

# FINAL JEE-MAIN EXAMINATION – JANUARY, 2024

(Held On Thursday 01<sup>st</sup> February, 2024)

TIME : 9 : 00 AM to 12 : 00 NOON

## MATHEMATICS

### SECTION-A

1. A bag contains 8 balls, whose colours are either white or black. 4 balls are drawn at random without replacement and it was found that 2 balls are white and other 2 balls are black. The probability that the bag contains equal number of white and black balls is:

- (1)  $\frac{2}{5}$  (2)  $\frac{2}{7}$   
(3)  $\frac{1}{7}$  (4)  $\frac{1}{5}$

Ans. (2)

Sol.

$$P(4W4B/2W2B) =$$

$$\frac{P(4W4B) \times P(2W2B/4W4B)}{P(2W6B) \times P(2W2B/2W6B) + P(3W5B) \times P(2W2B/3W5B) + \dots + P(6W2B) \times P(2W2B/6W2B)}$$

$$= \frac{\frac{1}{5} \times \frac{{}^4C_2 \times {}^4C_2}{{}^8C_4}}{\frac{1}{5} \times \frac{{}^2C_2 \times {}^6C_2}{{}^8C_4} + \frac{1}{5} \times \frac{{}^3C_2 \times {}^5C_2}{{}^8C_4} + \dots + \frac{1}{5} \times \frac{{}^6C_2 \times {}^2C_2}{{}^8C_4}}$$

$$= \frac{2}{7}$$

2. The value of the integral

$$\int_0^{\frac{\pi}{4}} \frac{xdx}{\sin^4(2x) + \cos^4(2x)} \text{ equals :}$$

- (1)  $\frac{\sqrt{2}\pi^2}{8}$  (2)  $\frac{\sqrt{2}\pi^2}{16}$   
(3)  $\frac{\sqrt{2}\pi^2}{32}$  (4)  $\frac{\sqrt{2}\pi^2}{64}$

Ans. (3)

## TEST PAPER WITH SOLUTION

Sol.  $\int_0^{\frac{\pi}{4}} \frac{xdx}{\sin^4(2x) + \cos^4(2x)}$

Let  $2x = t$  then  $dx = \frac{1}{2} dt$

$$I = \frac{1}{4} \int_0^{\frac{\pi}{2}} \frac{tdt}{\sin^4 t + \cos^4 t}$$

$$I = \frac{1}{4} \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - t\right) dt}{\sin^4\left(\frac{\pi}{2} - t\right) + \cos^4\left(\frac{\pi}{2} - t\right)}$$

$$I = \frac{1}{4} \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2} dt}{\sin^4 t + \cos^4 t} - I$$

$$2I = \frac{\pi}{8} \int_0^{\frac{\pi}{2}} \frac{dt}{\sin^4 t + \cos^4 t}$$

$$2I = \frac{\pi}{8} \int_0^{\frac{\pi}{2}} \frac{\sec^4 t dt}{\tan^4 t + 1}$$

Let  $\tan t = y$  then  $\sec^2 t dt = dy$

$$2I = \frac{\pi}{8} \int_0^{\infty} \frac{(1+y^2)dy}{1+y^4}$$

$$= \frac{\pi}{16} \int_0^{\infty} \frac{1 + \frac{1}{y^2}}{y^2 + \frac{1}{y^2}} dy$$

Put  $y - \frac{1}{y} = p$

$$I = \frac{\pi}{16} \int_{-\infty}^{\infty} \frac{dp}{p^2 + (\sqrt{2})^2}$$

$$= \frac{\pi}{16\sqrt{2}} \left[ \tan^{-1} \left( \frac{p}{\sqrt{2}} \right) \right]_{-\infty}^{\infty}$$

$$I = \frac{\pi^2}{16\sqrt{2}}$$



3. If  $A = \begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ ,  $C = ABA^T$  and  $X$

$= A^T C^2 A$ , then  $\det X$  is equal to :

(1) 243

(2) 729

(3) 27

(4) 891

Ans. (2)

Sol.

$$A = \begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix} \Rightarrow \det(A) = 3$$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \det(B) = 1$$

Now  $C = ABA^T \Rightarrow \det(C) = (\det(A))^2 \times \det(B)$

$$|C| = 9$$

Now  $|X| = |A^T C^2 A|$

$$= |A^T| |C|^2 |A|$$

$$= |A|^2 |C|^2$$

$$= 9 \times 81$$

$$= 729$$

4. If  $\tan A = \frac{1}{\sqrt{x(x^2+x+1)}}$ ,  $\tan B = \frac{\sqrt{x}}{\sqrt{x^2+x+1}}$

and

$$\tan C = (x^{-3} + x^{-2} + x^{-1})^{\frac{1}{2}}, 0 < A, B, C < \frac{\pi}{2}, \text{ then}$$

$A + B$  is equal to :

(1)  $C$

(2)  $\pi - C$

(3)  $2\pi - C$

(4)  $\frac{\pi}{2} - C$

Ans. (1)

Sol.

Finding  $\tan(A+B)$  we get

$$\Rightarrow \tan(A+B) =$$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{\sqrt{x(x^2+x+1)}} + \frac{\sqrt{x}}{\sqrt{x^2+x+1}}}{1 - \frac{1}{x^2+x+1}}$$

$$\Rightarrow \tan(A+B) = \frac{(1+x)(\sqrt{x^2+x+1})}{(x^2+x)(\sqrt{x})}$$

$$\frac{(1+x)(\sqrt{x^2+x+1})}{(x^2+x)(\sqrt{x})}$$

$$\tan(A+B) = \frac{\sqrt{x^2+x+1}}{x\sqrt{x}} = \tan C$$

$$A+B=C$$

5. If  $n$  is the number of ways five different employees can sit into four indistinguishable offices where any office may have any number of persons including zero, then  $n$  is equal to:

(1) 47

(2) 53

(3) 51

(4) 43

Ans. (3)

Sol.

Total ways to partition 5 into 4 parts are :

$$5, 0, 0, 0 \Rightarrow 1 \text{ way}$$

$$4, 1, 0, 0 \Rightarrow \frac{5!}{4!} = 5 \text{ ways}$$

$$3, 2, 0, 0 \Rightarrow \frac{5!}{3!2!} = 10 \text{ ways}$$

$$2, 2, 0, 1 \Rightarrow \frac{5!}{2!2!2!} = 15 \text{ ways}$$

$$2, 1, 1, 1 \Rightarrow \frac{5!}{2!(1!)^3 3!} = 10 \text{ ways}$$

$$3, 1, 1, 0 \Rightarrow \frac{5!}{3!2!} = 10 \text{ ways}$$

$$\text{Total} \Rightarrow 1+5+10+15+10+10 = 51 \text{ ways}$$



6. Let  $S = \{z \in \mathbb{C} : |z-1| = 1 \text{ and } (\sqrt{2}-1)(z+\bar{z}) - i(z-\bar{z}) = 2\sqrt{2}\}$ . Let  $z_1, z_2 \in S$  be such that  $|z_1| = \max_{z \in S} |z|$  and  $|z_2| = \min_{z \in S} |z|$ .

Then  $|\sqrt{2}z_1 - z_2|^2$  equals :

- (1) 1 (2) 4  
(3) 3 (4) 2

Ans. (4)

Sol. Let  $Z = x + iy$

$$\text{Then } (x-1)^2 + y^2 = 1 \rightarrow (1)$$

$$\& (\sqrt{2}-1)(2x) - i(2iy) = 2\sqrt{2}$$

$$\Rightarrow (\sqrt{2}-1)x + y = \sqrt{2} \rightarrow (2)$$

Solving (1) & (2) we get

$$\text{Either } x = 1 \text{ or } x = \frac{1}{2-\sqrt{2}} \rightarrow (3)$$

On solving (3) with (2) we get

$$\text{For } x = 1 \Rightarrow y = 1 \Rightarrow Z_2 = 1 + i$$

& for

$$x = \frac{1}{2-\sqrt{2}} \Rightarrow y = \sqrt{2} - \frac{1}{\sqrt{2}} \Rightarrow Z_1 = \left(1 + \frac{1}{\sqrt{2}}\right) + \frac{i}{\sqrt{2}}$$

Now

$$\begin{aligned} & |\sqrt{2}z_1 - z_2|^2 \\ &= \left| \left( \frac{1}{\sqrt{2}} + 1 \right) \sqrt{2} + i - (1+i) \right|^2 \\ &= (\sqrt{2})^2 \\ &= 2 \end{aligned}$$

7. Let the median and the mean deviation about the median of 7 observation 170, 125, 230, 190, 210, a, b be 170 and  $\frac{205}{7}$  respectively. Then the mean

deviation about the mean of these 7 observations is :

- (1) 31  
(2) 28  
(3) 30  
(4) 32

Ans. (3)

Sol. Median = 170  $\Rightarrow$  125, a, b, 170, 190, 210, 230

Mean deviation about

Median =

$$\frac{0+45+60+20+40+170-a+170-b}{7} = \frac{205}{7}$$

$$\Rightarrow a + b = 300$$

$$\text{Mean} = \frac{170+125+230+190+210+a+b}{7} = 175$$

Mean deviation

About mean =

$$\frac{50+175-a+175-b+5+15+35+55}{7} = 30$$

8. Let  $\vec{a} = -5\hat{i} + \hat{j} - 3\hat{k}, \vec{b} = \hat{i} + 2\hat{j} - 4\hat{k}$  and

$\vec{c} = \left( \left( (\vec{a} \times \vec{b}) \times \hat{i} \right) \times \hat{i} \right) \times \hat{i}$ . Then  $\vec{c} \cdot (-\hat{i} + \hat{j} + \hat{k})$  is equal to

- (1) -12 (2) -10  
(3) -13 (4) -15

Ans. (1)

Sol.  $\vec{a} = -5\hat{i} + \hat{j} - 3\hat{k}$

$$\vec{b} = \hat{i} + 2\hat{j} - 4\hat{k}$$

$$(\vec{a} \times \vec{b}) \times \hat{i} = (\vec{a} \cdot \hat{i})\vec{b} - (\vec{b} \cdot \hat{i})\vec{a}$$

$$= -5\vec{b} - \vec{a}$$

$$= \left( \left( (-5\vec{b} - \vec{a}) \times \hat{i} \right) \times \hat{i} \right)$$

$$= \left( \left( (-11\hat{j} + 23\hat{k}) \times \hat{i} \right) \times \hat{i} \right)$$

$$\Rightarrow (11\hat{k} + 23\hat{j}) \times \hat{i}$$

$$\Rightarrow (11\hat{j} - 23\hat{k})$$

$$\vec{c} \cdot (-\hat{i} + \hat{j} + \hat{k}) = 11 - 23 = -12$$



9. Let  $S = \{x \in \mathbb{R} : (\sqrt{3} + \sqrt{2})^x + (\sqrt{3} - \sqrt{2})^x = 10\}$ .

Then the number of elements in  $S$  is :

- (1) 4 (2) 0  
(3) 2 (4) 1

Ans. (3)

Sol.  $(\sqrt{3} + \sqrt{2})^x + (\sqrt{3} - \sqrt{2})^x = 10$

Let  $(\sqrt{3} + \sqrt{2})^x = t$

$$t + \frac{1}{t} = 10$$

$$t^2 - 10t + 1 = 0$$

$$t = \frac{10 \pm \sqrt{100 - 4}}{2} = 5 \pm 2\sqrt{6}$$

$$(\sqrt{3} + \sqrt{2})^x = (\sqrt{3} \pm \sqrt{2})^2$$

$$x = 2 \text{ or } x = -2$$

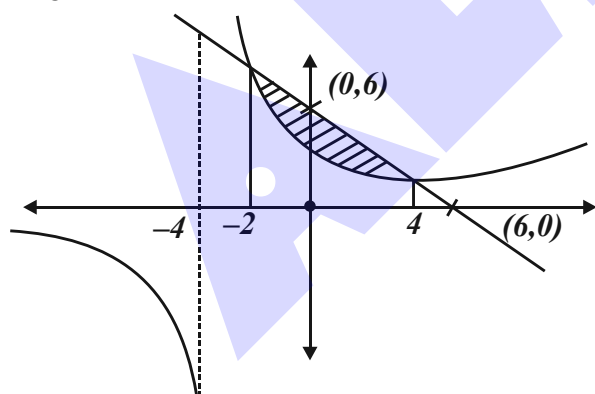
Number of solutions = 2

10. The area enclosed by the curves  $xy + 4y = 16$  and  $x + y = 6$  is equal to :

- (1)  $28 - 30 \log_e 2$  (2)  $30 - 28 \log_e 2$   
(3)  $30 - 32 \log_e 2$  (4)  $32 - 30 \log_e 2$

Ans. (3)

Sol.  $xy + 4y = 16$ ,  $x + y = 6$   
 $y(x + 4) = 16$  (1),  $x + y = 6$  (2)  
on solving, (1) & (2)  
we get  $x = 4$ ,  $x = -2$



$$\text{Area} = \int_{-2}^4 \left( (6-x) - \left( \frac{16}{x+4} \right) \right) dx$$

$$= 30 - 32 \ln 2$$

11. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  be defined as

$$f(x) = \begin{cases} \log_e x & , x > 0 \\ e^{-x} & , x \leq 0 \end{cases} \text{ and}$$

$$g(x) = \begin{cases} x & , x \geq 0 \\ e^x & , x < 0 \end{cases} \text{ Then, } g \circ f : \mathbb{R} \rightarrow \mathbb{R} \text{ is :}$$

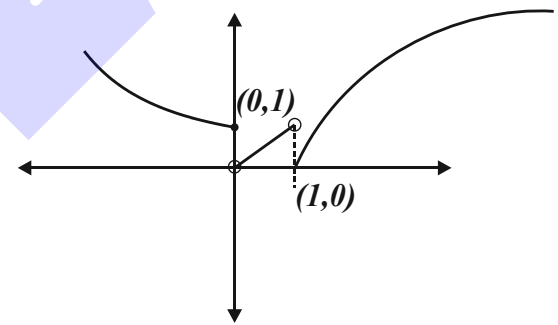
- (1) one-one but not onto  
(2) neither one-one nor onto  
(3) onto but not one-one  
(4) both one-one and onto

Ans. (2)

Sol.

$$g(f(x)) = \begin{cases} f(x), f(x) \geq 0 \\ e^{f(x)}, f(x) < 0 \end{cases}$$

$$g(f(x)) = \begin{cases} e^{-x}, (-\infty, 0] \\ e^{\ln x}, (0, 1) \\ \ln x, [1, \infty) \end{cases}$$



Graph of  $g(f(x))$

$g(f(x)) \Rightarrow$  Many one into

12. If the system of equations

$$2x + 3y - z = 5$$

$$x + \alpha y + 3z = -4$$

$$3x - y + \beta z = 7$$

has infinitely many solutions, then  $13 \alpha \beta$  is equal to

- (1) 1110 (2) 1120  
(3) 1210 (4) 1220

Ans. (2)



**Sol.** Using family of planes

$$2x + 3y - z - 5 = k_1(x + \alpha y + 3z + 4) + k_2(3x - y + \beta z - 7)$$

$$2 = k_1 + 3k_2, 3 = k_1\alpha - k_2, -1 = 3k_1 + \beta k_2, -5 = 4k_1 - 7k_2$$

On solving we get

$$k_2 = \frac{13}{19}, k_1 = \frac{-1}{19}, \alpha = -70, \beta = \frac{-16}{13}$$

$$13\alpha\beta = 13(-70)\left(\frac{-16}{13}\right) = 1120$$

**13.** For  $0 < \theta < \pi/2$ , if the eccentricity of the hyperbola  $x^2 - y^2 \operatorname{cosec}^2 \theta = 5$  is  $\sqrt{7}$  times eccentricity of the ellipse  $x^2 \operatorname{cosec}^2 \theta + y^2 = 5$ , then the value of  $\theta$  is :

- (1)  $\frac{\pi}{6}$  (2)  $\frac{5\pi}{12}$   
(3)  $\frac{\pi}{3}$  (4)  $\frac{\pi}{4}$

**Ans. (3)**

**Sol.**

$$e_h = \sqrt{1 + \sin^2 \theta}$$

$$e_e = \sqrt{1 - \sin^2 \theta}$$

$$e_h = \sqrt{7}e_e$$

$$1 + \sin^2 \theta = 7(1 - \sin^2 \theta)$$

$$\sin^2 \theta = \frac{6}{8} = \frac{3}{4}$$

$$\sin \theta = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{3}$$

**14.** Let  $y = y(x)$  be the solution of the differential equation  $\frac{dy}{dx} = 2x(x+y)^3 - x(x+y) - 1$ ,  $y(0) = 1$ .

Then,  $\left(\frac{1}{\sqrt{2}} + y\left(\frac{1}{\sqrt{2}}\right)\right)^2$  equals :

- (1)  $\frac{4}{4 + \sqrt{e}}$  (2)  $\frac{3}{3 - \sqrt{e}}$   
(3)  $\frac{2}{1 + \sqrt{e}}$  (4)  $\frac{1}{2 - \sqrt{e}}$

**Ans. (4)**

**Sol.**  $\frac{dy}{dx} = 2x(x+y)^3 - x(x+y) - 1$

$$x + y = t$$

$$\frac{dt}{dx} - 1 = 2xt^3 - xt - 1$$

$$\frac{dt}{2t^3 - t} = xdx$$

$$\frac{tdt}{2t^4 - t^2} = xdx$$

$$\text{Let } t^2 = z$$

$$\int \frac{dz}{2(2z^2 - z)} = \int xdx$$

$$\int \frac{dz}{4z\left(z - \frac{1}{2}\right)} = \int xdx$$

$$\ln \left| \frac{z - \frac{1}{2}}{z} \right| = x^2 + k$$

$$z = \frac{1}{2 - \sqrt{e}}$$

**15.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined as

$$f(x) = \begin{cases} \frac{a - b \cos 2x}{x^2} & ; x < 0 \\ x^2 + cx + 2 & ; 0 \leq x \leq 1 \\ 2x + 1 & ; x > 1 \end{cases}$$

If  $f$  is continuous everywhere in  $\mathbb{R}$  and  $m$  is the number of points where  $f$  is **NOT** differential then  $m + a + b + c$  equals :

- (1) 1 (2) 4  
(3) 3 (4) 2

**Ans. (4)**



**Sol.** At  $x = 1$ ,  $f(x)$  is continuous therefore,

$$f(1^-) = f(1) = f(1^+) \quad \dots(1)$$

$$f(1) = 3 + c \quad \dots(1)$$

$$f(1^+) = \lim_{h \rightarrow 0} 2(1+h) + 1$$

$$f(1^+) = \lim_{h \rightarrow 0} 3 + 2h = 3 \quad \dots(2)$$

from (1) & (2)

$$c = 0$$

at  $x = 0$ ,  $f(x)$  is continuous therefore,

$$f(0^-) = f(0) = f(0^+) \quad \dots(3)$$

$$f(0) = f(0^+) = 2 \quad \dots(4)$$

$f(0^-)$  has to be equal to 2

$$\lim_{h \rightarrow 0} \frac{a - b \cos(2h)}{h^2}$$

$$\lim_{h \rightarrow 0} \frac{a - b \left\{ 1 - \frac{4h^2}{2!} + \frac{16h^4}{4!} + \dots \right\}}{h^2}$$

$$\lim_{h \rightarrow 0} \frac{a - b + b \left\{ 2h^2 - \frac{2}{3}h^4 \dots \right\}}{h^2}$$

for limit to exist  $a - b = 0$  and limit is  $2b \quad \dots(5)$

from (3), (4) & (5)

$$a = b = 1$$

checking differentiability at  $x = 0$

$$\text{LHD : } \lim_{h \rightarrow 0} \frac{\frac{1 - \cos 2h}{h^2} - 2}{-h}$$

$$\lim_{h \rightarrow 0} \frac{1 - \left( 1 - \frac{4h^2}{2!} + \frac{16h^4}{4!} \dots \right) - 2h^2}{-h^3} = 0$$

$$\text{RHD : } \lim_{h \rightarrow 0} \frac{(0+h)^2 + 2 - 2}{h} = 0$$

Function is differentiable at every point in its domain

$$\therefore m = 0$$

$$m + a + b + c = 0 + 1 + 1 + 0 = 2$$

**16.** Let  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ,  $a > b$  be an ellipse, whose eccentricity is  $\frac{1}{\sqrt{2}}$  and the length of the latus rectum is  $\sqrt{14}$ . Then the square of the eccentricity of  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is :

$$(1) 3 \quad (2) 7/2$$

$$(3) 3/2 \quad (4) 5/2$$

**Ans. (3)**

**Sol.**

$$e = \frac{1}{\sqrt{2}} = \sqrt{1 - \frac{b^2}{a^2}} \Rightarrow \frac{1}{2} = 1 - \frac{b^2}{a^2}$$

$$\frac{2b^2}{a^2} = 14$$

$$e_H = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{1}{2}} = \sqrt{\frac{3}{2}}$$

$$(e_H)^2 = \frac{3}{2}$$

**17.** Let 3, a, b, c be in A.P. and 3, a - 1, b + 1, c + 9 be in G.P. Then, the arithmetic mean of a, b and c is :

$$(1) -4 \quad (2) -1$$

$$(3) 13 \quad (4) 11$$

**Ans. (4)**

**Sol.**

$$3, a, b, c \rightarrow \text{A.P.} \Rightarrow 3, 3+d, 3+2d, 3+3d$$

$$3, a-1, b+1, c+9 \rightarrow \text{G.P.} \Rightarrow 3, 2+d, 4+2d, 12+3d$$

$$a = 3 + d \quad (2+d)^2 = 3(4+2d)$$

$$b = 3 + 2d \quad d = 4, -2$$

$$c = 3 + 3d$$

$$\text{If } d = 4 \quad \text{G.P.} \Rightarrow 3, 6, 12, 24$$

$$a = 7$$

$$b = 11$$

$$c = 15$$

$$\frac{a+b+c}{3} = 11$$



18. Let  $C : x^2 + y^2 = 4$  and  $C' : x^2 + y^2 - 4\lambda x + 9 = 0$  be two circles. If the set of all values of  $\lambda$  so that the circles  $C$  and  $C'$  intersect at two distinct points, is  $\mathbf{R} - [a, b]$ , then the point  $(8a + 12, 16b - 20)$  lies on the curve :

- (1)  $x^2 + 2y^2 - 5x + 6y = 3$   
(2)  $5x^2 - y = -11$   
(3)  $x^2 - 4y^2 = 7$   
(4)  $6x^2 + y^2 = 42$

Ans. (4)

Sol.  $x^2 + y^2 = 4$

$$C(0, 0) \quad r_1 = 2$$

$$C'(2\lambda, 0) \quad r_2 = \sqrt{4\lambda^2 - 9}$$

$$|r_1 - r_2| < CC' < |r_1 + r_2|$$

$$|2 - \sqrt{4\lambda^2 - 9}| < |2\lambda| < 2 + \sqrt{4\lambda^2 - 9}$$

$$4 + 4\lambda^2 - 9 - 4\sqrt{4\lambda^2 - 9} < 4\lambda^2$$

$$\text{True } \lambda \in \mathbf{R} \dots (1)$$

$$4\lambda^2 < 4 + 4\lambda^2 - 9 + 4\sqrt{4\lambda^2 - 9}$$

$$5 < 4\sqrt{4\lambda^2 - 9} \text{ and } \lambda^2 \geq \frac{9}{4}$$

$$\frac{25}{16} < 4\lambda^2 - 9 \quad \lambda \in \left(-\infty, -\frac{3}{2}\right] \cup \left[\frac{3}{2}, \infty\right)$$

$$\frac{169}{64} < \lambda^2$$

$$\lambda \in \left(-\infty, -\frac{13}{8}\right) \cup \left(\frac{13}{8}, \infty\right) \dots (2)$$

from (1) and (2)  $\lambda \in$

$$\lambda \in \left(-\infty, -\frac{13}{8}\right) \cup \left(\frac{13}{8}, \infty\right) \Rightarrow \mathbf{R} - \left[-\frac{13}{8}, \frac{13}{8}\right]$$

$$\text{as per question } a = -\frac{13}{8} \text{ and } b = \frac{13}{8}$$

$\therefore$  required point is  $(-1, 6)$  with satisfies option (4)

19. If  $5f(x) + 4f\left(\frac{1}{x}\right) = x^2 - 2$ ,  $\forall x \neq 0$  and  $y = 9x^2f(x)$ ,

then  $y$  is strictly increasing in :

- (1)  $\left(0, \frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$   
(2)  $\left(-\frac{1}{\sqrt{5}}, 0\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$   
(3)  $\left(-\frac{1}{\sqrt{5}}, 0\right) \cup \left(0, \frac{1}{\sqrt{5}}\right)$   
(4)  $\left(-\infty, \frac{1}{\sqrt{5}}\right) \cup \left(0, \frac{1}{\sqrt{5}}\right)$

Ans. (2)

Sol.  $5f(x) + 4f\left(\frac{1}{x}\right) = x^2 - 2$ ,  $\forall x \neq 0 \dots (1)$

Substitute  $x \rightarrow \frac{1}{x}$

$$5f\left(\frac{1}{x}\right) + 4f(x) = \frac{1}{x^2} - 2 \dots (2)$$

On solving (1) and (2)

$$f(x) = \frac{5x^4 - 2x^2 - 4}{9x^2}$$

$$y = 9x^2f(x)$$

$$y = 5x^4 - 2x^2 - 4 \dots (3)$$

$$\frac{dy}{dx} = 20x^3 - 4x$$

for strictly increasing

$$\frac{dy}{dx} > 0$$

$$4x(5x^2 - 1) > 0$$

$$x \in \left(-\frac{1}{\sqrt{5}}, 0\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$$

20. If the shortest distance between the lines

$$\frac{x-\lambda}{-2} = \frac{y-2}{1} = \frac{z-1}{1} \text{ and } \frac{x-\sqrt{3}}{1} = \frac{y-1}{-2} = \frac{z-2}{1}$$

is 1, then the sum of all possible values of  $\lambda$  is :

- (1) 0  
(2)  $2\sqrt{3}$   
(3)  $3\sqrt{3}$   
(4)  $-2\sqrt{3}$

Ans. (2)





**Sol.** Passing points of lines  $L_1$  &  $L_2$  are

$$(\lambda, 2, 1) \& (\sqrt{3}, 1, 2)$$

$$S.D = \begin{vmatrix} \sqrt{3}-\lambda & -1 & 1 \\ -2 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$1 = \left| \frac{\sqrt{3}-\lambda}{\sqrt{3}} \right|$$

$$\lambda = 0, \lambda = 2\sqrt{3}$$

### SECTION-B

**21.** If  $x = x(t)$  is the solution of the differential equation  $(t+1)dx = (2x + (t+1)^4) dt$ ,  $x(0) = 2$ , then,  $x(1)$  equals \_\_\_\_\_.

**Ans. (14)**

**Sol.**  $(t+1)dx = (2x + (t+1)^4)dt$

$$\frac{dx}{dt} = \frac{2x + (t+1)^4}{t+1}$$

$$\frac{dx}{dt} - \frac{2x}{t+1} = (t+1)^3$$

$$I \cdot F = e^{-\int \frac{2}{t+1} dt} = e^{-2\ln(t+1)} = \frac{1}{(t+1)^2}$$

$$\frac{x}{(t+1)^2} = \int \frac{1}{(t+1)^2} (t+1)^3 dt + c$$

$$\frac{x}{(t+1)^2} = \frac{(t+1)^2}{2} + c$$

$$\Rightarrow c = \frac{3}{2}$$

$$x = \frac{(t+1)^4}{2} + \frac{3}{2}(t+1)^2$$

$$\text{put, } t = 1$$

$$x = 2^3 + 6 = 14$$

**22.** The number of elements in the set

$$S = \{(x, y, z) : x, y, z \in \mathbf{Z}, x + 2y + 3z = 42, x, y, z \geq 0\} \text{ equals } \underline{\hspace{2cm}}.$$

**Ans. (169)**

**Sol.**  $x + 2y + 3z = 42, \quad x, y, z \geq 0$

$$z = 0 \quad x + 2y = 42 \Rightarrow 22$$

$$z = 1 \quad x + 2y = 39 \Rightarrow 20$$

$$z = 2 \quad x + 2y = 36 \Rightarrow 19$$

$$z = 3 \quad x + 2y = 33 \Rightarrow 17$$

$$z = 4 \quad x + 2y = 30 \Rightarrow 16$$

$$z = 5 \quad x + 2y = 27 \Rightarrow 14$$

$$z = 6 \quad x + 2y = 24 \Rightarrow 13$$

$$z = 7 \quad x + 2y = 21 \Rightarrow 11$$

$$z = 8 \quad x + 2y = 18 \Rightarrow 10$$

$$z = 9 \quad x + 2y = 15 \Rightarrow 8$$

$$z = 10 \quad x + 2y = 12 \Rightarrow 7$$

$$z = 11 \quad x + 2y = 9 \Rightarrow 5$$

$$z = 12 \quad x + 2y = 6 \Rightarrow 4$$

$$z = 13 \quad x + 2y = 3 \Rightarrow 2$$

$$z = 14 \quad x + 2y = 0 \Rightarrow 1$$

Total : 169

**23.** If the Coefficient of  $x^{30}$  in the expansion of

$$\left(1 + \frac{1}{x}\right)^6 (1 + x^2)^7 (1 - x^3)^8; x \neq 0 \text{ is } \alpha, \text{ then } |\alpha| \text{ equals } \underline{\hspace{2cm}}.$$

**Ans. (678)**





**Sol.** coeff of  $x^{30}$  in  $\frac{(x+1)^6(1+x^2)^7(1-x^3)^8}{x^6}$

**coeff. of  $x^{36}$  in  $(1+x)^6(1+x^2)^7(1-x^3)^8$**

**General term**

$${}^6C_{r_1} {}^7C_{r_2} {}^8C_{r_3} (-1)^{r_3} x^{r_1+2r_2+3r_3}$$

$$r_1 + 2r_2 + 3r_3 = 36$$

Case-I :

$r_1$	$r_2$	$r_3$
0	6	8
2	5	8
4	4	8
6	3	8

 $r_1 + 2r_2 = 12$  (Taking  $r_3 = 8$ )

Case-II :

$r_1$	$r_2$	$r_3$
1	7	7
3	6	7
5	5	7

 $r_1 + 2r_2 = 15$  (Taking  $r_3 = 7$ )

Case-III :

$r_1$	$r_2$	$r_3$
4	7	6
6	6	6

 $r_1 + 2r_2 = 18$  (Taking  $r_3 = 6$ )

$$\text{Coeff.} = 7 + (15 \times 21) + (15 \times 35) + (35)$$

$$-(6 \times 8) - (20 \times 7 \times 8) - (6 \times 21 \times 8) + (15 \times 28) + (7 \times 28) = -678 = \alpha$$

$$|\alpha| = 678$$

- 24.** Let 3, 7, 11, 15, ..., 403 and 2, 5, 8, 11, ..., 404 be two arithmetic progressions. Then the sum, of the common terms in them, is equal to \_\_\_\_\_.

**Ans. (6699)**

**Sol.** 3, 7, 11, 15, ..., 403

2, 5, 8, 11, ..., 404

LCM (4, 3) = 12

11, 23, 35, ..., let (403)

$$403 = 11 + (n-1) \times 12$$

$$\frac{392}{12} = n-1$$

$$33 \cdot 66 = n$$

$$n = 33$$

$$\text{Sum } \frac{33}{2} (22 + 32 \times 12)$$

$$= 6699$$

- 25.** Let  $\{x\}$  denote the fractional part of  $x$  and  $f(x) = \frac{\cos^{-1}(1-\{x\}^2) \sin^{-1}(1-\{x\})}{\{x\} - \{x\}^3}$ ,  $x \neq 0$ . If  $L$  and  $R$  respectively denotes the left hand limit and the right hand limit of  $f(x)$  at  $x = 0$ , then  $\frac{32}{\pi^2} (L^2 + R^2)$  is equal to \_\_\_\_\_.

**Ans. (18)**

**Sol.** Finding right hand limit

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0+h)$$

$$= \lim_{h \rightarrow 0} f(h)$$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1-h^2) \sin^{-1}(1-h)}{h(1-h^2)}$$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1-h^2)}{h} \left( \frac{\sin^{-1} 1}{1} \right)$$

$$\text{Let } \cos^{-1}(1-h^2) = \theta \Rightarrow \cos \theta = 1-h^2$$

$$= \frac{\pi}{2} \lim_{\theta \rightarrow 0} \frac{\theta}{\sqrt{1-\cos \theta}}$$

$$= \frac{\pi}{2} \lim_{\theta \rightarrow 0} \frac{1}{\sqrt{\frac{1-\cos \theta}{\theta^2}}}$$

$$= \frac{\pi}{2} \frac{1}{\sqrt{1/2}}$$

$$R = \frac{\pi}{\sqrt{2}}$$



Now finding left hand limit

$$\begin{aligned} L &= \lim_{x \rightarrow 0^-} f(x) \\ &= \lim_{h \rightarrow 0} f(-h) \\ &= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1 - \{-h\}^2) \sin^{-1}(1 - \{-h\})}{\{-h\} - \{-h\}^3} \\ &= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1 - (-h+1)^2) \sin^{-1}(1 - (-h+1))}{(-h+1) - (-h+1)^3} \\ &= \lim_{h \rightarrow 0} \frac{\cos^{-1}(-h^2 + 2h) \sin^{-1} h}{(1-h)(1-(1-h)^2)} \\ &= \lim_{h \rightarrow 0} \left( \frac{\pi}{2} \right) \frac{\sin^{-1} h}{(1-(1-h)^2)} \\ &= \frac{\pi}{2} \lim_{h \rightarrow 0} \left( \frac{\sin^{-1} h}{-h^2 + 2h} \right) \\ &= \frac{\pi}{2} \lim_{h \rightarrow 0} \left( \frac{\sin^{-1} h}{h} \right) \left( \frac{1}{-h+2} \right) \\ L &= \frac{\pi}{4} \\ \frac{32}{\pi^2} (L^2 + R^2) &= \frac{32}{\pi^2} \left( \frac{\pi^2}{2} + \frac{\pi^2}{16} \right) \\ &= 16 + 2 \\ &= 18 \end{aligned}$$

26. Let the line  $L : \sqrt{2}x + y = \alpha$  pass through the point of the intersection  $P$  (in the first quadrant) of the circle  $x^2 + y^2 = 3$  and the parabola  $x^2 = 2y$ . Let the line  $L$  touch two circles  $C_1$  and  $C_2$  of equal radius  $2\sqrt{3}$ . If the centres  $Q_1$  and  $Q_2$  of the circles  $C_1$  and  $C_2$  lie on the  $y$ -axis, then the square of the area of the triangle  $PQ_1Q_2$  is equal to \_\_\_\_\_.

Ans. (72)

Sol.  $x^2 + y^2 = 3$  and  $x^2 = 2y$

$$y^2 + 2y - 3 = 0 \Rightarrow (y+3)(y-1) = 0$$

$$y = -3 \text{ or } y = 1$$

$$y = 1 \quad x = \sqrt{2} \Rightarrow P(\sqrt{2}, 1)$$

$p$  lies on the line

$$\sqrt{2}x + y = \alpha$$

$$\sqrt{2}(\sqrt{2}) + 1 = \alpha$$

$$\alpha = 3$$

For circle  $C_1$

$Q_1$  lies on  $y$  axis

Let  $Q_1 (0, \alpha)$  coordinates

$$R_1 = 2\sqrt{3} \text{ (Given)}$$

Line  $L$  act as tangent

Apply  $P = r$  (condition of tangency)

$$\Rightarrow \left| \frac{\alpha - 3}{\sqrt{3}} \right| = 2\sqrt{3}$$

$$\Rightarrow |\alpha - 3| = 6$$

$$\alpha - 3 = 6 \quad \text{or} \quad \alpha - 3 = -6$$

$$\Rightarrow \alpha = 9$$

$$\alpha = -3$$

$$\Delta PQ_1Q_2 = \frac{1}{2} \begin{vmatrix} \sqrt{2} & 1 & 1 \\ 0 & 9 & 1 \\ 0 & -3 & 1 \end{vmatrix}$$

$$= \frac{1}{2} (\sqrt{2}(12)) = 6\sqrt{2}$$

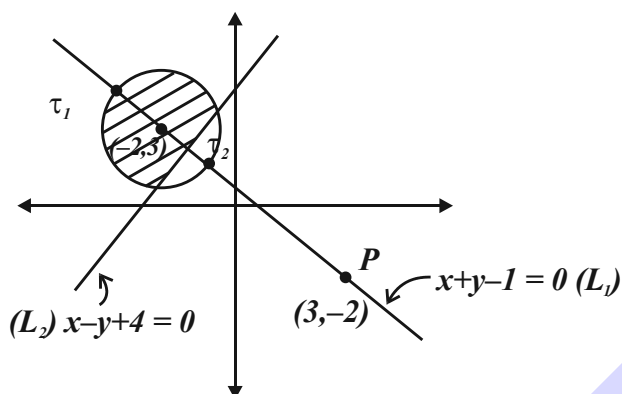
$$(\Delta PQ_1Q_2)^2 = 72$$



27. Let  $P = \{z \in \mathbb{C} : |z + 2 - 3i| \leq 1\}$  and  $Q = \{z \in \mathbb{C} : z(1+i) + \bar{z}(1-i) \leq -8\}$ . Let in  $P \cap Q$ ,  $|z - 3 + 2i|$  be maximum and minimum at  $z_1$  and  $z_2$  respectively. If  $|z_1|^2 + 2|z_2|^2 = \alpha + \beta\sqrt{2}$ , where  $\alpha, \beta$  are integers, then  $\alpha + \beta$  equals \_\_\_\_\_.

Ans. (36)

Sol.



Clearly for the shaded region  $z_1$  is the intersection of the circle and the line passing through P ( $L_1$ ) and  $z_2$  is intersection of line  $L_1$  &  $L_2$

$$\text{Circle : } (x + 2)^2 + (y - 3)^2 = 1$$

$$L_1 : x + y - 1 = 0$$

$$L_2 : x - y + 4 = 0$$

On solving circle &  $L_1$  we get

$$z_1 : \left(-2 - \frac{1}{\sqrt{2}}, 3 + \frac{1}{\sqrt{2}}\right)$$

On solving  $L_1$  and  $z_2$  is intersection of line  $L_1$  &  $L_2$

$$\text{we get } z_2 : \left(\frac{-3}{2}, \frac{5}{2}\right)$$

$$\begin{aligned} |z_1|^2 + 2|z_2|^2 &= 14 + 5\sqrt{2} + 17 \\ &= 31 + 5\sqrt{2} \end{aligned}$$

$$\text{So } \alpha = 31$$

$$\beta = 5$$

$$\alpha + \beta = 36$$

28. If  $\int_{-\pi/2}^{\pi/2} \frac{8\sqrt{2} \cos x dx}{(1 + e^{\sin x})(1 + \sin^4 x)} = \alpha\pi + \beta \log_e (3 + 2\sqrt{2})$ , where  $\alpha, \beta$  are integers, then  $\alpha^2 + \beta^2$  equals \_\_\_\_\_.

Ans. (8)

$$\text{Sol. } I = \int_{-\pi/2}^{\pi/2} \frac{8\sqrt{2} \cos x}{(1 + e^{\sin x})(1 + \sin^4 x)} dx$$

Apply king

$$I = \int_{-\pi/2}^{\pi/2} \frac{8\sqrt{2} \cos x (e^{\sin x})}{(1 + e^{\sin x})(1 + \sin^4 x)} dx \quad \dots (2)$$

adding (1) & (2)

$$2I = \int_{-\pi/2}^{\pi/2} \frac{8\sqrt{2} \cos x}{1 + \sin^4 x} dx$$

$$I = \int_0^{\pi/2} \frac{8\sqrt{2} \cos x}{1 + \sin^4 x} dx,$$

$$\sin x = t$$

$$I = \int_0^1 \frac{8\sqrt{2}}{1 + t^4} dt$$

$$I = 4\sqrt{2} \int_0^1 \left( \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} - \frac{1 - \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} \right) dt$$

$$I = 4\sqrt{2} \int_0^1 \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2} - \frac{\left(1 - \frac{1}{t^2}\right)}{\left(t + \frac{1}{t}\right)^2 - 2} dt$$

$$\text{Let } t - \frac{1}{t} = z \text{ \& } t + \frac{1}{t} = k$$



$$\begin{aligned}
 &= 4\sqrt{2} \left[ \int_{-\infty}^0 \frac{dz}{z^2 + 2} - \int_{\infty}^2 \frac{dk}{k^2 - 2} \right] \\
 &= 4\sqrt{2} \left[ \frac{1}{\sqrt{2}} \tan^{-1} \frac{z}{\sqrt{2}} \right]_{-\infty}^0 - \left[ \frac{1}{2\sqrt{2}} \ln \left( \frac{k - \sqrt{2}}{k + \sqrt{2}} \right) \right]_{\infty}^2 \\
 &= 4\sqrt{2} \left[ \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \left[ \ln \frac{2 - \sqrt{2}}{2 + \sqrt{2}} \right] \right] \\
 &= 2\pi + 2\ln(3 + 2\sqrt{2}) \\
 \alpha &= 2 \\
 \beta &= 2
 \end{aligned}$$

29. Let the line of the shortest distance between the lines

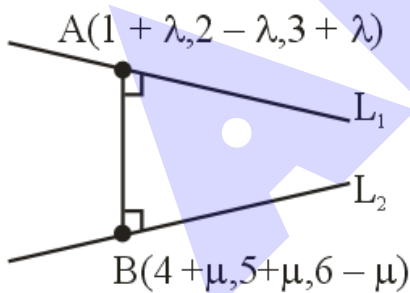
$$L_1 : \vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}) \text{ and}$$

$$L_2 : \vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(\hat{i} + \hat{j} - \hat{k})$$

intersect  $L_1$  and  $L_2$  at P and Q respectively. If  $(\alpha, \beta, \gamma)$  is the midpoint of the line segment PQ, then  $2(\alpha + \beta + \gamma)$  is equal to \_\_\_\_\_.

Ans. (21)

Sol.



$$\vec{b} = \hat{i} - \hat{j} + \hat{k} \text{ (DR's of } L_1)$$

$$\vec{d} = \hat{i} + \hat{j} - \hat{k} \text{ (DR's of } L_2)$$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 0\hat{i} + 2\hat{j} + 2\hat{k} \text{ (DR's of Line perpendicular to } L_1 \text{ and } L_2)$$

DR of AB line

$$= (0, 2, 2) = (3 + \mu - \lambda, 3 + \mu + \lambda, 3 - \mu - \lambda)$$

$$\frac{3 + \mu - \lambda}{0} = \frac{3 + \mu + \lambda}{2} = \frac{3 - \mu - \lambda}{2}$$

$$\text{Solving above equation we get } \mu = -\frac{3}{2} \text{ and } \lambda = \frac{3}{2}$$

$$\text{point A} = \left( \frac{5}{2}, \frac{1}{2}, \frac{9}{2} \right)$$

$$\text{B} = \left( \frac{5}{2}, \frac{7}{2}, \frac{15}{2} \right)$$

$$\text{Point of AB} = \left( \frac{5}{2}, 2, 6 \right) = (\alpha, \beta, \gamma)$$

$$2(\alpha + \beta + \gamma) = 5 + 4 + 12 = 21$$

30. Let  $A = \{1, 2, 3, \dots, 20\}$ . Let  $R_1$  and  $R_2$  two relation on A such that

$$R_1 = \{(a, b) : b \text{ is divisible by } a\}$$

$$R_2 = \{(a, b) : a \text{ is an integral multiple of } b\}.$$

Then, number of elements in  $R_1 - R_2$  is equal to \_\_\_\_\_.

Ans. (46)

$$\text{Sol. } n(R_1) = 20 + 10 + 6 + 5 + 4 + 3 + 2 + 2 + 2$$

$$+ 2 + \underbrace{1 + \dots + 1}_{10 \text{ times}}$$



$$n(R_1) = 66$$

$$R_1 \cap R_2 = \{(1,1), (2,2), \dots, (20,20)\}$$

$$n(R_1 \cap R_2) = 20$$

$$n(R_1 - R_2) = n(R_1) - n(R_1 \cap R_2)$$

$$= n(R_1) - 20$$

$$= 66 - 20$$

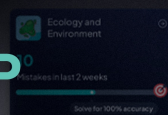
$$R_1 - R_2 = 46 \text{ Pair}$$

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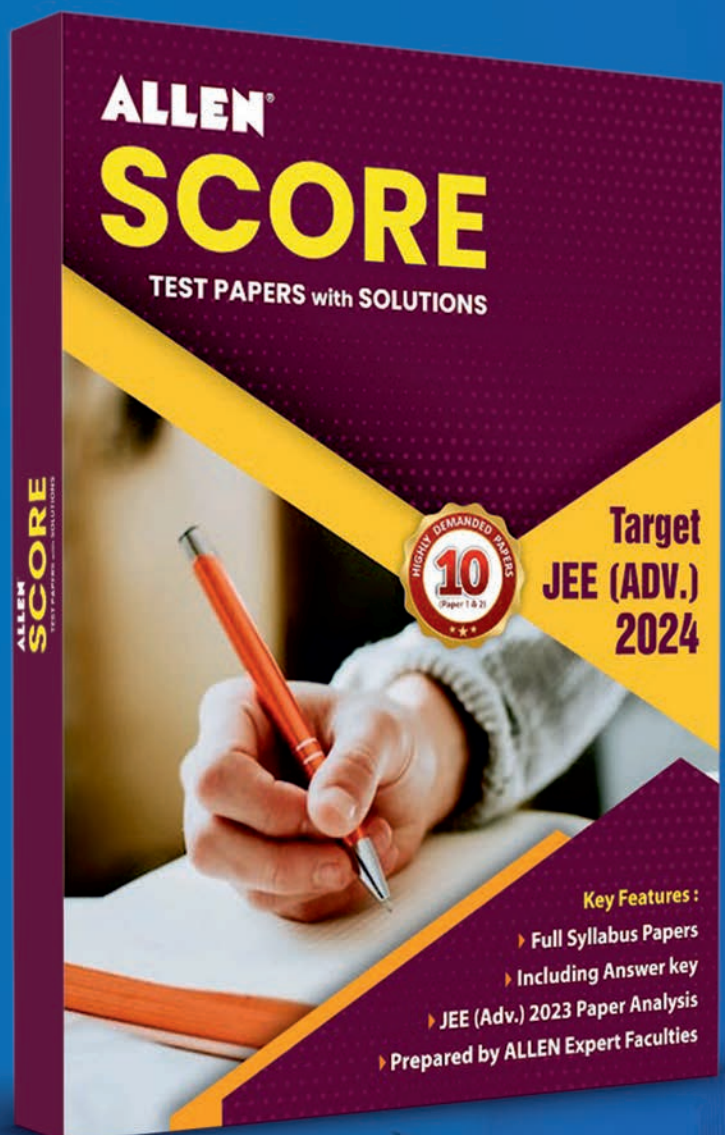


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