-a}^{-\varepsilon} \psi(x)\,dx + \int_{\varepsilon}^{a} \psi(x)\,dx$$

Since $\psi(x)$ is continuous at $x = 0$, we have

$$Pv\int_{-\infty}^{\infty} \phi(x)/x\,dx = \lim_{\varepsilon\to 0+}\left(\int_{-a}^{-\varepsilon} + \int_{\varepsilon}^{a}\right)\phi(x)/x\,dx = \int_{-a}^{a} \psi(x)\,dx$$

Thus

$$\langle Pv(1/x),\, \phi(x)\rangle = \int_{-a}^{a} \psi(x)\,dx \quad (1.3.8)$$

Clearly, $Pv(1/x)$ is linear on $\mathscr{D}(\mathbb{R})$. To show its continuity, assume that $\{\phi_n(x)\}_{n\in\mathbb{N}_0}$ is a sequence in $\mathscr{D}(\mathbb{R})$ which converges in $\mathscr{D}(\mathbb{R})$ to zero. Let supp $\phi_n(x) \subset [-a, a]$, $a > 0$ for all $n \in \mathbb{N}_0$. Then, as in the above, we can write

$$
\begin{aligned}
|\langle Pv(1/x), \phi_n(x)\rangle| &= \left|\int_{-a}^{a} \psi_n(x)\,dx\right| \\
&= \left|\int_{-a}^{a} \frac{\phi_n(x)-\phi(0)}{x}\,dx\right| \\
&\leq \int_{-a}^{a} \frac{dx}{|x|}\left|\int_0^x |\phi_n'(t)|\,dt\right| \\
&\leq \sup_{-a\leq t\leq a} |\phi_n'(t)| \int_{-a}^{a} \frac{dx}{|x|}|x| \\
&= 2a \sup_{-a\leq t\leq a} |\phi_n'(t)| \to 0 \text{ as } n\to\infty
\end{aligned}
$$

Therefore $Pv(1/x) \in \mathscr{D}'(\mathbb{R})$.

1.4 Operations on Distributions

In this section several operations, defined in a well-known way for functions, are extended to distributions in $\mathscr{D}'(\mathbb{R}^n)$.

(i) *Addition* The sum of two distributions f and g in $\mathscr{D}'(\mathbb{R}^n)$ is defined by

$$\langle f+g, \phi\rangle = \langle f, \phi\rangle + \langle g, \phi\rangle \ \forall\ \phi \in \mathscr{D}(\mathbb{R}^n)$$

Note that $f+g$ is a distribution as the right-hand side defines a continuous linear functional on $\mathscr{D}$.

(ii) *Multiplication by a constant* The multiplication of a distribution f by a constant c is defined by

$$\langle cf, \phi\rangle = \langle f, c\phi\rangle \ \forall\ \phi \in \mathscr{D}(\mathbb{R}^n)$$

Clearly cf is also a distribution in $\mathscr{D}'(\mathbb{R}^n)$.

Under the above two operations the space $\mathscr{D}'$ of distributions is a linear space.

(iii) *Translation of a distribution* The shift or translation of a distribution

$f \in \mathscr{D}'(\mathbb{R}^n)$ defined on the space $\mathbb{R}^n$ of the variable t by an amount $\tau \in \mathbb{R}^n$ is defined by

$$\langle f(t-\tau), \phi(t) \rangle := \langle f(t), \phi(t+\tau) \rangle, \ \phi \in \mathscr{D}(\mathbb{R}^n)$$

Note $f(t-\tau)$ is also a distribution.

This definition is motivated by the elementary rule of change of variable in the integral

$$\int_{-\infty}^{\infty} f(t-\tau)\, \phi(t)\, dt = \int_{-\infty}^{\infty} f(t)\, \phi(t+\tau)\, dt, \quad \phi \in \mathscr{D}(\mathbb{R})$$

when f is a locally integrable function on $\mathbb{R}$.

From the above definition it follows that

$$\langle \delta(t-\tau), \phi(t) \rangle = \langle \delta(t), \phi(t+\tau) \rangle = \phi(\tau), \ \phi \in \mathscr{D}(\mathbb{R}^n)$$

(iv) *Transpose of a distribution* The transpose $f(-t)$ of a distribution $f(t)$ is the distribution defined by

$$\langle f(-t), \phi(t) \rangle = \langle f(t), \phi(-t) \rangle, \ \phi \in \mathscr{D}(\mathbb{R}^n)$$

We say that the distribution $f(t)$ is *even* if

$$f(-t) = f(t)$$

and the distribution is *odd* if

$$f(-t) = -f(t)$$

(v) *Dilation of a distribution* The dilation of a distribution $f(t) \in \mathscr{D}'(\mathbb{R}^n)$ by $a > 0$ is defined by

$$\langle f(at), \phi(t) \rangle := \langle f(t), a^{-n} \phi(t/a) \rangle, \quad \phi \in \mathscr{D}(\mathbb{R}^n)$$

This definition is also motivated from the corresponding result on integrals, viz.

$$\int_{-\infty}^{\infty} f(at)\phi(t)\, dt = \int_{-\infty}^{\infty} f(t)\phi(t/a)\, dt/a, \ \phi \in \mathscr{D}(\mathbb{R})$$

in the case when $f(t)$ is a locally integrable function on $\mathbb{R}$.

1.5 Multiplication and Division of Distributions

Multiplication and division of distributions are difficult and involved problems. In general, the product of two distributions f and g is not defined.

It turns out that the product does not always exist within the framework of distributions as can be seen from the example $f(x) = g(x) = |x|^{-1/2} \in \mathscr{D}'(\mathbb{R}^n)$ but $f(x)g(x) = |x|^{-1} \notin \mathscr{D}'(\mathbb{R})$. However, distributions can always be multiplied by a smooth function ψ as described below.

Let $f \in \mathscr{D}'(\Omega)$ and $\psi \in C^\infty(\Omega)$. Define

$$\langle \psi f, \phi \rangle := \langle f, \psi\phi \rangle \ \forall \ \phi \in \mathscr{D}(\Omega) \tag{1.5.1}$$

Note that $\psi\phi \in \mathscr{D}(\Omega)$ because the product is infinitely differentiable and it vanishes outside the support of ϕ.

That ψf is a distribution can be shown as follows: Let $\phi_1, \phi_2 \in \mathscr{D}(\Omega)$ and $a, b \in \mathbb{C}$; then

$$\begin{aligned}\langle \psi f, a\phi_1 + b\phi_2 \rangle &= \langle f, \psi(a\phi_1 + b\phi_2) \rangle \\ &= a < f, \psi\phi_1 \rangle + b \langle f, \psi\phi_2 \rangle \\ &= a \langle \psi f, \phi_1 \rangle + b \langle \psi f, \phi_2 \rangle\end{aligned}$$

Therefore ψf is a linear functional on $\mathscr{D}(\Omega)$.

To show that it is continuous assume that $\{\phi_n\}_{n \in \mathbb{N}}$ is a sequence of test functions that converges in $\mathscr{D}$ to zero. Then using Leibnitz theorem we can show that $\{\psi\phi_n\}_{n \in \mathbb{N}}$ also converges in $\mathscr{D}$ to zero. Therefore,

$$\langle \psi f, \phi_n \rangle = \langle f, \psi\phi_n \rangle \to 0 \text{ as } n \to \infty$$

Thus ψf is a continuous linear functional on $\mathscr{D}$.

The division problem is the following: Let I denote the open interval $(0, \infty)$. Given $v \in \mathscr{D}'(I)$ and $f \in C^\infty(I)$, find a distribution $u \in \mathscr{D}'(I)$ such that $fu = v$. If $f \neq 0$ on I, then $1/f(x) \in C^\infty(I)$ and $1/|f(x)|$ is bounded away from zero on every compact subset of I, and we just have $u = v/f$. But if $f^{-1}\{0\} = \{x : f(x) = 0\}$ is not empty the problem is difficult. It has been solved by Hörmander (1958) and Łojasiewicz (1959) independently.

Example 1.5.1 The product $(\cos x)\delta$ is the distribution defined on $\mathscr{D}(\mathbb{R})$ by

$$\langle (\cos x)\delta, \phi \rangle = \langle \delta, (\cos x)\phi \rangle = (\cos 0)\ \phi(0) = \phi(0) \tag{1.5.2}$$

Also $$\langle (\sin x)\delta, \phi \rangle = \langle \delta, (\sin x)\ \phi \rangle = (\sin 0)\phi(0) = 0. \tag{1.5.3}$$

1.6 Local Properties of Distributions

It does not make sense to assign a value to a distribution at a given point but a meaning is given to its value on an open set.

Definition 1.6.1 For any $f \in \mathscr{D}'(\Omega)$ and any open subset G of Ω, we say that $f = 0$ on G if $\langle f, \phi \rangle = 0$ for every $\phi \in \mathscr{D}(G)$.

With this definition we can say that $f \in \mathscr{D}'(\Omega)$ is zero if $f = 0$ on Ω. Moreover, $f, g \in \mathscr{D}'(\Omega)$ are said to be equal if $f - g = 0$ on Ω; i.e.,

$$\langle f, \phi\rangle = \langle g, \phi\rangle \;\forall\; \phi \in \mathscr{D}(\Omega)$$

called *equality* of two distributions.

Definition 1.6.2 The *support* of $f \in \mathscr{D}'(\Omega)$ is the complement in Ω of the largest open subset of Ω where $f = 0$.

In other words, supp f = complement of $\{x : f = 0$ on a neighbourhood of $x\}$.

For example, the support of the Dirac delta distribution is the origin because $\langle \delta, \phi\rangle = 0$ for every ϕ in $\mathscr{D}(\Omega - \{0\})$.

Theorem 1.6.3 Let $f \in \mathscr{D}'(\Omega)$ and $\phi \in \mathscr{D}(\Omega)$. If the supports of f and ϕ are disjoint, then $\langle f, \phi\rangle = 0$.

Proof Let K = supp ϕ. By hypothesis, every point of the compact set K has an open neighbourhood on which $f = 0$. This collection of open neighbourhoods is an open convering of K. This open covering of the compact set K contains a finite subcovering $\Omega_1, \Omega_2, \ldots, \Omega_m$. Therefore, there exist C^∞-functions $\psi_1, \psi_2, \ldots, \psi_m$ subordinated to the covering $\{\Omega_1, \ldots, \Omega_m\}$, as in Theorem 1.2.5, such that

$$\sum_{i=1}^{m} \psi_i = 1 \text{ on a neighbourhood of } K.$$

Thus $$\langle f, \phi\rangle = \left\langle f, \sum_{i=1}^{m} \phi\psi_i \right\rangle = \sum_{i=1}^{m} \langle f, \phi\psi_i\rangle = 0$$

since $\phi\psi_i \in \mathscr{D}(\Omega_i)$ and $f = 0$ on Ω_i for $i = 1, \ldots, m$ by hypothesis.

Theorem 1.6.4 Let $\{G_\alpha : \alpha \in A\}$ be a collection of open sets in Ω, and let $f \in \mathscr{D}'(\Omega)$ be zero on every G_α. Then f is zero on the union $\cup_{\alpha\in A} G_\alpha$.

Proof Let $G = \cup G_\alpha$ and $\phi \in \mathscr{D}(G)$ with supp $\phi = K$. Since G_α is an open covering of the compact set K there exists an open subcovering of K, let it be denoted by $G_1, \ldots, G_m$ after relabeling, if necessary. For every $k = 1, 2, \ldots, m$ choose a compact set $K_k \subset G_k$ so that $K \subset \bigcup_{k=1}^{m} \dot{K}_k$.

Let $\psi_k \in \mathscr{D}(G_k)$, $0 \leq \psi_k \leq 1$ for every $k = 1, 2, \ldots, m$ and $\sum_{k=1}^{m} \psi_k = 1$ on a neighbourhood of K. This is a C^∞-partition of unity subordinated to the covering $\{G_1, \ldots, G_m\}$. Then $\phi = \sum_{k=1}^{m} \phi\psi_k$.

Since $\phi\psi_k \in \mathscr{D}(G_k)$ and $f = 0$ on G_k, we have

$$\langle f, \phi \rangle = \sum_{k=1}^{m} \langle f, \phi\psi_k \rangle = 0$$

1.7 A Boundedness Property

The following theorem provides a useful characterization of $\mathscr{D}'(\Omega)$.

Theorem 1.7.1 A linear functional f in $\mathscr{D}'(\Omega)$ is a distribution if and only if, for every compact set $K \subset \Omega$, there exist a nonnegative integer k and a finite constant $C > 0$ such that

$$|\langle f, \phi \rangle| \le C \sum_{|\alpha| \le k} \sup_{x \in K} |D^\alpha \phi(x)| \quad \forall\, \phi \in \mathscr{D}_K(\Omega) \tag{1.7.1}$$

Proof If the linear functional f satisfies (1.7.1) then it is also continuous on $\mathscr{D}(\Omega)$, because if $\{\phi_n\}_{n \in \mathbb{N}}$ with supp $\phi_n \subset K$, converges in $\mathscr{D}(\Omega)$ to zero then $\langle f, \phi_n \rangle \to 0$ as $n \to \infty$. Therefore f is a distribution.

Conversely, assume that f is a distribution and we can find a compact set $K \subset \Omega$ such that (1.7.1) is not true for all $C > 0$ and every $k \in \mathbb{N}$. Choose $C = k = j$, then we can find $\phi_j \in \mathscr{D}_K(\Omega)$ such that

$$|\langle f, \phi_j \rangle| \ge j \sum_{|\alpha| \le j} \sup_{x \in K} |D^\alpha\ \phi_j(x)|$$

Let us set $\psi_j = \phi_j / |\langle f, \phi_j \rangle|$, then $|\langle f, \psi_j \rangle| = 1$ and

$$1 \ge j \sum_{|\alpha| \le j} \sup_{K} |D^\alpha \psi_j|$$

Therefore $\sup_K |D^\alpha \psi_j| \le 1/j$ for $j \ge |\alpha|$ and supp $\psi_j =$ supp $\phi_j \subset K$. Now for every α the sequence $\{D^\alpha\ \psi_j\} \to 0$ in $\mathscr{D}(\Omega)$ and have $\langle f, \psi_j \rangle \to 0$, but $\langle f, \psi_j \rangle = 1$; which is a contradiction. Therefore, if f is a distribution, it satisfies (1.7.1).

Definition 1.7.2 The smallest value of k for which the inequality (1.7.1) holds for all K is called the *order* of the distribution f.

A distribution f generated by a locally integrable function has the order zero.

Exercises

1.1 Show that the function ϕ on $\mathbb{R}$ defined by

$$\phi(x) = \exp\left[-\left(\frac{1}{x^2} + \frac{1}{(x-a)^2}\right)\right], \quad 0 \le x \le a$$
$$= 0 \quad \text{elsewhere}$$

is in $\mathscr{D}(\mathbb{R})$.

1.2 Prove that $\mathscr{D}(\Omega)$ is dense in $C_0^{(k)}(\Omega)$ for every $k \in \mathbb{N}_0$, where $C_0^{(k)}(\Omega)$ denotes the space of k times continuously differentiable functions on Ω having compact support. Let $\mathscr{D}'^k$ denote the space of continuous linear functionals on $C_0^{(k)}(\Omega)$; prove that $\mathscr{D}'^k \subset \mathscr{D}'$.

1.3 If $f(x) \in C^\infty(\mathbb{R})$, show that for any given interval $a \le x \le b$ there exists a test function in $\mathscr{D}(\mathbb{R})$ that is identical to $f(t)$ over this interval.

1.4 Prove that the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ converges in $\mathscr{D}_K(\Omega)$ to zero if and only if the sequence of positive real numbers defined by

$$\gamma_{K,n}^k(\phi) = \sup_{x\in K, |\alpha|\le k} |D^\alpha \phi_n(x)|$$

converges to zero, as $n \to \infty$ for each $k \in \mathbb{N}_0$; see Chapter 8.

1.5 Determine which of the following functionals on $\mathscr{D}(\mathbb{R})$ are distributions:

(i) $\langle f, \phi\rangle = \sum_{k=0}^{\infty} \phi(k)$,

(ii) $\langle f, \phi\rangle = \sum_{k=0}^{\infty} \phi^{(k)}(0)$,

(iii) $\langle f, \phi\rangle = \sum_{k=0}^{\infty} \phi^{(k)}(k)$, $\forall\, \phi \in \mathscr{D}(\mathbb{R})$.

Write the order of each distribution.

1.6 Let $\psi \in C^\infty(\mathbb{R})$. Prove that

$$\langle \psi\delta, \phi\rangle = \psi(0)\,\phi(0)$$

and

$$\psi(x)\,\delta(x-\xi) = \psi(\xi)\,\delta(x-\xi)$$

1.7 Prove that

$$x^n Pf(1/x) = x^{n-1},\ n \in \mathbb{N}$$

1.8 Show that the finite part of the divergent integral $\int_{-\infty}^{\infty} \phi(x)/x^2\,dx$ is given by

$$Fp\int_{-\infty}^{\infty} \frac{\phi(x)}{x^2}\,dx = \lim_{\varepsilon\to 0+}\left[\int_{-\infty}^{-\varepsilon} \frac{\phi(x)}{x^2}\,dx + \int_{\varepsilon}^{\infty} \frac{\phi(x)}{x^2}\,dx - \frac{2\phi(0)}{\varepsilon}\right]$$

Prove that the map

$$\phi : \mathscr{D}(\mathbb{R}) \to Fp\int_{-\infty}^{\infty} \phi(x)/x^2\,dx$$

defines a distribution on $\mathbb{R}$, denoted by $Pf(1/x^2)$.

1.9 Let f be a locally integrable function on $\mathbb{R}^n\setminus\{0\}$ that satisfies

$$|f(x)| \le C/|x|^m \quad \forall\, x \in [-1, 1]$$

where $C > 0$ and $m \in \mathbb{N}$. Prove that there exists a distribution $f \in \mathscr{D}'(\mathbb{R}^n)$ such that

$$\langle f, \phi\rangle = \int_{\mathbb{R}^n} f(x)\,\phi(x)\,dx$$

for every $\phi \in \mathscr{D}(\mathbb{R}^n)$ with supp $\phi \subset \mathbb{R}^n \setminus \{0\}$.

1.10 Show that there is no distribution $f \in \mathscr{D}'(\mathbb{R})$ such that

$$\langle f, \phi \rangle = \int_{-\infty}^{\infty} \exp(x^{-2})\, \phi(x)\, dx$$

for all ϕ in $\mathscr{D}(\mathbb{R} \setminus \{0\})$.

2.1 Introduction

In this chapter we introduce the notion of convergence of a sequence distributions in $\mathscr{D}'(\Omega)$, and then define the convergence of the series of distributions. Many sequences which do not converge in the classical sense, have a limit in the distributional sense. On the other hand, there exist sequences of regular distributions which converge in $\mathscr{D}'$, whereas the corresponding sequences of functions do not converge pointwise anywhere. Moreover, a sequence may converge in the classical as well as distributional sense but the two limits may not be the same. Certain divergent series may have a meaning in the distributional sense.

2.2 Convergence of a Sequence of Distributions

Let Ω be an open subset of $\mathbb{R}^n$. Then the concept of distributional convergence, also called weak-convergence is as follows:

Definition 2.2.1 A sequence of distributions $\{f_j\}_{j\in\mathbb{N}}$ in $\mathscr{D}'(\Omega)$ is said to converge to zero if and only if, for every $\phi \in \mathscr{D}(\Omega)$, the sequence of numbers $\{\langle f_j, \phi\rangle\}$ converges to zero in the ordinary sense.

We say that $f_j \to f$ in $\mathscr{D}'(\Omega)$ if the sequence $\{f_j - f\}$ converges to zero in the sense of the above definition.

Theorem 2.2.2 The space of distributions is sequentially complete.

Proof Let $\{f_j\}_{j\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{D}'(\Omega)$. Then by Theorem 1.7.1,

$$|\langle f_j, \phi\rangle| \le C \sum_{|\alpha|\le k} \sup_{x\in K} |D^\alpha \phi(x)| \quad \forall \phi \in \mathscr{D}_K(\Omega) \tag{2.2.1}$$

for some $k \in \mathbb{N}$ and some $C > 0$. Since $\{\langle f_j, \phi\rangle\}$ is a Cauchy sequence of numbers in $\mathbb{C}$, it has a limit. Let f be defined by

$$\langle f, \phi\rangle = \lim \langle f_j, \phi\rangle \quad \forall \phi \in \mathscr{D}(\Omega)$$

Then clearly f is linear on $\mathscr{D}(\Omega)$. The continuity of f follows from (2.2.1). Thus f is a distribution in $\mathscr{D}'(\Omega)$.

Remark 2.2.3 In view of the above theorem it is not necessary to know the limit in order to say that the sequence $\{f_j\}_{j\in\mathbb{N}}$ converges in $\mathscr{D}'(\Omega)$. The limit of a sequence of distributions is unique.

When the sequence $\{f_j\}_{j\in\mathbb{N}}$ of distributions is generated by locally integrable functions, one must in general distinguish between convergence in the classical sense (pointwise convergence) and distributional convergence.

Example 2.2.4 Let $f_n(x) = e^{inx}$, $n \in \mathbb{N}$.
Then $\{f_n(x)\} \in \mathscr{D}'(\mathbb{R})$ converges pointwise only when $x = 2k\pi$, $k \in \mathbb{Z}$. But it converges to zero in the distributional sense. For, let $\phi \in \mathscr{D}(\mathbb{R})$. Then, by integration by parts, we have

$$\langle f_n, \phi\rangle = \int_{-\infty}^{\infty} e^{inx}\phi(x)\,dx = -\frac{1}{in}\int_{-\infty}^{\infty} e^{inx}\phi'(x)\,dx \to 0$$

as $n \to \infty$.

The following theorem characterizes, sequences of functions whose pointwise limit coincides with the limit in $\mathscr{D}'(\Omega)$.

Theorem 2.2.5 Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of locally integrable functions on Ω which converges to f a.e. in Ω, and let $|f_n| \le g$ for some $g \in L^1_{\text{loc}}(\Omega)$, then $f_n \to f$ in $\mathscr{D}'(\Omega)$ as $n \to \infty$.

Proof Let $\phi \in \mathscr{D}(\Omega)$, then

$$\langle f_n, \phi\rangle = \int_\Omega f_n\phi\,dx$$

But $\int_\Omega |f_n\phi|\,dx \le \int_\Omega |g\phi|\,dx$, hence using the Lebesgue's dominated covergence theorem, we have

$$\lim_{n\to\infty}\langle f_n, \phi\rangle = \int_\Omega f\phi\,dx = \langle f, \phi\rangle$$

In the following example we show that convergence a.e. for a sequence of locally integrable functions does not imply its convergence in $\mathscr{D}'(\mathbb{R})$.

Example 2.2.6 Assume that for $n \in \mathbb{N}$,

$$f_n(x) = \begin{cases} n^2 & |x| < 1/n \\ 0 & |x| \geq 1/n \end{cases}$$

Clearly, $f_n(x)$ converges to zero a.e.

Now, if we choose $\phi \in \mathscr{D}(\mathbb{R})$ such that $\phi(x) = 1$ for $x \in (-1, 1)$, then

$$\langle f_n, \phi \rangle = \int_{-1/n}^{1/n} n^2 \phi(x)\, dx = n^2 \int_{-1/n}^{1/n} dx = 2n$$

which does not converge.

In the next example we show that when the sequence of functions $\{f_n\}$ converges a.e. and in $\mathscr{D}'$, the two limits may not be equal.

Example 2.2.7 Let

$$f_n(x) = \begin{cases} n & |x| < 1/(2n) \\ 0 & |x| \geq 1/(2n)\ \forall\, n \in \mathbb{N} \end{cases}$$

Then
$$\int_{-\infty}^{\infty} f_n(x)\, dx = \int_{-1/(2n)}^{1/(2n)} n\, dx = 1$$

and $f_n(x) \to 0$ for every $x \neq 0$.

Moreover, for any $\phi \in \mathscr{D}(\mathbb{R})$, we have

$$\langle f_n, \phi \rangle = \int_{-1/(2n)}^{1/(2n)} n\phi(x)\, dx = \phi(0) + n \int_{-1/(2n)}^{1/(2n)} [\phi(x) - \phi(0)]\, dx$$

But
$$\left| n \int_{-1/(2n)}^{1/(2n)} [\phi(x) - \phi(0)]\, dx \right| \leq n \int_{-1/(2n)}^{1/(2n)} \left| \int_0^x \phi'(\xi)\, d\xi \right| dx$$

$$\leq n \sup_{\xi \in \mathbb{R}} |\phi'(\xi)| \int_{-1/(2n)}^{1/(2n)} |x|\, dx$$

$$= n \sup_{\xi \in \mathbb{R}} |\phi'(\xi)|\, 2 \int_0^{1/(2n)} |x|\, dx$$

$$= n \sup_{\xi \in \mathbb{R}} |\phi'(\xi)|\, 1/(4n^2) \to 0$$

Thus
$$\lim_{n \to \infty} \langle f_n, \phi \rangle = \phi(0) = \langle \delta, \phi \rangle \qquad \text{as } n \to \infty$$

Therefore $\{f_n\}$ converges in $\mathscr{D}'$ to δ.

A sequence of functions, such as $\{f_n\}$ in the above example which converges to δ in $\mathscr{D}'(\mathbb{R})$ is called a delta-convergent sequence. The following theorem provides a simple method of constructing such sequences.

Theorem 2.2.8 Let f be a non-negative, locally integrable function on $\mathbb{R}$ satisfying

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \tag{2.2.2}$$

and let

$$f_\alpha(x) = \alpha^{-1} f(x/\alpha), \quad \alpha > 0 \tag{2.2.3}$$

Then $f_\alpha \to \delta$ in $\mathscr{D}'(\mathbb{R})$ as $\alpha \to 0+$.

Proof Using (2.2.2), for any $\phi \in \mathscr{D}(\mathbb{R})$, we have

$$|\langle f_\alpha, \phi\rangle - \phi(0)| = \left| \int_{-\infty}^{\infty} f_\alpha(x)[\phi(x) - \phi(0)]\,dx \right|$$

$$\le \int_{|x|\le r} |f_\alpha(x)[\phi(x) - \phi(0)]|\,dx + \int_{|x|>r} |f_\alpha(x)[\phi(x)$$

$$- \phi(0)]|\,dx, \quad (r > 0)$$

$$\le \sup_{|x|\le r} |\phi(x) - \phi(0)| \int_{|x|\le r} f_\alpha(x)\,dx$$

$$+ \sup_{|x|\ge r} |\phi(x) - \phi(0)| \int_{|x|>r} f_\alpha(x)\,dx$$

$$\le \sup_{|x|\le r} |\phi(x) - \phi(0)| + M \int_{|\xi|>r/\alpha} f(\xi)\,d\xi$$

where $M = \sup_{\mathbb{R}} |\phi(x) - \phi(0)|$.

Let $\varepsilon > 0$ be given. Since ϕ is continuous at 0 we can make the first term on the right-hand side less than $\varepsilon/2$ by choosing r small. Now, keeping this r fixed and choosing α small enough we can make the last integral less than $\varepsilon/(2M)$.

Thus
$$\lim_{\alpha\to 0+} \langle f_\alpha, \phi\rangle = \phi(0) = \langle \delta, \phi\rangle$$

Example 2.2.9 Since

$$\frac{1}{\pi}\int_{-\infty}^{\infty} dx/(1 + x^2) = 1$$

we define

$$f_\alpha(x) = \alpha^{-1}\frac{1}{\pi(1+(x/\alpha)^2)} = \frac{\alpha}{\pi(x^2+\alpha^2)}, \quad \alpha > 0$$

Therefore, by the above theorem,

$$\alpha/(\pi(x^2+\alpha^2)) \to \delta(x) \text{ in } \mathscr{D}'(\mathbb{R})$$

Example 2.2.10 This example provides a motivation for the study of distributions as boundary values of analytic functions. The function $f(z) = 1/z$ is analytic in the upper and lower half planes. It attains distributional boundary values on the real line as given below:

$$\lim_{y\to 0+}\frac{1}{x \pm iy} = Pv\left(\frac{1}{x}\right) \pm i\pi\delta(x) \text{ in } \mathscr{D}'(\mathbb{R}) \tag{2.2.4}$$

The above relations are called *Sokhotski-Plemelj formulae.*

To prove (2.2.4) assume that $\phi \in \mathscr{D}(\mathbb{R})$ having support in $[-A, A]$, $A > 0$. Then putting

$$\phi(x) = \phi(0) + x\psi(x)$$

where $\psi(x)$ is continuous at 0 we have

$$\begin{aligned}\int_{-A}^{A}\frac{\phi(x)}{x+iy}\,dx &= \int_{-A}^{A}\frac{\phi(0)+x\psi(x)}{x+iy}\,dx\\ &= \phi(0)\int_{-A}^{A}\frac{dx}{x+iy} + \int_{-A}^{A}\frac{x\psi(x)}{x+iy}\,dx\\ &= T_1 + T_2 \quad \text{(say)}\end{aligned}$$

Now,

$$\begin{aligned}T_1 &= \phi(0)\int_{-A}^{A}\frac{dx}{x+iy} = \phi(0)\int_{-A}^{A}\frac{x-iy}{x^2+y^2}\,dx\\ &= 0 - iy2\int_{0}^{A}\frac{dx}{x^2+y^2} = -2i\phi(0)\tan^{-1}(A/y)\\ &\to -2i\,\phi(0)\pi/2 \quad \text{as } y \to 0+\\ &= -i\pi\,\langle\delta(x), \phi(x)\rangle\end{aligned}$$

Moreover,

$$T_2 = \int_{-A}^{A}\frac{x+iy-iy}{x+iy}\psi(x)\,dx = \int_{-A}^{A}\psi(x)\,dx - iy\int_{-A}^{A}\frac{\psi(x)}{x+iy}\,dx$$

Now, let $\sup_{|x|\le A} |\psi(x)| \le M$, then

$$|-iy|\left|\int_{-A}^{A} \frac{\psi(x)}{x+iy}\,dx\right| \le |y| \int_{-A}^{A} \frac{|\psi(x)|}{(x^2+y^2)^{1/2}}\,dx$$

$$\le |y| M \int_{-A}^{A} \frac{dx}{(x^2+y^2)^{1/2}}$$

$$= M2\,|y| \sinh^{-1}(A/y) \to 0 \text{ as } \quad y \to 0+$$

Therefore, as $y \to 0+$,

$$T_2 \to \int_{-A}^{A} \psi(x)\,dx = \langle Pv(1/x), \phi(x)\rangle$$

in view of Example 1.3.5. Thus

$$\lim_{y\to 0+} \left\langle \frac{1}{x+iy}, \phi(x)\right\rangle = \lim_{y\to 0+} \int_{-A}^{A} \frac{\phi(x)}{x+iy}\,dx$$

$$= \langle Pv(1/x), \phi(x)\rangle - i\pi \langle \delta(x), \phi(x)\rangle$$

The other result of (2.2.4) is similarly proved.

Adding and subtracting the two results we get

$$Pv(1/x) = \frac{1}{2} \lim_{y\to 0+} \left[\frac{1}{x+iy} + \frac{1}{x-iy}\right] = \lim_{y\to 0+} \frac{x}{x^2+y^2} \qquad (2.2.5)$$

$$\delta(x) = \frac{1}{2i\pi} \lim_{y\to 0+} \left[\frac{1}{x-iy} - \frac{1}{x+iy}\right] = \lim_{y\to 0+} \frac{1}{\pi}\left(\frac{y}{x^2+y^2}\right) \qquad (2.2.6)$$

2.3 Convergence of a Series of Distributions

Using the definition of convergence of the sequence $\{f_n\}_{n\in\mathbb{N}}$ of distributions in $\mathscr{D}'(\Omega)$ the convergence of the series $\sum_{n=1}^{\infty} f_n$ in $\mathscr{D}'(\Omega)$ is defined. Let us denote the partial sum of the series by

$$g_N = \sum_{n=1}^{N} f_n$$

If the sequence of partial sums $\{g_N\}_{N\in\mathbb{N}}$ converges in $\mathscr{D}'(\Omega)$, then the series $\sum_{n=1}^{\infty} f_n$ is said to converge in $\mathscr{D}'(\Omega)$.

Equivalently, the series $\sum_{n=1}^{\infty} f_n$ is said to converge in $\mathscr{D}'(\Omega)$ to the distribution f if for every $\phi \in \mathscr{D}(\Omega)$,

$$\lim_{N\to\infty}\left\langle \sum_{n=1}^{N} f_n,\phi\right\rangle = \langle f,\phi\rangle \tag{2.3.1}$$

Example 2.3.1 Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers such that

$$|a_n| \le Cn^p, \quad p \in \mathbb{N}$$

Then the series $\sum_{n=-\infty}^{\infty} a_n e^{2\pi inx}$ converges in $\mathscr{D}'(\mathbb{R})$ as can be seen as follows:

Let $\phi \in \mathscr{D}(\mathbb{R})$. Then, by integration by parts $p+2$ times, for $n \neq 0$, we get

$$\left|\left\langle \sum_{n=-N}^{N} a_n e^{2\pi inx},\phi(x)\right\rangle\right| = \left|\sum_{-N}^{N} a_n \langle e^{2\pi inx},\phi(x)\rangle\right|$$

$$= \left|\sum_{-N}^{N} a_n \int_{-\infty}^{\infty} e^{2\pi inx}\phi(x)\,dx\right|$$

$$= \left|\sum_{-N}^{N} a_n \left(\frac{-1}{2\pi in}\right)^{p+2} \int_{-\infty}^{\infty} e^{2\pi inx}\phi^{(p+2)}(x)\,dx\right|$$

$$\le \frac{C}{(2\pi)^{p+2}} \int_{-\infty}^{\infty} |\phi^{(p+2)}(x)|\,dx \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} n^{-2} < \infty$$

Therefore $\sum_{n=-\infty}^{\infty} a_n e^{2\pi inx}$ is convergent in $\mathscr{D}'(\mathbb{R})$, in view of Remark 2.2.3.

Example 2.3.2. The series of deltadistributions $\sum_{n=1}^{\infty} \delta(x-n)$ is convergent in $\mathscr{D}'(\mathbb{R})$. For let $\phi \in \mathscr{D}(\mathbb{R})$, then

$$\lim_{N\to\infty}\left\langle \sum_{n=1}^{N} \delta(x-n),\phi(x)\right\rangle = \lim_{N\to\infty} \sum_{n=1}^{N} \phi(n) \tag{2.3.2}$$

The compact support property of ϕ implies that for some $n_0(\phi)$, $\phi(n) = 0$ whenever $n > n_0$. Hence, there are only a finite number of terms in the series in the right-hand side of (2.3.2).

Exercises

2.1 Show that the following regular distributions converge in $\mathscr{D}'(\mathbb{R})$ to δ-distribution:

(a) $\dfrac{\sin xt}{\pi x}$ as $t \to \infty$;

(b) $\dfrac{n}{\sqrt{\pi}} e^{-n^2 x^2}$ as $n \to \infty$;

(c) $\dfrac{n}{2} e^{-n|x|}$ as $n \to \infty$.

2.2. Prove that series $\sum_{n=1}^{\infty} a^n \delta(x-n)$, $x \in \mathbb{R}$, converges in $\mathscr{D}'(\mathbb{R})$ for all $a > 0$.

2.3. Prove that the series $\sum_{n=1}^{\infty} e^{n^2} \delta(x-n)$, $x \in \mathbb{R}$, is convergent in $\mathscr{D}'(\mathbb{R})$.

2.4. Prove that the series $\sum_{n=1}^{\infty} n^a [\delta(x - 1/n) - \delta(x + 1/n)]$, $x \in \mathbb{R}$, converges in $\mathscr{D}'(\mathbb{R}\setminus\{0\})$ for all $a \in \mathbb{R}$, and it converges in $\mathscr{D}'(\mathbb{R})$ if $a < 0$.

3
Differentiation of Distributions

3.1 Introduction

Differentiation of distributions is another operation which is beyond the purview of the classical analysis. The definition of the derivative of a distribution is so general that the derivatives of all orders of a distribution are defined, in particular we have distributional derivatives of discontinuous functions also. Using distributional derivatives it is shown that an infinite series which is not convergent in the classical sense converges in the distributional sense. Applications of the distributional derivatives are enormous; some of these will be given in Chapters 6 and 7.

3.2 Distributional Derivative

Let $f \in C^1(\mathbb{R})$ and $\phi \in \mathscr{D}(\mathbb{R})$. Then, by integration by parts, we have

$$\begin{aligned}\langle f', \phi\rangle &= \int_{-\infty}^{\infty} f'(x)\,\phi(x)\,dx \\ &= f(x)\,\phi(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\phi'(x)\,dx \\ &= -\langle f, \phi'\rangle\end{aligned}$$

This motivativates the following:

Definition 3.2.1 Let Ω be an open subset of $\mathbb{R}^n$. Then the derivative Df of $f \in \mathscr{D}'(\Omega)$ is defined by

$$\langle Df, \phi\rangle = -\langle f, D\phi\rangle,\ \phi \in \mathscr{D}(\Omega) \tag{3.2.1}$$

Sometimes f' and $d/dx\, f(x)$ are also used to denote the derivative of $f \in \mathscr{D}'(\Omega)$.

If $\alpha \in \mathbb{N}_0^n$, then by induction, we define

$$\langle D^{\alpha}f, \phi\rangle = (-1)^{|\alpha|} \langle f, D^{\alpha}\phi\rangle, \ \phi \in \mathscr{D}(\Omega) \tag{3.2.2}$$

We note that the right-hand side of (3.2.2) is well defined for any multi-index α, because $\phi \in \mathscr{D}(\Omega)$, and represents a continuous linear functional on $\mathscr{D}(\Omega)$.

Example 3.2.2 (i) Let H denote the Heaviside unit function on $\mathbb{R}$ defined by

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

Then
$$\langle H', \phi\rangle = -\langle H, \phi'\rangle = -\int_0^{\infty} \phi'(t)\, dt$$
$$= \phi(0) = \langle \delta, \phi\rangle$$

Thus $H' = \delta$.

(ii) The signum function on $\mathbb{R}$ is defined by

$$\operatorname{sgn} x = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

An alternative form is $\operatorname{sgn} x = H(x) - H(-x)$. Hence, from (i) $D \operatorname{sgn} x = 2\delta(x)$.

Typical properties of the distributional derivatives are given in the following theorem which are, in general, not true for classical functions.

Theorem 3.2.3 (i) Derivative of a distribution is also a distribution.

(ii) The order of differentiation of partial derivatives can be changed.

Proof (i) Clearly $D^{\alpha}f$ defined by (3.2.2) is a linear functional on $\mathscr{D}(\Omega)$. To show that it is also continuous, let $\{\phi_n\}_{n\in N}$ be a sequence which converges in $\mathscr{D}(\Omega)$ to zero. Then $\{D^{\alpha}\phi_n\}$ also converges in $\mathscr{D}(\Omega)$ to zero. Hence

$$\langle D^{\alpha}f, \phi_n\rangle = (-1)^{|\alpha|} \langle f, D^{\alpha}\phi_n\rangle \to 0 \text{ as } n \to \infty$$

(ii) Let $\phi \in \mathscr{D}(\Omega)$; then by Schwarz's theorem,

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \frac{\partial^2 \phi}{\partial x_j \partial x_i}$$

in the classical sense. Therefore,

$$\left\langle \frac{\partial^2 f}{\partial x_i \partial x_j}, \phi \right\rangle = -\left\langle \frac{\partial f}{\partial x_j}, \frac{\partial \phi}{\partial x_i} \right\rangle$$

$$= (-1)^2 \left\langle f, \frac{\partial^2 \phi}{\partial x_j \partial x_i} \right\rangle = \left\langle f, \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right\rangle$$

$$= -\left\langle \frac{\partial f}{\partial x_i}, \frac{\partial \phi}{\partial x_j} \right\rangle = \left\langle \frac{\partial^2 f}{\partial x_j \partial x_i}, \phi \right\rangle \quad \blacksquare$$

Some other important properties of the distributional derivative are given by

Theorem 3.2.4 Differentiation is a continuous, linear operation in $\mathscr{D}'(\Omega)$ in the following sense:

Linearity $D^\alpha(af + bg) = a\, D^\alpha f + bD^\alpha g, f, g \in \mathscr{D}'$ and $a, b \in \mathbb{C}$.

Continuity If $f_n \to f$ in $\mathscr{D}'$, then $D^\alpha f_n \to D^\alpha f$ in $\mathscr{D}'\ \forall\ \alpha \in \mathbb{N}_0^n$.

Proof The linearity is trivial. To prove continuity, let $\phi \in \mathscr{D}(\Omega)$, then $D^\alpha \phi \in \mathscr{D}(\Omega)$.
Therefore,

$$\langle D^\alpha f_n, \phi\rangle = (-1)^{|\alpha|} \langle f_n, D^\alpha \phi\rangle \to (-1)^{|\alpha|} \langle f, D^\alpha \phi\rangle = \langle D^\alpha f, \phi\rangle \quad \blacksquare$$

Let us consider the series of distributions:

$$f = \sum_{n=1}^{\infty} f_n \text{ in } \mathscr{D}'(\mathbb{R})$$

Then, by linearity of D^k,

$$D^k \sum_{n=1}^{N} f_n = \sum_{n=1}^{N} D^k f_n$$

Hence, for any $\phi \in \mathscr{D}(\mathbb{R})$, we have

$$\langle D^k f, \phi\rangle = \langle f, (-1)^k D^k \phi\rangle = \lim_{N\to\infty} \sum_{n=1}^{N} \langle f_n, (-1)^k D^k \phi\rangle$$

$$= \lim_{N\to\infty} \left\langle D^k \left(\sum_{n=1}^{N} f_n \right), \phi \right\rangle$$

$$= \lim_{N\to\infty} \left\langle \sum_{n=1}^{N} D^k f_n, \phi \right\rangle$$

Therefore,

$$D^k f = \sum_{n=1}^{\infty} D^k f_n \text{ in } \mathscr{D}'$$

Thus, any infinite series of distributions can be differentiated term by term. In classical analysis term by term differentiation of an infinite series is not always possible.

3.3 Some Examples

The following examples demonstrate the procedure of computing distributional derivatives:

Example 3.3.1 Let us define on $\mathbb{R}$

$$x_+ = \begin{cases} x & x > 0 \\ 0 & x \le 0 \end{cases}$$

Then x_+ is not differentiable at $x = 0$ in the classical sense, but it can be differentiated in the distributional sense.

$$\begin{aligned}\langle x_+', \phi\rangle &= -\langle x_+, \phi'\rangle \quad \phi \in \mathscr{D}(\mathbb{R}) \\ &= -\int_0^\infty x\phi'(x)\,dx \\ &= -x\phi(x)\Big|_0^\infty + \int_0^\infty x\phi(x)\,dx \\ &= \int_{-\infty}^\infty H(x)\phi(x)\,dx = \langle H, \phi\rangle\end{aligned}$$

where H is the Heaviside function defined in Example 3.2.2. Thus

$$x_+' = H \tag{3.3.1}$$

Moreover, by Example 3.2.2,

$$\langle x_+'', \phi\rangle = \langle H', \phi\rangle = \langle \delta, \phi\rangle$$

Thus

$$x_+'' = H' = \delta \tag{3.3.2}$$

The derivative of δ is given by

$$\langle \delta', \phi\rangle = -\langle \delta, \phi'\rangle = -\phi'(0) \tag{3.3.3}$$

Example 3.3.2 Define the characteristic function of $[a, b]$ in $\mathbb{R}$ by

$$I_{[b,b]}(x) = \begin{cases} 1 & x \in [a, b] \\ 0 & x \in \mathbb{R}\setminus[a, b] \end{cases}$$

Then clearly $I_{[a,b]}(x) = H(x - a) - H(x - b)$.

Hence $$I'_{[b,b]}(x) = \delta(x-a) - \delta(x-b) = \delta_a - \delta_b$$

Example 3.3.3 The above procedure can be adapted to find the derivative of $H(x)x^{-\alpha}$ $(0 < \alpha < 1)$.

$$\left\langle \frac{d}{dx} H(x)x^{-\alpha}, \phi(x) \right\rangle = -\langle H(x)\,x^{-\alpha}, \phi'(x)\rangle,\ \phi \in \mathscr{D}(\mathbb{R})$$

$$= -\lim_{\varepsilon\to 0} \int_\varepsilon^\infty x^{-\alpha}\phi'(x)\,dx$$

$$= -\lim_{\varepsilon\to 0} \left[x^{-\alpha}\phi(x)\,|_\varepsilon^\infty + \alpha \int_\varepsilon^\infty x^{-\alpha-1}\phi(x)\,dx \right]$$

$$= -\lim_{\varepsilon\to 0} \left[-\varepsilon^{-\alpha}\phi(\varepsilon) + \alpha \int_\varepsilon^\infty x^{-\alpha-1}\phi(x)\,dx \right]$$

$$= -\alpha \lim_{\varepsilon\to 0} \int_\varepsilon^\infty x^{-\alpha-1}[\phi(x) - \phi(\varepsilon)]\,dx \quad (3.3.4)$$

Now, using (1.3.7) we can write

$$\left\langle \frac{d}{dx} (H(x)x^{-\alpha}), \phi(x) \right\rangle = -\alpha\,\langle Pf\,H(x)x^{-\alpha-1}, \phi(x)\rangle \quad (3.3.5)$$

Example 3.3.4 The function $\log|x|$, for $x \neq 0$, is integrable in a neighbourhood of the origin, can be seen as follows:

For every positive $\varepsilon < 1$, $|\,|x|^\varepsilon \log|x|\,| \to 0$ as $|x| \to 0$. Hence, in a neighbourhood of the origin $|\log|x|\,| \le C/|x|^\varepsilon$ $(x \neq 0)$ for some $C > 0$.

Therefore, $\log|x|$ is locally integrable on $\mathbb{R}$ and defines a distribution in $\mathscr{D}'(\mathbb{R})$. Its classical derivative

$$\frac{d}{dx}\log|x| = x^{-1},\ x \neq 0$$

does not define a distribution because it is not integrable near the origin; but its distributional derivative is a distribution according to Theorem 3.2.3. We investigate it.

For any $\phi \in \mathscr{D}(\mathbb{R})$ we have

$$\langle D\log|x|, \phi\rangle = -\langle \log|x|, \phi'\rangle$$

$$= -\int_{-\infty}^\infty \log|x|\,\phi'(x)\,dx$$

Since $\log|x|\,\phi'(x)$ is integrable in a neighbourhood of the origin, and ϕ has compact support, we can write

$$\begin{aligned}
\langle D \log|x|, \phi \rangle &= -\lim_{\varepsilon \to 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \log|x|\, \phi'(x)\, dx \\
&= -\lim_{\varepsilon \to 0} \left\{ \log|x|\, \phi(x) \Big|_{\varepsilon}^{-\varepsilon} - \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \phi(x) \frac{dx}{x} \right\} \\
&= \lim_{\varepsilon \to 0} \left[\varepsilon \log \varepsilon \left(\frac{\phi(\varepsilon) - \phi(-\varepsilon)}{\varepsilon} \right) \right. \\
&\qquad \left. + \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{\phi(x)}{x}\, dx \right]
\end{aligned}$$

The first term on the right-hand side vanishes because ϕ is differentiable at the origin and the second term tends to $\langle Pv(1/x), \phi(x) \rangle$ as $\varepsilon \to 0$; see Example 1.3.5. Thus

$$D \log|x| = Pv\,(1/x)$$

3.4 Derivative of the Product $f\psi$

The product of a distribution $f \in \mathscr{D}'(\mathbb{R})$ with a smooth function $\psi \in C^\infty(\mathbb{R})$ was defined in Section 1.5.

We show that the derivative of the product $f\psi \in \mathscr{D}'(\mathbb{R})$ obeys the ordinary product rule of derivatives of classical functions, viz.

$$D(f\psi) = (Df)\psi + f(D\psi) \tag{3.4.1}$$

Indeed, for $\phi \in \mathscr{D}(\mathbb{R})$, we have

$$\begin{aligned}
\langle D(f\psi), \phi \rangle &= -\langle f\psi, D\phi \rangle \\
&= -\langle f, \psi D\phi \rangle = -\langle f, D(\psi\phi) - \phi D\psi \rangle \\
&= -\langle f, D(\psi\phi) \rangle + \langle f, \phi D\psi \rangle \\
&= \langle Df, \psi\phi \rangle + \langle fD\psi, \phi \rangle \\
&= \langle \psi Df, \phi \rangle + \langle fD\psi, \phi \rangle \\
&= \langle \psi Df + fD\psi, \phi \rangle
\end{aligned}$$

More generally, by induction we can prove the Leibnitz type formula:

$$D^k(f\psi) = \sum_{r=0}^{k} \binom{k}{r} D^r f D^{k-r}\psi \tag{3.4.2}$$

for $f \in \mathscr{D}'(\mathbb{R})$ and $\psi \in C^\infty(\mathbb{R})$.

Example 3.4.1 For all $\phi \in \mathscr{D}(\mathbb{R})$, we have

$$\begin{aligned}\langle \psi D\delta, \phi\rangle &= \langle D\delta, \psi\phi\rangle = \langle \delta, -D(\psi\phi)\rangle = -D(\psi\phi)(0)\\ &= -\psi(0)\phi'(0) - \psi'(0)\phi(0)\\ &= \langle \psi(0)\delta' - \psi'(0)\delta, \phi\rangle\end{aligned}$$

Therefore, $$\psi D\delta = \psi(0)\delta' - \psi'(0)\delta$$

In particular,

$$xD\delta = -\delta,\ x^k D\delta = 0,\ k > 1$$

Moreover, for $r \in \mathbb{N}$ and $\phi \in \mathscr{D}(\mathbb{R})$, we have

$$\begin{aligned}\langle D^k(x^r\delta(x)), \phi(x)\rangle &= (-1)^k \langle x^r\delta(x), D^k\phi(x)\rangle\\ &= (-1)^k \langle \delta(x), x^r D^k\phi(x)\rangle\\ &= 0\end{aligned}$$

Thus $$D^k(x^r\delta(x)) = 0 \quad \forall\ r, k \in \mathbb{N}$$

3.5 Derivative of a Locally Integrable Function

Let f be a locally integrable function on $\mathbb{R}$ such that $\frac{d}{dx}f(x)$ exists in the ordinary sense everywhere except at isolated points $c_j, j = 1, \ldots, n$, where $f(c_j -)$ and $f(c_j +)$ exist. Then the distributional derivative

$$Df(x) = \frac{d}{dx}f(x) + \sum_{j=1}^{n} [f(c_j +) - f(c_j -)]\, \delta(x - c_j) \qquad (3.5.1)$$

To prove (3.5.1) assume that $\phi \in \mathscr{D}(\mathbb{R})$. Then for a point of discontinuity c, we have

$$\begin{aligned}\langle Df(x), \phi(x)\rangle &= -\langle f(x), D\phi(x)\rangle\\ &= -\int_{-\infty}^{c} f(x)\frac{d}{dx}\phi(x)\,dx - \int_{c}^{\infty} f(x)\frac{d}{dx}\phi(x)\,dx\\ &= [f(c+) - f(c-)]\,\phi(c)\\ &\quad + \int_{-\infty}^{c}\left(\frac{d}{dx}f(x)\right)\phi(x)\,dx + \int_{c}^{\infty}\left(\frac{d}{dx}f(x)\right)\phi(x)\,dx\\ &= [f(c+) - f(c-)]\,\langle \delta(x - c), \phi(x)\rangle\\ &\quad + \int_{-\infty}^{\infty}\left(\frac{d}{dx}f(x)\right)\phi(x)\,dx\end{aligned}$$

Hence $$Df(x) = [f(c+) - f(c-)]\ \delta(x-c) + \frac{d}{dx} f(x)$$

Now, replacing c by c_j and summing over j, we get

$$Df(x) = \sum_j [f(c_j+) - f(c_j-)]\ \delta(x-c_j) + \frac{d}{dx} f(x) \qquad \blacksquare$$

This formula provides a relation between classical and distributional derivatives.

An application of (3.5.1) is given in the following example:

Example 3.5.1 The function f defined on $\mathbb{R}$ such that

$$f(x) = \frac{1}{2}(1 - x/\pi),\ x \in [0, 2\pi]$$

is periodic with period 2π. It has jump discontinuities of magnitude 1 at the points $x = \pm 2m\pi$, $m \in \mathbb{N}_0$.

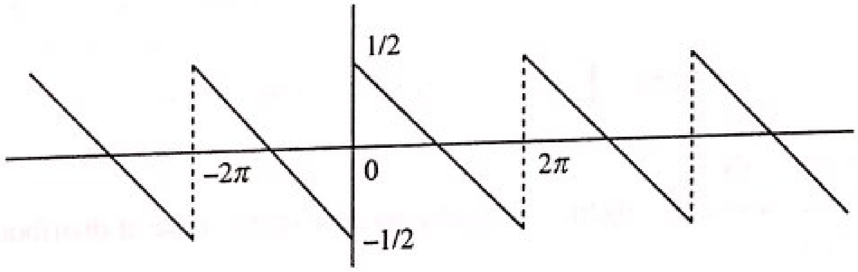

Fig. 3.1 **2π-periodicity of $f(x)$**

Therefore, in view of the relation (3.5.1)

$$Df(x) = -(2\pi)^{-1} + \sum_{m=-\infty}^{\infty} \delta(x - 2m\pi) \tag{3.5.2}$$

Moreover, we can find an ordinary Fourier series expansion of $f(x)$ as follows: Let

$$f(x) = \sum_{m=-\infty}^{\infty} \alpha_m e^{imx}$$

where

$$\alpha_m = (2\pi)^{-1} \int_0^{2\pi} f(x)\, e^{-imx}\, dx,\ m = \pm 1, \pm 2, \ldots$$

$$= (2\pi)^{-1} \int_0^{2\pi} \frac{1}{2}\left(1 - \frac{x}{\pi}\right) e^{-imx}\, dx = \frac{1}{2\pi i m}\ (m \neq 0)$$

and $$\alpha_0 = (2\pi)^{-1}\int_0^{2\pi}\frac{1}{2}\left(1-\frac{x}{\pi}\right)dx = 0$$

Thus $$f(x) = -(2\pi)^{-1}\, i \sum_{m=-\infty}^{\infty} e^{imx}/m,\; m \neq 0$$

which can be shown to be convergent in $\mathscr{D}'$ also. Indeed, if $\phi \in \mathscr{D}(\mathbb{R})$ having support in $[a, b]$, then for $m \neq 0$,

$$\left|\left\langle -(2\pi)^{-1} i \sum_{m=-M}^{M} e^{imx}/m,\, \phi(x)\right\rangle\right|$$

$$= \left| -(2\pi)^{-1} i \sum_m \frac{1}{m}\int_{-\infty}^{\infty} e^{imx}\phi(x)\,dx\right|$$

$$= \left|(2\pi)^{-1} i \sum_m \frac{1}{m}\int_a^b \phi'(x)\,\frac{e^{imx}}{im}\,dx\right|$$

$$\leq (2\pi)^{-1}\int_a^b |\phi'(x)|\,dx\left(\sum_{m=-M}^{M} 1/m^2\right) < \infty$$

for all $M \in \mathbb{N}$.

Hence, differentiating the series term by term in the sense of distributions, we obtain

$$Df(x) = (2\pi)^{-1}\sum_{m=-\infty}^{\infty} e^{imx},\; m \neq 0$$

$$= (2\pi)^{-1}\sum_{m=-\infty}^{\infty} e^{imx} - (2\pi)^{-1} \tag{3.5.2}$$

Comparing the two distributional derivatives (3.5.1) and (3.5.2) we have

$$\sum_{m=-\infty}^{\infty}\delta(x-2m\pi) = (2\pi)^{-1}\sum_{m=-\infty}^{\infty} e^{imx}$$

in the sense of equality in $\mathscr{D}'$. (3.5.3)

Exercises

3.1 Find the derivative of the following distributions:

(a) $\dfrac{d}{dx}(H(x)\log x)$

(b) $\dfrac{d}{dx}(\cos x\,\delta)$

(c) $\dfrac{d}{dx}\,(H(x)\sin^2 x)$

3.2 Prove that

(a) $x\delta'(x) = -\delta(x)$

(b) $e^x\delta(x) = \delta(x)$

(c) $(\sin ax)\,\delta'(x) = -a\delta(x)$

(d) $\dfrac{\partial}{\partial x}\,\delta(yx) = -\dfrac{1}{x^2}\,\delta(y), x > 0$

3.3 Show that

$$\frac{d}{dx}\text{ in }(x + i0) = Pf(1/x) - i\pi\delta(x)$$

3.4 Prove that

$$\frac{d}{dx}\,Pv(1/x) = -Fp(1/x^{2)}$$

3.5 Prove that

$$D(Fp\,x^{-n}) = n(Fp\,x^{-n-1}) - \delta^{(n)}(x)/n!,\ n \in \mathbb{N}$$

3.6 If f is a distribution of order m, prove that f' is of order $m + 1$.

3.7 Show that for $x \in \mathbb{R}$,

$$x^n\delta^{(m)}(x) = \begin{cases} 0 & m < n \\ (-1)^n\, n!\,\delta(x) & m = n \\ (-1)^n\,\dfrac{m!}{(m-n)!}\,\delta^{(m-n)}(x) & m > n \end{cases}$$

3.8 Prove that $\{(-1)^m\,\delta^{(m)}(x)\}_{m\in\mathbb{N}_0}$ and $\{x^n/n!\}_{n\in\mathbb{N}_0}$ form a biorthogonal set in the sense that

$$\left\langle (-1)^m\,\delta^{(m)}(x), \frac{x^n}{n!} \right\rangle = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

4.1 Introduction

The convolution product holds a central place among the various modes of composition of functions. The extension of the convolution to distributions has made it more powerful mathematical tool for solving the various approximation and boundary value problems. The convolution of the two distributions, in general, does not exist. If one of these distributions is of compact support then the convolution is always defined. In this chapter we study in brief distributions of compact support, direct product and convolution of distributions.

4.2 Distributions of Compact Support

Let Ω be an open subset of $\mathbb{R}^n$. The space of all those distributions in $\mathscr{D}'(\Omega)$ which have compact support is denoted by $\mathscr{E}'(\Omega)$. A sequence $\{f_j\}_{j\in\mathbb{N}}$ is said to converge in $\mathscr{E}'(\Omega)$ to a limit f if it converges in $\mathscr{D}'(\Omega)$ to f and if all the f_j have their supports contained in a fixed bounded domain $G \subset \Omega$. Then f will also be in $\mathscr{E}'(\Omega)$ with support in G. Clearly, $\delta \in \mathscr{E}'(\Omega)$.

An alternative characterisation, similar to distributions, can also be given. For this we need the following definition.

Definition 4.2.1 A sequence $\{\phi_j\}_{j\in\mathbb{N}} \in C^\infty(\Omega)$ is said to converge to zero in $C^\infty(\Omega)$ as $j \to \infty$ if, for each multi-index α, the $D^\alpha\phi_j$ converge to zero uniformly on every compact subset of Ω. With this concept of convergence the space $C^\infty(\Omega)$ is also denoted by $\mathscr{E}(\Omega)$ (in Schwartz's notation).

The following result will find several applications.

Theorem 4.2.2 $\mathscr{D}(\mathbb{R}^n)$ is a dense subspace of $C^\infty(\mathbb{R}^n)$.

Proof Let $\phi \in C^\infty(\mathbb{R}^n)$. Assume that $\rho \in C^\infty(\mathbb{R}^n)$ such that $\rho(x) = 1$ for $|x| \le 1$ and $\rho(x) = 0$ for $|x| > 2$.

Define

$$\theta_j(x) = \phi(x)\rho(x/j) \quad \text{for } j = 1, 2, \ldots,$$

Then $\theta_j(x) \in \mathscr{D}(\mathbb{R}^n)$ for all $j \in \mathbb{N}$ and the function

$$\phi_j(x) = \theta_j(x) - \phi(x) = \phi(x)(\rho(x/j) - 1)$$

belongs to $C^\infty(\mathbb{R}^n)$ and vanishes for $|x| \le j$.

To complete the proof we need to show that $\phi_j(x) \to 0$ in the space $C^\infty(\mathbb{R}^n)$ as $j \to \infty$. Let p be an arbitrarily fixed positive integer. Then for $j > p$ we have $\phi_j(x) = 0$ for $x \in \overline{B(p)} = \{x : |x| \le p\}$ and hence

$$\sup_{x \in \overline{B(p)}} |D^\beta \phi_j(x)| = 0 \quad \text{for all } \beta \in \mathbb{N}_0$$

Thus $\lim\limits_{j \to 0} \phi_j(x) = 0$ in $C^\infty(\mathbb{R}^n)$.

In fact, the proof can be suitably modified to show that $\mathscr{D}(\Omega)$ is dense in $C^\infty(\Omega)$, where Ω is any open subset of $\mathbb{R}^n$.

Let us define linearity and continuity of a functional on $C^\infty(\Omega)$ as in Section 1.3. The space of all continuous linear functional on $C^\infty(\Omega)$ is denoted by $\mathscr{E}'(\Omega)$; the elements of which are distributions of compact support in Ω. Using arguments similar to those used in the proof of (1.7.1) we can prove

Theorem 4.2.3 A distribution f is in $\mathscr{E}'(\Omega)$ if and only if there exist a constant $C > 0$, an integer $m \ge 0$ and a compact set $K \subset \Omega$ such that for every $\phi \in C^\infty(\Omega)$,

$$|\langle f, \phi \rangle| \le C \sum_{|\alpha| \le m} \sup_{x \in K} |D^\alpha \phi(x)| \tag{4.2.1}$$

Note that this compact set is not, in general, the support of f.

4.3 Direct Product of Distributions

Let $f(x)$ and $g(y)$ be locally integrable functions on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Then the function $f(x)g(y)$ is locally integrable on $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. It defines a regular distribution on $\mathscr{D}'(\mathbb{R}^{m+n})$.

$$\begin{aligned}\langle f(x)g(y), \phi(x, y)\rangle &= \int_{\mathbb{R}^m} f(x)\left(\int_{\mathbb{R}^n} g(y)\phi(x, y)\, dy\right) dx \\ &= \langle f(x), \langle g(y), \phi(x, y)\rangle\rangle, \ \phi(x, y) \in \mathscr{D}(\mathbb{R}^{m+n})\end{aligned} \tag{4.3.1}$$

Moreover,

$$\langle g(y)f(x), \phi(x,y)\rangle = \int g(y)\left(\int f(x)\phi(x,y)\,dx\right)dy$$

$$= \langle g(y), \langle f(x), \phi(x, y)\rangle\rangle \tag{4.3.2}$$

The above relations suggest the equality of (4.3.1) and (4.3.2).

Motivated by (4.3.1) we define the direct product (also called the tensor product) of the distributions $f(x) \in \mathscr{D}'(\mathbb{R}^m)$ and $g(y) \in \mathscr{D}'(\mathbb{R}^n)$ by

$$\langle f(x) \times g(y), \phi(x, y)\rangle := \langle f(x), \langle g(y), \phi(x, y)\rangle\rangle, \tag{4.3.3}$$

$$\phi(x, y) \in \mathscr{D}(\mathbb{R}^{m+n})$$

Theorem 4.3.1 The direct product $f(x) \times g(y)$ of distributions $f(x) \in \mathscr{D}'(\mathbb{R}^m)$ and $g(y) \in \mathscr{D}'(\mathbb{R}^n)$ is a distribution in $\mathscr{D}'(\mathbb{R}^{m+n})$.

To prove the theorem we need the following lemma, which justifies the above definition also.

Lemma 4.3.2 Let $g \in \mathscr{D}'(\mathbb{R}^n)$ and $\phi(x, y) \in \mathscr{D}(\mathbb{R}^{m+n})$. Then the function

$$\psi(x) := \langle g(y), \phi(x, y)\rangle \tag{4.3.4}$$

is in $\mathscr{D}(\mathbb{R}^m)$. Also, if the sequence $\{\phi_j(x, y)\} \to \phi(x, y)$ in $\mathscr{D}(\mathbb{R}^{m+n})$, then

$$\psi_j(x) = \{\langle g(y), \phi_j(x, y)\rangle\} \to \psi(x) \text{ in } \mathscr{D}(\mathbb{R}^m) \text{ as } j \to \infty.$$

Proof Let $K = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n : |x| \le a, |y| \le b;\ a, b > 0\}$ be the support of $\phi(x, y)$. Then it vanishes for $|x| > a$; so that $\psi(x)$ is also zero for $|x| > a$. Hence $\psi(x)$ is of compact support.

Now, we show that ψ is continuous. Let $\{x_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{R}^m$ which converges to x. Then, for any $\beta \in N_0^n$, $D_y^\beta \phi$ is uniformly continuous on $\mathbb{R}^{m+n}$ and so

$$\sup_{y\in\mathbb{R}^n} |D_y^\beta \phi(x_j, y) - D_y^\beta \phi(x, y)| \to 0 \quad \text{as } j \to \infty$$

It follows that $\phi(x_j, y) \to \phi(x, y)$ in $\mathscr{D}(\mathbb{R}^n)$. Since $g \in \mathscr{D}'(\mathbb{R}^n)$ its continuity implies that

$$\psi(x_j) = \langle g(y), \phi(x_j, y)\rangle \to \langle g(y), \phi(x, y)\rangle = \psi(x)$$

Thus ψ is continuous.

Next we show that

$$\frac{\partial}{\partial x_i}\psi(x) = \left\langle g(y), \frac{\partial}{\partial x_i}\phi(x, y)\right\rangle, \quad i = 0, 1, 2, 3, \ldots \tag{4.3.5}$$

Let x be a fixed point in $\mathbb{R}^m$, and set $h_i = (0, 0, \ldots, h, \ldots, 0)$, where h is located at the i-th place. Then, to prove (4.3.5) we need to show that

$$\left[\frac{1}{h}(\phi(x+h_i, y) - \phi(x, y)) - \frac{\partial}{\partial x_i}\phi(x, y)\right] \to 0$$

in $\mathscr{D}_y(\mathbb{R}^n)$ as $h \to 0$. Now, let $\phi^{(m)}(x, y)$ denote $D_y^m \phi(x, y)$ and let $\phi(x+h, y)$ denote $\phi(x_1, x_2, \ldots, x_i + h, \ldots, x_n, y)$. Then

$$\left[\frac{1}{h}(\phi^{(m)}(x+h_i, y) - \phi^{(m)}(x, y)) - \frac{\partial}{\partial x_i}\phi^{(m)}(x, y)\right]$$

$$= \left| \frac{1}{h}\int_{x_i}^{x_i+h} \left[\frac{\partial}{\partial u_i}\phi^{(m)}(x_i, \ldots, x_{i-1}, u_i, x_{i+1}, \ldots, x_n, y) - \frac{\partial}{\partial x_i}\phi^{(m)}(x, y)\right]\right| du_i$$

$$= \left| \frac{1}{h}\int_{x_i}^{x_i+h}\left(\int_{x_i}^{u_i}\frac{\partial^2}{\partial v_i^2}\phi^{(m)}(\ldots, v_i, \ldots, y)\, dv_i\right) du_i \right|$$

$$\leq \sup_{\mathbb{R}^{m+n}} \left|\frac{\partial^2}{\partial v_i^2}\phi^{(m)}(\ldots, v_i, \ldots, y)\right| \frac{1}{2}|h| \to 0$$

as $h \to 0$ uniformly for all $y \in \mathbb{R}^n$. This proves that

$$\frac{\psi(x+h_i) - \psi(x)}{h} - \left\langle g(y), \frac{\partial}{\partial x_i}\phi(x, y)\right\rangle \to 0 \quad \text{as } h \to 0$$

In other words,

$$\frac{\partial}{\partial x_i}\psi(x) = \langle g(y), \frac{\partial}{\partial x_i}\phi(x, y)\rangle$$

By repeated applications of the above technique, we can show that

$$D_x^{(k)}\psi(x) = \langle g(y), D_x^{(k)}\phi(x, y)\rangle \ \forall k \in \mathbb{N}_0^m \tag{4.3.6}$$

Thus $\psi(x) \in \mathscr{D}(\mathbb{R}^m)$ and the functional defined by (4.3.3) is meaningful.

Finally, to prove the last part of the lemma, assume that $\{\phi_j(x, y)\} \to 0$ in $\mathscr{D}(\mathbb{R}^{m+n})$ as $j \to \infty$. Define

$$\psi_j(x) = \langle g(y), \phi_j(x, y)\rangle$$

Our objective is to prove that $\psi_j(x) \to 0$ in $\mathscr{D}(\mathbb{R}^m)$ as $j \to \infty$. Note that

$$D_x^{(k)}\psi_j(x) = \langle g(y), D_x^{(k)}\phi_j(x, y)\rangle$$

If $\psi_j(x)$ does not tend to zero in $\mathscr{D}(\mathbb{R}^m)$ then for some fixed $k \in \mathbb{N}$, and $\varepsilon > 0$ there exists a sequence $\{x_j\}_{j \in \mathbb{N}}$ such that

$$|D_x^{(k)}\psi_j(x_j)| = |\langle g(y), D_x^{(k)}\phi_j(x, y)\rangle|_{x=x_j} \geq \varepsilon \tag{4.3.7}$$

for all $j = 1, 2, 3, \ldots$. If $\phi_j(x, y) \to 0$ in $\mathscr{D}(\mathbb{R}^{m+n})$ as $j \to \infty$, then there exists a compact set $K \subset \mathbb{R}^{m+n}$ containing the supports of all $\phi_j(x, y)$ and

$$\sup_{(x,y)\in\mathbb{R}^{m+n}} |D_x^{(k)}\phi_j(x, y)| \to 0 \quad \text{as } j \to \infty$$

Therefore, $\sup\limits_x D_x^k\phi_j(x, y) \to 0$ as $j \to \infty$ uniformly for all $y \in \mathbb{R}^n$, proving thereby $D_x^{(k)}\psi_j(x_j) \to 0$ as $j \to \infty$. This contradicts (4.3.7).

Proof of the Theorem Now, we consider (4.3.3). Since by the above lemma $\psi(x) = \langle g(y), \phi(x, y)\rangle$ is in $\mathscr{D}(\mathbb{R}^m)$, right-hand side of (4.3.3) is well defined. It is easy to see that $f \times g$ is a linear functional on $\mathscr{D}(\mathbb{R}^{m+n})$. Moreover, $\{\phi_j(x, y)\}$ tending to zero in $\mathscr{D}(\mathbb{R}^{m+n})$ implies that $\psi_j(x)$ tends to zero in $\mathscr{D}(\mathbb{R}^m)$, by the above lemma. Since f is a distribution, right-hand side of (4.3.3) $\to 0$ as $j \to \infty$. Therefore, $f \times g$ is continuous on $\mathscr{D}(\mathbb{R}^{m+n})$. Thus $f \times g \in \mathscr{D}'(\mathbb{R}^{m+n})$. ■

Example 4.3.3 Let us show that

$$\delta(x) \times \delta(y) = \delta(x, y) = \delta(y) \times \delta(x) \tag{4.3.8}$$

Let $\phi(x, y) \in \mathscr{D}(\mathbb{R}^{m+n})$. Then, by definition,

$$\begin{aligned}\langle \delta(x) \times \delta(y), \phi(x, y)\rangle &= \langle \delta(x), \langle \delta(y), \phi(x, y)\rangle\rangle \\ &= \langle \delta(x), \phi(x, 0)\rangle \\ &= \phi(0, 0) = \langle \delta(x, y), \phi(x, y)\rangle\end{aligned}$$

The proof of the second equality is similar.

4.4 Some Properties of the Direct Product

The direct product of distributions possesses many interesting properties. A few of them are given below:

As remarked after (4.3.2) and observed in (4.3.8), the direct product is commutative. To prove this result, in general form, we need the following lemma.

Lemma 4.4.1 The space of all test functions of the form

$$\phi(x, y) = \sum_{j=1}^{p} \theta_j(x)\psi_j(y) \tag{4.4.1}$$

where $\theta_j(x) \in \mathscr{D}(\mathbb{R}^m)$ and $\psi_j(y) \in \mathscr{D}(\mathbb{R}^n)$, is dense in $\mathscr{D}(\mathbb{R}^{m+n})$.

Proof Let the support of test function $\phi(x, y)$ be contained in $S = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n : |x| \le a, |y| \le a; a > 0\}$. Then by Weierstrass-theorem there exists a sequence of polynomials $\{p_j(x, y)\}_{j \in \mathbb{N}}$ which converges uniformly on $S' = \{(x, y) : |x| \le 2a, |y| \le 2a\}$ to $\phi(x, y)$. The derivatives of $p_j(x, y)$ also converge uniformly on S' to the corresponding derivatives of $\phi(x, y)$.

Now, let $\rho(x)$ be a test function that equals 1 on $|x| \le a$ and zero for $|x| \ge 2a$. Then $p_j(x, y)\rho(x)\rho(y) \in \mathscr{D}(\mathbb{R}^{m+n})$ is of the form (4.4.1) and

$$p_j(x, y)\rho(x)\rho(y) \to \phi(x, y)\rho(x)\rho(y) = \phi(x, y)$$

in $\mathscr{D}(\mathbb{R}^{m+n})$ as $j \to \infty$.

Property 1 *Commutativity* The direct product of two distributions $f \in \mathscr{D}'(\mathbb{R}^m)$ and $g \in \mathscr{D}'(\mathbb{R}^n)$ is commutative, i.e.,

$$f \times g = g \times f \tag{4.4.2}$$

Proof Let the test function $\phi(x, y)$ be given by (4.4.1). Then

$$\begin{aligned}
&\langle f(x) \times g(y), \sum_j \theta_j(x)\psi_j(y)\rangle \\
&= \sum_j \langle f(x), \langle g(y), \theta_j(x)\psi_j(y)\rangle\rangle \\
&= \sum_j \langle f(x), \theta_j(x)\, \langle g(y), \psi_j(y)\rangle\rangle \\
&= \sum_j \langle f(x), \theta_j(x)\rangle\, \langle g(y), \psi_j(y)\rangle
\end{aligned}$$

Similarly

$$\langle g(y) \times f(x), \sum_j \theta_j(x)\psi_j(y)\rangle = \sum_j \langle f(x), \theta_j(x)\rangle\, \langle g(y), \psi_j(y)\rangle$$

Since the test functions of the form (4.4.1) are dense in $\mathscr{D}(\mathbb{R}^{m+n})$ the general result follows:

Property 2 *Associativity* For $f(x) \in \mathscr{D}'(\mathbb{R}^m)$, $g(y) \in \mathscr{D}'(\mathbb{R}^n)$ and $h(z) \in \mathscr{D}'(\mathbb{R}^p)$ we have

$$f(x) \times [g(y) \times h(z)] = [f(x) \times g(y)] \times h(z) \tag{4.4.3}$$

Proof Let $\phi(x, y, z) \in D(\mathbb{R}^{m+n+p})$. Then

$$\begin{aligned}\langle f(x) \times [g(y) \times h(z)], \phi(x, y, z)\rangle &= \langle f(x), \langle g(y) \times h(z), \phi\rangle\rangle \\ &= \langle f(x), \langle g(y), \langle h(z), \phi\rangle\rangle\rangle \\ &= \langle f(x) \times g(y), \langle h(z), \phi\rangle\rangle \\ &= \langle [f(x) \times g(y)] \times h(z), \phi(x, y, z)\rangle\end{aligned}$$

Property 3 Differentiation

$$D_x^\alpha [f(x) \times g(y)] = D_x^\alpha f(x) \times g(y), \alpha \in \mathbb{N}^n \tag{4.4.4}$$

Proof Let $\phi(x, y) \in \mathscr{D}(\mathbb{R}^{m+n})$. Then

$$\begin{aligned}\langle D_x^\alpha [f(x) \times g(y)], \phi(x, y)\rangle &= (-1)^{|\alpha|} \langle f(x) \times g(y), D_x^\alpha \phi(x, y)\rangle \\ &= (-1)^{|\alpha|} \langle g(y), \langle f(x), D_x^\alpha \phi(x, y)\rangle\rangle \\ &= \langle g(y), \langle D_x^\alpha f(x), \phi(x, y)\rangle\rangle \\ &= \langle g(y) \times D_x^\alpha f(x), \phi(x, y)\rangle \\ &= \langle D_x^\alpha f(x) \times g(y), \phi(x, y)\rangle\end{aligned}$$

Property 4 *Continuity* If $f_j \to f$ in $\mathscr{D}'(\mathbb{R}^m)$, and $g \in \mathscr{D}'(\mathbb{R}^n)$, then

$$f_j \times g \to f \times g \text{ in } \mathscr{D}'(\mathbb{R}^{m+n}) \tag{4.4.5}$$

Proof Let $\phi(x, y) \in \mathscr{D}(\mathbb{R}^{m+n})$; then by lemma 4.3.2, $\psi(x) = \langle g(y), \phi(x, y)\rangle$ is in $\mathscr{D}(\mathbb{R}^m)$. Therefore,

$$\begin{aligned}\langle f_j(x) \times g(y), \phi(x, y)\rangle &= \langle f_j(x), \langle g(y), \phi(x, y)\rangle\rangle \\ &= \langle f_j(x), \psi(x)\rangle \\ &\to \langle f(x), \psi(x)\rangle \text{ as } j \to \infty \\ &= \langle f(x), \langle g(y), \phi(x, y)\rangle\rangle \\ &= \langle f(x) \times g(y), \phi(x, y)\rangle\end{aligned}$$

Property 5 *Support*

$$\operatorname{supp}(f \times g) = (\operatorname{supp} f) \times (\operatorname{supp} g) \tag{4.4.6}$$

Proof. Let $(x, y) \in \operatorname{supp} f \times \operatorname{supp} g$ and w be a neighbourhood of (x, y). Then there are open neighbourhoods U and V of x and y respectively such that $U \times V \subset W$. Since $x \in \operatorname{supp} f$, f does not vanish in U, i.e., there is a $\phi_1(x)$ in $\mathscr{D}(\mathbb{R}^m)$ such that $\operatorname{supp} \phi_1(x) \subset U$ and $\langle f(x), \phi_1(x)\rangle \neq 0$. Similarly, there is a $\phi_2(y)$ in $\mathscr{D}(\mathbb{R}^n)$ such that supp $\phi_2(y) \subset V$ and $\langle g(y), \phi_2(y)\rangle \neq 0$. Now $\phi_1(x)\phi_2(y) \in \mathscr{D}(\mathbb{R}^{m+n})$, supp $(\phi_1(x)\phi_2(y)) \subset U \times V \subset W$ and $\langle f(x)\ g(y),\ \phi_1(x)\phi_2(y)\rangle = \langle f(x),\ \phi_1(x)\rangle\ \langle g(y),\ \phi_2(y)\rangle \neq 0$. It follows that $(x, y) \in$ supp $(f \times g)$. Hence supp $f \times$ supp $g \subset$ supp $(f \times g)$.

Conversely, suppose that $(x, y) \notin$ supp $f \times$ supp g. Then $x \notin$ supp f or $y \notin$ supp g. For the sake of definiteness, assume that $x \notin$ supp f. Let U be an open neighbourhood of x which does not intersect the supp f; so that $\langle f(x), \phi(x)\rangle = 0$ for every $\phi \in \mathscr{D}(\mathbb{R}^m)$ whose support is contained in U. For a test function $\phi(x, y) \in \mathscr{D}(\mathbb{R}^{m+n})$ whose support is contained in $U \times \mathbb{R}^n$, (i.e., $\phi(x, y) = 0$ if $x \notin U$), we find that the support of $\psi(x) = \langle g(y), \phi(x, y)\rangle$ is contained in U. Hence

$$\langle f(x) \times g(y),\ \phi(x, y)\rangle = \langle f(x),\ \psi(x)\rangle = 0$$

It follows that $f(x) \times g(y)$ vanishes on $U \times \mathbb{R}^n$ and $(x, y) \notin$ supp $(f \times g)$. Hence supp $f \times$ supp $g \supset$ supp $(f \times g)$. This proves (4.4.6). ■

From (4.4.6) we infer that supp $(f \times g)$ consists of only those points (x, y) in $\mathbb{R}^{m+n}$ in which the first co-ordinate x belongs to supp f and the second coordinate y belongs to supp g.

4.5 Convolution

The convolution of two functions f and g on $\mathbb{R}^n$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy \tag{4.5.1}$$

provided that the integral exists. By a change of variable it follows that

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy = \int_{\mathbb{R}^n} g(y)f(x-y)\,dy = (g * f)(x)$$

If we assume that both functions are continuous, and one is of compact support, then $f * g$ exists and is continuous, and so determines a distribution which can be represented by

$$\langle f * g, \phi\rangle = \int_{\mathbb{R}^n} (f * g)(z)\phi(z)\,dz, \quad \phi \in \mathscr{D}(\mathbb{R}^n) \tag{4.5.2}$$

$$= \int \left[\int g(y)f(z-y)\,dy\right]\phi(z)\,dz$$

$$= \int g(y)\left[\int f(z-y)\phi(z)\,dz\right]dy$$

$$= \int g(y)\left[\int f(x)\phi(x+y)\,dx\right]dy$$

$$= \int\int f(x)g(y)\phi(x+y)\,dx\,dy$$

$$= \langle f(x)\times g(y),\ \phi(x+y)\rangle \tag{4.5.3}$$

Motivated by (4.5.3) we define the convolution of two distributions f and g by

$$\langle f*g,\ \phi\rangle = \langle f(x)\times g(y),\ \phi(x+y)\rangle \tag{4.5.4}$$

$$= \langle f(x),\ \langle g(y),\ \phi(x+y)\rangle\rangle, \quad \phi\in\mathscr{D}(\mathbb{R}^n) \tag{4.5.5}$$

We note that the above definition is not always valid because the support of $\phi(x+y)$ is not bounded. In fact if the support of $\phi(x)$ is contained in $\{x\in\mathbb{R}^n : |x|\le a,\ a>0\}$ then the support of $\phi(x+y)$ is contained in the $2n$-dimensional infinite band

$$\{(x,y): |x+y|\le a\}\subset\{(x,y): -a\le x_i+y_i\le a,\quad i=1,2,\ldots,n\}$$

For the legitimacy of the above definition we assume that one of the distributions f or g is of compact support. Of course, some other assumptions are also possible; see Zemanian [1965, p. 124].

If one of the distributions is of compact support, then from the relation $\operatorname{supp}(f\times g)=\operatorname{supp} f\times\operatorname{supp} g$, it follows that

$$\Omega = \operatorname{supp}(f\times g)\cap\operatorname{supp}\phi(x+y) \tag{4.5.6}$$

is bounded. For example, if $\operatorname{supp} f\subset\{x:|x|\le b$ and $\operatorname{supp}\phi\subset\{x:|x|\le a\}$, then

$$(\operatorname{supp} f\times\operatorname{supp} g)\cap\operatorname{supp}\phi(x+y)$$

$$=\{(x,y): |x|\le b,\ |x+y|\le a\}\subset\{(x,y): |x_i|\le b,$$

$$|x_i+y_i|\le a,\ i=1,2,\ldots,n\},$$

see Fig. 4.1.

Now, let $f\in\mathscr{E}'(\mathbb{R}^n)$, $g\in\mathscr{D}'(\mathbb{R}^n)$. Choose $\rho(x)\in\mathscr{D}(\mathbb{R}^n)$ such that $\rho(x)=1$ in a neighbourhood of support of f and zero outside some large domain. Then $\rho(x)\phi(x+y) := \theta(x,y)\in\mathscr{D}(\mathbb{R}^{2n})$. Hence, by Lemma 4.3.2, $\langle g(y),\ \theta(x,y)\rangle$ is an element of $\mathscr{D}(\mathbb{R}^n)$ and therefore we are justified in defining $f*g$ by

$$\langle f*g,\ \phi\rangle = \langle f(x),\ \langle g(y),\ \rho(x)\phi(x+y)\rangle\rangle$$

$$= \langle f(x),\ \rho(x)\,\langle g(y),\ \phi(x+y)\rangle\rangle \tag{4.5.7}$$

Since the value of a distirbution f depends on the value of the test

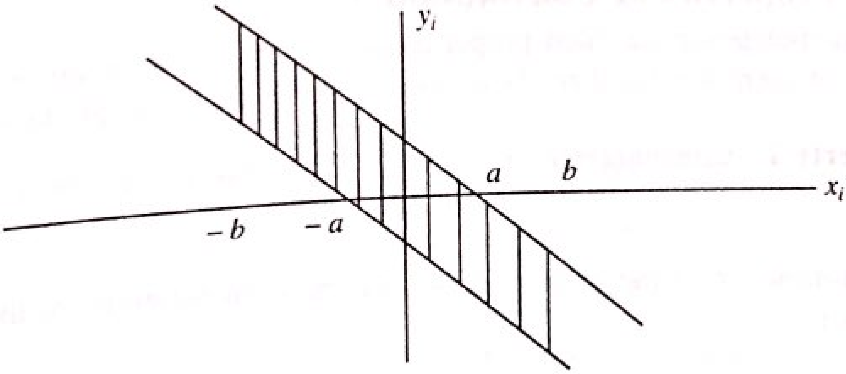

Fig. 4.1

function ϕ in a neighbourhood of the support of the distribution and is not altered by changing the value of ϕ outside any neighbourhood of the support f, definition (4.5.7) is independent of the choice of ρ. It is a standard convention to drop ρ and write

$$\langle f * g, \phi \rangle := \langle f(x), \langle g(y), \phi(x+y) \rangle \rangle \tag{4.5.8}$$

Theorem 4.5.1 Let $f \in \mathscr{E}'(\mathbb{R}^n)$ and $g \in \mathscr{D}'(\mathbb{R}^n)$; then $f * g$ is a distribution in $\mathscr{D}'(\mathbb{R}^n)$.

Proof For ρ and $\theta(x, y)$ defined as in the above we have

$$\langle f*g, \phi \rangle = \langle f(x) \times g(y), \theta(x, y) \rangle, \quad \phi \in \mathscr{D}(\mathbb{R}^n)$$

Since the direct product is a distribution, $f * g$ is linear on $\mathscr{D}(\mathbb{R}^n)$. Let $\{\phi_j(x)\}_{j \in \mathbb{N}}$ be a test function that converges in $\mathscr{D}(\mathbb{R}^n)$ to zero. Then $\rho(x)\phi_j(x+y) = \theta_j(x, y)$ is a test function that converges to zero in $\mathscr{D}(\mathbb{R}^{2n})$. Hence

$$\langle f * g, \phi_j \rangle = \langle f(x) \times g(y), \theta_j(x, y) \rangle \to 0 \text{ as } j \to \infty$$

Thus $f * g \in \mathscr{D}'(\mathbb{R}^n)$.

Example 4.5.2 Let us show that

$$\delta * f = f * \delta = f \tag{4.5.9}$$

Since δ is of compact support the above relation is valid for any f in $\mathscr{D}'(\mathbb{R}^n)$. Let $\phi \in \mathscr{D}(\mathbb{R}^n)$; then

$$\langle \delta * f, \phi \rangle = \langle \delta(x), \langle f(y), \phi(x+y) \rangle \rangle = \langle f(y), \phi(y) \rangle$$

Also,

$$\langle f * \delta, \phi \rangle = \langle f(x), \langle \delta(y), \phi(x+y) \rangle \rangle = \langle f(x), \phi(x) \rangle$$

4.6 Properties of Convolution

Convolution possesses many properties similar to those of the direct product. Some of them are not directly deducible from the direct product.

Property 1 *Commutavity* Let $f \in \mathscr{D}'(\mathbb{R}^n)$ and $g \in \mathscr{E}'(\mathbb{R}^n)$; then

$$f * g = g * f \tag{4.6.1}$$

This follows from the definition (4.5.7) and commutativity of the direct product.

Property 2 *Support* Let $f \in \mathscr{E}'(\mathbb{R}^n)$ and $g \in \mathscr{D}'(\mathbb{R}^n)$; then

$$\text{supp}\,(f * g) \subset \text{supp}\, f + \text{supp}\, g \tag{4.6.2}$$

Proof Let $A = \text{supp}\, f$ and $B = \text{supp}\, g$.

First we show that the set

$$A + B = \{x + y : x \in A, y \in B\}$$

is closed. Let $\{x_j + y_j\}_{j \in \mathbb{N}}$ be a sequence in $A + B$ which converges to a point a, where $x_j \in A$ and $y_j \in B$. Since A is compact, $\{x_j\}$ has a subsequence $\{x'_j\}$ which converges to $x \in A$. Since $\{x'_j\}$ and the subsequence $\{x'_j + y'_j\}$ are convergent their difference $\{y'_j\}$ converges to some y in B, since B is closed. Thus $a = x + y \in A + B$, which must be closed. Its complement $\Omega = \mathbb{R}^n \setminus (A + B)$ is open.

Now, for any $(x, y) \in \text{supp}\,(f \times g) = A \times B$ we have $x + y \in A + B$. If $\phi \in \mathscr{D}(\Omega)$, the support of $\phi(x + y)$ is contained in the open set $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x + y \in \Omega\}$. Therefore, supp $(f \times g)$ does not intersect the supp $\phi(x+y)$ for any $\phi \in \mathscr{D}(\Omega)$. Hence the relation $\langle f * g, \phi\rangle = \langle f(x) \times g(y), \phi(x + y)\rangle$ implies that $f * g$ vanishes on $\mathscr{D}(\Omega)$ and its support must be in $A + B$.

Note that if both $f, g \in \mathscr{E}'(\mathbb{R}^n)$ then $f * g \in \mathscr{E}'(\mathbb{R}^n)$.

Property 3 *Associativity* Let $f, g, h \in \mathscr{D}'(\mathbb{R}^n)$, and two of the three distributions be of compact support. Then

$$f * (g * h) = (f * g) * h \tag{4.6.3}$$

Proof Let us assume that $f, g \in \mathscr{E}'(\mathbb{R}^n)$. Then by Theorem 4.5.1 $f * (g * h)$ exists. Since by Property 2, $f * g$ is also of compact support, $(f * g) * h$ also exists in view of Theorem 4.5.1. We have

$$\begin{aligned}\langle f * (g * h), \phi\rangle &= \langle f(x) \times (g * h)(y), \phi(x + y)\rangle \\ &= \langle (g * h)(y), \langle f(x), \phi(x + y)\rangle\rangle\end{aligned}$$

$$= \langle g(y) \times h(z), \langle f(x), \phi(x + y + z)\rangle\rangle$$
$$= \langle f(x) \times g(y) \times h(z), \phi(x + y + z)\rangle \quad (4.6.4)$$

Similarly,

$$\langle (f * g) * h, \phi\rangle = \langle (f * g)(y) \times h(z), \phi(y + z)\rangle$$
$$= \langle f(x) \times g(y), \langle h(z), \phi(x + y + z)\rangle\rangle$$
$$= \langle f(x) \times g(y) \times h(z), \phi(x + y + z)\rangle \quad (4.6.5)$$

From the relations (4.6.4) and (4.6.5) the associativity follows.

Property 4 *Differentiation*

If the convolution $f * g$ exists then for $\alpha \in \mathbb{N}_0^n$,

$$D^\alpha f * g = D^\alpha(f * g) = f * D^\alpha g \quad (4.6.6)$$

Proof It is sufficient to prove that (4.6.6) holds for each partial derivative $\partial/\partial x_j$, $j = 1, \ldots, n$.

For $\phi \in \mathscr{D}(\mathbb{R}^n)$ we have

$$\langle \partial/\partial x_j(f * g), \phi\rangle = -\langle f * g, \partial/\partial x_j \phi\rangle = -\langle f \times g, \partial/\partial x_j \phi(x + y)\rangle$$
$$= -\langle f(x), \langle g(y), \partial/\partial x_j \phi(x + y)\rangle\rangle$$
$$= -\langle f(x), \langle g(y), \partial/\partial y_j \phi(x + y)\rangle\rangle$$
$$= \langle f(x), \langle \partial/\partial y_j g(y), \phi(x + y)\rangle\rangle$$
$$= \langle f * \partial/\partial x_j g, \phi\rangle \quad (4.6.7)$$

Using commutativity of convolution and the above technique we can write

$$\langle \partial/\partial x_j(f * g), \phi\rangle = \langle \partial/\partial x_j(g * f), \phi\rangle$$
$$= \langle (g * \partial/\partial x_j f), \phi\rangle$$
$$= \langle \partial/\partial x_j f * g, \phi\rangle \quad (4.6.8)$$

Combining (4.6.7) and (4.6.8) we get the desired result.

Property 5 *Continuity* In certain cases the convolution is a continuous operator. The following theorem deals with two such cases.

Theorem 4.6.1 (i) Let $f \in \mathscr{E}'(\mathbb{R}^n)$ and the sequence $\{g_j\}_{j\in\mathbb{N}}$ converge in $\mathscr{D}'(\mathbb{R}^n)$ to $g \in \mathscr{D}'(\mathbb{R}^n)$. Then $f * g_j \to f * g$ in $\mathscr{D}'(\mathbb{R}^n)$ as $j \to \infty$.

(ii) Let $f \in \mathscr{D}'(\mathbb{R}^n)$; and the sequence $\{g_j\}_{j\in\mathbb{N}}$ be such that all its elements have support in a fixed compact set K and $g_j \to g \in \mathscr{D}'(\mathbb{R}^n)$ in $\mathscr{D}'(\mathbb{R}^n)$. Then $f * g_j \to f * g$ in $\mathscr{D}'(\mathbb{R}^n)$ as $j \to \infty$.

Proof (i) Let $\phi \in \mathscr{D}(\mathbb{R}^n)$, then by Lemma 4.3.2, $\langle f(x), \phi(x+y)\rangle$ is in $\mathscr{D}(\mathbb{R}^n)$. Hence using (4.5.8) and (4.6.1) we have

$$\lim_{j\to\infty} \langle f * g_j, \phi\rangle = \lim_{j\to\infty} \langle g_j(y), \langle f(x), \phi(x+y)\rangle\rangle$$
$$= \langle g(y), \langle f(x), \phi(x+y)\rangle\rangle = \langle f * g, \phi\rangle$$

(ii) Let $\rho \in \mathscr{D}(\mathbb{R}^n)$ such that $\rho = 1$ in a neighbourhood of K. Then as in the above case, for any $\phi \in \mathscr{D}(\mathbb{R}^n)$, $\rho(y)\langle f(x), \phi(x+y)\rangle \in \mathscr{D}(\mathbb{R}^n)$ and we have

$$\lim_{j\to\infty} \langle f * g_j, \phi\rangle = \lim_{j\to\infty} \langle g_j(y), \rho(y)\langle f(x), \phi(x+y)\rangle\rangle$$
$$= \langle g(y), \rho(y) < f(x), \phi(x+y)\rangle\rangle$$
$$= \langle f * g, \phi\rangle$$

4.7 Regularization of Distributions

The convolution $f * \phi$ of a distribution $f \in \mathscr{D}'(\mathbb{R}^n)$ with some test function ϕ converts the distribution into an infinitely differentiable function. This process is called the *regularization* of distributions. The following theorem generalizes our earlier result given in Section 1.2.

Theorem 4.7.1 Let $f \in \mathscr{D}'(\mathbb{R}^n)$ and $\psi \in \mathscr{D}(\mathbb{R}^n)$; then

$$(f * \psi)(x) = \langle f(y), \psi(x-y)\rangle \in C^\infty(\mathbb{R}^n) \tag{4.7.1}$$

Proof Using the technique of proof of Lemma 4.3.2. it can be shown that $(f * \psi)(x) \in C^\infty(\mathbb{R}^n)$.

To prove the equality we may regard ψ as an element of $\mathscr{E}'(\mathbb{R}^n)$, and use the definition (4.5.8) and find that, for any $\phi \in \mathscr{D}(\mathbb{R}^n)$,

$$\langle f * \psi, \phi\rangle = \langle f(y) \times \psi(z), \phi(y+z)\rangle$$
$$= \langle f(y), \langle \psi(z), \phi(y+z)\rangle\rangle$$
$$= \left\langle f(y), \int_{\mathbb{R}^n} \psi(z)\phi(y+z)\,dz \right\rangle$$
$$= \left\langle f(y), \int_{\mathbb{R}^n} \phi(x)\psi(x-y)\,dx \right\rangle$$
$$= \langle f(y), \langle \phi(x), \psi(x-y)\rangle\rangle$$
$$= \langle f(y) \times \phi(x), \psi(x-y)\rangle$$
$$= \langle \phi(x), \times f(y), \psi(x-y)\rangle \qquad \text{(by (4.4.2))}$$

$$= \langle \phi(x), \langle f(y), \psi(x-y) \rangle \rangle$$

$$= \int_{\mathbb{R}^n} \langle f(y), \psi(x-y) \rangle \, \phi(x) \, dx$$

$$= \langle \langle f(y), \psi(x-y) \rangle, \phi(x) \rangle$$

This proves (4.7.1). ■

We can now prove the following important result.

Theorem 4.7.2 $\mathscr{D}(\mathbb{R}^n)$ is a dense subspace of $\mathscr{D}'(\mathbb{R}^n)$.

Proof We need to show that for each distribution f in $\mathscr{D}'(\mathbb{R}^n)$ there exists a sequence of test functions $\{f_j\}_{j \in \mathbb{N}}$ which converges to f in $\mathscr{D}'(\mathbb{R}^n)$. Let

$$\rho(x) = \begin{cases} 0 & |x| \geq 1 \\ \exp\left(\dfrac{1}{|x|^2 - 1}\right) & |x| < 1 \end{cases}$$

and set

$$\eta_j(x) = \frac{j^{|n|} \rho(jx)}{\displaystyle\int_{\mathbb{R}^n} \rho(x) \, dx}$$

Then $\{\eta_j(x)\}_{j \in \mathbb{N}}$ is a sequence of test functions in $\mathscr{D}(\mathbb{R}^n)$ which converges in $\mathscr{D}'$ to $\delta(x)$ by Theorem 2.2.8.

Let f be an arbitrary distribution in $\mathscr{D}'(\mathbb{R}^n)$. Then the regularizations $f(x) * \eta_j(x)$ are in $C^\infty(\mathbb{R}^n)$, and by the continuity property of the convolution (Theorem 4.6.1 (ii)), they converge in $\mathscr{D}'$ to $f * \delta = f$; see Example 4.5.2. Since $f(x) * \eta_j(x)$ may not be of the bounded support, we choose $\rho(x) \in \mathscr{D}(\mathbb{R}^n)$ such that $\rho(x) = 1$ for $|x| \leq 1$ and $\rho(x) = 0$ for $|x| \geq 2$, and define

$$f_j(x) := \rho(x/j)[f(x) * \eta_j(x)], \quad j = 1, 2, \ldots$$

Then regarding $\eta_j(x)$ as distributions in $\mathscr{E}'(\mathbb{R}^n)$ with supports contained in $|x| \leq 1/j$, for any $\phi \in \mathscr{D}(\mathbb{R}^n)$, we have $\rho(x/j)\phi(x) = \phi(x)$ for large j, and

$$\begin{aligned} \langle f_j(x), \phi(x) \rangle &= \langle \rho(x/j)[f(x) * \eta_j(x)], \phi(x) \rangle \\ &= \langle f(x) * \eta_j(x), \rho(x/j)\, \phi(x) \rangle \\ &= \langle f(x) * \eta_j(x), \phi(x) \rangle \to \langle f, \phi \rangle \quad \text{as } j \to \infty \end{aligned}$$ ■

4.8 Fundamental Solutions of Linear Differential Operators

One of the most important uses of convolution theory is to determine the particular solution of the differential equation

$$P(D)u = f \tag{4.8.1}$$

where $P(D) = \sum\limits_{|\alpha|\le m} a_\alpha D^\alpha$, $a_\alpha \in \mathbb{C}$ and f is a given distribution.

A distribution E such that

$$P(D)E = \delta \tag{4.8.2}$$

is called a *fundamental* (or *elementary*) solution of the operator $P(D)$. If E is known, a solution of (4.8.1) can be obtained when $E * f$ is defined.

Theorem 4.8.1 Let $f \in \mathscr{D}'(\mathbb{R}^n)$ and E be a fundamental solution of $P(D)$. Suppose that $E * f$ exists. Then $u_0 = E * f$ is a solution of (4.8.1). If u is any solution of (4.8.1), then $u = u_0 + v$ when v satisfies $P(D)v = 0$.

Proof Using (4.5.9) we can write

$$P(D)u_0 = \sum_{|\alpha|\le m} a_\alpha D^\alpha (E * f) = \left(\sum_\alpha a_\alpha D^\alpha E\right) * f = \delta * f = f$$

Therefore u_0 is a solution of (4.8.1).
Moreover, if u is any solution of (4.8.1), then $P(D)(u - u_0) = 0$.

Example 4.8.2 Let us now find a fundamental solution of the operator $\left(\dfrac{d}{dx}\right)^k$, $k \in \mathbb{N}$.

For $k = 1$, we have $\dfrac{dH}{dx} = \delta$, where H is the Heaviside unit function. Hence H is the fundamental solution of d/dx.

If $k \ge 2$, we obtain the fundamental solution of $\left(\dfrac{d}{dx}\right)^k$ by solving $\left(\dfrac{d}{dx}\right)^{k-1} E = H$. A solution is given by the formula

$$E_k = 1/(k-1)!\; x^{k-1} H(x)$$

Indeed by Leibnitz type formula (3.4.2) we have

$$\begin{aligned}
\left(\frac{d}{dx}\right)^{k-1} E_k &= \frac{1}{(k-1)!} \sum_{j=0}^{k-1} \binom{k-1}{j} (x^{k-1})^{(j)} H^{(k-1-j)} \\
&= H + \sum_{j=0}^{k-2} \binom{k-1}{j} (x^{k-1})^{(j)} \delta^{(k-2-j)} \\
&= H + \sum_{j=1}^{k-1} \binom{k-1}{j-1} (x^{k-1})^{(j-1)} \delta^{(k-1-j)} \\
&= H + \sum_{j=1}^{k-1} a_{j,k} x^{k-j} \delta^{(k-1-j)} \\
&= H
\end{aligned}$$

since $x^{k-j}\delta^{(k-1-j)} = 0$ by Chapter 3, Exercise 3.7.

Thus for all $k \in \mathbb{N}$,

$$\left(\frac{d}{dx}\right)^k \left[\frac{x^{k-1}H}{(k-1)!}\right] = \delta$$

Therefore particular solution of

$$\left(\frac{d}{dx}\right)^k u = f \tag{4.8.3}$$

where $f \in \mathscr{E}'(\mathbb{R})$, can be written as

$$u_0 = f * \frac{x^{k-1}}{(k-1)!} H \tag{4.8.4}$$

Hence the general solution of (4.8.3) is given by

$$u = \sum_{j=1}^{k} C_j x^{j-1} + f * \frac{x^{k-1}}{(k-1)!} H(x) \tag{4.8.5}$$

where the C_j are arbitrary constants.

We shall need the following identity in the sequel.

Green's Formula 4.8.3. Given $\varepsilon > 0$, let $\Omega_\varepsilon = \{x \in \mathbb{R}^n : |x| > \varepsilon\}$. For every $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and every $\phi \in \mathscr{D}(\mathbb{R}^n)$ we have

$$\int_{\Omega_\varepsilon} f(x)\,\Delta\phi(x)\,dx - \int_{\Omega_\varepsilon} \Delta f(x)\,\phi(x)\,dx$$

$$= \int_{|x|=\varepsilon} \left\{\phi(x)\frac{\partial f}{\partial r}(x) - f(x)\frac{\partial \phi}{\partial r}(x)\right\} d\sigma$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $\frac{\partial}{\partial r}$ is the radial derivative and $d\sigma$ denotes the surface element on the sphere $|x| = \varepsilon$.

Example 4.8.4 The fundamental solution of the Laplace operator $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$ in $\mathbb{R}^3$ is $-1/(4\pi r)$, where $r = |x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

Since the function $1/r$ is locally integrable in $\mathbb{R}^3$, for any $\phi \in \mathscr{D}(\mathbb{R}^3)$ we have

$$\langle \Delta(1/r), \phi(x, y, z)\rangle = \langle 1/r, \Delta\phi\rangle$$

$$= \int_{R^3} \frac{\Delta\phi}{r}\,dx = \lim_{\varepsilon \to 0+} \int_{|x| \geq \varepsilon} \frac{\Delta\phi}{r}\,dx$$

Assume that supp $\phi \subset \{x \in R^3 : |x| < R\}$ and apply Green's formula in the region bounded by the spheres $r = \varepsilon$ and $r = R$ and obtain

$$\int_{r \geq \varepsilon} \frac{\Delta \phi}{r} dx = \int_{r \geq \varepsilon} \phi \Delta \left(\frac{1}{r} \right) dx + \int_{r=\varepsilon} \left(\phi \frac{\partial}{\partial r} \left(\frac{1}{r} \right) - \frac{1}{r} \frac{\partial \phi}{\partial r} \right) d\sigma$$

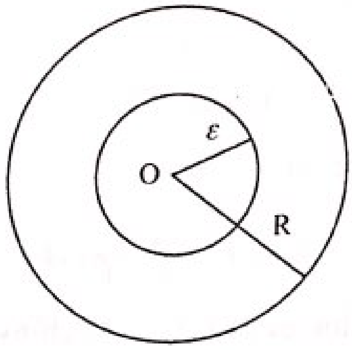

Fig. 4.2

The first term on the right vanishes since $\Delta(1/r) = 0$ at all points excluding the origin. Since ϕ is a smooth function with compact support, there exists $M > 0$ such that $|\partial\phi/\partial r| \leq M$ everywhere. Therefore

$$\left| \int_{r=\varepsilon} \frac{1}{r} \frac{\partial \phi}{\partial r} d\sigma \right| \leq M \frac{1}{\varepsilon} \int_{r=\varepsilon} d\sigma = M \frac{1}{\varepsilon} 4\pi\varepsilon^2 \to 0 \text{ as } \varepsilon \to 0+$$

Also,

$$\begin{aligned} \frac{1}{4\pi} \int_{r=\varepsilon} \phi \frac{d}{dr} \left(\frac{1}{r} \right) d\sigma &= \frac{-1}{4\pi\varepsilon^2} \int_{r=\varepsilon} \phi \, d\sigma \\ &= \frac{-1}{4\pi\varepsilon^2} \int_{r=\varepsilon} [\phi(x, y, z) - \phi(0, 0, 0)] \, d\sigma \\ &\quad - \frac{1}{4\pi\varepsilon^2} \int_{r=\varepsilon} \phi(0, 0, 0) \, d\sigma \end{aligned}$$

Since ϕ is uniformly continuous the first integral on the right-hand side tends to zero as $\varepsilon \to 0$. Therefore

$$\frac{1}{4\pi} \int_{r=\varepsilon} \phi \frac{d}{dr} \left(\frac{1}{r} \right) d\sigma \to -\phi(0) = -\langle \delta, \phi \rangle \text{ as } \varepsilon \to 0+$$

Thus
$$\langle \Delta(1/r), \phi \rangle = -4\pi \langle \delta, \phi \rangle \tag{4.8.6}$$

Hence $-1/(4\pi r)$ is a fundamental solution of the 3-dimensional Laplace operator Δ.

If $f \in \mathscr{D}'(\mathbb{R}^3)$, then the solution of the Poisson equation

$$\Delta u = f \tag{4.8.7}$$

is given by

$$u = (-1/4\pi r) * f \tag{4.8.8}$$

Moreover, if f is an integrable function on $\mathbb{R}^3$ with compact support, then we can write

$$u(x_1, x_2, x_3) = \frac{-1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y_1, y_2, y_3)\, dy_1 dy_2 dy_3}{\{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2\}^{1/2}} \tag{4.8.9}$$

Exercises

4.1 Prove that $\mathscr{D}(\mathbb{R}^m) \times \mathscr{D}(\mathbb{R}^n)$ is dense in $\mathscr{D}(\mathbb{R}^m \times \mathbb{R}^n)$.

4.2 Let $a \in C^\infty(\mathbb{R}^n)$; then prove that

$$a(x)\,[f(x) \times g(y)] = [a(x)f(x)] \times g(y), \quad f \in \mathscr{D}'(\mathbb{R}^m),\ g \in \mathscr{D}'(\mathbb{R}^n)$$

4.3 Prove that for $f \in \mathscr{D}'(\mathbb{R}^m)$ and $g \in \mathscr{D}'(\mathbb{R}^n)$,

$$(f \times g)(x + h, y) = f(x + h) \times g(y)$$

4.4 If $f \in \mathscr{E}'(\mathbb{R}^n)$ and $\phi \in C^\infty(\mathbb{R}^n)$ (or $f \in \mathscr{D}'(\mathbb{R}^n)$ and $\phi \in \mathscr{D}(\mathbb{R}^n)$), then prove that $(f * \phi)(0) = \langle f, \check{\phi} \rangle$, where $\check{\phi}(x) = \phi(-x)$.

4.5 If $f \in \mathscr{E}'(\mathbb{R}^n)$, $g \in \mathscr{D}'(\mathbb{R}^n)$ and $\phi \in \mathscr{D}(\mathbb{R}^n)$, prove that

$$\langle f * g, \phi \rangle = [f * (g * \check{\phi})]\,(0)$$

4.6 Prove that $\delta^{(m)} * f = f^{(m)}$ for $f \in \mathscr{D}'(\mathbb{R})$ and $m \in \mathbb{N}$.

4.7 Evaluate

(a) $e^{-|x|} * e^{-|x|}$

(b) $e^{-ax^2} * e^{-ax^2}$, $a > 0$

(c) $xH(x) * e^x H(x)$

4.8 For $p, q, m, n \in \mathbb{N}$, compute

$$f = [x^p \delta^{(q)}] * [x^m \delta^{(n)}]$$

Hint: Use Chapter 3, Exercise 3.7.)

4.9 Prove that

$$Pv(1/x) * Pv(1/x) = -\pi^2 \delta(x)$$

4.10 Let $f \in \mathscr{E}'(\mathbb{R})$ and $g \in \mathscr{D}'(\mathbb{R})$. Show that for $k \in \mathbb{N}$,

$$x^k (f * g) = \sum_{j=0}^{k} \binom{k}{j} (x^j f) * (x^{k-j} g)$$

4.11 Let $f \in \mathscr{E}'(\mathbb{R}^n)$ and $g \in \mathscr{D}'(\mathbb{R}^n)$. Then for every $a \in \mathbb{R}$, show that

$$e^{(a,x)}(f * g) = (e^{(a,x)} f) * (e^{(a,x)} g)$$

4.12. Prove that $D^m(f * g) = f^{(p)} * g^{(q)}$, where $p + q = m$, $p, q \in \mathbb{N}_0$, provided the convolution exists.

4.13. Prove that the convolution of two distributions f and g exists if the supports of both are bounded on the same side.

4.14. Determine the fundamental solution of the operator $d^2/dx^2 + \omega^2$, where ω is a nonzero constant.

4.15. Show that $E = (2\pi)^{-1} \log r$, $r = (x_1^2 + x_2^2)^{1/2}$, is a fundamental solution of the Laplace operator Δ on $\mathbb{R}^2$.

4.16. Show that the distribution

$$E(x, t) = (4\pi t)^{-1/2} H(t) \exp(-x^2/(4t))$$

where $H(t)$ is the Heaviside unit function, is a fundamental solution of the heat operator $\partial/\partial t - \partial^2/\partial x^2$.

5

Tempered Distributions and Fourier Transforms

5.1 Introduction

Tempered distributions are generalizations of L^p-functions, and well-suited for the study of Fourier transforms. These are continuous linear functionals on the space $\mathscr{S}(\mathbb{R}^n)$ of rapidly decreasing infinitely differentiable functions. The space of tempered distributions $\mathscr{S}'(\mathbb{R}^n)$, although an intermediate space between $\mathscr{E}'(\mathbb{R}^n)$ and $\mathscr{D}'(\mathbb{R}^n)$, possesses several useful and interesting properties, which are not possessed by many other distribution spaces. For example, the generalized Fourier transform maps $\mathscr{S}'$ onto itself. The function space $\mathscr{S}(\mathbb{R}^n)$ is an intermediate space between $\mathscr{D}(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^n)$. The classical Fourier transform maps $\mathscr{S}$ onto itself. Important properties of the spaces $\mathscr{S}$ and $\mathscr{S}'$ are given and Fourier transforms are studied on the spaces $\mathscr{S}(\mathbb{R}^n)$, $\mathscr{S}'(\mathbb{R}^n)$ and $\mathscr{E}'(\mathbb{R}^n)$. Many important properties of generalized Fourier transforms, including inversion, multiplication and convolution are investigated. Some applications are demonstrated.

5.2 The Space of Rapidly Decreasing Functions

In this section we introduce a very useful space of test functions whose derivatives decrease faster than $|x|^{-N}$ for all $N > 0$ as $|x| \to \infty$.

Definition 5.2.1 By $\mathscr{S}$ we denote the set of all rapidly decreasing functions $\phi \in C^\infty(\mathbb{R}^n)$ such that

$$\gamma_{\alpha,\beta}(\phi) = \sup_x |x^\alpha D^\beta \phi(x)| < \infty \tag{5.2.1}$$

for all multi-indices α and β.

$\mathscr{S}(\mathbb{R}^n)$ is a vector space over the field of complex numbers. From the above definition it follows that $\mathscr{S}(\mathbb{R}^n)$ is stable under differentiation, and under multiplication by polynomials. Moreover, from the definition it also follows that

$$\mathscr{D}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n),\ 1 \le p \le \infty$$

It can be easily shown that the function $\phi(x) = \exp(-a \mid x \mid^2)$, $a > 0$, belongs to $\mathscr{S}(\mathbb{R}^n)$, but $\phi(x) = e^{+x}$ and $\phi(x) = (1 + |x|^2)^{-1}$ are not members of $\mathscr{S}(\mathbb{R}^n)$.

A sequence of functions $\{\phi_k\}_{k\in \mathbb{N}}$ is said to converge in $\mathscr{S}$ to zero if every function $\phi_k \in \mathscr{S}$ and $\gamma_{\alpha,\beta}(\phi_k) \to 0$ as $k \to \infty$ for every $\alpha, \beta \in \mathbb{N}_0^n$. In other words, $\{\phi_k\}$ will converge to ϕ in $\mathscr{S}$ if and only if $\{\phi_k - \phi\}_{k\in\mathbb{N}_0}$ converges in $\mathscr{S}$ to zero.

The following lemma provides an alternative condition on $\phi \in \mathscr{S}(\mathbb{R}^n)$ instead of (5.2.1).

Lemma 5.2.2 A function $\phi \in C^\infty(\mathbb{R}^n)$ satisfies (5.2.1) if and only if

$$\tau_{m,\beta}(\phi) = \sup_{x\in\mathbb{R}^n} |(1 + |x|^2)^{m/2} D^\beta \phi(x)| < \infty \tag{5.2.2}$$

$\forall\, m \in \mathbb{N}_0, \beta \in \mathbb{N}_0^n$.

Proof Recall that $|x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$ and note that $|x^\alpha| = |x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}| = |x_1|^{\alpha_1} |x_2|^{\alpha_2} \ldots |x_n|^{\alpha_n}$ and $|x_i| \le |x|$, $i = 1, 2, \ldots, n$. Hence $|x_1|^{\alpha_1} |x_2|^{\alpha_2} \ldots |x_n|^{\alpha_n} \le |x|^{|\alpha|}$. Thus, we have

$$|x^\alpha| \le |x|^{|\alpha|} \le (1 + |x|^2)^{|\alpha|/2} \le (1 + |x|^2)^{m/2}\ \forall\, |\alpha| \le m \tag{5.2.3}$$

Therefore, if ϕ satisfies (5.2.2) then it satisfies (5.2.1) also.

Next, we assume that ϕ satisfies (5.2.1) for all $\alpha \in \mathbb{N}_0^n$. We note that for all $k \in \mathbb{N}_0$,

$$\begin{aligned} |x|^{2k} = (x_1^2 + x_2^2 + \ldots + x_n^2)^k &= \sum_{|\alpha|=k} a(\alpha, k)\, (x_1^2)^{\alpha_1} (x_2^2)^{\alpha_2} \ldots (x_n^2)^{\alpha_n} \\ &= \sum_{|\alpha|=k} a(\alpha, k)\, |x^\alpha|^2 \end{aligned}$$

where $a(\alpha, k) = k!/\alpha!$ are constant coefficients. Therefore, for $m \in \mathbb{N}_0$,

$$\begin{aligned} (1 + |x|^2)^m = \sum_{k=0}^m \binom{m}{k} |x|^{2k} &= \sum_{k=0}^m \sum_{|\alpha|=k} \binom{m}{k} a(\alpha, k)\, |x^\alpha|^2 \qquad (5.2.4) \\ &\le \left(\sum_{k=0}^m \sum_{|\alpha|=k} \left[\binom{m}{k} a(\alpha, k) \right]^{1/2} |x^\alpha| \right)^2 \end{aligned}$$

so that

$$(1+|x|^2)^{m/2} \le \sum_{k=0}^{m} \sum_{|\alpha|=k} \left[\binom{m}{k} a(\alpha, k)\right]^{1/2} |x^\alpha| \tag{5.2.5}$$

Consequently,

$$\tau_{m,\beta}(\phi) \le \sum_{k=0}^{m} \sum_{|\alpha|=k} \left[\binom{m}{k} a(\alpha, k)\right]^{1/2} \gamma_{\alpha,\beta}(\phi) \quad \blacksquare \tag{5.2.6}$$

In what follows we shall give many properties of the space $\mathscr{S}(\mathbb{R}^n)$ in the form of theorems.

Theorem 5.2.3 $\mathscr{D}$ is a dense subspace of $\mathscr{S}$, with the identity map from $\mathscr{D}$ into $\mathscr{S}$ continuous.

Proof We need to show that for each $\phi \in \mathscr{S}$ there exists a sequence $\{\phi_k\}_{k\in \mathbb{N}}$ in $\mathscr{D}$ such that ϕ_k converges in $\mathscr{S}$ to ϕ. Let

$$\rho(x) = \begin{cases} 0 & |x| \ge 1 \\ \exp\left(\dfrac{|x|^2}{|x|^2-1}\right) & |x| < 1 \end{cases} \tag{5.2.7}$$

Clearly, $\rho(x) \in \mathscr{D}(\mathbb{R}^n)$. For $\phi \in \mathscr{S}$ and each $k = 1, 2, \ldots,$ define

$$\phi_k(x) = \phi(x)\rho(x/k) \in \mathscr{D}(\mathbb{R}^n)$$

Therefore,

$$\begin{aligned} |x^\alpha[\phi_k^{(\beta)}(x) - \phi^{(\beta)}(x)]| &= |x^\alpha[\phi^{(\beta)}(x) - D^\beta(\phi(x)\,\rho(x/k))]| \\ &= |x^\alpha D^\beta[\phi(x)\{1-\rho(x/k)\}]| \\ &= \left| x^\alpha \sum_{\nu\le\beta} \binom{\beta}{\nu} D^{\beta-\nu}\phi(x)\, D^\nu\{1-\rho(x/k)\}\right| \end{aligned}$$

Now, $D^\nu\{1-\rho(x/k)\}$ converges to zero as $k \to \infty$ for $|\nu| = 0$, and for $|\nu| \ge 1$, $D^\nu\{1-\rho(x/k)\} = k^{-|\nu|}\rho^{|\nu|}(x/k) \to 0$ uniformly for all x in a bounded domain of $\mathbb{R}^n$. Therefore,

$$\gamma_{\alpha,\beta}(\phi_k - \phi) \le \sum_{\nu\le\beta} \binom{\beta}{\nu} \gamma_{\alpha,\beta-\nu}(\phi) \sup_x |D^\nu\{1-\rho(x/k)\}| \to 0$$

as $k \to \infty$. Thus $\mathscr{D}$ is dense in $\mathscr{S}$.

Next, we show that the convergence in $\mathscr{D}$ implies convergence in $\mathscr{S}$. Let $\{\phi_k(x)\}_{k\in\mathbb{N}}$ converge in $\mathscr{D}$ to zero, then all ϕ_k vanish outside the same bounded domain $|x| = R > 1$; so that

$$\sup_x |x^\alpha \phi_k^{(\beta)}(x)| \leq R^{|\alpha|} \sup_x |\phi_k^{(\beta)}(x)| \to 0 \text{ as } k \to \infty \tag{5.2.8}$$

Therefore, $\{\phi_k\}$ converges in $\mathscr{S}$ to zero. ■

Theorem 5.2.4 $\mathscr{S}(\mathbb{R}^n)$ is a dense subspace of $C^\infty(\mathbb{R}^n)$, with the identity map from $\mathscr{S}(\mathbb{R}^n)$ into $C^\infty(\mathbb{R}^n)$ continuous.

Proof From Theorem 4.2.2 we know that $\mathscr{D}(\mathbb{R}^n)$ is dense in $C^\infty(\mathbb{R}^n)$. This implies that $\mathscr{S}(\mathbb{R}^n)$ is dense in $C^\infty(\mathbb{R}^n)$ because we have the inclusion:

$$\mathscr{D}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$$

In order to prove that the identity map from $\mathscr{S}(\mathbb{R}^n)$ to $C^\infty \mathbb{R}^n)$ is continuous assume that $\{\phi_j\}_{j\in \mathbb{N}}$ is a sequence which converges to zero in $\mathscr{S}(\mathbb{R}^n)$. Then for every compact set $K \subset \mathbb{R}^n$,

$$\sup_K |D^k\phi_j(x)| \leq \sup_{x\in\mathbb{R}^n} |(1+|x|^2)^{1/2m} D^k\phi_j(x)| \to 0 \tag{5.2.9}$$

as $j \to \infty$ for all $m, k \in \mathbb{N}_0^n$, which implies the desired continuity. ■

Theorem 5.2.5 $\mathscr{S}(\mathbb{R}^n)$ is a dense subspace of $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, with the identity map from $\mathscr{S}$ into L^p continuous.

Proof Since, by Theorem 1.2.4, $\mathscr{D}$ is dense in L^p, and $\mathscr{D} \subset \mathscr{S}$, the space $\mathscr{S}$ is dense in L^p.

Now, let $\phi \in \mathscr{S}(\mathbb{R}^n)$. Then by definition $(1+|x|^2)^m\phi(x) \in \mathscr{S}$ for every positive integer m; so $\phi \in L^p$. If we now let $\phi_k \to 0$ in $\mathscr{S}$, then

$$\sup_{x\in\mathbb{R}^n} (1+|x|^2)^m |\phi_k(x)|^p \to 0$$

for every m as $k \to \infty$. Since for $m > n/2$, $(1+|x|^2)^{-m}$ is integrable, we have a positive constant M such that

$$\|\phi_k\|_p^p = \int_{\mathbb{R}^n} (1+|x|^2)^m |\phi_k(x)|^p (1+|x|^2)^{-m}\, dx$$

$$\leq M \sup_{x\in\mathbb{R}^n} (1+|x|^2)^m |\phi_k(x)|^p$$

Therefore $\phi_k \to 0$ in $L^p(\mathbb{R}^n)$. ■

5.3 The Space of Tempered Distributions

This is the space of all continuous and linear functionals on $\mathscr{S}(\mathbb{R}^n)$; the continuity and linearity being defined in the same way as in Section 1.3. The members of $\mathscr{S}'$ are called *tempered distributions* or *distributions of slow growth.*

A sequence $\{f_j\}_{j\in\mathbb{N}}$ in $\mathscr{S}'$ is said to converge to f $\in \mathscr{S}'$ if $\langle f_j, \phi\rangle \to \langle f, \phi\rangle$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$ as $j \to \infty$.

Let f be an element in $\mathscr{S}'$. Since $\mathscr{D}$ is a subspace of $\mathscr{S}$, $\langle f, \phi\rangle$ is defined for $\phi \in \mathscr{D}$. Clearly, f is a linear functional on $\mathscr{D}$. Also, since convergence in $\mathscr{D}$ implies convergence in $\mathscr{S}$, $\{\langle f, \phi_j\rangle\}$ converges to zero whenever $\{\phi_j\}$ converges in $\mathscr{D}$ to zero. Thus f is a distribution in $\mathscr{D}'$. Hence $\mathscr{S}' \subset \mathscr{D}'$.

Also, if we assume that $f \in \mathscr{E}'(\mathbb{R}^n)$, then $\langle f, \phi\rangle$ is defined by $\phi \in C^\infty(\mathbb{R}^n)$. But $\mathscr{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$, hence $\langle f, \phi\rangle$ is defined for $\phi \in \mathscr{S}(\mathbb{R}^n)$ also, and f is a linear functional on $\mathscr{S}$. From (5.2.2) and Definition 4.2.1 it follows that the convergence in $\mathscr{S}$ implies convergence in $C^\infty(\mathbb{R}^n)$, hence $\{\langle f, \phi_j\rangle\} \to 0$ whenever $\{\phi_j\}$ converges to 0 in $\mathscr{S}$. Therefore $f \in \mathscr{S}'(\mathbb{R}^n)$.

Altogether we have proved the following

Theorem 5.3.1 We have the inclusions

$$\mathscr{E}'(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n) \subset \mathscr{D}'(\mathbb{R}^n)$$

with the identity maps from $\mathscr{S}'$ to $\mathscr{D}'$ and from $\mathscr{S}'$ to $\mathscr{D}'$ continuous.

We can show by means of an example that $\mathscr{S}'$ is a proper subspace of $\mathscr{D}'$. For, the series

$$f(t) = \sum_{n=1}^{\infty} e^{n^2}\delta(t-n)$$

converges in $\mathscr{D}'$ and defines a distribution in $\mathscr{D}'$, but it does not converge in $\mathscr{S}'$. To see this, let $\phi \in \mathscr{D}(\mathbb{R})$, then

$$\langle f, \phi\rangle = \sum_{n=1}^{\infty} e^{n^2}\phi(n)$$

Since ϕ is of compact support the last series possesses only a finite number of terms. On the other hand, if we choose $\phi(t) = e^{-1/2t^2} \in \mathscr{S}(\mathbb{R})$, then the last series does not converge. Hence $f(t) \notin \mathscr{S}'(\mathbb{R})$.

Example 5.3.2 $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, is a subspace of $\mathscr{S}'(\mathbb{R}^n)$.

For, let $f \in L^p(\mathbb{R}^n)$; then for every $\phi \in \mathscr{S}(\mathbb{R}^n)$,

$$|\langle f, \phi\rangle| = \left|\int f\phi\, dx\right| \le \|f\|_{L^p}\|\phi\|_{L^q} < \infty, \ (1/p + 1/q = 1)$$

since $\phi \in \mathscr{S} \subset L^q$. Moreover, if $\{\phi_k\}$ converges in $\mathscr{S}$ to zero then from Theorem 5.2.5 it also converges in L^q to zero; the above inequality implies that $\langle f, \phi_k\rangle \to 0$. Hence f is continuous on $\mathscr{S}$. Obviously, it is linear on $\mathscr{S}$.

Thus $f \in \mathscr{S}'$. Therefore every $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, defines a tempered distribution by means of the relation

$$\langle f, \phi \rangle = \int_{\mathbb{R}^n} f(t)\, \phi(t)\, dt,\ \phi \in \mathscr{S}(\mathbb{R}^n)$$

Example 5.3.3 Every polynomial $P(x) = \sum_{|\alpha| \le m} a_\alpha x^\alpha$, $a_\alpha \in \mathbb{C}$, defines a tempered distribution by

$$\langle f, \phi \rangle = \int_{\mathbb{R}^n} P(x)\, \phi(x)\, dx,\ \phi \in \mathscr{S}(\mathbb{R}^n),$$

because

$$|\langle f, \phi \rangle| \le \sum_{|\alpha| \le m} |a_\alpha| \int_{\mathbb{R}^n} |x^\alpha \phi(x)|\, dx$$

$$\le \sum_{|\alpha| \le m} |a_\alpha| \int_{\mathbb{R}^n} (1 + |x|^2)^{m/2} |\phi(x)|\, dx$$

$$\le \sum_{|\alpha| \le m} |a_\alpha|\, \tau_{m+2n,0}(\phi) \int_{\mathbb{R}^n} (1 + |x|^2)^{-n}\, dx < \infty$$

The continuity of f follows from this inequality. The linearity is obvious.

Example 5.3.4 Let f be a continuous function on $\mathbb{R}$ such that for every $m \in \mathbb{N}$,

$$(1 + |x|^2)^{-m/2} |f(x)|\, dx < \infty \tag{5.3.1}$$

then f defines a tempered distribution. Let us set

$$\langle f, \phi \rangle = \int_{\mathbb{R}^n} f(x)\, \phi(x)\, dx,\ \phi \in \mathscr{S}(\mathbb{R}^n)$$

Then

$$|\langle f, \phi \rangle| \le \int_{\mathbb{R}^n} |(1 + |x|^2)^{-m/2} f(x)|\, |(1 + |x|^2)^{m/2} \phi(x)|\, dx,\ m \in \mathbb{N}$$

$$\le C \int_{\mathbb{R}^n} |(1 + |x|^2)^{1/2m+n} \phi(x)|\, dx/(1 + |x|^2)^n$$

$$\le C'\, \tau_{m+2n,0}(\phi)$$

for certain positive constant C'. Using this inequality we can show the continuity; the linearity is trivial.

Any function f satisfying (5.3.1) is called *slowly increasing function* at infinity, or a *tempered function*; which justifies the name tempered distribution.

Example 5.3.5 Any distributional derivative of a continuous tempered function defines a tempered distribution. For, let $g = D^\alpha f$, where f is a tempered function as in Example 5.3.4; then

$$\langle g, \phi \rangle = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x) \cdot D^\alpha \phi(x)\, dx \ \forall \ \phi \in \mathscr{S}$$

Now, the conclusion follows from Example 5.3.4.

Example 5.3.6 The function $f(x) = e^x \cos(e^x)$, $x \in \mathbb{R}$, is not a slowly increasing function at ∞, still it defines a tempered distribution.

There does not exist any $m \in \mathbb{N}$ such that $| x |^{-m} e^x | \cos(e^x) |$ is bounded as $x \to \infty$. But for any $\phi \in \mathscr{S}(\mathbb{R})$,

$$\begin{aligned}
\left| \int_{-\infty}^{\infty} f(x)\, \phi(x)\, dx \right| &= \left| \int_{-\infty}^{\infty} e^x \cos(e^x)\, \phi(x)\, dx \right| \\
&= \left| [\sin(e^x)\phi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \sin(e^x)\phi'(x)\, dx \right| \\
&= \left| \int_{-\infty}^{\infty} \sin(e^x)\phi'(x)\, dx \right| \\
&\le \int_{-\infty}^{\infty} (1 + x^2)\, |\, \phi'(x)\, |\, dx/(1 + x^2) \\
&\le \pi \sup_{\mathbb{R}} (1 + x^2)\, |\, \phi'(x)\, |
\end{aligned}$$

Thus f defines a tempered distribution.

5.4 Multipliers in $\mathscr{S}'(\mathbb{R}^n)$

In Section 1.5 we have seen that the product of $f \in \mathscr{D}'(\mathbb{R}^n)$ with $\psi \in C^\infty(\mathbb{R}^n)$ is always defined and $\psi f \in \mathscr{D}'(\mathbb{R}^n)$. In general, the product of $f \in \mathscr{S}'(\mathbb{R}^n)$ with $\psi \in C^\infty(\mathbb{R}^n)$ is not defined, as can be seen by taking $\psi(x) = \exp(\,|\,x\,|^2)$. However, the product ψf can be defined if ψ belongs to space of multipliers θ_M of $\mathscr{S}'(\mathbb{R}^n)$ defined as follows:

Definition 5.4.1 By θ_M we denote the space of all infinitely differentiable functions ψ such that for each $\alpha \in \mathbb{N}^n$ there exists a polynomial $P_\alpha(x)$ such that

$$| D^\alpha \psi(x) | \le | P_\alpha(x) |, \ \forall\ x \in \mathbb{R}^n \tag{5.4.1}$$

As an example, we can take $\psi(x) = \exp(i\,|\,x\,|^2)$.

Let $f \in \mathscr{S}'(\mathbb{R}^n)$ and $\psi \in \theta_M$. Define

$$\langle \psi f, \phi \rangle := \langle f, \psi\phi \rangle \ \forall\ \phi \in \mathscr{S}(\mathbb{R}^n) \tag{5.4.2}$$

To justify the above definition we need to show that $\psi\phi \in \mathscr{S}(\mathbb{R}^n)$. Indeed, using Leibnitz theorem we have

$$D^\alpha(\psi\phi) = \sum_{\nu \le \alpha} \binom{\alpha}{\nu} (D^\nu \psi)(D^{\alpha-\nu}\,\phi)$$

Since ψ satisfies condition (5.4.1) there exists a positive integer N such that

$$| D^\nu \psi(x) | \le C(1 + |\,x\,|^2)^N \ \forall\ x \in \mathbb{R}^n,\ \forall\ \nu \le \alpha$$

Therefore, by inequality (5.2.3)

$$| x^\beta D^\alpha(\psi\phi) | \le C \sum_{\nu \le \alpha} \binom{\alpha}{\nu} (1 + |\,x\,|^2)^N\, | x^\beta D^{\alpha-\nu}\phi(x) |$$

$$\le C \sum_{\nu \le \alpha} \binom{\alpha}{\nu} (1 + |\,x\,|^2)^{1/2(2N+|\beta|)}\, | D^{\alpha-\nu}\phi(x) |$$

so that

$$\gamma_{\beta,\alpha}(\psi\phi) \le C \sum_{\nu \le \alpha} \binom{\alpha}{\nu} \tau_{2N+|\beta|,\alpha-\nu}(\phi) \tag{5.4.3}$$

Hence $\psi\phi \in \mathscr{S}(\mathbb{R}^n)$. From this inequality we also conclude that the map $\phi \to \psi\phi$ is continuous from $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}(\mathbb{R}^n)$.

Now, we show that $\psi f \in \mathscr{S}'(\mathbb{R}^n)$. Let $\{\phi_j\}_{j \in \mathbb{N}}$ converge in $\mathscr{S}$ to zero, then by inequality (5.4.3) we conclude that $\{\psi\phi_j\}$ converges in $\mathscr{S}$ to zero. Since $f \in \mathscr{S}'$, the right-hand side of (5.4.2) converges to zero, which in turn implies the continuity of ψf. As ψf is also linear on $\mathscr{S}$, it is a tempered distribution.

Next, to show that the map $f \to \psi f \in \mathscr{S}'(\mathbb{R}^n)$ is continuous, assume that $\{f_j\}_{j \in \mathbb{N}}$ converges to f in $\mathscr{S}'$, so that $\{\langle f_j, \psi\phi \rangle\}$ converges to $\langle f, \psi\phi \rangle$. Then (5.4.2) implies that $\langle \psi f_j, \phi \rangle \to \langle \psi f, \phi \rangle \ \forall\ \phi \in \mathscr{S}$.

Altogether we have proved:

Theorem 5.4.2 Let $\psi \in \theta_M$; then the map $f \to \psi f$ is continuous from $\mathscr{S}'$ into $\mathscr{S}'$.

5.5 The Fourier Transform on $L^1(\mathbb{R}^n)$

The Fourier transform of a function $\phi \in L^1(\mathbb{R}^n)$, is defined by

$$\hat{\phi}(\xi) = (\mathcal{F}\phi)(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-i(x,\xi)}\, dx,\ \xi \in \mathbb{R}^n \tag{5.5.1}$$

where $(x, \xi) = \sum_{i=1}^{n} x_i \xi_i$. Clearly,

$$|\hat{\phi}(\xi)| \le \int_{\mathbb{R}^n} |\phi(x)|\, dx = \|\phi\|_{L^1}$$

Moreover, if $\{\xi_k\}$ is a sequence in $\mathbb{R}^n$ which converges to ξ, then

$$|\hat{\phi}(\xi_k) - \hat{\phi}(\xi)| \le \int_{\mathbb{R}^n} |f(x)|\, |e^{-i(\xi_k, x)} - e^{-i(\xi, x)}|\, dx \to 0$$

as $\xi_k \to \xi$ by Lebesgue's convergence theorem. Thus ϕ is a bounded continuous function on $\mathbb{R}^n$. In general, $\hat{\phi}$ is not integrable. For example, let $\phi(x) = 1$ on $(0, 1)$ and zero elsewhere. Then $\hat{\phi}(\xi) = 2 \sin \xi/\xi \notin L^1(\mathbb{R})$.

If $\hat{\phi}$ also happens to be integrable one can express ϕ in terms of $\hat{\phi}$ by *Fourier's inversion formula*

$$\phi(x) = (\mathcal{F}^{-1}\hat{\phi})(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x,\xi)}\, \hat{\phi}(\xi) d\xi,\ x \in \mathbb{R}^n \tag{5.5.2}$$

When $f, \phi \in L^1(\mathbb{R}^n)$ we also have $f\hat{\phi} \in L^1(\mathbb{R}^n)$, since $\hat{\phi}$ is bounded, and

$$\int_{\mathbb{R}^n} f(x)\hat{\phi}(x)\, dx = \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} \phi(\xi)\, e^{-i(x,\xi)}\, d\xi\, dx$$

$$= \int_{\mathbb{R}^n} \phi(\xi) \left(\int_{\mathbb{R}^n} f(x)\, e^{-i(x,\xi)}\, dx \right) d\xi$$

by Fubini's theorem. Therefore

$$\int_{\mathbb{R}^n} f(x)\hat{\phi}(x)\, dx = \int_{\mathbb{R}^n} \hat{f}(\xi)\, \phi(\xi)\, d\xi \tag{5.5.3}$$

This is called the *Parseval formula* for Fourier transforms. Another form of this formula is

$$\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{\phi}(\xi)\, d\xi = (2\pi)^n \int_{\mathbb{R}^n} f(t)\, \phi(-t)\, dt \tag{5.5.4}$$

An immediate consequence of the above formula is

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi = (2\pi)^n \int_{\mathbb{R}^n} |f(x)|^2 dx, f \in L^2(\mathbb{R}^n) \quad (5.5.5)$$

From (5.5.1) and (5.5.2) the following relation follows:

$$\mathscr{F}^{-1}[f(x)] = (2\pi)^{-n} \mathscr{F}[\check{f}(\xi)](x) \quad (5.5.6)$$

where $\check{f}(\xi) = f(-\xi)$

If $f \in L^1(\mathbb{R}^n)$ and $x^\alpha f \in L^1(\mathbb{R}^n)$, for each n-tuple α with $\alpha_i \geq 0$, $i \in \mathbb{N}$, then

$$D^\alpha(f^\wedge(\xi)) = ((-ix)^\alpha f)^\wedge(\xi) \quad (5.5.7)$$

where $D^\alpha = (\partial/\partial\xi_i)^{\alpha_1} \dots (\partial/\partial\xi_n)^{\alpha_n}$.

Also, if $f \in L^1(\mathbb{R}^n)$ and $D^\alpha f \in L^1(\mathbb{R}^n)$ for each n-tuple α with $\alpha_i \geq 0$, $i \in \mathbb{N}$, then

$$(D^\alpha \hat{f})(\xi) = (i\xi)^\alpha \hat{f}(\xi) \quad (5.5.8)$$

where $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$.

5.5.1 The Fourier Transform on $\mathscr{S}(R^n)$

To extend the Fourier transform to tempered distributions we study it at first on the space $\mathscr{S}(\mathbb{R}^n)$. The basic result is the following

Theorem 5.5.1 The Fourier transform is a continuous isomorphism from $\mathscr{S}(\mathbb{R}^n)$ onto $\mathscr{S}(\mathbb{R}^n)$; its inverse, given by (5.5.2), is also a continuous isomorphism from $\mathscr{S}(\mathbb{R}^n)$ onto $\mathscr{S}(\mathbb{R}^n)$.

Proof (i) Let $\phi(x) \in \mathscr{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$; then its Fourier transform

$$\hat{\phi}(\xi) = (\mathscr{F}f)(\xi) = \int_{\mathbb{R}^n} e^{-i(\xi,x)} \phi(x)\, dx$$

exists. Since the integral is uniformly convergent with respect to ξ we may differentiate within the integral sign and get

$$\partial/\partial\xi_j \hat{\phi}(\xi) = \int_{\mathbb{R}^n} e^{-i(\xi,x)} (-i\, x_j)\phi(x)\, dx$$

Since $x^\alpha\phi(x) \in \mathscr{S}$ for each $\alpha \in \mathbb{N}_0^n$, we may apply the above procedure repeatedly and get

$$\begin{aligned} D_\xi^\alpha \hat{\phi}(\xi) &= \int_{\mathbb{R}^n} e^{-i(\xi,x)} (-1)^{|\alpha|}\, x^\alpha \phi(x)\, dx \\ &= \mathscr{F}[(-ix)^\alpha \phi(x)](\xi) \end{aligned} \quad (5.5.9)$$

Now, integrating by parts with respect to each of the variable x_j, $j = 1, \ldots, n$, and noting that for $\phi \in \mathscr{S}$,

$$(\partial/\partial x_j)^{\beta_j}(x^\alpha \phi(x)) \to 0$$

as $x_j \to \pm\infty$, we obtain

$$(i\xi)^\beta D^\alpha \hat{\phi}(\xi) = \int_{\mathbb{R}^n} e^{-i(\xi,x)} D_x^\beta\, [(-i)^{|\alpha|} x^\alpha \phi(x)]\, dx$$

$$= \mathscr{F}[D_x^\beta((-i)^{|\alpha|} x^\alpha \phi(x)]\,(\xi) \tag{5.5.10}$$

Hence

$$|\,\xi^\beta D^\alpha \hat{\phi}(\xi)\,| = \left| \int_{\mathbb{R}^n} \sum_{\nu \le \beta} \binom{\beta}{\nu} (D^\nu x^\alpha)(D^{\beta-\nu}\phi(x))\, dx \right|$$

$$\le \sum_{\nu \le \beta} a(\alpha, \nu) \int_{\mathbb{R}^n} |\, x^{\alpha-\nu} D^{\beta-\nu}\phi(x)\,|\, dx$$

where $a(\alpha, \nu)$ is a positive constant. Using the inequality (5.2.3) we get

$$\gamma_{\beta,\alpha}(\hat{\phi}(\xi)) \le \sum_{\nu \le \beta} a(\alpha, \nu) \int_{\mathbb{R}^n} |\,(1 + |\,x\,|^2)^{1/2|\alpha-\nu|} D^{\beta-\nu}\phi(x)\,|\, dx$$

$$\le \sum_{\nu \le \beta} a(\alpha, \nu) \int_{\mathbb{R}^n} |(1+|\,x\,|^2)^{1/2|\alpha-\nu|+n} D^{\beta-\nu}\phi(x)|/(1+|\,x\,|^2)^n$$

$$\le \sum_{\nu \le \beta} a(\alpha, \nu)\, \tau_{|\alpha-\nu|+2n,\beta-\nu}(\phi(x)) \int_{\mathbb{R}^n} dx/(1 + |\,x\,|^2)^n$$

Since the last integral is finite we can write

$$\gamma_{\beta,\alpha}(\hat{\phi}) \le \sum_{\nu \le \beta} C(\alpha, \nu, n)\, \tau_{|\alpha-\nu|+2n,\beta-\nu}(\phi) \tag{5.5.11}$$

where $C(\alpha, \nu, n)$ is an other constant. This proves that $\hat{\phi} \in \mathscr{S}(\mathbb{R}^n)$. Also, from (5.5.1) and (5.5.2) we see that for all $\phi \in \mathscr{S}$,

$$\mathscr{F}^{-1}\mathscr{F}\phi = \phi = \mathscr{F}\mathscr{F}^{-1}\phi$$

It follows that $\mathscr{F}$ is a one-one function of $\mathscr{S}$ onto itself. $\mathscr{F}$ is clearly a linear map of $\mathscr{S}$ onto itself. To show that it is continuous assume that the sequence $\{\phi_j\}_{j\in\mathbb{N}}$ converges in $\mathscr{S}(\mathbb{R}^n)$ to zero, then from (5.5.11) it follows that $\gamma_{\beta,\alpha}(\hat{\phi}_j) \to 0$ as $j \to \infty$. This shows the continuity of the Fourier transform. Similarly, we can show that the inverse Fourier transform (5.5.2) is also a continuous linear map from $\mathscr{S}$ onto $\mathscr{S}$. This completes the proof. ∎

5.6 The Fourier Transform on $\mathscr{S}'(\mathbb{R}^n)$

Motivated by the Parseval formula (5.5.3) we define the Fourier transform of tempered distributions. For the sake of convenience the Fourier transform of a distribution f will also be denoted by $\mathscr{F}f = \hat{f}$.

Definition 5.6.1 The Fourier transform $\mathscr{F}f$ of $f \in \mathscr{S}'(\mathbb{R}^n)$ is defined by

$$\langle \mathscr{F}f, \phi \rangle := \langle f, \mathscr{F}\phi \rangle, \ \phi \in \mathscr{S}(\mathbb{R}^n) \tag{5.6.1}$$

Since by Theorem 5.5.1, $\mathscr{F}\phi \in \mathscr{S}$ whenever $\phi \in \mathscr{S}$, the right-hand side of (5.6.1) is well defined. Also, if $\{\phi_j\}$ converges in $\mathscr{S}$ to zero, then by continuity of the Fourier transform $\{\mathscr{F}\phi_j\} \to 0$, and so the right-hand side of (5.6.1) converges to zero, which in turn implies that $\{\langle \mathscr{F}f, \phi_j \rangle\} \to 0$ as $j \to \infty$. Thus $\mathscr{F}f$ is continuous on $\mathscr{S}$. Clearly $\mathscr{F}f$ is linear on $\mathscr{S}$.

Similarly the inverse Fourier transform $\mathscr{F}^{-1}f$ of $f \in \mathscr{S}'(\mathbb{R}^n)$ is defined by

$$\langle \mathscr{F}^{-1}f, \phi \rangle := \langle f, \mathscr{F}^{-1}\phi \rangle, \ \phi \in \mathscr{S}(\mathbb{R}^n) \tag{5.6.2}$$

It can be shown, as in the above, that $\mathscr{F}^{-1}f \in \mathscr{S}'(\mathbb{R}^n)$. Thus we have

Theorem 5.6.2 The Fourier transform of a tempered distribution is a continuous isomorphism from $\mathscr{S}'(\mathbb{R}^n)$ onto $\mathscr{S}'(\mathbb{R}^n)$.

From (5.6.1) and (5.6.2) it follows that for $f \in \mathscr{S}'$ and $\phi \in \mathscr{S}$,

$$\langle f, \phi \rangle = \langle \mathscr{F}f, \mathscr{F}^{-1}\phi \rangle = \langle f, \mathscr{F}\mathscr{F}^{-1}\phi \rangle$$

so that

$$\mathscr{F}^{-1}\mathscr{F}f = f \tag{5.6.3}$$

Similarly, $\mathscr{F}\mathscr{F}^{-1}f = f$. Thus, $\mathscr{F}$ and $\mathscr{F}^{-1}$ are one-one maps of $\mathscr{S}'$ onto itself.

Sometimes, motivated by the other Parseval formula (5.5.4), the Fourier transform of $f \in \mathscr{S}'$ is defined by

$$\langle \mathscr{F}f, \mathscr{F}\phi \rangle = (2\pi)^n \langle f, \check{\phi} \rangle, \ \phi \in \mathscr{S} \tag{5.6.4}$$

where $\check{\phi}(x) = \phi(-x)$. This gives

$$\langle f, \phi \rangle = (2\pi)^{-n} \langle \mathscr{F}f, \mathscr{F}\check{\phi} \rangle, \ \phi \in \mathscr{S} \tag{5.6.5}$$

Also, since

$$\mathscr{F}^{-1}[\phi] = (2\pi)^{-n}\mathscr{F}[\check{\phi}], \ \phi \in \mathscr{S} \tag{5.6.6}$$

the definition (5.6.2) yields

$$\mathscr{F}^{-1}[f] = (2\pi)^{-n}(\mathscr{F}\,\check{\phi}),\ f \in \mathscr{S}' \tag{5.6.7}$$

It is worthwhile to note that the Fourier transform and inverse Fourier transform of tempered distributions are linear and continuous operations from $\mathscr{S}'$ onto itself. The linearity is obvious. To show the continuity, assume that $\{f_j\}_{j\in\mathbb{N}}$ converges in $\mathscr{S}'(\mathbb{R}^n)$ to f. Since $\mathscr{S}'$ is closed under convergence $f \in \mathscr{S}'$, and hence $\mathscr{F}f \in \mathscr{S}'$. Therefore,

$$\langle \mathscr{F} f_j, \phi\rangle = \langle f_j, \mathscr{F}\phi\rangle \to \langle f, \mathscr{F}\phi\rangle = \langle \mathscr{F}f, \phi\rangle\ \forall\ \phi \in \mathscr{S}$$

which shows the continuity of $\mathscr{F}$. The case of inverse Fourier transform is similar.

Now, let

$$g = \sum_{j=1}^{\infty} g_j \text{ in } \mathscr{S}' \tag{5.6.8}$$

Then, for each $\phi \in \mathscr{S}(\mathbb{R}^n)$ we have

$$\langle \mathscr{F}g, \phi\rangle = \langle g, \mathscr{F}\phi\rangle = \lim_{N\to\infty}\left\langle \sum_{j=1}^{N} g_j, \mathscr{F}\phi\right\rangle$$

$$= \lim_{N\to\infty}\left\langle \sum_{j=1}^{N} \mathscr{F}g_j, \phi\right\rangle$$

Thus
$$\mathscr{F}g = \sum_{j=1}^{\infty} \mathscr{F}g_j \text{ in } \mathscr{S}' \tag{5.6.9}$$

Consequently, the Fourier transform may be applied to the series in (5.6.8) term by term, which is not true, in general, in the classical sense.

Example 5.6.3

$$\mathscr{F}[\delta(x-a)](\xi) = e^{-i(\xi,a)},\ x \in \mathbb{R}^n,\ a \in \mathbb{R} \tag{5.6.10}$$

Indeed, for $\phi \in \mathscr{S}(\mathbb{R}^n)$, we have

$$\begin{aligned}\langle \mathscr{F}[\delta(x-a)], \phi\rangle &= \langle \delta(x-a), \mathscr{F}\phi\rangle \\ &= (\mathscr{F}\phi)(a) \\ &= \int_{\mathbb{R}^n} \phi(\xi)\, e^{-i(\xi,a)}\, d\xi = \langle e^{-i(a,\xi)}, \phi(\xi)\rangle\end{aligned}$$

Setting $a = 0$ in (5.6.10) we get

$$\mathscr{F}[\delta] = 1 \tag{5.6.11}$$

Applying (5.6.3) to (5.6.10) we get

$$\delta(x-a) = \mathscr{F}^{-1}[e^{-i(\xi,a)}]$$

to which an application by (5.6.7) yields

$$\delta(x-a) = (2\pi)^{-n}(\mathscr{F}\, e^{-i(\xi,a)})^{\vee}$$

so that

$$\mathscr{F}\,[e^{-i(x,a)}] = (2\pi)^n\delta(\xi + a) \tag{5.6.12}$$

Putting $a = 0$ in (5.6.12), we get

$$\mathscr{F}\,[1] = (2\pi)^n\delta$$

Example 5.6.4

$$\mathscr{F}\,[\delta^{(k)}(x-a)]\,(\xi) = (i\xi)^k e^{-i(\xi,a)},\ x \in \mathbb{R}^n, a \in \mathbb{R}, k \in \mathbb{N}_0 \tag{5.6.13}$$

For any $\phi \in \mathscr{S}$, we have

$$\begin{aligned}\langle \mathscr{F}\,[\delta^{(k)}(x-a)](\xi), \phi(\xi)\rangle &= \langle \delta^{(k)}(x-a), (\mathscr{F}\phi)(x)\rangle \\ &= (-1)^k\langle \delta(x-a), D^k(\mathscr{F}\phi)(x)\rangle \\ &= (-1)^k\mathscr{F}[(-i)^k x^k\phi(x)](a) \text{ by (5.5.9)} \\ &= (i)^k\int_{-\infty}^{\infty} e^{-i(a,\xi)}\xi^k\phi(\xi)\,d\xi \\ &= \langle (i\xi)^k e^{-i(a,\xi)}, \phi(\xi)\rangle\end{aligned}$$

Applying inversion formula (5.6.3) and proceeding as in the previous example one can prove that

$$\mathscr{F}\,[(ix)^k e^{-i(x,a)}](\xi) = (2\pi)^n\delta^{(k)}(\xi + a) \tag{5.6.14}$$

Example 5.6.5 Every periodic locally integrable function on $\mathbb{R}$ with period 2π can be written in the form of Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

which converges in $\mathscr{S}'(\mathbb{R})$. Taking Fourier transform and using (5.6.10) we obtain

$$(\mathscr{F}f(\xi) = \sum_{n=-\infty}^{\infty} C_n(2\pi)^n\delta(\xi + n)$$

Thus $\mathscr{F}f$ is a tempered distribution concentrated on the countable set of points $\xi = \pm\, n,\ n \in \mathbb{N}_0$.

Example 5.6.6 (*Poisson Summation Formula*) From (3.5.3) we know that

$$\sum_{m=-\infty}^{\infty} \delta(x - 2m\pi) = (2\pi)^{-1} \sum_{m=-\infty}^{\infty} e^{imx} \quad in\ \mathscr{D}'(\mathbb{R})$$

Both series can be shown to be convergent in $\mathscr{S}'(\mathbb{R})$ also. Applying Fourier transform to both sides, using formula (5.6.10) and the continuity property of the Fourier transform we can write

$$\mathscr{F}\left[\sum_m \delta(x - 2m\pi)\right] = \sum_m \delta(\xi - m).$$

Therefore for any $\phi \in \mathscr{S}$,

$$\langle \sum_m \delta(\xi - m), \phi(\xi) \rangle = \langle \sum_m \delta(x - 2m\pi), (\mathscr{F}\phi)(x) \rangle,$$

whence
$$\sum_{m=-\infty}^{\infty} \phi(m) = \sum_{m=-\infty}^{\infty} \hat{\phi}(2m\pi).$$

5.7 Properties of the Fourier Transform on $\mathscr{S}'(\mathbb{R}^n)$

The following formulae hold for test functions in $\mathscr{S}$ and distributions in $\mathscr{S}'$.

(i) Differentiation of a Fourier transform:

$$D^\alpha \mathscr{F}[f] = (-i)^{|\alpha|} \mathscr{F}[x^\alpha f], \alpha \in \mathbb{N}_0^n \qquad (5.7.1)$$

Indeed, for $\phi \in \mathscr{S}(\mathbb{R}^n)$,

$$\begin{aligned}
\langle D^\alpha \mathscr{F}[f]\,(\xi), \phi(\xi) \rangle &= (-1)^{|\alpha|} \langle \mathscr{F}[f]\,(\xi), D^\alpha \phi(\xi) \rangle \\
&= (-1)^{|\alpha|} \langle f(x), \mathscr{F}[D^\alpha \phi(\xi)]\,(x) \rangle \\
&= (-1)^{|\alpha|} \langle f(x), (ix)^\alpha \hat{\phi}(x) \rangle \quad \text{by (5.5.8)} \\
&= \langle \mathscr{F}\,[(-ix)^\alpha f(x)]\,(\xi), \phi(\xi) \rangle.
\end{aligned}$$

(ii) The Fourier transform of a derivative:

$$\mathscr{F}[D^\alpha f] = (i\xi)^\alpha \mathscr{F}[f], \ \alpha \in \mathbb{N}_0^n. \qquad (5.7.2)$$

For, let $\phi \in \mathscr{S}(\mathbb{R}^n)$; then

$$\begin{aligned}
\langle \mathscr{F}[D^\alpha f]\,(\xi), \phi(\xi) \rangle &= \langle D^\alpha f(x), \hat{\phi}(x) \rangle \\
&= (-1)^{|\alpha|} \langle f(x), D^\alpha \hat{\phi}(x) \rangle \\
&= \langle f(x), (ix)^\alpha \hat{\phi}(x) \rangle \quad \text{by (5.5.7)} \\
&= \langle \mathscr{F}[(ix)^\alpha f(x)]\,(\xi), \phi(\xi) \rangle.
\end{aligned}$$

(iii) The Fourier transform of a translation:

$$\mathscr{F}[f(x-x_0)] = e^{-i(\xi, x_0)}\, \mathscr{F}[f]. \tag{5.7.3}$$

(iv) The translation of a Fourier transform:

$$(\mathscr{F} f)\,(\xi + \xi_0) = \mathscr{F}[e^{-i(\xi_0, x)} f]\,(\xi). \tag{5.7.4}$$

Setting $f = 1$ in (5.7.1) we can show that

$$\mathscr{F}\,[x^\alpha] = i^{|\alpha|}\, D^\alpha \mathscr{F}\,[1] = (2\pi)^n\, i^{|\alpha|}\, D^\alpha \delta(\xi). \tag{5.7.5}$$

Also, setting $f = \delta$ in (5.7.2) we get

$$\mathscr{F}\,[D^\alpha \delta] = (i\xi)^\alpha \mathscr{F}\,[\delta] = i^{|\alpha|}\, \xi^\alpha. \tag{5.7.6}$$

5.8 Convolution Theorem in $\mathscr{S}'(\mathbb{R}^n)$

The so-called convolution theorem

$$\mathscr{F}(f * g) = (\mathscr{F} f)\,(\mathscr{F} g) \tag{5.8.1}$$

plays a central role in the classical theory of Fourier transforms. It has important applications in differential equations. The relation holds if, for example, when f, g are in $\mathscr{S}(\mathbb{R}^n)$, in which case $f * g$ is also in $\mathscr{S}(\mathbb{R}^n)$. This can be proved as follows:

Using Fubini's theorem we have

$$\begin{aligned}
\mathscr{F}\,[f * g]\,(\xi) &= \int_{\mathbb{R}^n} e^{-i(x,\xi)} (f * g)(x)\, dx \\
&= \int_{\mathbb{R}^n} e^{-i(x,\xi)} \left\{ \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy \right\} dx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i(x-y,\xi)}\, e^{-i(y,\xi)} f(y)\, g(x-y)\, dx\, dy \\
&= \int_{\mathbb{R}^n} e^{-i(y,\xi)} \left\{ \int_{\mathbb{R}^n} e^{-i(x-y,\xi)}\, g(x-y)\, dx \right\} f(y)\, dy \\
&= \hat{g}(\xi) \cdot \int_{\mathbb{R}^n} e^{-i(y,\xi)} f(y)\, dy = \hat{f}(\xi) \cdot \hat{g}(\xi) \in \mathscr{S}(\mathbb{R}^n),
\end{aligned}$$

by (5.4.3). Then by inverse Fourier transform we conclude that $f * g \in \mathscr{S}$. Similarly, for $f, g \in \mathscr{S}(\mathbb{R}^n)$ we have

$$\mathscr{F}^{-1}(f * g)\,(\xi) = (2\pi)^n\, \mathscr{F}^{-1}[f]\,(\xi)\, \mathscr{F}^{-1}[g]\,(\xi) \in \mathscr{S}(\mathbb{R}^n). \tag{5.8.2}$$

A generalization of (5.8.1) to distributions is now given. Let $f \in \mathscr{S}'(\mathbb{R}^n)$ and $g \in \mathscr{S}(\mathbb{R}^n)$. Then motivated by the definition of convolution given in section 4.5, we define the convolution $f*g$ as a functional on $\mathscr{S}$, given by

$$\langle f*g, \phi\rangle = \langle f(x), \psi(x)\rangle, \tag{5.8.3}$$

where $\psi(x) = \langle g(y), \phi(x+y)\rangle$, $\phi \in \mathscr{S}(\mathbb{R}^n)$

$$= \int_{\mathbb{R}^n} g(y)\,\phi(x+y)\,dy = \int_{\mathbb{R}^n} \phi(y)\,g(y-x)dy$$

$$= \int_{\mathbb{R}^n} \phi(y)\check{g}(x-y)\,dy = (\phi * \check{g})(x) \in \mathscr{S}(\mathbb{R}^n).$$

Hence $f*g$ is well-defined by (5.8.3). Clearly, $f*g$ is a linear functional on $\mathscr{S}$. To show that it is continuous also; assume that $\{\phi_j\}_{j\in\mathbb{N}}$ is a sequence which converges in $\mathscr{S}$ to zero. Then

$$\psi_j(x) = (\phi_j * \check{g})(x) = \mathscr{F}^{-1}(\mathscr{F}(\phi_j * \check{g}))(x)$$

$$= \mathscr{F}^{-1}[(\mathscr{F}\phi_j)(\mathscr{F}\check{g})] \to 0$$

by continuity of Fourier transform on $\mathscr{S}$. Therefore,

$$\langle f*g, \phi_j\rangle = \langle f(x), \psi_j(x)\rangle \to 0 \text{ as } j \to \infty.$$

Thus $f*g \in \mathscr{S}'$.

We can now give an extension of (5.8.1) to distributions.

Theorem 5.8.1 Let $f \in \mathscr{S}'(\mathbb{R}^n)$ and $g \in \mathscr{S}(\mathbb{R}^n)$; then

$$\mathscr{F}(f*g) = (\mathscr{F}f)(\mathscr{F}g). \tag{5.8.4}$$

Proof Let $\phi \in \mathscr{S}(\mathbb{R}^n)$; then

$$\langle \mathscr{F}(f*g), \phi\rangle = \langle f*g, \mathscr{F}\phi\rangle = \langle f(x), \langle g(y), (\mathscr{F}\phi)(x+y)\rangle\rangle$$

$$= \langle f, (\mathscr{F}\phi) * \check{g}\rangle.$$

Now, using (5.8.1) and the inversion formula (5.5.2), we have

$$\mathscr{F}\phi * \check{g} = \mathscr{F}\phi * \mathscr{F}(\mathscr{F}^{-1}\check{g}) = (2\pi)^{-n}\,\mathscr{F}\phi * \mathscr{F}(\mathscr{F}g)$$

$$= (2\pi)^{-n}\,\mathscr{F}[\mathscr{F}^{-1}(\mathscr{F}\phi * \mathscr{F}\hat{g})] = \mathscr{F}(\phi\hat{g})$$

by (5.8.2). Hence

$$\langle \mathscr{F}(f*g), \phi\rangle = \langle f, \mathscr{F}(\phi\mathscr{F}g)\rangle = \langle \mathscr{F}f, \phi\mathscr{F}g\rangle = \langle (\mathscr{F}f)(\mathscr{F}g), \phi\rangle \quad \blacksquare$$

Problem 5.8.2 If $f \in \mathscr{S}'(\mathbb{R}^n)$ and $\phi \in \mathscr{S}(\mathbb{R}^n)$, define

$$(f * \phi)(x) = \langle f(y), \phi(x-y) \rangle.$$

Using the method of proof of Lemma 4.3.2 prove that $f * \phi \in C^\infty(\mathbb{R}^n)$.

5.9 The Fourier Transform on $\mathscr{E}'(\mathbb{R}^n)$

Since $e^{-i(x,\xi)} \in C^\infty(\mathbb{R}^n)$, the Fourier transform of $f \in \mathscr{E}'(\mathbb{R}^n)$ can be defined by

$$\hat{f}(\xi) = (\mathscr{F} f)(\xi) = \langle f(x), e^{-i(x,\xi)} \rangle \tag{5.9.1}$$

$$= \langle f(x), \lambda(x)\, e^{-i(x,\xi)} \rangle, \tag{5.9.2}$$

where $\lambda \in \mathscr{D}(\mathbb{R}^n)$ such that $\lambda(x) = 1$ in a neighbourhood K of the support of f.

Note that the value of $\hat{f}(\xi)$ is independent of the choice of λ in (5.9.2) We can show that

$\hat{f}(\xi) \in C^\infty(\mathbb{R}^n)$ and

$$D^\alpha \hat{f}(\xi) = \langle f(x), \left(\frac{\partial}{\partial \xi}\right)^\alpha (\lambda(x) e^{-i(x,\xi)}) \rangle \ \forall \alpha \in \mathbb{N}_0^n \tag{5.9.3}$$

Indeed, proceeding as in the proof of Lemma 4.3.2 we can show that

$$\frac{\partial}{\partial \xi_i} \hat{f}(\xi) = \langle f(x), \frac{\partial}{\partial \xi_i} \phi(x, \xi) \rangle,$$

where $\phi(x, \xi) = \lambda(x) e^{-i(x,\xi)} \in C^\infty(\mathbb{R}^{2n})$. If we assume that the supp $\lambda \subset \{x : |x| \le R, R > 0\}$, then $\phi(x, \xi) = 0$ for $|x| > R$. In this case we find that for $0 < |h| < 1$,

$$\left| \frac{1}{h} \{(D_x^m \phi)(x, \xi_1, \ldots, \xi_i + h, \ldots, \xi_n) - (D_x^m \phi)(x, \xi)\} - \frac{\partial}{\partial \xi_i} \phi^{(m)}(x, \xi) \right|$$

$$\le \sup_{|x| \le R} \left| \frac{\partial^2}{\partial \vartheta_i^2} (D_x^m \phi)(x, \xi_1, \ldots, \vartheta_i, \ldots, \xi_n) \right| \frac{1}{2} |h| \to 0 \text{ as } h \to 0,$$

$$-|\xi_i| - |h| \le \vartheta_i \le |\xi_i| + |h|$$

from which the result follows. The general result follows by induction on i.

In fact we can show that $\hat{f}(\xi) \in \theta_M(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$.

Theorem 5.9.1 The Fourier transform of $f \in \mathscr{E}'(\mathbb{R}^n)$ is an element of $\theta_M(\mathbb{R}^n)$.

Proof In view of the boundedness property (4.2.1), for any neighbourhood K of f which is a compact subset of $\mathbb{R}^n$ there exist a positive constant C and a non-negative integer m such that

$$|D^\alpha \hat{f}(\xi)| \le C \sum_{|\beta|\le m} \sup_{x\in K} |D_x^\beta (\lambda(x) x^\alpha \cdot e^{-i(x,\xi)})|,$$

to which an application of the Leibnitz theorem gives

$$|D^\alpha \hat{f}(\xi)| \le C \sum_{|\beta|\le m} \sup_{x\in K} \left| \sum_{\nu\le\beta} \binom{\beta}{\nu} D_x^\nu (\lambda(x) x^\alpha) \times (-i)^{\beta-\nu} \xi^{\beta-\nu} e^{-i(x,\xi)} \right|.$$

Let
$$\sup_{x\in K} |D_x^\nu (\lambda(x) x^\alpha)| = a(\alpha, \nu).$$

Then, using (5.2.3) we get

$$|D^\alpha \hat{f}(\xi)| \le C \sum_{|\beta|\le m} \left| \sum_{\nu\le\beta} \binom{\beta}{\nu} a(\alpha, \nu) (1 + |\xi|)^{|\beta-\nu|} \right|$$
$$\le D(\alpha, m) (1 + |\xi|)^m,$$

where $D(\alpha, m)$ is a constant. Thus $\hat{f}(\xi) \in \theta_M(\mathbb{R}^n)$. ■

For computation purposes the definition (5.9.1) is very handy. For instance, $\delta \in \mathscr{E}'(\mathbb{R}^n)$ and

$$(\mathscr{F}\delta)(\xi) = \langle \delta(x), e^{-i(x,\xi)} \rangle = 1.$$

The above theorem is useful in finding a structural formula for elements in $\mathscr{D}'(\mathbb{R}^n)$.

Theorem 5.9.2 Let $f \in \mathscr{D}'(\mathbb{R}^n)$, and Ω be a bounded open subset of $\mathbb{R}^n$. Then there exist $g \in C(\mathbb{R}^n)$ and an integer $N \ge 0$ such that $f = (1 - \Delta)^N g$ on Ω, where $\Delta = \sum_{j=1}^{n} (\partial/\partial x_j)^2$.

Proof Let us choose a function $\rho \in \mathscr{D}(\mathbb{R}^n)$ such that $\rho(x) = 1$ on Ω; then $f = \rho f$ on Ω. As $\rho f \in \mathscr{E}'(\mathbb{R}^n)$ by Theorem 5.9.1 its Fourier transform is a continuous (indeed C^∞-function) of polynomial growth. Hence there exists a non-negative integer N such that

$$h(\xi) = (1 + |\xi|^2)^{-N} (\rho f)^\wedge(\xi) \in L^1(\mathbb{R}^n).$$

Thus $\mathscr{F}^{-1}(h) = g$(say), is a bounded continuous function on $\mathbb{R}^n$; see Section 5.5. But by the derivative rule (5.7.2),

$$\mathscr{F}[(1-\Delta)^N g] = (1 + |\xi|^2)^N h.$$

Therefore, $(\rho f)^\wedge = ((1-\Delta)^N g)$. This yields $f = \rho f = (1-\Delta)^N g$ on Ω as desired.

Corollary 5.9.3 If $f \in \mathscr{E}'(\mathbb{R}^n)$, then there exists an integer $m \geq 0$ and a set of continuous functions f_α, $|\alpha| \leq m$, such that

$$f = \sum_{|\alpha| \leq m} D^\alpha f_\alpha .$$

Since $\mathscr{S}' \subset \mathscr{D}'$, the convolution of $f \in \mathscr{S}'$ and $g \in \mathscr{E}'$ exists by Theorem 4.5.1: and we have the following version of the convolution theorem for distributions.

Theorem 5.9.4 Let $f \in \mathscr{S}'(\mathbb{R}^n)$ and $g \in \mathscr{E}'(\mathbb{R}^n)$. Then $f * g \in \mathscr{S}'(\mathbb{R}^n)$ and

$$\mathscr{F}(f * g) = (\mathscr{F} f)(\mathscr{F} g) \tag{5.9.4}$$

in the sense of equality in $\mathscr{S}'$.

Proof Since $\mathscr{F} g \in \theta_M$ and $\mathscr{F} f \in \mathscr{S}'$, the product $(\mathscr{F} f)(\mathscr{F} g) \in \mathscr{S}'$. Therefore there exists $h \in \mathscr{S}'(\mathbb{R}^n)$ such that $\hat{h} = \hat{f}\hat{g}$. Our problem is now to determine h.

Let $\phi \in \mathscr{S}(\mathbb{R}^n)$; then using the Fourier inversion formula (5.6.5) we have

$$\begin{aligned} \langle h, \phi \rangle &= (2\pi)^{-n} \langle (\mathscr{F} f)(\mathscr{F} g), \mathscr{F} \check{\phi} \rangle \\ &= (2\pi)^{-n} \langle \hat{f}, \hat{g}\, \mathscr{F} \check{\phi} \rangle. \end{aligned} \tag{5.9.5}$$

This holds, in particular, when $\phi \in \mathscr{D} \subset \mathscr{S}$. Then by convolution formula (5.8.1),

$$\hat{g}(\mathscr{F} \check{\phi}) = \mathscr{F}(g * \check{\phi}).$$

Hence from (5.9.5),

$$\begin{aligned} \langle h, \phi \rangle &= (2\pi)^{-n} \langle \hat{f}, (g * \check{\phi}) \rangle \\ &= \langle f, \check{g} * \phi \rangle = \langle f(x), \langle \check{g}(y), \phi(x-y) \rangle \rangle \text{ by (4.7.1)} \\ &= \langle f(x), \langle g(y), \phi(x+y) \rangle \rangle = \langle f * g, \phi \rangle, \ \phi \in \mathscr{D}(\mathbb{R}^n). \end{aligned}$$

Thus $h = f * g \in \mathscr{D}'(\mathbb{R}^n)$, and as $h = \mathscr{S}'(\mathbb{R}^n)$ the theorem follows. ∎

5.10 Applications

The Fourier transform of distributions possesses many applications.

Applications to a partial differential equation and a convolution equation are given in this section.

1. Let us consider the partial differential equation

$$(1 - \Delta)\, u = f \tag{5.10.1}$$

where $f \in \mathscr{S}'(\mathbb{R}^n)$ and $\Delta = \sum_{i=1}^{n} (\partial/\partial x_i)^2$. Applying Fourier transform to both sides we get

$$(1 + |\xi|^2)\, \hat{u}(\xi) = \hat{f}(\xi).$$

Hence $\hat{u}(\xi) = (1 + |\xi|^2)^{-1}\ \hat{f}(\xi) \in \mathscr{S}'(\mathbb{R}^n)$, because $(1 + |\xi|^2)^{-1}$ is a multiplier in $\mathscr{S}'$. Now, an application of the inverse Fourier transform gives the solution

$$u(x) = \mathscr{F}^{-1}[(1 + |\xi|^2)^{-1}\ \hat{f}(\xi)]. \tag{5.10.2}$$

2. Let $g \in \mathscr{S}(\mathbb{R}^n)$ and $f \in \mathscr{S}'(\mathbb{R}^n)$ be given; consider the problem of finding $u \in \mathscr{S}'$ such that

$$u * g = f \text{ in } \mathscr{S}'. \tag{5.10.3}$$

Application of the Fourier transform and the relation (5.9.4) yields

$$\hat{u}\hat{g} = \hat{f}.$$

If v be such that

$$v\,\hat{g} = \hat{f} \text{ in } \mathscr{S}';$$

then $\hat{u} = v$ and so

$$u = \mathscr{F}^{-1}[v]$$

is the solution of (5.10.3).

Exercises

5.1 Using formula (5.6.10) compute the Fourier transform of cos ax and sin ax for $a > 0$ and $x \in \mathbb{R}$

5.2 Define the signum function on $\mathbb{R}$ by

$$\operatorname{sign} x = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0. \end{cases}$$

Prove that (i) $D(\operatorname{sgn} x) = 2$

(ii) $\mathscr{F}[\operatorname{sgn} x] = -2i\, Pv\,(1/\xi)$

(iii) $\mathscr{F}[H] = -i\, Pv\,(1/\xi) + \pi\delta$
(iv) $\mathscr{F}[Pv\,(1/x)] = -i\pi \operatorname{sgn} \xi$.

5.3 Prove that $\mathscr{F}\left[\exp\left(-\frac{1}{2}|x|^2\right)\right] = (2\pi)^{n/2} \exp\left(-\frac{1}{2}|\xi|^2\right)$.

5.4 If $\theta(x) \in \theta_M$, prove that $D^\beta\theta(x) \in \theta_M$ for all $\beta \in \mathbb{N}_0^n$.

5.5 If $\phi \in C^\infty(\mathbb{R}^n)$ such that $\phi f \in \mathscr{S}(\mathbb{R}^n)$ for all $f \in \mathscr{S}$, prove that $\phi \in \theta_M$.

5.6 Let $f \in \mathscr{D}'(\mathbb{R}^n)$ and $\phi \in \mathscr{D}(\mathbb{R}^n)$, prove that $\phi f \in \mathscr{E}'(\mathbb{R}^n)$.

5.7 Prove that

$$\mathscr{F}^{-1}(f * g) = (2\pi)^n\,(\mathscr{F}^{-1}f)\,(\mathscr{F}^{-1}g),\ f, g \in \mathscr{S}(\mathbb{R}^n).$$

5.8 If $\phi \in \mathscr{S}$ and $f \in \mathscr{S}'$, prove that

$$\mathscr{F}(\phi f) = (2\pi)^{-n}(\mathscr{F}\phi) * (\mathscr{F}f),$$

and

$$\mathscr{F}^{-1}(\phi * f) = (2\pi)^n\,(\mathscr{F}^{-1}\phi)\,(\mathscr{F}^{-1}g).$$

5.9 Let $\sigma(x) = (1 + |x|^2)^{m/2}$, $m \in \mathbb{R}$. Prove that

$$|D^\beta\sigma(x)| \le C_{m,\beta}\,(1 + |x|)^{m-\beta}\ \forall\, x \in \mathbb{R}^n,$$

Hence $\sigma(x) \in \theta_M$ (Hint: See Wong (1991), pp 29–30).

5.10 Using definition (5.6.1) show that

$$\mathscr{F}[G_s(x)]\,(\xi) = (1 + |\xi|^2)^{-s/2},\ s > 0,\ x, \xi \in \mathbb{R}^n$$

where

$$G_s(x) = 2^{-s/2}(\Gamma(s/2))^{-1}\int_0^\infty \exp(-r/2 - |x|^2/(2r))\, r^{-(n-s)/2}\, dr/r.$$

(Hint: See Wong (1991), pp. 92–94).