FINAL JEE–MAIN EXAMINATION – APRIL, 2023

(Held On Tuesday 11th April, 2023) **TIME : 9 : 00 AM to 12 : 00 NOON**

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

1. The value of the integral $\int_{-\log_e 2}^{\log_e 2} e^x\left(\log_e\left(e^x+\sqrt{1+e^{2x}}\right)\right)dx$ is equal to

(1) $\log_e\left(\frac{2\left(2+\sqrt{5}\right)}{\sqrt{1+\sqrt{5}}}\right)-\frac{\sqrt{5}}{2}$

(2) $\log_e\left(\frac{\sqrt{2}\left(3-\sqrt{5}\right)^2}{\sqrt{1+\sqrt{5}}}\right)+\frac{\sqrt{5}}{2}$

(3) $\log_e\left(\frac{\left(2+\sqrt{5}\right)^2}{\sqrt{1+\sqrt{5}}}\right)+\frac{\sqrt{5}}{2}$

(4) $\log_e\left(\frac{\sqrt{2}\left(2+\sqrt{5}\right)^2}{\sqrt{1+\sqrt{5}}}\right)-\frac{\sqrt{5}}{2}$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $I=\int_{-\ln 2}^{\ln 2} e^x\left(\ln\left(e^x+\sqrt{1+e^{2x}}\right)\right)dx$

Put $e^x=t \Rightarrow e^x dx=dt$

$$I=\int_{1/2}^{2}\ln\left(t+\sqrt{1+t^2}\right)dt$$

Applying integration by parts.

$$=\left[t\ln\left(t+\sqrt{1+t^2}\right)\right]_{\frac{1}{2}}^{2}-\int_{1/2}^{2}\frac{t}{t+\sqrt{1+t^2}}\left(1+\frac{2t}{2\sqrt{1+t^2}}\right)dt$$

$$=2\ln\left(2+\sqrt{5}\right)-\frac{1}{2}\ln\left(\frac{1+\sqrt{5}}{2}\right)-\int_{1/2}^{2}\frac{t}{\sqrt{1+t^2}}dt$$

$$=2\ln\left(2+\sqrt{5}\right)-\frac{1}{2}\ln\left(\frac{1+\sqrt{5}}{2}\right)-\frac{\sqrt{5}}{2}$$

$$=\ln\left(\frac{\left(2+\sqrt{5}\right)^2}{\left(\frac{\sqrt{5}+1}{2}\right)^{\frac{1}{2}}}\right)-\frac{\sqrt{5}}{2}$$

2. If equation of the plane that contains the point (–2,3,5) and is perpendicular to each of the planes $2x+4y+5z=8$ and $3x-2y+3z=5$ is $\alpha x+\beta y+\gamma z+97=0$ then $\alpha+\beta+\gamma=$

(1) 18

(2) 17

(3) 16

(4) 15

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. The equation of plane through (–2,3,5) is

a(x+2) + b(y-3) + c (z-5) = 0

it is perpendicular to 2x+4y+5z=8 & 3x-2y+3z=5

$\therefore$ $2a+4b+5c=0$

$3a-2b+3c=0$

$\therefore$ $\frac{a}{\begin{vmatrix}4 & 5\\ -2 & 3\end{vmatrix}}=\frac{-b}{\begin{vmatrix}2 & 5\\ 3 & 3\end{vmatrix}}=\frac{c}{\begin{vmatrix}2 & 4\\ 3 & -2\end{vmatrix}}$

$\Rightarrow$ $\frac{a}{22}=\frac{b}{9}=\frac{c}{-16}$

$\therefore$ Equation of Plane is

$22\left(x+2\right)+9\left(y-3\right)-16\left(z-5\right)=0$

$\Rightarrow$ $22x+9y-16z+97=0$

Comparing with $\alpha x+\beta y+\gamma x+97=0$

We get $\alpha+\beta+\gamma=22+9-16=15$

3. Let R be a rectangle given by the lines x = 0, x = 2, y = 0 and y = 5. Let $A(\alpha,0)$ and $B(0,\beta), \alpha \in [0,2]$ and $\beta \in [0,5]$, be such that the line segment AB divides the area of the rectangle R in the ratio 4:1. Then, the mid-point of AB lies on a

(1) parabola

(2) hyberbola

(3) straight line

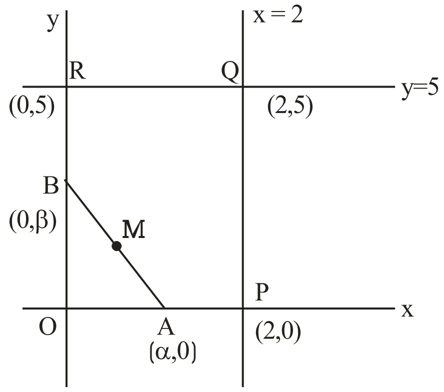
(4) circle

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $\dfrac{ar(OPQR)}{or(OAB)} = \dfrac{4}{1}$

Let M be the mid-point of AB.



$M(h,k) \equiv \left(\dfrac{\alpha}{2}, \dfrac{\beta}{2}\right)$

$\Rightarrow \dfrac{10 - \frac{1}{2}\alpha\beta}{\frac{1}{2}\alpha\beta} = 4$

$\Rightarrow \dfrac{5}{2}\alpha\beta = 10 \Rightarrow \alpha\beta = 4$

$\Rightarrow (2h)(2K) = 4$

$\therefore$ Locus of M is xy = 1

Which is a hyperbola.

4. Let sets A and B have 5 elements each. Let the mean of the elements in sets A and B be 5 and 8 respectively and the variance of the elements in sets A and B be 12 and 20 respectively A new set C of 10 elements is formed by subtracting 3 from each element of A and adding 2 to each element of B. Then the sum of the mean and variance of the elements of C is ________.

(1) 32

(2) 38

(3) 40

(4) 36

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $\omega \quad A = \{a_1, a_2, a_3, a_4, a_5\}$

$B = \{b_1, b_2, b_3, b_4, b_5\}$

Given, $\sum_{i=1}^{5} a\,i = 25, \ \sum_{i=1}^{5} b\,i = 40$

$$\frac{\sum_{i=1}^{5} a_i^2}{5} - \left(\frac{\sum_{i=1}^{5} a_i}{5}\right)^2 = 12, \quad \frac{\sum_{i=1}^{5} b_i^2}{5} - \left(\frac{\sum_{i=1}^{5} b_i}{5}\right)^2 = 20$$

$$\Rightarrow \sum_{i=1}^{5} a_i^2 = 185 \quad , \quad \sum_{i=1}^{5} b_i^2 = 420$$

Now, $C = \{C_1, C_2, ...C_{10}\}$

s.f. $C_i = \underset{\text{First five elements}}{a_i = 3}$ or $\underset{\text{Last five elements}}{b_i + 2}$

$\therefore$ Mean of C, $\bar{C} = \dfrac{\left(\sum a_i - 15\right) + \left(\sum b_i + 10\right)}{10}$

$\bar{C} = \dfrac{10 + 50}{10} = 6$

$\therefore \quad \sigma^2 = \dfrac{\sum_{i=1}^{10} C_i^2}{10} = \left(\bar{C}\right)^2$

$= \dfrac{\sum (a_i - 3)^2 + \sum (b_i + 2)^2}{10} - (6)^2$

$= \dfrac{\sum a_i^2 + \sum b_i^2 - 6\sum a_i + 4\sum b_i + 65}{10} - 36$

$= \dfrac{185 + 420 - 150 + 160 + 65}{10} - 36$

$= 32$

$\therefore$ Mean + Variance $= \bar{C} + \sigma^2 = 6 + 32 = 38$

5. Let $f(x)=[x^2-x]+|-x+[x]|$, where $x\in\mathbb{R}$ and $[t]$ denotes the greatest integer less than or equal to t. Then, f is

(1) continuous at x = 0, but not continuous at x = 1

(2) continuous at x = 0 and x = 1

(3) not continuous at x = 0 and x = 1

(4) continuous at x = 1, but not continuous at x = 0

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. Here $f(x)=[x(x-1)]+\{x\}$

$f(o^+)=-1+0=-1$ $f(o)=0$	$f(1^+)=0+0=0$ $f(1)=0$ $f(1^-)=-1+1=0$

$\therefore$ f(x) is continuous at x = 1, discontinuous at x = 0

6. The number of triplets (x, y, z). where x, y, z are distinct non negative integers satisfying x + y + z = 15, is

(1) 80

(2) 114

(3) 92

(4) 136

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. x + y + z = 15

Total no. of solution = ${}^{15+3-1}C_{3-1}=136$ …(1)

Let $x=y\neq z$

$2x+z=15\Rightarrow z=15-2t$

$\Rightarrow r\in\{0,1,2,...7\}-\{5\}$

$\therefore$ 7 solutions

$\therefore$ there are 21 solutions in which exactly

Two of $x_1\ y_1\ z$ are equal …(2)

There is one solution in which x=y=z …(3)

Required answer = 136–21–1 = 114

7. For any vector $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, with $10|a_i|<1, i=1,2,3$, consider the following statements:

(A) : max $\{|a_1|,|a_2|,|a_3|\}\leq|\vec{a}|$

(B) : $|\vec{a}|\leq 3\max\{|a_1|,|a_2|,|a_3|\}$

(1) Only (B) is true

(2) Only (A) is true

(3) Neither (A) nor (B) is true

(4) Both (A) and (B) are true

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. Without loss of generality

Let $|a_1|\leq|a_2|\leq|a_3|$

$|\vec{a}|^2=|a_1|^2+|a_2|^2+|a_3|^2\geq(a_3)^2$

$\Rightarrow |\vec{a}|\geq|a_3|=\max\{|a_1|,|a_2|,|a_3|\}$

A is true

$|\vec{a}|^2=|a_1|^2+|a_2|^2+|a_3|^2\leq|a_3|^2+|a_3|^2+|a_3|^2$

$\Rightarrow |\vec{a}|^2\leq 3|a_3|^2$

$\Rightarrow |\vec{a}|\leq\sqrt{3}|a_3|=\sqrt{3}\max\{|a_1|,|a_2|,|a_3|\}$

$\leq 3\max\{|a_1|,|a_2|,|a_3|\}$

(2) is true

8. Let w_1 be the point obtained by the rotation of $z_1 = 5 + 4i$ about the origin through a right angle in the anticlockwise direction, and w_2 be the point obtained by the rotation of $z_2 = 3 + 5i$ about the origin through a right angle in the clockwise direction. Then the principal argument of $w_1 - w_2$ is equal to

(1) $-\pi+\tan^{-1}\frac{33}{5}$

(2) $-\pi-\tan^{-1}\frac{33}{5}$

(3) $-\pi+\tan^{-1}\frac{8}{9}$

(4) $\pi-\tan^{-1}\frac{8}{9}$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $W_1 = z_1 i = (5+4i)i = -4+5i$ ...(i)

$W_2 = z_2(-i) = (3+5i)(-i) = 5-3i$ ...(2)

$W_1 - W_2 = -9+8i$

Principal argument = $\pi - \tan^{-1}\left(\frac{8}{9}\right)$

9. An organization awarded 48 medals in event 'A', 25 in event 'B' and 18 in event 'C'. If these medals went to total 60 men and only five men got medals in all the three events, then, how many received medals in exactly two of three events?

(1) 10

(2) 9

(3) 21

(4) 15

Official Ans. by NTA (3)

Allen Ans. (3)

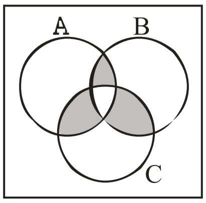
Sol. $|A| = 48$

$|B| = 25$

$|C| = 18$

$|A \cup B \cup C| = 60$ [Total]

$|A \cap B \cap C| = 5$



$|A \cup B \cup C| = \sum|A| - \sum|A \cap B| + |A \cap B \cap C|$

$\Rightarrow \sum|A \cap B| = 48+25+18+5-60$

$= 36$

No. of men who received exactly 2 medals

$= \sum|A \cap B| - 3|A \cap B \cap C|$

= 36 – 15

= 21

10. Let $S = \{M = [a_{ij}], a_{ij} \in \{0,1,2\}, 1 \le i, j \le 2\}$ be a sample space and $A = \{M \in S : M \text{ is invertible}\}$ be an event. Then P(A) is equal to

(1) $\frac{50}{81}$

(2) $\frac{47}{81}$

(3) $\frac{49}{81}$

(4) $\frac{16}{27}$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $M\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a, b, c, d, $\in \{0,1,2\}$

$n(s) = 3^4 = 81$

we first bound $p(\bar{A})$

$|m| = 0 \Rightarrow ad = bc$

$ad = bc = 0 \Rightarrow$ no. of (a,b,c,d) = $(3^2–2^2)^2 = 25$

$ad = bc = 1 \Rightarrow$ no. of (a,b,c,d) = $1^2 = 1$

$ad = bc = 2 \Rightarrow$ no. of (a,b,c,d) = $2^2 = 4$

$ad = bc = 4 \Rightarrow$ no. of (a,b,c,d) = $1^2 = 1$

: $P(\bar{A}) = \frac{31}{81} \Rightarrow p(A) = \frac{50}{81}$

11. Consider ellipses $E_k : kx^2 + k^2y^2 = 1, k = 1, 2, ..., 20$. Let C_k be the circle which touches the four chords joining the end points (one on minor axis and another on major axis) of the ellipse E_k, If r_k is the radius of the circle C_k, then the value of $\sum_{k=1}^{20} \frac{1}{r_k^2}$ is

(1) 3080

(2) 3210

(3) 3320

(4) 2870

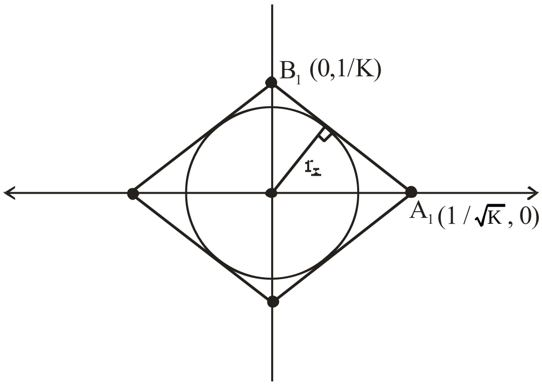
Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $Kx^2+K^2y^2=1$

$$\frac{x^2}{1/K}+\frac{y^2}{1/K^2}=1$$

Now



Equation of

$$A_1B_2;\frac{x}{1/\sqrt{K}}+\frac{y}{1/K}=1\Rightarrow\sqrt{K}x+Ky=1$$

$r_K=\perp r$ distance of (0,0) from line A_1B_1

$$r_k=\left|\frac{(0+0-1)}{\sqrt{K+K^2}}\right|=\frac{1}{\sqrt{K+K^2}}$$

$$\frac{1}{r_K^2}=K+K^2\Rightarrow\sum_{k=1}^{20}\frac{1}{r_K^2}=\sum_{K=1}^{20}\left(K+K^2\right)$$

$$=\sum_{K=1}^{20}K+\sum_{K=1}^{20}K^2$$

$$=\frac{20\times21}{2}+\frac{20.21.41}{6}$$

$$=210+10\times7\times41$$

$$=210+2870$$

$$=3080$$

12. The number of integral solutions x of $\log_{\left(x+\frac{7}{2}\right)}\left(\frac{x-7}{2x-3}\right)^2\geq0$ is

(1) 6 (2) 8

(3) 5 (4) 7

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $\log_{x+\frac{7}{2}}\left(\frac{x-7}{2x-3}\right)^2\geq0$

Feasible region : $x+\frac{7}{2}>0\Rightarrow x>-\frac{7}{2}$

And $x+\frac{7}{2}\neq1\Rightarrow x\neq-\frac{5}{2}$

And $\frac{x-7}{2x-3}\neq0$ and $2x-3\neq0$

$\Downarrow$ $\Downarrow$

$x\neq7$ $x\neq\frac{3}{2}$

Taking intersection : $x\in\left(\frac{-7}{2},\infty\right)-\left\{-\frac{5}{2},\frac{3}{2},7\right\}$

Now $\log_a b\geq0$ if $a>1$ and $b\geq1$

Or

$a\in(0,1)$ and $b\in(0,1)$

C – I; $x+\frac{7}{2}>1$ and $\left(\frac{x-7}{2x-3}\right)^2\geq1$

$x>-\frac{5}{2}$ $(2x-3)^2-(x-7)^2\leq0$

$(2x-3+n-7)(2x-3-x+7)\leq0$

$(3x-10)(x+4)\leq0$

$x\in\left[-4,\frac{10}{3}\right]$

Intersection : $x\in\left(\frac{-5}{2},\frac{10}{3}\right]$

C-II $x+\frac{7}{2}\in(0,1)$ and $\left(\frac{x-7}{2x-3}\right)^2\in(0,1)$

$0<x+\frac{7}{2}<1$ $\left(\frac{x-7}{2x-3}\right)^2<1$

$-\frac{7}{2}<x<\frac{-5}{2}$ $(x-7)^2<(2x-3)^2$

$x\in(-\infty,-4)\cup\left(\frac{10}{3},\infty\right)$

No common values of x.

Hence intersection with feasible region

We get $x\in\left(\frac{-5}{2},\frac{10}{3}\right]-\left\{\frac{3}{2}\right\}$

Integral value of x are {–2,–1,0,1,2,3}

No. of integral values = 6

13. Area of the region $\{(x,y): x^2+(y-2)^2 \le 4,\ x^2 \ge 2y\}$ is

(1) $2\pi - \frac{16}{3}$ (2) $\pi - \frac{8}{3}$

(3) $\pi + \frac{8}{3}$ (4) $2\pi + \frac{16}{3}$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $x^2+(y-2)^2 \le 2^2$ and $x^2 \ge 2y$

Solving circle and parabola simultaneously :

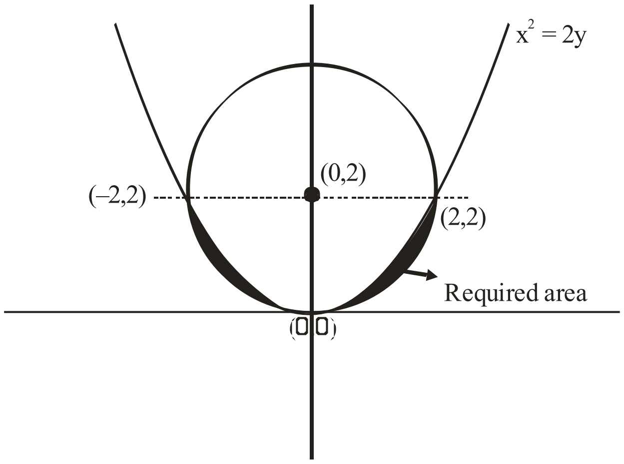
$2y+y^2-4y+4=4$

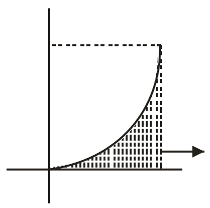
$y^2-2y=0$

$y=0,2$

Put $y = 2$ in $x^2 = 2y \rightarrow x = \pm 2$

$\Rightarrow$ (2,2) and (–2,2)





$= 2\times 2 - \frac{1}{4}\cdot \pi \cdot 2^2 = 4-\pi$

Required area $= 2\left[\int_0^2 \frac{x^2}{2}dx - (4-\pi)\right]$

$= 2\left[\left.\frac{x^3}{6}\right|_0^2 - 4 + \pi\right]$

$= 2\left[\frac{4}{3}+\pi-4\right]$

$= 2\left[\pi - \frac{8}{3}\right]$

$= 2\pi - \frac{16}{6}$

14. Let $f:[2,4]\rightarrow \mathbb{R}$ be a differentiable function such that $(x\log_e x)f'(x)+(\log_e x)f(x)+f(x)\ge 1$, $x\in[2,4]$ with $f(2)=\frac{1}{2}$ and $f(4)=\frac{1}{4}$.

Consider the following two statements:

(A) : $f(x)\le 1$, for all $x\in[2,4]$

(B) : $f(x)\ge \frac{1}{8}$, for all $x\in[2,4]$

Then,

(1) Only statement (B) is true

(2) Neither statement (A) nor statement (B) is true

(3) Both the statement (A) and (B) are true

(4) Only statement (A) is true

Official Ans. by NTA (3)

Allen Ans. (Bonus)

Sol. $x\,\ell nx f'(x)+\ln x\, f(x)+f(x)\ge I, x\in[2,4]$

And $f(2)=\frac{1}{2}, f(4)=\frac{1}{4}$

Now $x\ln x\frac{dy}{dx}+(\ln+1)y\ge 1$

$\frac{d}{dx}(y\cdot x\ln x)\ge 1$

$\frac{d}{dx}(f(x).x\ln x)\ge 1$

$\Rightarrow \frac{d}{dx}(x\ln x f(x)-x)\ge 0,\ x\in[2,4]$

$\Rightarrow$ The function g(x) = $x\ln x f(x)-x$ is increasing in [2.4]

And $g(2)=2\ln 2 f(2)-2=\ln 2-2$

$g(4)=4\ln 4 f(4)-4=\ln 4-4$

$=2(\ln 2-2)$

Now $g(2)\le g(x)\le g(4)$

$\ln 2-2\le x\ln x f(x)-x\le 2(\ln 2-2)$

$\frac{\ln 2-2}{x\ln x}+\frac{1}{\ln x}\le f(x)\le \frac{2(\ln 2-2)}{x\ln x}+\frac{1}{\ln x}$

Now for $x \in [2,4]$

$$\frac{2(\ell n2-2)}{x\ln x}+\frac{1}{\ell n\,x}<\frac{2(\ln 2-2)}{2\ln 2}+\frac{1}{\ln 2}=1-\frac{1}{\ln 2}<1$$

$\Rightarrow f(x)\le 1$ for $x\in[2,4]$

Also for $x \in [2,4]$:

$$\frac{\ln 2-2}{x\ln x}+\frac{1}{\ln x}\ge\frac{\ln 2-2}{4\ln 4}+\frac{1}{\ln 4}=\frac{1}{8}+\frac{1}{2\ln 2}>\frac{1}{8}$$

$\Rightarrow f(x)\ge\frac{1}{8}$ for $x\in[2,4]$

Hence both A and B are true.

LMVT on (yx (lnx)) not satisfied.

Hence no such function exists.

Therefore it should be bonus.

15. Let y = y(x) be a solution curve of the differential equation, $(1-x^2y^2)dx = ydx + xdy$.

If the line x = 1 intersects the curve y = y(x) at y = 2 and the line x = 2 intersects the curve $y = y(x)$ at $y = \alpha$, then a value of α is

(1) $\frac{3e^2}{2(3e^2-1)}$

(2) $\frac{3e^2}{2(3e^2+1)}$

(3) $\frac{1-3e^2}{2(3e^2+1)}$

(4) $\frac{1+3e^2}{2(3e^2-1)}$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $(1-x^2y^2)dx = ydx + x\,dy,\ y(1)=2$

$$y(2)=\propto\,=\,?$$

$$dx=\frac{d(xy)}{1-(xy)^2}$$

$$\int dx=\int\frac{d(xy)}{1-(xy)^2}$$

$$x=\frac{1}{2}\ln\left|\frac{1+xy}{1-xy}\right|+C$$

Put x = 1 and y = 2 :

$$1=\frac{1}{2}\ln\left|\frac{1+2}{1-2}\right|+C$$

$$C=1-\frac{1}{2}\ln 3$$

Now put x = 2 :

$$2=\frac{1}{2}\ln\left|\frac{1+2\alpha}{1-2\alpha}\right|+1-\frac{1}{2}\ln 3$$

$$1+\frac{1}{2}\ln 3=\frac{1}{2}\ln\left|\frac{1+2\alpha}{1-2\alpha}\right|$$

$$2+\ln 3=\ln\left(\frac{1+2\alpha}{1-2\alpha}\right)$$

$$\left|\frac{1+2\alpha}{1-2\alpha}\right|=3e^2$$

$$\frac{1+2\alpha}{1-2\alpha}=3e^2\ ,\ -3e^2$$

$$\frac{1+2\alpha}{1-2\alpha}=3e^2\Rightarrow\alpha=\frac{3e^2-1}{2(3e^2+1)}$$

And $\frac{1+2\alpha}{1-2\alpha}=-3e^2\Rightarrow\alpha=\frac{3e^2+1}{2(3e^2-1)}$

16. Let A be a 2 × 2 matrix with real entries such that $A' = \alpha A + I$, where $\alpha \in \mathbb{R}-\{-1,1\}$. If det $(A^2 - A) = 4$, then the sum of all possible values of α is equal to

(1) 0 (2) $\frac{3}{2}$

(3) $\frac{5}{2}$ (4) 2

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $A^T = \alpha A + I$

$A = \alpha A^T + I$

$A = \alpha(\alpha A + I) + I$

$A = \alpha^2 A + (\alpha+1)I$

$A(1-\alpha^2) = (\alpha+1)I$

$A = \dfrac{I}{1-\alpha}$ ...(1)

$|A| = \dfrac{1}{(1-\alpha)^2}$ ...(2)

$|A^2 - A| = |A||A-I|$ ...(3)

$A - I = \dfrac{I}{1-\alpha} - I = \dfrac{\alpha}{1-\alpha} I$

$|A - I| = \left(\dfrac{\alpha}{1-\alpha}\right)^2$ ...(4)

Now $|A^2 - A| = 4$

$|A||A-I| = 4$

$\Rightarrow \dfrac{1}{(1-\alpha)^2} \dfrac{\alpha^2}{(1-\alpha)^2} = 4$

$\Rightarrow \dfrac{\alpha}{(1-\alpha)^2} = \pm 2$

$\Rightarrow 2(1-\alpha)^2 = \pm\alpha$

(C_1) $2(1-\alpha)^2 = \alpha$ $2\alpha^2 - 5\alpha + 2 = 0 <^{\alpha_1}_{\alpha_2}$ $\alpha_1 + \alpha_2 = \dfrac{5}{2}$	(C_2) $2(1-\alpha)^3 = -\alpha$ $2\alpha^2 - 3\alpha + 2 = 0$ $\alpha \notin R$

Sum of value of $\alpha = \dfrac{5}{2}$

17. Let (α, β, γ) be the image of the point P(2, 3, 5) in the plane 2x + y – 3z = 6. Then $\alpha + \beta + \gamma$ is equal to

(1) 10

(2) 5

(3) 12

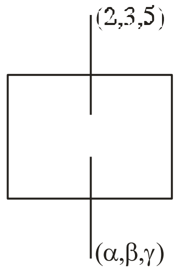
(4) 9

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $\dfrac{\alpha-2}{2} = \dfrac{\beta-3}{1} = \dfrac{\gamma-5}{-3} = -2\left(\dfrac{2x2+3-3\times5-6}{2^2+1^2+1-3^2}\right) = 2$

$\dfrac{\alpha-2}{2} = 2$ $\alpha = 6$	$\beta - 3 = 2$ $\beta = 5$	$\gamma - 5 = -6$ $\gamma = -1$



$\alpha + \beta + \gamma = 10$

18. Let $\vec{a}$ be a non-zero vector parallel to the line of intersection of the two planes described by $\hat{i} + \hat{j}, \hat{i} + k$ and $\hat{i} - \hat{j}, \hat{j} - k$. If θ is the angle between the vector $\vec{a}$ and the vector $\vec{b} = 2\hat{i} - 2\hat{j} + k$ and $\vec{a} \cdot \vec{b} = 6$ then the ordered pair $\left(\theta, |\vec{a} \times \vec{b}|\right)$ is equal to

(1) $\left(\dfrac{\pi}{4}, 3\sqrt{6}\right)$

(2) $\left(\dfrac{\pi}{3}, 3\sqrt{6}\right)$

(3) $\left(\dfrac{\pi}{3}, 6\right)$

(4) $\left(\dfrac{\pi}{4}, 6\right)$

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $\vec{n}_1$ and $\vec{n}_2$ are normal vector to the plane $\hat{i} + \hat{j}, \hat{i} + \hat{k}$ and $\hat{i} - \hat{j}; \hat{j} - \hat{k}$ respectively

$\vec{n}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{j} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \hat{i} - \hat{j} - \hat{k}$

$\vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{j} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \hat{i} + \hat{j} + \hat{k}$

$\vec{a} = \lambda|\vec{n}_2 \times \vec{n}_2|$

$$= \lambda \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \lambda\left(-2\hat{j}+2\hat{k}\right)$$

$$\vec{a}\cdot\vec{b} = \lambda|0+4+2| = 6$$

$$\Rightarrow \lambda = 1$$

$$\vec{\alpha} = -2\hat{j}+2\hat{k}$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|a||b|}$$

$$\cos\theta = \frac{6}{2\sqrt{2}\times 3} = \frac{1}{\sqrt{2}}$$

$$\theta = \frac{\pi}{4}$$

Now $\quad |\vec{a}.\vec{b}|^2 + |\vec{a}\times\vec{b}|^2 = |a|^2|b|^2$

$$36 + |\vec{a}\times b^2| = 8\times 9 = 72$$

$$|\vec{a}\times b|^2 = 36$$

$$|\vec{a}\times\vec{b}| = 6$$

19. The number of elements in the set $S = \{\theta \in [0,2\pi] : 3\cos^4\theta - 5\cos^2\theta - 2\sin^2\theta + 2 = 0\}$ is

(1) 10 (2) 8

(3) 9 (4) 12

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $3\cos^4\theta - 5\cos^2\theta - 2\sin^6\theta + 2 = 0$

$\Rightarrow \quad 3\cos^4\theta - 3\cos^2\theta - 2\cos^2\theta - 2\sin^6\theta + 2 = 0$

$\Rightarrow \quad 3\cos^4\theta - 3\cos^2\theta + 2\sin^2\theta - 2\sin^6\theta = 0$

$\Rightarrow \quad 3\cos^2\theta\left(\cos^2\theta - 1\right) + 2\sin^2\theta\left(\sin^4\theta - 1\right) = 0$

$\Rightarrow \quad -3\cos^2\theta\sin^2\theta + 2\sin^2\theta\left(1+\sin^2\theta\right)\cos^2\theta - 1$

$\Rightarrow \quad \sin^2\theta\cos^2\theta\left(2 + 2\sin^2\theta - 3\right) = 0$

$\Rightarrow \quad \sin^2\theta\cos^2\theta\left(2\sin^2\theta - 1\right) = 0$

(C1) $\sin^2\theta = 0 \rightarrow$ 3 solution ; $\theta = \{0, \pi, 2\pi\}$

(C2) $\cos^2\theta = 0 \rightarrow$ 2 solution ; $\theta = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$

(C3) $\sin^2\theta = \frac{1}{2} \rightarrow$ 4 solution ; $\theta = \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

No. of solution = 9

20. Let $x_1, x_2, \ldots, x_{100}$ be in an arithmetic progression, with $x_1 = 2$ and their mean equal to 200. If $y_i = i\left(x_i - i\right), 1 \le i \le 100$, then the mean of $y_1, y_2, \ldots, y_{100}$ is .

(1) 10101.50

(2) 10051.50

(3) 10049.50

(4) 10100

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. Mean $= 200$

$$\Rightarrow \frac{\frac{100}{2}\left(2\times 2 + 99d\right)}{100} = 200$$

$$\Rightarrow 4 + 99d = 400$$

$$\Rightarrow d = 4$$

$$y_i = i\left(xi - i\right)$$

$$= i\left(2 + (i-1)4 - i\right) = 3i^2 - 2i$$

$$\text{Mean} = \frac{\sum y_i}{100}$$

$$= \frac{1}{100}\sum_{i=1}^{100} 3i^2 - 2i$$

$$= \frac{1}{100}\left\{\frac{3\times 100\times 101\times 201}{6} - \frac{2\times 100\times 101}{2}\right\}$$

$$= 101\left\{\frac{201}{2} - 1\right\} = 101\times 99.5$$

$$= 10049\cdot 50$$

SECTION-B

21. The mean of the coefficients of $x, x^2, ...x^7$ in the binomial expansion of $(2 + x)^9$ is _______ .

Official Ans. by NTA (2736)

Allen Ans. (2736)

Sol. Coefficient of $x = {}^9C_1 2^8$

Of $x^2 = {}^9C_2 2^7$

Of $x^7 = {}^9C_7 \cdot 2^2$

$$\text{Mean} = \frac{{}^9C_1 \cdot 2^8 + {}^9C_2 \cdot 2^7 ..... + {}^9C_7 \cdot 2^2}{7}$$

$$= \frac{(1+2)^9 - {}^9C_0 \cdot 2^9 - {}^9C_8 \cdot 2^1 - {}^9C_9}{7}$$

$$= \frac{3^9 - 2^9 - 18 - 1}{7}$$

$$= \frac{19152}{7} = 2736$$

22. Let $S = 109 + \frac{108}{5} + \frac{107}{5^2} + ... + \frac{2}{5^{107}} + \frac{1}{5^{108}}$. Then the value of $(16S - (25)^{-54})$ is equal to _______ .

Official Ans. by NTA (2175)

Allen Ans. (2175)

Sol. $S = 109 + \frac{108}{5} + \frac{107}{5^2} ..... + \frac{1}{5^{108}}$

$$\frac{S}{5} = \frac{109}{5} + \frac{108}{5^2} ...... + \frac{2}{5^{108}} + \frac{1}{5^{109}}$$

$$\frac{4S}{5} = 109 - \frac{1}{5} - \frac{1}{5^2} ...... - \frac{1}{5^{108}} - \frac{1}{5^{109}}$$

$$= 109 - \left(\frac{1}{5}\frac{\left(1 - \frac{1}{5^{109}}\right)}{\left(1 - \frac{1}{5}\right)}\right)$$

$$= 109 - \frac{1}{4}\left(1 - \frac{1}{5^{109}}\right)$$

$$= 109 - \frac{1}{4} + \frac{1}{4} \times \frac{1}{5^{109}}$$

$$s = \frac{5}{4}\left(109 - \frac{1}{4} + \frac{1}{4.5^{109}}\right)$$

$$16S = 20 \times 109 - 5 + \frac{1}{5^{108}}$$

$$16S - (25)^{-54} = 2180 - 5 = 2175$$

23. For m, n > 0, let $\alpha(m,n) = \int_0^2 t^m (1+3t)^n\, dt$. If $11\alpha(10,6) + 18\alpha(11,5) = p(14)^6$, then p is equal to _______ .

Official Ans. by NTA (32)

Allen Ans. (32)

Sol. $\alpha(m,n) = \int_0^2 t^m (1+3t)^n\, dt$

If $11\alpha(10,6) + 18\alpha(11,5) = p(14)^6$ then P

$$= 11\int_0^2 \underset{\text{II}}{t^{10}}\, \underset{\text{I}}{(1+3t)^6} + 10\int^2 t^{11}(1+3t)^5\, dt$$

$$= 11\left[(1+3t)^6 \cdot \frac{t^{11}}{11} - \int 6(1+3t)^5 \cdot 3\frac{t^{11}}{11}\right]_0^2 + 18\int_0^2 t^{11}(1+3t)^5\, dt$$

$$= \left(t^{11}(1+3t)^6\right)_0^2$$

$$= 2^{11}(7)^6$$

$$= 2^5(14)^6$$

$$= 32(14)^6$$

24. In an examination, 5 students have been allotted their seats as per their roll numbers. The number of ways, in which none of the students sits on the allotted seat, is _______ .

Official Ans. by NTA (44)

Allen Ans. (44)

Sol. Derangement of 5 students

$$D_5 = 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$$

$$= 120\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right)$$

$$= 60 - 20 + 5 - 1$$

$$= 40 + 4$$

$$= 44$$

25. Let a line l pass through the origin and be perpendicular to the lines

$$l_1 : \vec{r} = \left(\hat{i} - 11\hat{j} - 7\hat{k}\right) + \lambda\left(\hat{i} + 2\hat{j} + 3\hat{k}\right), \lambda \in \mathbb{R}$$

and $l_2 : \vec{r} = \left(-\hat{i} + \hat{k}\right) + \mu\left(2\hat{i} + 2\hat{j} + \hat{k}\right), \mu \in \mathbb{R}$.

If P is the point of intersection of l and l_1, and Q(α, β, γ) is the foot of perpendicular from P on l_2, then $9(\alpha+\beta+\gamma)$ is equal to_______ .

Official Ans. by NTA (5)

Allen Ans. (5)

Sol. Let $\ell = \left(0\hat{i} + 0\hat{j} + 0\hat{k}\right) + \gamma\left(a\hat{i} + b\hat{j} + c\hat{k}\right)$

$= \gamma\left(a\hat{i} + b\hat{j} + c\hat{k}\right)$

$$a\hat{i} + b\hat{j} + c\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 2 & 2 & 1 \end{vmatrix}$$

$= \hat{i}(2-6) - \hat{j}(1-6) + \hat{k}(2-4)$

$= -4\hat{i} - 5\hat{j} - 2\hat{k}$

$\ell = \gamma\left(-4\hat{i} + 5\hat{j} - 2\hat{k}\right)$

P is intersection of ℓ and ℓ_1

$-4\gamma = 1 + \lambda, 5\gamma = -11 + 2\lambda, -2\gamma = -7 + 3\lambda$

By solving there equation $\gamma = -1$, P (4,–5,2)

Let $Q(-1+2\mu, 2\mu, 1+\mu)$

$\overrightarrow{PQ} \cdot \left(2\hat{i} + 2\hat{j} + \hat{k}\right) = 0$

$-2 + 4\mu + 4\mu + 1 + \mu = 0$

$9\mu = 1$

$\mu = \dfrac{1}{9}$

$Q\left(\dfrac{-7}{9}, \dfrac{2}{9}, \dfrac{10}{9}\right)$

$9(\alpha + \beta + \gamma) = 9\left(\dfrac{-7}{9} + \dfrac{2}{9} + \dfrac{10}{9}\right)$

$= 5$

26. The number of integral terms in the expansion of $\left(3^{\frac{1}{2}} + 5^{\frac{1}{4}}\right)^{680}$ is equal to

Official Ans. by NTA (171)

Allen Ans. (171)

Sol. The number of integral term in the expression of $\left(3^{\frac{1}{2}} + 5^{\frac{1}{4}}\right)^{680}$ is equal to

General term $= {}^{680}C_r \left(3^{\frac{1}{2}}\right)^{680-r} \left(5^{\frac{1}{4}}\right)^r$

$= {}^{680}C_r\, 3^{\frac{680-r}{2}}\, 5^{\frac{r}{4}}$

Value's of r, where $\dfrac{r}{4}$ goes to integer

r = 0, 4, 8, 12, ……….680

All value of r are accepted for $\dfrac{680-r}{2}$ as well so

No of integral terms = 171.

27. The number of ordered triplets of the truth values of p, q and r such that the truth value of the statement $(p \vee q) \wedge (p \vee r) \Rightarrow (q \vee r)$ is True, is equal to _______ .

Official Ans. by NTA (7)

Allen Ans. (7)

Sol.

p	**q**	**r**	**Pvq**	**Pvr**	**(pvq) ∧ (pvr)**	**qvr**	**(pvq) ∧ (pvr) → qvr**
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	T	T	T
T	F	F	T	T	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	F	T	F	T	T
F	F	F	F	F	F	F	T

Hence total no of ordered triplets are 7

28. Let $H_n = \frac{x^2}{1+n} - \frac{y^2}{3+n} = 1, n \in N$. Let k be the smallest even value of n such that the eccentricity of H_k is a rational number. If l is length of the latus return of H_k, then $21l$ is equal to ________

Official Ans. by NTA (306)

Allen Ans. (306)

Sol. $Hn \Rightarrow \frac{x^2}{1+n} - \frac{y^2}{3+n} = 1$

$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{3+n}{1+n}} = \sqrt{\frac{2n+4}{n+1}}$

$e = \sqrt{\frac{2n+4}{n+1}}$

$n = 48$ (smallest even value for which $e \in Q$)

$\boxed{e = \frac{10}{7}}$

$a^2 = n+1 \quad b^2 = n+3$

$= 49 \quad , \quad = 51$

1 = length of LR $= \frac{2b^2}{a}$

$L = 2 \cdot \frac{51}{7}$

$1 = \frac{102}{7}$

$\boxed{21\ell = 306}$

29. If a and b are the roots of equation $x^2 - 7x - 1 = 0$, then the value of $\frac{a^{21} + b^{21} + a^{17} + b^{17}}{a^{19} + b^{19}}$ is equal to ________ .

Official Ans. by NTA (51)

Allen Ans. (51)

Sol. $x^2 - 7x - 1 = 0 <^{a}_{b}$

By newton's theorem

$S_{n+2} - 7S_{n+1} - S_n = 0$

$S_{21} - 7S_{20} - S_{19} = 0$

$S_{20} - 7S_{19} - S_{18} = 0$

$S_{19} - 7S_{18} - S_{17} = 0$

$\frac{S_{21} + S_{17}}{S_{19}} = \frac{S_{21} + (S_{19} - 7S_{18})}{S_{19}}$

$= \frac{S_{21} + S_{19} - 7(S_{20} - 7S_{19})}{S_{19}}$

$= \frac{50S_{19} + (S_{21} - 7S_{20})}{S_{19}}$

$= 51 \cdot \frac{S_{19}}{S_{19}} = \boxed{51}$

30. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 3 \\ 1 & c & 0 \end{bmatrix}$, where $a, c \in R$. If $A^3 = A$ and the positive value of a belongs to the interval (n – 1 , n], where n ∈ N, then n is equal to ________ .

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $A = \begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 3 \\ 1 & c & 0 \end{bmatrix}$

$A^3 = A$

$A^2 = \begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 3 \\ 1 & c & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 3 \\ 1 & c & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} a+2 & 2c & 3 \\ 3 & a+3c & 2a \\ ac & 1 & 2+3c \end{bmatrix}$

$A^3 = \begin{bmatrix} a+2 & 2c & 3 \\ 3 & a+3c & 2a \\ ac & a & 2+3c \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 3 \\ 1 & c & 0 \end{bmatrix}$

$A^3 = \begin{bmatrix} 2ac+3 & a+2+3c & 2a+4+6c \\ a(a+3c)+2a & 3+2ac & 6+3a+9c \\ a+2+3c & ac+c(2+3c) & 2ac+3 \end{bmatrix}$

Given $A^3 = A$

$2ac + 3 = 0$ ...(1) and $a+2+3c = 1$

$a + 1 + 3c = 0$

$a + 1 - \frac{9}{2a} = 0$

$2a^2 + 2a - 9 = 0$

$f(1) < 0, f(2) > 0$

$a \in (1, 2]$ $\qquad \boxed{n = 2}$