Natural Algorithms for Flow Problems

Damian Straszak* Nisheeth K. Vishnoi†

Abstract

In the last few years, there has been a significant interest in the computational abilities of *Physarum polycephalum* (a slime mold). This arose from a remarkable experiment which showed that this organism can compute shortest paths in a maze [10]. Subsequently, the workings of Physarum were mathematically modeled as a dynamical system and algorithms inspired by this model were proposed to solve several graph problems: shortest paths, flows, and linear programs to name a few. Indeed, computer scientists have initiated a rigorous study of these dynamics and a first step towards this was taken by [1,2] who proved that the Physarum dynamics for the shortest path problem are efficient (when edge-lengths are polynomially bounded). In this paper, we take this further: we prove that the discrete time Physarum-dynamics can also efficiently solve the uncapacitated min-cost flow problems on undirected and directed graphs; problems that are non-trivial generalizations of shortest path. This raises the tantalizing possibility that nature, via evolution, developed algorithms that efficiently solve some of the most complex computational problems, about a billion years before we did.

1 Introduction

In a striking experiment, [10] showed that a single-celled slime-mold, *Physarum polycephalum*, could solve the shortest path problem on a maze. This sparked interest in the computational abilities of this organism. Roughly speaking, when the organism is distributed throughout a maze with "food" only at the entry and the exit points of the maze, it quickly reaches an equilibrium where it occupies only the shortest path in the maze connecting the two endpoints. Subsequently, the inner workings of Physarum were mathematically modeled as continuous time dynamical systems and natural discretizations of these dynamics were proposed as algorithms to solve several graph problems: shortest paths, flows, and linear programs to name a few (see [5, 14]).

While these dynamical systems were determined by modeling Physarum as a "tubular network" through which slime flows, they can be equivalently described using the language of electrical networks. At a high-level, all Physarum dynamics maintain *resistances* for edges, compute electrical flows based on these resistances and, subsequently, update the resistances based on these flows until the cost of the flow is minimized.

To be concrete, let us describe the Physarum dynamics for solving the shortest path problem on undirected graphs proposed by [14]. Given an undirected graph $G = (V, E)$ with positive edge lengths $l \in \mathbb{R}^E_{>0}$ on which one desires to compute the shortest path from a vertex s to t, they introduced variables x_e for each edge which, in turn, determines a flow q_e across each edge corresponding exactly to the electrical flow obtained by setting the resistance of each edge to $l_e/x_e(t)$ while pumping in one unit of current at s and taking it out from t. Importantly, the vector x and the flow vector q were related by the set of differential equations

$$\dot{x}_e(t) = |q_e(t)| - x_e(t)$$

for all $e \in E$. Qualitatively, the resistance increases if the magnitude of the current is high and reduces otherwise. The starting resistances are all chosen to be some positive quantity such as 1. The authors of [5] presented natural extensions of these dynamics to the more general transshipment problem and also introduced a natural *directed* version of the Physarum dynamics for versions of these problems for directed graphs. Recall that in the transshipment problem, one is given demands b_v for each vertex and the goal is to find a minimum cost flow in G which satisfies the demands. The only change to the dynamical system is that it now computes the electrical flow q in the graph corresponding to the demand vector b. The directed Physarum dynamics differ from the undirected dynamics in that the evolution of x can now also depend on the *sign* of the flow:

$$\dot{x}_e(t) = q_e(t) - x_e(t)$$

for all e.

Several questions arise: do solutions to these differential equations exist? If so, what do they converge to? While earlier papers [7, 8] observed that the solution to the above differential equations for the shortest

*Damian Straszak, École Polytechnique Fédérale de Lausanne (EPFL). Email: damian.straszak@epfl.ch

†Nisheeth K. Vishnoi, École Polytechnique Fédérale de Lausanne (EPFL). Email: nisheeth.vishnoi@epfl.ch

path problem indeed appears to converge to the shortest path,[1] a proof of this fact was first provided in [2]. The authors also proved convergence for the *undirected* transshipment problem. Assuming the existence of a positive solution to the corresponding differential equations, it was proved in [5] that the directed Physarum dynamics converge to the optimal solution of the directed transshipment problem. However, these results gave no indication of the time to convergence, i.e., the time it takes for the solution of the dynamical system to come close in value to the optimal solution. While the notion of continuous time is equally interesting, in this paper we restrict ourselves to studying these dynamics in *discrete time*. Here, it becomes natural to study the number of *steps* taken by the following natural (Euler) discretizations (with the step-length parameter $h > 0$) before they come close to some optimal solution:

$$x_e(k+1) - x_e(k) = h(|q_e(k)| - x_e(k)).$$

For the directed case:

$$x_e(k+1) - x_e(k) = h(q_e(k) - x_e(k)).$$

This brings us to the question central to this paper:

Could the discrete-time versions of these Physarum dynamics be efficient?

To define efficiency we have to define the input size. While it would be nice to have a notion of efficiency similar to that of a Turing machine, there is evidence that this may be impossible for such dynamics and we assume that the costs and demands are given in unary; one can construct a simple example to show that the convergence time to the optimal solution necessarily depends on the magnitude of the input numbers (see Section B).

Mathematically, to prove that the discretizations of the above-mentioned dynamics result in efficient algorithms we need to establish the following three things: 1) that the $x_e(k)$s remain positive (for the dynamics to make sense the resistances have to be positive) 2) that the dynamics converge to the min-cost flow and 3) that we can choose a large enough step-size to guarantee efficiency. A step towards this question was taken by [1] who showed that, for the undirected shortest path problem, the discretization of the original Physarum dynamics using Euler's method can compute a $(1+\varepsilon)$-approximation to the length of the shortest path in a graph with n vertices and longest edge-length L, in time $\mathrm{poly}(n, L, 1/\varepsilon)$.

[1] All of these results assume that the shortest path is unique. We discuss this issue later in the introduction.

Given that not only the shortest path problem, but many fundamental flow/matching problems, reduce to the transshipment problem [4], it remained an important open problem if the discrete Physarum dynamics are efficient for the undirected/directed transshipment problems. Unlike in the world of algorithms, in the world of Physarum, one cannot rely on the efficiency of the shortest path dynamics to deduce that the dynamics for the transshipment problem is efficient. Moreover, there is no known *gadget* reduction from the transshipment problem to the shortest path problem. Thus, there is no a priori reason for the Physarum dynamics for the transshipment to be efficient.

Mathematically, the problems (1)-(3) mentioned above have varying difficulties depending on the directed or undirected dynamics. They are generally harder in the directed setting (e.g., proving (1) is straightforward for the undirected shortest path case, while for the directed shortest path it is not), and are simpler for the shortest path case; see Section 2.

The main contribution of this paper is that the discrete Physarum dynamics for *both* the undirected and the directed versions of the transshipment problem are efficient. This opens up the tantalizing possibility that nature, via evolution, has developed algorithms that can efficiently solve some of the most complex computational problems, and beat us by about a billion years; see [3] for more examples!

1.1 Our results. In the undirected transshipment problem, we are given an undirected graph $G = (V, E)$, a demand vector $b \in \mathbb{Z}^V$ and a cost vector $c \in \mathbb{Z}^E_{>0}$. Assume that $\sum_v b_v = 0$ and let $B \in \mathbb{R}^{V \times E}$ be any signed incidence matrix of G. The goal is to find a flow vector $f \in \mathbb{R}^E$ which satisfies the demand ($Bf = b$) and minimizes the cost of the unsigned flow: $\sum_{e \in E} c_e |f_e|$. The directed transshipment problem is similar except that the graph G is directed and the goal is to find a non-negative flow which satisfies the demand and minimizes the cost $\sum_{e \in E} c_e f_e$. We use "opt" to denote the optimal cost of the instance which should be clear from the context.

The undirected Physarum-based dynamical system for this problem has variables $x \in \mathbb{R}^E_{>0}$ and $q \in \mathbb{R}^E$ with a specified $x(0) > 0$. The equations are as follows: $\dot{x}_e(t) = |q_e(t)| - x_e(t)$ for all $e \in E$ where q_e is the electrical flow in the network with edge conductances $x_e(t)/c_e$ for all $e \in E$ and with demand b_v for every vertex $v \in V$. The directed Physarum-based dynamical system is similar with the equations above replaced by: $\dot{x}_e(t) = q_e(t) - x_e(t)$ for all $e \in E$. The discrete versions of the Physarum dynamics are obtained by replacing the differential equations above by the following recurrences for the undirected and the directed version respectively:

$$x_e(k+1) = (1-h)x_e(k) + h|q_e(k)| \quad (1.1)$$

$$x_e(k+1) = (1-h)x_e(k) + hq_e(k) \quad (1.2)$$
$$\text{for all } k \in \mathbb{N} \text{ and } e \in E$$

Here $h \in (0,1)$ is the *step-size*. As in the continuous version, $q(k) \in \mathbb{R}^E$ is the electrical flow calculated in step k with respect to edge conductances $x_e(k)/c_e$.

Our first result concerns the discrete undirected process (1.1) for the undirected transshipment problem. Consider a feasible instance (G, b, c), where G is an n-node undirected graph, $b_P := \sum_{v:b_v>0} b_v$ is the total demand and $C := \max_{e\in E} c_e$ is the maximum cost.

Theorem 1.1 **(Main theorem- undirected transshipment problem).** *Given $0 < \varepsilon < 1/2$, let $h := \varepsilon/10nC$. Suppose we run the discrete undirected process (1.1) corresponding to a feasible instance (G, b, c), with step size h, from the starting point $x_e(0) = b_P$ for every $e \in E$. Let $x(k)$ be the point obtained after k iterations. Then, it suffices to take $k = O\left(nC \ln(nCb_P)/\varepsilon^3\right)$ to guarantee that* $\mathrm{opt} \le c^\top x(k) \le (1+\varepsilon)\mathrm{opt}$. *Further, one can also find a flow f such that* $\sum_e c_e|f_e| \le (1+\varepsilon)\mathrm{opt}$ *in the same time.*

Note that this theorem also improves the previous bound [1] on the number of iterations on the undirected shortest path dynamics by a multiplicative factor of n/m, where m is the number of edges in G.

Our second result concerns the directed transshipment problem. The directed case turns out to be harder to analyze than the undirected case and our result is also of a slightly different nature. For example, here, proving that $x_e(k) > 0$ for all k, which is important to assure that the discrete dynamics (1.2) is well defined turns out to be non-trivial. Further, to prove a good bound on the convergence time, we need a trick - *preconditioning*.[2] Given the input instance, we modify it slightly and then run the discrete directed dynamics on it. Our preconditioning, which we describe shortly, allows us to prove fast convergence to the optimal solution of the original instance.

The preconditioning is as follows: suppose (G_0, b, c_0), with $G_0 = (V, E_0)$, is a feasible input instance to the directed transshipment problem. We construct a new instance (G, b, c) and an initial point $x(0) \in \mathbb{Z}^E_{>0}$. Let $G = (V, E)$ for $E := E_0 \cup E'$ where E' consists of new edges for every pair of vertices u, v with $b_u < 0$ and $b_v > 0$, i.e., $E' := \{(u, v) : b_u < 0 \wedge b_v > 0\}$ (note that E may be a multiset). The costs of old edges remain the same, while for a new edge $e' \in E'$ we set $c_{e'} = nC$, where C is the maximum cost over $e \in E_0$. For the initial vector we assign $x_e(0) = 1$ for all $e \in E_0$ and for $e' = (u, v) \in E'$ we set $x_{e'} := 2m \cdot |b_u| \cdot b_v$. Note that by adding E' to G_0 we do not introduce any solution with lower cost than the optimal solution in G_0. As before, let $C := \max_{e\in E_0} c_e$ and b_P be the total demand.

Theorem 1.2 **(Main theorem- directed transshipment problem).** *Given $0 < \varepsilon < 1/3$, let $h := 1/4n^5C^2b_P$. Suppose we run the discrete undirected process (1.2) corresponding to (G, b, c) (as above) with step size h, from the starting point as specified above. Then, it suffices to take $k = O\left(nCb_P \ln(mb_P/\varepsilon)/h\right)$ steps to guarantee that $\|x(k) - f^\star\|_\infty < \varepsilon$ for some optimal solution $f^\star$.*

It is interesting to note that unlike in the undirected case, h does not depend upon ε and the overall running time of the algorithm depends logarithmically on $1/\varepsilon$. Further, the guarantees in the statement of the theorem is with respect to the original graph and not the preconditioned one. Note that this theorem also improves the previous bound on the number of iterations on the directed shortest path dynamics. We note that the preconditioning step is computationally easy and may not be required. Complete proofs of Theorem 1.1 and Theorem 1.2 are presented in Section 4 and Section 5 respectively.

Discussion and open problems. We start by noting that neither the bit lengths of the numbers, uniqueness, nor the positivity of cost pose major problems. The bit length analysis can be done akin to [1]. If one is interested in just the value of the optimal solution, uniqueness is not required. If one desires the optimal flow, one can appeal to the Isolation Lemma [9]. Recall that the cost vector is also required to be positive in order to make sense of the electrical flow. Perturbing the costs a bit allows us to achieve this. Thus, by a standard reduction (see [4]), we can efficiently solve the capacitated min-cost flow problem with non-negative costs.

Notice that these algorithms inspired by Physarum dynamics are novel and can be converted into efficient algorithms (for bounded costs and demands) using the near-linear Laplacian solvers of Spielman-Teng [12]; giving yet another instance of the "Laplacian paradigm" [11, 13, 15]. A natural question is if they (or simple modifications of them) are competitive with the current best algorithms for the same problems. Towards this, it becomes important to understand to what extent can we improve the bounds on the number of iterations in Theorems 1.1 and 1.2.

While we can construct simple examples (such as that in Section B) that demonstrate that the dependence

[2]This is a different notion of preconditioning than what is usually used in numerical linear algebra.

on C has to be polynomial if one desires to get close to an optimal solution, it seems plausible that one can get within $(1+\varepsilon)$ in *value* in time which depends logarithmically on C and polynomially in $1/\varepsilon$ and is an important direction to explore.

Finally, an extension of the directed Physarum dynamics to the even more general setting of linear programming was presented by [6]. It is a challenging open problem to extend our Theorem 1.2 to this setting.

2 Technical overview

Recall that the main goal of our study is to understand whether the discrete Physarum dynamics (1.1) and (1.2) indeed converge to the optimal solutions of the underlying flow problems, and if so, whether they lead to efficient algorithms. One can view them as processes, which maintain edge resistances at every step and use electrical flows to update them in a certain way. The first question which needs to be answered is whether these processes are in fact well defined. In other words whether the update can be performed at every step. Since we are computing electrical flows with respect to our current conductance values $x(k)$ one needs to assure that $x_e(k) > 0$ for every step k and every $e \in E$. This turns out to be easy for the undirected dynamics (1.1) (since we are using $|q(k)|$ in the update), however it is not at all obvious for the directed case (1.2). One can show that this positivity issue is highly related to the problem of bounding the maximum potential difference between vertices in the electrical network at various stages of the process. In fact a bound on the maximum potential difference implies existence of $h > 0$ such that the x's produced by the process (1.2) will always stay positive. For the shortest path case one can easily obtain a bound on the maximum potential difference by noting that it must occur between the starting and target vertex of the underlying instance. Moreover, in this case, the potential difference between the sink and the source is exactly equal to the energy of the electrical flow! This quantity can be bounded easily.

In the case of general flow problems that occur in transshipment problems, the connection between the energy and maximum potential difference disappears. The pair of nodes between which the maximum is attained can change during the process multiple times and we cannot adapt the approach used for the shortest path to work in the general case. Further, once we know that indeed we can iterate those processes (1.1) and (1.2) without getting stuck, we may ask what do they converge to. We prove that, as claimed in our main theorems, they converge to the optimal solutions of the underlying flow problems. However, several new ideas and techniques are required. Let us discuss first the undirected variant in detail, then we move to the directed one.

Given an instance of the undirected transshipment problem with any optimal solution $f^\star$, we choose the incidence matrix B such that $f^\star \geq 0$. To prove Theorem 1.1, start by observing that in the undirected case, since we are updating $x(k)$ by adding a positive flow, we easily get $0 < x(k) \leq b_P$ for all $k \geq 1$ as long as $0 < x(0) \leq b_P$. Drawing from the expectation that $x(k) \to f^\star$, we track the progress of the dynamics using a natural potential function

$$\mathcal{V}(k) = \sum_e c_e x_e(k).$$

If our expectation was indeed true then $\mathcal{V}(k)$ should be at least opt for all k and would tend to opt as k increases. Here, opt is the cost of $f^\star$, i.e., $\sum_e c_e f_e^\star$. Towards this, one tries to understand how $\mathcal{V}(k)$ changes with k. It turns out that $\Delta\mathcal{V}(k) = \mathcal{V}(k+1) - \mathcal{V}(k)$ can be upper bounded by, roughly, $\mathcal{E}(k) - \mathcal{V}(k)$, where $\mathcal{E}(k)$ is the energy of the electrical flow $q(k)$ corresponding to conductances $x_e(k)/c_e$. Due to the fact that electrical flows minimize energy among all flows with the same demands, it can be shown that $\mathcal{E}(k) \leq \mathcal{V}(k)$. Thus, $\Delta\mathcal{V}(k) \leq 0$. We would be done if we could prove that in fact $\Delta\mathcal{V}(k) \leq -\varepsilon$, or $\mathcal{E}(k) < \mathcal{V}(k) - \varepsilon$. However, since $\mathcal{V}(k)$ is lower bounded by opt, we cannot hope this will happen at every step. Moreover, $\mathcal{E}(k)$ can come very close to $\mathcal{V}(k)$, thus the process could slow down. To track progress when this might happen we introduce a barrier function

$$\mathcal{B}(k) = \sum_e f_e^\star c_e \ln x_e(k).$$

$\mathcal{B}(k)$ achieves its maximum for $k = 0$ and thus remains bounded. The choice of $f^\star$ in this barrier function shows up when one tries to understand $\mathcal{B}(k+1) - \mathcal{B}(k)$ which equals

$$\begin{aligned}
&\sum_e f_e^\star c_e \ln \frac{x_e(k+1)}{x_e(k)} \\
&= \sum_e f_e^\star c_e \ln \left(1 + h \cdot \frac{|q_e(k)| - x_e(k)}{x_e(k)}\right) \\
&= \sum_e f_e^\star c_e \ln \left(1 + h \left(|\Delta_e p(k)|/c_e - 1\right)\right)
\end{aligned}$$

where h is the step size in the discretization and $|\Delta_e p(k)|$ is the potential difference across the edge e induced by the flow $q(k)$. Thus, if $h\left(|\Delta_e p(k)|/c_e - 1\right) \ll 1$, we could replace the ln-term in the expression above by $h\left(|\Delta_e p(k)|/c_e - 1\right)$. This would in turn imply that

$$\mathcal{B}(k+1) - \mathcal{B}(k) \approx h \sum_e f_e^\star |\Delta_e p(k)| - h \cdot \text{opt}.$$

Now, note that

$$\sum_{e\in E} f_e^\star |\Delta_e p(k)| \ge \sum_{e\in E} f_e^\star \Delta_e p(k) = {f^\star}^\top B^\top p(k)$$
$$= (Bf^\star)^\top p(k) = b^\top p(k) = \mathcal{E}(k).$$

Thus, using the optimality of $f^\star$ we obtain that

$$\mathcal{B}(k+1) - \mathcal{B}(k) \approx h(\mathcal{E}(k) - \text{opt}).$$

Thus, whenever $\mathcal{E}(k) \gg \text{opt}$, $-\mathcal{B}(k)$ reduces. Since, while $\mathcal{V}(k) \gg \text{opt}$, either $\mathcal{E}(k) \ll \mathcal{V}(k)$ or $\mathcal{E}(k) \gg \text{opt}$ must occur, by taking a potential function which combines $\mathcal{V}(k)$ and $-\mathcal{B}(k)$, we always make progress!

However, this depended on $h \cdot {}^{|\Delta_e p(k)|}/_{c_e} \ll 1$, which brings us to the next key issue: how to bound the maximum potential difference, $\max_e |\Delta_e p(k)|$ across all iterations? For the case of the undirected shortest path problem, this is easily derived using the fact that this maximum occurs between the source and the sink of the shortest path which can be bounded by the energy, which in turn can be bounded by $m \cdot C$ where $C = \max_e c_e$. For our transshipment problem, we need a different argument; our approach additionally allows us to improve the upper bound to $n \cdot C$. From what we said above, the choice of $h \approx {}^1/_{nC}$ is also dictated by this constraint and determines the number of iterations required in the theorem.

The proof for bounding the drop in potential proceeds by sorting the vertices according to their potential and then showing that the potential jump between any two neighboring vertices in this embedding can be at most C. The proof of the latter involves looking at the sweep-cut induced by these two vertices and deploying elementary properties of electrical flows, the maxflow-mincut theorem and the existence of, what we call x-capacitated flows: an unsigned flow f (with $Bf = b$) is said to be x-capacitated if $|f| \le x$. This definition of *x-capacitated flows* turn out to be a useful abstraction and also enables proofs of several other technical facts. This completes the sketch of the proof and the reader is referred to Section 4 for more details.

The proof of Theorem 1.2 poses additional challenges when compared to the undirected case, and we highlight a few. Positivity is a concern here since, as the conductances can become negative (see Section C). Even assuming that the input instance is feasible, we start by noting that since the electrical flows $q_e(k)$ are added with their sign, it is no longer clear why $x_e(k) > 0$ other than when $k = 0$. Here, the notion of x-capacitated flows comes to our rescue: in the directed setting, a flow is said to be x-capacitated if $Bf = b$ and $0 \le f \le x$. While in the undirected case if we choose an $x(0)$ such that there is an $x(0)$-capacitated flow then it follows that there is an $x(k)$-capacitated flow for all k. This is no longer true in the directed case and presents itself as a significant hurdle. It is still possible to prove (though the argument is quite involved) that for every initial point $x(0)$, there is some $h > 0$ for which the discretization gives a well defined sequence of positive vectors $x(k)$, which converges to the optimal solution. However, they key problem is that the step length h needs to be very small, in order to get such a result. In other words, the resulting algorithm in not efficient. The reason for this is the chaotic behavior of the process in the initial steps. It is not clear how one could provide an initial point which makes the process "smooth" enough. This is where preconditioning comes in: for the preconditioned instance, one can show that if $x(j)$s are positive for $j = 0, 1, \dots, k$ then there is an $x(k)$-capacitated flow. Such a flow, in turn, allows us to prove by a variant of the sweep-cut argument that the maximum potential difference corresponding to $q(k)$ remains bounded. This in turn allows us to prove that $x(k+1)$ is positive and we can continue. The potential bounding step requires h to be roughly at most ${}^1/_{n\tilde{C}}$ where $\tilde{C}$ is the largest cost in the preconditioned instance.

What about the number of iterations required to converge? The key is to start by noting that

$$x_e(k) = x_e(0)(1-h)^k + (1 - (1-h)^k)\bar{q}_e$$

where $\bar{q}(k)$ is a certain *geometric time average* of the flows $q(j)$s, giving more importance to the newer flows than the older ones. Thus, as k increases, $x(k) \approx \bar{q}(k)$. Thus $x(k)$, for a large enough k is nearly a flow and, similarly, $\bar{q}(k)$ is a flow but can have a small negative component. The results of [1,5] then allow us to round this flow to an optimal flow provided for every non-optimal flow g among the vertices of the flow polytope, there is an edge e supported in g for which $x_e(k)$ is tiny after a small number of iterations. This, via standard arguments, allows us to complete the proof of the theorem.

To establish this property for any non-optimal vertex flow g, we consider the barrier function $\mathcal{B}_g(k) = \sum_e g_e c_e \ln x_e(k)$. With a bit of effort, it can be shown that $\mathcal{B}_g(k) \to -\infty$. In fact one can show that $\mathcal{B}_g(k) \lesssim -hk + \tilde{C} b_P$. On the other hand if $x_e(k) > \varepsilon$ for all edges e with $g_e > 0$, then we get a lower bound of $\tilde{C} b_P \varepsilon$. Thus, for $k \approx {}^{\tilde{C} b_P}/_h$, this cannot happen and there must be an edge e where $g_e > 0$ but $x_e(k) < \varepsilon$. This completes an overview of the proof; details appear in Section 5.

3 Preliminaries

3.1 Combinatorial flows, electrical flows and their properties. For a directed graph G and a demand vector $b \in \mathbb{Z}^V$ with $\sum_{v\in V} b_v = 0$, we consider flows $f \in \mathbb{R}^E$ satisfying $Bf = b$, where B is the incidence matrix of G. A flow f is called *non-negative* if $f \geq 0$. If f does not contain a directed cycle in its support, then f is called a *non-circular flow*. *Basic flows* are defined by vertices of the polytope $\{f : Bf = b, f \geq 0\}$.

The mincut-maxflow theorem in this setting says that in a graph (V, E) with edge capacities $x \in \mathbb{R}^E_{\geq 0}$ there exists a non-negative flow f respecting the capacities: $0 \leq f \leq x$ if and only if for every partition of V into S and $\bar{S}$, we have $\sum_{e \in E(\bar{S}, S)} x_e \geq b_S$, where $b_S = \sum_{v \in S} b_v$.

In this paper we are particularly interested in electrical flows. These are flows defined with respect to a conductance vector $w \in \mathbb{R}^E_{>0}$. Denote $W = \mathsf{Diag}(w_1, w_2, \ldots, w_m)$. q is called an electrical flow if it is the unique flow minimizing the quadratic form $\mathcal{E}(q) = q^\top W^{-1} q$. Equivalently, let $p \in \mathbb{R}^V$ be the potential vector obtained as a solution to the following Laplacian system: $Lp = b$ (where $L = BWB^\top$ is the Laplacian matrix). Then q is the flow induced by p, i.e., $q_{uv} = w_{uv}(p_u - p_v)$. The following are some folklore facts about electrical flows, basic and non-circular flows.

Fact 3.1. *Suppose q is the electrical flow in the graph $G = (V, E)$ for vertex demands $b \in \mathbb{Z}^V$ and conductances $w \in \mathbb{R}^E_{>0}$. Let $p \in \mathbb{R}^V$ be the corresponding potential vector.*

1. *If $b_P := \sum_{v: b_v > 0} b_v$ is the total demand, then $|q_e| \leq b_P$ for every edge $e \in E$.*
2. *The energy of the flow: $\mathcal{E}(q) := \sum_{e \in E} q_e^2/w_e$ is equal to $b^\top p$.*

Fact 3.2. *Suppose g is a basic flow in the directed graph $G = (V, E)$ for vertex demands $b \in \mathbb{Z}^V$. Then $g \in \mathbb{Z}^E_{\geq 0}$ and $g_e \leq b_P$, where $b_P := \sum_{v: b_v > 0} b_v$ is the total demand.*

Fact 3.3. *Suppose g is a non-circular flow in the directed graph $G = (V, E)$ for vertex demands $b \in \mathbb{R}^V$. Then g can be expressed as a convex combination of at most $|E|$ basic flows.*

4 Discrete dynamics for the undirected transshipment problem

In this section we prove Theorem 1.1. The section consists of two parts. The first one introduces and proves the most important properties of the undirected dynamics. The second part contains the actual proof.

4.1 Discrete, undirected dynamics and their properties. We consider the undirected transshipment problem (also known as undirected uncapacitated minimum cost flow). In this problem an undirected graph $G = (V, E)$ is given, together with a demand vector $b \in \mathbb{Z}^V$ and a cost vector $c \in \mathbb{Z}^E_{>0}$. The goal is to find a flow f satisfying demands b, which minimizes $\sum_{e \in E} |f_e| c_e$. We assume that G is a connected graph and that the set of feasible solutions is non-empty. We denote by $f^\star \in \mathbb{Z}^E$ an arbitrary but fixed optimal solution, its cost is denoted by opt. We fix an orientation of edges in G, this can be arbitrary, so we assume for convenience that $f^\star \geq 0$. For an edge $e = (u, v)$ we denote the potential drop $p_u - p_v$ by $\Delta_e p$. Finally, let $C = \max_{e \in E} c_e$.

For the continuous variant of the undirected physarum dynamics it was established in [2] that for every $x(0) \in \mathbb{R}^E_{>0}$, the solution $x(t)$ converges to the set of optimal solutions. In our result we prove convergence and give bounds on the convergence rate in the discrete variant. Lemmas 4.1, 4.2, 4.3 from the following text were introduced first in the paper [2] and then used in [1]. We generalize them to the min cost flow setting. Let us begin with a simple but important lemma. Starting with $x_e(0) = b_P$ for all e, it tells us that the sequence $x(1), x(2), \ldots$ is well defined.

Lemma 4.1 (**Positivity of the solution**). *For every $k \geq 0$ and for every $e \in E$ we have $x_e(k) > 0$.*

Proof. We proceed by induction. For $k = 0$ we have $x_e(0) = b_P > 0$. It remains to perform the induction step:

$$x_e(k+1) = (1-h)x_e(k) + h|q_e(k)| \geq (1-h)x_e(k) > 0.$$

□

The next lemma shows the existence of x-capacitated flows at every step and plays an important role in the proofs.

Lemma 4.2 (**x-capacitated flows**). *For every step $k \geq 0$ there exists a flow f in G satisfying the demands b and respecting the capacities $x(k)$, that is $|f_e| \leq x_e(k)$ for every $e \in E$.*

Proof. To show that there is such a flow, we will verify the corresponding cut condition. For every partition $V = S \cup \bar{S}$ with $b_S > 0$ we need to show that the total capacity of edges going between S and $\bar{S}$ is at least b_S.

Let us fix a partition $V = S \cup \bar{S}$ with $b_S > 0$. Let E_S be the set of edges going between those two sets. By induction, we show that for every $k \geq 0$:

$$\sum_{e \in E_S} x_e(k) \geq b_S.$$

Start with $k = 0$. Since the instance is feasible there has to be some edge between S and $\bar{S}$. The initial value of every edge is $b_P \geq b_S$. Let us now assume that the induction hypothesis holds for some k, we will conclude that it holds for $k+1$ as well.

$$\sum_{e \in E_S} x_e(k+1) = (1-h) \sum_{e \in E_S} x_e(k) + h \sum_{e \in E_S} |q_e(k)|$$

The right hand side is a convex combination of two numbers, which we know are at least b_S. The first one because of induction hypothesis. The second one because the flow q satisfies demands b, therefore it has to send at least b_S units between S and $\bar{S}$. □

We define three main quantities which are used to keep track of the convergence.

Definition 4.1 (**Potential functions**).

1. Cost:
$$\mathcal{V}(k) := c^\top x(k) = \sum_{e \in E} c_e x_e(k).$$

2. Energy:
$$\mathcal{E}(k) := q(k)^\top W^{-1}(k) q(k) = \sum_{e \in E} \frac{c_e}{x_e(k)} q_e^2(k).$$

3. Barrier:
$$\mathcal{B}(k) := \sum_{e \in E} f_e^\star c_e \ln x_e(k).$$

Cost and energy potentials were introduced for the shortest path problem in [2]. A function similar to our barrier function was first used in [7,8] for shortest path. The cost potential intuitively indicates how close $x(k)$ is to the optimal solution. Hence, we want it to drop as fast as possible. The drop rate of $\mathcal{V}(k)$ is approximately captured by the difference $\mathcal{V}(k) - \mathcal{E}(k)$ (the larger the difference, the faster the drop). The barrier function helps keep track of this difference. Intuitively, if $\mathcal{E}(k)$ is big, the barrier is increasing; this cannot last for long as $\mathcal{B}(k)$ is bounded from above.

Lemma 4.3 (**Properties of potentials**). *The following hold for every step $k \geq 0$.*

1. $\mathcal{E}(k) \leq \mathcal{V}(k)$,
2. $\text{opt} \leq \mathcal{V}(k)$,
3. $\Delta\mathcal{V}(k) = \mathcal{V}(k+1) - \mathcal{V}(k) \leq 0$.

Proof. For brevity we will write $x_e, q_e, \ldots$ instead of $x_e(k), q_e(k), \ldots$. To prove the first and second part we use Lemma 4.3. Let f be the flow, which existence is guaranteed. As always we use W for the matrix $\mathsf{Diag}\left(\frac{x_1(k)}{c_1}, \ldots, \frac{x_m(k)}{c_m}\right)$. Since q minimizes the energy, we have:

$$\mathcal{E}(k) = q^\top W^{-1} q \leq f^\top W^{-1} f$$

furthermore $|f| \leq x$, hence:

$$f^\top W^{-1} f \leq x^\top W^{-1} x$$

but

$$x^\top W^{-1} x = \sum_{e \in E} x_e \frac{c_e}{x_e} x_e = c^\top x = \mathcal{V}(k)$$

hence the first inequality. The second follows by noting that:

$$\text{opt} \leq \sum_{e \in E} c_e |f_e| \leq \sum_{e \in E} c_e x_e = \mathcal{V}(k).$$

For the third part, we calculate:

$$\begin{aligned}\mathcal{V}(k+1) - \mathcal{V}(k) =& h(c^\top |q| - c^\top x) \\ =& h(x^\top W^{-1}|q| - x^\top W^{-1} x).\end{aligned}$$

By the Cauchy-Schwarz inequality:

$$x^\top W^{-1}|q| \leq \left(x^\top W^{-1} x\right)^{1/2} \left(q^\top W^{-1} q\right)^{1/2}.$$

In conclusion:

(4.3)
$$\mathcal{V}(k+1) - \mathcal{V}(k) \leq h\mathcal{V}(k)^{1/2} \left(\mathcal{E}(k)^{1/2} - \mathcal{V}(k)^{1/2}\right) \leq 0.$$

□

In the next lemma we present the key for our convergence analysis. The vertex potentials induced in the electrical network at certain time steps behave in a rather unpredictable way. Still, we are able to uniformly bound the maximum potential difference induced over all of the steps.

Lemma 4.4 (**Bounded potentials**). *For all $k \geq 0$ we have* $\max_{u,v \in V} |p_u(k) - p_v(k)| \leq nC$.

For the case when the instance is a shortest $s-t$ path problem, one can prove a statement similar to Lemma 4.4 by the following argument. The maximum potential difference is $p_s - p_t$, which in this case is the same as the electrical flow energy. As Lemma 4.3 (1) asserts, the energy is bounded, hence the bound on maximum potential difference follows. This argument unfortunately does not extend to general electrical networks. We cannot predict which vertex has the lowest potential and which one the highest; typically this can change between timesteps. We give an argument based on the existence of an x-capacitated flow.

Proof of Lemma 4.4. Fix k. For brevity we omit the argument (k) from $p(k)$, $q(k)$, etc. Sort all the potentials in non-decreasing order and pick two neighbouring ones p_u, p_v. In other words, take u, v, such that $p_u \le p_v$ and for all $w \in V$ either $p_w \le p_u$ or $p_w \ge p_v$. We show that $p_v - p_u \le C$.

Assume the contrary: $p_v - p_u > C$. Recall that C is the maximum cost c_e over all $e \in E$. We define a partition of V into two sets $S, \bar{S}$: $S = \{w \in V : p_w \le p_u\}$, $\bar{S} = \{w \in V : p_w \ge p_v\}$. Let E_S be the set of edges going between S and $\bar{S}$. Since the graph is connected we know that $E_S \neq \varnothing$. Therefore, there is some non-zero flow going from $\bar{S}$ to S (recall that $\bar{S}$ has vertices with higher potentials than those in S). In other words $b_S = \sum_{v \in S} b_v > 0$. Since q is a valid flow, we have: $\sum_{e \in E_S} |q_e| = b_S$. On the other hand:

$$\sum_{e \in E_S} |q_e| = \sum_{e \in E_S} \left| \frac{\Delta_e p x_e}{c_e} \right| \overset{|\Delta_e p| > C}{>} \sum_{e \in E_S} \left| \frac{C x_e}{c_e} \right| \ge \sum_{e \in E_S} |x_e| \ge b_S.$$

We arrive at a contradiction. The last inequality follows from Lemma 4.2, the existence of a flow implies the cut condition for $(S, \bar{S})$. $\square$

4.2 Proof of convergence. In this section we analyze the convergence of the discrete process for the undirected transshipment problem. The analysis is based on ideas developed in [1] for the undirected shortest path problem. However, it is not straightforward to generalize the analysis to minimum cost flow. One of the difficulties is to obtain a uniform upper bound on the maximum potential difference (see Lemma 4.4). The obtained bound also plays a key role in constructing an algorithm, which applied to the shortest path problem has a better running time than the one in [1].

To prove Theorem 1.1 we analyze the following potential function: $\phi(k) := 13 \ln \mathcal{V}(k) - \frac{\varepsilon}{\text{opt}} \mathcal{B}(k)$. The intuitive meaning of ϕ is as follows: we know that $\mathcal{V}(k)$ is non-increasing with k, so $\ln \mathcal{V}(k)$ is non-increasing as well, however we may not guarantee a big drop of $\mathcal{V}(k)$ in every step, that is why we have the second term; $\mathcal{B}(k)$ gets bigger at such steps. The crucial lemma we would like to show asserts that ϕ drops significantly at every step:

Lemma 4.5 (**Potential drop**). *For every k with $\mathcal{V}(k) > (1+\varepsilon)\text{opt}$ we have $\Delta\phi(k) \le -{}^{h\varepsilon^2}/_{30}$.*

Before we give a proof of the above lemma, let us first conclude Theorem 1.1 from it. For this, we need a simple lemma.

Lemma 4.6 (**Bounded conductances**). *For every k and every edge $e \in E$ we have $x_e(k) \le b_P$.*

Proof. We prove it by induction. For $k = 0$ we have $x_e(0) = b_P$. In the induction step:

$$x_e(k+1) = (1-h)x_e(k) + h|q_e(k)| \le (1-h)b_P + hb_P = b_P$$

the inequality $|q_e| \le b_P$ follows from a property of electrical flows 3.1 (1). $\square$

Proof of Theorem 1.1 assuming Lemma 4.5. We control the drop of $\mathcal{V}(k) = c^\top x(k)$ by analyzing $\phi(k)$. At the beginning, for $k = 0$, we have

$$\mathcal{V}(0) = \sum_{e \in E} c_e x_e(0) = b_P \sum_{e \in E} c_e \le b_P m C.$$

Hence

$$\phi(0) \le 13 \ln b_P m C - {}^{\varepsilon}/_{\text{opt}} \mathcal{B}(0).$$

Let us now provide a lower bound on the value of ϕ. From Lemma 4.3 (2) we know that $\mathcal{V}(k) \ge \text{opt} \ge 1$ for every k. Moreover, by Lemma 4.6 the maximum possible value of $\mathcal{B}$ is $\mathcal{B}(0)$. Hence $\phi(k) \ge \ln 1 - {}^{\varepsilon}/_{\text{opt}} \cdot \mathcal{B}(0) = -{}^{\varepsilon}/_{\text{opt}} \cdot \mathcal{B}(0)$. Lemma 4.5 guarantees that until $\mathcal{V}(k)$ drops below $(1+\varepsilon)\text{opt}$, ϕ always drops by at least ${}^{h\varepsilon^2}/_{30}$. Because of our bounds on $\phi(0)$ and $\phi(k)$, this drop cannot continue for a long time, it stops after $O\left({}^{\ln(b_P m C)}/_{\varepsilon^2 h}\right)$ steps.

This already proves that $\text{opt} \le c^\top x(k) \le \text{opt}(1+\varepsilon)$ for $k = \Omega\left({}^{\ln(b_P m C)}/_{\varepsilon^2 h}\right)$. We still need to show how to obtain a solution f with the guarantee $Bf = b$ and $c^\top |f| \le \text{opt}(1+\varepsilon)$. To this end we can keep an auxiliary vector $f(k) \in \mathbb{R}^E$ in addition to $x(k)$ and update it according to the rule:

$$f(k+1) = (1-h)f(k) + hq(k).$$

Suppose that $f(0)$ satisfies $Bf(0) = b$ and $|f(0)| \le x(0)$. This is easy to obtain by taking $f(0)$ to be any electrical flow in G (e.g. with respect to uniform resistances). Then since $q(k)$ satisfies $Bq(k) = b$, we can prove by induction that $Bf(k) = b$ for every $k \ge 0$. Moreover, by induction we can prove that for every k, $|f(k)| \le x(k)$:

$$\begin{aligned} |f(k+1)| =& |(1-h)f(k) + hq(k)| \\ \le& (1-h)|f(k)| + h|q(k)| \\ \le& (1-h)x(k) + h|q(k)| = x(k+1). \end{aligned}$$

Hence at every step k we have that $f(k)$ is a flow and $|f(k)| \le x(k)$. In particular $c^\top |f| \le \text{opt}(1+\varepsilon)$ for $k = \Omega\left({}^{\ln(b_P m C)}/_{\varepsilon^2 h}\right)$. $\square$

We come back to the proof of Lemma 4.5, we split it into two cases and deal with them separately. They are expressed in the following facts.

Fact 4.7. *If* ${}^{\mathcal{E}(k)}/_{\mathcal{V}(k)} < (1 - {}^{\varepsilon}/_{3})$ *then* $\Delta\phi(k) \leq -{}^{h\varepsilon^2}/_{30}$.

Fact 4.8. *If* $\mathcal{E}(k) > (1 + {}^{\varepsilon}/_{3})\mathrm{opt}$ *then* $\Delta\phi(k) \leq -{}^{h\varepsilon^2}/_{30}$.

Proof of Fact 4.7. We will show that $\ln \mathcal{V}$ always drops and $\mathcal{B}$ may increase only a little. Let us first look at $\ln \mathcal{V}(k+1) - \ln \mathcal{V}(k)$. By a reasoning as in the proof of Lemma 4.3, we get (see 4.3):

$$\begin{aligned}\frac{\mathcal{V}(k+1)}{\mathcal{V}(k)} &\leq \left(1 + h\left(\frac{\mathcal{E}(k)}{\mathcal{V}(k)}\right)^{1/2} - h\right)\\ &< \left(1 + h\left(\sqrt{1-\varepsilon/3} - 1\right)\right)\\ &\leq 1 - h\varepsilon/6.\end{aligned}$$

Where we used the assumption and the inequality $\sqrt{1 - {}^{\varepsilon}/_{3}} \leq 1 - {}^{\varepsilon}/_{6}$. We obtain:

$$\ln \mathcal{V}(k+1) - \ln \mathcal{V}(k) = \ln \frac{\mathcal{V}(k+1)}{\mathcal{V}(k)} \overset{(A.1)}{\leq} -\frac{h\varepsilon}{6}.$$

Consider now $\Delta\mathcal{B}(k) := \mathcal{B}(k+1) - \mathcal{B}(k)$:

$$\begin{aligned}\Delta\mathcal{B}(k) &= \sum_{e\in E} f_e^\star c_e \ln \frac{x_e(k+1)}{x_e(k)}\\ &= \sum_{e\in E} f_e^\star c_e \ln \frac{(1-h)x_e(k) + h|q_e(k)|}{x_e(k)}\\ &\geq \sum_{e\in E} f_e^\star c_e \ln(1-h)\\ &\overset{(A.3)}{\geq} -2h \sum_{e\in E} f_e^\star c_e\\ &= -2h \cdot \mathrm{opt}.\end{aligned}$$

Putting these two pieces together yields:

$$\phi(k+1) - \phi(k) \leq -\frac{13h\varepsilon}{6} + 2h\varepsilon = -\frac{h\varepsilon}{6} \leq -\frac{h\varepsilon^2}{30}.$$

□

Proof of Fact 4.8. We know that $\mathcal{V}(k)$ is non-increasing with k, it remains to show that $\mathcal{B}$ will increase by a considerable amount. We have

$$\Delta\mathcal{B}(k) = \sum_{e\in E} f_e^\star c_e \ln\left(1 + h\left(\frac{|q_e(k)| - x_e(k)}{x_e(k)}\right)\right).$$

Note that by Lemma 4.4:

$$\begin{aligned}\left|h\left(\frac{|q_e(k)| - x_e(k)}{x_e(k)}\right)\right| &= |h|\left|\frac{|\Delta_e p(k)|}{c_e} - 1\right|\\ &\leq hnC = \frac{\varepsilon}{10} \leq \frac{1}{2}.\end{aligned}$$

Hence we can apply the inequality (A.2), by which we get:

$$\begin{aligned}\Delta\mathcal{B}(k) &\overset{(A.2)}{\geq} \sum_{e\in E} f_e^\star c_e h\left(\frac{|q_e(k)| - x_e(k)}{x_e(k)}\right)\\ &\quad - \sum_{e\in E} f_e^\star c_e \left[h\left(\frac{|q_e(k)| - x_e(k)}{x_e(k)}\right)\right]^2.\end{aligned} \tag{4.4}$$

Let us analyze the second term of the RHS of (4.4)

$$\begin{aligned}&\sum_{e\in E} f_e^\star c_e \left[h\left(\frac{|q_e(k)| - x_e(k)}{x_e(k)}\right)\right]^2\\ &= h^2 \sum_{e\in E} f_e^\star c_e \left|\frac{|\Delta_e p(k)|}{c_e} - 1\right|^2\\ &\overset{4.4}{\leq} h^2 \sum_{e\in E} f_e^\star c_e \left|\frac{|\Delta_e p(k)|}{c_e} - 1\right| \cdot nC\\ &= \frac{h\varepsilon}{10} \sum_{e\in E} f_e^\star c_e \left|\frac{|\Delta_e p(k)|}{c_e} - 1\right|\\ &\leq \frac{h\varepsilon}{10} \sum_{e\in E} f_e^\star |\Delta_e p(k)| + \frac{h\varepsilon}{10}\mathrm{opt}.\end{aligned}$$

The first term in the RHS of (4.4) is

$$\sum_{e\in E} f_e^\star c_e h\left(\frac{|q_e(k)| - x_e(k)}{x_e(k)}\right) = h\sum_{e\in E} f_e^\star |\Delta_e p(k)| - h\cdot\mathrm{opt}.$$

We continue now with (4.4)

$$\begin{aligned}\Delta\mathcal{B}(k) &\overset{(A.2)}{\geq} \sum_{e\in E} f_e^\star c_e h\left(\frac{|q_e(k)| - x_e(k)}{x_e(k)}\right)\\ &\quad - \sum_{e\in E} f_e^\star c_e \left[h\left(\frac{|q_e(k)| - x_e(k)}{x_e(k)}\right)\right]^2\\ &\geq h\sum_{e\in E} f_e^\star |\Delta_e p(k)| - h\cdot\mathrm{opt}\\ &\quad - \frac{h\varepsilon}{10}\sum_{e\in E} f_e^\star |\Delta_e p(k)| - \frac{h\varepsilon}{10}\mathrm{opt}\\ &\geq h\left(1 - \frac{\varepsilon}{10}\right)\sum_{e\in E} f_e^\star |\Delta_e p(k)| - h\left(1 + \frac{\varepsilon}{10}\right)\mathrm{opt}.\end{aligned}$$

The remaining thing to note now is:

$$\begin{aligned}\sum_{e\in E} f_e^\star |\Delta_e p(k)| &\geq \sum_{e\in E} f_e^\star \Delta_e p(k) = f^{\star\top} B^\top p(k)\\ &= (Bf^\star)^\top p(k) = b^\top p(k) = \mathcal{E}(k).\end{aligned}$$

Where the last equality follows from a property of electrical flows 3.1 (2). Finally, by the assumption that

$\mathcal{E}(k) > (1 + \varepsilon/3)\text{opt}$, we get

$$\begin{aligned}
\Delta\mathcal{B}(k) &\geq h\left(1 - \frac{\varepsilon}{10}\right)\sum_{e\in E} f_e^\star |\Delta_e p(k)| - h\left(1 + \frac{\varepsilon}{10}\right)\text{opt} \\
&\geq h\left(1 - \frac{\varepsilon}{10}\right)\left(1 + \frac{\varepsilon}{3}\right)\text{opt} - h\left(1 + \frac{\varepsilon}{10}\right)\text{opt} \\
&= \text{opt}\cdot h\left(\frac{\varepsilon}{3} - \frac{\varepsilon}{10} - \frac{\varepsilon}{10} - \frac{\varepsilon^2}{30}\right) \\
&\geq \frac{\text{opt}\cdot h\varepsilon}{30}.
\end{aligned}$$

This implies the following drop of ϕ:

$$\phi(k+1) - \phi(k) \leq -\frac{\varepsilon}{\text{opt}}\left(\mathcal{B}(k+1) - \mathcal{B}(k)\right) \leq -\frac{h\varepsilon^2}{30}.$$

□

We conclude by deducing Lemma 4.5 from Facts 4.7 and 4.8.

Proof of Lemma 4.5. If ε is small enough and $\mathcal{V}(k) > (1+\varepsilon)\text{opt}$ then obviously either $\frac{\mathcal{E}(k)}{\mathcal{V}(k)} < (1 - \varepsilon/3)$ or $\mathcal{E}(k) > (1 + \varepsilon/3)\text{opt}$ and we use Fact 4.7 or Fact 4.8 respectively. □

5 Discrete dynamics for the directed transshipment problem

In this section we prove Theorem 1.2. The first subsection is devoted to proving that the dynamics run on the preconditioned instance behaves nicely. In particular, we show that it is well defined. In the second subsection we give the proof of convergence.

5.1 Properties of the preconditioned instance. We now study properties of the preconditioned instance introduced in Section 1.1. Let (G, b, c) be the preconditioned instance and $x(0)$ the corresponding initial vector. Recall that $C = \max_{e\in E_0} c_e$, denote also $\widetilde{C} = \max_{e\in E} c_e = nC$. In the following text, for an edge $e = (u, v)$ we denote the potential drop $p_u - p_v$ by $\Delta_e p$. We start by proving that the sequence $x(k)$ is well defined for all $k \in \mathbb{N}$. As mentioned in Section 2, even this basic property is not straightforward to obtain in the directed case (see also Appendix C). In the next lemma, we prove a key property of the preconditioned instance: existence of x-capacitated flows. It implies a bound on the maximum potential difference and positivity of the conductance vector.

Lemma 5.1 (**Properties of the preconditioned instance**). *Let (G, b, c) be the preconditioned instance and $x(0)$ be the corresponding initial vector. Suppose that $h < 1/n\widetilde{C}+1$ Then, for every $k \in \mathbb{N}$:*

1. *Positivity: $x_e(k) > 0$ for every $e \in E$,*
2. *Bounded potentials: $\max_{u,v\in V} |p_u(k) - p_v(k)| \leq n\widetilde{C}$,*
3. *x-capacitated flows: there is a vector $f \in \mathbb{R}^E$ with $0 \leq f \leq x(k)$ such that $Bf = b$.*

Proof of Lemma 5.1. We prove this lemma by joint induction on the listed properties. Towards this end, we start by proving some useful implications between the properties. First we prove that whenever both properties (1) and (3) hold for some k, then the property (2) holds as well.

$(1)_k, (3)_k \Rightarrow (2)_k$**:** Let f be such a flow, respecting the capacities x and satisfying demands b. By the mincut-maxflow theorem this means that whenever we have a partition of V into two sets $S, \bar{S}$, such that $b_S = \sum_{v\in S} b_v > 0$ then $\sum_{e\in E(\bar{S},S)} x_e \geq b_S$. Sort all of the potentials in non-decreasing order and pick two neighbouring ones p_u, p_v. In other words, take u, v, such that $p_u \leq p_v$ and for all $w \in V$ either $p_w \leq p_u$ or $p_w \geq p_v$. We show that $p_v - p_u \leq \widetilde{C}$.

Assume the contrary: $p_v - p_u > \widetilde{C}$. Recall that $\widetilde{C}$ is the maximum cost c_e over all $e \in E$. We define a partition of V into two sets $S = \{w \in V : p_w \leq p_u\}$ and $\bar{S}$. Let 1_S be the indicator vector of the set S. We know that $Bq = b$, multiplying both sides by $1_S^\top$ yields $1_S^\top Bq = 1_S^\top b = b_S$. Thus,

$$b_S = \sum_{e\in E(\bar{S},S)} q_e - \sum_{e\in E(S,\bar{S})} q_e.$$

Note that since electrical flow goes always from bigger potential to lower potential, for $e \in E(\bar{S}, S)$ we have $q_e > 0$ and for $e \in E(S, \bar{S})$ we have $q_e < 0$. Moreover, there is at least one edge between $S, \bar{S}$, since the graph is connected, hence $b_S > 0$. We have:

$$\begin{aligned}
b_S &\geq \sum_{e\in E(\bar{S},S)} q_e = \sum_{e\in E(\bar{S},S)} \frac{x_e \Delta_e p}{c_e} > \sum_{e\in E(\bar{S},S)} \frac{x_e \widetilde{C}}{c_e} \\
&\geq \sum_{e\in E(\bar{S},S)} x_e \geq b_S.
\end{aligned}$$

The last inequality follows from mincut-maxflow theorem. We have reached a contradiction, hence $p_v - p_u \leq \widetilde{C}$. Consequently $\max_{u,v\in V} |p_u - p_v| \leq n\widetilde{C}$. Note that we have implicitly used $(1)_k$ to reason about potentials (if we do not have $(1)_k$ the electrical flow could be not well defined). The next step is to argue that property (2) for step k implies (1) for step $k+1$.

$(1)_k, (2)_k \Rightarrow (1)_{k+1}$: Pick any edge $e \in E$. We have:

$$\begin{aligned} x_e(k+1) &= (1-h)x_e(k) + hq_e(k) \\ &= (1-h)x_e(k) + h\frac{x_e\Delta_e p}{c_e} \\ &\geq x_e(k)(1-h(1+n\widetilde{C})) > 0. \end{aligned}$$

The last implication we would like to argue about is the following: whenever $x(0), \ldots, x(k)$ are well defined (i.e., (1) holds) then (3) is true for step k.

$(1)_0, (1)_1, \ldots, (1)_k \Rightarrow (3)_k$: To argue that a certain flow exists, we show lower bounds on cut capacities. Pick any partition of V into $S, \bar{S}$ with $b_S > 0$. We need to show that: $\sum_{e\in E(\bar{S},S)} x_e(k) \geq b_S$. We make use of the quantity

$$x_S(j) := \sum_{e\in E(\bar{S},S)} x_e(j) - \sum_{e\in E(S,\bar{S})} x_e(j).$$

Clearly $\sum_{e\in E(\bar{S},S)} x_e(k) \geq x_S(k)$, so it is enough for us to show $x_S(k) \geq b_S$. It turns out that for $x_S(j)$ we can obtain a nice formula, let us first compute:

$$\begin{aligned} x_S(j+1) &= \sum_{e\in E(\bar{S},S)} x_e(j+1) - \sum_{e\in E(S,\bar{S})} x_e(j+1) \\ &= (1-h)x_S(j) + h\left(\sum_{e\in E(\bar{S},S)} q_e(j) - \sum_{e\in E(S,\bar{S})} q_e(j)\right) \\ &= (1-h)x_S(j) + h1_S^\top Bq = (1-h)x_S(j) + hb_S. \end{aligned}$$

This yields a recursive formula for $x_S(j)$. We can solve it and obtain:

$$x_S(j) = (1-h)^j x_S(0) + (1-(1-h)^j)b_S.$$

Thus $x_S(j)$ is a convex combination of $x_S(0)$ and b_S. Hence showing that $x_S(j) \geq b_S$ is in fact equivalent to showing $x_S(0) \geq b_S$. We count separately the contribution of edges from E_0 and E' to $x_S(0)$. The first part is simple:

$$\sum_{e\in E_0(\bar{S},S)} x_e(0) - \sum_{e\in E_0(S,\bar{S})} x_e(0) \geq -m.$$

From now on we focus on the new edges: E'. Let S_- be the set $\{v \in S : b_v < 0\}$ and $S_+ = \{v \in S : b_v > 0\}$, similarly we define $\bar{S}_-$ and $\bar{S}_+$. Note that $b_{S_+} = -b_{\bar{S}_-}$ and $b_{S_-} = -b_{\bar{S}_+}$, also clearly $b_S = b_{S_+} + b_{S_-}$. From the definition of $x(0)$, one can see that:

$$\sum_{e\in E'(\bar{S},S)} x_e(0) = 2m \cdot |b_{\bar{S}_-}| \cdot b_{S_+}$$
$$\sum_{e\in E'(S,\bar{S})} x_e(0) = 2m \cdot |b_{S_-}| \cdot b_{\bar{S}_+}.$$

Hence, the contribution of edges from E' to $x_S(0)$ is:

$$\begin{aligned} &2m(|b_{\bar{S}_-}| \cdot b_{S_+} - |b_{S_-}| \cdot b_{\bar{S}_+}) = 2m(b_{S_+}^2 - b_{S_-}^2) \\ =&2m(b_{S_+} + b_{S_-})(b_{S_+} - b_{S_-}) \geq 2mb_S. \end{aligned}$$

Hence $x_S(0) \geq 2mb_S - m \geq b_S$ (we used $b_S \geq 1$) which concludes the proof.

After showing the three implications above, one can easily prove the lemma by joint induction. In the base case it is enough to observe that $(1)_0$ holds. Then $(3)_0$ follows immediately, which in turn yields $(2)_0$. The three conditions allow us to conclude $(1)_1$ and so on. $\square$

By Lemma 5.1 we know that the sequence $\{x(k)\}_{k\in\mathbb{N}}$ is well defined and we can reason about its properties. From now on we always assume that we work with the preconditioned instance and h is small enough, so that the conclusion of Lemma 5.1 holds.

5.2 Proof of convergence. In this subsection we present a proof of convergence of the discrete, directed process. Recall that we work constantly with the preconditioned instance. Let us first state some preliminary properties:

Lemma 5.2 (Bounds for conductances). *For every $k \geq 0$ the following hold:*

1. $x_e(k) = x_e(0)(1-h)^k + h\sum_{j=0}^{k-1}(1-h)^{k-j}q_e(j)$,
2. $x_e(k) = x_e(0)\prod_{j=0}^{k-1}\left(1 + h\left(\Delta_e p(j)/c_e - 1\right)\right)$,
3. $x_e(k) \leq 2mb_P^2$.

Proof. Proof of Lemma 5.2 Formulas (1) and (2) can be proved by induction. We proceed with (3).
In the proof we will use a property of electrical flows 3.1 (1), which implies that $q_e(k) \leq b_P$. We have:

$$x_e(k+1) = (1-h)x_e(k) + hq_e(k) \leq (1-h)x_e(k) + hb_P.$$

By induction, it follows that:

$$x_e(k) \leq (1-h)^k x_e(0) + (1-(1-h)^k)b_P \leq 2mb_P^2.$$

$\square$

Since it is not convenient to work with $x(k)$ (we would prefer $x(k)$ to be a flow), we prove that for big enough k there is always a non-negative flow $f(k)$ which is close to $x(k)$. Then we argue that for big k the resulting flow $f(k)$ is close to the optimal solution, hence $x(k)$ is close as well.

Lemma 5.3 (**Rounding flows, [5]**). *Let g be an arbitrary flow. Let $F \subseteq E$ be a set of edges, satisfying $w := \sum_{e\in F} |g_e| < 1$ and $g_e \geq 0$ for all $e \in E \setminus F$. Then there exists a non-negative flow f, such that $supp(f) \subseteq (supp(g) \setminus F)$ and $\|f-g\|_\infty \leq w$.*

We omit the proof. Now, we show that for large k, $x(k)$ is an almost non-negative flow.

Lemma 5.4 (**Almost non-negative flow**). *For every ε with $0 < \varepsilon < 1$ and for every $k > {}^{10\ln(nmb_P/\varepsilon)}/_h$ there exists a non-circular flow $f \geq 0$ such that: $\|x(k) - f\|_\infty < \varepsilon$.*

Proof of Lemma 5.4. Denote:

$$\bar{q}(k) = h\sum_{j=0}^{k-1}(1-h)^{k-j}q(j),$$
$$\bar{p}(k) = \sum_{j=0}^{k-1} p(j),$$
$$X_0 = \max_{e\in E} x_e(0).$$

We will find a non-circular, non-negative flow f which satisfies $\|f - \bar{q}(k)\|_\infty < \varepsilon/2$, our choice of k implies $\|x(k) - \bar{q}(k)\|_\infty < \varepsilon/2$ (as we will show) and the result follows. First note that by Lemma 5.2 (1):

$$\|x(k) - \bar{q}(k)\|_\infty = \max_{e\in E} x_e(0)(1-h)^k \leq X_0(1-h)^k.$$

For this to become smaller than $\varepsilon/2$ it suffices to take $k > \frac{3\ln(X_0/\varepsilon)}{h}$. By taking $k > \frac{10\ln(X_0 nm/\varepsilon)}{h}$ we can take it down to at most $\frac{\varepsilon}{4(m^2+n)}$.

Now assume that $k > \frac{10\ln(nmX_0/\varepsilon)}{h}$ and consider the set of edges

$$F = \{e \in E : \bar{q}(e) \leq 0\} \cup \{e \in E : \Delta_e \bar{p}(k) \leq 0\}.$$

We intend to apply Lemma 5.3 for h equal to $\bar{q}(k)$, for this we need to bound the quantity $\sum_{e\in F} |\bar{q}_e(k)|$. If e satisfies $\bar{q}(e) \leq 0$ then by Lemma 5.2 part (1) we have:

$$|\bar{q}_e(k)| \leq x_e(0)(1-h)^k < \frac{\varepsilon}{4(m+n^2)} < \frac{\varepsilon}{m+n^2}.$$

Similarly if $\Delta_e \bar{p}(k) \leq 0$, then by Lemma 5.2 (2):

$$\begin{aligned} x_e(k) &= x_e(0)\prod_{j=0}^{k-1}\left(1 + h\left(\frac{\Delta_e p(j)}{c_e} - 1\right)\right) \\ &\leq x_e(0)\exp\left(h\sum_{j=0}^{k-1}\left(\frac{\Delta_e p(j)}{c_e} - 1\right)\right) \\ &= x_e(0)\exp\left(-hk + \frac{h}{c_e}\Delta_e\bar{p}(k)\right) \\ &\leq x_e(0)\exp(-hk) \\ &< \frac{\varepsilon}{2(m+n^2)}. \end{aligned}$$

We also know that for such big k, $|x_e(k) - \bar{q}_e(k)| < \frac{\varepsilon}{2(m+n^2)}$ holds, so:

$$|q_e(k)| < \frac{\varepsilon}{2(m+n^2)} + \frac{\varepsilon}{4(m+n^2)} \leq \frac{\varepsilon}{m+n^2}.$$

By the above calculation we obtain:

$$\sum_{e\in F} |\bar{q}_e(k)| < |F|\frac{\varepsilon}{2(m+n^2)} \leq \frac{\varepsilon}{2}.$$

Thus we can apply Lemma 5.3 and obtain a flow f. Note that the obtained flow satisfies:

- $f \geq 0$,
- f is non-circular. Let $\gamma \geq 0$ be a directed cycle, i.e, $B\gamma = 0$ then
  $$\sum_{e\in E} \Delta_e \bar{p}(k)\gamma_e = \bar{p}(k)^\top B\gamma = 0$$
  therefore there must be some edge $e \in E$ with $\gamma_e > 0$ and $\Delta_e \bar{p}(k) \leq 0$, hence $e \in F$, which implies $f_e = 0$,
- $\|f - \bar{q}(k)\|_\infty < \frac{\varepsilon}{2}$, as Lemma 5.3 guarantees.

□

From now on for simplicity we assume that there exists a unique optimal solution $f^\star$ to the underlying instance. For every step k let $f(k)$ be the non-negative, non-circular flow which is guaranteed to exist by Lemma 5.4. Our goal is to show that for k large enough $\|f(k) - f^\star\|_\infty$ is small, this in turn would imply our goal: that $x(k)$ is close to $f^\star$. We proceed with two technical lemmas.

Lemma 5.5 (**Optimality criterion**). *Let $\varepsilon \in (0,1)$. Let f be a non-negative, non-circular flow. Suppose that for every basic flow $g \neq f^\star$ there exists an edge $e \in E$ with $g_e > 0$, such that $f_e < {}^{\varepsilon}/_{2b_P(m+n^2)}$. Then $\|f - f^\star\|_\infty < \varepsilon$.*

Proof of Lemma 5.5. Since f is non-circular, by Fact 3.3, it can be written as a convex combination of basic flows:

$$f = \sum_{i=1}^{M} \alpha_i f_i$$

where $M \le |E| \le n^2 + m$. We assume that f_i are pairwise distinct in this decomposition and $f_1 = f^\star$. Take g to be any of f_i for some $i \ge 2$. From the hypothesis there is $e \in E$ such that $g_e > 0$ and $f_e < \frac{\varepsilon}{2b_P(m+n^2)}$. Recall that g is a basic flow, hence by Fact 3.2 it satisfies: $1 \le g_e \le b_P$.

$$\alpha_i \le \alpha_i g_e = \alpha_i f_{ie} \le \sum_{i=1}^{M} \alpha_i f_{ie} = f_e < \frac{\varepsilon}{2b_P(m+n^2)}.$$

This implies that $\sum_{i=2}^{M} \alpha_i < \frac{\varepsilon}{2b_P}$, hence $\alpha_1 > 1 - \frac{\varepsilon}{2b_P}$. Moreover:

$$\begin{aligned}\left\| \sum_{i=2}^{M} \alpha_i f_i \right\|_\infty &\le \frac{\varepsilon}{2b_P(m+n^2)} \sum_{i=2}^{M} \|f_i\|_\infty \\ &\le \frac{M b_P \cdot \varepsilon}{2b_P(m+n^2)} \le \frac{\varepsilon}{2}.\end{aligned}$$

To finish the proof it remains to argue that $\|\alpha_1 f_1 - f^\star\|_\infty < \frac{\varepsilon}{2}$. Indeed

$$\begin{aligned}\|\alpha_1 f_1 - f^\star\|_\infty &= \|\alpha_1 f^\star - f^\star\|_\infty \\ &= |1-\alpha_1| \cdot \|f^\star\|_\infty < \frac{\varepsilon}{2b_P} b_P = \frac{\varepsilon}{2}.\end{aligned}$$

□

Lemma 5.6 (**Decay of non-optimal flows**). *Let $\varepsilon \in (0,1)$ and g be any non-optimal basic flow. Suppose $h < 1/2nb_P(n\widetilde{C}+\widetilde{C})^2$. For every $k > 12\widetilde{C}b_P/h \ln(mb_P/\varepsilon)$ there is an edge $e \in E$ such that $g_e > 0$ and $x_e(k) < \varepsilon$.*

Proof of Lemma 5.6. Fix ε and g as in the statement. We consider the following parametrized barrier function:

$$\mathcal{B}_g(k) := \sum_{e \in E} g_e c_e \ln x_e(k).$$

The idea is to show that $\mathcal{B}_g(k)$ gets very small for k large enough. To this end consider $\Delta\mathcal{B}_g(k) := \mathcal{B}_g(k+1) - \mathcal{B}_g(k)$:

$$\begin{aligned}\Delta\mathcal{B}_g(k) &= \sum_{e\in E} g_e c_e \ln \frac{x_e(k+1)}{x_e(k)} \\ &= \sum_{e\in E} g_e c_e \ln\left(1 + h\left(\frac{\Delta_e p(k)}{c_e} - 1\right)\right) \\ &\le h \sum_{e\in E} g_e c_e \frac{\Delta_e p(k)}{c_e} - hc^\top g.\end{aligned}$$

Observe that

$$\begin{aligned}\sum_{e\in E} g_e c_e \frac{\Delta_e p(k)}{c_e} &= \sum_{e\in E} g_e \Delta_e p(k) = g^\top B^\top p(k) \\ &= (Bg)^\top p(k) = b^\top p(k).\end{aligned}$$

Moreover, by the property 3.1 (2), we know that $b^\top p(k) = \mathcal{E}(k)$. Hence:

$$\Delta\mathcal{B}_g(k) \le h\mathcal{E}(k) - hc^\top g.$$

Let us now look how $\mathcal{B}_{f^\star}(k) := \sum_{e\in E} f_e^\star c_e \ln x_e(k)$ changes. We will use the inequality $\ln(1+\alpha) \ge \alpha - \alpha^2$, valid for $|\alpha| \le \frac{1}{2}$.

$$\begin{aligned}\Delta\mathcal{B}_{f^\star}(k) &= \sum_{e\in E} f_e^\star c_e \ln\left(1 + h\left(\frac{\Delta_e p(k)}{c_e} - 1\right)\right) \\ &\ge \sum_{e\in E} f_e^\star c_e \left(h\left(\frac{\Delta_e p(k)}{c_e} - 1\right) - h^2\left(\frac{\Delta_e p(k)}{c_e} - 1\right)^2\right).\end{aligned}$$

The first term in the above sum is easy to deal with:

$$\begin{aligned}&h\sum_{e\in E} f_e^\star c_e \left(\frac{\Delta_e p(k)}{c_e} - 1\right) \\ =&h\sum_{e\in E} f_e^\star c_e \frac{\Delta_e p(k)}{c_e} - h\sum_{e\in E} f_e^\star c_e \\ =&h\mathcal{E}(k) - hc^\top f^\star.\end{aligned}$$

We need to bound the second term:

$$\begin{aligned}&\sum_{e\in E} f_e^\star c_e \left(\frac{\Delta_e p(k)}{c_e} - 1\right)^2 \\ =&\sum_{e\in E} f_e^\star \left|\Delta_e p(k) - c_e\right| \left|\frac{\Delta_e p(k)}{c_e} - 1\right| \\ \le&(n\widetilde{C}+\widetilde{C})^2 \sum_{e\in E} f_e^\star \\ \le&(n\widetilde{C}+\widetilde{C})^2 n b_P.\end{aligned}$$

Now we can continue our estimate of $\Delta\mathcal{B}_{f^\star}(k)$:

$$\begin{aligned}\Delta\mathcal{B}_{f^\star}(k) &\ge h\mathcal{E}(k) - hc^\top f^\star - h^2(n\widetilde{C}+\widetilde{C})^2 n b_P \\ &\ge h\mathcal{E}(k) - hc^\top f^\star - \frac{h}{2}.\end{aligned}$$

By combining our bounds on $\Delta\mathcal{B}_g(k)$ and $\Delta\mathcal{B}_{f^\star}(k)$ we obtain:

$$\Delta\mathcal{B}_g(k) \le -hc^\top g + hc^\top f^\star + \frac{h}{2} + \Delta\mathcal{B}_{f^\star}(k).$$

Since both $f^\star$ and g are integral solutions $c^\top g \geq c^\top f^\star + 1$, this implies in particular that:

$$\Delta\mathcal{B}_g(k) \leq -\frac{h}{2}(c^\top g - c^\top f^\star) + \Delta\mathcal{B}_{f^\star}(k)$$

By expanding, we obtain:

$$\begin{aligned}(5.5)\quad \mathcal{B}_g(k) &\leq -\frac{hk}{2}(c^\top g - c^\top f^\star) + \mathcal{B}_{f^\star}(k) - \mathcal{B}_{f^\star}(0) + \mathcal{B}_g(0)\\ &= -\frac{hk}{2}(c^\top g - c^\top f^\star) + \mathcal{B}_{f^\star}(k) + \mathcal{B}_g(0).\end{aligned}$$

Before we proceed, let us first bound $\mathcal{B}_{f^\star}(k) + \mathcal{B}_g(0)$, Lemma 5.2 (3) tells us that $x_e(j) \leq 2mb_P^2$ for every j, hence:

$$\begin{aligned}\mathcal{B}_g(0) &= \sum_{e\in E} g_e c_e \ln x_e(0)\\ &\leq \ln\left(2mb_P^2\right)\sum_{e\in E} g_e c_e = c^\top g \cdot \ln\left(2mb_P^2\right).\end{aligned}$$

A similar bound applies to $\mathcal{B}_{f^\star}(k)$:

$$\mathcal{B}_{f^\star}(k) \leq c^\top f^\star \cdot \ln\left(2mb_P^2\right).$$

Combining them:

$$(5.6)\qquad \mathcal{B}_{f^\star}(k) + \mathcal{B}_g(0) \leq (c^\top g + c^\top f^\star)\ln\left(2mb_P^2\right).$$

Suppose now that $x_e(k) \geq \varepsilon$ for all e with $g_e > 0$. We will show that k has to be small. We have:

$$(5.7)\quad \mathcal{B}_g(k) = \sum_{e\in E} g_e c_e \ln x_e(k) \geq \sum_{e\in E} g_e c_e \ln\varepsilon \geq \widetilde{C} b_P \ln\varepsilon.$$

Combining (5.5), (5.6) and (5.7) we have:

$$\widetilde{C} b_P \ln\varepsilon \leq -\frac{hk}{2}(c^\top g - c^\top f^\star) + (c^\top g + c^\top f^\star)\ln\left(2mb_P^2\right).$$

Hence:

$$\frac{k}{2} \leq \frac{\widetilde{C} b_P \ln 1/\varepsilon}{h(c^\top g - c^\top f^\star)} + \frac{(c^\top g + c^\top f^\star)\ln\left(2mb_P^2\right)}{h(c^\top g - c^\top f^\star)}.$$

Since $c^\top g - c^\top f^\star \geq 1$, the first term is bounded by $h^{-1}\widetilde{C} b_P \ln 1/\varepsilon$. The second term is bounded by $h^{-1}(2c^\top f^\star + 1)\ln\left(2mb_P^2\right)$. Note that $c^\top f^\star \leq \widetilde{C} b_P$, since in the optimal solution the cost per one unit of flow is at most $\widetilde{C}$. Altogether this gives a bound:

$$k \leq \frac{12\widetilde{C} b_P}{h}\ln\left(\frac{mb_P}{\varepsilon}\right).$$

□

At this point we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We prove the theorem in the special case when there is only one optimal solution $f^\star$. The general case requires some minor adjustments in the statements of lemmas.

Fix any $\varepsilon \in (0, 1)$. Let $\delta = \frac{\varepsilon}{4b_P(m+n^2)}$. We choose h to be $\frac{1}{4nb_P(n\widetilde{C})^2} < \frac{1}{2nb_P(n\widetilde{C}+\widetilde{C})^2}$ (to make sure the hypothesis of Lemma 5.6 is satisfied). Fix $k > \frac{12\widetilde{C}b_P}{h}\ln\left(\frac{mb_P}{\delta}\right) = \frac{12\widetilde{C}b_P}{h}\ln\left(4b_P^2(m+n^2)m/\varepsilon\right)$ and use Lemma 5.6 (with δ in place of ε). We are guaranteed that for every basic non-optimal flow g, there is an edge $e \in E$ such that $g_e > 0$ and $x_e(k) < \delta$.

Because k is big enough we can use Lemma 5.4 (again with δ in place of ε), to obtain a non-circular flow $f \geq 0$ with $\|f - x(k)\|_\infty < \delta$. Hence for every basic non-optimal flow g, there is an edge $e \in E$ such that $g_e > 0$ and $f_e < 2\delta = \frac{\varepsilon}{2b_P(m+n^2)}$. By Lemma 5.5, $\|f - f^\star\|_\infty < \frac{\varepsilon}{2}$, hence:

$$\begin{aligned}\|x(k) - f^\star\|_\infty &\leq \|f^\star - f\|_\infty + \|f - x(k)\|_\infty\\ &\leq \frac{\varepsilon}{2} + \delta < \varepsilon.\end{aligned}$$

□

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A Elementary inequalities

The following well known inequalities are used throughout the paper.

$$\ln(1+\alpha) \le \alpha \qquad \text{for every } \alpha \in \mathbb{R}. \tag{A.1}$$

$$\ln(1+\alpha) \ge \alpha - \alpha^2 \qquad \text{for every } \alpha \in [-1/2, 1/2]. \tag{A.2}$$

$$\ln(1-\alpha) \ge -2\alpha \qquad \text{for every } \alpha \in [0, 1/2]. \tag{A.3}$$

B Lower bounds on convergence time

In the paper [1] authors obtain an algorithm for computing $(1+\varepsilon)-$approximation to the length of the shortest path in a graph with running time $\mathrm{poly}(n, L, 1/\varepsilon)$, where L is the maximum length of an edge in the graph. They ask whether the running time can be improved to make the algorithm run in polynomial time, in particular if the dependence on L can be made $\log(L)$ by performing a better analysis of the algorithm. We give a partial answer to this question by showing that using the current discretization technique it is not possible to obtain $\mathrm{poly}(n, \log L, \log 1/\varepsilon)$ running time. More precisely, we prove that whenever $\varepsilon = O(1/L)$, the running time will depend at least linearly on L. Note that this still leaves a possibility that $\mathrm{poly}(n, \log L, 1/\varepsilon)$ is achievable.

We start by defining our graph. We take $V = \{s, t\}$ and two parallel edges $e_1 = (s, t)$ with length $l_1 = L - 1$ and $e_2 = (s, t)$ with length $l_2 = L$ (L is a parameter). By solving symbolically the linear system defining the electrical flow one obtains the following system:

$$\begin{cases} x_1(k+1) = x_1(k)\left(1 + h\left(\frac{l_2}{x_1(k)l_2 + x_2(k)l_1} - 1\right)\right) \\ x_2(k+1) = x_2(k)\left(1 + h\left(\frac{l_1}{x_1(k)l_2 + x_2(k)l_1} - 1\right)\right) \\ x_1(0) = x_2(0) = 1. \end{cases} \tag{B.4}$$

Remark B.1. In this example we use the directed variant of the dynamics ($\dot{x} = q - x$). However, one can show that $|q_e(k)| = q_e(k)$ for every step k. Hence the equations $\dot{x} = q - x$ and $\dot{x} = |q| - x$ are equivalent, implying that both models give the same behavior for this case.

Remark B.2. The instance we work with can be modified in many possible ways to yield bigger graphs where the same phenomenon of slow convergence shows up.

Note that the optimal solution is $f^\star = \begin{pmatrix}1\\0\end{pmatrix}$. We show that the time to reach some close neighborhood of the optimal solution depends polynomially on L. More precisely:

Lemma B.1. *For $x(k)$, a solution to the system B.4, to guarantee $\|x(k) - f^\star\|_\infty < \frac{1}{4}$ we need to take $k = \Omega\left(\frac{L}{h}\right)$.*

The lemma says basically that it takes $\Omega(L)$ time to get very close to the optimal solution, even if the step length is maximum possible, i.e, h is close to 1.

Proof. By adding first two equations in (B.4) one gets:

$$x_1(k+1) + x_2(k+1) = (1-h)(x_1(k) + x_2(k)) + h.$$

This implies in particular that $1 \le x_1(k) + x_2(k) \le 2$ for all k. We keep track of the ratio $\frac{x_1(k)}{x_2(k)}$. At the beginning it is 1. For every point $y \ge 0$ with $\|y - f^\star\| < \frac{1}{4}$, the ratio $\frac{y_1}{y_2}$ is at least 2 (even at least 3). Hence, it suffices to show, that for a long time the ratio $\frac{x_1(k)}{x_2(k)}$ remains upper bounded by 2.

Suppose the ratio at step k is $(1+\varepsilon)$ for some $0 \le \varepsilon < 1$. We will give a bound on the ratio at step $(k+1)$. Let us denote $s := x_1(k)l_2 + x_2(k)l_1$. One can see that

$L-1 \le s \le 2L$. We calculate:

$$\begin{aligned}\frac{x_1(k+1)}{x_2(k+1)} =& \frac{x_1(k)\left(1+h\left(\frac{l_2}{x_1(k)l_2+x_2(k)l_1}-1\right)\right)}{x_2(k)\left(1+h\left(\frac{l_1}{x_1(k)l_2+x_2(k)l_1}-1\right)\right)}\\ =&(1+\varepsilon)\frac{1+h\left(\frac{l_2}{s}-1\right)}{1+h\left(\frac{l_1}{s}-1\right)}.\end{aligned}$$

It remains to upper bound $\frac{1+h\left(\frac{l_2}{s}-1\right)}{1+h\left(\frac{l_1}{s}-1\right)}$. To this end, we use the estimate: $\frac{1+\alpha}{1+\beta} \approx 1+\alpha-\beta$, which is valid, whenever $|\alpha-\beta|$ is small. More precisely, the additive error in this estimate is $\left|\frac{\beta(\alpha-\beta)}{1+\beta}\right|$. We take $\alpha = h\left(\frac{l_2}{s}-1\right)$ and $\beta = h\left(\frac{l_1}{s}-1\right)$. By straightforward computations one can see that in our case the error is $O(h^2/L)$. Hence we get:

$$\begin{aligned}&1+h\left(\frac{l_2}{s}-1\right)-h\left(\frac{l_1}{s}-1\right)+O\left(\frac{h^2}{L}\right)\\ =&O\left(\frac{h}{s}\right)+O\left(\frac{h^2}{L}\right)=O\left(\frac{h}{L}\right).\end{aligned}$$

Consequently

$$\frac{x_1(k+1)}{x_2(k+1)} = (1+\varepsilon)\left(1+O\left(\frac{h}{L}\right)\right) = 1+\varepsilon+O\left(\frac{h}{L}\right).$$

In every step the ratio increases by at most $O(h/L)$, hence it takes $\Omega(L/h)$ steps to reach 2 from 1. $\square$

C An example for non-positivity

In this section we would like to give some evidence that it is not clear that the directed version of the Physarum dynamics is well defined at all timesteps. It could potentially happen that at some point we will reach a conductance vector x, for which the electrical flow q is not well defined (i.e, x has some negative or zero entries). In the paper [5] the focus was on proving convergence and the existence was assumed. This issue however turns out to be non-trivial. Authors of [1] point this out and prove existence for the case of shortest path problem. They also give an example of an infeasible instance for which the system has a solution only for finite time. This demonstrates that in fact feasibility is crucial for the existence of a solution. For completeness we give here the example from [1]. We present a slightly modified version to emphasize that there is no simple workaround. We will focus on the continuous variant of the dynamics, it is a bit more convenient to work with. However, exactly the same principle shows up for the discrete version.

We consider the directed $s-t$ shortest path problem on the (multi)graph $G=(V,E)$ with $V=\{s,t\}$ and $E=\{e_1,e_2\}$. We choose e_1, e_2 to be two copies of the same edge (t,s). G is a multigraph which has no $s-t$ path. We set $c_{e_1}=c_{e_2}=1$. The demand vector is $b_s=-1$, $b_t=1$. For brevity we will write x_1, x_2 instead of x_{e_1}, x_{e_2}. Note that the electrical flow is not well defined for $(x_1,x_2)=(0,0)$ (for many reasons: the Laplacian is a zero-matrix, the energy of every flow is the same, so every flow is minimizing the energy,...). Hence, we cannot include this point in the domain $\Omega \subseteq \mathbb{R}^2$ of our dynamical system. The equations for the dynamical system are as follows:

$$\begin{aligned}\dot{x}_1 &= -\frac{x_1}{x_1+x_2}-x_1,\\ \dot{x}_2 &= -\frac{x_2}{x_1+x_2}-x_2.\end{aligned}$$

Consider the initial condition $x(0)=\begin{pmatrix}1\\1\end{pmatrix}$. One can see that the solution is given by the formula $x_1(t)=x_2(t)=\frac{3}{2}e^{-t}-\frac{1}{2}$, however it is defined only on the interval $[0,\ln 3)$. We see that

$$\lim_{t\to\ln 3} x(t) = \begin{pmatrix}0\\0\end{pmatrix}.$$

This point does not belong (and as discussed before, cannot belong) to the domain. Therefore the solution cannot be extended in any way. One can show that the initial condition $\begin{pmatrix}1\\1\end{pmatrix}$ is not special in any sense. For any positive initial condition, the system tends to $\begin{pmatrix}0\\0\end{pmatrix}$ in finite time.