JEE–MAIN EXAMINATION – JANUARY 2025

(HELD ON WEDNESDAY 22^{nd} JANUARY 2025) **TIME : 3 : 00 PM TO 6 : 00 PM**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. Let α, β, γ and δ be the coefficients of x^7, x^5, x^3 and x respectively in the expansion of $\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$, x > 1. If u and v satisfy the equations

$\alpha u + \beta v = 18$,

$\gamma u + \delta v = 20$,

then u + v equals :

(1) 5 (2) 4

(3) 3 (4) 8

Ans. **(1)**

Sol. $\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$

$= 2\{{}^5C_0.x^5 + {}^5C_2.x^3(x^3-1) + {}^5C_4.x(x^3-1)^2\}$

$= 2\{5x^7 + 10x^6 + x^5 - 10x^4 - 10x^3 + 5x\}$

$\Rightarrow \alpha = 10, \beta = 2, \gamma = -20, \delta = 10$

Now, $10u + 2v = 18$

$-20u + 10v = 20$

$\Rightarrow u = 1, v = 4$

$u + v = 5$

2. In a group of 3 girls and 4 boys, there are two boys B_1 and B_2. The number of ways, in which these girls and boys can stand in a queue such that all the girls stand together, all the boys stand together, but B_1 and B_2 are not adjacent to each other, is :

(1) 144 (2) 72

(3) 96 (4) 120

Ans. **(1)**

Sol. Total – when B_1 and B_2 are together

$= 2!(3!\,4!) - 2!\,(3!(3!\,2!)) = 144$

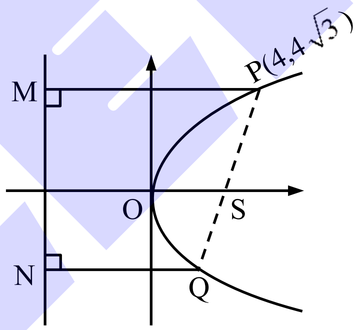
3. Let $P\left(4,4\sqrt{3}\right)$ be a point on the parabola $y^2 = 4ax$ and PQ be a focal chord of the parabola. If M and N are the foot of perpendiculars drawn from P and Q respectively on the directrix of the parabola, then the area of the quadrilateral PQMN is equal to:

(1) $\frac{263\sqrt{3}}{8}$ (2) $17\sqrt{3}$

(3) $\frac{343\sqrt{3}}{8}$ (4) $\frac{34\sqrt{3}}{3}$

Ans. **(3)**

Sol.



$(4, 4\sqrt{3})$ lies on $y^2 = 4ax$

$\Rightarrow 48 = 4a.4$

$4a = 12$

$\Rightarrow y^2 = 12x$ is equation of parabola

Now, parameter of P is $t_1 = \frac{2}{\sqrt{3}} \Rightarrow$ Parameters of Q is $t_2 = -\frac{\sqrt{3}}{2} \Rightarrow Q\left(\frac{9}{4}, -3\sqrt{3}\right)$

Area of trapezium PQNM

$= \frac{1}{2}MN.(PM + QN)$

$= \frac{1}{2}MN.(PS + QS)$

$= \frac{1}{2}MN.\ PQ$

$= \frac{1}{2}7\sqrt{3}.\frac{49}{4} = (343)\frac{\sqrt{3}}{8} = 3$

4. For a 3 × 3 matrix M, let trace (M) denote the sum of all the diagonal elements of M. Let A be a 3 × 3 matrix such that $|A| = \frac{1}{2}$ and trace (A) = 3. If B = adj(adj(2A)), then the value of |B| + trace (B) equals:

(1) 56 (2) 132

(3) 174 (4) 280

Ans. **(4)**

Sol. $|A|=\frac{1}{2}$, trace(A) = 3, $B = \text{adj}(\text{adj}(2A))=|2A|^{n-2}(2A)$

$n = 3, B = |2A|(2A) = 2^3.|A|(2A) = 8A$

$|B| = |8A| = 8^3.|A| = 2^8 = 256$

trace(B) = 8 trace(A) = 24

|B| + trace(B) = 280

5. Suppose that the number of terms in an A.P. is 2k, k∈N. If the sum of all odd terms of the A.P. is 40, the sum of all even terms is 55 and the last term of the A.P. exceeds the first term by 27, then k is equal to

(1) 5 (2) 8

(3) 6 (4) 4

Ans. **(1)**

Sol. $a_1, a_2, a_3, \ldots\ldots, a_{2k} \rightarrow$ A.P.

$\sum_{r=1}^{k} a_{2r-1} = 40$, $\sum_{r=1}^{k} a_{2r} = 55$, $a_{2k} - a_1 = 27$

$\frac{k}{2}[2a_1 + (k-1)2d] = 40$, $\frac{k}{2}[2a_2 + (k-1)2d] = 55$,

$d = \frac{27}{2k-1}$

$a_1 = \frac{40}{k} - (k-1)d = \frac{55}{k} - kd$

$d = \frac{15}{k} \Rightarrow \frac{27}{2k-1} = \frac{15}{k} \Rightarrow 9k = 10k - 5$

$\therefore k = 5$

6. Let a line pass through two distinct points P(–2, –1, 3) and Q, and be parallel to the vector $3\hat{i}+2\hat{j}+2\hat{k}$. If the distance of the point Q from the point R(1, 3, 3) is 5, then the square of the area of ΔPQR is equal to:

(1) 136 (2) 140

(3) 144 (4) 148

Ans. **(1)**

Sol. $\overrightarrow{PQ}$ parallel to $3\hat{i}+2\hat{j}+2\hat{k}$, R(1, 3, 3)

$\Rightarrow Q(3\lambda - 2, 2\lambda - 1, 2\lambda + 3), \lambda \in R - \{0\}$

$|\overrightarrow{QR}| = 5 = \sqrt{(3\lambda-3)^2+(2\lambda-4)^2+(2\lambda)^2}$

$\therefore 17\lambda^2 - 34\lambda + 25 = 25 \Rightarrow \lambda = 2 (\because \lambda \neq 0)$

$\therefore Q(4, 3, 7), P(-2, -1, 3), R(1, 3, 3)$

Area of $\Delta PQR = [PQR] = \frac{1}{2}|\overrightarrow{PQ}\times\overrightarrow{PR}|$

$$[PQR] = \frac{1}{2}\left\|\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 6 & 4 & 4\\ 3 & 4 & 0\end{vmatrix}\right\| = \left\|\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 2 & 2\\ 3 & 4 & 0\end{vmatrix}\right\|$$

$[PQR] = |-8\hat{i}+6\hat{j}+6\hat{k}| = \sqrt{136}$

$\therefore [PQR]^2 = 136$

7. If $\lim_{x\to\infty}\left(\left(\frac{e}{1-e}\right)\left(\frac{1}{e}-\frac{x}{1+x}\right)\right)^x = \alpha$, then the value of $\frac{\log_e \alpha}{1+\log_e \alpha}$ equals :

(1) e (2) e^{-2}

(3) e^2 (4) e^{-1}

Ans. **(1)**

Sol. $\alpha = \lim_{x\to\infty}\left(\left(\frac{e}{1-e}\right)\left(\frac{1}{e}-\frac{x}{1+x}\right)\right)^x$ (1^∞ form)

$\therefore \alpha = e^L$

Where $L = \lim_{x\to\infty} x\left(\left(\frac{e}{1-e}\right)\left(\frac{1}{e}-\frac{x}{1+x}\right)-1\right)$

$\Rightarrow L = \lim_{x\to\infty}\left(\frac{e}{1-e}\right)x\left(\frac{1}{e}-\frac{x}{1+x}-\left(\frac{1-e}{e}\right)\right)$

$\Rightarrow L = \frac{e}{1-e}\lim_{x\to\infty} x\left(1-\frac{x}{1+x}\right)$

$\Rightarrow L = \frac{e}{1-e}\lim_{x\to\infty}\frac{x}{x+1}$

$\Rightarrow L = \frac{e}{1-e}.1$

$\Rightarrow L = \frac{e}{1-e}$

$\therefore \alpha = e^{\frac{e}{1-e}} \Rightarrow \log\alpha = \frac{e}{1-e}$

$\therefore$ Required value $= \frac{\frac{e}{1-e}}{1+\frac{e}{1-e}} = e$

8. Let $f(x) = \int_0^{x^2} \frac{t^2 - 8t + 15}{e^t} dt$, $x \in \mathbf{R}$. Then the numbers of local maximum and local minimum points of f, respectively, are :

(1) 2 and 3 (2) 3 and 2
(3) 1 and 3 (4) 2 and 2

Ans. (1)

Sol. $f'(x) = \left(\frac{x^4 - 8x^2 + 15}{e^{x^2}}\right)(2x)$

$$= \frac{(x^2-3)(x^2-5)(2x)}{e^{x^2}}$$

$$= \frac{(x-\sqrt{3})(x+\sqrt{3})(x-\sqrt{5})(x+\sqrt{5})2x}{e^{x^2}}$$

Sign scheme: $-$ | $-\sqrt{5}$ | $+$ | $-\sqrt{3}$ | $-$ | 0 | $+$ | $\sqrt{3}$ | $-$ | $\sqrt{5}$ | $+$

Maxima at $x \in \{-\sqrt{3}, \sqrt{3}\}$

Minima at $x \in \{-\sqrt{5}, 0, \sqrt{5}\}$

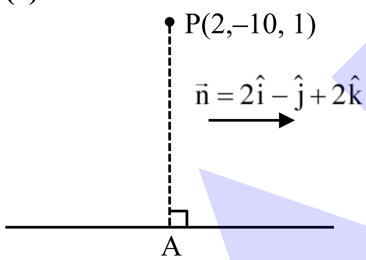
2 points of maxima and 3 points of minima.

9. The perpendicular distance, of the line $\frac{x-1}{2} = \frac{y+2}{-1} = \frac{z+3}{2}$ from the point P(2, –10,1), is:

(1) 6 (2) $5\sqrt{2}$
(3) $3\sqrt{5}$ (4) $4\sqrt{3}$

Ans. (3)

Sol.



$$\frac{x-1}{2} = \frac{y+2}{-1} = \frac{z+3}{2} = \lambda \text{ (let)}$$

$(2\lambda + 1, -\lambda - 2, 2\lambda - 3)$

$\because \overrightarrow{PA}.\vec{n} = 0$

$\Rightarrow (2\lambda - 1)2 + (-\lambda + 8)(-1) + (2\lambda - 4)2 = 0$

$\Rightarrow 4\lambda - 2 + \lambda - 8 + 4\lambda - 8 = 0$

$\Rightarrow 9\lambda - 18 = 0 \Rightarrow \lambda = 2$

$\therefore$ A(5, –4, 1)

$\therefore AP = \sqrt{3^2 + 6^2 + 0^2} = \sqrt{45} = 3\sqrt{5}$

10. If x = f(y) is the solution of the differential equation

$$(1 + y^2) + \left(x - 2e^{\tan^{-1} y}\right)\frac{dy}{dx} = 0,\ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

with f(0) = 1, then $f\left(\frac{1}{\sqrt{3}}\right)$ is equal to :

(1) $e^{\pi/4}$ (2) $e^{\pi/12}$
(3) $e^{\pi/3}$ (4) $e^{\pi/6}$

Ans. (4)

Sol. $\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{2e^{\tan^{-1} y}}{1+y^2}$

I.F. $= e^{\tan^{-1} y}$

$$xe^{\tan^{-1} y} = \int \frac{2(e^{\tan^{-1} y})^2 dy}{1+y^2}$$

Put $\tan^{-1} y = t$, $\frac{dy}{1+y^2} = dt$

$xe^{\tan^{-1} y} = \int 2e^{2t} dt$

$xe^{\tan^{-1} y} = e^{2\tan^{-1} y} + c$

$x = e^{\tan^{-1} y} + ce^{-\tan^{-1} y}$

$\because$ y = 0, x = 1

$1 = 1 + c \Rightarrow c = 0$

$y = \frac{1}{\sqrt{3}}$, $x = e^{\pi/6}$

11. If $\int e^x \left(\frac{x\sin^{-1} x}{\sqrt{1-x^2}} + \frac{\sin^{-1} x}{(1-x^2)^{3/2}} + \frac{x}{1-x^2}\right) dx = g(x) + C$, where C is the constant of integration, then $g\left(\frac{1}{2}\right)$ equals :

(1) $\frac{\pi}{6}\sqrt{\frac{e}{2}}$ (2) $\frac{\pi}{4}\sqrt{\frac{e}{2}}$
(3) $\frac{\pi}{6}\sqrt{\frac{e}{3}}$ (4) $\frac{\pi}{4}\sqrt{\frac{e}{3}}$

Ans. (3)

Sol. $\because \frac{d}{dx}\left(\frac{x\sin^{-1} x}{\sqrt{1-x^2}}\right) = \frac{\sin^{-1} x}{(1-x^2)^{3/2}} + \frac{x}{1-x^2}$

$$\Rightarrow \int e^x \left(\frac{x\sin^{-1} x}{\sqrt{1-x^2}} + \frac{\sin^{-1} x}{(1-x^2)^{3/2}} + \frac{x}{1-x^2}\right) dx$$

$$= e^x . \frac{x\sin^{-1} x}{\sqrt{1-x^2}} + c = g(x) + C$$

Note : assuming $g(x) = \dfrac{xe^x \sin^{-1} x}{\sqrt{1-x^2}}$

$$g(1/2) = \frac{e^{1/2}}{2} \cdot \frac{\frac{\pi}{6}\times 2}{\sqrt{3}} = \frac{\pi}{6}\sqrt{\frac{e}{3}}$$

Comment : In this question we will not get a unique function g(x), but in order to match the answer we will have to assume $g(x) = \dfrac{xe^x \sin^{-1} x}{\sqrt{1-x^2}}$.

12. Let α_θ and β_θ be the distinct roots of $2x^2 + (\cos\theta)x - 1 = 0$, $\theta \in (0, 2\pi)$. If m and M are the minimum and the maximum values of $\alpha_\theta^4 + \beta_\theta^4$, then 16(M + m) equals :

(1) 24 (2) 25

(3) 27 (4) 17

Ans. **(2)**

Sol. $(\alpha^2 + \beta^2)^2 - 2\,\alpha^2\beta^2$

$[(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$

$$\left[\frac{\cos^2\theta}{4} + 1\right]^2 - 2.\frac{1}{4}$$

$$\left(\frac{\cos^2\theta}{4} + 1\right)^2 - \frac{1}{2}$$

$$M = \frac{25}{16} - \frac{1}{2} = \frac{17}{16}$$

$$m = \frac{1}{2},\ 16(M + m) = 25$$

13. Let A = {1, 2, 3, 4} and B = {1, 4, 9, 16}. Then the number of many-one functions $f : A \to B$ such that $1 \in f(A)$ is equal to :

(1) 127 (2) 151

(3) 163 (4) 139

Ans. **(2)**

Sol. Total = 4^4

One-one = 4!

Many-one = 256 – 24 = 232

Many-one which $1 \notin f(A)$

= 3.3.3.3 = 81

232 – 81 = 151

14. If the system of linear equations :

x + y + 2z = 6,

2x + 3y + az = a + 1,

–x – 3y + bz = 2b,

where a, b $\in$ **R**, has infinitely many solutions, then 7a + 3b is equal to :

(1) 9 (2) 12

(3) 16 (4) 22

Ans. **(3)**

Sol. $$\Delta = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 3 & a \\ -1 & -3 & b \end{vmatrix} = 0$$

$\Rightarrow 2a + b - 6 = 0$ ....(1)

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 3 & a+1 \\ -1 & -3 & 2b \end{vmatrix} = 0$$

$\Rightarrow a + b - 8 = 0$ ....(2)

Solving (1) + (2)

a = –2, b = 10

$\Rightarrow 7a + 3b = 16$

15. Let $\vec{a}$ and $\vec{b}$ be two unit vectors such that the angle between them is $\frac{\pi}{3}$. If $\lambda\vec{a} + 2\vec{b}$ and $3\vec{a} - \lambda\vec{b}$ are perpendicular to each other, then the number of values of λ in [–1, 3] is :

(1) 3 (2) 2

(3) 1 (4) 0

Ans. **(4)**

Sol. $\hat{a}.\hat{b} = \frac{1}{2}$

Now $\left(\lambda\hat{a} + 2\hat{b}\right).\left(3\hat{a} - \lambda\hat{b}\right) = 0$

$3\lambda\hat{a}.\hat{a} - \lambda^2\hat{a}.\hat{b} + 6\hat{a}.\hat{b} - 2\lambda\hat{b}.\hat{b} = 0$

$3\lambda - \frac{\lambda^2}{2} + 3 - 2\lambda = 0$

$\lambda^2 - 2\lambda - 6 = 0$

$\lambda = 1 \pm \sqrt{7}$

$\Rightarrow$ number of values = 0

16. Let E : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, a > b and H : $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$. Let the distance between the foci of E and the foci of H be $2\sqrt{3}$. If a – A = 2, and the ratio of the eccentricities of E and H is $\frac{1}{3}$, then the sum of the lengths of their latus rectums is equal to :

(1) 10 (2) 7

(3) 8 (4) 9

Ans. **(3)**

Sol. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ foci are (ae, 0) and (–ae, 0)

$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ foci are (Ae′, 0) and (–Ae′, 0)

$\Rightarrow 2ae = 2\sqrt{3} \Rightarrow ae = \sqrt{3}$

and $2Ae' = 2\sqrt{3} \Rightarrow Ae' = \sqrt{3}$

$\Rightarrow ae = Ae' \Rightarrow \frac{e}{e'} = \frac{A}{a}$

$\Rightarrow \frac{1}{3} = \frac{A}{a} \Rightarrow a = 3A$

Now $a - A = 2 \Rightarrow a - \frac{a}{3} - 2 \Rightarrow a = 3$ and $A = 1$

$Ae = \sqrt{3} \Rightarrow e = \frac{1}{\sqrt{3}}$ and $e' = \sqrt{3}$

$b^2 = a^2(1 - e^2)$

$b^2 = 6$

and $B^2 = A^2((e')^2 - 1) = (2) \Rightarrow B^2 = 2$

sum of LR = $\frac{2b^2}{a} + \frac{2B^2}{A} = 8$

17. If A and B are two events such that $P(A \cap B) = 0.1$, and $P(A|B)$ and $P(B|A)$ are the roots of the equation $12x^2 - 7x + 1 = 0$, then the value of $\frac{P(\bar{A} \cup \bar{B})}{P(\bar{A} \cap \bar{B})}$ is:

(1) $\frac{5}{3}$ (2) $\frac{4}{3}$

(3) $\frac{9}{4}$ (4) $\frac{7}{4}$

Ans. **(3)**

Sol. $12x^2 - 7x + 1 = 0$

$x = \frac{1}{3}, \frac{1}{4}$

Let $P\left(\frac{A}{B}\right) = \frac{1}{3}$ & $P\left(\frac{B}{A}\right) = \frac{1}{4}$

$\frac{P(A \cap B)}{P(B)} = \frac{1}{3}$ & $\frac{P(A \cap B)}{P(A)} = \frac{1}{4}$

$\Rightarrow P(B) = 0.3$

& $P(A) = 0.4$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.3 + 0.4 - 0.1 = 0.6$

Now $\frac{P(\bar{A} \cup \bar{B})}{P(\bar{A} \cap \bar{B})} = \frac{P(\overline{A \cap B})}{P(\overline{A \cup B})}$

$= \frac{1 - P(A \cap B)}{1 - P(A \cup B)} = \frac{1 - 0.1}{1 - 0.6} = \frac{9}{4}$

18. The sum of all values of θ ∈ [0, 2π] satisfying $2\sin^2\theta = \cos 2\theta$ and $2\cos^2\theta = 3\sin\theta$ is

(1) $\frac{\pi}{2}$ (2) 4π

(3) $\frac{5\pi}{6}$ (4) π

Ans. **(4)**

Sol. $2\sin^2\theta = \cos 2\theta$

$2\sin^2\theta = 1 - 2\sin^2\theta$

$4\sin^2\theta = 1$

$\sin^2\theta = \frac{1}{4}$

$\sin\theta = \pm\frac{1}{2}$

$2\cos^2\theta = 3\sin\theta$

$2 - 2\sin^2\theta + 3\sin\theta - 2 = 0$

$(2\sin\theta - 1)(2\sin\theta - 2) = 0$

$\sin\theta = \frac{1}{2}$

so common equation which satisfy both equations is $\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ $(\theta \in [0, 2\pi])$

Sum $= \pi$

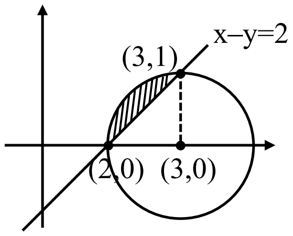
19. Let the curve $z(1 + i) + \bar{z}(1-i) = 4$, $z \in C$, divide the region $|z - 3| \le 1$ into two parts of areas α and β. Then $|\alpha - \beta|$ equals :

(1) $1+\frac{\pi}{2}$ (2) $1+\frac{\pi}{3}$

(3) $1+\frac{\pi}{4}$ (4) $1+\frac{\pi}{6}$

Ans. **(1)**

Sol.



Let $z = x + iy$

$(x + iy)(1 + i) + (x - iy)(1 - i) = 4$

$x + ix + iy - y + x - ix - iy - y = 4$

$2x - 2y = 4$

$x - y = 2$

$|z - 3| \le 1$

$(x - 3)^2 + y^2 \le 1$

Area of shaded region $= \frac{\pi.1^2}{4} - \frac{1}{2}.1.1 = \frac{\pi}{4} - \frac{1}{2}$

Area of unshaded region inside the circle

$= \frac{3}{4}\pi.1^2 + \frac{1}{2}.1.1 = \frac{3\pi}{4} + \frac{1}{2}$

$\therefore$ difference of area $= \left(\frac{3\pi}{4}+\frac{1}{2}\right)-\left(\frac{\pi}{4}-\frac{1}{2}\right)$

$= \frac{\pi}{2}+1$

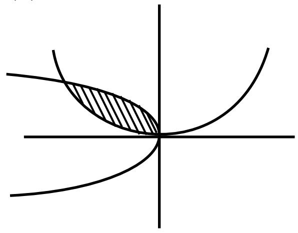
20. The area of the region enclosed by the curves $y = x^2 - 4x + 4$ and $y^2 = 16 - 8x$ is :

(1) $\frac{8}{3}$ (2) $\frac{4}{3}$

(3) 5 (4) 8

Ans. **(1)**

Sol.



$y = (x - 2)^2$, $y^2 = 8(x - 2)$

$y = x^2$, $y^2 = -8x$

$= \frac{16ab}{3} = \frac{16\times\frac{1}{4}\times 2}{3} = \frac{8}{3}$

SECTION-B

21. Let $y = f(x)$ be the solution of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2-1} = \frac{x^6+4x}{\sqrt{1-x^2}}$, $-1 < x < 1$ such that $f(0) = 0$. If $6\int_{-1/2}^{1/2} f(x)dx = 2\pi - \alpha$ then α^2 is equal to ______.

Ans. **(27)**

Sol. I.F. $e^{-\frac{1}{2}\int\frac{2x}{1-x^2}dx} = e^{-\frac{1}{2}\ell n(1-x^2)} = \sqrt{1-x^2}$

$y\times\sqrt{1-x^2} = \int(x^6+4x)dx = \frac{x^7}{7}+2x^2+c$

Given $y(0) = 0 \Rightarrow c = 0$

$y = \frac{\frac{x^7}{7}+2x^2}{\sqrt{1-x^2}}$

Now, $6\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\frac{x^7}{7}+2x^2}{\sqrt{1-x^2}}dx = 6\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{2x^2}{\sqrt{1-x^2}}dx$

$= 24\int_0^{\frac{1}{2}}\frac{x^2}{\sqrt{1-x^2}}dx$

Put $x = \sin\theta$

$dx = \cos\theta \, d\theta$

$$= 24\int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\cos\theta}\cos\theta d\theta$$

$$= 24\int_0^{\frac{\pi}{6}} \left(\frac{1-\cos 2\theta}{2}\right) d\theta = 12\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{6}}$$

$$= 12\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$$

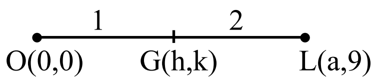
$$= 2\pi - 3\sqrt{3}$$

$$\alpha^2 = (3\sqrt{3})^2 = 27$$

22. Let A(6, 8), B(10 cosα, –10 sinα) and C (–10 sinα, 10 cosα), be the vertices of a triangle. If L(a, 9) and G(h, k) be its orthocenter and centroid respectively, then (5a – 3h + 6k + 100 sin2α) is equal to___________

Ans. (145)

Sol. All the three points A, B, C lie on the circle $x^2 + y^2 = 100$ so circumcentre is (0, 0)



$$\frac{a+0}{3} = h \Rightarrow a = 3h$$

and $\dfrac{9+0}{3} = k \Rightarrow k = 3$

also centroid $\dfrac{6+10\cos\alpha - 10\sin\alpha}{3} = h$

$\Rightarrow 10(\cos\alpha - \sin\alpha) = 3h - 6$ ....(i)

and $\dfrac{8+10\cos\alpha - 10\sin\alpha}{3} = k$

$\Rightarrow 10(\cos\alpha - \sin\alpha) = 3k - 8 = 9 - 8 = 1$....(ii)

on squaring $100(1 - \sin 2\alpha) = 1$

$\Rightarrow 100\sin 2\alpha = 99$

from equ. (i) and (ii) we get $h = \dfrac{7}{3}$

Now $5a - 3h + 6k + 100 \sin 2\alpha$

$= 15h - 3h + 6k + 100 \sin 2\alpha$

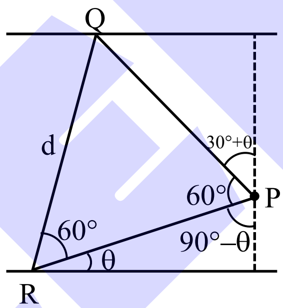
$= 12 \times \dfrac{7}{3} + 18 + 99$

$= 145$

23. Let the distance between two parallel lines be 5 units and a point P lie between the lines at a unit distance from one of them. An equilateral triangle PQR is formed such that Q lies on one of the parallel lines, while R lies on the other. Then $(QR)^2$ is equal to _________.

Ans. (28)

Sol.



$PR = \text{cosec}\theta$, $PQ = 4\sec(30 + \theta)$

For equilateral

$d = PR = PQ$

$\Rightarrow \cos(\theta + 30°) = 4\sin\theta$

$$\Rightarrow \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta = 4\sin\theta$$

$$\Rightarrow \tan\theta = \frac{1}{3\sqrt{3}}$$

$QR^2 = d^2 = \text{cosec}^2\theta = 28$

24. If $\displaystyle\sum_{r=1}^{30} \frac{r^2\left({}^{30}C_r\right)^2}{{}^{30}C_{r-1}} = \alpha \times 2^{29}$, then α is equal to ______.

Ans. (465)

Sol. $\sum_{r=1}^{30} \frac{r^2({}^{30}C_r)^2}{{}^{30}C_{r-1}}$

$= \sum_{r=1}^{30} r^2\left(\frac{31-r}{r}\right).\frac{30!}{r!(30-r)!}$

$\left(\because \frac{{}^{30}C_r}{{}^{30}C_{r-1}} = \frac{30-r+1}{r} = \frac{31-r}{r}\right)$

$= \sum_{r=1}^{30} \frac{(31-r)30!}{(r-1)!(30-r)!}$

$= 30\sum_{r=1}^{30} \frac{(31-r)29!}{(r-1)!(30-r)!}$

$= 30\sum_{r=1}^{30} (30-r+1)\,{}^{29}C_{30-r}$

$= 30\left(\sum_{r=1}^{30}(31-r)\,{}^{29}C_{30-r} + \sum_{r=1}^{30} {}^{29}C_{30-r}\right)$

$= 30(29 \times 2^{28} + 2^{29}) = 30(29 + 2)2^{28}$

$= 15 \times 31 \times 2^{29}$

$= 465(2^{29})$

$\alpha = 465$

25. Let A = {1, 2, 3}. The number of relations on A, containing (1, 2) and (2, 3), which are reflexive and transitive but not symmetric, is ______.

Ans. **(3)**

Sol. Transitivity

$(1, 2) \in R, (2, 3) \in R \Rightarrow (1, 3) \in R$

For reflexive $(1, 1), (2, 2)\ (3, 3) \in R$

Now (2, 1), (3, 2), (3, 1)

(3, 1) cannot be taken

(1) (2, 1) taken and (3, 2) not taken

(2) (3, 2) taken and (2, 1) not taken

(3) Both not taken

therefore 3 relations are possible.