FINAL JEE–MAIN EXAMINATION – APRIL, 2023

(Held On Tuesday 11th April, 2023) **TIME : 3 : 00 PM to 6 : 00 PM**

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

1. If $\begin{vmatrix} x+1 & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda^2 \end{vmatrix} = \frac{9}{8}(103x+81)$, then λ, $\frac{\lambda}{3}$ are the roots of the equation

(1) $4x^2 + 24x - 27 = 0$ (2) $4x^2 - 24x + 27 = 0$

(3) $4x^2 + 24x + 27 = 0$ (4) $4x^2 - 24x - 27 = 0$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. Put x = 0

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^2 \end{vmatrix} = \frac{9}{8}\times 81$$

$$\lambda^3 = \frac{9^3}{8} \therefore \lambda = \frac{9}{2}$$

$$\therefore \frac{\lambda}{3} = \frac{3}{2}$$

$$\therefore \text{Required equation is : } x^2 - x\left(\frac{9}{2}+\frac{3}{2}\right)x + \frac{27}{4} = 0$$

$$4x^2 - 24x + 27 = 0$$

2. Let the line passing through the points, P (2, – 1, 2) and Q (5, 3, 4) meet the plane x – y + z = 4 at the point R. Then the distance of the point R from the plane x + 2y + 3z + 2 = 0 measured parallel to the line $\frac{x-7}{2} = \frac{y+3}{2} = \frac{z-2}{1}$ is equal to

(1) $\sqrt{31}$ (2) $\sqrt{189}$

(3) $\sqrt{61}$ (4) 3

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. Line : $\frac{x-5}{3} = \frac{y-3}{4} = \frac{z-4}{2} = \lambda$

$$R(3\lambda+5, 4\lambda+3, 2\lambda+4)$$

$$\therefore 3\lambda + 5 - 4\lambda - 3 + 2\lambda + 4 = 4$$

$$\lambda + 6 = 4 \therefore \lambda = -2$$

$$\therefore R \equiv (-1, -5, 0)$$

Line : $\frac{x+1}{2} = \frac{y+5}{2} = \frac{z-0}{1} = \mu$

Point $T = (2\mu-1, 2\mu-5, \mu)$

It lies on plane

$$2\mu - 1 + 2(2\mu-5) + 3\mu + 2 = 0$$

$$\mu = 1$$

$$\therefore T = (1, -3, 1)$$

$$\therefore RT = 3$$

3. If the 1011th term from the end in the binomial expansion of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^{2022}$ is 1024 times 1011th term from the beginning, then |x| is equal to

(1) 12 (2) 8

(3) 10 (4) 15

Official Ans. by NTA (3)

Allen Ans. (BONUS)

Sol. T_{1011} from beginning $= T_{1010+1}$

$$= {}^{2022}C_{1010}\left(\frac{4x}{5}\right)^{1012}\left(\frac{-5}{2x}\right)^{1010}$$

T_{1011} from end

$$= {}^{2022}C_{1010}\left(\frac{-5}{2x}\right)^{1012}\left(\frac{4x}{5}\right)^{1010}$$

Given : ${}^{2022}C_{1010}\left(\frac{-5}{2x}\right)^{1012}\left(\frac{4x}{5}\right)^{1010}$

$$= 2^{10}\cdot {}^{2022}C_{1010}\left(\frac{-5}{2x}\right)^{1010}\left(\frac{4x}{5}\right)^{1012}$$

$$\left(\frac{-5}{2x}\right)^2 = 2^{10}\left(\frac{4x}{5}\right)^2$$

$$x^4 = \frac{5^4}{2^{16}}$$

$$|x| = \frac{5}{16}$$

4. Let the function $f : [0, 2] \rightarrow R$ be defined as

$$f(x)=\begin{cases} e^{\min\{x^2, x-[x]\}}, & x\in[0,1) \\ e^{[x-\log_e x]}, & x\in[1,2] \end{cases}$$

where [t] denotes the greatest integer less than or equal to t. Then the value of the integral $\int_0^2 xf(x)dx$ is

(1) $2e - 1$ (2) $1+\frac{3e}{2}$

(3) $2e-\frac{1}{2}$ (4) $(e-1)\left(e^2+\frac{1}{2}\right)$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. Minimum $\{x^2, \{x\}\} = x^2$; $x\in[0,1)$

$$[x-\log_e x]=1 ; x\in[1,2)$$

$$\therefore f(x)=\begin{cases} e^{x^2} ; x\in[0,1) \\ e ; x\in[1,2) \end{cases}$$

$$\int_0^2 xf(x)dx=\int_0^1 xe^{x^2}dx+\int_1^2 ex\ dx$$

$$=\frac{1}{2}(e-1)+\frac{1}{2}(4-1)e$$

$$=2e-\frac{1}{2}$$

5. Let y = y (x) be the solution of the differential equations $\frac{dy}{dx}+\frac{5}{x(x^5+1)}y=\frac{(x^5+1)^2}{x^7}, x>0$. If $y(1) = 2$, then $y(2)$ is equal to

(1) $\frac{637}{128}$ (2) $\frac{679}{128}$

(3) $\frac{693}{128}$ (4) $\frac{697}{128}$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $I.F=e^{\int\frac{5dx}{x(x^5+1)}}=e^{\int\frac{5x^{-6}dx}{(x^{-5}+1)}}$

Put, $1+x^{-5}=t\Rightarrow -5x^{-6}dx=dt$

$$\Rightarrow e^{\int\frac{-dt}{t}}=\frac{1}{t}=\frac{x^5}{1+x^5}$$

$$y\cdot\frac{x^5}{1+x^5}=\int\frac{x^5}{(1+x^5)}\times\frac{(1+x^5)^2}{x^7}dx$$

$$=\int x^3dx+\int x^{-2}dx$$

$$y\cdot\frac{x^5}{1+x^5}=\frac{x^4}{4}-\frac{1}{x}+c$$

Given that : $x = 1 \Rightarrow y = 2$

$$2\cdot\frac{1}{2}=\frac{1}{4}-1+c$$

$$c=\frac{7}{4}$$

$$y\cdot\frac{x^5}{1+x^5}=\frac{x^4}{4}-\frac{1}{x}+\frac{7}{4}$$

Now put, $x = 2$

$$y\cdot\left(\frac{32}{33}\right)=\frac{21}{4}$$

$$y=\frac{693}{128}$$

6. If four distinct points with position vectors $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ are coplanar; then $[\vec{a}\,\vec{b}\,\vec{c}]$ is equal to

(1) $[\vec{d}\,\vec{c}\,\vec{a}]+[\vec{b}\,\vec{d}\,\vec{a}]+[\vec{c}\,\vec{d}\,\vec{b}]$

(2) $[\vec{d}\,\vec{b}\,\vec{a}]+[\vec{a}\,\vec{c}\,\vec{d}]+[\vec{d}\,\vec{b}\,\vec{c}]$

(3) $[\vec{a}\,\vec{d}\,\vec{b}]+[\vec{d}\,\vec{c}\,\vec{a}]+[\vec{d}\,\vec{b}\,\vec{c}]$

(4) $[\vec{b}\,\vec{c}\,\vec{d}]+[\vec{d}\,\vec{a}\,\vec{c}]+[\vec{d}\,\vec{b}\,\vec{a}]$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $\vec{a},\vec{b},\vec{c},\vec{d}$ are coplanar points.

$\vec{b}-\vec{a},\vec{c}-\vec{a},\vec{d}-\vec{a}$ are coplanar vectors.

So, $[\vec{b}-\vec{a}\ \ \vec{c}-\vec{a}\ \ \vec{d}-\vec{a}]=0$

$$(\vec{b}-\vec{a})\cdot((\vec{c}-\vec{a})\times(\vec{d}-\vec{a}))=0$$

$$[\vec{b}\ \vec{c}\ \vec{d}]-[\vec{b}\ \vec{c}\ \vec{a}]-[\vec{b}\ \vec{a}\ \vec{d}]-[\vec{a}\ \vec{c}\ \vec{d}]=0$$

$$\Rightarrow[\vec{a}\,\vec{b}\,\vec{c}]=[\vec{c}\,\vec{d}\,\vec{b}]+[\vec{b}\,\vec{d}\,\vec{a}]+[\vec{d}\ \vec{c}\ \vec{a}]$$

7. If $f:\mathbb{R}\rightarrow\mathbb{R}$ be a continuous function satisfying

$$\int_0^{\pi/2} f(\sin 2x)\cdot\sin x dx+\alpha\int_0^{\pi/4} f(\cos 2x)\cdot\cos x\,dx=0,$$

then α is equal to

(1) $-\sqrt{3}$ (2) $\sqrt{2}$

(3) $\sqrt{3}$ (4) $-\sqrt{2}$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $I=\int_0^{\frac{\pi}{4}} f(\sin 2x)\sin x\,dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} f(\sin 2x)\sin x\,dx$

$$+\alpha\int_0^{\frac{\pi}{4}} f(\cos 2x)\cos x\,dx=0$$

Apply king in first part and put $x-\frac{\pi}{4}=t$ in second part.

$$I=\int_0^{\frac{\pi}{4}} f(\cos 2x)\sin\left(\frac{\pi}{4}-x\right)dx+\int_0^{\frac{\pi}{4}} f(\cos 2t)\sin\left(\frac{\pi}{4}+t\right)dt$$

$$+\alpha\int_0^{\frac{\pi}{4}} f(\cos 2x)\cos x\,dx = 0$$

$$I=\int_0^{\frac{\pi}{4}} f(\cos 2x)\left[2\sin\frac{\pi}{4}\cdot\cos x+\alpha\cos x\right]dx=0$$

$$I=\left(\alpha+\sqrt{2}\right)\int_0^{\frac{\pi}{4}} f(\cos 2x)\cos x\,dx=0$$

$\therefore \alpha=-\sqrt{2}$

8. If the system of linear equations

$7x+11y+\alpha z=13$

$5x+4y+7z=\beta$

$175x+194y+57z=361$

has infinitely many solutions, then $\alpha+\beta+2$ is equal to

(1) 4 (2) 3

(3) 5 (4) 6

Official Ans. by NTA (1)
Allen Ans. (1)

Sol. $7x+11y+\alpha z=13$ ...... (i)

$5x+4y+7z=\beta$ ...... (ii)

$175x+194y+57z=361$ ..... (iii)

(i) × 10 + (ii) × 21 – (iii)

$z(10\alpha+147-57)=130+21\beta-361$

$\therefore 10\alpha+90=0$

$\alpha=-9$

$130-361+21\beta=0$

$\beta=11$

$\alpha+\beta+2=4$

9. The domain of the function

$f(x)=\dfrac{1}{\sqrt{[x]^2-3[x]-10}}$ is (where [x] denotes the greatest integer less than or equal to x)

(1) $(-\infty,-2)\cup(5,\infty)$ (2) $(-\infty,-3]\cup[6,\infty)$

(3) $(-\infty,-2)\cup[6,\infty)$ (4) $(-\infty,-3]\cup(5,\infty)$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $[x]^2-3[x]-10>0$

$[x]<-2 \text{ or } [x]>5$

10. Let P be the plane passing through the points (5, 3, 0), (13, 3, –2) and (1, 6, 2). For $\alpha\in N$, if the distances of the points A (3, 4, α) and B (2, α, a) from the plane P are 2 and 3 respectively, then the positive value of a is

(1) 6 (2) 4

(3) 3 (4) 5

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 8 & 0 & -2\\ 4 & -3 & -2\end{vmatrix}=\hat{i}(-6)+8\hat{j}-24\hat{k}$

Normal of the plane = $3\hat{i}-4\hat{j}+12\hat{k}$

Plane : $3x-4y+12z=3$

Distance from A (3, 4, α)

$\left|\dfrac{9-16+12\alpha-3}{13}\right|=2$

$\alpha=3$

$\alpha=-8(\text{rejected})$

Distance from B (2, 3, a)

$\left|\dfrac{6-12+12a-3}{13}\right|=3$

$a=4$

11. The converse of the statement $((\sim p)\wedge q)\Rightarrow r$ is

(1) $(\sim r)\Rightarrow p\wedge q$ (2) $(\sim r)\Rightarrow((\sim p)\wedge q)$

(3) $((\sim p)\vee q)\Rightarrow r$ (4) $(p\vee(\sim q))\Rightarrow(\sim r)$

Official Ans. by NTA (4)
Allen Ans. (4)

Sol. Converse of $((\sim p)\wedge q)\Rightarrow r$

$\equiv r\Rightarrow(\sim p\wedge q)$

$\equiv\sim r\vee(\sim p\wedge q)$

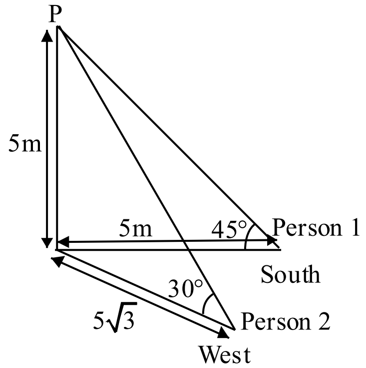
$\equiv\sim r\vee(p\vee\sim q)\equiv(p\vee\sim q)\Rightarrow\sim r$

12. The angle of elevation of the top P of a tower from the feet of one person standing due South of the tower is 45° and from the feet of another person standing due west of the tower is 30°. If the height of the tower is 5 meters, then the distance (in meters) between the two persons is equal to

(1) 10 (2) 5

(3) $5\sqrt{5}$ (4) $\frac{5}{2}\sqrt{5}$

Official Ans. by NTA (1)
Allen Ans. (1)



Sol.
Distance = 10 (By Pythagoras theorem)

13. Let a, b, c and d be positive real numbers such that a + b + c + d = 11. If the maximum value of $a^5b^3c^2d$ is 3750β, then the value of β is

(1) 90 (2) 110

(3) 55 (4) 108

Official Ans. by NTA (1)
Allen Ans. (1)

Sol. $$\frac{5\left(\frac{a}{5}\right)+3\left(\frac{b}{3}\right)+2\left(\frac{c}{2}\right)+d}{11} \geq \left(\frac{a^5b^3c^2d}{5^5 3^3 2^2}\right)^{1/11}$$

$$1 \geq \left(\frac{a^5b^3c^2d}{5^5 3^3 2^2}\right)^{1/11}$$

$\beta = 90$

14. If the radius of the largest circle with centre (2, 0) inscribed in the ellipse $x^2 + 4y^2 = 36$ is r, then $12r^2$ is equal to

(1) 72 (2) 115

(3) 92 (4) 69

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $(x-2)^2 + y^2 = r^2$

Solving with ellipse, we get

$$(x-2)^2 + \frac{36-x^2}{4} = r^2$$

$3x^2 - 16x + 52 - 4r^2 = 0$

$D = 0 \Rightarrow 4r^2 = \frac{92}{3}$

15. Let the mean of 6 observation 1, 2, 4, 5, x and y be 5 and their variance be 10. Then their mean deviation about the mean is equal to

(1) $\frac{10}{3}$ (2) $\frac{7}{3}$

(3) 3 (4) $\frac{8}{3}$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $x + y = 18 \ \{\because \text{mean} = 5\}$ ..... (i)

$$10 = \frac{1+4+16+25+x^2+y^2}{6} - 25$$

$x^2 + y^2 = 164$ ...... (ii)

By solving (i) and (ii)

x = 8, y = 10

$$\text{M.D.}(\bar{x}) = \frac{\sum|x_i - \bar{x}|}{6} = \frac{8}{3}$$

16. The sum of the coefficients of three consecutive terms in the binomial expansion of $(1+x)^{n+2}$, which are in the ratio 1 : 3 : 5, is equal to

(1) 25 (2) 63

(3) 41 (4) 92

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. ${}^{n+2}C_{r-1} : {}^{n+2}C_r : {}^{n+2}C_{r+1} = 1:3:5$

$$\frac{{}^{n+2}C_{r-1}}{{}^{n+2}C_r} = \frac{1}{3}$$

$n = 4r - 3$ ...... (i)

$$\frac{{}^{n+2}C_r}{{}^{n+2}C_{r+1}} = \frac{3}{5}$$

$8r - 1 = 3n$ ...... (ii)

From, (i) and (ii)

r = 2 and n = 5

Required sum = 63

17. If the letters of the word MATHS are permuted and all possible words so formed are arranged as in a dictionary with serial numbers, then the serial number of the word THAMS is

(1) 103 (2) 104

(3) 101 (4) 102

Official Ans. by NTA (1)
Allen Ans. (1)
Sol. $4 \times 4! + 1 \times 3! + 1 = 103$

18. For $a \in C$, let $A = \{z \in C : \text{Re}(a + \bar{z}) > \text{Im}(\bar{a} + z)\}$ and $B = \{z \in C : \text{Re}(a + \bar{z}) < \text{Im}(\bar{a} + z)\}$. Then among the two statements :
(S1) : If Re (A), Im (A) > 0, then the set A contains all the real numbers

(S2) : If Re (A), Im (A) < 0, then the set B contains all the real numbers,
(1) Only (S1) is true (2) both are false

(3) Only (S2) is true (4) Both are true

Official Ans. by NTA (2)
Allen Ans. (2)

Sol. Let $a = x_1 + iy_1$ $z = x + iy$

Now $\text{Re}(a + \bar{z}) > \text{Im}(\bar{a} + z)$

$\therefore x_1 + x > -y_1 + y$

$x_1 = 2, y_1 = 10, x = -12, y = 0$

Given inequality is not valid for these values.

S1 is false.

Now $\text{Re}(a + \bar{z}) < \text{Im}(\bar{a} + z)$

$x_1 + x < -y_1 + y$

$x_1 = -2, y_1 = -10, x = 12, y = 0$

Given inequality is not valid for these values.

S2 is false.

19. Let A = {1, 3, 4, 6, 9} and B = {2, 4, 5, 8, 10}. Let R be a relation defined on A × B such that R = $\{((a_1, b_1),(a_2, b_2)) : a_1 \le b_2 \text{ and } b_1 \le a_2\}$. Then the number of elements in the set R is

(1) 26 (2) 160

(3) 180 (4) 52

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. Let $a_1 = 1 \Rightarrow 5$ choices of b_2

$a_1 = 3 \Rightarrow 4$ choices of b_2

$a_1 = 4 \Rightarrow 4$ choices of b_2

$a_1 = 6 \Rightarrow 2$ choices of b_2

$a_1 = 9 \Rightarrow 1$ choices of b_2

For (a_1, b_2) 16 ways .

Similarly, $b_1 = 2 \Rightarrow 4$ choices of a_2

$b_1 = 4 \Rightarrow 3$ choices of a_2

$b_1 = 5 \Rightarrow 2$ choices of a_2

$b_1 = 8 \Rightarrow 1$ choices of a_2

Required elements in R = 160

20. Let f and g be two functions defined by

$$f(x) = \begin{cases} x+1, & x<0 \\ |x-1|, & x \ge 0 \end{cases} \text{ and } g(x) = \begin{cases} x+1, & x<0 \\ 1, & x \ge 0 \end{cases}.$$

Then (gof) (x) is

(1) Differentiable everywhere

(2) Continuous everywhere but not differentiable exactly at one point

(3) Not continuous at x = – 1

(4) Continuous everywhere but not differentiable at x = 1

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $$f(x) = \begin{cases} x+1, & x<0 \\ 1-x, & 0 \le x < 1 \\ x-1, & 1 \le x \end{cases}$$

$$g(x) = \begin{cases} x+1, & x<0 \\ 1, & x \ge 0 \end{cases}$$

$$g(f(x)) = \begin{cases} x+2, & x<-1 \\ 1, & x \ge -1 \end{cases}$$

$\therefore g(f(x))$ is continuous everywhere

$g(f(x))$ is not differentiable at x = – 1

Differentiable everywhere else

SECTION-B

21. The number of points, where the curve $f(x) = e^{8x} - e^{6x} - 3e^{4x} - e^{2x} + 1,\ x \in \mathbb{R}$ cuts x-axis, is equal to

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. Let $e^{2x} = t$

$$\Rightarrow t^4 - t^3 - 3t^2 - t + 1 = 0$$

$$\Rightarrow t^2 + \frac{1}{t^2} - \left(t + \frac{1}{t}\right) - 3 = 0$$

$$\Rightarrow \left(t + \frac{1}{t}\right)^2 - \left(t + \frac{1}{t}\right) - 5 = 0$$

$$\Rightarrow t + \frac{1}{t} = \frac{1+\sqrt{21}}{2}$$

Two real values of t.

22. Let the probability of getting head for a biased coin be $\frac{1}{4}$. It is tossed repeatedly until a head appears. Let N be the number of tosses required. If the probability that the equation $64x^2 + 5Nx + 1 = 0$ has no real root is $\frac{p}{q}$, where p and q are co-prime, then q – p is equal to

Official Ans. by NTA (27)

Allen Ans. (27)

Sol. $64x^2+5Nx+1=0$

$D=25N^2-256<0$

$\Rightarrow N^2<\frac{256}{25}\Rightarrow N<\frac{16}{5}$

$\therefore N=1,2,3$

$\therefore \text{Probability}=\frac{1}{4}+\frac{3}{4}\times\frac{1}{4}+\frac{3}{4}\times\frac{3}{4}\times\frac{1}{4}=\frac{37}{64}$

$\therefore q-p=27$

23. Let $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}-\hat{k}$. If $\vec{c}$ is a vector such that $\vec{a}\cdot\vec{c}=11$, $\vec{b}\cdot(\vec{a}\times\vec{c})=27$ and $\vec{b}\cdot\vec{c}=-\sqrt{3}|\vec{b}|$, then $|\vec{a}\times\vec{c}|^2$ is equal to

Official Ans. by NTA (285)

Allen Ans. (285)

Sol. $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}, \vec{b}=\hat{i}+\hat{j}-\hat{k}$

$\vec{b}\cdot(\vec{a}\times\vec{c})=27,\ \vec{a}\cdot\vec{b}=0$

$\vec{b}\times(\vec{a}\times\vec{c})=-3\vec{a}$

Let θ be angle between $\vec{b}, \vec{a}\times\vec{c}$

Then $|\vec{b}|\cdot|\vec{a}\times\vec{c}|\sin\theta=3\sqrt{14}$

$|\vec{b}|\cdot|\vec{a}\times\vec{c}|\cos\theta=27$

$\Rightarrow \sin\theta=\frac{\sqrt{14}}{\sqrt{95}}$

$\therefore |\vec{b}|\times|\vec{a}\times\vec{c}|=3\sqrt{95}$

$\Rightarrow |\vec{a}\times\vec{c}|=\sqrt{3}\times\sqrt{95}$

24. Let $S=\left\{z\in C-\{i,2i\}:\frac{z^2+8iz-15}{z^2-3iz-2}\in R\right\}$. If $\alpha-\frac{13}{11}i\in S, \alpha\in\mathbb{R}-\{0\}$, then $242\alpha^2$ is equal to

Official Ans. by NTA (1680)

Allen Ans. (1680)

Sol. $\left(\frac{z^2+8iz-15}{z^2-3iz-2}\right)\in R$

$\Rightarrow 1+\frac{(11iz-13)}{(z^2-3iz-2)}\in R$

Put $z=\alpha-\frac{13}{11}i$

$\Rightarrow (z^2-3iz-2)$ is imaginary

Put $z=x+iy$

$\Rightarrow (x^2-y^2+2xyi-3ix+3y-2)\in$ Imaginary

$\Rightarrow \text{Re}(x^2-y^2+3y-2+(2xy-3x)i)=0$

$\Rightarrow x^2-y^2+3y-2=0$

$x^2=y^2-3y+2$

$x^2=(y-1)(y-2)\ \therefore z=\alpha-\frac{13}{11}i$

Put $x=\alpha,\ y=\frac{-13}{11}$

$\alpha^2=\left(\frac{-13}{11}-1\right)\left(\frac{-13}{11}-2\right)$

$\alpha^2=\frac{(24\times35)}{121}$

$242\alpha^2=48\times35=1680$

25. For $k\in\mathbb{N}$, if the sum of the series $1+\frac{4}{k}+\frac{8}{k^2}+\frac{13}{k^3}+\frac{19}{k^4}+\ldots$ is 10, then the value of k is

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $10 = 1 + \frac{4}{k} + \frac{8}{k^2} + \frac{13}{k^3} + \frac{19}{k^4} + ...\text{upto}\infty$

$9 = \frac{4}{k} + \frac{8}{k^2} + \frac{13}{k^3} + \frac{19}{k^4} + ...\text{upto}\infty$

$\frac{9}{k} = \frac{4}{k^2} + \frac{8}{k^3} + \frac{13}{k^4} + ...\text{upto}\infty$

$S = 9\left(1 - \frac{1}{k}\right) = \frac{4}{k} + \frac{4}{k^2} + \frac{5}{k^3} + \frac{6}{k^4} + ...\text{upto}\infty$

$\frac{S}{k} = \qquad \frac{4}{k^2} + \frac{4}{k^3} + \frac{5}{k^4} + ...\text{upto}\infty$

$\left(1 - \frac{1}{k}\right)S = \frac{4}{k} + \frac{1}{k^3} + \frac{1}{k^4} + \frac{1}{k^5} + ...\infty$

$9\left(1 - \frac{1}{k}\right)^2 = \frac{4}{k} + \frac{\frac{1}{k^3}}{\left(1 - \frac{1}{k}\right)}$

$9(k-1)^3 = 4k(k-1) + 1$

$k = 2$

26. Let A = {1, 2, 3, 4, 5} and B = {1, 2, 3, 4, 5, 6}. Then the number of functions $f : A \to B$ satisfying f(1) + f(2) = f(4) – 1 is equal to

Official Ans. by NTA (360)

Allen Ans. (360)

Sol. $f(1) + f(2) + 1 = f(4) \le 6$

$f(1) + f(2) \le 5$

Case (i) $f(1) = 1 \Rightarrow f(2) = 1,2,3,4 \Rightarrow 4\,\text{mappings}$

Case (ii) $f(1) = 2 \Rightarrow f(2) = 1,2,3 \Rightarrow 3\,\text{mappings}$

Case (iii) $f(1) = 3 \Rightarrow f(2) = 1,2 \Rightarrow 2\,\text{mappings}$

Case (iv) $f(1) = 4 \Rightarrow f(2) = 1 \Rightarrow 1\,\text{mapping}$

$f(5)\,\&\,f(6)$ both have 6 mappings each

Number of functions $= (4+3+2+1)\times 6 \times 6 = 360$

27. Let the tangent to the parabola $y^2 = 12x$ at the point $(3, \alpha)$ be perpendicular to the line 2x + 2y = 3. Then the square of distance of the point (6, – 4) from the normal to the hyperbola $\alpha^2x^2 - 9y^2 = 9\alpha^2$ at its point $(\alpha - 1, \alpha + 2)$ is equal to

Official Ans. by NTA (116)

Allen Ans. (116)

Sol. $\because P(3,\alpha)$ lies on $y^2 = 12x$

$\Rightarrow \alpha = \pm 6$

But, $\left.\frac{dy}{dx}\right|_{(3,\alpha)} = \frac{6}{\alpha} = 1 \Rightarrow \alpha = 6\,(\alpha = -6\,\text{reject})$

Now, hyperbola $\frac{x^2}{9} - \frac{y^2}{36} = 1$, normal at $Q(\alpha-1, \alpha+2)$ is $\frac{9x}{5} + \frac{36y}{8} = 45$

$\Rightarrow 2x + 5y - 50 = 0$

Now, distance of (6, –4) from 2x + 5y – 50 = 0 is equal to

$\left|\frac{2(6) - 5(4) - 50}{\sqrt{2^2 + 5^2}}\right| = \frac{58}{\sqrt{29}}$

$\Rightarrow \text{Square of distance} = 116$

28. Let the line $\ell : x = \frac{1-y}{-2} = \frac{z-3}{\lambda}, \lambda \in \mathbb{R}$ meet the plane P : x + 2y + 3z = 4 at the point (α, β, γ). If the angle between the line ℓ and the plane P is $\cos^{-1}\left(\sqrt{\frac{5}{14}}\right)$, then $\alpha + 2\beta + 6\gamma$ is equal to

Official Ans. by NTA (11)

Allen Ans. (11)

Sol. $\ell : x = \frac{y-1}{2} = \frac{z-3}{\lambda}, \lambda \in \mathbb{R}$

DR's of line ℓ $(1, 2, \lambda)$

DR's of normal vector of plane P : x + 2y + 3z = 4 are (1, 2, 3)

Now, angle between line ℓ and plane P is given by

$$\sin\theta = \left|\frac{1+4+3\lambda}{\sqrt{5+\lambda^2}\cdot\sqrt{14}}\right| = \frac{3}{\sqrt{14}}\left(\text{given} \cos\theta = \sqrt{\frac{5}{14}}\right)$$

$$\Rightarrow \lambda = \frac{2}{3}$$

Let variable point on line ℓ is $\left(t, 2t+1, \frac{2}{3}t+3\right)$

lies on plane P.

$\Rightarrow t = -1$

$\Rightarrow \left(-1, -1, \frac{7}{3}\right) \equiv (\alpha, \beta, \gamma)$

$\Rightarrow \alpha + 2\beta + 6\gamma = 11$

29. If the line $\ell_1 : 3y - 2x = 3$ is the angular bisector of the lines $\ell_2 : x - y + 1 = 0$ and $\ell_3 : \alpha x + \beta y + 17 = 0$, then $\alpha^2 + \beta^2 - \alpha - \beta$ is equal to

Official Ans. by NTA (348)

Allen Ans. (348)

Sol. Point of intersection of $\ell_1 : 3y - 2x = 3$

$\ell_2 : x - y + 1 = 0$ is $P \equiv (0,1)$

Which lies on $\ell_3 : \alpha x + \beta y + 17 = 0$,

$\Rightarrow \boxed{\beta = -17}$

Consider a random point $Q \equiv (-1,0)$

on $\ell_2 : x - y + 1 = 0$, image of Q about

$\ell_2 : x - y + 1 = 0$ is $Q' \equiv \left(\frac{-17}{13}, \frac{6}{13}\right)$ which is calculated by formulae

$$\frac{x-(-1)}{2} = \frac{y-0}{-3} = -2\left(\frac{-2+3}{13}\right)$$

Now, Q' lies on $\ell_3 : \alpha x + \beta y + 17 = 0$

$\Rightarrow \boxed{\alpha = 7}$

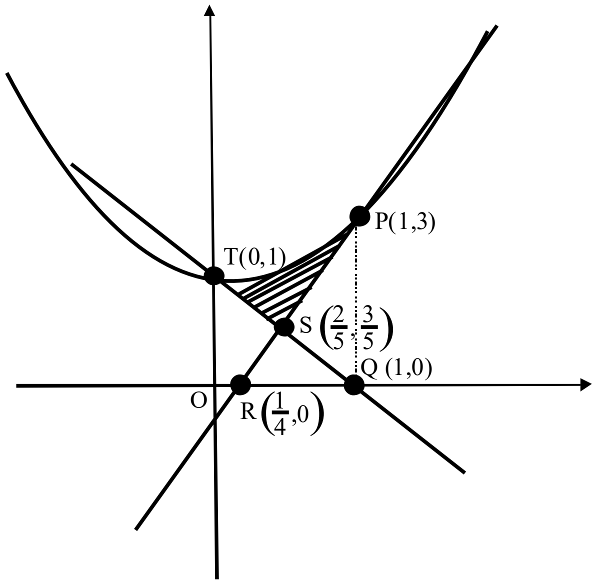
Now, $\alpha^2 + \beta^2 - \alpha - \beta = 348$

30. If A is the area in the first quadrant enclosed by the curve $C : 2x^2 - y + 1 = 0$, the tangent to C at the point (1, 3) and the line x + y = 1, then the value of 60A is

Official Ans. by NTA (16)

Allen Ans. (16)

Sol.



$y = 2x^2 + 1$

Tangent at (1, 3)

$y = 4x - 1$

$A = \int_0^1 (2x^2+1)dx$ – area of (ΔQOT) – area of (ΔPQR) + area of (ΔQRS)

$$A = \left(\frac{2}{3}+1\right) - \frac{1}{2} - \frac{9}{8} + \frac{9}{40} = \frac{16}{60}$$