FINAL JEE–MAIN EXAMINATION – JANUARY, 2023

(Held On Monday 30th January, 2023) **TIME : 9 : 00 AM to 12 : 00 NOON**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

61. Let $A=\begin{pmatrix} m & n \\ p & q \end{pmatrix}, d=|A|\neq 0 \quad |A-d(\mathrm{Adj}\,A)|=0$. Then

(1) $(1+d)^2=(m+q)^2$

(2) $1+d^2=(m+q)^2$

(3) $(1+d)^2=m^2+q^2$

(4) $1+d^2=m^2+q^2$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $A=\begin{bmatrix} m & n \\ p & q \end{bmatrix}, \quad |A-d(\mathrm{adj}\,A)|=0$

$\Rightarrow \quad |A-d(\mathrm{adj}\,A)|=\left|\begin{bmatrix} m & n \\ p & q \end{bmatrix}-d\begin{bmatrix} q & -n \\ -p & m \end{bmatrix}\right|$

$=\begin{vmatrix} m-qd & n(1+d) \\ p(1+d) & q-md \end{vmatrix}=0$

$\Rightarrow \quad (m-qd)(q-md)-np(1+d)^2=0$

$\Rightarrow \quad mq-m^2d-q^2d+mqd^2-np(1+d)^2=0$

$\Rightarrow \quad (mq-np)+d^2(mq-np)-d(m^2+q^2+2np)=0$

$\Rightarrow \quad d+d^3-d((m+q)^2-2d)=0$

$\Rightarrow \quad 1+d^2=(m+q)^2-2d$

$\Rightarrow \quad (1+d)^2=(m+q)^2$

$\therefore$ Option (1) is correct.

62. The line l_1 passes through the point (2,6,2) and is perpendicular to the plane $2x+y-2z=10$. Then the shortest distance between the line l_1 and the line $\frac{x+1}{2}=\frac{y+4}{-3}=\frac{z}{2}$ is :

(1) 7

(2) $\frac{19}{3}$

(3) $\frac{19}{3}$

(4) 9

Official Ans. by NTA (4)

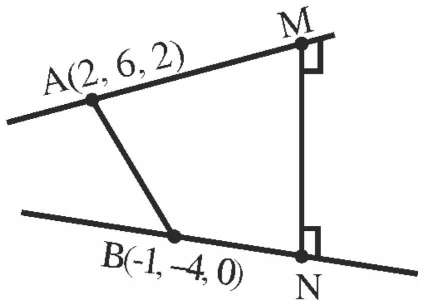
Allen Ans. (4)

Sol. Line ℓ, is given by

$L_1:\frac{x-2}{2}=\frac{y-6}{1}=\frac{z-2}{-2}$

Given,

$L_2:\frac{x+1}{2}=\frac{y+4}{-3}=\frac{z}{2}$



Shortest distance $=\left|\frac{\overrightarrow{AB}\cdot\overrightarrow{MN}}{MN}\right|$

$\overrightarrow{AB}=3\hat{i}+10\hat{j}+2\hat{k}$

$\overrightarrow{MN}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 2 & -3 & 2 \end{vmatrix}=-4\hat{i}-8\hat{j}-8\hat{k}$

$MN=\sqrt{16+64+64}=12$

$\therefore$ Shortest distance $=\left|\frac{-12-80-16}{12}\right|=9$

$\therefore$ Option (4) is correct.

63. If an unbiased die, marked with –2, –1, 0, 1, 2, 3 on its faces, is through five times, then the probability that the product of the outcomes is positive, is :

(1) $\frac{881}{2592}$

(2) $\frac{521}{2592}$

(3) $\frac{440}{2592}$

(4) $\frac{27}{288}$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. Either all outcomes are positive or any two are negative.

Now, p = P (positive) $= \frac{3}{6} = \frac{1}{2}$

q = p (negative) $= \frac{2}{6} = \frac{1}{3}$

Required probability

$$= {}^5C_5\left(\frac{1}{2}\right)^5 + {}^5C_2\left(\frac{1}{3}\right)^2\left(\frac{1}{2}\right)^3 + {}^5C_4\left(\frac{1}{3}\right)^4\left(\frac{1}{2}\right)^1$$

$$= \frac{521}{2592}$$

$\therefore$ Option (2) is correct.

64. Let the system of linear equations

$$x + y + kz = 2$$
$$2x + 3y - z = 1$$
$$3x + 4y + 2z = k$$

have infinitely many solutions. Then the system

$$(k+1)x + (2k-1)y = 7$$

$(2k+1)x + (k+5)y = 10$ has :

(1) infinitely many solutions

(2) unique solution satisfying x – y = 1

(3) no solution

(4) unique solution satisfying x + y = 1

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $\begin{vmatrix} 1 & 1 & k \\ 2 & 3 & -1 \\ 3 & 4 & 2 \end{vmatrix} = 0$

$\Rightarrow$ 1(10) – 1(7) + k(–1) – 0

$\Rightarrow$ k = 3

For k = 3, 2nd system is

4x + 5y = 7 .....(1)

and 7x + 8y = 10 .....(2)

Clearly, they have a unique solution

(2) – (1) $\Rightarrow$ 3x + 3y = 3

$\Rightarrow$ x + y = 1

65. If $\tan 15° + \frac{1}{\tan 75°} + \frac{1}{\tan 105°} + \tan 195° = 2a$, then the value of $\left(a + \frac{1}{a}\right)$ is :

(1) 4 (2) $4 - 2\sqrt{3}$

(3) 2 (4) $5 - \frac{3}{2}\sqrt{3}$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. Option (1)

$$\tan 15° = 2 - \sqrt{3}$$

$$\frac{1}{\tan 75°} = \cot 75° = 2 - \sqrt{3}$$

$$\frac{1}{\tan 105°} = \cot(105°) = -\cot 75° = \sqrt{3} - 2$$

$$\tan 195° = \tan 15° = 2 - \sqrt{3}$$

$\therefore$ $2(2-\sqrt{3}) = 2a \Rightarrow a = 2 - \sqrt{3}$

$\Rightarrow$ $a + \frac{1}{a} = 4$

66. Suppose $f: R \to (0, \infty)$ be a differentiable function such that $5f(x+y) = f(x)\cdot f(y), \forall x, y \in R$. If $f(3) = 320$, then $\sum_{n=0}^{5} f(n)$ is equal to :

(1) 6875 (2) 6575

(3) 6825 (4) 6528

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. Option (3)

$$5f(x+y) = f(x)\cdot f(y)$$

$$5f(0) = f(0)^2 \Rightarrow f(0) = 5$$

$$5f(x+1) = f(x)\cdot f(1)$$

$\Rightarrow$ $\frac{f(x+1)}{f(x)} = \frac{f(1)}{5}$

$\Rightarrow$ $\frac{f(1)}{f(0)}\cdot\frac{f(2)}{f(1)}\cdot\frac{f(3)}{f(2)} = \left(\frac{f(1)}{5}\right)^3$

$\Rightarrow$ $\frac{320}{5} = \frac{(f(1))^3}{5^3} \Rightarrow f(1) = 20$

$\therefore$ $5f(x+1) = 20\cdot f(x) \Rightarrow f(x+1) = 4f(x)$

$$\sum_{n=0}^{5} f(n) = 5 + 5.4 + 5.4^2 + 5.4^3 + 5.4^4 + 5.4^5$$

$$= \frac{5\left[4^6 - 1\right]}{3} = 6825$$

67. If $a_n = \frac{-2}{4n^2-16n+15}$, then $a_1 + a_2 + \ldots + a_{25}$ is equal to :

(1) $\frac{51}{144}$ (2) $\frac{49}{138}$

(3) $\frac{50}{141}$ (4) $\frac{52}{147}$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. Option (3)

If $a_n = \frac{-2}{4n^2-16n+15}$ then $a_1 + a_2 + \ldots\ldots a_{25}$

$$\Rightarrow \sum_{n=1}^{25} a_n = \sum \frac{-2}{4n^2-16n+15}$$

$$= \sum \frac{-2}{4n^2-6n-10n+15}$$

$$= \sum \frac{-2}{2n(2n-3)-5(2n-3)}$$

$$= \sum \frac{-2}{(2n-3)(2n-5)}$$

$$= \sum \frac{1}{2n-3} - \frac{1}{2n-5}$$

$$= \frac{1}{47} - \frac{1}{(-3)}$$

$$= \frac{50}{141}$$

68. If the coefficient of x^{15} in the expansion of $\left(ax^3 + \frac{1}{bx^{\frac{1}{3}}}\right)^{15}$ is equal to the coefficient of x^{-15} in the expansion of $\left(ax^{\frac{1}{3}} - \frac{1}{bx^3}\right)^{15}$, where a and b are positive real numbers, then for each such ordered pair (a, b) :

(1) a = b (2) ab = 1

(3) a = 3b (4) ab = 3

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. Option (2)

Coefficient Of x^{15} in $\left(ax^3 + \frac{1}{bx^{1/3}}\right)^{15}$

$$T_{r+1} = {}^{15}C_r \left(ax^3\right)^{15-r} \left(\frac{1}{bx^{1/3}}\right)^r$$

$$45 - 3r - \frac{r}{3} = 15$$

$$30 = \frac{10r}{3}$$

$$r = 9$$

Coefficient of $x^{15} = {}^{15}C_9\, a^6\, b^{-9}$

Coefficient of x^{-15} in $\left(ax^{1/3} - \frac{1}{bx^3}\right)^{15}$

$$T_{r+1} = {}^{15}C_r \left(ax^{1/3}\right)^{15-r} \left(-\frac{1}{bx^3}\right)^r$$

$$5 - \frac{r}{3} - 3r = -15$$

$$\frac{10r}{3} = 20$$

$$r = 6$$

Coefficient $= {}^{15}C_6\, a^9 \times b^{-6}$

$$\Rightarrow \frac{a^9}{b^6} = \frac{a^6}{b^9}$$

$$\Rightarrow a^3b^3 = 1 \Rightarrow ab = 1$$

69. If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero vectors and $\hat{n}$ is a unit vector perpendicular to $\vec{c}$ such that $\vec{a} = \alpha\vec{b} - \hat{n}, (\alpha \neq 0)$ and $\vec{b}.\vec{c} = 12$, then $\left|\vec{c} \times \left(\vec{a} \times \vec{b}\right)\right|$ is equal to :

(1) 15

(2) 9

(3) 12

(4) 6

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $\hat{n} \perp \vec{c}$ $\quad \vec{a} = \alpha\vec{b} - \vec{n}$

$$\vec{b}\cdot\vec{c} = 12$$

$$\vec{a}\cdot\vec{c} = \alpha(\vec{b}\cdot\vec{c}) - \vec{n}\cdot\vec{c}$$

$$\vec{a}\cdot\vec{c} = \alpha(\vec{b}\cdot\vec{c})$$

$$|\vec{c}\times(\vec{a}\times\vec{b})| = |(\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}|$$

$$= |(\vec{c}\cdot\vec{b})\vec{a} - \alpha(\vec{b}\cdot\vec{c})\vec{b}|$$

$$= |(\vec{c}\cdot\vec{b})|\,|\vec{a} - \alpha\vec{b}|$$

$$= 12\times(|\vec{n}|)$$

$$= 12\times 1$$

$$= 12$$

70. The number of points on the curve $y = 54x^5 - 135x^4 - 70x^3 + 180x^2 + 210x$ at which the normal lines are parallel to $x + 90y + 2 = 0$ is :

(1) 2 (2) 3

(3) 4 (4) 0

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. Normal of line is parallel to line $x + 90y + 2 = 0$

$$m_N = -\frac{1}{90}$$

$$-\left(\frac{dx}{dy}\right)_{(x_1y_1)} = -\frac{1}{90} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1y_1)} = 90$$

Now,

$$\frac{dy}{dx} = 270x^4 - 540x^3 - 210x^2 + 360x + 210 = 90$$

$$\Rightarrow x = 1, 2, \frac{-2}{3}, \frac{-1}{3}$$

(4) normals

71. Let $y = x + 2, 4y = 3x + 6$ and $3y = 4x + 1$ be three tangent lines to the circle $(x-h)^2 + (y-k)^2 = r^2$. Then $h + k$ is equal to :

(1) 5

(2) $5(1+\sqrt{2})$

(3) 6

(4) $5\sqrt{2}$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $L_1 : y = x + 2,\ L_2 : 4y = 3x + 6,\ L_3 : 3y = 4x + 1$

Bisector of lines L_2 & L_3

$$\frac{4x-3y+1}{5} = \pm\left(\frac{3x-4y+6}{5}\right)$$

(+) $4x - 3y + 1 = 3x - 4y + 6$

$x + y = 5$

Centre lies on Bisector of $4x - 3y + 1 = 0$ &

(0) $3x - 4y + 6 = 0$

$\Rightarrow h + k = 5$

72. Let the solution curve $y = y(x)$ of the differential equation $\frac{dy}{dx} - \frac{3x^5\tan^{-1}(x^3)}{(1+x^6)^{\frac{3}{2}}}y = 2x \exp\left\{\frac{x^3 - \tan^{-1}x^3}{\sqrt{(1+x)^6}}\right\}$ pass through the origin. Then $y(1)$ is equal to :

(1) $\exp\left(\frac{4-\pi}{4\sqrt{2}}\right)$ (2) $\exp\left(\frac{\pi-4}{4\sqrt{2}}\right)$

(3) $\exp\left(\frac{1-\pi}{4\sqrt{2}}\right)$ (4) $\exp\left(\frac{4+\pi}{4\sqrt{2}}\right)$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $$\frac{dy}{dx} + \left(\frac{-3x^5\tan^{-1}x^3}{(1+x^6)^{3/2}}\right)y = 2e^{\left\{\frac{x-\tan x}{\sqrt{1+x^6}}\right\}}$$

$$\text{I.F.} = e^{\int\frac{-3x^5\tan^{-1}x^3}{(1+x^6)^{3/2}}dx}$$

$$= e^{\frac{\tan^{-1}x^3 - x^3}{\sqrt{1+x^6}}}$$

Solution of differential equation

$$y\cdot e^{\frac{\tan^{-1}x^3 - x^3}{\sqrt{1+x^6}}} = \int 2xe^{\left(\frac{x^3-\tan^{-1}x^3}{\sqrt{1+x^6}}\right)}\cdot e^{\left(\frac{\tan^{-1}(x^3)-x^3}{\sqrt{1+x^6}}\right)}dx$$

$$= \int 2x\,dx + c$$

$$y\cdot e^{\frac{\tan^{-1}x^3 - x^3}{\sqrt{1+x^6}}} = x^2 + c$$

Also it passes through origin

$c = 0$

$$y(1)\cdot e^{\frac{\tan^{-1}(1)-1}{\sqrt{2}}} = 1$$

$$y(1)\cdot e^{\frac{\frac{\pi}{4}-1}{\sqrt{2}}} = 1$$

$$y(1)\cdot e^{\frac{\pi-4}{4\sqrt{2}}} = 1$$

$$y(1) = \frac{1}{e^{\frac{\pi-4}{4\sqrt{2}}}} = e^{\frac{4-\pi}{4\sqrt{2}}}$$

73. Let a unit vector OP make angle α, β, γ with the positive directions of the co-ordinate axes OX, OY, OZ respectively, where $\beta \in \left(0, \frac{\pi}{2}\right)$ OP is perpendicular to the plane through points (1, 2, 3), (2, 3, 4) and (1, 5, 7), then which one of the following is true ?

(1) $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ and $\gamma \in \left(\frac{\pi}{2}, \pi\right)$

(2) $\alpha \in \left(0, \frac{\pi}{2}\right)$ and $\gamma \in \left(0, \frac{\pi}{2}\right)$

(3) $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ and $\gamma \in \left(0, \frac{\pi}{2}\right)$

(4) $\alpha \in \left(0, \frac{\pi}{2}\right)$ and $\gamma \in \left(\frac{\pi}{2}, \pi\right)$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. Equation of plane :-

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 1 & 1 & 1 \\ 0 & 3 & 4 \end{vmatrix} = 0$$

$$\Rightarrow [x-1]-4[y-2]+3[z-3]=0$$

$$\Rightarrow x-4y+3z=2$$

D.R's of normal of plane <1, –4, 3>

D.C's of $\left\langle \pm\frac{1}{\sqrt{26}}, \mp\frac{4}{\sqrt{26}}, \pm\frac{3}{\sqrt{26}} \right\rangle$

$$\cos\beta = \frac{4}{\sqrt{26}}$$

$$\cos\alpha = \frac{-1}{\sqrt{26}} \qquad \frac{\pi}{2} < \alpha < \pi$$

$$\cos\gamma = \frac{-3}{\sqrt{26}} \qquad \frac{\pi}{2} < \gamma < \pi$$

Ans. : (1)

74. If [t denotes the greatest integer ≤ 1, then the value of $\frac{3(e-1)^2}{e}\int_1^2 x^2 e^{[x]+[x^3]}dx$ is :

(1) $e^9 - e$ (2) $e^8 - e$

(3) $e^7 - 1$ (4) $e^8 - 1$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $\int_1^2 x^2 e^{[x^3]+1} dx$

$x^3 = t$

$3x^2\,dx = dt$

$$= \frac{e}{3}\int_1^8 e^{[t]}dt$$

$$= \frac{e}{3}\left\{\int_1^2 e\,dt + \int_2^3 e^2\,dt + ... + \int_7^8 e^7\,dt\right\}$$

$$= \frac{e}{3}(e + e^2 + ... + e^7)$$

$$= \frac{e^2}{3}(1 + e + ... + e^6) = \frac{e^2}{3}\frac{(e^7-1)}{(e-1)}$$

$$\frac{3(e-1)}{e}\int_1^2 x^2 \times e^{[x]+[x^3]}dx = \frac{3}{e}(e-1)\times\frac{e^2}{3}\frac{(e^7-1)}{(e-1)}$$

$$= e(e^7-1)$$

$$= e^8 - e$$

Ans. : (2)

75. If P(h,k) be point on the parabola $x = 4y^2$, which is nearest to the point Q(0,33), then the distance of P from the directrix of the parabola $y^2 = 4(x+y)$ is equal to :

(1) 2 (2) 4

(3) 8 (4) 6

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. Equation of normal

$y = -tx + 2at + at^3$

$$y = -tx + \frac{2}{16}t + \frac{1}{16}t^3$$

It passes through (0, 33)

$$33 = \frac{t}{8} + \frac{t^3}{16}$$

$t^3 + 2t - 528 = 0$

$(t - 8)(t^2 + 8t + 66) = 0$

$t = 8$

$$P(at^2, 2at) = \left(\frac{1}{16}\times 64,\ 2\times\frac{1}{16}\times 8\right) = (4, 1)$$

Parabola :

$y^2 = 4(x + y)$

$$\Rightarrow y^2 - 4y = 4x$$

$$\Rightarrow (y-2)^2 = 4(x+1)$$

Equation of directrix :-

$x + 1 = -1$

$x = -2$

Distance of point = 6

Ans. : (4)

76. A straight line cuts off the intercepts $OA = a$ and $OB = b$ on the positive directions of x–axis and y–axis respectively. If the perpendicular from origin O to this line makes an angle of $\frac{\pi}{6}$ with positive direction of y–axis and the area of ΔOAB is $\frac{98}{3}\sqrt{3}$, then $a^2 - b^2$ is equal to:

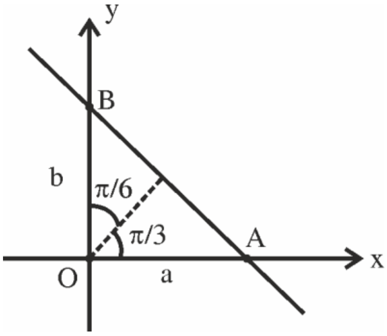
(1) $\frac{392}{3}$ (2) 196

(3) $\frac{196}{3}$ (4) 98

Official Ans. by NTA (1)

Allen Ans. (1)

Sol.



Equation of straight line : $\frac{x}{a} + \frac{y}{b} = 1$

Or $x\cos\frac{\pi}{3} + y\sin\frac{\pi}{3} = p$

$\frac{x}{2} + \frac{y\sqrt{3}}{2} = p$

$\frac{x}{3p} + \frac{y}{2p} = 1$

Comparing both : $a = 2p, b = \frac{2p}{\sqrt{3}}$

Now area of $\Delta OAB = \frac{1}{2}\cdot ab = \frac{98}{3}\cdot\sqrt{3}$

$\frac{1}{2}\cdot 2p\cdot\frac{2p}{\sqrt{3}} = \frac{98}{3}\cdot\sqrt{3}$

$p^2 = 49$

$a^2 - b^2 = 4p^2 - \frac{4p^2}{3} = \frac{2}{3}4p^2$

$= \frac{8}{3}\cdot 49 = \frac{392}{3}$

77. The coefficient of x^{301} in $(1+x)^{500} + x(1+x)^{499} + x^2(1+x)^{498} + ... + x^{500}$ is:

(1) ${}^{501}C_{302}$ (2) ${}^{500}C_{301}$

(3) ${}^{500}C_{300}$ (4) ${}^{501}C_{200}$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $(1+x)^{500} + x(1+x)^{499} + x^2(1+x)^{498} + ... + x^{500}$

$$= (1+x)^{500}\cdot\left\{\frac{1-\left(\frac{x}{1+x}\right)^{501}}{1-\frac{x}{1+x}}\right\}$$

$$= (1+x)^{500}\frac{\left((1+x)^{501} - x^{501}\right)}{(1+x)^{501}}\cdot(1+x)$$

$$= (1+x)^{501} - x^{501}$$

Coefficient of x^{301} in $(1+x)^{501} - x^{501}$ is given by ${}^{501}C_{301} = {}^{501}C_{200}$

78. Among the statements:

(S1) $((p\vee q)\Rightarrow r)\Leftrightarrow(p\Rightarrow r)$

(S2) $((p\vee q)\Rightarrow r)\Leftrightarrow((p\Rightarrow r)\vee(q\Rightarrow r))$

(1) Only (S1) is a tautology

(2) Neither (S1) nor (S2) is a tautology

(3) Only (S2) is a tautology

(4) Both (S1) and (S2) are tautologies

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $S_1 \equiv ((p\vee q)\Rightarrow r)\Leftrightarrow(p\Rightarrow r)$

p	q	r	$p\vee q$	$(p\vee q)\Rightarrow r$	$p\Rightarrow r$	$((p\vee q)\Rightarrow r)\Leftrightarrow(p\Rightarrow r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	T	T	T	T
F	T	T	T	T	T	T
T	F	F	T	F	F	T
F	T	F	T	F	T	T
F	F	F	T	T	T	T
F	F	F	F	T	T	T

$S_2 \equiv (p \vee q) \Rightarrow r \Leftrightarrow ((p \Rightarrow r) \vee (q \Rightarrow r))$

p	q	r	$(p\vee q)\Rightarrow r$	$p\Rightarrow r$	$q\Rightarrow r$	$(p\Rightarrow r\vee(q\Rightarrow r))$	S2
T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	T
T	F	T	T	T	T	T	T
F	T	T	T	T	T	T	T
T	F	F	F	T	T	T	F
F	T	F	F	F	F	T	F
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

S2 $\rightarrow$ not a tautology

79. The minimum number of elements that must be added to the relation $R=\{(a,b),(b,c)\}$ on the set $\{a,b,c\}$ so that it becomes symmetric and transitive is:

(1) 4 (2) 7

(3) 5 (4) 3

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. *For Symmetric* $(a,b),(b,c)\in R$

$\Rightarrow (b,a),(c,b)\in R$

For Transitive $(a,b),(b,c)\in R$

$\Rightarrow (a,c)\in R$

Now

1. *Symmetric*

$\therefore (a,c)\in R \Rightarrow (c,a)\in R$

2. *Transitive*

$\therefore (a,b),(b,a)\in R$

$\Rightarrow (a,a)\in R$ & $(b,c),(c,b)\in R$

$\Rightarrow (b,b)$ & $(c,c)\in R$

$\therefore$ *Elements to be added*

$$\left\{\begin{matrix}(b,a),(c,b),(a,c),(c,a) \\ ,(a,a),(b,b),(c,c)\end{matrix}\right\}$$

Number of elements to be added $=7$

80. If the solution of the equation $\log_{\cos x}\cot x + 4\log_{\sin x}\tan x = 1, x\in\left(0,\frac{\pi}{2}\right)$, is $\sin^{-1}\left(\frac{\alpha+\sqrt{\beta}}{2}\right)$, where α, β are integers, then $\alpha+\beta$ is equal to:

(1) 3

(2) 5

(3) 6

(4) 4

Official Ans. by NTA (4)

Allen Ans. (4)

Sol.

$\log_{\cos x}\cot x + 4\log_{\sin x}\tan x = 1$

$\Rightarrow \dfrac{\ln\cos x - \ln\sin x}{\ln\cos x} + 4\dfrac{\ln\sin x - \ln\cos x}{\ln\sin x} = 1$

$\Rightarrow (\ln\sin x)^2 - 4(\ln\sin x)(\ln\cos x) + 4(\ln\cos x)^2 = 1$

$\Rightarrow \ln\sin x = 2\ln\cos x$

$\Rightarrow \sin^2 x + \sin x - 1 = 0 \Rightarrow \sin x = \dfrac{-1+\sqrt{5}}{2}$

$\therefore \alpha+\beta = 4$

Correct option (4)

SECTION-B

81. Let $S=\{1,2,3,4,5,6\}$. Then the number of one–one functions $f:S\rightarrow P(S)$, where $P(S)$ denote the power set of S, such that $f(n)\subset f(m)$ where $n<m$ is ______.

Official Ans. by NTA (3240)

Allen Ans. (3240)

Sol. Let S={1,2,3,4,5,6}, then the number of one-one functions, f:S⟶P(S), where P(S) denotes the power set of S, such that f(n)<f(m) where n<m is

$n(S)=6$

$$P(S)=\left\{\begin{matrix}\phi,\{1\},\ldots\{6\},\{1,2\},\ldots, \\ \{5,6\},\ldots,\{1,2,3,4,5,6\}\end{matrix}\right\}$$

– 64 *elements*

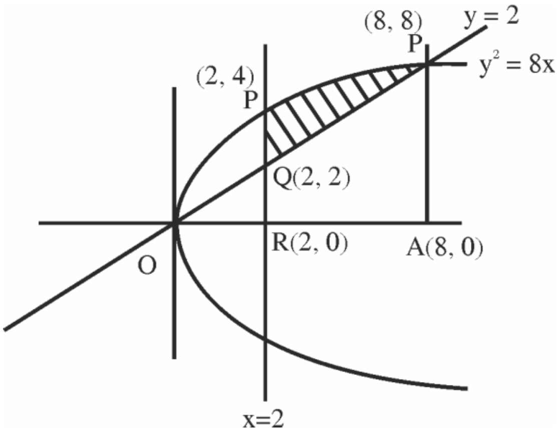
case – 1
f (6) = S i.e. 1 option,
f (5) = any 5 element subset A of S i.e. 6 options,
f (4) = any 4 element subset B of A i.e. 5 options,
f (3) = any 3 element subset C of B i.e. 4 options,
f (2) = any 2 element subset D of C i.e. 3 options,
f (1) = any 1 element subset E of D or empty subset i.e. 3 options,
Total functions = 1080
Case – 2
f (6) = any 5 element subset A of S i.e. 6 options,
f (5) = any 4 element subset B of A i.e. 5 options,
f' (4) = any 3 element subset C of B i.e. 4 options,
f (3) = any 2 element subset D of C i.e. 3 options,
f' (2) = any 1 element subset E of D i.e. 2 options,
f (1) = empty subset i.e. 1 option
Total functions = 720
Case – 3
f(6)= S
f(5) = any 4 element subset A of' S i.e. 15 options,
f(4) = any 3 element subset B of A i.e. 4 options,
f(3) = any 2 element subset C of B i.e. 3 options,
f(2) = any 1 element subset D of C i.e. 2 options,
f(1) = empty subset i.e. 1 option
Total functions = 360
Case – 4
f(6) = S
f(5) = any 5 element subset A of S i.e. 6 options,
f(4) = any 3 element subset B of A i.e. 10 options,
f(3) = any 2 element subset C of B i.e. 3 options,
f(2) = any 1 element subset D of C i.e. 2 options,
f(1) = empty subset i.e. 1 option
Total functions = 360
Case – 5
f(6) = S
f(5) = any 5 element subset A of S i.e. 6 options,
f(4) = any 4 element subset B of A i.e. 5 options,
f(3) = any 2 element subset C of B i.e. 6 options,
f(2) = any 1 element subset D of C i.e. 2 options,
f(1) = empty subset i.e. 1 option
Total functions = 360
Case – 6
f(6) = S
f(5) = any 5 element subset A of S i.e. 6 options,
f(4) = any 4 element subset B of A i.e. 5 options,
f(3) = any 3 element subset C of B i.e. 4 options,
f(2) = any 1 element subset D of C i.e. 3 options,
f(1) = empty subset i.e. 1 option
Total functions = 360
$\therefore$ Number of such functions = 3240

82. Let α be the area of the larger region bounded by the curve $y^2 = 8x$ and the lines $y = x$ and $x = 2$, which lies in the first quadrant. Then the value of 3α is equal to __________.

Official Ans. by NTA (22)

Allen Ans. (22)

Sol.



$y = x$

& $y^2 = 8x$

Solving it

$x^2 = 8x$

$\therefore$ $x = 0, 8$

$\therefore$ $y = 0, 8$

x = 2 will intersect occur at

$y^2 = 16 \Rightarrow \quad y = \pm 4$

$\therefore$ Area of shaded

$$= \int_2^8 \left(\sqrt{8x} - x\right)dx = \int_2^8 \left(2\sqrt{2}\sqrt{x} - x\right)dx$$

$$= \left[2\sqrt{2}\cdot\frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^8$$

$$= \left(\frac{4\sqrt{2}}{3}\cdot 2^{9/2} - 32\right) - \left(\frac{4\sqrt{2}}{3}\cdot 2^{9/2} - 2\right)$$

$$= \frac{128}{3} - 32 - \frac{16}{3} + 2 = \frac{112 - 90}{3} = \frac{22}{3} = A$$

$\therefore 3A = 22$

83. If $\lambda_1 < \lambda_2$ are two values of λ such that the angle between the planes $P_1 : \vec{r}\left(3\hat{i} - 5\hat{j} + k\right) = 7$ and $P_2 : \vec{r}\cdot\left(\lambda\hat{i} + \hat{j} - 3k\right) = 9$ is $\sin^{-1}\left(\frac{2\sqrt{6}}{5}\right)$, then the square of the length of perpendicular from the point $(38\lambda_1, 10\lambda_2, 2)$ to the plane P_1 is _____.

Official Ans. by NTA (315)

Allen Ans. (315)

Sol. $P_1 = \vec{r}\cdot\left(3\hat{i} - 5\hat{j} + k\right) = 7$

$P_2 = \vec{r}\cdot\left(\lambda\hat{i} + \hat{j} - 3k\right) = 9$

$\theta = \sin^{-1}\left(\frac{2\sqrt{6}}{5}\right)$

$\Rightarrow \sin\theta = \frac{2\sqrt{6}}{5}$

$\therefore \cos\theta = \frac{1}{5}.$

$\cos\theta = \frac{\vec{r}\cdot\vec{r}}{|\vec{r}_1||\vec{r}_2|}$

$= \frac{(3i - 5j + K)(\lambda i + j - 3K)}{\sqrt{35}\cdot\sqrt{\lambda^2 + 10}}$

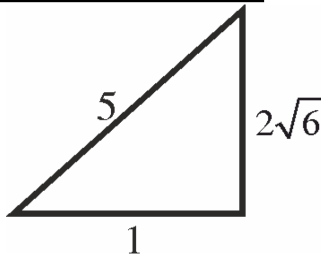
$\frac{1}{5} = \left|\frac{3\lambda - 8}{\sqrt{35}\cdot\sqrt{\lambda^2 + 10}}\right|$

Square $\Rightarrow \frac{1}{25} = \frac{9\lambda^2 + 64 - 48\lambda}{35\left(\lambda^2 + 10\right)}$

$\Rightarrow 19\lambda^2 - 120\lambda + 125 = 0$

$\Rightarrow 19\lambda^2 - 95\lambda - 25\lambda + 125 = 0$

$\Rightarrow x = 5, \frac{25}{19}$



Perpendicular distance of point

$(38\lambda_1, 10\lambda_2, 2) \equiv (50, 50, 2)$ from plane P_1

$= \frac{|3\times 50 - 5\times 50 + 2 - 7|}{\sqrt{35}} = \frac{105}{\sqrt{35}}$

Square $= \frac{105\times 105}{35} = 315$

84. Let $z = 1 + i$ and $z_1 = \frac{1 + i\bar{z}}{\bar{z}(1 - z) + \frac{1}{z}}$. Then $\frac{12}{\pi}$ $\arg(z_1)$ is equal to _______ .

Official Ans. by NTA (9)

Allen Ans. (9)

Sol. $z = 1 + i$

$z_1 = \frac{1 + i\bar{z}}{\bar{z}(1 - z) + \frac{1}{z}}$

$z_1 = \frac{1 + i(1 - i)}{(1 - i)(1 - 1 - i) + \frac{1}{1 + i}}$

$= \frac{1 + i - i^2}{(1 - i)(-i) + \frac{1 - i}{2}}$

$= \frac{2 + i}{\frac{-3i - 1}{2}} = \frac{4 + 2i}{-3i - 1}$

$= \frac{-(4 + 2i)(3i - 1)}{(3i)^2 - (1)^2}$

$\text{Arg}(z_1) = \frac{3\pi}{4}$

$\therefore \frac{12}{\pi}\arg(z_1) = \frac{12}{\pi}\times\frac{3\pi}{4} = 9$

85. $\lim\limits_{x\to 0}\frac{48}{x^4}\int_0^x \frac{t^3}{t^6+1}dt$ is equal to ______.

Official Ans. by NTA (12)

Allen Ans. (12)

Sol. $48\lim\limits_{x\to 0}\frac{\int_0^x \frac{t^3}{t^6+1}dt}{x^4}\ \left(\frac{0}{0}\right)$

Applying L' Hospitals Rule

$$48\lim_{x\to 0}\frac{x^3}{x^6+1}\times\frac{1}{4x^3}$$

$= 12$

86. The mean and variance of 7 observations are 8 and 16 respectively. If one observation 14 is omitted a and b are respectively mean and variance of remaining 6 observation, then $a+3b-5$ is equal to ______.

Official Ans. by NTA (37)

Allen Ans. (37)

Sol. $\frac{x_1+x_2+....+x_7}{7}=8$

$$\frac{x_1+x_2+x_3....+x_6+14}{7}=8$$

$$\Rightarrow x_1+x_2+....+x_6=42$$

$$\therefore \frac{x_1+x_2....+x_6}{6}=\frac{42}{6}=7=a$$

$$\frac{\Sigma x_i^2}{7}-8^2=16$$

$$\Sigma xi^2=560$$

$$\Rightarrow x_1^2+x_2^2+...+x_6^2=364$$

$$b=\frac{x_1^2+x_2^2+....+x_6^2}{6}-7^2$$

$$=\frac{364}{6}-49$$

$$b=\frac{70}{6}$$

$$a+3b-5=7+3\times\frac{70}{6}-5$$

$= 37$

87. If the equation of the plane passing through the point $(1,1,2)$ and perpendicular to the line $x-3y+2z-1=0$ $4x-y+z$ is $Ax+By+Cz=1$, then $140(C-B+A)$ is equal to _____.

Official Ans. by NTA (15)

Allen Ans. (15)

Sol. $x-3y+2z-1=0$

$$4x-y+z=0$$

$$\vec{n}_1\times\vec{n}_2=\begin{vmatrix}\hat{i} & \hat{j} & k\\ 1 & -3 & 2\\ 4 & -1 & 1\end{vmatrix}$$

$$=-\hat{i}+7\hat{j}+11k$$

Dr^s of normal to the plane is –1, 7, 11

Equation of plane :

$$-1(x-1)+7(y-1)+11(z-2)=0$$

$$-x+7y+11z=28$$

$$\frac{-1}{28}x+\frac{7y}{28}+\frac{11z}{28}=1$$

$$Ax+By+Cz=1$$

$$140(C-B+A)=140\left(\frac{11}{28}-\frac{7}{28}-\frac{1}{28}\right)$$

$$=140\times\frac{3}{28}=15$$

88. Let $\sum\limits_{n=0}^{\infty}\frac{n^3((2n)!)+(2n-1)(n!)}{(n!)((2n)!)}=ae+\frac{b}{e}+c$, where $a,b,c\in\mathbb{Z}$ and $e=\sum\limits_{n=0}^{\infty}\frac{1}{n!}$ Then a^2-b+c is equal to ________.

Official Ans. by NTA (26)

Allen Ans. (26)

Sol. $\sum_{n=0}^{\infty}\frac{n^3((2n)!)+(2n-1)(n!)}{(n!)((2n)!)}$

$$=\sum_{n=0}^{\infty}\frac{1}{(n-3)!}+\sum_{n=0}^{\infty}\frac{3}{(n-2)!}$$

$$+\sum_{n=0}^{\infty}\frac{1}{(n-1)!}+\sum_{n=0}^{\infty}\frac{1}{(2n-1)!}-\sum_{n=0}^{\infty}\frac{1}{(2n)!}$$

$$=e+3e+e+\frac{1}{2}\left(e-\frac{1}{e}\right)-\frac{1}{2}\left(e+\frac{1}{e}\right)$$

$$=5e-\frac{1}{e}$$

$\therefore a^2-b+c=26$

89. Number of 4–digit numbers (the repetition of digits is allowed) which are made using the digits 1, 2, 3 and 5, and are divisible by 15, is equal to ______

Official Ans. by NTA (21)

Allen Ans. (21)

Sol. For number to be divisible by 15, last digit should be 5 and sum of digits must be divisible by 3.

Possible combinations are

1	2	1	5

Numbers = 3

2	2	3	5

Numbers = 3

3	3	1	5

Numbers = 3

1	1	5	5

Numbers = 3

2	3	5	5

Numbers = 6

3	5	5	5

Numbers = 3

Total Numbers = 21

90. Let $f^1(x)=\frac{3x+2}{2x+3}, x\in R-\left\{\frac{-3}{2}\right\}$

For $n\geq 2$, define $f^n(x)=f^1 0 f^{n-1}(x)$.

If $f^5(x)=\frac{ax+b}{bx+a}, \gcd(a,b)=1$, then $a+b$ is equal to ________.

Official Ans. by NTA (3125)

Allen Ans. (3125)

Sol. $f^1(x)=\frac{3x+2}{2x+3}$

$$\Rightarrow f^2(x)=\frac{13x+12}{12x+13}$$

$$\Rightarrow f^3(x)=\frac{63x+62}{62x+63}$$

$$\therefore f^5(x)=\frac{1563x+1562}{1562x+1563}$$

$a+b=3125$