2] a.

$$x_1 + 2x_2 + 3x_3 = 1$$

given nearest $(-1, 0, 1)^T$

point in the plane $(x_1\ x_2\ x_3)$

$$\therefore\ (x_1+1)^2 + (x_2-0)^2 + (x_3-1)^2$$

$$= (x_1+1)^2 + x_2^2 + (x_3-1)^2$$

[2 Eqs. , 3 unknowns]

$$\Rightarrow\ x_1 + 2x_2 + 3x_3 = 1$$

{unconstrained}

$$x_1 = 1 - 2x_2 - 3x_3 \quad \text{——— ①}$$

$$\text{——— ②}$$

$$f(x) = (x_1+1)^2 + x_2^2 + (x_3-1)^2$$

$$H = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_3} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \dfrac{\partial^2 f}{\partial x_2 \partial x_3} \\ \dfrac{\partial^2 f}{\partial x_3 \partial x_1} & \dfrac{\partial^2 f}{\partial x_3 \partial x_2} & \dfrac{\partial^2 f}{\partial x_3^3} \end{vmatrix}$$

Substituting ① in ②

$$= (1 - 2x_2 - 3x_3 + 1)^2 + x_2^2 + (x_3-1)^2$$

$$= (-2x_2 - 3x_3 + 2)^2 + x_2^2 + (x_3-1)^2$$

$$\frac{\partial f}{\partial x_2} = 2(-2x_2 - 3x_3 + 2)(-2) + 2x_2 + 0$$

$$= (-4x_2 - 6x_3 + 4)(-2) + 2x_2 + 0$$

$$= +8x_2 + 12x_3 - 8 + 2x_2$$

$$= 10x_2 + 12x_3 - 8 = 0 \quad \text{—— } (3)$$

$$\frac{\partial f}{\partial x_3} = 2(-2x_2 - 3x_3 + 2)(-3) + 0 + 2(x_3 - 1)$$

$$= (-4x_2 - 6x_3 + 4)(-3) + \cancel{8}\, 2x_3 - 2$$

$$= +12x_2 + 18x_3 - 12 + 2x_3 - 2$$

$$= 20x_3 + 12x_2 - 14 = 0 \quad \text{—— } (4)$$

Solving (3) and 4

$$10x_2 + 12x_3 - 8 = 0 \qquad 2(5x_2 + 6x_3 - 4) = 0$$

$$12x_2 + 20x_3 - 14 = 0 \qquad 2(6x_2 + 10x_3 - 7) = 0$$

$$6(5x_2 + 6x_2 - 4) = 0$$

$$5(6x_2 + 10x_3 - 7) = 0$$

$$30x_2 + 36x_3 - \overset{24}{\cancel{x}} = 0$$

$$- \quad 30x_2 \underset{-}{+} 50x_3 + 35 = 0$$

$$-14x_3 = +11$$

$$x_3 = 11/14$$

$$\therefore \quad x_2 = -1/7$$

$\therefore$ Subs.. x_2 and x_3

$$x_1 = 1 - 2x_2 + 3x_3$$

$$= 1 - 2\left(\frac{-1}{7}\right) + 3\left(\frac{11}{14}\right)$$

$$= \frac{-15}{14}$$

For Hessian

$$\frac{\partial^2 f}{\partial x_2^2} = 10 \qquad \frac{\partial^2 f}{\partial x_2 \partial x_3} = 12$$

$$\frac{\partial^2 f}{\partial x_3^2} = 20 \qquad \cancel{\frac{\partial^2 f}{\partial}}$$

{ for (3 and 4 Eq) }

$$H = \begin{vmatrix} 10 & 12 \\ 12 & 20 \end{vmatrix}$$

$H = 56$ $\therefore$ +ve definite Hessian

Thus the func. is convex

For eigen values

$$\det \left| H - \lambda I \right| = 0$$

$$= \left| \begin{matrix} 10 & 12 \\ 12 & 20 \end{matrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$= (10 - \lambda)(20 - \lambda) - 144$$

$= 200 - 10\lambda - 20\lambda + \lambda^2 - 144 = 0$

$= \lambda^2 - 30\lambda + 56$

$\therefore \lambda = 28, 2$

as $\lambda_i > 0$

min $\lambda, 2 > 0$ $\therefore$ the func. is convex everywhere

3] a.

To prove $af(x) + bg(x)$ is convex function

for $a > 0$ and $b > 0$

As, ~~for~~ $f(x)$ and $g(x)$ to be convex they

are p.s.d (positive semi definite)

$\therefore F : x \rightarrow \mathbb{R}$ $\lambda \in (0,1)$

$F(\lambda x_1 + (1-\lambda)x_2) \leq \lambda F(x_1) + (1-\lambda) F(x_2)$ — ①

and $G(\lambda x_1 + (1-\lambda)x_2) \leq \lambda G(x_1) + (1-\lambda) G(x_2)$

as $a > 0$ & $b > 0$

$a(F(\lambda x_1 + (1-\lambda)x_2)) \leq a((\lambda F(x_1) + (1-\lambda) F(x_2)))$

$af(x) + bg(x)$

$\rightarrow$ $af(\lambda x_1 + (1-\lambda)x_2) + b\,G(\lambda x_1 + (1-\lambda)x_2) \leq a(\lambda F(x_1) + (1-\lambda)$
$F(x_2)) + b(\lambda g(x_1) + (1-\lambda) g(x_2))$ — ②

② function is always greater than the first function ①.

(or) As $f(x)$ and $G(x)$ are p.s.d

$F(x) \geq 0 \quad \forall x \in \mathbb{R}$

$G(x) \geq 0$

will also be p.s.d

$aF(x) + bG(x) \geq 0$

$\therefore$ its convex

b. To prove

$F(G(x))$ is convex function

necessary

~~by using second Order condition~~

assume $\Delta^* = f(g(x))$

on diff. further

$U \cdot V = U'V + V'U$

$= f'(g(x)) \cdot g'(x)$

$= f''(g(x))\, g'(x)\, g'(x) + g''(x)\, f'(g(x))$

$= \underline{f''(g(x))}\, \underline{(g'(x))^2} + g''(x)\, \underline{f'(g(x))}$

$= \cancel{f'(x)\,(g(x))^2 + g(x)''\, f'(x)}$

$g''(x) \geq 0 \,\|\, g'(x)^2 \geq 0 \,\|\, f''(g(x)) \geq 0$

$\therefore f'(x)$ has to be ≥ 0

Thus its convex