FINAL JEE–MAIN EXAMINATION – JANUARY, 2024

(Held On Saturday 27th January, 2024) **TIME : 9 : 00 AM to 12 : 00 NOON**

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

1. ${}^{n-1}C_r = (k^2-8)\,{}^nC_{r+1}$ if and only if :

(1) $2\sqrt{2} < k \le 3$ (2) $2\sqrt{3} < k \le 3\sqrt{2}$

(3) $2\sqrt{3} < k < 3\sqrt{3}$ (4) $2\sqrt{2} < k < 2\sqrt{3}$

Ans. (1)

Sol. ${}^{n-1}C_r = (k^2-8)\,{}^nC_{r+1}$

$$\underbrace{r+1 \ge 0,\ \ r \ge 0}_{r\ge 0}$$

$$\frac{{}^{n-1}C_r}{{}^nC_{r+1}} = k^2-8$$

$$\frac{r+1}{n} = k^2-8$$

$$\Rightarrow k^2-8>0$$

$$\left(k-2\sqrt{2}\right)\left(k+2\sqrt{2}\right)>0$$

$$k \in \left(-\infty, -2\sqrt{2}\right)\cup\left(2\sqrt{2}, \infty\right) \qquad \ldots(I)$$

$$\therefore\ n \ge r+1,\ \frac{r+1}{n} \le 1$$

$$\Rightarrow k^2-8 \le 1$$

$$k^2-9 \le 0$$

$$-3 \le k \le 3 \qquad \ldots.(II)$$

From equation (I) and (II) we get

$$k \in \left[-3, -2\sqrt{2}\right)\cup\left(2\sqrt{2}, 3\right]$$

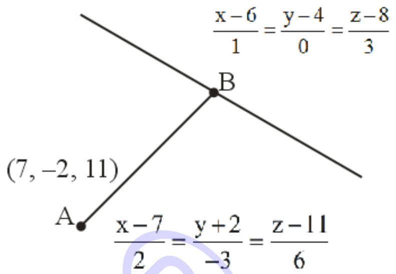
2. The distance, of the point (7, –2, 11) from the line $\frac{x-6}{1}=\frac{y-4}{0}=\frac{z-8}{3}$ along the line $\frac{x-5}{2}=\frac{y-1}{-3}=\frac{z-5}{6}$, is :

(1) 12 (2) 14

(3) 18 (4) 21

Ans. (2)

Sol. B = (2λ +7, –3λ – 2, 6λ + 11)



Point B lies on $\frac{x-6}{1}=\frac{y-4}{0}=\frac{z-8}{3}$

$$\frac{2\lambda+7-6}{1}=\frac{-3\lambda-2-4}{0}=\frac{6\lambda+11-8}{3}$$

$$-3\lambda-6=0$$

$$\lambda=-2$$

$$B \Rightarrow (3, 4, -1)$$

$$AB=\sqrt{(7-3)^2+(4+2)^2+(11+1)^2}$$

$$=\sqrt{16+36+144}$$

$$=\sqrt{196}=14$$

3. Let x = x(t) and y = y(t) be solutions of the differential equations $\frac{dx}{dt}+ax=0$ and $\frac{dy}{dt}+by=0$ respectively, a, b ∈ R. Given that x(0) = 2; y(0) = 1 and 3y(1) = 2x(1), the value of t, for which x(t) = y(t), is :

(1) $\log_{\frac{2}{3}} 2$ (2) $\log_4 3$

(3) $\log_3 4$ (4) $\log_{\frac{4}{3}} 2$

Ans. (4)

Sol. $\frac{dx}{dt}+ax=0$

$\frac{dx}{x}=-adt$

$\int\frac{dx}{x}=-a\int dt$

$\ln|x|=-at+c$

at $t=0$, $x=2$

$\ln 2=0+c$

$\ln x=-at+\ln 2$

$\frac{x}{2}=e^{-at}$

$x=2e^{-at}$ ....(i)

$\frac{dy}{dt}+by=0$

$\frac{dy}{y}=-bdt$

$\ln|y|=-bt+\lambda$

$t=0$, $y=1$

$0=0+\lambda$

$y=e^{-bt}$ ....(ii)

According to question

$3y(1)=2x(1)$

$3e^{-b}=2(2e^{-a})$

$e^{a-b}=\frac{4}{3}$

For $x(t)=y(t)$

$\Rightarrow 2e^{-at}=e^{-bt}$

$2=e^{(a-b)t}$

$2=\left(\frac{4}{3}\right)^t$

$\log_{\frac{4}{3}}2=t$

4. If (a, b) be the orthocentre of the triangle whose vertices are $(1, 2)$, $(2, 3)$ and $(3, 1)$, and $I_1=\int_a^b x\sin\left(4x-x^2\right)dx$, $I_2=\int_a^b \sin\left(4x-x^2\right)dx$, then $36\frac{I_1}{I_2}$ is equal to :

(1) 72 (2) 88

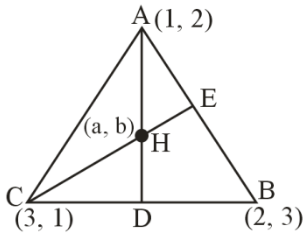
(3) 80 (4) 66

Ans. (1)

Sol. Equation of CE

$y-1=-(x-3)$

$x+y=4$



orthocentre lies on the line $x+y=4$

so, $a+b=4$

$I_1=\int_a^b x\sin\left(x(4-x)\right)dx$ ...(i)

Using king rule

$I_1=\int_a^b (4-x)\sin\left(x(4-x)\right)dx$ ...(ii)

(i) + (ii)

$2I_1=\int_a^b 4\sin\left(x(4-x)\right)dx$

$2I_1=4I_2$

$I_1=2I_2$

$\frac{I_1}{I_2}=2$

$\frac{36I_1}{I_2}=72$

5. If A denotes the sum of all the coefficients in the expansion of $(1-3x+10x^2)^n$ and B denotes the sum of all the coefficients in the expansion of $(1+x^2)^n$, then :

(1) $A=B^3$ (2) $3A=B$

(3) $B=A^3$ (4) $A=3B$

Ans. (1)

Sol. Sum of coefficients in the expansion of

$(1-3x+10x^2)^n=A$

then $A=(1-3+10)^n=8^n$ (put $x=1$)

and sum of coefficients in the expansion of

$(1+x^2)^n=B$

then $B=(1+1)^n=2^n$

$A=B^3$

6. The number of common terms in the progressions 4, 9, 14, 19, ... , up to 25^{th} term and 3, 6, 9, 12, ..., up to 37^{th} term is :

(1) 9 (2) 5

(3) 7 (4) 8

Ans. (3)

Sol. 4, 9, 14, 19, …., up to 25^{th} term

$T_{25} = 4 + (25 - 1)\,5 = 4 + 120 = 124$

3, 6, 9, 12, …, up to 37^{th} term

$T_{37} = 3 + (37 - 1)3 = 3 + 108 = 111$

Common difference of I^{st} series $d_1 = 5$

Common difference of II^{nd} series $d_2 = 3$

First common term = 9, and

their common difference = 15 (LCM of d_1 and d_2)

then common terms are

9, 24, 39, 54, 69, 84, 99

7. If the shortest distance of the parabola $y^2 = 4x$ from the centre of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$ is d, then d^2 is equal to :

(1) 16 (2) 24

(3) 20 (4) 36

Ans. (3)

Sol. Equation of normal to parabola

$y = mx - 2m - m^3$

this normal passing through center of circle (2, 8)

$8 = 2m - 2m - m^3$

$m = -2$

So point P on parabola $\Rightarrow (am^2, -2am) = (4, 4)$

And C = (2, 8)

$PC = \sqrt{4+16} = \sqrt{20}$

$d^2 = 20$

8. If the shortest distance between the lines $\frac{x-4}{1} = \frac{y+1}{2} = \frac{z}{-3}$ and $\frac{x-\lambda}{2} = \frac{y+1}{4} = \frac{z-2}{-5}$ is $\frac{6}{\sqrt{5}}$, then the sum of all possible values of λ is :

(1) 5 (2) 8

(3) 7 (4) 10

Ans. (2)

Sol. $\frac{x-4}{1} = \frac{y+1}{2} = \frac{z}{-3}$

$\frac{x-\lambda}{2} = \frac{y+1}{4} = \frac{z-2}{-5}$

the shortest distance between the lines

$$= \left|\frac{(\vec{a}-\vec{b})\cdot(\vec{d_1}\times\vec{d_2})}{|\vec{d_1}\times\vec{d_2}|}\right|$$

$$= \left|\frac{\begin{vmatrix} \lambda-4 & 0 & 2 \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix}}{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix}}\right|$$

$$= \left|\frac{(\lambda-4)(-10+12)-0+2(4-4)}{|2\hat{i}-1\hat{j}+0\hat{k}|}\right|$$

$$\frac{6}{\sqrt{5}} = \left|\frac{2(\lambda-4)}{\sqrt{5}}\right|$$

$3 = |\lambda - 4|$

$\lambda - 4 = \pm 3$

$\lambda = 7, 1$

Sum of all possible values of λ is = 8

9. If $\int_0^1 \frac{1}{\sqrt{3+x}+\sqrt{1+x}}dx = a + b\sqrt{2} + c\sqrt{3}$, where a, b, c are rational numbers, then 2a + 3b – 4c is equal to :

(1) 4 (2) 10

(3) 7 (4) 8

Ans. (4)

Sol. $\int_0^1 \frac{1}{\sqrt{3+x}+\sqrt{1+x}}dx = \int_0^1 \frac{\sqrt{3+x}-\sqrt{1+x}}{(3+x)-(1+x)}dx$

$$\frac{1}{2}\left[\int_0^1 \sqrt{3+x}\,dx - \int_0^1 \left(\sqrt{1+x}\right)dx\right]$$

$$\frac{1}{2}\left[2\frac{(3+x)^{\frac{3}{2}}}{3}-\frac{2(1+x)^{\frac{3}{2}}}{3}\right]_0^1$$

$$\frac{1}{2}\left[\frac{2}{3}\left(8-3\sqrt{3}\right)-\frac{2}{3}\left(2^{\frac{3}{2}}-1\right)\right]$$

$$\frac{1}{3}\left[8-3\sqrt{3}-2\sqrt{2}+1\right]$$

$$=3-\sqrt{3}-\frac{2}{3}\sqrt{2}=a+b\sqrt{2}+c\sqrt{3}$$

$$a=3,\ b=-\frac{2}{3},\ c=-1$$

$2a + 3b - 4c = 6 - 2 + 4 = 8$

10. Let S = {1, 2, 3, ... , 10}. Suppose M is the set of all the subsets of S, then the relation

R = {(A, B): $A \cap B \neq \phi$; A, B $\in$ M} is :

(1) symmetric and reflexive only

(2) reflexive only

(3) symmetric and transitive only

(4) symmetric only

Ans. (4)

Sol. Let S = {1, 2, 3, …, 10}

R = {(A, B): $A \cap B \neq \phi$; A, B $\in$ M}

For Reflexive,

M is subset of 'S'

So $\phi \in M$

for $\phi \cap \phi = \phi$

$\Rightarrow$ but relation is $A \cap B \neq \phi$

So it is not reflexive.

For symmetric,

ARB $\quad A \cap B \neq \phi$,

$\Rightarrow$ BRA $\quad \Rightarrow B \cap A \neq \phi$,

So it is symmetric.

For transitive,

If A = {(1, 2), (2, 3)}

B = {(2, 3), (3, 4)}

C = {(3, 4), (5, 6)}

ARB & BRC but A does not relate to C

So it not transitive

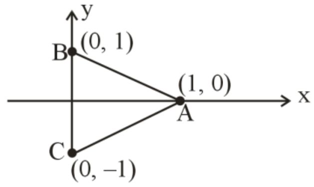
11. If S = {z ∈ C : |z – i| = |z + i| = |z–1|}, then, n(S) is:

(1) 1 (2) 0

(3) 3 (4) 2

Ans. (1)

Sol. |z – i| = |z + i| = |z – 1|



ABC is a triangle. Hence its circum-centre will be the only point whose distance from A, B, C will be same.

So n(S) = 1

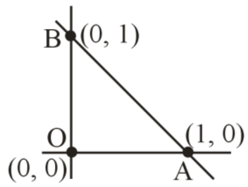
12. Four distinct points (2k, 3k), (1, 0), (0, 1) and (0, 0) lie on a circle for k equal to :

(1) $\frac{2}{13}$ (2) $\frac{3}{13}$

(3) $\frac{5}{13}$ (4) $\frac{1}{13}$

Ans. (3)

Sol. (2k, 3k) will lie on circle whose diameter is AB.



(x – 1) (x) + (y – 1) (y) = 0

$x^2 + y^2 - x - y = 0$ …(i)

Satisfy (2k, 3k) in (i)

$(2k)^2 + (3k)^2 - 2k - 3k = 0$

$13k^2 - 5k = 0$

$$k=0,\ k=\frac{5}{13}$$

$$\text{hence } k=\frac{5}{13}$$

13. Consider the function.

$$f(x)=\begin{cases}\dfrac{a(7x-12-x^2)}{b|x^2-7x+12|} & , \quad x<3 \\ 2^{\frac{\sin(x-3)}{x-[x]}} & , \quad x>3 \\ b & , \quad x=3\end{cases}$$

Where [x] denotes the greatest integer less than or equal to x. If S denotes the set of all ordered pairs (a, b) such that f(x) is continuous at x = 3, then the number of elements in S is :

(1) 2 (2) Infinitely many

(3) 4 (4) 1

Ans. (4)

Sol. $f(3^-)=\dfrac{a}{b}\dfrac{(7x-12-x^2)}{|x^2-7x+12|}$ (for f(x) to be cont.)

$\Rightarrow f(3^-)=\dfrac{-a}{b}\dfrac{(x-3)(x-4)}{(x-3)(x-4)}; x<3 \Rightarrow \dfrac{-a}{b}$

Hence $f(3^-)=\dfrac{-a}{b}$

Then $f(3^+)=2^{\lim\limits_{x\to3^+}\left(\frac{\sin(x-3)}{x-3}\right)}=2$ and

f(3) = b.

Hence $f(3)=f(3^+)=f(3^-)$

$\Rightarrow b=2=-\dfrac{a}{b}$

b = 2, a = –4

Hence only 1 ordered pair (–4, 2).

14. Let $a_1, a_2, \ldots.. a_{10}$ be 10 observations such that $\sum\limits_{k=1}^{10}a_k=50$ and $\sum\limits_{\forall k<j}a_k\cdot a_j=1100$. Then the standard deviation of $a_1, a_2, .., a_{10}$ is equal to :

(1) 5 (2) $\sqrt{5}$

(3) 10 (4) $\sqrt{115}$

Ans. (2)

Sol. $\sum\limits_{k=1}^{10}a_k=50$

$a_1+a_2+\ldots+a_{10}=50$ ….(i)

$\sum\limits_{\forall k<j}a_ka_j=1100$ ....(ii)

If $a_1+a_2+\ldots+a_{10}=50$.

$(a_1+a_2+\ldots+a_{10})^2=2500$

$\Rightarrow \sum\limits_{i=1}^{10}a_i^2+2\sum\limits_{k<j}a_ka_j=2500$

$\Rightarrow \sum\limits_{i=1}^{10}a_i^2=2500-2(1100)$

$\sum\limits_{i=1}^{10}a_i^2=300$, Standard deviation 'σ'

$=\sqrt{\dfrac{\sum a_i^2}{10}-\left(\dfrac{\sum a_i}{10}\right)^2}=\sqrt{\dfrac{300}{10}-\left(\dfrac{50}{10}\right)^2}$

$=\sqrt{30-25}=\sqrt{5}$

15. The length of the chord of the ellipse $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$, whose mid point is $\left(1,\dfrac{2}{5}\right)$, is equal to :

(1) $\dfrac{\sqrt{1691}}{5}$ (2) $\dfrac{\sqrt{2009}}{5}$

(3) $\dfrac{\sqrt{1741}}{5}$ (4) $\dfrac{\sqrt{1541}}{5}$

Ans. (1)

Sol. Equation of chord with given middle point.

$T=S_1$

$\dfrac{x}{25}+\dfrac{y}{40}=\dfrac{1}{25}+\dfrac{1}{100}$

$\dfrac{8x+5y}{200}=\dfrac{8+2}{200}$

$y=\dfrac{10-8x}{5}$ …(i)

$$\frac{x^2}{25}+\frac{(10-8x)^2}{400}=1 \quad \text{(put in original equation)}$$

$$\frac{16x^2+100+64x^2-160x}{400}=1$$

$4x^2 - 8x - 15 = 0$

$$x=\frac{8\pm\sqrt{304}}{8}$$

$$x_1=\frac{8+\sqrt{304}}{8};\ x_2=\frac{8-\sqrt{304}}{8}$$

Similarly, $y=\frac{10-18\pm\sqrt{304}}{5}=\frac{2\pm\sqrt{304}}{5}$

$$y_1=\frac{2-\sqrt{304}}{5};\ y_2=\frac{2+\sqrt{304}}{5}$$

Distance $=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

$$=\sqrt{\frac{4\times304}{64}+\frac{4\times304}{25}}=\frac{\sqrt{1691}}{5}$$

16. The portion of the line 4x + 5y = 20 in the first quadrant is trisected by the lines L_1 and L_2 passing through the origin. The tangent of an angle between the lines L_1 and L_2 is :

(1) $\frac{8}{5}$ (2) $\frac{25}{41}$

(3) $\frac{2}{5}$ (4) $\frac{30}{41}$

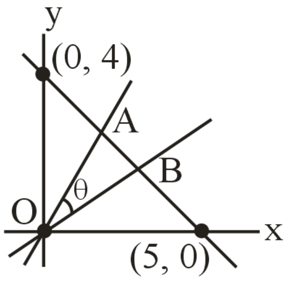
Ans. (4)

Sol. Co-ordinates of $A=\left(\frac{5}{3},\frac{8}{3}\right)$

Co-ordinates of $B=\left(\frac{10}{3},\frac{4}{3}\right)$

Slope of OA = $m_1=\frac{8}{5}$

Slope of OB = $m_2=\frac{2}{5}$



$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$\tan\theta=\frac{\frac{6}{5}}{1+\frac{16}{25}}=\frac{30}{41}$$

$$\tan\theta=\frac{30}{41}$$

17. Let $\vec{a}=\hat{i}+2\hat{j}+\hat{k}$, $\vec{b}=3(\hat{i}-\hat{j}+\hat{k})$. Let $\vec{c}$ be the vector such that $\vec{a}\times\vec{c}=\vec{b}$ and $\vec{a}\cdot\vec{c}=3$. Then $\vec{a}\cdot\left(\left(\vec{c}\times\vec{b}\right)-\vec{b}-\vec{c}\right)$ is equal to :

(1) 32 (2) 24

(3) 20 (4) 36

Ans. (2)

Sol. $\vec{a}\cdot\left[\left(\vec{c}\times\vec{b}\right)-\vec{b}-\vec{c}\right]$

$\vec{a}\cdot\left(\vec{c}\times\vec{b}\right)-\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{c}$ ....(i)

given $\vec{a}\times\vec{c}=\vec{b}$

$\Rightarrow \left(\vec{a}\times\vec{c}\right)\cdot\vec{b}=\vec{b}\cdot\vec{b}=\left|\vec{b}\right|^2=27$

$\Rightarrow \vec{a}\cdot\left(\vec{c}\times\vec{b}\right)=\left[\vec{a}\ \vec{c}\ \vec{b}\right]=\left(\vec{a}\times\vec{c}\right)\cdot\vec{b}=27$ ...(ii)

Now $\vec{a}\cdot\vec{b}=3-6+3=0$ ...(iii)

$\vec{a}\cdot\vec{c}=3$ ...(iv) (given)

By (i), (ii), (iii) & (iv)

27 – 0 – 3 = 24

18. If $a = \lim_{x\to 0}\frac{\sqrt{1+\sqrt{1+x^4}}-\sqrt{2}}{x^4}$ and

$b = \lim_{x\to 0}\frac{\sin^2 x}{\sqrt{2}-\sqrt{1+\cos x}}$, then the value of ab^3 is :

(1) 36 (2) 32 (3) 25 (4) 30

Ans. (2)

Sol. $a = \lim_{x\to 0}\frac{\sqrt{1+\sqrt{1+x^4}}-\sqrt{2}}{x^4}$

$$= \lim_{x\to 0}\frac{\sqrt{1+x^4}-1}{x^4\left(\sqrt{1+\sqrt{1+x^4}}+\sqrt{2}\right)}$$

$$= \lim_{x\to 0}\frac{x^4}{x^4\left(\sqrt{1+\sqrt{1+x^4}}+\sqrt{2}\right)\left(\sqrt{1+x^4}+1\right)}$$

Applying limit $a = \frac{1}{4\sqrt{2}}$

$$b = \lim_{x\to 0}\frac{\sin^2 x}{\sqrt{2}-\sqrt{1+\cos x}}$$

$$= \lim_{x\to 0}\frac{\left(1-\cos^2 x\right)\left(\sqrt{2}+\sqrt{1+\cos x}\right)}{2-(1+\cos x)}$$

$$b = \lim_{x\to 0}(1+\cos x)\left(\sqrt{2}+\sqrt{1+\cos x}\right)$$

Applying limits $b = 2\left(\sqrt{2}+\sqrt{2}\right) = 4\sqrt{2}$

Now, $ab^3 = \frac{1}{4\sqrt{2}}\times\left(4\sqrt{2}\right)^3 = 32$

19. Consider the matrix $f(x) = \begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$.

Given below are two statements :

Statement I: f(–x) is the inverse of the matrix f(x).

Statement II: f(x) f(y) = f(x + y).

In the light of the above statements, choose the correct answer from the options given below

(1) Statement I is false but Statement II is true

(2) Both Statement I and Statement II are false

(3) Statement I is true but Statement II is false

(4) Both Statement I and Statement II are true

Ans. (4)

Sol. $f(-x) = \begin{bmatrix}\cos x & \sin x & 0\\ -\sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$

$$f(x)\cdot f(-x) = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = I$$

Hence statement- I is correct

Now, checking statement II

$$f(y) = \begin{bmatrix}\cos y & -\sin y & 0\\ \sin y & \cos y & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$f(x)\cdot f(y) = \begin{bmatrix}\cos(x+y) & -\sin(x+y) & 0\\ \sin(x+y) & \cos(x+y) & 0\\ 0 & 0 & 1\end{bmatrix}$$

$\Rightarrow f(x)\cdot f(y) = f(x+y)$

Hence statement-II is also correct.

20. The function f : N – {1} → N; defined by f(n) = the highest prime factor of n, is :

(1) both one-one and onto

(2) one-one only

(3) onto only

(4) neither one-one nor onto

Ans. (4)

Sol. $f : N - \{1\} \to N$

f(n) = The highest prime factor of n.

f(2) = 2

f(4) = 2

⇒ many one

4 is not image of any element

⇒ into

Hence many one and into

Neither one-one nor onto.

SECTION-B

21. The least positive integral value of α, for which the angle between the vectors $\alpha\hat{i}-2\hat{j}+2\hat{k}$ and $\alpha\hat{i}+2\alpha\hat{j}-2\hat{k}$ is acute, is _____.

Ans. (5)

Sol. $\cos\theta = \dfrac{(\alpha\hat{i}-2\hat{j}+2\hat{k})\cdot(\alpha\hat{i}+2\alpha\hat{j}-2\hat{k})}{\sqrt{\alpha^2+4+4}\ \sqrt{\alpha^2+4\alpha^2+4}}$

$\cos\theta = \dfrac{\alpha^2-4\alpha-4}{\sqrt{\alpha^2+8}\ \sqrt{5\alpha^2+4}}$

$\Rightarrow \alpha^2-4\alpha-4>0$

$\Rightarrow \alpha^2-4\alpha+4>8 \qquad \Rightarrow (\alpha-2)^2>8$

$\Rightarrow \alpha-2>2\sqrt{2}$ or $\alpha-2<-2\sqrt{2}$

$\alpha>2+2\sqrt{2}$ or $\alpha<2-2\sqrt{2}$

$\alpha\in(-\infty,-0.82)\cup(4.82,\infty)$

Least positive integral value of $\alpha \Rightarrow 5$

22. Let for a differentiable function $f:(0,\infty)\to R$, $f(x)-f(y)\ge \log_e\left(\dfrac{x}{y}\right)+x-y,\ \forall\ x, y \in (0,\infty)$.

Then $\sum_{n=1}^{20} f'\left(\dfrac{1}{n^2}\right)$ is equal to _____.

Ans. (2890)

Sol. $f(x)-f(y)\ge \ln x-\ln y+x-y$

$\dfrac{f(x)-f(y)}{x-y}\ge\dfrac{\ln x-\ln y}{x-y}+1$

Let $x>y$

$\lim_{y\to x} f'(x^-)\ge\dfrac{1}{x}+1 \qquad .....(1)$

Let $x<y$

$\lim_{y\to x} f'(x^+)\le\dfrac{1}{x}+1 \qquad .....(2)$

$f'(x^-)=f'(x^+)$

$f'(x)=\dfrac{1}{x}+1$

$f'\left(\dfrac{1}{x^2}\right)=x^2+1$

$\sum_{x=1}^{20}(x^2+1)=\sum_{x-1}^{20}x^2+20$

$=\dfrac{20\times21\times41}{6}+20$

$=2890$

23. If the solution of the differential equation $(2x+3y-2)\,dx+(4x+6y-7)\,dy=0$, $y(0)=3$, is $\alpha x+\beta y+3\log_e|2x+3y-\gamma|=6$, then $\alpha+2\beta+3\gamma$ is equal to _____.

Ans. (29)

Sol. $2x+3y-2=t \qquad\qquad 4x+6y-4=2t$

$2+3\dfrac{dy}{dx}=\dfrac{dt}{dx} \qquad\qquad 4x+6y-7=2t-3$

$\dfrac{dy}{dx}=\dfrac{-(2x+3y-2)}{4x+6y-7}$

$\dfrac{dt}{dx}=\dfrac{-3t+4t-6}{2t-3}=\dfrac{t-6}{2t-3}$

$\int\dfrac{2t-3}{t-6}\,dt=\int dx$

$\int\left(\dfrac{2t-12}{t-6}+\dfrac{9}{t-6}\right)\cdot dt=x$

$2t+9\ln(t-6)=x+c$

$2(2x+3y-2)+9\ln(2x+3y-8)=x+c$

$x=0,\ y=3$

$c=14$

$4x+6y-4+9\ln(2x+3y-8)=x+14$

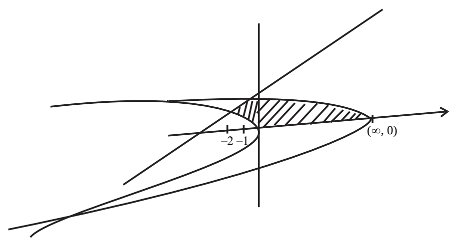
$x+2y+3\ln(2x+3y-8)=6$

$\alpha=1,\ \beta=2,\ \gamma=8$

$\alpha+2\beta+3\gamma=1+4+24=29$

24. Let the area of the region $\{(x, y):\ x-2y+4\ge0,\ x+2y^2\ge0,\ x+4y^2\le8,\ y\ge0\}$ be $\dfrac{m}{n}$, where m and n are coprime numbers. Then m + n is equal to _____.

Ans. (119)



Sol.

$A=\int_0^1\left[(8-4y^2)-(-2y^2)\right]dy+$

$\int_1^{3/2}\left[(8-4y^2)-(2y-4)\right]dy$

$=\left[8y-\dfrac{2y^3}{3}\right]_0^1+\left[12y-y^2-\dfrac{4y^3}{3}\right]_1^{3/2}=\dfrac{107}{12}=\dfrac{m}{n}$

$\therefore\ m+n=119$

25. If

$$8=3+\frac{1}{4}(3+p)+\frac{1}{4^2}(3+2p)+\frac{1}{4^3}(3+3p)+....\infty,$$

then the value of p is _____.

Ans. (9)

Sol. $8=\frac{3}{1-\frac{1}{4}}+\frac{p\cdot\frac{1}{4}}{\left(1-\frac{1}{4}\right)^2}$

(sum of infinite terms of A.G.P $=\frac{a}{1-r}+\frac{dr}{(1-r)^2}$)

$\Rightarrow \frac{4p}{9}=4 \Rightarrow p=9$

26. A fair die is tossed repeatedly until a six is obtained. Let X denote the number of tosses required and let a = P(X = 3), b = P(X ≥ 3) and c = P(X ≥ 6 |X > 3). Then $\frac{b+c}{a}$ is equal to _____.

Ans. (12)

Sol. $a=P(X=3)=\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}=\frac{25}{216}$

$b=P(X\geq 3)=\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}+\left(\frac{5}{6}\right)^3\cdot\frac{1}{6}+\left(\frac{5}{6}\right)^4\cdot\frac{1}{6}+......$

$=\frac{\frac{25}{216}}{1-\frac{5}{6}}=\frac{25}{216}\times\frac{6}{1}=\frac{25}{36}$

$P(X\geq 6)=\left(\frac{5}{6}\right)^5\cdot\frac{1}{6}+\left(\frac{5}{6}\right)^6\cdot\frac{1}{6}+.......$

$=\frac{\left(\frac{5}{6}\right)^5\cdot\frac{1}{6}}{1-\frac{5}{6}}=\left(\frac{5}{6}\right)^5$

$c=\frac{\left(\frac{5}{6}\right)^5}{\left(\frac{5}{6}\right)^3}=\frac{25}{36}$

$\frac{b+c}{a}=\frac{\left(\frac{5}{6}\right)^2+\left(\frac{5}{6}\right)^2}{\left(\frac{5}{6}\right)^2\cdot\frac{1}{6}}=12$

27. Let the set of all a ∈ R such that the equation $\cos 2x+a\sin x=2a-7$ has a solution be [p, q] and $r=\tan 9°-\tan 27°-\frac{1}{\cot 63°}+\tan 81°$, then pqr is equal to _____.

Ans. (48)

Sol. cos2x + a·sinx = 2a – 7

$a(\sin x-2)=2(\sin x-2)(\sin x+2)$

$\sin x=2,\quad a=2(\sin x+2)$

$\Rightarrow a\in[2, 6]$

p = 2 q = 6

r = tan 9° + cot 9° – tan27 – cot27

$r=\frac{1}{\sin 9\cdot\cos 9}-\frac{1}{\sin 27\cdot\cos 27}$

$=2\left[\frac{4}{\sqrt{5}-1}-\frac{4}{\sqrt{5}+1}\right]$

r = 4

p . q . r = 2 × 6 × 4 = 48

28. Let $f(x)=x^3+x^2f'(1)+xf''(2)+f'''(3)$, x ∈ R. Then $f'(10)$ is equal to _____.

Ans. (202)

Sol. $f(x) = x^3 + x^2 \cdot f'(1) + x \cdot f''(2) + f'''(3)$

$f'(x)=3x^2+2xf'(1)+f''(2)$

$f''(x)=6x+2f'(1)$

$f'''(x)=6$

$f'(1) = –5,\ f''(2) = 2,\ f'''(3) = 6$

$f(x) = x^3 + x^2 \cdot (–5) + x \cdot (2) + 6$

$f'(x) = 3x^2 – 10x + 2$

$f'(10) = 300 – 100 + 2 = 202$

29. Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $B = [B_1, B_2, B_3]$, where B_1, B_2, B_3 are column matrices, and $AB_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$,

$$AB_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \quad AB_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

If $\alpha = |B|$ and β is the sum of all the diagonal elements of B, then $\alpha^3 + \beta^3$ is equal to _____.

Ans. (28)

Sol. $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $\qquad B = [B_1, B_2, B_3]$

$$B_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \quad B_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

$$AB_1 = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 1, y_1 = -1, z_1 = -1$

$$AB_2 = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

$x_2 = 2, y_2 = 1, z_2 = -2$

$$AB_3 = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$x_3 = 2, y_3 = 0, z_3 = -1$

$$B = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

$\alpha = |B| = 3$

$\beta = 1$

$\alpha^3 + \beta^3 = 27 + 1 = 28$

30. If α satisfies the equation $x^2 + x + 1 = 0$ and $(1 + \alpha)^7 = A + B\alpha + C\alpha^2$, $A, B, C \geq 0$, then $5(3A - 2B - C)$ is equal to _____.

Ans. (5)

Sol. $x^2 + x + 1 = 0 \Rightarrow x = \omega, \omega^2 = \alpha$

Let $\alpha = \omega$

Now $(1 + \alpha)^7 = -\omega^{14} = -\omega^2 = 1 + \omega$

$A = 1, B = 1, C = 0$

$\therefore 5(3A - 2B - C) = 5(3 - 2 - 0) = 5$