

FINAL JEE-MAIN EXAMINATION – JANUARY, 2023

(Held On Sunday 29th January, 2023)

TIME : 9 : 00 AM to 12 : 00 NOON

MATHEMATICS

TEST PAPER WITH SOLUTION

SECTION-A

61. The domain of $f(x) = \frac{\log_{(x+1)}(x-2)}{e^{2\log_e x} - (2x+3)}, x \in \mathbb{R}$ is

- (1) $\mathbb{R} - \{1-3\}$ (2) $(2, \infty) - \{3\}$
(3) $(-1, \infty) - \{3\}$ (4) $\mathbb{R} - \{3\}$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $x - 2 > 0 \Rightarrow x > 2$

$$x + 1 > 0 \Rightarrow x > -1$$

$$x + 1 \neq 1 \Rightarrow x \neq 0 \text{ and } x > 0$$

Denominator

$$x^2 - 2x - 3 \neq 0$$

$$(x - 3)(x + 1) \neq 0$$

$$x \neq -1, 3$$

So Ans $(2, \infty) - \{3\}$

62. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that

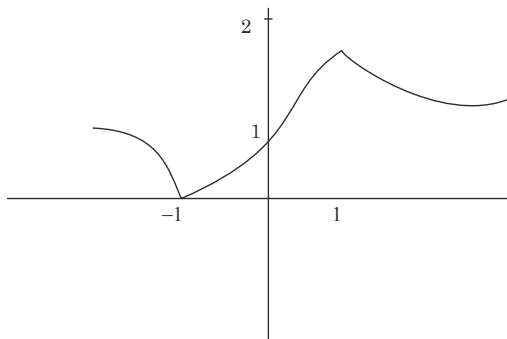
$$f(x) = \frac{x^2 + 2x + 1}{x^2 + 1}. \text{ Then}$$

- (1) $f(x)$ is many-one in $(-\infty, -1)$
(2) $f(x)$ is many-one in $(1, \infty)$
(3) $f(x)$ is one-one in $[1, \infty)$ but not in $(-\infty, \infty)$
(4) $f(x)$ is one-one in $(-\infty, \infty)$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol.



$$f(x) = \frac{(x+1)^2}{x^2 + 1} = 1 + \frac{2x}{x^2 + 1}$$

$$f(x) = 1 + \frac{2}{x + \frac{1}{x}}$$

63. For two non-zero complex number z_1 and z_2 , if $\operatorname{Re}(z_1 z_2) = 0$ and $\operatorname{Re}(z_1 + z_2) = 0$, then which of the following are possible ?

- (A) $\operatorname{Im}(z_1) > 0$ and $\operatorname{Im}(z_2) > 0$
(B) $\operatorname{Im}(z_1) < 0$ and $\operatorname{Im}(z_2) > 0$
(C) $\operatorname{Im}(z_1) > 0$ and $\operatorname{Im}(z_2) < 0$
(D) $\operatorname{Im}(z_1) < 0$ and $\operatorname{Im}(z_2) < 0$

Choose the correct answer from the options given below :

- (1) B and D (2) B and C
(3) A and B (4) A and C

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $z_1 = x_1 + iy_1$

$$z_2 = x_2 + iy_2$$

$$\operatorname{Re}(z_1 z_2) = x_1 x_2 - y_1 y_2 = 0$$

$$\operatorname{Re}(z_1 + z_2) = x_1 + x_2 = 0$$

x_1 & x_2 are of opposite sign

y_1 & y_2 are of opposite sign

64. Let $\lambda \neq 0$ be a real number. Let α, β be the roots of the equation $14x^2 - 31x + 3\lambda = 0$ and α, γ be the roots of the equation $35x^2 - 53x + 4\lambda = 0$. Then

$\frac{3\alpha}{\beta}$ and $\frac{4\alpha}{\gamma}$ are the roots of the equation :

- (1) $7x^2 + 245x - 250 = 0$
(2) $7x^2 - 245x + 250 = 0$
(3) $49x^2 - 245x + 250 = 0$
(4) $49x^2 + 245x + 250 = 0$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $14x^2 - 31x + 3\lambda = 0$

$$\alpha + \beta = \frac{31}{14} \dots(1) \text{ and } \alpha\beta = \frac{3\lambda}{14} \dots(2)$$

$$35x^2 - 53x + 4\lambda = 0$$

$$\alpha + \gamma = \frac{53}{35} \dots(3) \text{ and } \alpha\gamma = \frac{4\lambda}{35} \dots(4)$$

$$\frac{(2)}{(4)} \Rightarrow \frac{\beta}{\gamma} = \frac{3 \times 35}{4 \times 14} = \frac{15}{8} \Rightarrow \beta = \frac{15}{8}\gamma$$

$$(1) - (3) \Rightarrow \beta - \gamma = \frac{31}{14} - \frac{53}{35} = \frac{155 - 106}{70} = \frac{7}{10}$$

$$\frac{15}{8}\gamma - \gamma = \frac{7}{10} \Rightarrow \gamma = \frac{4}{5}$$

$$\Rightarrow \beta = \frac{15}{8} \times \frac{4}{5} = \frac{3}{2}$$

$$\Rightarrow \alpha = \frac{31}{14} - \beta = \frac{31}{14} - \frac{3}{2} = \frac{5}{7}$$

$$\Rightarrow \lambda = \frac{14}{3} \alpha\beta = \frac{14}{3} \times \frac{5}{7} \times \frac{3}{2} = 5$$

so, sum of roots $\frac{3\alpha}{\beta} + \frac{4\alpha}{\gamma} = \left(\frac{3\alpha\gamma + 4\alpha\beta}{\beta\gamma} \right)$

$$= \frac{\left(3 \times \frac{4\lambda}{35} + 4 \times \frac{3\lambda}{14} \right)}{\beta\gamma} = \frac{12\lambda(14 + 35)}{14 \times 35\beta\gamma}$$

$$= \frac{49 \times 12 \times 5}{490 \times \frac{3}{2} \times \frac{4}{5}} = 5$$

Product of roots

$$= \frac{3\alpha}{\beta} \times \frac{4\alpha}{\gamma} = \frac{12\alpha^2}{\beta\gamma} = \frac{12 \times \frac{25}{49}}{\frac{3}{2} \times \frac{4}{5}} = \frac{250}{49}$$

So, required equation is $x^2 - 5x + \frac{250}{49} = 0$

$$\Rightarrow 49x^2 - 245x + 250 = 0$$

65. Consider the following system of questions

$$\alpha x + 2y + z = 1$$

$$2\alpha x + 3y + z = 1$$

$$3x + \alpha y + 2z = \beta$$

For some $\alpha, \beta \in \mathbb{R}$. Then which of the following is NOT correct.

(1) It has no solution if $\alpha = -1$ and $\beta \neq 2$

(2) It has no solution for $\alpha = -1$ and for all $\beta \in \mathbb{R}$

(3) It has no solution for $\alpha = 3$ and for all $\beta \neq 2$

(4) It has a solution for all $\alpha \neq -1$ and $\beta = 2$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $D = \begin{vmatrix} \alpha & 2 & 1 \\ 2\alpha & 3 & 1 \\ 3 & \alpha & 2 \end{vmatrix} = 0 \Rightarrow \alpha = -1, 3$

$$D_x = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 1 & 1 \\ \alpha & 2 & \beta \end{vmatrix} = 0 \Rightarrow \beta = 2$$

$$D_y = \begin{vmatrix} \alpha & 1 & 1 \\ 2\alpha & 1 & 1 \\ 3 & 2 & \beta \end{vmatrix} = 0$$

$$D_z = \begin{vmatrix} \alpha & 2 & 1 \\ 2\alpha & 3 & 1 \\ 3 & \alpha & \beta \end{vmatrix} = 0$$

$$\beta = 2, \alpha = -1$$

$$\alpha = -1, \beta = 2 \text{ Infinite solution}$$

66. Let α and β be real numbers. Consider a 3×3 matrix A such that $A^2 = 3A + \alpha I$. If $A^4 = 21A + \beta I$, then

(1) $\alpha = 1$ (2) $\alpha = 4$

(3) $\beta = 8$ (4) $\beta = -8$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $A^2 = 3A + \alpha I$

$$A^3 = 3A^2 + \alpha A$$

$$A^3 = 3(3A + \alpha I) + \alpha A$$

$$A^3 = 9A + \alpha A + 3\alpha I$$

$$A^4 = (9 + \alpha)A^2 + 3\alpha A$$

$$= (9 + \alpha)(3A + \alpha I) + 3\alpha A$$

$$= A(27 + 6\alpha) + \alpha(9 + \alpha)$$

$$\Rightarrow 27 + 6\alpha = 21 \Rightarrow \alpha = -1$$

$$\Rightarrow \beta = \alpha(9 + \alpha) = -8$$

67. Let $x = 2$ be a root of the equation $x^2 + px + q = 0$

$$\text{and } f(x) = \begin{cases} \frac{1 - \cos(x^2 - 4px + q^2 + 8q + 16)}{(x - 2p)^4}, & x \neq 2p \\ 0, & x = 2p \end{cases}$$

Then $\lim_{x \rightarrow 2p^+} [f(x)]$

where $[\cdot]$ denotes greatest integer function, is

(1) 2

(2) 1

(3) 0

(4) -1

Official Ans. by NTA (3)

Allen Ans. (3)

Sol.

$$\lim_{x \rightarrow 2p^+} \left(\frac{1 - \cos(x^2 - 4px + q^2 + 8q + 16)}{(x^2 - 4px + q^2 + 8q + 16)^2} \right) \left(\frac{(x^2 - 4px + q^2 + 8q + 16)^2}{(x - 2p)^2} \right)$$

$$\lim_{h \rightarrow 0} \frac{1}{2} \left(\frac{(2p + h)^2 - 4p(2p + h) + q^2 + 8q + 16}{h^2} \right)^2 = \frac{1}{2}$$

Using L'Hospital's

$$\lim_{x \rightarrow 2p^+} [f(x)] = 0$$

68. Let $f(x) = x + \frac{a}{\pi^2 - 4} \sin x + \frac{b}{\pi^2 - 4} \cos x$,

$x \in \mathbb{R}$ be a function which satisfies

$$f(x) = x + \int_0^{\pi/2} \sin(x + y) f(y) dy. \text{ Then } (a + b)$$

is equal to

(1) $-\pi(\pi + 2)$

(2) $-2\pi(\pi + 2)$

(3) $-2\pi(\pi - 2)$

(4) $-\pi(\pi - 2)$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $f(x) = x + \int_0^{\pi/2} (\sin x \cos y + \cos x \sin y) f(y) dy$

$$f(x) = x + \int_0^{\pi/2} ((\cos y f(y) dy) \sin x + (\sin y f(y) dy) \cos x) \dots (1)$$

On comparing with

$$f(x) = x + \frac{a}{\pi^2 - 4} \sin x + \frac{b}{\pi^2 - 4} \cos x, \quad x \in \mathbb{R} \text{ then}$$

$$\Rightarrow \frac{a}{\pi^2 - 4} = \int_0^{\pi/2} \cos y f(y) dy \dots (2)$$

$$\Rightarrow \frac{b}{\pi^2 - 4} = \int_0^{\pi/2} \sin y f(y) dy \dots (3)$$

Add (2) and (3)

$$\frac{a + b}{\pi^2 - 4} = \int_0^{\pi/2} (\sin y + \cos y) f(y) dy \dots (4)$$

$$\frac{a + b}{\pi^2 - 4} = \int_0^{\pi/2} (\sin y + \cos y) f\left(\frac{\pi}{2} - y\right) dy \dots (5)$$

Add (4) and (5)

$$\frac{2(a + b)}{\pi^2 - 4} = \int_0^{\pi/2} (\sin y + \cos y) \left(\frac{\pi}{2} + \frac{(a + b)}{\pi^2 - 4} (\sin y + \cos y) \right) dy$$

$$= \pi + \frac{a + b}{\pi^2 - 4} \left(\frac{\pi}{2} + 1 \right)$$

$$(a + b) = -2\pi(\pi + 2)$$

69. Let $A = \{(x, y) \in \mathbb{R}^2 : y \geq 0, 2x \leq y \leq \sqrt{4 - (x - 1)^2}\}$

and $B = \{(x, y) \in \mathbb{R} \times \mathbb{R} : 0 \leq y \leq \min\{2x, \sqrt{4 - (x - 1)^2}\}\}$

Then the ratio of the area of A to the area of B is

(1) $\frac{\pi - 1}{\pi + 1}$

(2) $\frac{\pi}{\pi - 1}$

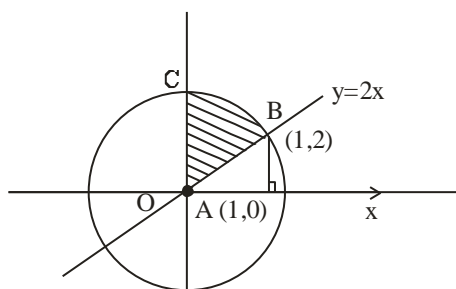
(3) $\frac{\pi}{\pi + 1}$

(4) $\frac{\pi + 1}{\pi - 1}$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $y^2 + (x-1)^2 = 4$

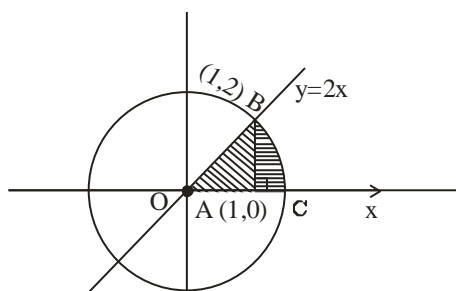


shaded portion = circular (OABC)

$$- \text{Ar}(\Delta OAB)$$

$$= \frac{\pi(4)}{4} - \frac{1}{2}(2)(1)$$

$$A = (\pi - 1)$$



Area B = Ar (ΔAOB) + Area of arc of circle (ABC)

$$= \frac{1}{2}(1)(2) + \frac{\pi(2)^2}{4} = \pi + 1$$

$$\frac{A}{B} = \frac{\pi - 1}{\pi + 1}$$

70. Let Δ be the area of the region

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 21, y^2 \leq 4x, x \geq 1\}.$$

Then

$$\frac{1}{2} \left(\Delta - 21 \sin^{-1} \frac{2}{\sqrt{7}} \right) \text{ is equal to}$$

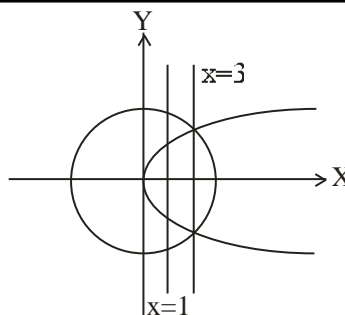
$$(1) 2\sqrt{3} - \frac{1}{3} \quad (2) \sqrt{3} - \frac{2}{3}$$

$$(3) 2\sqrt{3} - \frac{2}{3} \quad (4) \sqrt{3} - \frac{4}{3}$$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol.



$$\text{Area} = 2 \int_1^3 2\sqrt{x} dx + 2 \int_3^{\sqrt{21}} \sqrt{21-x^2} dx$$

$$\Delta = \frac{8}{3} (3\sqrt{3} - 1) + 21 \sin^{-1} \left(\frac{2}{\sqrt{7}} \right) - 6\sqrt{3}$$

$$\frac{1}{2} \left(\Delta - 21 \sin^{-1} \left(\frac{2}{\sqrt{7}} \right) \right) = \frac{2\sqrt{3} - \frac{8}{3}}{2}$$

$$= \sqrt{3} - \frac{4}{3}$$

71. A light ray emits from the origin making an angle 30° with the positive x-axis. After getting reflected by the line $x + y = 1$, if this ray intersects x-axis at Q, then the abscissa of Q is

$$(1) \frac{2}{(\sqrt{3}-1)} \quad (2) \frac{2}{3+\sqrt{3}}$$

$$(3) \frac{2}{3-\sqrt{3}} \quad (4) \frac{\sqrt{3}}{2(\sqrt{3}+1)}$$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. Slope of reflected ray = $\tan 60^\circ = \sqrt{3}$

$$\text{Line } y = \frac{x}{\sqrt{3}} \text{ intersect } y + x = 1 \text{ at } \left(\frac{\sqrt{3}}{\sqrt{3}+1}, \frac{1}{\sqrt{3}+1} \right)$$

Equation of reflected ray is

$$y - \frac{1}{\sqrt{3}+1} = \sqrt{3} \left(x - \frac{\sqrt{3}}{\sqrt{3}+1} \right)$$

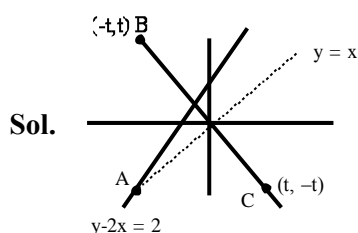
$$\text{Put } y = 0 \Rightarrow x = \frac{2}{3+\sqrt{3}}$$

72. Let B and C be the two points on the line $y + x = 0$ such that B and C are symmetric with respect to the origin. Suppose A is a point on $y - 2x = 2$ such that $\triangle ABC$ is an equilateral triangle. Then, the area of the $\triangle ABC$ is

- (1) $3\sqrt{3}$ (2) $2\sqrt{3}$
(3) $\frac{8}{\sqrt{3}}$ (4) $\frac{10}{\sqrt{3}}$

Official Ans. by NTA (3)

Allen Ans. (3)



At A $x = y$

$$y - 2x = 2$$

$$(-2, -2)$$

Height from line $x + y = 0$

$$h = \frac{4}{\sqrt{2}}$$

$$\text{Area of } \Delta = \frac{\sqrt{3}}{4} \frac{h^2}{\sin^2 60} = \frac{8}{\sqrt{3}}$$

73. Let the tangents at the points A (4, -11) and B(8, -5) on the circle $x^2 + y^2 - 3x + 10y - 15 = 0$, intersect at the point C. Then the radius of the circle, whose centre is C and the line joining A and B is its tangent, is equal to

- (1) $\frac{3\sqrt{3}}{4}$ (2) $2\sqrt{13}$
(3) $\sqrt{13}$ (4) $\frac{2\sqrt{13}}{3}$

Official Ans. by NTA (4)

Allen Ans. (4)

- Sol. Equation of tangent at A (4, -11) on circle is

$$\Rightarrow 4x - 11y - 3 \left(\frac{x+4}{2} \right) + 10 \left(\frac{y-11}{2} \right) - 15 = 0$$

$$\Rightarrow 5x - 12y - 152 = 0 \dots (1)$$

Equation of tangent at B (8, -5) on circle is

$$\Rightarrow 8x - 5y - 3 \left(\frac{x+8}{2} \right) + 10 \left(\frac{y-5}{2} \right) - 15 = 0$$

$$\Rightarrow 13x - 104 = 0 \Rightarrow x = 8$$

$$\text{put in (1)} \Rightarrow y = \frac{28}{3}$$

$$r = \left| \frac{3.8 + \frac{2.28}{3} - 34}{\sqrt{13}} \right| = \frac{2\sqrt{13}}{3}$$

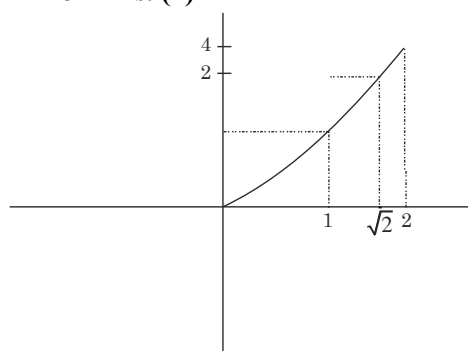
74. Let $[x]$ denote the greatest integer $\leq x$. Consider the function $f(x) = \max \{x^2, 1+[x]\}$. Then the value of the integral $\int_0^2 f(x) dx$ is :

- (1) $\frac{5+4\sqrt{2}}{3}$ (2) $\frac{8+4\sqrt{2}}{3}$
(3) $\frac{1+5\sqrt{2}}{3}$ (4) $\frac{4+5\sqrt{2}}{3}$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol.



$$\begin{aligned} A &= \int_0^1 1 \cdot dx + \int_1^{\sqrt{2}} 2 dx + \int_{\sqrt{2}}^2 x^2 dx \\ &= 1 + 2\sqrt{2} - 2 + \frac{8}{3} - \frac{2\sqrt{2}}{3} \\ &= \frac{5}{3} + \frac{4\sqrt{2}}{3} \end{aligned}$$

75. If the vectors $\vec{a} = \lambda\hat{i} + \mu\hat{j} + 4\hat{k}$, $\vec{b} = 2\hat{i} + 4\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} + 3\hat{j} + \hat{k}$ are coplanar and the projection of \vec{a} on the vector \vec{b} is $\sqrt{54}$ units, then the sum of all possible values of $\lambda + \mu$ is equal to

- (1) 0 (2) 6
(3) 24 (4) 18

Official Ans. by NTA (3)

Allen Ans. (3)

Sol.
$$\begin{vmatrix} \lambda & \mu & 4 \\ -2 & 4 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$$\lambda(10) = \mu(2) + 4(-14) = 0$$

$$10\lambda - 2\mu = 56$$

$$5\lambda - \mu = 28 \quad \dots(1)$$

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \sqrt{54}$$

$$\frac{-2\lambda + 4\mu - 8}{\sqrt{24}} = \sqrt{54}$$

$$-2\lambda + 4\mu - 8 = \sqrt{54} \times 24 \quad \dots(2)$$

By solving equation (1) & (2)

$$\Rightarrow \lambda + \mu = 24$$

76. Fifteen football players of a club-team are given 15 T-shirts with their names written on the backside. If the players pick up the T-shirts randomly, then the probability that at least 3 players pick the correct T-shirt is

- (1) $\frac{5}{24}$ (2) $\frac{2}{15}$
(3) $\frac{1}{6}$ (4) $\frac{5}{36}$

Official Ans. by NTA (3)

Allen Ans. (Bonus)

Sol.

$$\text{Required probability} = 1 - \frac{D_{(15)} + {}^{15}C_1 \cdot D_{(14)} + {}^{15}C_2 \cdot D_{(13)}}{15!}$$

$$\text{Taking } D_{(15)} \text{ as } \frac{15!}{e}$$

$$D_{(14)} \text{ as } \frac{14!}{e}$$

$$D_{(13)} \text{ as } \frac{13!}{e}$$

$$\text{We get, } 1 - \left(\frac{\frac{15!}{e} + 15 \cdot \frac{14!}{e} + \frac{15 \times 14}{2} \times \frac{13!}{e}}{15!} \right)$$

$$= 1 - \left(\frac{1}{e} + \frac{1}{e} + \frac{1}{2e} \right) = 1 - \frac{5}{2e} \approx .08$$

77. Let $f(\theta) = 3 \left(\sin^4 \left(\frac{3\pi}{2} - \theta \right) + \sin^4 (3\pi + \theta) \right) - 2(1 - \sin^2 2\theta)$ and

$$S = \left\{ \theta \in [0, \pi] : f'(\theta) = -\frac{\sqrt{3}}{2} \right\}. \text{ If } 4\beta = \sum_{\theta \in S} \theta,$$

then $f(\beta)$ is equal to

- (1) $\frac{11}{8}$ (2) $\frac{5}{4}$
(3) $\frac{9}{8}$ (4) $\frac{3}{2}$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol.

$$f(\theta) = 3 \left(\sin^4 \left(\frac{3\pi}{2} - \theta \right) + \sin^4 (3\pi + \theta) \right) - 2(1 - \sin^2 2\theta)$$

$$S = \left\{ \theta \in [0, \pi] : f'(\theta) = -\frac{\sqrt{3}}{2} \right\}$$

$$\Rightarrow f(\theta) = 3(\cos^4 \theta + \sin^4 \theta) - 2\cos^2 2\theta$$

$$\Rightarrow f(\theta) = 3 \left(1 - \frac{1}{2} \sin^2 2\theta \right) - 2\cos^2 2\theta$$

$$\Rightarrow f(\theta) = 3 - \frac{3}{2} \sin^2 2\theta - 2\cos^2 2\theta$$

$$= \frac{3}{2} - \frac{1}{2} \cos^2 2\theta = \frac{3}{2} - \frac{1}{2} \left(\frac{1 + \cos 4\theta}{2} \right)$$

$$f(\theta) = \frac{5}{4} - \frac{\cos 4\theta}{4}$$

$$f'(\theta) = \sin 4\theta$$

$$\Rightarrow f'(\theta) = \sin 4\theta = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow 4\theta = n\pi + (-1)^n \frac{\pi}{3}$$

$$\Rightarrow \theta = \frac{n\pi}{4} + (-1)^n \frac{\pi}{12}$$

$$\Rightarrow \theta = \frac{\pi}{12}, \left(\frac{\pi}{4} - \frac{\pi}{12}\right), \left(\frac{\pi}{2} + \frac{\pi}{12}\right), \left(\frac{3\pi}{4} - \frac{\pi}{12}\right)$$

$$\Rightarrow 4\beta = \frac{\pi}{4} + \frac{\pi}{2} + \frac{3\pi}{4} = \frac{3\pi}{2}$$

$$\Rightarrow \beta = \frac{3\pi}{8} \Rightarrow f(\beta) = \frac{5}{4} - \frac{\cos \frac{3\pi}{2}}{4} = \frac{5}{4}$$

78. If p, q and r are three propositions, then which of the following combination of truth values of p, q and r makes the logical expression $\{(p \vee q) \wedge ((\sim p) \vee r)\} \rightarrow ((\sim q) \vee r)$ false?

- (1) p = T, q = F, r = T (2) p = T, q = T, r = F
(3) p = F, q = T, r = F (4) p = T, q = F, r = F

Official Ans. by NTA (3)

Allen Ans. (3)

Sol.

	p	q	r	$(p \vee q) \wedge ((\sim p) \vee r)$	$\sim q \vee r$
(1)	T	F	T	T	T
(2)	T	T	F	F	F
(3)	F	T	F	T	F
(4)	T	F	F	F	T

Option (3) $(p \vee q) \wedge (\sim q \vee r) \rightarrow (\sim p \vee r)$ will be False.

79. There rotten apples are mixed accidentally with seven good apples and four apples are drawn one by one without replacement. Let the random variable X denote the number of rotten apples. If μ and σ^2 represent mean and variance of X, respectively, then $10(\mu^2 + \sigma^2)$ is equal to

- (1) 20
(2) 250
(3) 25
(4) 30

Official Ans. by NTA (1)

Allen Ans. (1)

Sol.

X	P(x)	XP(x)	X ² P(x)
0	1/6	0	0
1	1/2	1/2	1/2
2	3/10	6/10	12/10
3	1/30	1/10	9/30

$$\sum xP(x) = \frac{6}{2} = \mu$$

$$\sigma^2 = \sum x^2 P(x) - \mu^2$$

$$\sigma^2 + \mu^2 = 0 + \frac{1}{2} + \frac{12}{10} + \frac{9}{30} = 2$$

$$10(\sigma^2 + \mu^2) = 20 \text{ Ans.}$$

80. Let $y = f(x)$ be the solution of the differential equation $y(x+1) dx - x^2 dy = 0$, $y(1) = e$. Then

$\lim_{x \rightarrow 0^+} f(x)$ is equal to

- (1) 0 (2) $\frac{1}{e}$
(3) e^2 (4) $\frac{1}{e^2}$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $\frac{x+1}{x^2} dx = \frac{dy}{y}$

$$\ln x - \frac{1}{x} = \ln y + c$$

$$(1, e)$$

$$c = -2$$

$$\ln x - \frac{1}{x} = \ln y - 2$$

$$y = e^{\ln x - \frac{1}{x} + 2}$$

$$\lim_{x \rightarrow 0^+} e^{\ln x - \frac{1}{x} + 2}$$

$$= e^{-\infty}$$

$$= 0$$

SECTION-B

81. Let the co-ordinates of one vertex of ΔABC be $A(0, 2, \alpha)$ and the other two vertices lie on the line $\frac{x+\alpha}{5} = \frac{y-1}{2} = \frac{z+4}{3}$. For $\alpha \in \mathbb{Z}$, if the area of ΔABC is 21 sq. units and the line segment BC has length $2\sqrt{21}$ units, then α^2 is equal to ____.

Official Ans. by NTA (9)

Allen Ans. (9)

Sol. A. $(0, 2, \alpha)$

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ (-\alpha, 1, -4) & B & C (5i + 2j + 3k) \end{array}$$

$$\left| \frac{1}{2} \cdot 2\sqrt{21} \cdot \begin{vmatrix} i & j & k \\ \alpha & 1 & \alpha+4 \\ 5 & 2 & 3 \end{vmatrix} \right| = 21\sqrt{21}$$

$$\sqrt{(2\alpha+5)^2 + (2\alpha+20)^2 + (2\alpha-5)^2} = \sqrt{21}\sqrt{38}$$

$$\Rightarrow 12\alpha^2 + 80\alpha + 450 = 798$$

$$\Rightarrow 12\alpha^2 + 80\alpha - 348 = 0$$

$$\Rightarrow \alpha = 3 \Rightarrow \alpha^2 = 9$$

82. Let the equation of the plane P containing the line $x+10 = \frac{8-y}{2} = z$ be $ax + by + 3z = 2(a+b)$ and the distance of the plane P from the point $(1, 27, 7)$ be c. Then $a^2 + b^2 + c^2$ is equal to ____.

Official Ans. by NTA (355)

Allen Ans. (355)

Sol. The line $\frac{x+10}{1} = \frac{y-8}{-2} = \frac{z}{1}$ have a point $(-10, 8, 0)$

with d. r. $(1, -2, 1)$

\therefore the plane $ax + by + 3z = 2(a+b)$

$$\Rightarrow b = 2a$$

& dot product of d.r.'s is zero

$$\therefore a - 2b + 3 = 0$$

$$\therefore a = 1 \text{ \& } b = 2$$

Distance from $(1, 27, 7)$ is

$$c = \frac{1+54+21-6}{\sqrt{14}} = \frac{70}{\sqrt{14}} = 5\sqrt{14}$$

$$\therefore a^2 + b^2 + c^2 = 1 + 4 + 350 = 355$$

83. Suppose f is a function satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{N}$ and $f(1) = \frac{1}{5}$. If

$$\sum_{n=1}^m \frac{f(n)}{n(n+1)(n+2)} = \frac{1}{12}, \text{ then } m \text{ is equal to } \underline{\hspace{2cm}}.$$

Official Ans. by NTA (10)

Allen Ans. (10)

$$\text{Sol. } \therefore f(1) = \frac{1}{5} \therefore f(2) = f(1) + f(1) = \frac{2}{5}$$

$$f(2) = \frac{2}{5} \quad f(3) = f(2) + f(1) = \frac{3}{5}$$

$$f(3) = \frac{3}{5}$$

$$\therefore \sum_{n=1}^m \frac{f(n)}{n(n+1)(n+2)}$$

$$= \frac{1}{5} \sum_{n=1}^m \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$$

$$= \frac{1}{5} \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{m+1} - \frac{1}{m+2} \right)$$

$$= \frac{1}{5} \left(\frac{1}{2} - \frac{1}{m+2} \right) = \frac{m}{10(m+2)} = \frac{1}{12}$$

$$\therefore m = 10$$

84. Let a_1, a_2, a_3, \dots be a GP of increasing positive numbers. If the product of fourth and sixth terms is 9 and the sum of fifth and seventh terms is 24, then $a_1 a_9 + a_2 a_4 a_9 + a_5 + a_7$ is equal to ____.

Official Ans. by NTA (60)

Allen Ans. (60)

$$\text{Sol. } a_4 \cdot a_6 = 9 \Rightarrow (a_5)^2 = 9 \Rightarrow a_5 = 3$$

$$\& a_5 + a_7 = 24 \Rightarrow a_5 + a_5 r^2 = 24 \Rightarrow (1+r^2) = 8 \Rightarrow r = \sqrt{7}$$

$$\Rightarrow a = \frac{3}{49}$$

$$\Rightarrow a_1 a_9 + a_2 a_4 a_9 + a_5 + a_7 = 9 + 27 + 3 + 21 = 60$$

85. Let \vec{a} , \vec{b} and \vec{c} be three non-zero non-coplanar vectors. Let the position vectors of four points A, B, C and D be $\vec{a} - \vec{b} + \vec{c}$, $\lambda\vec{a} - 3\vec{b} + 4\vec{c}$, $-\vec{a} + 2\vec{b} - 3\vec{c}$ and $2\vec{a} - 4\vec{b} + 6\vec{c}$ respectively. If \overline{AB} , \overline{AC} and \overline{AD} are coplanar, then λ is :

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $\overline{AB} = (\lambda - 1)\vec{a} - 2\vec{b} + 3\vec{c}$

$$\overline{AC} = 2\vec{a} + 3\vec{b} - 4\vec{c}$$

$$\overline{AD} = \vec{a} - 3\vec{b} + 5\vec{c}$$

$$\begin{vmatrix} \lambda - 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 1)(15 - 12) + 2(-10 + 4) + 3(6 - 3) = 0$$

$$\Rightarrow (\lambda - 1) = 1 \Rightarrow \lambda = 2$$

86. If all the six digit numbers $x_1 x_2 x_3 x_4 x_5 x_6$ with $0 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6$ are arranged in the increasing order, then the sum of the digits in the 72th number is _____.

Official Ans. by NTA (32)

Allen Ans. (32)

Sol. $\begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & & & & \\ \hline \end{array} = {}^7C_4 = 35$

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 3 & & & & \\ \hline \end{array} = {}^6C_4 = 15$$

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 4 & & & & \\ \hline \end{array} = {}^5C_4 = 5$$

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 5 & & & & \\ \hline \end{array} = {}^4C_4 = 1$$

$$\begin{array}{|c|c|c|c|c|c|} \hline 2 & 3 & & & & \\ \hline \end{array} = {}^6C_4 = 15$$

71 words

$$245678 \rightarrow 72^{\text{th}} \text{ word}$$

$$2 + 4 + 5 + 6 + 7 + 8 = 32$$

87. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function that satisfies the relation $f(x + y) = f(x) + f(y) - 1$, $\forall x, y \in \mathbb{R}$. If $f'(0) = 2$, then $|f(-2)|$ is equal to _____.

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $f(x + y) = f(x) + f(y) - 1$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = f'(0) = 2$$

$$f'(x) = 2 \Rightarrow dy = 2dx$$

$$y = 2x + C$$

$$x = 0, y = 1, C = 1$$

$$y = 2x + 1$$

$$|f(-2)| = |-4 + 1| = |-3| = 3$$

88. If the co-efficient of x^9 in $\left(\alpha x^3 + \frac{1}{\beta x}\right)^{11}$ and the co-efficient of x^{-9} in $\left(\alpha x - \frac{1}{\beta x^3}\right)^{11}$ are equal, then $(\alpha\beta)^2$ is equal to _____.

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. Coefficient of x^9 in $\left(\alpha x^3 + \frac{1}{\beta x}\right)^{11} = {}^{11}C_6 \cdot \frac{\alpha^5}{\beta^6}$

\therefore Both are equal

$$\therefore \frac{11}{C_6} \cdot \frac{\alpha^5}{\beta^6} = -\frac{11}{C_5} \cdot \frac{\alpha^6}{\beta^5}$$

$$\Rightarrow \frac{1}{\beta} = -\alpha$$

$$\Rightarrow \alpha\beta = -1$$

$$\Rightarrow (\alpha\beta)^2 = 1$$

89. Let the coefficients of three consecutive terms in the binomial expansion of $(1 + 2x)^n$ be in the ratio 2 : 5 : 8. Then the coefficient of the term, which is in the middle of these three terms, is _____.

Official Ans. by NTA (1120)

Allen Ans. (1120)

Sol. $t_{r+1} = {}^nC_r (2x)^r$

$$\Rightarrow \frac{{}^nC_{r-1} (2)^{r-1}}{{}^nC_r (2)^r} = \frac{2}{5}$$

$$\Rightarrow \frac{\frac{n!}{(r-1)!(n-r+1)!}}{\frac{n!}{r!(n-r)!}} = \frac{2}{5}$$

$$\Rightarrow \frac{r}{n-r+1} = \frac{4}{5} \Rightarrow 5r = 4n - 4r + 4$$

$$\Rightarrow 9r = 4(n+1) \quad \dots (1)$$

$$\Rightarrow \frac{{}^nC_r (2)^r}{{}^nC_{r+1} (2)^{r+1}} = \frac{5}{8}$$

$$\Rightarrow \frac{\frac{n!}{r!(n-r)!}}{\frac{n!}{(r+1)!(n-r-1)!}} = \frac{5}{4} \Rightarrow \frac{r+1}{n-r} = \frac{5}{4}$$

$$\Rightarrow 4r + 4 = 5n - 5r \Rightarrow 5n - 4 = 9r \quad \dots (2)$$

From (1) and (2)

$$\Rightarrow 4n + 4 = 5n - 4 \Rightarrow n = 8$$

$$(1) \Rightarrow r = 4$$

so, coefficient of middle term is

$${}^8C_4 2^4 = 16 \times \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 16 \times 70 = 1120$$

- 90.** Five digit numbers are formed using the digits 1, 2, 3, 5, 7 with repetitions and are written in descending order with serial numbers. For example, the number 77777 has serial number 1. Then the serial number of 35337 is _____.

Official Ans. by NTA (1436)

Allen Ans. (1436)

- Sol.** No of 5 digit numbers starting with digit 1
 $= 5 \times 5 \times 5 \times 5 = 625$
 No of 5 digit numbers starting with digit 2
 $= 5 \times 5 \times 5 \times 5 = 625$
 No of 5 digit numbers starting with 31
 $= 5 \times 5 \times 5 = 125$
 No of 5 digit numbers starting with 32
 $= 5 \times 5 \times 5 = 125$
 No of 5 digit numbers starting with 33
 $= 5 \times 5 \times 5 = 125$
 No of 5 digit numbers starting with 351
 $= 5 \times 5 = 25$
 No of 5 digit numbers starting with 352
 $= 5 \times 5 = 25$
 No of 5 digit numbers starting with 3531
 $= 5$
 No of 5 digit numbers starting with 3532
 $= 5$
 Before 35337 will be 4 numbers,
 So rank of 35337 will be 1690

So, in descending order serial number will be
 $3125 - 1690 + 1 = 1436$