FINAL JEE–MAIN EXAMINATION – APRIL, 2023

(Held On Saturday 08th April, 2023) **TIME : 9 : 00 AM to 12 : 00 NOON**

MATHEMATICS **TEST PAPER WITH SOLUTION**

SECTION-A

1. Let $I(x)=\int \frac{(x+1)}{x(1+xe^x)^2}dx,\ x>0,$

If $\lim_{x\to\infty} I(x)=0$, then I(1) is equal to

(1) $\frac{e+1}{e+2}-\log_e(e+1)$

(2) $\frac{e+1}{e+2}+\log_e(e+1)$

(3) $\frac{e+2}{e+1}+\log_e(e+1)$

(4) $\frac{e+2}{e+1}-\log_e(e+1)$

Official Ans. by NTA (4)
Allen Ans. (4)

Sol. $I(x)=\int \frac{xe^x+e^x}{xe^x(1+xe^x)^2}dx$

Put $1+xe^x=t$

$I(x)=\int \frac{1}{(t-1)t^2}dt=\frac{1}{t}+\ln\left|\frac{t-1}{t}\right|+C$

$\because \lim_{x\to\infty} I(x)=0 \ \therefore C=0$

$I(1)=\frac{e+2}{e+1}-\ln(1+e)$

2. If the equation of the plane containing the line $x+2y+3z-4=0 \ =2x+y-z+5$ and perpendicular to the plane $\vec{r}=(\hat{i}-\hat{j})+\lambda(\hat{i}+\hat{j}+\hat{k})+\mu(\hat{i}-2\hat{j}+3\hat{k})$ is $ax+by+cz=4$, then (a–b + c) is equal to

(1) 20 (2) 24

(3) 22 (4) 18

Official Ans. by NTA (3)
Allen Ans. (3)

Sol. D.R's of line $\vec{n}_1=-5\hat{i}+7\hat{j}-3\hat{k}$

D.R's of normal of second plane

$\vec{n}_2=5\hat{i}-2\hat{j}-3\hat{k}$

$\vec{n}_1\times\vec{n}_2=-27\hat{i}-30\hat{j}-25\hat{k}$

A point on the required plane is $\left(0,-\frac{11}{5},\frac{14}{5}\right)$

The equation of required plane is

$27x + 30y + 25z = 4$

$\therefore\ a - b + c = 22$

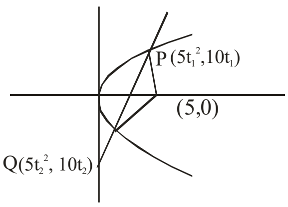
3. Let R be the focus of the parabola $y^2=20x$ and the line $y=mx+c$ intersect the parabola at two points P and Q. Let the point G(10, 10) be the centroid of the triangle PQR. If c–m = 6 , then $(PQ)^2$ is

(1) 325 (2) 317

(3) 296 (4) 346

Official Ans. by NTA (1)
Allen Ans. (1)

Sol.



$10t_1 + 10t_2 = 30$

$\Rightarrow m=\frac{2}{t_1+t_2}=\frac{2}{3}$

$C=m+6=\frac{20}{3}$

$PQ=\frac{4\sqrt{a^2-amc}\sqrt{1+m^2}}{m^2}=\sqrt{325}$

4. Let $C(\alpha,\beta)$ be the circumcenter of the triangle formed by the lines

$4x+3y=69$

$4y-3x=17$ and

$x+7y=61$

Then $(\alpha-\beta)^2+\alpha+\beta$ is equal to

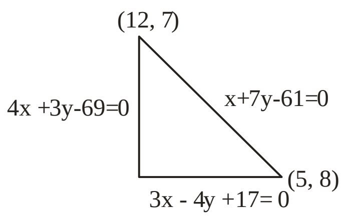
(1) 18 (2) 17

(3) 16 (4) 15

Official Ans. by NTA (2)

Allen Ans. (2)

Sol.



$\Rightarrow$ Circumcentre $\left(\frac{17}{2},\frac{15}{2}\right)$

$\Rightarrow (\alpha-\beta)^2+\alpha+\beta=17$

5. Let $P=\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$, $A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ and

$Q=PQP^T$. If $P^TQ^{2007}P=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, then

$2a+b-3c-4d$ equal to

(1) 2007 (2) 2005

(3) 2006 (4) 2004

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $PP^T=I$

$P^TQ^{2007}P=A^{2007}$

$=\begin{bmatrix}1 & 2007\\ 0 & 1\end{bmatrix}\Rightarrow 2a+b-3c-4d=2005$

6. Let α,β,γ be the three roots of the equation $x^3+bx+c=0$. If $\beta\gamma=1=-\alpha$, then $b^3+2c^3-3\alpha^3-6\beta^3-8\gamma^3$ is equal to

(1) 21 (2) $\frac{169}{8}$

(3) 19 (4) $\frac{155}{8}$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $\alpha\beta\gamma=-c$

$\alpha=-c$

$c=1$

since $\alpha^3+b\alpha+c=0$

$\Rightarrow(-1)^3+b(-1)+1=0$

$b=0$

$\therefore x^3+1=0$

$x=-1,-\omega,-\omega^2$

$b^3+2c^3-3\alpha^3-6\beta^3-8\gamma^3=19$

7. The number of ways , in which 5 girls and 7 boys can be seated at a round table so that no two girls sit together, is

(1) $126(5!)^2$

(2) $7(360)^2$

(3) 720

(4) $7(720)^2$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. 7 boys can be seated in 6! ways

now girls will be placed in gaps

$\therefore$ total ways $=6!\times{}^7C_5\times5!$

$=126\,(5!)^2$

8. In a bolt factory, machines A, B and C manufacture respectively 20%, 30% and 50% of the total bolts. Of their output 3, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product. If the bolt drawn is found the defective, then the probability that it is manufactured by the machine C is

(1) $\frac{2}{7}$ (2) $\frac{9}{28}$

(3) $\frac{5}{14}$ (4) $\frac{3}{7}$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $P\left(\frac{C}{D}\right)=\frac{0.5\times0.02}{0.2\times0.03+0.3\times0.04+0.5\times0.02}$

$=\frac{5}{14}$

9. The number of arrangements of the letter of the word "INDEPENDENCE" in which all the vowels always occur together is

(1) 16800 (2) 14800

(3) 18000 (4) 33600

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. Vowels: I, 4E

Consonants: 3N, 2D, P, C

Total ways of arrangements taking vowels together

$=\frac{8!}{3!2!}\times\frac{5!}{4!}$

= 16800

10. Let $f(x)=\frac{\sin x+\cos x-\sqrt{2}}{\sin x-\cos x}, x\in[0,\pi]-\left\{\frac{\pi}{4}\right\}$.

Then $f\left(\frac{7\pi}{12}\right)f''\left(\frac{7\pi}{12}\right)$ is equal to

(1) $\frac{-2}{3}$ (2) $\frac{2}{9}$

(3) $-\frac{1}{3\sqrt{3}}$ (4) $\frac{-2}{3\sqrt{3}}$

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $f(x)=\frac{\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x-1}{\frac{1}{\sqrt{2}}\sin x-\frac{1}{\sqrt{2}}\cos x}$

$=\frac{\sin\left(x+\frac{\pi}{4}\right)-\sin\left(\frac{\pi}{2}\right)}{\sin\left(x-\frac{\pi}{4}\right)}$

$=-\tan\left(\frac{x}{2}-\frac{\pi}{8}\right)$

$f'(x)=-\frac{1}{2}\sec^2\left(\frac{x}{2}-\frac{\pi}{8}\right)$

$f''(x)=-\frac{1}{2}\sec^2\left(\frac{x}{2}-\frac{\pi}{8}\right).\tan\left(\frac{x}{2}-\frac{\pi}{8}\right)$

$f\left(\frac{7\pi}{12}\right).f''\left(\frac{7\pi}{12}\right)=\frac{2}{9}$

11. If the points with vectors $\alpha\hat{i}+10\hat{j}+13\hat{k}$, $6\hat{i}+11\hat{j}+11\hat{k}$, $\frac{9}{2}\hat{i}+\beta\hat{j}-8\hat{k}$ are collinear, then $(19\alpha-6\beta)^2$ is equal to

(1) 36

(2) 16

(3) 25

(4) 49

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $\overrightarrow{AB}\parallel\overrightarrow{BC}$

$\frac{6-\alpha}{-\frac{3}{2}}=\frac{1}{\beta-11}=\frac{2}{19}$

$6\beta=123,\ 19\alpha=117$

12. If the coefficients of the three consecutive terms in the expansion of $(1+x)^n$ are in the ratio 1 : 5 : 20, then the coefficient of the fourth term is

(1) 3654 (2) 1827

(3) 5481 (4) 2436

Official Ans. by NTA (1)
Allen Ans. (1)

Sol. $\dfrac{{}^{n}C_{r-1}}{{}^{n}C_{r}} = \dfrac{1}{5}, \dfrac{{}^{n}C_{r}}{{}^{n}C_{r+1}} = \dfrac{1}{4}$

$\dfrac{r}{n-r+1} = \dfrac{1}{5}, \dfrac{r+1}{n-r} = \dfrac{1}{4}$

$n = 29$

$T_4 = {}^{29}C_3$

13. Let $S_k = \dfrac{1+2+.....+K}{K}$ and

$\sum\limits_{j=1}^{n} S_j^2 = \dfrac{n}{A}\left(Bn^2 + Cn + D\right)$, where A, B, C, $D \in N$ and A has least value. Then

(1) A + B is divisible by D

(2) A + B = 5 (D–C)

(3) A + C + D is not divisible by B

(4) A + B + C + D is divisible by 5

Official Ans. by NTA (1)
Allen Ans. (1)

Sol. $S_k = \dfrac{k+1}{2}$

$\sum S_j^2 = \dfrac{1}{4}\left(2^2 + 3^2 + ... + (n+1)^2\right)$

$= \dfrac{2n^3 + 9n^2 + 13n}{24}$

14. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$. If $|\text{adj}(\text{adj}(\text{adj}2A))| = (16)^n$, then n is equal to

(1) 10 (2) 9

(3) 12 (4) 8

Official Ans. by NTA (1)
Allen Ans. (1)

Sol. $|\text{adj}(\text{adj}(\text{adj}2A)| = |2A|^{(k-1)^3}$, k is order of matrix

$= 16^{10}$

15. Negation of $(p \Rightarrow q) \Rightarrow (q \Rightarrow p)$ is

(1) $(\sim p) \vee q$ (2) $(\sim q) \wedge p$

(3) $q \wedge (\sim p)$ (4) $p \vee (\sim q)$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $(\sim p \vee q) \rightarrow (\sim q \vee p)$

$= \sim(\sim p \vee q) \vee (\sim q \vee p)$

$= (p \wedge \sim q) \vee (\sim q \vee p)$

$\therefore$ negation is $q \wedge \sim p$ (from venn diagram)

16. The shortest distance between the lines $\dfrac{x-4}{4} = \dfrac{y+2}{5} = \dfrac{z+3}{3}$ and $\dfrac{x-1}{3} = \dfrac{y-3}{4} = \dfrac{z-4}{2}$ is

(1) $3\sqrt{6}$ (2) $6\sqrt{3}$

(3) $6\sqrt{2}$ (4) $2\sqrt{6}$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. Shortest distance $= \left|\dfrac{(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right| = 3\sqrt{6}$

17. The area of the region $\{(x, y) : x^2 \le y \le 8 - x^2, y \le 7\}$ is

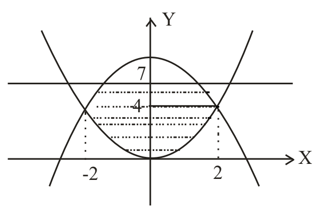
(1) 21 (2) 18

(3) 24 (4) 20

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $2\left(\int_0^4 \sqrt{y}\, dy + \int_4^7 \sqrt{8-y}\, dy\right) = 20$



18. Let the number of elements in sets A and B be five and two respectively. Then the number of subsets of A × B each having at least 3 and at most 6 element is :

(1) 792 (2) 752

(3) 782 (4) 772

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $n(A \times B) = 10$

${}^{10}C_3 + {}^{10}C_4 + {}^{10}C_5 + {}^{10}C_6 = 792$

19. $\lim_{x\to 0}\left(\left(\frac{1-\cos^2(3x)}{\cos^3(4x)}\right)\left(\frac{\sin^3(4x)}{(\log_e(2x+1))^5}\right)\right)$ is equal to _____

(1) 9 (2) 18

(3) 15 (4) 24

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $$\lim_{x\to 0}\left(\left(\frac{\frac{\sin^2(3x)}{(3x)^2}}{\cos^3(4x)}\right)\left(\frac{\frac{\sin^3(4x)}{(4x)^3}}{\left(\frac{\log_e(2x+1)}{2x}\right)^5}\right)\times\frac{(3x)^2\times(4x)^3}{(2x)^5}\right)$$

$= 18$

20. If for $z = \alpha + i\beta, |z+2| = z + 4(1+i)$, then $\alpha+\beta$ and $\alpha\beta$ are the roots of the equation

(1) $x^2+7x+12=0$

(2) $x^2+3x-4=0$

(3) $x^2+2x-3=0$

(4) $x^2+x-12=0$

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $\sqrt{(\alpha+2)^2+\beta^2} = (\alpha+4)+i(\beta+4)$

$\Rightarrow \beta = -4, \alpha = 1$

$\therefore x^2 + 7x + 12 = 0$

SECTION-B

21. Let [t] denotes the greatest integer $\le t$. Then $\frac{2}{\pi}\int_{\pi/6}^{5\pi/6}(8[\operatorname{cosec} x]-5[\cot x])dx$ is equal to

Official Ans. by NTA (14)

Allen Ans. (14)

Sol. $I = \frac{2}{\pi}\int_{\pi/6}^{5\pi/6}(8[\operatorname{cosec}x]-5[\cot x])dx$

$= \frac{2}{\pi}\int_{\pi/6}^{5\pi/6}(8[\operatorname{cosec}x]-5[\cot(\pi-x)])dx$

$2I = \frac{4}{\pi}\int_{\pi/6}^{5\pi/6}8[\operatorname{cosec}x]dx$

$-\frac{10}{\pi}\int_{\pi/6}^{5\pi/6}([\cot x]+[-\cot x])dx$

$2I = \frac{4}{\pi}\times 8\times\frac{4\pi}{6}+\frac{10}{\pi}\times\frac{4\pi}{6}$

$I = 14$

22. Let [t] denotes the greatest integer $\le t$. If the constant term in the expansion of $\left(3x^2-\frac{1}{2x^5}\right)^7$ is α , then $[\alpha]$ is equal to ____

Official Ans. by NTA (1275)

Allen Ans. (1275)

Sol. For constant term 14 - 7r = 0

r = 2

$\therefore$ constant term is ${}^7C_2 3^5\left(-\frac{1}{2}\right)^2$ or $\alpha = \frac{5103}{4}$

$[\alpha] = 1275$

23. Let $\vec{a}=6\hat{i}+9\hat{j}+12\hat{k}$, $\vec{b}=\alpha\hat{i}+11\hat{j}-2\hat{k}$ and $\vec{c}$ be vectors such that $\vec{a}\times\vec{c}=\vec{a}\times\vec{b}$. If $\vec{a}.\vec{c}=-12$, $\vec{c}.\left(\hat{i}-2\hat{j}+\hat{k}\right)=5$, then $\vec{c}.\left(\hat{i}+\hat{j}+\hat{k}\right)$ is equal to ________.

Official Ans. by NTA (11)

Allen Ans. (11)

Sol. $\vec{a}\times\left(\vec{c}-\vec{b}\right)=\vec{0}\Rightarrow\vec{a}\parallel\vec{c}-\vec{b}$

$\vec{c}=\vec{b}+\lambda\vec{a}$

$\vec{a}\,.\,\vec{c}=\vec{a}\,.\,\vec{b}+\lambda|\vec{a}|^2=-12$

$6\alpha+261\lambda=-87$

$\vec{c}.\left(\hat{i}-2\hat{j}+\hat{k}\right)=5$

$\left(\vec{b}+\lambda\vec{a}\right).\left(\hat{i}-2\hat{j}+\hat{k}\right)=5$

$\Rightarrow\alpha=29,\ \lambda=-1$

24. The largest natural number n such that 3^n divides 66! is _____ .

Official Ans. by NTA (31)

Allen Ans. (31)

Sol. $\left[\frac{66}{3}\right]+\left[\frac{66}{3^2}\right]+\left[\frac{66}{3^3}\right]=22+7+2=31$

25. If a_n is the greatest term in the sequence $a_n=\frac{n^3}{n^4+147}, n=1,2,3.......$, then α is equal to ____

Official Ans. by NTA (5)

Allen Ans. (5)

Sol. $a'(n)=\frac{(3n^2)(n^4+147)-n^3(4n^3)}{(n^4+147)^2}$

$a'(n)=0$ or $n=\sqrt{21}$

$a_4=\frac{64}{403}$

$a_5=\frac{125}{772}$ which is largest

26. Let $A=\{0,3,4,6,7,8,9,10\}$ and R be the relation defined on A such that $R=\{(x,y)\in A\times A: x-y$ is odd positive integer or $x-y=2\}$. The minimum number of elements that must be added to the relation R, so that it is a symmetric relation, is equal to _____.

Official Ans. by NTA (19)

Allen Ans. (19)

Sol. 5 even numbers and 3 odd numbers

$\therefore\ {}^5C_1\times{}^3C_1+4=19$

27. Consider a circle $C_1: x^2+y^2-4x-2y=\alpha-5$. Let its mirror image in the line $y=2x+1$ be another circle $C_2: 5x^2+5y^2-10fx-10gy+36=0$. Let r be the radius of C_2. Then $\alpha+r$ is equal to _____.

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. Mirror image of centre of $C_1(2,1)$ in $y=2x+1$ is centre of $C_2\left(-\frac{6}{5},\frac{13}{5}\right)$

$\therefore C_2$ is $x^2+y^2+\frac{12}{5}x-\frac{26}{5}y+\frac{36}{5}=0$

$r_2=1$ and $\alpha=1\Rightarrow\alpha+r_2=2$

28. If the solution curve of the differential equation

$$\left(y-2\log_e x\right)dx+\left(x\log_e x^2\right)dy=0,\ x>1$$

passes through the points $\left(e,\frac{4}{3}\right)$ and $\left(e^4,\alpha\right)$, then α is equal to ____.

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $\frac{dy}{dx}+\frac{y}{2x\ln x}=\frac{1}{x}$

$\text{I.F.}=e^{\int\frac{1}{2x\ln x}dx}=\sqrt{\ln x}$

$y\sqrt{\ln x}=\int\frac{1}{x}\sqrt{\ln x}\,dx$

Put $\ln x=t^2 \Rightarrow \frac{1}{x}dx=2t\,dt$

$\Rightarrow y\sqrt{\ln x}=\int 2t^2dt$

$y\sqrt{\ln x}=\frac{2(\ln x)^{\frac{3}{2}}}{3}+\frac{2}{3}$

(e^4,α) satisfies curve

$\therefore \alpha=3$

29. Let λ_1,λ_2 be the values of λ for which the points $\left(\frac{5}{2},1,\lambda\right)$ and $\left(-2,0,1\right)$ are at equal distance from the plane $2x+3y-6z+7=0$. if $\lambda_1>\lambda_2$, then the distance of the point $\left(\lambda_1-\lambda_2,\lambda_2,\lambda_1\right)$ from the line $\frac{x-5}{1}=\frac{y-1}{2}=\frac{z+7}{2}$ is ____.

Official Ans. by NTA (9)

Allen Ans. (9)

Sol. $\left|\frac{5+3-6\lambda+7}{\sqrt{49}}\right|=\left|\frac{-4+0-6+7}{\sqrt{49}}\right|$

$\Rightarrow \lambda_1=3,\ \lambda_2=2$

Shortest distance $=\left|\frac{\left|(\vec{a}_2-\vec{a}_1)\times\vec{b}\right|}{|\vec{b}|}\right|=9$

30. Let the mean and variance of 8 numbers x, y, 10, 12, 6, 12, 4, 8, be 9 and 9.25 respectively. If $x>y$, then $3x-2y$ is equal to ___.

Official Ans. by NTA (25)

Allen Ans. (25)

Sol. Mean $=\frac{x+y+52}{8}=9\Rightarrow x+y=20$

Variance $=\frac{x^2+y^2+504}{8}-9^2=9.25$

$\Rightarrow x^2+y^2=218$

$\therefore\ x=13,\ y=7$ $\qquad\Rightarrow 3x-2y=25$