CS1.404 (Spring 2024)
Optimization Methods
Assignment - 3

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KKT condition verification -

Test case -0

Test case 0

1

① Trid functions:-

$$f(x) = \sum_{i=1}^{d} (x_i - 1)^2 - \sum_{i=2}^{d} x_{i-1} x_i$$

Constraints:

$$x_1^2 - 2x_2 \le 0$$

for two variable $f(x)$ becomes,

$$f(x) = (x_1 - 1)^2 + (x_2 - 1)^2 - x_1 x_2$$

$$\nabla f(x) = \begin{bmatrix} 2(x_1 - 1) - x_2 \\ 2(x_2 - 1) - x_1 \end{bmatrix}$$

[from the python program,
$x^* = [x_1, x_2] = [2, 2]$
$\lambda = 0$]

$$\nabla h(x) = \begin{bmatrix} 2x_1 \\ -2 \end{bmatrix}$$

KKT conditions are -

a→ $\nabla f(x^*) + \sum_{j=1}^{l} \lambda_j \nabla h_j(x^*) = 0$

$$\Rightarrow \begin{bmatrix} 2(x_1 - 1) - x_2 \\ 2(x_2 - 1) - x_1 \end{bmatrix} + \lambda \begin{bmatrix} 2x_1 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2(2-1) - 2 \\ 2(2-1) - 2 \end{bmatrix} + 0 \begin{bmatrix} 2*2 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 0$$

b→ $\lambda_j h_j(x^*) = 0$; $j = 1, \dots l$

$$\Rightarrow \lambda \cancel{\begin{bmatrix} 2x_1 \\ -2 \end{bmatrix}} \Rightarrow 0 * (x_1^2 - 2x_2)$$

$$\Rightarrow 0 \begin{bmatrix} 4 \\ -2 \end{bmatrix} \Rightarrow 0$$

$$\Rightarrow 0$$

c→ $\lambda_j \ge 0$; $j = 1 \dots l$

$$\boxed{\lambda = 0 \ge 0}$$

→ KKT condition is verified.

Test case -1

Test case-1

Trid function:

$$f(x) = \sum_{i=1}^{d} (x_i - 1)^2 - \sum_{i=2}^{d} x_{i-1} x_i$$

constraints:-

$$x_1^2 - x_2^2 + 1 \leq 0$$

for two variable, $f(x)$ becomes:-

$$f(x) = (x_1 - 1)^2 + (x_2 - 1)^2 - x_1 x_2$$

from the python program:-

$$x^* = [x_1, x_2] = [1.899, 2.146]$$

$$\lambda = 0.09166$$

$$\nabla f(x) = \begin{bmatrix} 2(x_1 - 1) - x_2 \\ 2(x_2 - 1) - x_1 \end{bmatrix}$$

$$\nabla h(x) = \begin{bmatrix} 2x_1 \\ -2x_2 \end{bmatrix}$$

KKT conditions are-

(a) $\nabla f(x^*) + \sum_{j=1}^{l} \lambda_j \nabla h_j(x^*) = 0$

$$\Rightarrow \begin{bmatrix} 2(x_1 - 1) - x_2 \\ 2(x_2 - 1) - x_1 \end{bmatrix} + \lambda \begin{bmatrix} 2x_1 \\ -2x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1.899 - 1) - 2.146 \\ 2(2.146 - 1) - 1.899 \end{bmatrix} + 0.09166 \begin{bmatrix} 2 * 1.899 \\ -2 * 2.146 \end{bmatrix}$$

$$= \begin{bmatrix} -0.348 \\ 0.393 \end{bmatrix} + \begin{bmatrix} 0.348 \\ -0.393 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(b) $\lambda_j h_j(x^*) = 0$

$$\Rightarrow 0.09166 (x_1^2 - x_2^2 + 1)$$

$$\Rightarrow 0.09166 (1.899^2 - 2.146^2 + 1)$$

$$\Rightarrow 0.09166 (3.60 - 4.6 + 1)$$

$$\Rightarrow 0.09166 (0)$$

$$\Rightarrow 0$$

(c) $\lambda_j \geq 0; \; j = 1, \ldots l$

$$\lambda = 0.09166 \geq 0$$

Test case -2

Test case -2

Trid function:-
$$f(x) = \sum_{i=1}^{d} (x_i - 1)^2 - \sum_{i=2}^{d} x_{i-1} x_i$$

Constraints:-
$$-1 - x_1 \le 0$$
$$x_1 - 1 \le 0$$
$$-1 - x_2 \le 0$$
$$x_2 - 1 \le 0$$

for two variable, $f(x)$ becomes
$$f(x) = (x_1 - 1)^2 + (x_2 - 1)^2 - x_1 x_2$$

$$\nabla f(x) = \begin{bmatrix} 2(x_1 - 1) - x_2 \\ 2(x_2 - 1) - x_1 \end{bmatrix}$$

from the python program:-
$$x^* = [x_1, x_2] = [1, 1]$$
$$\lambda = [0, 1, 0, 1]$$

$$\nabla h_1(x) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \nabla h_2(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\nabla h_3(x) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \nabla h_4(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

KKT Condition are!

ⓐ $\nabla f(x^*) + \sum_{j=1}^{l} \lambda_j \nabla h_j(x^*) = 0$

$$\Rightarrow \begin{bmatrix} 2(x_1 - 1) - x_2 \\ 2(x_2 - 1) - x_1 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2(1-1) - 1 \\ 2(1-1) - 1 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

ⓑ $\lambda_j h_j(x^*) = 0 \,;\, j = 1 \ldots l$

$\lambda_1(-1 - x_1) = 0$
$\Rightarrow 0(-1-1)$
$\Rightarrow 0$

$\lambda_2(x_1 - 1) = 0$
$\Rightarrow 1(1-1)$
$\Rightarrow 0$

$\lambda_3(-1 - x_2) = 0$
$\Rightarrow 0(-1-1)$
$\Rightarrow 0$

$\lambda_4(x_2 - 1) = 0$
$\Rightarrow 1(1-1)$
$\Rightarrow 0$

ⓒ $\lambda_j \ge 0 ;\, j = 1, \ldots l$
$\lambda_1 = 0 \ge 0$, $\lambda_2 = 1 \ge 0$, $\lambda_3 = 0 \ge 0$, $\lambda_4 = 1 \ge 0$

Test case -3

Test case 3

Tri'd function:-

$$f(x) = \sum_{i=1}^{d} (x_i - 1)^2 - \sum_{i=2}^{d} x_{i-1} x_i$$

Constraints:-

$$-x_1 \leq 0$$
$$x_1 - 3 \leq 0$$
$$-x_2 \leq 0$$
$$x_2 - 3 \leq 0$$

for two variable, f(x) becomes,

$$f(x) = (x_1 - 1)^2 + (x_2 - 1)^2 - x_1 x_2$$

$$\nabla f(x) = \begin{bmatrix} 2(x_1 - 1) - x_2 \\ 2(x_2 - 1) - x_1 \end{bmatrix}$$

from the python Program:-

$$x^* = [x_1, x_2] = [2, 2]$$
$$\lambda = [0, 0, 0, 0]$$

$$\nabla h_1(x) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \nabla h_2(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\nabla h_3(x) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \nabla h_4(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

KKT conditions are:

ⓐ $\nabla f(x^*) + \sum_{j=1}^{l} \lambda_j \nabla h_j(x^*) = 0$

$$= \begin{bmatrix} 2(x_1 - 1) - x_2 \\ 2(x_2 - 1) - x_1 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(2-1) - 2 \\ 2(2-1) - 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

ⓑ $\lambda_j h_j(x^*) = 0, \; j = 1 \dots l$

$\lambda_1(-x_1) = 0$	$\lambda_2(x_1 - 3) = 0$	$\lambda_3(-x_2) = 0$	$\lambda_4(x_2 - 3) = 0$
$\Rightarrow 0 * (-2)$	$\Rightarrow 0(2-3)$	$\Rightarrow 0(-2)$	$\Rightarrow 0(2-3)$
$\Rightarrow 0$	$\Rightarrow 0$	$\Rightarrow 0$	$\Rightarrow 0$

ⓒ $\lambda_j \geq 0; \; j = 1 \dots l$

$$\lambda_1 = 0 \geq 0, \quad \lambda_2 = 0 \geq 0, \quad \lambda_3 = 0 \geq 0, \quad \lambda_4 = 0 \geq 0$$

Test case -4

Test case -4

Test function:

$$f(x) = \sum_{i=1}^{d} (x_i - 1)^2 - \sum_{i=2}^{d} x_{i-1} x_i$$

constraints:

$$3 - x_1 \leq 0$$
$$x_1 - 4 \leq 0$$
$$3 - x_2 \leq 0$$
$$x_2 - 4 \leq 0$$

for two variable, $f(x)$ becomes,

$$f(x) = (x_1 - 1)^2 + (x_2 - 1)^2 - x_1 x_2$$

$$\nabla f(x) = \begin{bmatrix} 2(x_1 - 1) - x_2 \\ 2(x_2 - 1) - x_1 \end{bmatrix}$$

from the python Program:-

$$x^* = [x_1, x_2] = [3, 3]$$
$$\lambda = [1, 0, 1, 0]$$

$$\nabla h_1(x) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \nabla h_2(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\nabla h_3(x) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \nabla h_4(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

KKT Conditions are-

ⓐ $\nabla f(x^*) + \sum_{j=1}^{l} \lambda_j \nabla h_j(x^*) = 0$

$$\Rightarrow \begin{bmatrix} 2(x_1 - 1) - x_2 \\ 2(x_2 - 1) - x_1 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2(3-1) - 3 \\ 2(3-1) - 3 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

ⓑ $\lambda_j h_j(x^*) = 0; \ j = 1 \ldots l$

$\Rightarrow \lambda_1(3 - x_1) = 0$
$\Rightarrow 1(3-3)$
$\Rightarrow 0$

$\lambda_2(x_1 - 4) = 0$
$\Rightarrow 0(x_1 - 4)$
$\Rightarrow 0$

$\lambda_3(3 - x_2) = 0$
$\Rightarrow 1(3-3)$
$\Rightarrow 0$

$\lambda_4(x_2 - 4) = 0$
$\Rightarrow 0(x_2 - 4)$
$\Rightarrow 0$

ⓒ $\lambda_j \geq 0; \ j = 1 \ldots l$

$\lambda_1 = 1 \geq 0, \ \lambda_2 = 0 \geq 0, \ \lambda_3 = 1 \geq 0, \ \lambda_4 = 0 \geq 0$

Test case -5

Test case-5

Matyas function:-

$$f(x) = 0.26\,(x_1^2 + x_2^2) - 0.48\,x_1 * x_2$$

Constraints:-

$$-x_1 \le 0$$
$$x_1 - 1 \le 0$$
$$-x_2 \le 0$$
$$x_2 - 1 \le 0$$

from the python program:-

$x^* = [0.011, 0.011]$

$\lambda = [0, 0, 0, 0]$

$$\nabla f(x) = \begin{bmatrix} 0.52x_1 - 0.48x_2 \\ 0.52x_2 - 0.48x_1 \end{bmatrix}$$

$$\nabla h_1(x) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \nabla h_2(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\nabla h_3(x) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \nabla h_4(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

KKT conditions are:-

(a) $\nabla f(x^*) + \sum_{j=1}^{l} \lambda_j \nabla h_j(x^*) = 0$

$$= \begin{bmatrix} 0.52x_1 - 0.48x_2 \\ 0.52x_2 - 0.48x_1 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0.52 * 0.011 - 0.48 \times 0.011 \\ 0.52 * 0.011 - 0.48 * 0.011 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0.00044 \\ 0.00044 \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(b) $\lambda_j h_j(x^*) = 0 \; ; \; j = 1, \ldots l$

$\lambda_1(-x_1) = 0 \Rightarrow 0$

$\lambda_2(x_1 - 1) = 0 \Rightarrow 0(x_1 - 1) \Rightarrow 0$

$\lambda_3(-x_2) = 0 \Rightarrow 0(-x_2) \Rightarrow 0$

$\lambda_4(x_2 - 1) \Rightarrow 0(x_2 - 1) \Rightarrow 0$

(c) $\lambda_j \geq 0 \; ; \; j = 1, \ldots l$

$\lambda_1 = 0 \geq 0, \; \lambda_2 = 0 \geq 0, \; \lambda_3 = 0 \geq 0, \; \lambda_4 = 0 \geq 0$

Test case -6

Test case - 6

Matyas function:-

$$f(x) = 0.26\,(x_1^2 + x_2^2) - 0.48\,x_1 * x_2.$$

Constraints :-

$$1 - x_1 \le 0$$
$$x_1 - 2 \le 0$$
$$1 - x_2 \le 0$$
$$x_2 - 2 \le 0$$

$$\nabla f(x) = \begin{bmatrix} 0.52x_1 - 0.48x_2 \\ 0.52x_2 - 0.48x_1 \end{bmatrix}$$

from the python program:

$x^* = [1.004, 1.004]$

$\lambda = [0.03460, 0, 0.03460, 0]$

$$\nabla h_1(x) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \nabla h_2(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \nabla h_3(x) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \nabla h_4(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

KKT Condition are:-

(a) $\nabla f(x^*) + \sum_{j=1}^{l} \lambda_j \nabla h_j(x^*) = 0$

$$= \begin{bmatrix} 0.52x_1 - 0.48x_2 \\ 0.52x_2 - 0.48x_1 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.52 * 1.004 - 0.48 * 1.004 \\ 0.52 * 1.004 - 0.48 * 1.004 \end{bmatrix} + 0.0346 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.0346 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.04008 \\ 0.04008 \end{bmatrix} + \begin{bmatrix} -0.0346 \\ -0.0346 \end{bmatrix}$$

$$= \begin{bmatrix} 0.005 \\ 0.005 \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(b) $\lambda_j h_j(x^*) = 0;\ j = 1, \dots l$

$$\lambda_1 (1 - x_1) = 0 \Rightarrow 0.0346(1 - 0.004) \Rightarrow 0.0001 \approx 0$$

$$\lambda_2 (x_1 - 2) = 0 \Rightarrow 0\,(x_1 - 2) \Rightarrow 0$$

$$\lambda_3 (1 - x_2) = 0 \quad 0.0346\,(1 - 1.004) \Rightarrow 0.0001 \approx 0$$

$$\lambda_4 (x_2 - 2) = 0 \Rightarrow 0\,(x_2 - 2) \Rightarrow 0$$

(c) $\lambda_j \ge 0 \;;\ j = 1 \dots l$

$$\lambda_1 = 0.0346 \ge 0, \quad \lambda_2 = 0 \ge 0, \quad \lambda_3 = 0.0346 \ge 0, \quad \lambda_4 = 0 \ge 0$$

Test case-7

Test Case-7

Matyas function:-

$$f(x) = 0.26(x_1^2 + x_2^2) - 0.48x_1 x_2$$

Constraints:-

$$-1 - x_1 \le 0$$
$$x_1 + 0.5 \le 0$$
$$-0.5 - x_2 \le 0$$
$$x_2 - 0.5 \le 0$$

$$\nabla f(x) = \begin{bmatrix} 0.52x_1 - 0.48x_2 \\ 0.52x_2 - 0.48x_1 \end{bmatrix}$$

from the python program:-

$x^* = [-0.5, 0.462]$

$\lambda = [0, 0.0384, 0, 0]$

$\nabla h_1(x) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $\nabla h_2(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\nabla h_3(x) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $\nabla h_4(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

KKT Condition are:-

(a) $\nabla f(x^*) + \sum_{j=1}^{4} \lambda_j \nabla h_j(x^*) = 0$

$$\Rightarrow \begin{bmatrix} 0.52x_1 - 0.48x_2 \\ 0.52x_2 - 0.48x_1 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0.52 * -0.5 - 0.48 * -0.462 \\ 0.52 * -0.462 - 0.48 * -0.5 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 0.0384 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -0.03824 \\ 0.00024 \end{bmatrix} + \begin{bmatrix} 0.0384 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 \\ 0.0024 \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(b) $\lambda_j h_j(x^*) = 0 \; ; \; j = 1 \ldots l$

$\lambda_1(-1 - x_1) = 0$
$\Rightarrow 0(-1 - x_1)$
$\Rightarrow 0$

$\lambda_2(x_1 + 0.5)$
$\Rightarrow 0.0384(-0.5 + 0.5)$
$\Rightarrow 0$

$\lambda_3(-0.5 - x_2) = 0$
$\Rightarrow 0(-0.5 - x_2)$
$\Rightarrow 0$

$\lambda_4(x_2 - 0.5) = 0$
$\Rightarrow 0(x_2 - 0.5)$
$\Rightarrow 0$

(c) $\lambda_j \ge 0 \; ; \; j = 1 \ldots l$

$\lambda_1 = 0 \ge 0, \; \lambda_2 = 0.0384 \ge 0, \; \lambda_3 = 0 \ge 0, \; \lambda_4 = 0 \ge 0$