Prashant Kolkur

# Final Exam

1\. a) $P(\text{First card is ace}) = \frac{4}{52} = \frac{1}{13}$

b) $P(\text{First and second are aces}) = \frac{\cancel{4}}{\cancel{52}_{13}} \times \frac{3}{\cancel{51}_{17}} = \frac{1}{221}$

(c) $P(\text{Second card is an ace}) = P(1^{st}\ \&\ 2^{nd}\ \text{ace}) + P(1^{st}\ \text{not ace and}\ 2^{nd}\ \text{ace})$

$$= \frac{1}{13 \times 17} + \frac{\cancel{48}^{\cancel{12}\,4}}{\cancel{52}_{13}} \times \frac{4}{\cancel{51}_{17}}$$

$$= \frac{1}{13 \times 17} + \frac{16}{13 \times 17}$$

$$= \frac{\cancel{17}}{13 \times \cancel{17}}$$

$$= \frac{1}{13}$$

(d) $P(\text{First ace} \mid \text{heart}) = \frac{P(\text{First ace} \cap \text{heart})}{P(\text{heart})}$

$$= \frac{1/\cancel{52}}{13/\cancel{52}}$$

$$= \frac{1}{13}$$

(e) $P(2^{nd}\ \text{ace} \mid 1^{st}\ \text{ace}) = \frac{P(2^{nd}\ \text{ace} \cap 1^{st}\ \text{ace})}{P(1^{st}\ \text{ace})}$

$$= \frac{\cancel{4/52} \times \cancel{3}/\cancel{51}\ 17}{\cancel{4/52}}$$

$$= \frac{1}{17}$$

(3) (a) $P = \frac{1}{5!} = \frac{1}{120}$

(b) $P(1,1,1) + P(2,2,2) + P(3,3,3) + P(4,4,4) + P(5,5,5) + P(6,6,6)$

$= \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} + \text{-----}$

$= 6 \times \left[\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}\right]$

$= \frac{1}{36}$

(c) $P = 20\% = 0.2$

E(Expected number of trips) $= \frac{1}{P} = \frac{1}{0.2} = 5$

(d) Total population $= 100$ (~~let~~ lets say)

Men $= 40$, Left handed men $= 20\%$ of $40 = \frac{20}{100} \times 40 = 8$

Women $= 60$, Left handed women $= 10\%$ of $60 = 0.1 \times 60 = 6$

P(female / left handed) $= \frac{6}{8+6} = \frac{6}{14} = \frac{3}{7}$

3) (e) $P(\text{medicine}/\text{feels good}) = ?$

$P(\text{medicine}) = \frac{2}{10}$

$P(\text{placebos}) = \frac{8}{10}$

$P(\text{feels good}) = P(\text{feels good}/\text{medicine}) \times P(\text{medicine})$
$+ P(\text{feels good}/\text{placebo}) \times P(\text{placebo})$

$= \frac{3}{\cancel{4}_2} \times \frac{\cancel{2}}{10} + \frac{1}{2} \times \frac{8}{10}$

$= \frac{3}{20} + \frac{8}{20}$

$= \frac{11}{20}$

$\therefore P(\text{medicine}/\text{feels good}) = \frac{P(\text{feels good}/\text{medicine}) \times P(\text{medicine})}{P(\text{feels good})}$

$= \frac{\frac{3}{\cancel{4}2} \times \frac{\cancel{2}}{10}}{\frac{11}{20}}$

$= \frac{3}{11}$

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4) a) $E(X) = \sum x\, P(x)$

$$= 1\times\frac{1}{12} + 2\times\frac{1}{12} + 3\times\frac{1}{12} + 4\times\frac{1}{4} + 5\times\frac{1}{4} + 6\times\frac{1}{4}$$

$$= (1+2+3)\frac{1}{12} + (4+5+6)\frac{1}{4}$$

$$= \frac{\cancel{6}^{1}}{\cancel{12}_{2}} + \frac{15}{4}$$

$$= \frac{17}{4} = 4.25$$

(b) $Var(X) = \sum x^2 p(x) - (E(x))^2$

$$= (1^2+2^2+3^2)\frac{1}{12} + (4^2+5^2+6^2)\frac{1}{4} - \left(\frac{17}{4}\right)^2$$

$$= \frac{\cancel{14}^{7}}{\cancel{12}_{6}} + \frac{77}{4} - \frac{289}{16}$$

$$= 7\left[\frac{1}{6} + \frac{11}{4}\right] - \frac{289}{16}$$

$$= 7\left[\frac{4+66}{24}\right] - \frac{289}{16}$$

$$= 7\left[\frac{\cancel{70}^{35}}{\cancel{24}_{12}}\right] - \frac{289}{16}$$

$$= \frac{1}{4}\left[\frac{245}{3} - \frac{289}{4}\right]$$

$$= \frac{1}{4}\left[\frac{980-867}{12}\right]$$

$$= \frac{113}{48} = 2.3541$$

4) (c) $E(Z) = E(Z_1) + E(Z_2) + \cdots E(Z_{100})$

$= 100 \times 4.25$

$= 425$

(d) $Var(Z) = Var(Z_1) + Var(Z_2) + \cdots Var(Z_{100})$ since rolling die hundred times are independent events

$= 100 \times 2.354$

$= 235.4$

(5) $X = \{-1, 1\}$

$E(x_1) = 0 = \sum x_i P(x_i)$

$= (-1) P(-1) + (1) P(1) = 0$

$P(-1) = P(1)$

$\therefore P(-1) = 0.5$

$P(1) = 0.5$

$\sigma^2_{x_1} = \sum x^2 P(x) - (E(x_1))^2$

$= (-1)^2 P(-1) + 1^2 P(1) - 0^2$

$= P(-1) + P(1) - 0$

$= 0.5 + 0.5 - 0$

$\sigma^2_{x_1} = 1$

$\sigma_{x_1} = \sqrt{1} = 1$

$E(X_2) = 0.5 = \sum x P(x)$

$= (-1)P(-1) + (1)P(1) = 0.5$

$-P(-1) + P(1) = 0.5$

and $P(-1) + P(1) = 1$

$\therefore \quad -P(\cancel{-1}) + P(1) = 0.5$

$P(\cancel{-1}) + P(1) = 1$

$2P(1) = 1.5$

$P(1) = 0.75$

$P(-1) = 1 - 0.75$

$P(-1) = 0.25$

$\sigma^2_{X_2} = \sum x^2 P(x) - (E(X_2))^2$

$= (-1)^2 P(-1) + 1^2 P(1) - 0.5^2$

$= 0.25 + 0.75 - 0.25$

$= 1 - 0.25$

$= 0.75$

$\sigma_{X_2} = \sqrt{0.75} = 0.866$

$corr = 0.25 = \frac{cov}{\sigma_{X_1} \times \sigma_{X_2}}$

$cov = 0.25 \times 1 \times 0.866$

$cov = 0.2165$

$\therefore$ Mean $= \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$

$$\text{cov} = \begin{bmatrix} 1 & 0.2165 \\ 0.2165 & 0.75 \end{bmatrix}$$

(6) $E(X) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\quad \text{cov}(X) = \begin{bmatrix} 5 & -3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} = M$

$M\lambda = \lambda u$

where $\lambda$ = eigen values

$u$ = eigen vectors

$(M - \lambda I)u = 0$

$\therefore |M - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & -3 & 0 \\ -3 & 5-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0$$

$(5-\lambda)[(5-\lambda)(4-\lambda) - 0] + 3(-3(4-\lambda) - 0) + 0 = 0$

$(5-\lambda)(5-\lambda)(4-\lambda) - 9(4-\lambda) = 0$

$(4-\lambda)((5-\lambda)^2 - 9) = 0$

$\therefore 4 - \lambda = 0$ & $(5-\lambda)^2 - 9 = 0$

$\boxed{\lambda = 4}$

$\lambda^2 - 10\lambda + 25 - 9 = 0$

$\lambda^2 - 10\lambda + 16 = 0$

$\lambda^2 - 8\lambda - 2\lambda + 16 = 0$

$\boxed{\lambda = 8}$ $\boxed{\lambda = 2}$

$\therefore$ Eigen values $= \lambda = 2, 4, 8$

$\lambda = 2$, $\begin{bmatrix} 5-\lambda & -3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0$

$$\begin{bmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3u_1 - 3u_2 \\ -3u_1 + 3u_2 \\ 2u_3 \end{bmatrix} = 0$$

$\therefore 3u_1 - 3u_2 = 0$

$\boxed{u_1 = u_2}$

$2u_3 = 0$

$\boxed{u_3 = 0}$

Lets take $u_1 = u_2 = 1$

$\therefore U = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\hat{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\underline{\lambda = 4}$, $\begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0$

$$\begin{bmatrix} u_1 - 3u_2 \\ -3u_1 + u_2 \\ 0 \times u_3 \end{bmatrix} = 0$$

$(u_1 - 3u_2 = 0) \times 3$

$-3u_1 + u_2 = 0$

$$\begin{aligned} 3u_1 - 9u_2 &= 0 \\ -3u_1 + u_2 &= 0 \\ \hline -8u_2 &= 0 \end{aligned}$$

$\boxed{u_2 = 0}$

$u_1 = 3u_2$

$\boxed{u_1 = 0}$

$0 \times u_3 = 0$

Lets take $u_3 = 1$.

$\therefore u = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 8$, $\begin{bmatrix} -3 & -3 & 0 \\ -3 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0$

$\begin{bmatrix} -3u_1 - 3u_2 \\ -3u_1 - 3u_2 \\ -4u_3 \end{bmatrix} = 0$

$-3u_1 - 3u_2 = 0$ $\qquad$ $-4u_3 = 0$

$\boxed{u_1 = -u_2}$ $\qquad$ $\boxed{u_3 = 0}$

Let take $\boxed{u_1 = 1}$ Then

$\boxed{u_2 = -1}$

$\therefore \lambda = 8 \qquad u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\hat{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\therefore$ Eigen vectors

(b) $\lambda = 2$, Eigen vector $= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 4$, Eigen vector $= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 8$, Eigen vector $= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

(c) It would project on

$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(d) $x = (4, 0, 2)$

$[4 \; 0 \; 2] \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} [4 + 0 + 0] = \frac{4}{\sqrt{2}}$

$[4 \; 0 \; 2] \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = [0 + 0 + 2] = 2$

Resulting 2-d projection $= \left( \frac{4}{\sqrt{2}}, 2 \right)$

$= (2\sqrt{2}, 2)$

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(c) (e) Reconstruction of $x$

$$\left[\frac{4}{\sqrt{2}} \;\; 2\right] \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\frac{4}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + 0 \;,\; \frac{4}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} + 0 \;\; 0+2\right]$$

$$[2, -2, 2]$$

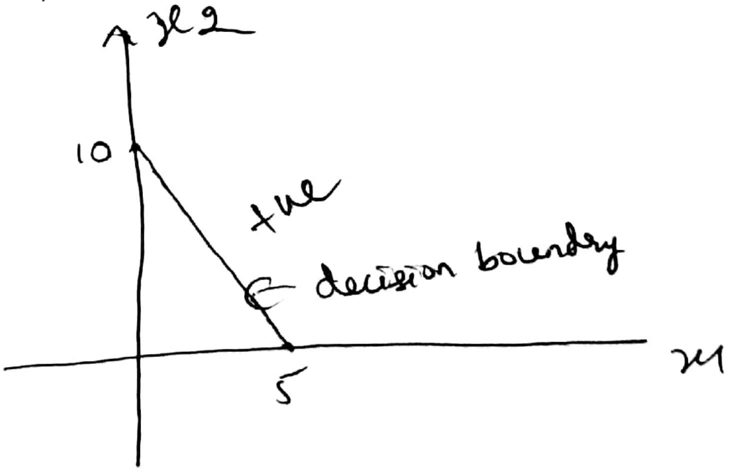
(7) $w \cdot x \geq \theta$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq 10$$

$$2x_1 + x_2 \geq 10$$

$x_1 = 0, \quad x_2 \geq 10$

$x_2 = 0, \quad x_1 \geq 5$



7 (a) $n = 100$

$p = 0.4$

$\mu = p = 0.4$

$$\sigma = \sqrt{\frac{p(1-p)}{n}}$$

$$= \sqrt{\frac{0.4 \times 0.6}{100}}$$

$$= 0.04899$$

95% confi~~g~~dence $= \mu \pm 2\sigma$

$\cancel{0.4 \pm 2 \times 0.04899}$

$\cancel{[0.4 - 2 \times 0.04899,\ 0.4 + 0.04899]}$

$= \cancel{[0.3\ldots,\ 0.4]}$

$0.4 \pm 2 \times 0.04899$

$= [0.4 - 2 \times 0.04899,\ 0.4 + 2 \times 0.04899]$

$= [0.302,\ 0.498]$

(b) $2\sigma = 0.01$

$\sigma = 0.05$

$$\sqrt{\frac{p(1-p)}{n}} = 0.05$$

$$n = \frac{0.4 \times 0.6}{(0.005)^2} = 9600$$

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(9) $n = 100$

$\mu = 12.2$

$\sigma = 5.4$

$N\left(\hat{\mu}, \frac{\sigma^2}{n}\right)$

$\hat{\sigma} = \frac{\sigma}{\sqrt{n}} = \frac{5.4}{\sqrt{100}} = 0.54$

$\therefore \mu \pm 2\sigma$ (95% confidence interval)

$12.2 \pm 2 \times 0.54$

$12.2 \pm 1.08$

$= [11.12, 13.28]$

(10) (a) Null hypothesis:- mean SAT scores of Genius academy students and local high school should be same.

(b) $n = 100$

$\mu_1 = 1930$, $\sigma_1 = 150$, $N = (n\mu_1, n\sigma^2)$

$\mu_2 = 1860$, $\sigma_2 = 200$, $N(n\mu_2, n\sigma^2)$

$\hat{\mu}_1 = n\mu_1 = 100 \times 1930$

$\hat{\sigma}_1 = \sqrt{100 \times 150^2} = 1500$

$\hat{\mu}_2 = n\mu_2 = 100 \times 1860$

$\hat{\sigma}_2 = \sqrt{100 \times 200^2} = 2000$

$\therefore \sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$

$= \sqrt{1500^2 + 2000^2}$

$= 2500$

$$Z = \frac{\text{Observed} - \text{Expected}}{\sigma}$$

$= \cancel{\frac{100 \times 1930 - 100 \times 1980}{2500}}$ $\frac{100 \times 1860 - 100 \times 1930}{2500}$

$= -2.8$

(c) p value $= 0.002555$ (For significance level $= 0.05$)

p value $< 0.05$

Conclusion :- This is a strong evidence ~~agg~~ against null hypothesis ~~again~~