Types of Matrices

- A square matrix has the same number of rows as columns.

eg- $\begin{bmatrix} -2 & 8 \\ 3 & 1 \end{bmatrix}$ is a 2×2 square matrix

- The zero matrix has all elements equal to 0.

- A diagonal matrix is a square matrix that has all elements equal to 0 except for those on the leading diagonal.

eg- $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ is a 3×3 diagonal matrix

lagging diagonal. leading diagonal

Types of matrices (2)

- Another special type of matrix is the identity matrix
  - An identity matrix is a diagonal matrix with all its diagonal elements equal to 1.

eg- $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a 2×2 identity matrix.

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a 3×3 identity matrix.

→ Multiplying a 2×2 matrix A by I leaves A unchanged.

~~That~~ is, $\boxed{AI = A}$

Matrix arithmetic 1.

→ Matrix with the same dimensions can be added and substracted.

↳ We simply add or substract corresponding elements.

eg- Find A+B, B-C and A-B+C

$A = \begin{bmatrix} -2 & 8 \\ 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$ $C = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}$

$A+B = \begin{bmatrix} -2 & 8 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -2+3 & 8+6 \\ 3+2 & 1+4 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 14 \\ 5 & 5 \end{bmatrix}$

$$B-C = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3-4 & 6-2 \\ 2-(-1) & 4-3 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 3 & 1 \end{bmatrix}$$

$$A-B+C = \begin{bmatrix} -2 & 8 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -2-3+4 & 8-6+2 \\ 3-2+(-1) & 1-4+3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix}$$

* Multiplication by a Scalar

- Any matrix can be multiplied by a number k (a Scalar).

- We simply multiply each element of the matrix by k.

eg- Find 7M and -0.5M given $M = \begin{bmatrix} 2 & 0.5 & 6 \\ -3 & 0 & 1 \end{bmatrix}$

$$7M = 7\begin{bmatrix} 2 & 0.5 & 6 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 3.5 & 42 \\ -21 & 0 & 7 \end{bmatrix}$$

$$-\frac{1}{2}M = -\frac{1}{2}\begin{bmatrix} 2 & 0.5 & 6 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -0.25 & -3 \\ 1.5 & 0 & -0.5 \end{bmatrix}$$

Matrix ~~math~~ multiplication

- We can multiply matrix A by matrix B to form the product AB provided that number of columns of A = number of rows.

↳ The product AB will be another matrix C, with the same number of rows as A and the same number of columns as B.

$$A \times B = C$$

Order → $p \times q$ $q \times r$ $p \times r$

p = rows, q = columns

$p \times r$ ↳ matrix C having p rows and r columns.

eg- $A = \begin{bmatrix} -2 & 8 & 1 \\ 3 & 1 & 6 \end{bmatrix}$ $\underbrace{2 \times 3}$ $B = \begin{bmatrix} 1 & 2 \\ -4 & 3 \\ -2 & 5 \end{bmatrix}$ $\underbrace{3 \times 2}$

$A \times B$ = will be of order 2×2.

BA will be of order 3×3.

Notice that in General : $AB \neq BA$

To form a product, we multiply the rows of the first matrix by the columns of the second matrix.

$$A = \begin{bmatrix} -2 & 8 & 1 \\ 3 & 1 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ -4 & 3 \\ -2 & 5 \end{bmatrix}$$

find the products AB and BA

$$AB = \begin{bmatrix} -2 & 8 & 1 \\ 3 & 1 & 6 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ -4 & 3 \\ -2 & 5 \end{bmatrix} =$$

$$\begin{bmatrix} -2\times1 + 8\times-4 + 1\times-2 & -2\times2 + 8\times3 + 1\times5 \\ 3\times1 + 1\times-4 + 6\times-2 & 3\times2 + 1\times3 + 6\times5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 - 32 - 2 & -4 + 24 + 5 \\ 3 - 4 - 12 & 6 + 3 + 30 \end{bmatrix} = \begin{bmatrix} -36 & 25 \\ -13 & 39 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ -4 & 3 \\ -2 & 5 \end{bmatrix} \times \begin{bmatrix} -2 & 8 & 1 \\ 3 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1\times-2 + 2\times3 & 1\times8 + 2\times1 & 1\times1 + 2\times6 \\ -4\times-2 + 3\times3 & -4\times8 + 3\times1 & -4\times1 + 3\times6 \\ -2\times-2 + 5\times3 & -2\times8 + 5\times1 & -2\times1 + 5\times6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 + 6 & 8 + 2 & 1 + 12 \\ 8 + 9 & -32 + 3 & -4 + 18 \\ 4 + 15 & -16 + 5 & -2 + 30 \end{bmatrix} = \begin{bmatrix} 4 & 10 & 13 \\ 17 & -29 & 14 \\ 19 & -11 & 28 \end{bmatrix}$$

→ Sometimes it is possible to find AB but not BA.

→ If matrix A is 2×3 and matrix B is 3×1 then AB can be formed and will be 2×1

→ However, matrix BA cannot be formed since $1\neq2$.

$\boxed{\text{it is not possible to divide two matrices together.}}$

Inverse of a 2x2 matrix

- A matrix can be inverted if its determinant is not zero.

→ $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

→ Consider this 2×2 matrix A; its determinant is

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (a\times b) - (b\times c)$$

- If det $A\neq0$, then A is a non-singular matrix, and inverse exists.
- If det $A=0$, then A is a singular matrix, and inverse doesn't exists.

Find the determinant of a matrix A and State whether it is invertible or not.

$$A = \begin{bmatrix} -2 & 8 \\ 3 & 1 \end{bmatrix}$$

$\det(A) = (-2 \times 1) - (3 \times 8) = -2 - 24 = -26$

$\because \det(A) \neq 0$, so non-singular and invertible.

Calculate inverse of 2×2 matrix.

↳ Only square, non-singular matrix have an inverse.

↳ The inverse of a 2×2 matrix is another matrix, A^{-1}

→ Consider matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

↳ The inverse is given by $\boxed{A^{-1} = \frac{1}{ad - bc} \underbrace{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{\text{Adjoint}}}$

→ The inverse of a matrix has the properties $AA^{-1} = I$

$A^{-1}A = I$.

Find the inverse of matrix $A = \begin{bmatrix} 2 & 0.5 \\ 1 & 0 \end{bmatrix}$

$$A^{-1} = \frac{1}{2 \times 0 - 0.5 \times 1} \begin{bmatrix} 0 & -0.5 \\ -1 & 2 \end{bmatrix}$$

$$\Rightarrow \frac{1}{0 - 0.5} \begin{bmatrix} 0 & -0.5 \\ -1 & 2 \end{bmatrix}$$

$$\Rightarrow -2 \begin{bmatrix} 0 & -0.5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix}$$

→ Solving equations with matrices

→ Suppose A is a non-singular 2×2 matrix such that

$$AX = B$$

where B and X are both 2×1 matrices.

- We can then write

$A^{-1}AX = A^{-1}B$ i.e. we multiply on the left by A^{-1}.

$IX = A^{-1}B$ $(\because AA^{-1} = I)$.

Hence, $X = A^{-1}B$ since $IX = X$

Solving simultaneous linear equations

→ We can replace two linear simultaneous eq.

$ax + by = p$

$cx + dy = q$

by the single matrix eq = $\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} p \\ q \end{bmatrix}}_{B}$

So, providing A is non-singular, we can obtain the solution matrix

$\begin{bmatrix} x \\ y \end{bmatrix}$ using $X = A^{-1}B$

$$\boxed{\text{i.e} = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}}$$

eg - Use matrices to solve the following pair of simultaneous eq.

$2x + y = 9$

$x - 2y = -8$

[Q] » $\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 9 \\ -8 \end{bmatrix}}_{B}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-4 - 1} \begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 9 \\ -8 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-5} \begin{bmatrix} -2 \times 9 + (-1) \times (-8) \\ -1 \times 9 + 2 \times (-8) \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-5} \begin{bmatrix} -18 + 8 \\ -9 - 16 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-5} \begin{bmatrix} -10 \\ -25 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$\therefore (x, y) = (2, 5)$

Q. $y + 1 = 2x$

$2y + x = 8$

$2x - y = 1$

$x + 2y = 8$

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4 - (-1)} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2}{5}\times 1 + \frac{1}{5}\times 8 \\ -\frac{1}{5}\times 1 + \frac{2}{5}\times 8 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{2}{5} + \frac{8}{5} \\ -\frac{1}{5} + \frac{16}{5} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{10}{5} \\ \frac{15}{5} \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\therefore (x, y) = (2, 3)$$