FINAL JEE–MAIN EXAMINATION – APRIL, 2023

(Held On Saturday 08th April, 2023) **TIME : 3 : 00 PM to 6 : 00 PM**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. Let the mean and variance of 12 observations be $\frac{9}{2}$ and 4 respectively. Later on, it was observed that two observations were considered as 9 and 10 instead of 7 and 14 respectively. If the correct variance is $\frac{m}{n}$, where m and n are co-prime, then m+n is equal to

(1) 316

(2) 314

(3) 317

(4) 315

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. Given mean $(\bar{x}) = \frac{9}{2}$

$$\bar{x}_{new} = \frac{12 \times \frac{9}{2} + 7 + 14 - 9 - 10}{12} = \frac{14}{3} \quad \ldots\ldots\text{(i)}$$

Given, $\sigma^2 = 4$

$$\sigma^2 = \frac{\sum x_i^2}{12} - \left(\frac{9}{2}\right)^2$$

$$4 = \frac{\sum x_i^2}{12} - \frac{81}{4}$$

$$\frac{\sum x_i^2}{12} = \frac{97}{4}$$

$$\sum x_i^2 = 291$$

Now,

$$\sum (x_i^2)_{new} = 291 - 9^2 - 10^2 + 7^2 + 14^2 = 355$$

$$\therefore \sigma^2_{new} = \frac{\sum (x_i^2)_{new}}{12} - (\bar{x}_{new})^2$$

$$\sigma^2_{new} = \frac{355}{12} - \left(\frac{14}{3}\right)^2 = \frac{281}{36} \ (\text{from eq.(i)})$$

2. Let a_n be the n^{th} term of the series 5 + 8 + 14 + 23 + 35 + 50 + … and $S_n = \sum_{k=1}^{n} a_k$. Then $S_{30} - a_{40}$ is equal to

(1) 11310

(2) 11280

(3) 11290

(4) 11260

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $S_n = 5 + 8 + 14 + 23 + \ldots. + T_n$

$S_n = \quad 5 + 8 + 14 + \ldots\ldots + T_{n-1} + T_n$

$- \quad -$

$T_n = 5 + (3 + 6 + 9 + \ldots.$ to $(n - 1)$ terms$)$

$$T_n = 5 + \frac{n-1}{2}(6 + (n-2)3) = 5 + \frac{3}{2}(n-1)n$$

$$T_n = \frac{1}{2}(3n^2 - 3n + 10) = a_n$$

$$S_n = \sum a_k = \frac{1}{2}\left[3\frac{(n)(n+1)(2n+1)}{6} - 3\frac{n(n+1)}{2} + 10n\right]$$

$$S_n = \frac{n}{2}(n^2 + 9)$$

$S_{30} = 13635$ & $a_{40} = 2345$

$\therefore \ S_{30} - a_{40} = 11290$

3. Let P be the plane passing through the line $\frac{x-1}{1} = \frac{y-2}{-3} = \frac{z+5}{7}$ and the point (2, 4, –3). If the image of the point (–1, 3, 4) in the plane P is (α, β, γ), then $\alpha + \beta + \gamma$ is equal to

(1) 12

(2) 11

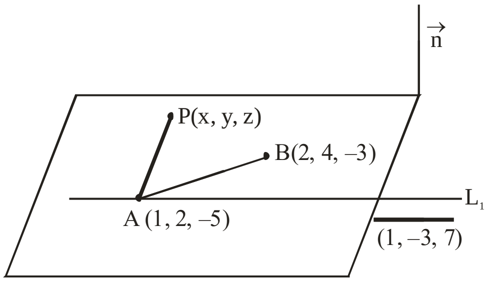
(3) 9

(4) 10

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. Vector $\perp$ to plane is given by



$\vec{n} = \lambda\,((1, 2, 2) \times (1, -3, 7))$

$\vec{n} = \lambda(4\hat{i} - \hat{j} - \hat{k})$

Eq. of plane is given by

$\overrightarrow{AP} \perp \vec{n} \Rightarrow \overrightarrow{AP} \cdot \vec{n} = 0$

$\Rightarrow \left((x-1)\hat{i} + (y-2)\hat{j} + (z+5)\hat{k}\right) \cdot \left(4\hat{i} - \hat{j} - \hat{k}\right) = 0$

$\Rightarrow 4x - y - z - 7 = 0$

Image of point (–1, 3, 4) in plane 4x – y – z – 7 = 0, is given by

$\dfrac{\alpha+1}{4} = \dfrac{\beta-3}{-1} = \dfrac{\gamma-4}{-1} = -2\left(\dfrac{4(-1)-3-4-7}{4^2+1^2+1^2}\right)$

$\alpha = 7\ ;\ \beta = 1\ ;\ \gamma = 2$

$\alpha + \beta + \gamma = 10$

4. Let A = $\left\{\theta \in (0, 2\pi) : \dfrac{1+2i\sin\theta}{1-i\sin\theta} \text{ is purely imaginary}\right\}$.

Then the sum of the elements in A is

(1) π

(2) 2π

(3) 4π

(4) 3π

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. Let $z = \dfrac{1+2i\sin\theta}{1-i\sin\theta} \times \dfrac{1+i\sin\theta}{1+i\sin\theta}$

$z = \dfrac{(1+2i\sin\theta)(1+i\sin\theta)}{1+\sin^2\theta}$

For purely imaginary Re(Z) = 0

$\therefore \dfrac{1-2\sin^2\theta}{1+\sin^2\theta} = 0$

$\sin^2\theta = \dfrac{1}{2}$

$\sin\theta = \pm\dfrac{1}{\sqrt{2}}$

$\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

Sum of the elements in A = 4π

5. The absolute difference of the coefficients of x^{10} and x^7 in the expansion of $\left(2x^2 + \dfrac{1}{2x}\right)^{11}$ is equal to

(1) $12^3 - 12$ (2) $11^3 - 11$

(3) $10^3 - 10$ (4) $13^3 - 13$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $T_{r+1} = {}^{11}C_r (2x^2)^{11-r}\left(\dfrac{1}{2x}\right)^r$

$= {}^{11}C_r\, 2^{11-2r} x^{22-3r}$

Coeff. of x^{10} (r = 4) = ${}^{11}C_4 . 2^3$

Coeff. of x^7 (r = 5) = ${}^{11}C_5 . 2^1$

Absolute difference of coefficients of x^7 & x^{10}

$= \left|{}^{11}C_5 \times 2^1 - {}^{11}C_4 \times 2^3\right|$

$= 12^3 - 12$

6. If the number of words, with or without meaning, which can be made using all the letters of the word MATHEMATICS in which C and S do not come together, is (6!)k, then k is equal to

(1) 1890 (2) 945

(3) 2835 (4) 5670

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. / M / A / T / H / E / M / A / T / I /

Arrange remaining 9 letters and put C and S in any 2 gaps out of 10 gaps.

i.e. $\dfrac{9!}{2!\times 2!\times 2!} \times {}^{10}C_2 \times 2! = (6!)\,k$ (Given)

k = 5670

7. Let S be the set of all values of $\theta \in [-\pi, \pi]$ for which the system of linear equations

$x + y + \sqrt{3}z = 0$

$-x + (\tan\theta)y + \sqrt{7}z = 0$

$x + y + (\tan\theta)z = 0$

has non-trivial solution. Then $\dfrac{120}{\pi}\sum\limits_{\theta \in s}\theta$ is equal to

(1) 40 (2) 10

(3) 20 (4) 30

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. For non-trivial solution D = 0

$$D=\begin{vmatrix}1 & 1 & \sqrt{3}\\ -1 & \tan\theta & \sqrt{7}\\ 1 & 1 & \tan\theta\end{vmatrix}=0$$

$$(\tan\theta-\sqrt{3})(\tan\theta+1)=0$$

$$\tan\theta=\sqrt{3},-1$$

$$\theta=\frac{-2\pi}{3},\frac{\pi}{3},\frac{-\pi}{4},\frac{3\pi}{4}$$

$$\frac{120}{\pi}\sum_{\theta\in S}\theta=20$$

8. If the probability that the random variable X takes values x is given by $P(X = x) = k\,(x + 1)3^{-x}$, x = 0, 1, 2, 3… , where k is a constant, then $P(X \geq 2)$ is equal to

(1) $\frac{7}{27}$ (2) $\frac{11}{18}$

(3) $\frac{7}{18}$ (4) $\frac{20}{27}$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $\sum P=1\Rightarrow k\left(1+2.3^{-1}+3.3^{-2}+....\right)=1$

$$\Rightarrow k=\frac{4}{9}$$

Now, $P(X \geq 2) = 1 - P(X = 0) - P(X = 1)$

$$=1-\left(k+\frac{2k}{3}\right)=\frac{7}{27}$$

9. The value of $36\,(4\cos^2 9^\circ - 1)(4\cos^2 27^\circ - 1)(4\cos^2 81^\circ - 1)(4\cos^2 243^\circ - 1)$ is

(1) 54 (2) 18

(3) 27 (4) 36

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. As we know

$$4\cos^2\theta-1=\frac{\sin 3\theta}{\sin\theta}$$

Value of the above expression will be

$$=36\,.\frac{\sin 27^\circ}{\sin 9^\circ}\cdot\frac{\sin 81^\circ}{\sin 27^\circ}\cdot\frac{\sin 243^\circ}{\sin 81^\circ}\cdot\frac{\sin 729^\circ}{\sin 243^\circ}$$

$$=36\cdot\frac{\sin 729^\circ}{\sin 9^\circ}=36$$

10. The integral $\int\left(\left(\frac{x}{2}\right)^x+\left(\frac{2}{x}\right)^x\right)\log_2 x\,dx$ is equal to

(1) $\left(\frac{x}{2}\right)^x+\left(\frac{2}{x}\right)^x+C$ (2) $\left(\frac{x}{2}\right)^x-\left(\frac{2}{x}\right)^x+C$

(3) $\left(\frac{x}{2}\right)^x\log_2\left(\frac{x}{2}\right)+C$ (4) $\left(\frac{x}{2}\right)^x\log_2\left(\frac{2}{x}\right)+C$

Official Ans. by NTA (2)

Allen Ans. (Bonus)

Sol. **If all 2 replace by e then question is correct and solvable by taking substitution $\left(\frac{x}{e}\right)^x=t$.**

11. The area of the quadrilateral ABCD with vertices A(2, 1, 1), B(1, 2, 5), C (–2, –3, 5) and D (1, –6, –7) is equal to

(1) 48 (2) $8\sqrt{38}$

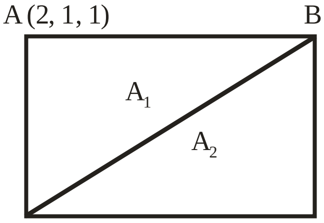
(3) 54 (4) $9\sqrt{38}$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol.

A (2, 1, 1) B (1, 2, 5)



D (1, –6, –7) C (–2, –3, 5)

$$\overrightarrow{AB}\equiv(-1,1,4)$$

$$\overrightarrow{AD}\equiv(-1,-7,-8)$$

$$\overrightarrow{AB}\times\overrightarrow{AD}=\begin{vmatrix}i & j & k\\ -1 & 1 & 4\\ -1 & -7 & -8\end{vmatrix}$$

$$=20\hat{i}-12\hat{j}+8\hat{k}$$

$$A_1=\frac{1}{2}\sqrt{(20)^2+(-12)^2+(8)^2}=2\sqrt{38}$$

$$\overrightarrow{CB}\times\overrightarrow{CD}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 5 & 0\\ 3 & -3 & -12\end{vmatrix}=-60\hat{i}+36\hat{j}-24\hat{k}$$

$$A_2=\frac{1}{2}\sqrt{(60)^2+(36)^2+(-24)^2}=6\sqrt{38}$$

$$\therefore\ \text{Area}=A_1+A_2=8\sqrt{38}$$

12. For a, b $\in$ Z and $|a - b| \le 10$, let the angle between the plane P: $ax + y - z = b$ and the line $l : x - 1 = a - y = z + 1$ be $\cos^{-1}\left(\frac{1}{3}\right)$. If the distance of the point (6, –6, 4) from the plane P is $3\sqrt{6}$, then $a^4 + b^2$ is equal to

(1) 25

(2) 85

(3) 48

(4) 32

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. Line $l : x - 1 = a - y = z + 1$

Line : $\vec{r} = (\hat{i} + a\hat{j} - \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$

P : $ax + y - z = b$; $\vec{n} = (a\hat{i} + \hat{j} - \hat{k})$

So, we have to find angle between plane & line.

$\sin\theta = \cos(90 - \theta) = a$

Given, $\theta = \cos^{-1}\left(\frac{1}{3}\right)$

$$\sin\theta = \left|\frac{a-1-1}{\sqrt{3}\sqrt{a^2+2}}\right| = \frac{2\sqrt{2}}{3}$$

$\Rightarrow 8(a^2 + 2) = 3(a - 2)^2$

$a = -2 \ \& \ \frac{-2}{5}$; $a \in I$

Distance of point

(6, – 6, 4) from plane P

$$= \left|\frac{6a-6-4-b}{\sqrt{a^2+2}}\right| = 3\sqrt{6}$$

Taking $a = -2$

$(b + 22) = 18$

$b = -4$

Hence, $a^4 + b^2 = 32$

13. $25^{190} - 19^{190} - 8^{190} + 2^{190}$ is divisible by

(1) 34 but not by 14

(2) both 14 and 34

(3) neither 14 nor 34

(4) 14 but not by 34

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $25^{190} - 19^{190} - 8^{190} + 2^{190}$

$(25^{190} - 19^{190}) - (8^{190} - 2^{190})$ is divisible by 6

also $(25^{190} - 8^{190}) - (19^{190} - 2^{190})$ is divisible by 17

$\therefore$ Given expression is divisible by 34 but not by 14.

14. Let the vectors $\vec{u}_1 = \hat{i} + \hat{j} + a\hat{k}, \vec{u}_2 = \hat{i} + b\hat{j} + \hat{k}$ and $\vec{u}_3 = c\hat{i} + \hat{j} + \hat{k}$ be coplanar. If the vectors $\vec{v}_1 = (a+b)\hat{i} + c\hat{j} + c\hat{k}$, $\vec{v}_2 = a\hat{i} + (b+c)\hat{j} + a\hat{k}$ and $\vec{v}_3 = b\hat{i} + b\hat{j} + (c+a)\hat{k}$ are also coplanar, then 6 (a + b + c) is equal to

(1) 0

(2) 6

(3) 12

(4) 4

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. For coplanar $\Delta = 0$

$$\begin{vmatrix} 1 & 1 & a \\ 1 & b & 1 \\ c & 1 & 1 \end{vmatrix} = 0 \Rightarrow a + b + c = 2 + abc \ ......(i)$$

$$\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} = 0 \Rightarrow abc = 0$$

$\therefore$ From eq.(i) we get a + b + c =2.

15. Let O be the origin and OP and OQ be the tangents to the circle $x^2 + y^2 - 6x + 4y + 8 = 0$ at the point P and Q on it. If the circumcircle of the triangle OPQ passes through the point $\left(\alpha, \frac{1}{2}\right)$, then a value of α is

(1) $\frac{3}{2}$

(2) $\frac{5}{2}$

(3) 1

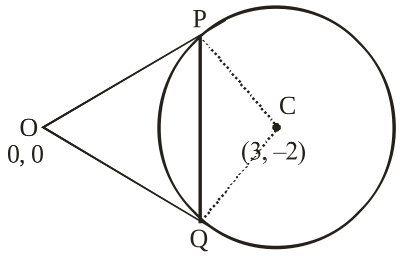
(4) $-\frac{1}{2}$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $x^2 + y^2 - 6x + 4y + 8 = 0$

centre $\equiv (3, -2)$



as O, P, C, Q are concyclic and OC being the diameter, eq^n of circumcircle is [diametric form]

$(x - 0)(x-3) + (y - 0)(y + 2) = 0$

$\left(\alpha, \frac{1}{2}\right)$ lies on the circle

$$(\alpha)(\alpha-3)+\left(\frac{1}{2}\right)\left(\frac{1}{2}+2\right)=0$$

$$\Rightarrow \alpha = \frac{1}{2}, \frac{5}{2}$$

16. The negation of $(p \wedge (\sim q)) \vee (\sim p)$ is equivalent to

(1) $p \wedge q$

(2) $p \wedge (\sim q)$

(3) $p \wedge (q \wedge (\sim p))$

(4) $p \vee (q \vee (\sim p))$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $(p \wedge (\sim q)) \vee (\sim p)$

$= (p \vee (\sim p)) \wedge ((\sim q) \vee (\sim p))$

$= t \wedge \sim (q \wedge p)$ (Demorgan's law)

$= \sim (q \wedge p)$

Negation of $\sim (q \wedge p)$ is $q \wedge p$ or $p \wedge q$

17. If $\alpha > \beta > 0$ are the roots of the equation $ax^2 + bx + 1 = 0$, and

$$\lim_{x\to\frac{1}{\alpha}}\left(\frac{1-\cos(x^2+bx+a)}{2(1-\alpha x)^2}\right)^{\frac{1}{2}}=\frac{1}{k}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)$$,then k is equal to

(1) 2β

(2) 2α

(3) α

(4) β

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. α, β are roots of $ax^2 + bx + 1 = 0$

$\frac{1}{\alpha}, \frac{1}{\beta}$ are roots of $x^2 + bx + a = 0$,

(by transformation)

$$x^2 + bx + a = \left(x - \frac{1}{\alpha}\right)\left(x - \frac{1}{\beta}\right)$$

$$\lim_{x\to\frac{1}{\alpha}}\left[\frac{1-\cos\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)}{2(1-\alpha x)^2}\right]^{\frac{1}{2}} = L$$

$$\left(\text{By using } \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}=\frac{1}{2}\right)$$

$$\Rightarrow \left[\frac{\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^2}{4\alpha^2}\right]^{\frac{1}{2}} = L$$

$$\Rightarrow \frac{\frac{1}{\beta}-\frac{1}{\alpha}}{2\alpha} = L$$

Comparing $k = 2\alpha$

18. If $A = \begin{bmatrix} 1 & 5 \\ \lambda & 10 \end{bmatrix}$, $A^{-1} = \alpha A + \beta I$ and $\alpha + \beta = -2$, then $4\alpha^2 + \beta^2 + \lambda^2$ is equal to:

(1) 12

(2) 10

(3) 19

(4) 14

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $A = \begin{bmatrix} 1 & 5 \\ \lambda & 10 \end{bmatrix}$

$A^{-1} = \alpha A + \beta I$

$\alpha + \beta = -2$

$$A^{-1} = \frac{1}{10-5\lambda}\begin{bmatrix} 10 & -5 \\ -\lambda & 1 \end{bmatrix} = \alpha\begin{bmatrix} 1 & 5 \\ \lambda & 10 \end{bmatrix} + \beta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Comparing we get

$\lambda = 3$

$\alpha = \frac{1}{5}$

$\beta = \frac{-11}{5}$

$4\alpha^2 + \beta^2 + \lambda^2 = 14$

19. Let A(0,1), B(1, 1) and C(1, 0) be the mid – points of the sides of a triangle with incentre at the point D. If the focus of the parabola $y^2 = 4ax$ passing through D is $\left(\alpha+\beta\sqrt{2},0\right)$, where α and β are rational numbers, then $\frac{\alpha}{\beta^2}$ is equal to

(1) 6

(2) 8

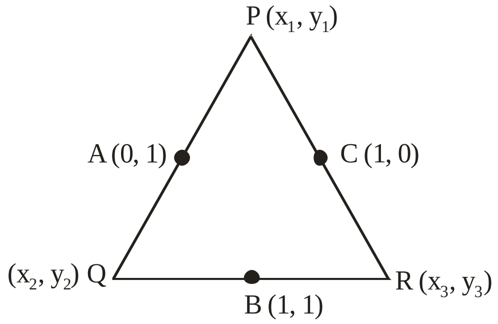
(3) 12

(4) $\frac{9}{2}$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol.



By mid point theorem, we get

$x_1 = 0, x_2 = 0, x_3 = 2$; $y_1 = 0, y_2 = 2, y_3 = 0$

Incentre of ΔPQR ($PQ = 2$, $QR = 2\sqrt{2}$, $PR = 2$)

is D $\left(\frac{4}{4+2\sqrt{2}},\frac{4}{4+2\sqrt{2}}\right)$

parabola $y^2 = 4ax$ passes through D

we get $a = \frac{1}{4+2\sqrt{2}} = \frac{1}{2}-\frac{\sqrt{2}}{4} = \left(\alpha+\beta\sqrt{2},0\right)$

(Given)

$\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{4}$

20. Let A = {1, 2, 3, 4, 5, 6, 7}.Then the relation R = {(x,y) $\in$ A $\times$ A : x + y =7}is

(1) transitive but neither symmetric nor reflexive

(2) reflexive but neither symmetric nor transitive

(3) an equivalence relation

(4) symmetric but neither reflexive nor transitive

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $A = \{1, 2, 3, 4, 5, 6, 7\}$

$R = \{(x, y) \in A \times A : x + y = 7\}$

$x + y = 7$

$y = 7 - x$

$R = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$

$(a, b) \in R \quad \Rightarrow (b, a) \in R$

$\Rightarrow$ Relation is symmetric

SECTION-B

21. Let [t] denote the greatest integer function. If $\int_0^{2.4}\left[x^2\right]dx = \alpha+\beta\sqrt{2}+\gamma\sqrt{3}+\delta\sqrt{5}$, then $\alpha+\beta+\gamma+\delta$ is equal to ____.

Official Ans. by NTA (6)

Allen Ans. (Bonus)

Sol. **Reason : It should be given that $\alpha,\beta,\gamma,\delta \in Q$**

$$\int_0^{2.4}[x^2]dx$$

$$= \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1dx + \int_{\sqrt{2}}^{\sqrt{3}} 2dx + \int_{\sqrt{3}}^{\sqrt{4}} 3\,dx + \int_{\sqrt{4}}^{\sqrt{5}} 4\,dx + \int_{\sqrt{5}}^{2.4} 5\,dx$$

$$= 9-\sqrt{2}-\sqrt{3}-\sqrt{5}$$

$\therefore\ \alpha = 9, \beta = -1, \gamma = -1, \delta = -1$

$\therefore\ \alpha+\beta+\gamma+\delta = 6$

22. Let k and m be positive real numbers such that the function $f(x)=\begin{cases}3x^2+k\sqrt{x+1}, & 0<x<1\\ mx^2+k^2, & x\geq 1\end{cases}$ is differentiable for all x > 0. Then $\frac{8f'(8)}{f'\left(\frac{1}{8}\right)}$ is equal to ____.

Official Ans. by NTA (309)

Allen Ans. (309)

Sol. $f'(x) = \begin{cases} 6x + \dfrac{k}{2\sqrt{x+1}}, & 0 < x < 1 \\ 2mx \quad , & x > 1 \end{cases}$

f(x) is differentiable at all $x > 0$

$\Rightarrow$ f(x) is continuous and differentiable at x = 1

$\Rightarrow 3 + \sqrt{2}k = m + k^2$ and $6 + \dfrac{k}{2\sqrt{2}} = 2m$

$\Rightarrow 3 + \sqrt{2}k = 3 + \dfrac{k}{4\sqrt{2}} + k^2$

$\Rightarrow k = \dfrac{7}{4\sqrt{2}}, m = \dfrac{103}{32}$

Now, $\dfrac{8f'(8)}{f'\left(\frac{1}{8}\right)} = \dfrac{8 \times \frac{103}{16} \times 8}{\frac{6}{8} + \dfrac{7}{4\sqrt{2} \times 2 \times \frac{3}{\sqrt{8}}}} = 309$

23. Let $0 < z < y < x$ be three real numbers such that $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in an arithmetic progression and x, $\sqrt{2}y$, z are in a geometric progression. If xy + yz + zx = $\frac{3}{\sqrt{2}}$xyz, then $3(x+y+z)^2$ is equal to____

Official Ans. by NTA (150)

Allen Ans. (150)

Sol. $\dfrac{2}{y} = \dfrac{1}{x} + \dfrac{1}{z} \quad , \quad 2y^2 = xz$

$\dfrac{xy + yz + zx}{xyz} = \dfrac{3}{\sqrt{2}}$

$\Rightarrow \dfrac{1}{z} + \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{3}{\sqrt{2}}$

$\Rightarrow y = \sqrt{2}$

$\Rightarrow \dfrac{2}{y} = \dfrac{1}{x} + \dfrac{1}{z} \quad , \quad 2y^2 = xz$

$\Rightarrow x + z = 4\sqrt{2} \quad , \quad 4 = xz$

$\Rightarrow x = 2(\sqrt{2} + 1)$

$\Rightarrow z = \dfrac{4}{2(\sqrt{2}+1)} = 2(\sqrt{2} - 1)$

Now, $3(x + y + z)^2 = 3(5\sqrt{2})^2 = 150$

24. If domain of the function

$\log_e\left(\dfrac{6x^2 + 5x + 1}{2x - 1}\right) + \cos^{-1}\left(\dfrac{2x^2 - 3x + 4}{3x - 5}\right)$ is (α, β) $\cup (\gamma, \delta]$, then $18(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)$ is equal to ____

Official Ans. by NTA (20)

Allen Ans. (20)

Sol. $D_f : \dfrac{6x^2 + 5x + 1}{2x - 1} > 0, \dfrac{2x^2 - 3x + 4}{3x - 5} \geq -1, \dfrac{2x^2 - 3x + 4}{3x - 5} \leq 1$

$D_f : \left(\dfrac{-1}{2}, \dfrac{-1}{3}\right) \cup \left(\dfrac{1}{2}, \dfrac{1}{\sqrt{2}}\right]$

25. Let m and n be the numbers of real roots of the quadratic equations $x^2 - 12x + [x] + 31 = 0$ and $x^2 - 5|x + 2| - 4 = 0$ respectively, where [x] denotes the greatest integer $\leq x$. Then $m^2 + mn + n^2$ is equal to ____.

Official Ans. by NTA (9)

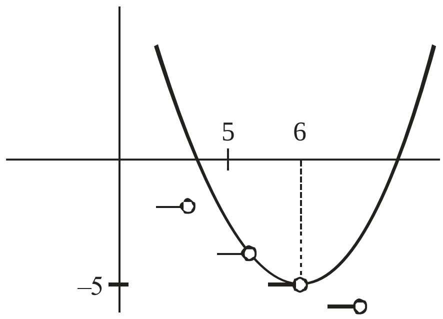
Allen Ans. (9)

Sol. $x^2 - 12x + [x] + 31 = 0$

$x^2 - 12x + 31 = -[x]$

$(x - 6)^2 - 5 = -[x]$

By graph



zero point of intersection, m = 0

$x^2 - 5|x + 2| - 4 = 0$

case-I : $x < -2$

$x^2 + 5x + 6 = 0$

$x = -3, -2$ (rejected)

case-II : $x \geq -2$

$x^2 - 5x - 14 = 0$

$x = 7, -2$

No. of solution (n) = 3

So $m^2 + mn + n^2 = 9$

26. The ordinates of the points P and Q on the parabola with focus (3, 0) and directrix x = –3 are in the ratio 3 : 1. If R(α, β) is the point of intersection of the tangents to the parabola at P and Q, then $\frac{\beta^2}{\alpha}$ is equal to____:

Official Ans. by NTA (16)

Allen Ans. (16)

Sol. Given parabola : $y^2 = 12x$

Let P : $(3t_1^2, 6t_1)$ & Q : $(3t_2^2, 6t_2)$

$\frac{t_1}{t_2} = 3 \quad \Rightarrow t_1 = 3t_2$

Point of intersection of tangent (α, β)

$\alpha = 3t_1 \cdot t_2 = 9t_2^2$

$\beta = 3(t_1 + t_2) = 12t_2$

Now, $\frac{\beta^2}{\alpha} = \frac{144t_2^2}{9t_2^2} = 16$

27. Let the solution curve x = x(y), $0 < y < \frac{\pi}{2}$, of the differential equation $(\log_e(\cos y))^2\cos y\, dx - (1 + 3x \log_e(\cos y))\sin y\, dy = 0$ satisfy $x\left(\frac{\pi}{3}\right) = \frac{1}{2\log_e 2}$. If $x\left(\frac{\pi}{6}\right) = \frac{1}{\log_e m - \log_e n}$, where m and n are co-prime, then mn is equal to

Official Ans. by NTA (12)

Allen Ans. (12)

Sol. $(\log_e(\cos y))^2\cos y\, dx - (1+ 3x \log_e(\cos y))\sin y\, dy = 0$

$$\frac{dx}{dy} - \frac{3\sin y}{\cos y(\log_e \cos y)}x = \frac{\sin y}{(\log_e \cos y)^2 \cdot \cos y}$$

I.F = $e^{\int \frac{-3\sin y}{\cos y(\log_e \cos y)}dy}$

Put $\ell n(\cos y) = t$

I.F = $e^{\int \frac{3}{t}dt} = (\ell n \cos y)^3$

$$x.(\log_e \cos y)^3 = \int (\log_e \cos y)^3 \cdot \frac{\sin y}{(\log_e \cos y)^2 \times \cos y}dy$$

$$x.(\log_e \cos y)^3 = \frac{-(\log_e \cos y)^2}{2} + c$$

Given, $x\left(\frac{\pi}{3}\right) = \frac{1}{2\log_e 2}$

c = 0

$x = \frac{-1}{2\ell n(\cos y)}$

$x\left(\frac{\pi}{6}\right) = \frac{1}{\ell n 4 - \ell n 3}$

m = 4, n = 3

Hence, m.n = 12

28. Let P_1 be the plane 3x – y – 7z = 11 and P_2 be the plane passing through the points (2, –1, 0), (2, 0, –1), and (5, 1, 1). If the foot of the perpendicular drawn from the point (7, 4, –1)on the line of intersection of the planes P_1 and P_2 is (α, β, γ), then α + β + γ is equal to ____.

Official Ans. by NTA (11)

Allen Ans. (11)

Sol. Given,

$P_1 : 3x - y - 7z = 11 \; ; \; \vec{n}_1 = (3, -1, -7)$

$$P_2 : \begin{vmatrix} x-2 & y+1 & z-0 \\ 2-2 & 0+1 & -1-0 \\ 5-2 & 1+1 & 1-0 \end{vmatrix} = 0$$

$\Rightarrow x - y - z = 3 \quad ; \quad \vec{n}_2 = (1, -1, -1)$

Vector along line of intersection is $\vec{n} = \vec{n}_1 \times \vec{n}_2$

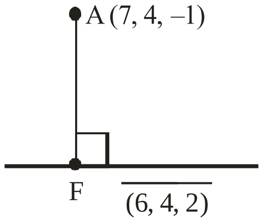
$\vec{n} = 6\hat{i} + 4\hat{j} + 2\hat{k}$

We need a point on L.O.I. : put z = 0 in plane equations, solving eq. we get x = 4, y = 1

Required line of intersection

$$L : \frac{x-4}{6} = \frac{y-1}{4} = \frac{z-0}{2} = \lambda \text{(let)}$$

Any point on line $F \equiv (6\lambda+4, 4\lambda+1, 2\lambda)$



F being foot of perpendicular from A

$$\overrightarrow{AF}.\vec{n} = 0 \Rightarrow \lambda = \frac{1}{2}$$

$F \equiv (7, 3, 1) \equiv (\alpha, \beta, \gamma)$

29. Let R = {a, b, c, d, e}and S = {1, 2, 3, 4}. Total number of onto function $f : R \to S$ such that $f(a) \neq 1$, is equal to _____.

Official Ans. by NTA (384)

Allen Ans. (180)

Sol. Total no. of onto function provided $f(a) \neq 1$

= Total no. of onto function – No. of onto function when f(a) = 1

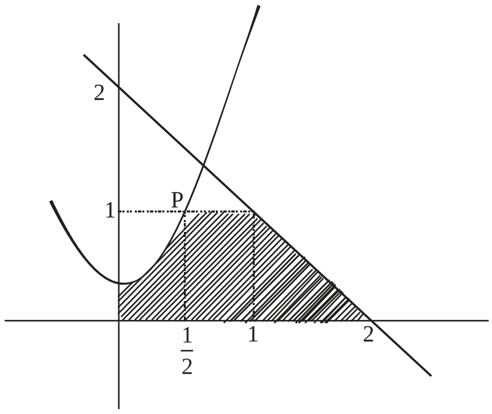
$$= \frac{5!}{2!3!} \times 4! - \left(\frac{4!}{2!2!} \times 3! + 4!\right) = 180$$

30. Let the area enclosed by the lines x + y = 2, y = 0, x = 0 and the curve $f(x) = \min\left\{x^2 + \frac{3}{4}, 1 + [x]\right\}$ where [x] denotes the greatest integer ≤ x, be A. Then the value of 12A is _____

Official Ans. by NTA (17)

Allen Ans. (17)

Sol.



Shaded region is the required area

$$\text{Area} = \int_0^{\frac{1}{2}} \left(x^2 + \frac{3}{4}\right) dx + \left(\frac{1}{2} \times 1\right) + \left(\frac{1}{2} \times 1 \times 1\right)$$

$$= \frac{17}{12}$$

Thus 12A = 17