Prashant Kolkur

Worksheet 7

1. $x = (1,2,3)$

unit vector of $x = \left(\frac{1}{\sqrt{1^2+2^2+3^2}}, \frac{2}{\sqrt{1^2+2^2+3^2}}, \frac{3}{\sqrt{1^2+2^2+3^2}}\right)$

$= \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

2. $(1,1)$

orthogonal vectors $= x \cdot y = 0$

$(1,1)\,(y_1, y_2) = 0$

$y_1 + y_2 = 0$

$y_1 = 1, \; y_2 = -1$

$y_1 = -1, \; y_2 = 1$

unit vectors $= \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

3. $x \in \mathbb{R}^d$

$x = (x_1, x_2, \ldots x_d)$ d-dimension

$x \cdot x = 25$

$x_1^2 + x_2^2 + x_3^2 + \ldots x_d^2 = 25$

4) $f(x) = 2x_1 - x_2 + 6x_3$

$x \in R^3$

$w \cdot x$

$w = (2, -1, 6)$

8) $x = (1, 3, 5)$

$$x^T x = [1, 3\ 5] \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$= [1 + 9 + 25]$$

$$= [35]$$

$$x x^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} [1\ 3\ 5]$$

$$= \begin{bmatrix} 1 & 3 & 5 \\ 3 & 9 & 15 \\ 5 & 15 & 25 \end{bmatrix}$$

9) Vectors $x, y \in \mathbb{R}^d$, Length $= 2$

$\theta =$ angle between x and $y = ?$

$x^T y = 2.$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix} [y_1 \; y_2 \; -- \; y_d] = 2$$

$x_1 y_1 + x_2 y_2 + -- x_d y_d = 2$

$x \cdot y = 2$

$$\cos\theta = \frac{x \cdot y}{\|x\| \|y\|} = \frac{2}{2 \times 2} = \frac{1}{2}$$

$\theta = \cos^{-1}\left(\frac{1}{2}\right)$

$\theta = 60^\circ$

10) $f : \mathbb{R}^3 \to \mathbb{R}$

$f(x) = 3x_1^2 + 2x_1 x_2 - 4x_1 x_3 + 6x_3^2$

$x^T M x$

$M =$ symmetric

$$M = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 0 & 0 \\ -2 & 0 & 6 \end{bmatrix}$$

12) $A = \text{diag}(1,2,3,4,5,6,7,8)$

$|A| = 1\times2\times3\times4\times5\times6\times7\times8 = 8! = 40320$

$$A^{-1} = \begin{bmatrix} 1 & & & & & & & \\ & 1/2 & & & & 0 & & \\ & & 1/3 & & & & & \\ & & & 1/4 & & & & \\ & & & & 1/5 & & & \\ & 0 & & & & 1/6 & & \\ & & & & & & 1/7 & \\ & & & & & & & 1/8 \end{bmatrix}$$

14) $A = \begin{pmatrix} 1 & 2 \\ 3 & z \end{pmatrix}$ is singular

$|A| = 0$

$1\times z - 2\times3 = 0$

$$\boxed{z = 6}$$

Prashant Kolkur

Worksheet 6

1) a) Uncorrelated - Its not necessary that a heavy car is expensive

b) Positively correlated - More the seats then more the space and car is bigger and heavier

c) Negatively correlated - As the car age increases, its market price decreases.

2) $W = 0.9H$

corr = ?

Let μ_H be the mean of all husband ages

$\therefore \mu_W = 0.9\mu_H$

$\mathrm{Var}(H) = \sigma^2_H = \dfrac{\Sigma(H-\mu_H)^2}{N}$ where N = total number of data.

$$\mathrm{Var}(W) = \sigma^2_W = \frac{\Sigma(W-\mu_W)^2}{N}$$

$$= \frac{\Sigma(0.9H - 0.9\mu_H)^2}{N}$$

$$= (0.9)^2 \frac{\Sigma(H-\mu_H)^2}{N}$$

$$\sigma^2_W = (0.9)^2 \sigma^2_H$$

$$\mathrm{cov}(H,W) = \frac{\Sigma(H-\mu_H)(W-\mu_W)}{N}$$

$$= \frac{\Sigma(H-\mu_H)(0.9H - 0.9\mu_H)}{N}$$

$$= \frac{0.9\,\Sigma(H-\mu_H)^2}{N}$$

$$\mathrm{corr}(H,W) = \frac{\mathrm{cov}(H,W)}{\mathrm{std}(H) \times \mathrm{std}(W)}$$

$$= \frac{0.9\,\Sigma(H-\mu_H)^2/N}{\sqrt{\frac{\Sigma(H-\mu_H)^2}{N}} \times \sqrt{\frac{(0.9)^2\,\Sigma(H-\mu_H)^2}{N}}}$$

$$= \frac{\cancel{0.9}\,\cancel{\Sigma(H-\mu_H)^2/N}}{\cancel{0.9} \times \cancel{\Sigma(H-\mu_H)^2/N}}$$

$$\boxed{\mathrm{corr}(H,W) = 1}$$

3) $\mu_x = 2$ $std(x) = \sigma_x = 1$

(a) $\mu_y = 2$ $std(y) = \sigma_y = 0.5$

$corr(x,y) = -0.5$

Bivariate Gaussian Distribution $= (\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, cov(x,y))$

$\mu_x = 2$

$\mu_y = 2$

$\sigma_x^2 = 1^2 = 1$

$\sigma_y^2 = (0.5)^2 = 0.25$

$cov(x,y) = \sigma_x \sigma_y \times corr(x,y)$

$= 1 \times 0.5 \times (-0.5)$

$= -0.25$

$\Sigma = cov(matrix) = \begin{bmatrix} 1 & -0.25 \\ -0.25 & 0.25 \end{bmatrix}$

~~Meot~~ $\mu = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

3(b) $\mu_x = 1 \quad \sigma_x = 1$

$\mu_y = 1 \quad \sigma_y = 1$

$\sigma_x^2 = 1 \quad \sigma_y^2 = 1$

$$cov(x,y) = \frac{\Sigma(x-\mu_x)\,\Sigma(y-\mu_y)}{N}$$

$$= \frac{\Sigma(x-\mu_x)^2}{N} \quad (x = y \text{ given})$$

$$= \sigma_x^2$$

$$= 1$$

$$corr(x,y) = \frac{cov(x,y)}{\sigma_x \sigma_y} = \frac{1}{1 \times 1} = 1$$

$$\Sigma = Cov = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mu = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

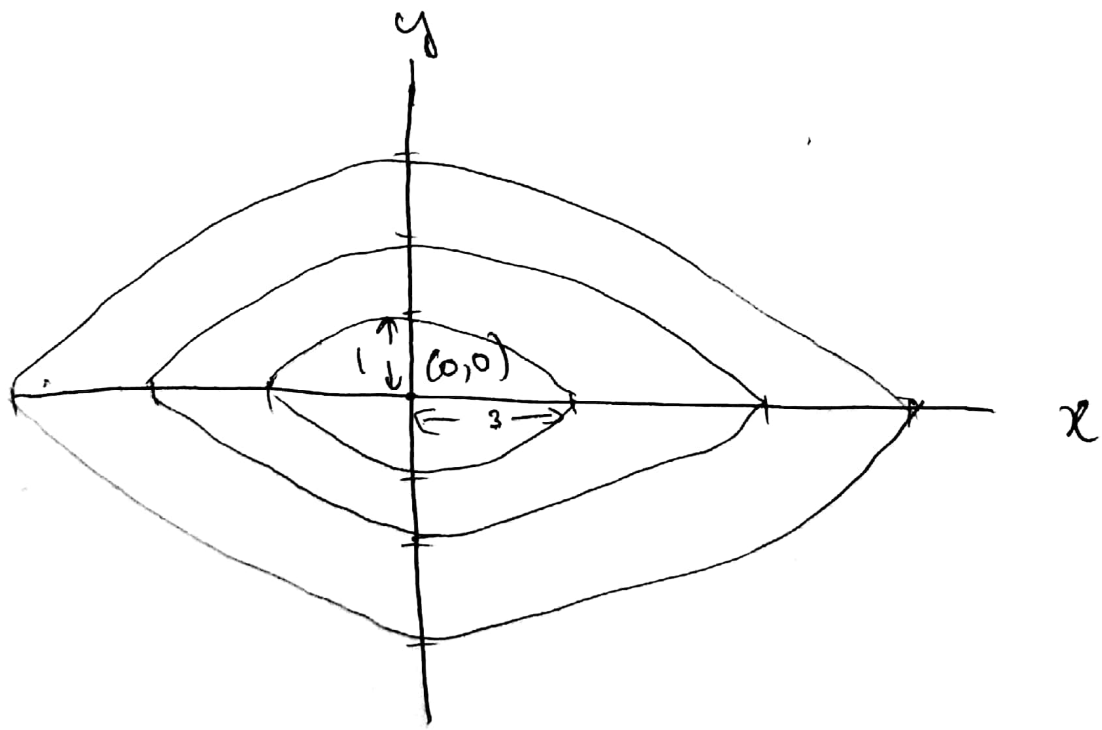
4) (a) $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Sigma = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$

$var(x) = \sigma_x^2 = 9$ $\sigma_x = 3$

$var(y) = \sigma_y^2 = 1$ $\sigma_y = 1$

$cov(x, y) = 0$

$$corr(x, y) = \frac{cov(x, y)}{\sigma_x \times \sigma_y} = 0$$



4) b) $\mu_x = 0$ $\qquad \mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\mu_y = 0$

$$\Sigma = \begin{pmatrix} 1 & -0.75 \\ -0.75 & 1 \end{pmatrix}$$

$\sigma_x^2 = 1 \qquad \sigma_y^2 = 1$

$\sigma_x = 1 \qquad \sigma_y = 1$

$\mathrm{cov}(x,y) = -0.75$

$$\mathrm{corr}(x,y) = \frac{\mathrm{cov}(x,y)}{\sigma_x \times \sigma_y} = \frac{-0.75}{1 \times 1} = -0.75$$