4.

$f(x_1) \geq f(x_0) + g_{x_0}^T (x_1 - x_0)$ where $f(x) : x \in \mathbb{R}$

for a func. to be convex we have,

$$f(x + \lambda(y-x)) \leq (1-\lambda) f(x) + (y) \quad \forall \lambda \in [0,1]$$

dividing both sides by λ;

$$\frac{f(x + \lambda(y-x))}{\lambda} \leq \frac{(1-\lambda) f(x) + (y)}{\lambda}$$

$$f(y) \geq f(x) + \frac{f(x) + \lambda(y-x)}{\lambda} - f(x)$$

by taking $z = \lambda x + (1-\lambda) y)$ $\lambda \in (0,1)$

$$f(x) \geq f(z) + f'(z)(x-z)$$

$$f(y) \geq f(z) + f''(z)(y-z) \Rightarrow \lambda f(x) + (1-\lambda) y \geq f(z)$$

$\therefore$ F is convex

e.g:

consider a function $f(x) = 3x_1^2 + x_2^2 + 4x_1x_2 + 1$

gradient, $g = \begin{bmatrix} 6x_1 + 4x_2 \\ 2x_2 + 4x_1 \end{bmatrix}$

$[x_1, x_2] =$

$g_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ saddle point

$F(x) = 3 + 4 + 1 = 6$

$F(x_0) + g_0^T (\cancel{x_1}^{0} - x_0) = 0$

As $F(x) > F(x_0) + g_0^T (x - x_0)$