FINAL JEE–MAIN EXAMINATION – APRIL, 2023

(Held On Monday 10th April, 2023) **TIME : 3 : 00 PM to 6 : 00 PM**

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

1. Let f be a continuous function satisfying $\int_0^{t^2}\left(f(x)+x^2\right)dx=\frac{4}{3}t^3,\ \forall\ t>0$. Then $f\left(\frac{\pi^2}{4}\right)$ is equal to :

(1) $\pi\left(1-\frac{\pi^3}{16}\right)$

(2) $-\pi^2\left(1+\frac{\pi^2}{16}\right)$

(3) $-\pi\left(1+\frac{\pi^3}{16}\right)$

(4) $\pi^2\left(1-\frac{\pi^2}{16}\right)$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $\int_0^{t^2}\left(f(x)+x^2\right)dx=\frac{4}{3}t^3,\ \forall\ t>0$

$\left(f(t^2)+t^4\right)=2t$

$f(t^2)=2t-t^4$

$t=\frac{\pi}{2}\Rightarrow f\left(\frac{\pi^2}{4}\right)=\frac{2\pi}{2}-\frac{\pi^4}{16}$

$=\pi-\frac{\pi^4}{16}=\pi\left(1-\frac{\pi^3}{16}\right)$

2. Eight persons are to be transported from city A to city B in three cars of different makes. If each car can accommodate at most three persons, then the number of ways, in which they can be transported, is:

(1) 3360

(2) 1680

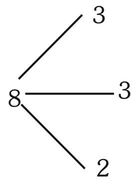
(3) 560

(4) 1120

Official Ans. by NTA (1)

Allen Ans. (2)

Sol.



Ways $=\frac{8!}{3!3!2!2!}\times 3!$

$=\frac{8\times7\times6\times5\times4}{4}$

$= 56 \times 30$

$= 1680$

3. For, $\alpha,\beta,\gamma,\delta\in\mathbb{N}$, if

$$\int\left(\left(\frac{x}{e}\right)^{2x}+\left(\frac{e}{x}\right)^{2x}\right)\log_e x\,dx=\frac{1}{\alpha}\left(\frac{x}{e}\right)^{\beta x}-\frac{1}{\gamma}\left(\frac{e}{x}\right)^{\delta x}+C,$$

Where $e=\sum_{n=0}^{\infty}\frac{1}{n!}$ and C is constant of integration, then $\alpha + 2\beta + 3\gamma - 4\delta$ is equal to:

(1) 1

(2) –4

(3) –8

(4) 4

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $\left(x=e^{\ln x}\right)$

$\int\left(\left(\frac{x}{e}\right)^{2x}+\left(\frac{e}{x}\right)^{2x}\right)\log_e x\,dx=\int\left[e^{2(x\ln x-x)}+e^{-2(x\ln x-x)}\right]\ln x\,dx$

$x \ln x - x = t$

$\ln x.dx = dt$

$\int\left(e^{2t}+e^{-2t}\right)dt$

$\frac{e^{2t}}{2}-\frac{e^{-2t}}{2}+C$

$=\frac{1}{2}\left(\frac{x}{e}\right)^{2x}-\frac{1}{2}\left(\frac{e}{x}\right)^{2x}+C$

$\alpha = \beta = \gamma = \delta = 2$

$\alpha + 2\beta + 3\gamma - 4\delta = 4$

4. Let the image of the point P(1, 2, 6) in the plane passing through the points A(1, 2, 0), B(1, 4, 1) and C(0, 5, 1) be Q (α, β, γ). Then $(\alpha^2+\beta^2+\gamma^2)$ is equal to :

(1) 65

(2) 70

(3) 76

(4) 62

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. Equation of plane $A(x-1)+B(y-2)+C(z-0)=0$

Put (1, 4, 1) $\Rightarrow 2B + C = 0$

Put (0, 5, 1) $\Rightarrow -A + 3B + C = 0$

Sub : $B - A = 0 \Rightarrow A = B,\ C = -2B$

$1(x-1)+1(y-2)-2(z-0)=0$

$x+y-2z-3=0$

Image is (α, β, γ) pt $\equiv (1, 2, 6)$

$$\frac{\alpha-1}{1}=\frac{\beta-2}{1}=\frac{\gamma-6}{-2}=\frac{-2(1+2-12-3)}{6}$$

$$\frac{\alpha-1}{1}=\frac{\beta-2}{1}=\frac{\gamma-6}{-2}=4$$

$\alpha=5, \beta=6, \gamma=-2 \Rightarrow \alpha^2+\beta^2+\gamma^2$

$=25+36+4=65$

5. Let A = {2, 3, 4} and B = {8, 9, 12}. Then the number of elements in the relation

R = $\{((a_1, b_1), (a_2, b_2)) \in (A \times B, A \times B) : a_1$ divides b_2 and a_2 divides $b_1\}$ is :

(1) 36

(2) 12

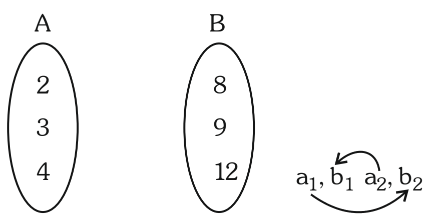
(3) 18

(4) 24

Official Ans. by NTA (1)

Allen Ans. (1)

Sol.



a_1 divides b_2

Each element has 2 choices

$\Rightarrow 3 \times 2 = 6$

a_2 divides b_1

Each element has 2 choices

$\Rightarrow 3 \times 2 = 6$

Total = 6 × 6 = 36

6. If $A=\frac{1}{5!6!7!}\begin{bmatrix} 5! & 6! & 7! \\ 6! & 7! & 8! \\ 7! & 8! & 9! \end{bmatrix}$, then $|adj(adj(2A))|$ is equal to :

(1) 2^8

(2) 2^{12}

(3) 2^{20}

(4) 2^{16}

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $|adj\,adj(2A)|=|2A|^{(n-1)^2}$

$=|2A|^4$

$=(2^3|A|)^4$

$= 2^{12}|A|^4 \Rightarrow 2^{16}$

$$|A| = \frac{1}{5!6!7!}5!6!\begin{vmatrix} 1 & 6 & 42 \\ 1 & 7 & 56 \\ 1 & 8 & 72 \end{vmatrix}$$

$R_3 \to R_3 \to R_2$

$R_2 \to R_2 \to R_1$

$$|A| = \begin{vmatrix} 1 & 8 & 42 \\ 0 & 1 & 14 \\ 0 & 1 & 16 \end{vmatrix} = 2$$

7. Let A be the point (1, 2) and B be any point on the curve $x^2+y^2=16$. If the centre of the locus of the point P, which divides the line segment AB in the ratio 3 : 2 is the point C (α, β), then the length of the line segment AC is

(1) $\frac{6\sqrt{5}}{5}$

(2) $\frac{4\sqrt{5}}{5}$

(3) $\frac{2\sqrt{5}}{5}$

(4) $\frac{3\sqrt{5}}{5}$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. A(1, 2) P(h,k) B(4cosθ, 4sinθ)

$\frac{12\cos\theta + 2}{5} = h \quad \Rightarrow \quad 12\cos\theta = 5h - 2$

$\frac{12\sin\theta + 4}{5} = k \quad \Rightarrow \quad 12\sin\theta = 5k \ \ 4$

Sq & add :

144 = (5h – 2)2 + (5k – 4)2

$\left(x - \frac{2}{5}\right)^2 + \left(y - \frac{4}{5}\right)^2 = \frac{144}{25}$

Centre $\equiv \left(\frac{2}{5}, \frac{4}{5}\right) \equiv (\alpha, \beta)$

$AC = \sqrt{\left(1 - \frac{2}{5}\right)^2 + \left(2 - \frac{4}{5}\right)^2}$

$= \sqrt{\frac{9}{25} + \frac{36}{25}} = \frac{\sqrt{45}}{5} = \frac{3\sqrt{5}}{5}$

8. Let a die be rolled n times. Let the probability of getting odd numbers seven times be equal to the probability of getting odd numbers nine times. If the probability of getting even numbers twice is $\frac{k}{2^{15}}$, then k is equal to :

(1) 30

(2) 90

(3) 15

(4) 60

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. P(odd number 7 times) = P(odd number 9 times)

$^nC_7\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^{n-7} = {}^nC_9\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^{n-9}$

$^nC_7 = {}^nC_9$

$\Rightarrow \ n = 16$

Required

$P = {}^{16}C_2 \times \left(\frac{1}{2}\right)^{16}$

$= \frac{16 \cdot 15}{2} \times \frac{1}{2^{16}} = \frac{15}{2^{13}}$

$\Rightarrow \ \frac{60}{2^{15}} \ \Rightarrow \ k = 60$

9. Let $g(x) = f(x) + f(1-x)$ and $f''(x) > 0, x \in (0,1)$. If g is decreasing in the interval (0, α) and increasing in the interval (α, 1), then $\tan^{1}(2\alpha) + \tan^{-1}\left(\frac{1}{\alpha}\right) + \tan^{-1}\left(\frac{\alpha+1}{\alpha}\right)$ is equal to :

(1) $\frac{3\pi}{2}$

(2) π

(3) $\frac{5\pi}{4}$

(4) $\frac{3\pi}{4}$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $g(x) = f(x) + f(1-x)$ & $f''(x) > 0, x \in (0, 1)$

$g'(x) = f'(x) - f'(1-x) = 0$

$\Rightarrow \quad f'(x) = f'(1-x)$

$x = 1 - x$

$x = \frac{1}{2}$

$g'(x) = 0$

at $x = \frac{1}{2}$

$g''(x) = f''(x) + f''(1-x) > 0$

g is concave up

hence $\alpha = \frac{1}{2}$

$\tan^{-1}2\alpha + \tan^{-1}\frac{1}{\alpha} + \tan^{-1}\frac{\alpha+1}{\alpha}$

$\Rightarrow \quad \tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$

10. Let a circle of radius 4 be concentric to the ellipse $15x^2 + 19y^2 = 285$. Then the common tangents are inclined to the minor axis of the ellipse at the angle.

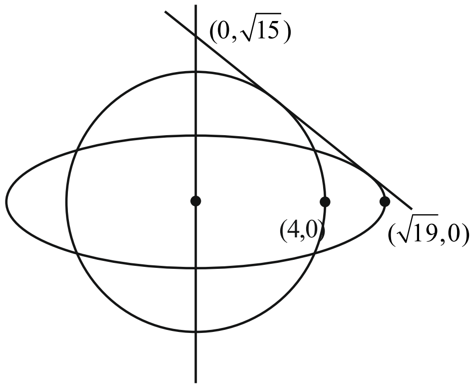
(1) $\frac{\pi}{4}$ (2) $\frac{\pi}{3}$

(3) $\frac{\pi}{12}$ (4) $\frac{\pi}{6}$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $\frac{x^2}{19}+\frac{y^2}{15}=1$



Let tang be

$y=mx\pm\sqrt{19m^2+15}$

$mx-y\pm\sqrt{19m^2+15}=0$

Parallel from (0, 0) = 4

$\left|\frac{\pm\sqrt{19m^2+15}}{\sqrt{m^2+1}}\right|=4$

$19m^2+15=16m^2+16$

$3m^2=1$

$m=\pm\frac{1}{\sqrt{3}}$

$\theta=\frac{\pi}{6}$ with x-axis

Required angle $\frac{\pi}{3}$.

11. Let $\vec{a}=2\hat{i}+7\hat{j}-\hat{k}$, $\vec{b}=3\hat{i}+5\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}+2\hat{k}$. Let $\vec{d}$ be a vector which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c}\cdot\vec{d}=12$. Then $\left(-\hat{i}+\hat{j}-\hat{k}\right)\cdot(\vec{c}\times\vec{d})$ is equal to

(1) 48

(2) 42

(3) 44

(4) 24

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $\vec{a}=2\hat{i}+7\hat{j}-\hat{k}$

$\vec{b}=3\hat{i}+5\hat{k}$

$\vec{c}=\hat{i}-\hat{j}+2\hat{k}$

$$\vec{d}=\lambda(\vec{a}\times\vec{b})=\lambda\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 7 & -1\\ 3 & 0 & 5\end{vmatrix}$$

$\vec{d}=\lambda(35\hat{i}-13\hat{j}-21\hat{k})$

$\lambda(35+13-42)=12$

$\lambda=2$

$\vec{d}=2(35\hat{i}-13\hat{j}-21\hat{k})$

$(\hat{i}+\hat{j}-\hat{k})(\vec{c}\times\vec{d})$

$$=\begin{vmatrix}-1 & 1 & -1\\ 1 & -1 & 2\\ 70 & -26 & -42\end{vmatrix}=44$$

12. If $S_n=4+11+21+34+50+\ldots$ to n terms, then $\frac{1}{60}(S_{29}-S_9)$ is equal to

(1) 226 (2) 220

(3) 223 (4) 227

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $S_n=4+11+21+34+50+\ldots+n$ terms

Difference are in A.P.

Let $T_n=an^2+bn+c$

$T_1=a+b+c=4$

$T_2=4a+2b+c=11$

$T_3=9a+3b+c=21$

By solving these 3 equations

$a=\frac{3}{2}, b=\frac{5}{2}, c=0$

So $T_n=\frac{3}{2}n^2+\frac{5}{2}n$

$S_n=\Sigma T_n$

$=\frac{3}{2}\Sigma n^2+\frac{5}{2}\Sigma n$

$=\frac{3}{2}\frac{n(n+1)(2n+1)}{6}=\frac{5}{2}\frac{(n)(n+1)}{2}$

$=\frac{n(n+1)}{4}[2n+1+5]$

$S_n=\frac{n(n+1)}{4}(2n+6)=\frac{n(n+1)(n+3)}{2}$

$\frac{1}{60}\left(\frac{29\times30\times32}{2}-\frac{9\times10\times12}{2}\right)=223$

13. If the points P and Q are respectively the circumcentre and the orthocentre of a ΔABC, then $\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}$ is equal to

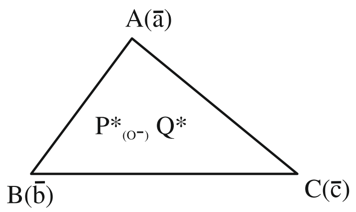
(1) $2\overrightarrow{QP}$ (2) $\overrightarrow{QP}$

(3) $2\overrightarrow{PQ}$ (4) $\overrightarrow{PQ}$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol.



$\overline{PA}+\overline{PB}+\overline{PC}=\bar{a}+\bar{b}+\bar{c}$

$\overline{PG}=\dfrac{\bar{a}+\bar{b}+\bar{c}}{3}$

$\Rightarrow \bar{a}+\bar{b}+\bar{c}=3\overline{PG}=\overline{PQ}$

Ans. (4)

14. The statement $\sim[p \vee (\sim(p \wedge q))]$ is equivalent to

(1) $(\sim(p \wedge q)) \wedge q$

(2) $\sim(p \wedge q)$

(3) $\sim(p \vee q)$

(4) $(p \wedge q) \wedge (\sim p)$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $\sim[pv(\sim(p \wedge q))]$

$\sim p \wedge (p \wedge q)$

15. Let $S=\left\{x \in\left(-\frac{\pi}{2},\frac{\pi}{2}\right): 9^{1-\tan^2 x}+9^{\tan^2 x}=10\right\}$ and $\beta=\sum\limits_{x\in s}\tan^2\left(\frac{x}{3}\right)$, then $\frac{1}{6}(\beta-14)^2$ is equal to

(1) 32

(2) 8

(3) 64

(4) 16

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. Let $9^{\tan^2 x}=P$

$\frac{9}{P}+P=10$

$P^2-10P+9=0$

$(P-9)(P-1)=0$

$P=1, 9$

$9^{\tan^2 x}=1,\ 9^{\tan^2 x}=9$

$\tan^2 x=0, \tan^2 x=1$

$x=0,\pm\frac{\pi}{4} \quad \therefore x\in\left(-\frac{\pi}{2},\frac{p}{2}\right)$

$\beta=\tan^2(0)+\tan^2\left(+\frac{\pi}{12}\right)+\tan^2\left(-\frac{\pi}{12}\right)$

$=0+2(\tan 15°)^2$

$2(2-\sqrt{3})^2$

$2(7-4\sqrt{3})$

Than $\frac{1}{6}(14-8\sqrt{3}-14)^2=32$

16. If the coefficients of x and x^2 in $(1 + x)^p (1 - x)^q$ are 4 and –5 respectively, then 2p + 3q is equal to

(1) 63

(2) 69

(3) 66

(4) 60

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $(1+x)^P(1-x)^q$

$\left(1+px+\frac{p(p-1)}{2!}x^2+\ldots\right)$

$\left(1-qx+\frac{q(q-1)}{2!}x^2-\ldots\right)$

$p-q=4$

$\frac{p(p-1)}{2}+\frac{q(q-1)}{2}-pq=-5$

$p^2+q^2-p-q-2pq=-10$

$(q+4)^2+q^2-(q+4)-q-2(4+q)q=-10$

$q^2+8q+16-q^2-q-4-q-8q-2q^2=-10$

$-2q=-22$

$q=11$

$p=15$

$2(15)+3(11)$

$30+33=63$

17. Let the line $\frac{x}{1}=\frac{6-y}{2}=\frac{z+8}{5}$ intersect the lines $\frac{x-5}{4}=\frac{y-7}{3}=\frac{z+2}{1}$ and $\frac{x+3}{6}=\frac{3-y}{3}=\frac{z-6}{1}$ at the points A and B respectively. Then the distance of the mid-point of the line segment AB from the plane $2x-2y+z=14$ is

(1) 4
(2) $\frac{10}{3}$
(3) 3
(4) $\frac{11}{3}$

Official Ans. by NTA (1)
Allen Ans. (1)

Sol. $\frac{x}{1}=\frac{y-6}{-2}=\frac{z+8}{5}=\lambda$ .... (1)

$\frac{x-5}{4}=\frac{y-7}{3}=\frac{z+2}{1}=\mu$ ....(2)

$\frac{x+3}{4}=\frac{y-3}{-3}=\frac{z-6}{1}=\gamma$ .... (3)

Intersection of (1) & (2) "A"

$(\lambda,-2\lambda+6,5\lambda-8)$ & $(4\mu+5,3\mu+7,\mu-2)$

$\lambda=1,\ \mu=-1$

A(1, 4, –3)

Intersection of (1) & (3) "B"

$(\lambda,-2\lambda+6,5\lambda-8)$ & $(6\gamma-3,-3\gamma+3,\gamma+6)$

$\lambda=3$

$\gamma=1$

B(3, 0, 7)

Mid point of A & B $\Rightarrow$ (2, 2, 2)

Perpendicular distance from the plane

$2x-2y+z=14$

$\Rightarrow \left|\frac{2(2)-2(2)+2-14}{\sqrt{4+4+1}}\right|=4$

18. Let $S=\left\{z=x+iy:\frac{2z-3i}{4z+2i}\text{ is a real number}\right\}$.

Then which of the following is <u>NOT</u> correct?

(1) $y+x^2+y^2\neq-\frac{1}{4}$
(2) $x=0$
(3) $(x,y)=\left(0,-\frac{1}{2}\right)$
(4) $y\in\left(-\infty,-\frac{1}{2}\right)\cup\left(-\frac{1}{2},\infty\right)$

Official Ans. by NTA (3)
Allen Ans. (3)

Sol. $\frac{2z-3i}{qz+2i}\in R$

$\frac{2(x+iy)-3i}{4(x+it)+2i}=\frac{2x+(2y-3)i}{4x+(4y+2)i}\times\frac{4x-(4y+2)i}{4x-(4y+2)i}$

$4x(2y-3)-2x(4y+2)=0$

$x=0$ $\qquad y\neq-\frac{1}{2}$

Ans. = 3

19. Let the number $(22)^{2022}+(2022)^{22}$ leave the remainder α when divided by 3 and β when divided by 7. Then $(\alpha^2+\beta^2)$ is equal to

(1) 10
(2) 5
(3) 20
(4) 13

Official Ans. by NTA (2)
Allen Ans. (2)

Sol. $(22)^{2022}+(2022)^{22}$

divided by 3

$(21+1)^{2022}+(2022)^{22}$

$=3k+1$ $\qquad(\alpha=1)$

Divided by 7

$(21+1)^{2022}+(2023-1)^{22}$

$7k+1+1$ $\qquad(\beta=2)$

$7k+2$

So $\alpha^2+\beta^2\Rightarrow 5$

20. Let μ be the mean and σ be the standard deviation of the distribution

x_i	0	1	2	3	4	5
f_i	$k+2$	$2k$	k^2-1	k^2-1	k^2+1	$k-3$

where $\sum f_i=62$. if [x] denotes the greatest integer $\leq x$, then $[\mu^2+\sigma^2]$ is equal

(1) 8
(2) 7
(3) 6
(4) 9

Official Ans. by NTA (1)
Allen Ans. (1)

Sol. $\sum f_i = 62$

$\Rightarrow$ $3k^2 + 16k - 12k - 64 = 0$

$\Rightarrow$ $k = \text{or } -\frac{16}{3}$ (rejected)

$$\mu = \frac{\sum f_i x_i}{\sum f_i}$$

$$\mu = \frac{8 + 2(15) + 3(15) + 4(17) + 5}{62} = \frac{156}{62}$$

$$\sigma^2 = \sum f_i x_i^2 - \left(\sum f_i x_i\right)^2$$

$$= \frac{8 \times 1^2 + 15 \times 13 + 17 \times 16 + 25}{62} - \left(\frac{156}{62}\right)^2$$

$$\sigma^2 = \frac{500}{62} - \left(\frac{156}{62}\right)^2$$

$$\sigma^2 + \mu^2 = \frac{500}{62}$$

$$\left[\sigma^2 + \mu^2\right] = 8$$

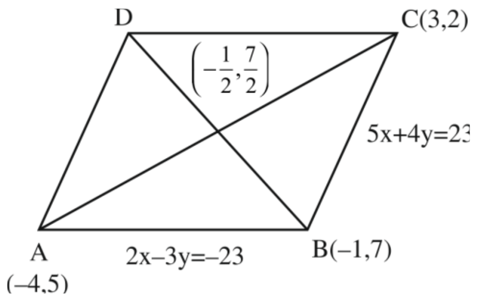
SECTION-B

21. Let the equations of two adjacent sides of a parallelogram ABCD be $2x - 3y = -23$ and $5x + 4y = 23$. If the equation of its one diagonal AC is $3x + 7y = 23$ and the distance of A from the other diagonal is d, then $50\ d^2$ is equal to _____ .

Official Ans. by NTA (529)

Allen Ans. (529)

Sol.



A & C point will be $(-4, 5)$ & $(3, 2)$

mid point of AC will be $\left(-\frac{1}{2}, \frac{7}{2}\right)$

equation of diagonal BD is

$$y - \frac{7}{2} = \frac{\frac{7}{2}}{-\frac{1}{2}}\left(x + \frac{1}{2}\right)$$

$\Rightarrow$ $7x + y = 0$

Distance of A from diagonal BD

$$= d = \frac{23}{\sqrt{50}}$$

$\Rightarrow$ $50d^2 = (23)^2$

$50d^2 = 529$

22. Let S be the set of values of λ, for which the system of equations

$6\lambda x - 3y + 3z = 4\lambda^2$,

$2x + 6\lambda y + 4z = 1$,

$3x + 2y + 3\lambda z = \lambda$ has no solution. Then $12 \sum_{\lambda \in S} |\lambda|$ is equal to _________ .

Official Ans. by NTA (24)

Allen Ans. (24)

Sol. $\Delta = \begin{vmatrix} 6\lambda & -3 & 3 \\ 2 & 6\lambda & 4 \\ 3 & 2 & 3\lambda \end{vmatrix} = 0$ (For No Solution)

$2\lambda\ (9\lambda^2 - 4) + (3\lambda - 6) + (2 - 9\lambda) = 0$

$18\lambda^3 - 14\lambda - 4 = 0$

$(\lambda - 1)(3\lambda + 1)(3\lambda + 2) = 0$

$\Rightarrow \lambda = 1, -1/3, -2/3$

For each λ, $\Delta_1 = \begin{vmatrix} 6\lambda & -3 & 4\lambda^2 \\ 2 & 6\lambda & 1 \\ 3 & 2 & \lambda \end{vmatrix} \neq 0$

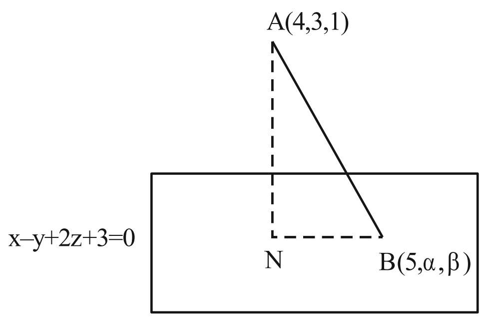
Ans. $12\left(1 + \frac{1}{3} + \frac{2}{3}\right) = 24$

23. Let the foot of perpendicular from the point A(4, 3, 1) on the plane $P : x - y + 2z + 3 = 0$ be N. If $B(5, \alpha, \beta)$, $\alpha, \beta \in \mathbb{Z}$ is a point on plane P such that the area of the triangle ABN in $3\sqrt{2}$, then $\alpha^2 + \beta^2 + \alpha\beta$ is equal to _________ .

Official Ans. by NTA (7)

Allen Ans. (7)

Sol.



$AN = \sqrt{6}$

$5 - \alpha + 2\beta + 3 = 0$

$\Rightarrow$ $\alpha = 8 + 2\beta$ …. (1)

N is given by

$$\frac{x-4}{1}=\frac{y-3}{-1}=\frac{z-1}{2}=\frac{-(4-3+2+3)}{1+1+4}$$

$\Rightarrow$ $x = 3, y = 4, z = -1$

$\Rightarrow$ N is $(3, 4, -1)$

$$BN=\sqrt{4+(\alpha-4)^2+(\beta+1)^2}$$

$$=\sqrt{4+(2\beta+4)^2+(\beta+1)^2}$$

Area of $\Delta ABN = \frac{1}{2}AN\times BN = 3\sqrt{2}$

$\Rightarrow$ $\frac{1}{2}\times\sqrt{6}\times BN = 3\sqrt{2}$

$BN = 2\sqrt{3}$

$\Rightarrow$ $4 + (2\beta + 4)^2 + (\beta + 1)^2 = 12$

$(2\beta + 4)^2 + (\beta + 1)^2 - 8 = 0$

$5\beta^2 + 18\beta + 9 = 0$

$(5\beta + 3)(\beta + 3) = 0$

$\beta = -3$

$\Rightarrow$ $\alpha = 2$

$\Rightarrow$ $\alpha^2 + \beta^2 + \alpha\beta = 9 + 4 - 6 = 7$

24. Let quadratic curve passing through the point $(-1, 0)$ and touching the line $y = x$ at $(1, 1)$ be $y = f(x)$. Then the x-intercept of the normal to the curve at the point $(\alpha, \alpha + 1)$ in the first quadrant is _____ .

Official Ans. by NTA (11)

Allen Ans. (11)

Sol. $f(x) = (x + 1)(ax + b)$

$1 = 2a + 2b$ ____(1)

$f'(x) = (ax + b) + a(x + 1)$

$1 = (3a + b)$ _____(2)

$\Rightarrow b = 1/4, a = 1/4$

$$f(x)=\frac{(x+1)^2}{4}$$

$$f'(x)=\frac{x}{2}+\frac{1}{2} \qquad \alpha+1=\frac{(\alpha+1)^2}{4}, \alpha>-1$$

$\alpha + 1 = 4$

$\alpha = 3$

normal at $(3, 4)$

$$y-4=-\frac{1}{2}(x-3)$$

$y = 0$ $\qquad$ $x = 8 + 3$

Ans. 11

25. Let the tangent at any point P on a curve passing through the points $(1, 1)$ and $\left(\frac{1}{10},100\right)$, intersect positive x-axis and y-axis at the points A and B respectively. If $PA : PB = 1 : k$ and $y = y(x)$ is the solution of the differential equation $e^{\frac{dy}{dx}} = kx + \frac{k}{2}$, $y(0) = k$, then $4y(1) - 5\log_e 3$ is equal to _____ .

Official Ans. by NTA (6)

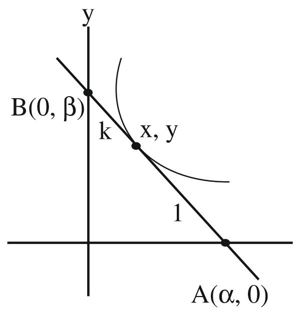
Allen Ans. (5) (answer is $4+\ell n3$)

Sol. equation of tangent at P (x, y)

$$Y-y=\frac{dy}{dx}(X-x)$$

$Y = 0$

$$X=\frac{-ydx}{dy}+x$$



$$\frac{k\alpha+0}{k+1}=x,\ \alpha=\frac{k+1}{k}x$$

$$\frac{k+1}{k}x=-y\frac{dx}{dy}+x$$

$$x+\frac{x}{k}=-y\frac{dx}{dy}+x$$

$$x\frac{dy}{dx}+ky=0 \qquad \frac{dy}{dx}+\frac{k}{x}y=0$$

$y.\ x^k = C$

$C = 1$

$$100.\left(\frac{1}{10}\right)^k=1$$

$K = 2$

$$\frac{dy}{dx}=\ell n(2x+1)$$

$$y=\frac{2x+1}{2}(\ell n(2x+1)-1)+c$$

$$2=\frac{1}{2}(0-1)+C$$

$$C=2+\frac{1}{2}=\frac{5}{2}$$

$$y(1)=\frac{3}{2}(\ell n3-1)+\frac{5}{2}$$

$$=\frac{3}{2}\ell n3+1$$

$4y(1) = 6\ell n3 + 4$

$4y(1) - 5\ell n3 = 4 + \ell n3$

26. Suppose a_1, a_2, 2, a_3, a_4 be in an arithmetico-geometric progression. If the common ratio of the corresponding geometric progression is 2 and the sum of all 5 terms of the arithmetico-geometric progression is $\frac{49}{2}$, then a_4 is equal to ________ .

Official Ans. by NTA (16)

Allen Ans. (16)

Sol. $\frac{(a-2d)}{4}, \frac{(a-d)}{2}$, a, 2(a + d), 4(a + 2d)

a = 2

$$\left(\frac{1}{4}+\frac{1}{2}+1+6\right)\times 2+(-1+2+8)d=\frac{49}{2}$$

$$2\left(\frac{3}{4}+7\right)+9d=\frac{49}{2}$$

$$9d=\frac{49}{2}-\frac{62}{4}=\frac{98-62}{4}=9$$

d = 1

$\Rightarrow a_4 = 4\ (a + 2d)$

$= 16$

27. If the domain of the function $f(x)=\sec^{-1}\left(\frac{2x}{5x+3}\right)$ is $[\alpha,\beta)\cup(\gamma,\delta]$, then $|3\alpha+10(\beta+\gamma)+21\delta|$ is equal to ________ .

Official Ans. by NTA (24)

Allen Ans. (24)

Sol. $f(x) = \sec^{-1}\frac{2x}{5x+3}$

$$\left|\frac{2x}{5x+3}\right|$$

$$\left|\frac{2x}{5x+3}\right|\geq 1\Rightarrow |2x|\geq|5x+3|$$

$(2x)^2 - (5x+3)^2 \geq |5x+3|$

$(7x + 3)\ (-3x - 3) \geq 0$

$$\begin{array}{ccc} - & + & - \\ \hline & -1 & -\frac{3}{7} \end{array}$$

$\therefore$ domain $\left[-1,\frac{-3}{5}\right)\cup\left(\frac{-3}{5},\frac{-3}{7}\right]$

$$\alpha=-1,\ \beta=\frac{-3}{5},\ \ \gamma=\frac{-3}{5},\ \ \delta=\frac{-3}{7}$$

$3\alpha + 10\ (\beta + \gamma) + 21\delta = -3$

$$-3+10\left(\frac{-6}{5}\right)+\left(\frac{-3}{7}\right)21=-24$$

28. The sum of all the four-digit numbers that can be formed using all the digits 2, 1, 2, 3 is equal to ________ .

Official Ans. by NTA (26664)

Allen Ans. (26664)

Sol. **2, 1, 2, 3**

$\_\ \_\ \_\ \ \underline{1}\ \ \frac{3!}{2!}=3$

$\_\ \_\ \_\ \ \underline{2}\ \ 3!=6$

$\_\ \_\ \_\ \ \underline{3}\ \ \frac{3!}{2!}=3$

Sum of digits of unit place = 3 × 1 + 6 × 2 + 3 × 3 = 24

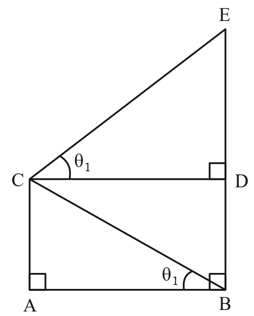
$\therefore$ required sum

= 24 ×1000 + 24 × 100 + 24 × 10 + 24 × 1

= 24 ×1111

Ans ; 26664

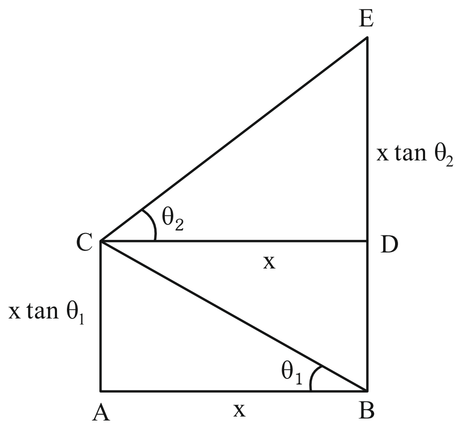
29. In the figure, $\theta_1+\theta_2=\frac{\pi}{2}$ and $\sqrt{3}(BE)=4(AB)$. If the area of ΔCAB is $2\sqrt{3}-3$ unit^2, when $\frac{\theta_2}{\theta_1}$ is the largest, then the perimeter (in unit) of ΔCED is equal to ___________.



Official Ans. by NTA (6)

Allen Ans. (6)

Sol.



$\sqrt{3}$ BE = 4 AB

Ar (ΔCAB) = $2\sqrt{3}-3$

$\frac{1}{2}x^2\tan\theta_1 = 2\sqrt{3}-3$

BE = BD + DE

= x ($\tan\theta_1 + \tan\theta_2$)

BE = AB ($\tan\theta_1 + \cot\theta_1$)

$\frac{4}{\sqrt{3}}$ $\tan\theta_1 + \cot\theta_1 \Rightarrow \tan\theta_1 = \sqrt{3}, \frac{1}{\sqrt{3}}$

$\theta_1 = \frac{\pi}{6}$ $\qquad$ $\theta_2 = \frac{\pi}{3}$

$\theta_1 = \frac{\pi}{3}$ $\qquad$ $\theta_2 = \frac{\pi}{6}$

as $\frac{\theta_2}{\theta_1}$ is largest $\therefore$ $\theta_1 = \frac{\pi}{6}$ $\theta_2 = \frac{\pi}{3}$

$$\therefore x^2 = \frac{\left(2\sqrt{3}-3\right)\times 2}{\tan\theta_1} = \frac{\sqrt{3}\left(2-\sqrt{3}\right)\times 2}{\tan\frac{\pi}{6}}$$

$x^2 = 12-6\sqrt{3} = \left(3-\sqrt{3}\right)^2$

$x = 3-\sqrt{3}$

Perimeter of ΔCED

= CD + DE + CE

$= 3\sqrt{3}+\left(3-\sqrt{3}\right)\sqrt{3}+\left(3-\sqrt{3}\right)\times 2 = 6$

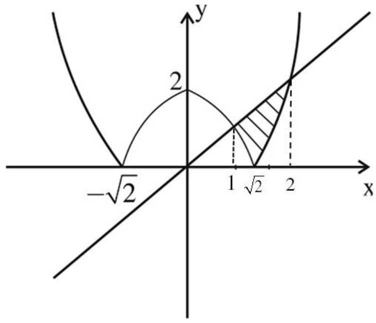
Ans : 6

30. If the area of the region $\left\{(x,y) : \left|x^2-2\right| \le y \le x\right\}$ is A, then $6A+16\sqrt{2}$ is equal to ________ .

Official Ans. by NTA (27)

Allen Ans. (27)

Sol. $|x^2-2| \le y \le x$



$$A = \int_1^{\sqrt{2}}(x-(2-x^2))dx + \int_{\sqrt{2}}^{2}(x-(x^2-2))dx$$

$$= \left(1-2\sqrt{2}+\frac{2\sqrt{2}}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right)+\left(2-\frac{8}{3}+4\right)-\left(1-\frac{2\sqrt{2}}{3}+2\sqrt{2}\right)$$

$$= -4\sqrt{2}+\frac{4\sqrt{2}}{3}+\frac{7}{6}+\frac{10}{3} = \frac{-8\sqrt{2}}{3}+\frac{9}{2}$$

$6A = -16\sqrt{2}+27 \therefore 6A+16\sqrt{2} = 27$

Ans : 27